Convex blocks and a coarse Dobrushin chain: a Poincaré bound for the core of the double-well path measure with constants polynomial in $\hbar$

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2026-10-17 (revised 2026-11-14; text revised 2026-11-15)

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Abstract

We consider the Euclidean path measure of the quartic double well $V(z) = (z^2-1)^2/8$ on a time lattice of spacing $\varepsilon = 1/2$. We restrict it to the box $[-x_c, 0]^k$ with $x_c = 0.89$ and take both outer boundary values in the well, $\alpha, \beta \le -x_c$. This "core" measure carries the weight $e^{-S/\hbar}$ and the Dirichlet form $(\hbar/\varepsilon)\sum_i |\partial_i f|^2$. We prove three Poincaré bounds on paper. Theorem B: a block-heat-bath path coupling and Ollivier's contraction-to-gap criterion turn a block Poincaré constant C_B and a Dobrushin–Shlosman type average influence $\delta < 1$ into $C_P \le C_B/(1-\delta)$ for every k. Theorem BP: for blocks of length at most 15 and boundary values in $(-\infty, 0]$, $C_B \le 9.18 + c_1\hbar^{-8}$ with $\ln c_1 \le 89.38$, for all $\hbar \in (0, 1]$. Here a hidden convexity of the action in the variables z_i^2 and a one-dimensional Hardy-type bound remove the exponential $e^{c/\hbar}$. Theorem CC: for the coarse chain $y_m = z_{8m}$ with Dobrushin column sum $\rho_{\mathrm{col}}(\hbar) < 1$, $C_P \le 9.18 + 455.84\, C_{\mathrm{site}}(\hbar)/(1 - \rho_{\mathrm{col}}(\hbar))$ with $C_{\mathrm{site}} \le e^{80.79}\hbar^{-8}$, uniformly in k. Theorems BP and CC also use the Bakry–Émery criterion on convex domains and the Brascamp–Lieb variance inequality, which are standard results and are cited. The hypothesis $\rho_{\mathrm{col}} < 1$ is established for every $\hbar \in (0, 1]$. On $[10^{-5}, 1]$ we use an interval-arithmetic certificate (directed outward rounding) over 239 intervals of $\hbar$ that cover the range without gaps; it gives $\rho_{\mathrm{col}} \le 0.400$. On $(0, 10^{-5}]$ the argument is on paper, up to one one-dimensional lower bound checked by interval arithmetic; it gives $\rho_{\mathrm{col}} \le 0.874$ (Proposition PIN). Consequently $C_P \le 9.18 + e^{88.99}\hbar^{-8}$ for all $\hbar \in (0, 1]$, uniformly in k, under the stated assumptions of the certificate and two cited inequalities. The algebraic steps are written in Lean 4; the analysis is on paper. The model is one-dimensional; nothing is claimed about four-dimensional lattice gauge theory.

1 Introduction

A Poincaré inequality for a Gibbs measure in a double-well potential typically has a constant of order $e^{\Delta/\hbar}$, with Δ the barrier height (Kramers). Restricting to a single well removes the barrier, but the restricted measure is not log-concave: the potential V has $V'' < 0$ on $(-1/\sqrt{3}, 0]$ and the box reaches up to the top of the barrier at 0. The standard routes from a Dobrushin–Shlosman condition to a Poincaré or log-Sobolev inequality bound the block constants by the Holley–Stroock perturbation lemma [HS]. They therefore pay a factor exponential in the interaction strength $1/(\varepsilon\hbar)$ (for instance the block constant $c_0 e^{4|X|\|\Phi\|}$ in [SZ]; see also [Ma]). The aim of this paper is a Poincaré constant for the single-well core that is *polynomial* in $\hbar$ and uniform in the length k.

A Poincaré bound of this form for the core (the *core Poincaré bound* below) is one of the inputs on which a conditional argument for the double-well path measure rests (§6.7). A companion note [KSR2] proves the opposite direction: an inverse squared spectral gap bounds the Langevin Poincaré constant of Euclidean path measures *from below*. The present paper concerns the *upper* bound for the core, which is a different object. The two results are not used in each other's proofs.

What the paper does.

- Theorem 3.1 (§3) is an abstract step: a path coupling for the block heat bath, combined with Ollivier's criterion [Ol, Prop. 29, Cor. 30], gives $C_P \le C_B/(1-\delta)$. No step introduces $e^{1/(\varepsilon\hbar)}$.
- Theorem 4.5 (§4) bounds the block constant polynomially in $\hbar$.
- Theorem 5.3 (§5) replaces the block chain by a one-site Dobrushin chain on every eighth site. It needs neither Theorem 3.1 nor Theorem 4.5 as stated, only their ingredients. Its single numerical hypothesis is $\rho_{\mathrm{col}}(\hbar) < 1$.
- Proposition 6.5 (§6) establishes $\rho_{\mathrm{col}}(\hbar) < 1$ for every $\hbar \in (0,1]$. Its statement names the strength of each part: interval arithmetic on $[10^{-5},1]$, paper plus one one-dimensional interval-arithmetic lower bound on $(0,10^{-5}]$ (Proposition 6.4).

Changes from the version of 2026-10-15. That version certified $\rho_{\mathrm{col}} < 1$ at 23 values of $\hbar$ by a double-precision computation with outward margins (not interval arithmetic) and left the values between the points, and $\hbar < 10^{-5}$, as a numerical reading. The present version replaces the point certificate by an interval-arithmetic certificate over 239 intervals covering $[10^{-5},1]$ (§6.3), adds Proposition 6.4 for $\hbar \le 10^{-5}$, and extends the conclusion to all of $(0,1]$ (Corollary 6.6). Two Lean modules were added (§7).

2 Setting

Fix $\varepsilon = 1/2$, $V(z) = (z^2-1)^2/8$, $x_c = 0.89$ and $\hbar \in (0,1]$. For $k \ge 1$ and outer boundary values $\alpha, \beta \in \mathbb{R}$ put $z_0 = \alpha$, $z_{k+1} = \beta$ and

$$S(z) = \sum_{i=0}^{k} \frac{(z_{i+1}-z_i)^2}{2\varepsilon} + \sum_{i=1}^{k} \varepsilon V(z_i), \qquad \mu_k^{\alpha,\beta,\hbar}(dz) = \frac{1}{Z}\, e^{-S(z)/\hbar}\, \mathbf{1}_{\Omega_k}(z)\, dz, \quad \Omega_k = [-x_c, 0]^k.$$

The Dirichlet form is $\mathcal{E}(f) = (\hbar/\varepsilon) \sum_{i=1}^{k} \mu |\partial_i f|^2$ (Neumann conditions on $\partial\Omega_k$), and $C_P(\mu)$ is the least constant with $\mathrm{Var}_\mu f \le C_P(\mu)\, \mathcal{E}(f)$ for Lipschitz f. The *core* is the family $Q_k = \mu_k^{\alpha,\beta,\hbar}$ with

$$\textbf{(END)} \qquad \alpha, \beta \le -x_c,$$

that is, both outer neighbours sit in the well. For $a > 0$ the boundary term is not compatible with the hidden convexity below. The restriction (END) is part of every statement.

The Hessian of S in the z coordinates is $\varepsilon^{-1} T_n + \varepsilon \operatorname{diag} V''(z_i)$ with $T_n = \mathrm{tridiag}(-1, 2, -1)$, and $V'' \ge V''(0) = -1/2$. Hence on any block of n consecutive free sites, for arbitrary real boundary values,

$$\operatorname{Hess} S \;\ge\; \kappa(n)\, I, \qquad \kappa(n) := \tfrac{15}{4} - 4\cos\!\big(\tfrac{\pi}{n+1}\big). \tag{1}$$

Numerically $\kappa(6) = 0.1461$, $\kappa_0 := \kappa(7) = 0.054482$ and $\kappa(8) = -0.00877$. Blocks of length ≤ 7 are uniformly convex in z; length 8 is the first that is not. Every construction below cuts the chain into pieces of length ≤ 7.

Hidden convexity (K′). For $z \le 0$ put $\rho_i = z_i^2$. Then $(z_i - z_j)^2 = \rho_i + \rho_j - 2\sqrt{\rho_i \rho_j}$ is jointly convex in (ρ_i, ρ_j) (the geometric mean is concave) and $\varepsilon V(-\sqrt{\rho}) = \varepsilon(\rho - 1)^2/8$ is convex. Hence, for boundary values $a, b \le 0$, $\tilde{S}(\rho) := S(-\sqrt{\rho})$ is convex, jointly in (ρ, a^2, b^2); for fixed a, b it is $(\varepsilon/4)$-strongly convex in $\rho \in [0, x_c^2]^n$, and its minimiser is unique (Lemma A.1, Appendix A). Its algebraic core is in Lean (§7). The hidden convexity is used below only through partial minimisation: if $\tilde{S}$ is jointly convex then $\sigma \mapsto \min_{\rho'} \tilde{S}(\rho', \sigma)$ is convex.

The coarse chain. For $k \ge 8$ let $M = \lfloor k/8 \rfloor$, $y_m := z_{8m}$ $(1 \le m \le M)$, and let $B_0 = \{1, \dots, 7\}$, $B_m = \{8m+1, \dots, \min(8m+7, k)\}$ be the blocks between coarse sites. With the coarse sites frozen, the blocks are independent (Markov property) and each is κ_0-convex by (1). The law ν of y is a nearest-neighbour chain on $[-x_c, 0]^M$ with effective pair interaction $w_B(a, b) = -\hbar \ln \int e^{-S_B(z; a, b)/\hbar} dz$. The conditional law $\nu(dy_m \mid y_{m-1} = a,\, y_{m+1} = b)$ is the marginal of the middle site z_8 of a chain $z_1, \dots, z_N$, $N = 8 + r$, with boundary values $z_0 = a \in [-x_c, 0]$ and $z_{N+1} = b$. In the interior, $r = 7$ and $b \in [-x_c, 0]$; at the right end of the chain, $r \in \{0, \dots, 7\}$ and $b = \beta \le -x_c$. The *influence* of this conditional law (a Dobrushin coefficient [Do]) is

$$\rho(a, b; r) := \partial_a \mathbb{E}[z_8 \mid a, b] = \frac{\mathrm{Cov}(z_1, z_8)}{\varepsilon \hbar},$$
$$\rho_{\mathrm{col}}(\hbar) := \sup_m \sum_{m' = m \pm 1} \sup_{a,b} \rho_{m' \leftarrow m} \;\le\; 2 \sup_{r,a,b} \rho(a, b; r).$$

What is shown and what is not. Theorems 3.1, 4.5 and 5.3 are proved on paper, with cited ingredients listed in Table 2. Theorem 5.3 is conditional on $\rho_{\mathrm{col}}(\hbar) < 1$. That hypothesis is established for every $\hbar \in (0, 1]$ (Proposition 6.5): on $[10^{-5}, 1]$ under the assumptions (A1)–(A5) of the interval-arithmetic certificate (§6.3), and on $(0, 10^{-5}]$ on paper, up to one one-dimensional lower bound checked by interval arithmetic. Accordingly, for the core in which both outer values lie in the well (the compartment-side core), the core Poincaré bound moves from an assumption to a statement that is conditional on the assumptions of the certificate and on two cited inequalities (Table 2). This bound does not become a paper theorem. In the box-side core of that argument, an outer neighbour lies beyond the summit and the conditional law is bistable with constants of type $e^{\Delta S/\hbar}$. This core is not covered by the hypothesis $\alpha, \beta \le -x_c$ and is not addressed here. The four-dimensional lattice gauge problem that motivates the question is not addressed (§9).

3 Theorem B: block heat bath and contraction

Fix a block length L and put $B_j = [j, j + L - 1] \cap [1, k]$ for $j = 2 - L, \dots, k$, so that there are $N = k + L - 1$ blocks, truncated blocks included, and every site lies in exactly L of them. Let E_B be the conditional expectation given the coordinates outside B, and $P = N^{-1} \sum_j E_{B_j}$ the block heat bath. For a site y and a block B having y as a boundary site, let $J_B(y) = \sup_s \sum_{i \in B} \partial_s \mathbb{E}_B^{s,\cdot}[z_i]$, the supremum running also over the other boundary value of B, and

$$\delta := \sup_y \frac{J_{\mathrm{right}}(y) + J_{\mathrm{left}}(y)}{L}.$$

Theorem 3.1 (Theorem B: from block constants to the chain)**.** *Let $\alpha, \beta \le -x_c$ and $\hbar > 0$. Assume* (BP) *every block B_j (length $\le L$, all boundary values in $(-\infty, 0]$) satisfies $\mathrm{Var}_{E_B}(f) \le C_B(\hbar/\varepsilon) \sum_{i \in B} E_B |\partial_i f|^2$, and* (DS) *$\delta < 1$. Then for every $k \ge 1$,*

$$C_P(\mu_k^{\alpha, \beta, \hbar}) \le \frac{C_B}{1 - \delta}.$$

The proof uses the FKG lattice condition and Holley's criterion for monotone coupling [Ho], [Pr] (standard results, cited) and Ollivier's criterion [Ol] (checked against the arXiv version).

Proof. Monotone coupling. The cross term of $-S/\hbar$ is $+z_i z_{i+1}/(\varepsilon\hbar)$, supermodular, and the reference measure is a product of intervals; the block measures satisfy the FKG lattice condition and the density ratio between boundary values $s < s'$ is $\exp((s'-s)z_{\text{first}}/(\varepsilon\hbar))$, increasing. By Holley's criterion and Strassen's theorem there is a coupling with $z \le z'$ coordinatewise, and then $W_1(\mu_B^s, \mu_B^{s'}) = \sum_i (\mathbb{E}^{s'} z_i - \mathbb{E}^s z_i) \le |s'-s|\, J_B(y)$.

Path coupling. If z, z' differ only at the site y, by d, then one step of P chooses a block containing y (L choices; the two configurations are coupled to agree), a block having y as a boundary site (two choices; the discrepancy is at most $d + J_B(y)d$), or another block (discrepancy d). Hence $W_1(P(z,\cdot), P(z',\cdot)) \le (1-\kappa)d$ with $\kappa = (L/N)(1-\delta)$. For general z, z', change one coordinate at a time (the box is a product) and use the triangle inequality.

Contraction to gap. P is self-adjoint and positive in $L^2(\mu)$. Since Ω_k is compact, $\|P^n f - \mu f\|_2 \le \operatorname{diam}(\Omega_k)(1-\kappa)^n \|f\|_{\text{Lip}}$ for Lipschitz f, so the spectral measure of $f - \mu f$ has no mass in $(1-\kappa, 1]$. By density of Lipschitz functions, $\langle f, (I-P)f\rangle \ge \kappa \operatorname{Var} f$ for all f [Ol, Prop. 29, Cor. 30].

Comparison. $\langle f, (I-P)f\rangle = N^{-1}\sum_j \mu(\operatorname{Var}_{B_j} f) \le (C_B L/N)\,\mathcal{E}(f)$ by (BP), since each site lies in L blocks. Combining, $\operatorname{Var} f \le C_B(1-\delta)^{-1}\mathcal{E}(f)$. □

Remark 3.2. The ingredients are of known types: path coupling under a Dobrushin–Shlosman type condition [Ma], comparison of block dynamics, and Ollivier's criterion. We do not know whether this combination appears in the literature in this form. In the references we were able to read, the block constant is controlled by a Holley–Stroock perturbation, which is where $e^{c/(\varepsilon\hbar)}$ enters. Here that step is the hypothesis (BP), and Theorem 4.5 supplies it. Truncated blocks enter the supremum in δ.

The algebraic steps are in Lean (§7): the one-step contraction inequality, the passage from spectral weight to gap, and the final comparison. The quantity δ itself is not bounded on paper in this paper; Theorem 5.3 below avoids it.

4 Theorem BP: block constants polynomial in $\hbar$

For a block of length $L \le 15$ with boundary values $a, b \in (-\infty, 0]$ write $\mu_{a,b} \propto e^{-S/\hbar}\mathbf{1}_{[-x_c,0]^L}$ and $C_B(\hbar) := \sup_{L\le 15,\, a,b\le 0} C_P(\mu_{a,b})$.

Lemma 4.1 (short blocks; BE_7)**.** *If $n \le 7$, then for every $\hbar > 0$ and all real boundary values, $C_P(\mu_{a,b}) \le \varepsilon/\kappa(n) \le \varepsilon/\kappa_0 = 9.1774$.*

Proof. By (1) the potential $S/\hbar$ is $(\kappa(n)/\hbar)$-strongly convex on the convex set $[-x_c, 0]^n$. The Bakry–Émery criterion for a strongly log-concave measure restricted to a convex set (Neumann boundary) gives $\operatorname{Var} f \le (\hbar/\kappa(n))\int |\nabla f|^2 d\mu$ [BGL], [BL, Thm. 4.1]. The criterion is standard and is cited; its application to the box $[-x_c, 0]^n$ through (1) is the step of this paper. □

Lemma 4.2 (variance split at the middle site)**.** *Let $8 \le L \le 15$, $j = \lceil L/2\rceil$, let m be the law of z_j and C_m its Poincaré constant for $\operatorname{Var}_m g \le C_m \int |g'|^2 dm$. Then $C_P(\mu_{a,b}) \le \varepsilon/\kappa_0 + (4/(\varepsilon\kappa_0^2))\, C_m/\hbar$, with $4/(\varepsilon\kappa_0^2) = 2695.2$.*

Proof. Both sides of j have length ≤ 7. By the law of total variance, $\operatorname{Var}_\mu f = m(\operatorname{Var}_{\mu_s} f) + \operatorname{Var}_m g$, $g(s) = \mu_s(f)$, where μ_s is the law with $z_j = s$ frozen, a product of two κ_0-convex blocks. Lemma 4.1 and tensorisation give the first term $\le (\hbar/\kappa_0)A$, $A = \sum_{i\ne j}\mu|\partial_i f|^2$. Differentiating, $g'(s) = \mu_s(\partial_j f) + \operatorname{Cov}_{\mu_s}(f, h)/(\varepsilon\hbar)$ with $h = z_{j-1} + z_{j+1}$. Cauchy–Schwarz

and $\mathrm{Var}_{\mu_s} h \le 2\hbar/\kappa_0$ give $|g'|^2 \le 2\mu_s|\partial_j f|^2 + 4A_s/(\varepsilon^2\kappa_0^2)$. Integrate against m and rescale to $\mathcal{E}$. □

Lemma 4.3 (one-dimensional Hardy-type bound; H1D)**.** *Let $m(dt) = Z^{-1}e^{-F(t)/\hbar}dt$ on $I = [-\ell, 0]$ with F continuous, and $\Delta := \sup_{y\le s\le x}\big[F(s) - \max(F(x), F(y))\big]$. Then $C_m \le 2\ell^2 e^{\Delta/\hbar}$.*

Proof. $\mathrm{Var}_m\, g = \iint_{y<x}(g(x)-g(y))^2 dm\, dm$ and $(g(x)-g(y))^2 \le \int_y^x |g'|^2 p \int_y^x p^{-1}$. By Fubini $\mathrm{Var}_m\, g \le \int |g'(t)|^2 p(t) K(t) dt$ with $K(t) = \iint_{y\le t\le x}\int_y^x p(x)p(y)/p(s)\, ds\, dx\, dy$. For $y \le s \le x$, $p(x)p(y)/p(s) \le e^{\Delta/\hbar}(p(x)+p(y))$, and $\iint_{y\le t\le x}(x-y)(p(x)+p(y)) \le 2\ell^2$. □

If F is quasi-convex ($\Delta = 0$), $C_m \le 2\ell^2$ independently of $\hbar$: the factor $e^{c/\hbar}$ can only come from a second valley of the free energy. The discrete form of Lemma 4.3 is in Lean; the Fubini step is on paper. We do not use Muckenhoupt's criterion [Mu].

Lemma 4.4 (quasi-convex free energy plus small oscillation; QC)**.** *Let $n = L - 1 \le 14$, $F_\hbar(s) = -\hbar \log \int e^{-S(z',s)/\hbar} dz'$ the free energy of the middle site, $F_0(s) = \min_{z'} S(z', s)$ and $R = F_\hbar - F_0$. Then* (a) *F_0 is quasi-convex on $[-x_c, 0]$, and* (b) *$R(s) - R(s') \le \hbar \log(K_1 \hbar^{-n/2})$ for all s, s', where $K_1 = (2\pi/\kappa_0)^{n/2} G^n e^{n(1+\Lambda/(2G^2))}$, $G = 6x_c/\varepsilon + \varepsilon \max|V'| = 10.776$, $\Lambda = 4/\varepsilon + \varepsilon \max V'' = 8.344$. Consequently $\Delta(F_\hbar) \le \hbar \log(K_1\hbar^{-n/2})$.*

Proof. (a) $F_0(s) = \tilde{F}_0(s^2)$ with $\tilde{F}_0(\sigma) = \min_{\rho'} \tilde{S}(\rho', \sigma)$ convex by (K′) and partial minimisation. The map $s \mapsto s^2$ is monotone on $[-x_c, 0]$, so the sublevel sets of F_0 are intervals. (b) Let $z^{(s)}$ be the minimiser (unique by Lemma 4.1), $g = \nabla_{z'} S(z^{(s)}, s)$ and $d = z' - z^{(s)}$. The KKT condition gives $\langle g, d\rangle \ge 0$ on the box, and $(\kappa_0/2)|d|^2 + \langle g, d\rangle \le S - F_0 \le (\Lambda/2)|d|^2 + \langle g, d\rangle$. The lower bound gives $\int e^{-(S-F_0)/\hbar} \le (2\pi\hbar/\kappa_0)^{n/2}$. On the cube obtained by moving each coordinate inward by $\hbar/G$ one has $\langle g, d\rangle \le n\hbar$, whence $\int e^{-(S-F_0)/\hbar} \ge (\hbar/G)^n e^{-n(1+\Lambda\hbar/(2G^2))}$ when $|a|, |b| \le 3x_c$. For $|a| > 3x_c$ the first coordinate of the minimiser is pinned at $-x_c$ and the same estimate holds with the first direction rescaled by $\hbar/g_1$. A quasi-convex function plus an oscillation of size D has barrier at most D. □

Theorem 4.5 (Theorem BP: Poincaré bound for blocks of length at most 15)**.** *For all $\hbar \in (0, 1]$, $L \le 15$ and $a, b \in (-\infty, 0]$,*

$$C_P(\mu_{a,b}) \le \frac{\varepsilon}{\kappa_0} + c_1\hbar^{-8}, \qquad \frac{\varepsilon}{\kappa_0} = 9.1774, \quad \ln c_1 \le 89.38,$$

and $C_P(\mu_{a,b}) \le \varepsilon/\kappa(L) \le 9.18$ if $L \le 7$. The bound uses (K′) *and Lemma 4.1, whose convex-domain Bakry–Émery step is cited.*

Proof. Lemmas 4.3 and 4.4 give $C_m \le 2x_c^2 K_1 \hbar^{-n/2}$, and Lemma 4.2 gives $C_P \le \varepsilon/\kappa_0 + (8x_c^2/(\varepsilon\kappa_0^2))K_1\hbar^{-1-n/2}$. For $n \le 14$ and $\hbar \le 1$, $\hbar^{-1-n/2} \le \hbar^{-8}$ and K_1 is largest at $n = 14$: $\ln(8x_c^2/(\varepsilon\kappa_0^2)) = 8.359$ and $\ln K_1 = 33.234 + 33.283 + 14.503 = 81.020$, so $\ln c_1 \le 89.379$. □

The constant is crude. (K′) is used once, in Lemma 4.4(a), to exclude a second valley of $F_\hbar$. Without it $\Delta = O(1)$ would be possible and the bound would be of Kramers type $e^{\Delta/\hbar}$. With Theorem 3.1, Theorem 4.5 gives $C_P(Q_k) \le (9.18 + c_1\hbar^{-8})/(1-\delta)$ whenever the averaged influence δ for $L = 15$ is < 1. We do not certify δ here and use Theorem 5.3 instead.

5 Theorem CC: the coarse Dobrushin chain

Let $M_7 = \varepsilon^{-1}T_7 - (\varepsilon/2)I$ (the worst-case Hessian of a block of length 7), so $\lambda_{\min}(M_7) = \kappa_0$, and

$$g_7 := (M_7^{-1})_{11} = (M_7^{-1})_{77} = \tfrac{1832324}{1770735} = 1.03478,$$

computed by the continued fraction of the tridiagonal Schur recursion (value in Lean). For a block B of length $n \le 7$, $(M_n^{-1})_{11} \le g_7$.

Lemma 5.1 (stochastic monotonicity of the coarse conditional laws)**.** *$\nu(\cdot \mid a, b)$ is increasing in a in the likelihood-ratio order, so that $W_1(\nu(\cdot \mid a, b), \nu(\cdot \mid a', b)) = |\mathbb{E}_{a'} y_m - \mathbb{E}_a y_m| \le \sup \rho \cdot |a' - a|$.*

Proof. The law of z_8 given a is the image of the Markov chain of Lemma 6.2 below, whose kernels are increasing in their starting point in the likelihood-ratio order (Lemma 6.3(a)). Composition of TP_2 kernels is TP_2 [Ka]. Hence the law of z_8 is increasing in a, and the monotone coupling realises W_1 as the difference of means. □

Theorem 5.2 (coarse chain)**.** *If $\rho_{\mathrm{col}} < 1$, then for every Lipschitz g on $[-x_c, 0]^M$,*

$$\operatorname{Var}_\nu g \le \frac{C_{\mathrm{site}}}{1 - \rho_{\mathrm{col}}} \frac{\hbar}{\varepsilon} \sum_m \nu |\partial_{y_m} g|^2, \qquad C_{\mathrm{site}} := \sup_{m,a,b} \frac{\varepsilon}{\hbar} C_P(\nu(\cdot \mid a, b)).$$

Proof. This is Theorem 3.1 with blocks of one coarse site. Let $P = M^{-1} \sum_m P_m$ be the one-site heat bath. If y, y' differ only at m, by d, choosing m couples them to agree, choosing $m' = m \pm 1$ costs at most $d + \rho_{m' \leftarrow m} d$ by Lemma 5.1, and other choices leave d. Hence $W_1(P(y, \cdot), P(y', \cdot)) \le (1 - (1 - \rho_{\mathrm{col}})/M) d$. Ollivier's criterion gives $\langle g, (I - P) g \rangle \ge ((1 - \rho_{\mathrm{col}})/M) \operatorname{Var}_\nu g$, and $\langle g, (I - P) g \rangle = M^{-1} \sum_m \nu(\operatorname{Var}_m g) \le M^{-1} C_{\mathrm{site}} (\hbar/\varepsilon) \sum_m \nu |\partial_m g|^2$. □

In Theorem 5.2, $C_P(\nu(\cdot \mid a, b))$ is the Poincaré constant of the one-site law in the sense $\operatorname{Var} g \le C \int |g'|^2$, as C_m in Lemma 4.2, and not the constant for the Dirichlet form of §2. The conditional law $\nu(\cdot \mid a, b)$ is the marginal of the middle site of a chain of at most 15 sites whose two halves have length ≤ 7. Lemmas 4.3 and 4.4 (with (K′) extended to the boundary values, §2) therefore give

$$C_{\mathrm{site}} \le 2\varepsilon x_c^2 K_1(14)\, \hbar^{-8} = e^{80.79} \hbar^{-8}. \tag{2}$$

Theorem 5.3 (Theorem CC: core Poincaré bound, conditional on (DOB))**.** *Let $\varepsilon = 1/2$, $V(z) = (z^2 - 1)^2/8$, $x_c = 0.89$, $\hbar \in (0, 1]$ and $\alpha, \beta \le -x_c$. Assume* (DOB) *$\rho_{\mathrm{col}}(\hbar) < 1$. Assume moreover the cited statements* BE_7 *(Lemma 4.1) and* BL*: for a measure on a convex set with* $\mathrm{Hess}(S/\hbar) \ge M/\hbar$, $\operatorname{Var}\langle e, z \rangle \le \hbar \langle e, M^{-1} e \rangle$ *[BL, Thm. 4.1] (a standard result, cited). Then for every $k \ge 1$,*

$$C_P(Q_k) \le \frac{\varepsilon}{\kappa_0} + c_6 \frac{C_{\mathrm{site}}(\hbar)}{1 - \rho_{\mathrm{col}}(\hbar)} \le 9.18 + \frac{455.84\, e^{80.79}\, \hbar^{-8}}{1 - \rho_{\mathrm{col}}(\hbar)}, \qquad c_6 := \max\Big(3, \frac{6 g_7}{\varepsilon^2 \kappa_0}\Big) = 455.84.$$

In particular, if $\rho_{\mathrm{col}}(\hbar) \le 1/2$ then $C_P(Q_k) \le 9.18 + e^{87.61} \hbar^{-8}$, uniformly in k.

Proof. For $k \le 7$ the core is one block and Lemma 4.1 gives $C_P \le 9.18$. Let $k \ge 8$, write $A = \sum_{i \text{ not coarse}} \mu |\partial_i f|^2$ and $B = \sum_m \mu |\partial_{8m} f|^2$, so $\mathcal{E}(f) = (\hbar/\varepsilon)(A + B)$.

1. $\operatorname{Var} f = \nu(\operatorname{Var}(f \mid y)) + \operatorname{Var}_\nu g$, $g(y) = \mathbb{E}[f \mid y]$.
2. Given y the blocks are independent and κ_0-convex; by Lemma 4.1 and tensorisation, $\nu(\operatorname{Var}(f \mid y)) \le (\varepsilon/\kappa_0)(\hbar/\varepsilon) A$.
3. $\partial_{y_m} g = \mathbb{E}[\partial_{8m} f \mid y] + [\operatorname{Cov}(f, z_{8m-1} \mid y) + \operatorname{Cov}(f, z_{8m+1} \mid y)]/(\varepsilon\hbar)$, since $\partial_{y_m} S = (2y_m - z_{8m-1} - z_{8m+1})/\varepsilon + \varepsilon V'(y_m)$ and terms depending on y_m alone do not contribute to covariances.
4. $X = z_{8m \pm 1}$ depends on one block B only, so $\operatorname{Cov}(f, X \mid y) = \operatorname{Cov}(f_B, X \mid y)$ with $f_B = \mathbb{E}[f \mid y, x_B]$. Then $\operatorname{Var}(f_B \mid y) \le (\hbar/\kappa_0) A_B$ (block Poincaré and Jensen) and $\operatorname{Var}(X \mid y) \le \hbar g_7$ (BL), so $(\operatorname{Cov}/(\varepsilon\hbar))^2 \le g_7 A_B/(\varepsilon^2 \kappa_0)$.

5. $(u+v+w)^2 \le 3(u^2+v^2+w^2)$ and the fact that each block touches at most two coarse sites give $\sum_m \nu|\partial_m g|^2 \le 3B + (6g_7/(\varepsilon^2\kappa_0))A \le c_6(A+B)$.

6. Theorem 5.2: $\mathrm{Var}_\nu\, g \le (C_{\mathrm{site}}/(1-\rho_{\mathrm{col}}))\, c_6\, (\hbar/\varepsilon)(A+B)$.

7. Adding 2. and 6. gives the claim; (2) gives the numerical form, and $\ln(c_6 e^{80.79}) = 86.9121$, so $86.9121 + \ln 2 < 87.61$. □

Theorem 5.3 uses neither Theorem 3.1 with $L = 15$ nor Theorem 4.5 as stated. It uses their ingredients: path coupling and Lemmas 4.1, 4.3, 4.4. The factor ρ_{col} enters only through $1/(1-\rho_{\mathrm{col}})$. Its numerical role is that of a single hypothesis, examined next.

6 The Dobrushin certificate

6.1 Why a certificate

A paper bound for ρ via a sup of $\partial_s \mathbb{E}[z_1 \mid z_8 = s]$ times $\mathrm{Var}(z_8)/(\varepsilon\hbar)$ does not close at small $\hbar$, and the curvature route cannot close it in principle:

Proposition 6.1 (curvature floor)**.** *For all boundary values with both neighbours of the middle site in the box and all $s \in [-x_c, 0]$, $F_\hbar''(s) \le 2/\varepsilon + \varepsilon \max V'' = 4.3441$. Hence every bound of the form $\mathrm{Var}_m \le \hbar/\kappa$ with $\kappa \le \inf F_\hbar''$ gives $\mathrm{Var}_m\,/(\varepsilon\hbar) \ge 0.4604$.*

Proof. $F_\hbar'' = \mathbb{E}[\partial_s^2 S \mid s] - \mathrm{Var}(\partial_s S \mid s)/\hbar \le 2/\varepsilon + \varepsilon V''(s)$, and $\max_{[-x_c,0]} V'' = V''(-x_c) = 0.68815$. □

At small $\hbar$ the middle site is pressed against the wall $-x_c$ by the slope $\varepsilon V'(-x_c) = 0.0463$, and its variance is of order $\hbar^2$. This is a boundary effect that curvature bounds do not see. The certificate below follows the law of the chain step by step instead.

6.2 Two lemmas

Let $\lambda = 1/(\varepsilon\hbar)$, $K(x) = e^{-\lambda x^2/2}$ and $[lo, hi] = [-x_c, 0]$. Define backward functions $g_N(v) = e^{-\varepsilon V(v)/\hbar} K(b-v)$, $g_i(v) = e^{-\varepsilon V(v)/\hbar} \int K(w-v) g_{i+1}(w)\, dw$.

Lemma 6.2 (influence recursion; Lemma R)**.** *Given $z_0 = a$, the chain $(z_1, \dots, z_N)$ is Markov with kernels $P_i(u, dv) = K(v-u)g_i(v)dv / \int K(v'-u)g_i(v')dv'$ (Doob h-transform). Put $f_8 = \mathrm{id}$, $f_{i-1} = P_i f_i$; then $\mathbb{E}[z_8 \mid a] = f_0(a)$, $\rho = f_0'(a)$, and*

$$f_{i-1}'(u) = \lambda \int f_i'(t)\, k_i(t;u)\, dt, \qquad k(t) := \mathrm{Cov}(V, \mathbf{1}\{V > t\}) = \bar F(t) \int_{lo}^{t} F + F(t) \int_{t}^{hi} \bar F \;\ge 0,$$

where $V \sim P_i(u, \cdot)$ with distribution function F and $\bar F = 1 - F$.

Proof. $\partial_u (P_i f)(u) = \lambda\, \mathrm{Cov}_{P_i(u,\cdot)}(V, f(V))$ because $\partial_u \log K(v-u) = \lambda(v-u)$; and $f(V) = f(lo) + \int f'(t)\mathbf{1}\{V > t\}dt$ gives $\mathrm{Cov}(V, f(V)) = \int f'(t)k(t)dt$ (Hoeffding [Hoe]). □

By induction $f_i' \ge 0$, without appeal to FKG. If $f_i' \le \Phi_i$ then $f_{i-1}' \le \lambda \int \Phi_i k_i$, and an upper bound for k follows from two-sided bounds $F_{lo} \le F \le F_{hi}$ alone, as a sum of products of nonnegative terms with no cancellation. Explicitly, for $t \in [t_j, t_{j+1}]$,

$$k(t) \le (1 - F_{lo}(t_j)) \int_{lo}^{t_{j+1}} F_{hi} + F_{hi}(t_{j+1}) \int_{t_j}^{hi} (1 - F_{lo}),$$

and the lower bound has the same form with the roles of F_{lo} and F_{hi} exchanged. For $V \equiv 0$ and an unbounded box the product of the eight factors is $\prod_{i=1}^{8} (16-i)/(17-i) = 1/2$, the exact value for the Gaussian bridge.

Lemma 6.3 (sandwiching; Lemma M)**.** (a) *$P_i(u,\cdot)$ is increasing in u in the likelihood-ratio order.* (b) *$g_i^{(b)}(v)$ is TP_2 in (v,b), so $P_i^{(b)}(u,\cdot)$ is increasing in b; the limit $b\to-\infty$ is the chain with $z_N = lo$.* (c) *Write $P_i(u,dw)\propto\exp(\lambda uw-\lambda w^2+\chi_i(w))$. If on each cell of a partition $\chi_i'\in[s_j^-,s_j^+]$, then replacing χ_i by the piecewise linear functions with slopes $s_j^\pm$ (jumping only upwards, resp. downwards, at nodes) gives laws $L^-\le P_i(u,\cdot)\le L^+$ in the likelihood-ratio order.*

Proof. (a) The density is $e^{\lambda uv}$ times a function of v. (b) g_N is $e^{\lambda bv}$ times functions of v and of b; $K(w-v)$ is TP_2; compositions of TP_2 kernels are TP_2 by the 2×2 Cauchy–Binet formula [Ka]. (c) The log-ratio of the densities is monotone on each cell and across nodes. □

The Gaussian factor $e^{\lambda uw-\lambda w^2}$ times an exponential of a linear function is integrated exactly on each cell by the error function, so the strong push towards the wall at small $\hbar$ costs nothing in cell width. The remaining coarseness is the slope range $s_j^+-s_j^-$.

The certificate inequality. On a cell $[u_l,u_{l+1}]$ of a and a cell $[b_{lo},b_{hi}]$ of b one has $F_{L_i^+(u_{l+1})}\le F_{P_i}\le F_{L_i^-(u_l)}$, with cell masses given by closed erf formulas. Inserting these into Lemma 6.2 and running Φ_i backward as cellwise constants yields

$$\sup_{a\in[-x_c,0],\,b\in[b_{lo},b_{hi}]}\rho(a,b)\le\max_l\Phi_0(l).$$

The variable a is covered by cells continuously and b by the endpoint slopes of its cells. No reading between grid points of a or b remains. Reversing the roles gives a lower bound, used only as a check.

6.3 The certificate in interval arithmetic

Every quantity that enters an inequality of the certificate is enclosed in an interval with outward rounding. The implementation (Python, double precision) rests on the following assumptions, which are part of the statement of Proposition 6.5(a).

(A1) IEEE 754 double precision with round-to-nearest; after each of the five basic operations the result r is widened to $r\pm(|r|2^{-51}+2^{-1074})$, which contains the exact result.

(A2) Constants and the coefficient tables are computed with the interval type of `mpmath` [mp] and stored as enclosures.

(A3) erfcx on $[0,16)$ is evaluated from degree-8 Taylor tables at the nodes $k/32$ with an enclosure of the remainder (the derivatives are completely monotone); for $x\ge16$ from the asymptotic series, whose remainder is bounded by the first omitted term [DLMF, §7.12(i)]; erf for $|x|\le0.3$ from the alternating series, and otherwise from erfcx.

(A4) Polynomials and sums are evaluated in floating point once, with an a priori bound on the rounding error of Horner's rule and of summation [Hi, (5.3)] added to the enclosure.

(A5) The box end is the double $x_c=\mathrm{fl}(0.89)=0.89000000000000001\ldots$; the certificate is a statement about the chain with this value of x_c.

Under (A1)–(A5) every number reported as an upper bound encloses the corresponding exact quantity of the chain with $x_c=\mathrm{fl}(0.89)$. The implementation was tested against the enclosure property on random inputs, with no violation in $4\cdot10^4$ cases. As negative controls, versions with one component deliberately broken (the widening removed, the remainder of exp or of erfcx dropped, the summation bound removed) were seen to violate this property. The enclosures are not a proof inside a proof assistant: the interval evaluations are outside Lean, and what is in Lean is the algebra that turns the table of intervals into the statement about the window (§7).

Intervals of $\hbar$: the centred form. Let $I = [\hbar_1, \hbar_2]$, $\delta = \hbar_2/\hbar_1 - 1$, $\lambda_k = 1/(\varepsilon\hbar_k)$ and $\lambda_0 = \sqrt{\lambda_1\lambda_2}$, so that $|\lambda - \lambda_0| \le (\delta/2)\lambda_0$ for $\lambda \in [\lambda_2, \lambda_1]$. The slope of the log-density of one step $P_i(u, \cdot)$ is $\lambda G_\lambda(w)$ with $G_\lambda(w) = u - 2w + M_\lambda(w) - \varepsilon^2 V'(w)$, where M_λ is the mean of the next site. In the decomposition

$$\lambda G = \lambda_0(u - 2w) + (\lambda - \lambda_0)\, G + \lambda_0(M - \varepsilon^2 V')$$

the first term is the reference Gaussian, integrated exactly by the error function as before. The width in $\hbar$ enters only through $(\lambda - \lambda_0)G$, whose size on a cell is at most $(\delta/2)\lambda_0 \max|G|$: the relative size $\delta/2$ of the true slope, with no power of λ. The corners of the product of the two intervals give the slope enclosure $s_j^{\pm}$ on each cell, and Lemma 6.3(c) applies unchanged (Lean: `centered_split`, `geomean_dev`, `mul_le_corners`). A cell of b is handled by two chains, with b at its lower and upper end (Lemma 6.3(b)), and within each chain only $\lambda \in I$ is enclosed. The recursion is run with λ_1 for the upper bound and λ_2 for the lower bound (the lower bound is a check only).

The table. The range $[10^{-5}, 1]$ is covered by 239 intervals, fixed in advance (no adaptation during the run). Their relative widths $\delta = \hbar_2/\hbar_1 - 1$ range from $5 \cdot 10^{-4}$ (next to $\hbar = 10^{-2}$) to 0.63. The covering has no gaps (Lean: `geometric_cover_count`, `cover_le`, `join_windows`). For each interval the certificate is run for every geometry of the coarse chain (interior $r = 7$ with the b-cells of §6.6; ends $r = 0, \dots, 7$ with $b \in (-\infty, -x_c]$), and the largest upper bound over the geometries is recorded. The run took 33 CPU-hours on one machine. The record (one entry per interval with its settings, cells, bounds and the worst geometry) was checked independently for completeness, for the absence of gaps, and for identical settings across all 239 entries.

6.4 Small $\hbar$ on paper

For $\hbar \le 10^{-5}$ the coarse sites are pressed against the wall $-x_c$ and the influence is controlled on paper.

Proposition 6.4 (PIN: the coarse chain for $\hbar \le 10^{-5}$)**.** *Let $\lambda = 1/(\varepsilon\hbar)$, $L_g := 1980/1217$, $h_{77} := 3664648/1770735$, $D_0 := 0.023126$ and $d(\hbar) := D_0 - 2\sqrt{7 h_{77}\varepsilon\hbar}$. For every $0 < \hbar \le 10^{-5}$ and every geometry of the coarse chain,*

$$0 \le \rho(a, b; r) \le \frac{2L_g\,\varepsilon\hbar}{d(\hbar)^2} + \textit{(tail)} \le 0.437, \qquad \textit{hence} \quad \rho_{\mathrm{col}}(\hbar) \le 0.874 < 1.$$

The bound is increasing in $\hbar$ and $\rho = O(\hbar)$ as $\hbar \to 0$. The value D_0 is a lower bound, checked by one-dimensional interval arithmetic, of the classical force $D(u) = 2u - m_7(u) - m_9(u) + \varepsilon^2 V'(u)$ on $[-x_c, -0.8]$ in the worst geometry; everything else is on paper, with Brascamp–Lieb for linear functionals as the cited ingredient; the comparison principle for the classical minimisers is Lemma B.1 (Appendix B).

Proof (outline). (L0) Convexity of short bridges. $\varepsilon^2 V'' \ge -1/8$, so the Hessian of a bridge of n sites with fixed ends is at least $H_0^{(n)} = \mathrm{tridiag}(-1, 15/8, -1)$, whose leading principal minors D_n are positive for $n \le 7$ and negative for $n = 8$ (Lean: `D_pos_le_seven`, `D_eight_neg`; $D_7 = 1770735/2097152$). Bridges of at most 7 sites are uniformly convex; the coarse spacing 8 confines the non-convex region to the inside of a bridge. *(L1)* Given $z_8 = u$ the two halves are independent bridges and a enters only through $e^{-\lambda(z_1 - a)^2/2}$, so $\rho = \lambda\,\mathrm{Cov}(z_8, g(z_8))$ with $g(u) = \mathbb{E}[z_1 \mid z_8 = u]$ increasing (TP_2), hence $\rho \ge 0$. *(L2) Brascamp–Lieb on the bridge.* $g'(u) = \lambda\,\mathrm{Cov}(z_1, z_7) \le \lambda\,\mathrm{Var}(z_1 + z_7)/4 \le L_g$ with $L_g = (h_{11} + h_{77} + 2h_{17})/4$, $h_{ij} = ((H_0^{(7)})^{-1})_{ij}$ (rational values in Lean: `h11_value`, `h17_value`, `Lg_value`). Thus

$\rho \le \lambda L_g \operatorname{Var}(z_8) \le \lambda L_g \mathbb{E}[(z_8 + x_c)^2]$. *(L3) Integration by parts on the box.* For $\mu \propto e^{-\lambda U}$ on a convex box with minimiser x^*, the divergence theorem (faces by Fubini, corners of measure zero, boundary term ≥ 0 by convexity) gives $\mathbb{E}\langle \nabla U, X - x^* \rangle \le n/\lambda$. With $\operatorname{Hess} U \ge H_0$ this yields $\mathbb{E}[z_7] \le m_7 + \sqrt{7h_{77}/\lambda}$ for the classical minimiser m_7 of the left bridge, and likewise on the right. *(L4), (L5)* The marginal density of z_8 is $\propto e^{-\lambda W(u)}$ with $W'(u) = 2u - \mathbb{E}_L[z_7] - \mathbb{E}_R[z_9] + \varepsilon^2 V'(u)$. By the comparison principle of Lemma B.1 (the minimiser of a bridge is nondecreasing in its boundary values), m_7 and m_9 are largest in the worst geometry ($a = 0$; $b = 0$ in the interior, $b = -x_c$ at the ends), so $W' \ge D(u) - 2e$ with $e = \sqrt{7h_{77}\varepsilon\hbar}$. On $[-x_c, -0.8]$, $D \ge D_0$ and the conditional law is stochastically dominated by an exponential of rate λd, so $\mathbb{E}[(z_8 + x_c)^2] \le 2/(\lambda d)^2$ there. On $[-0.8, 0]$ the mass is at most $x_c^3 \lambda M e^{-\lambda\Delta}$ with $\Delta \ge 0.0041$ at $\hbar = 10^{-5}$ and $\lambda\Delta \ge 819$, numerically zero. Assembling (the main term $2L_g\varepsilon\hbar/d^2$ in Lean: `assemble`, `final_bound`; the tail on paper): $\rho \le 2L_g\varepsilon\hbar/d^2 + \text{tail} \le 0.437$ at $\hbar = 10^{-5}$ ($e = 0.00851$, $d = 0.00610$), and the bound increases with $\hbar$. □

The loss against the measured values (about $1.4 \cdot 10^{-4}$ at $\hbar = 10^{-5}$) is three orders of magnitude. This loss comes from the $\sqrt{n}$ of (L3) and from not using the pinning of z_7. At $\hbar = 7 \cdot 10^{-6}$ the bound gives $\rho_{\mathrm{col}} \le 0.29$, and at $5 \cdot 10^{-6}$ it gives 0.13.

6.5 Proposition DOB

Proposition 6.5 (DOB: Dobrushin condition for the coarse chain)**.** (a) [interval arithmetic under (A1)–(A5)] *For every $\hbar \in [10^{-5}, 1]$ and every geometry of the coarse chain (interior $r = 7$, $b \in [-x_c, 0]$; ends $r = 0, \dots, 7$, $b \in (-\infty, -x_c]$) and every $a \in [-x_c, 0]$, $\rho(a, b; r) \le 0.1997$; hence $\rho_{\mathrm{col}}(\hbar) \le 0.400$. The 239 intervals all pass with largest upper bound 0.199666, attained in the interior geometry on $[1.325 \cdot 10^{-4}, 1.433 \cdot 10^{-4}]$.*

(b) [paper, plus one one-dimensional interval-arithmetic lower bound] *For every $\hbar \in (0, 10^{-5}]$, $\rho_{\mathrm{col}}(\hbar) \le 0.874$ (Proposition 6.4).*

(c) [paper, plus one floating-point value; not needed below] *Under the convention that for $k \equiv 0 \pmod 8$ the coarse sites are $y_m = z_{8m-1}$ (so that both neighbours of every coarse site lie in the box), for $\hbar \in [0.27, 1]$ one has $\sup \rho \le (V_{\mathrm{in}}(0.27)/(0.27\,\varepsilon))^2$ and hence $\rho_{\mathrm{col}} \le 0.858$, provided $V_{\mathrm{in}}(0.27) \le 0.0884$. Here $V_{\mathrm{in}}(\hbar) = \min_\theta \max(\mathbb{E}_{\nu_R}(X-\theta)^2, \mathbb{E}_{\nu_L}(X-\theta)^2)$ with $\nu_R \propto e^{-(x^2/\varepsilon + \varepsilon V(x))/\hbar}$ on $[\theta, 0]$ and $\nu_L \propto e^{-((x+x_c)^2/\varepsilon + \varepsilon V(x))/\hbar}$ on $[-x_c, \theta]$. The value $V_{\mathrm{in}}(0.27) = 0.08744$ was computed in floating point by the trapezoidal rule, without interval arithmetic.*

In particular $\rho_{\mathrm{col}}(\hbar) < 1$ for every $\hbar \in (0, 1]$.

Part (a) is the output of the certificate of §6.3. Its proof is the computation, whose inputs, outputs and hashes are distributed with the paper (§7). Table 1 summarises the record by decades of $\hbar$. The lower bounds (the same inequality reversed) are reported as a consistency check; in every interval lower $\le$ upper. The interval bounds are coarser than the point bounds of the previous version, which is the price of the width δ. They are not comparable cell by cell, since the cells differ. The method of the previous version (double precision with outward margins) was also run on the same 239 intervals with the same cells. For $\hbar \ge 10^{-4}$ the two upper bounds agree to within $4 \cdot 10^{-4}$ relatively. Below 10^{-4} the previous method is larger by up to 20% and, in four intervals near $3 \cdot 10^{-5}$ where its worst geometry is an end geometry, by up to a factor 3.4. In 203 of the 239 intervals the interval-arithmetic bound is the larger one, by at most $3.3 \cdot 10^{-4}$ relatively. Those values are not used.

Proof of (c). For a positive density p on $[l, r]$ with $(\log p)' \le c_R$ on (θ, r) and $(\log p)' \ge -c_L$ on (l, θ), $\mathbb{E}_p(X-\theta)^2 \le \max(\mathbb{E}_{\nu_R}(X-\theta)^2, \mathbb{E}_{\nu_L}(X-\theta)^2)$, by the Chebyshev order inequality on each side (the ratio p/ν decreases in $|X - \theta|$ while $(X-\theta)^2$ increases). For the middle site,

$-\partial_s S \le -2s/\varepsilon - \varepsilon V'(s)$ (both neighbours at 0) and $-\partial_s S \ge -[2(s+x_c)/\varepsilon + \varepsilon V'(s)]$ (both at the wall), which are the densities ν_R, ν_L. The same bounds hold for z_1 and z_7 in the left block. With the Hoeffding representation $\rho = (\varepsilon\hbar)^{-1} \int \varphi'(t) H(t)\, dt$, $\varphi(s) = \mathbb{E}[z_1 \mid z_8 = s]$, $H \ge 0$, $\int H = \mathrm{Var}_m$ [Hoe], and $\varphi' = \mathrm{Cov}(z_1, z_7 \mid s)/(\varepsilon\hbar) \le V_{\mathrm{in}}/(\varepsilon\hbar)$ by Cauchy–Schwarz, one gets $\sup \rho \le (V_{\mathrm{in}}(\hbar)/(\varepsilon\hbar))^2$. The comparison densities are monotone in $\hbar$ in the likelihood-ratio order, so $V_{\mathrm{in}}(\hbar)/(\varepsilon\hbar)$ is non-increasing and the value at $\hbar = 0.27$ suffices. The inequality $2(0.0884/0.135)^2 < 0.86$ is checked in Lean. The use of the convention for $k \equiv 0 \pmod 8$ in Theorem 5.3 only needs block lengths ≤ 7 and at most 14 sites around a coarse site, which the shifted division keeps. □

$\hbar$	intervals	worst interval	u-cells	upper	geometry	lower	CPU h
$[10^{-5}, 10^{-4}]$	44	$[8.846 \cdot 10^{-5},\ 9.313 \cdot 10^{-5}]$	2047	0.1989	interior	$1.3 \cdot 10^{-5}$	20.7
$[10^{-4}, 10^{-3}]$	38	$[1.325 \cdot 10^{-4},\ 1.433 \cdot 10^{-4}]$	1577	0.1997	interior	$3.3 \cdot 10^{-5}$	6.1
$[10^{-3}, 10^{-2}]$	52	$[1.201 \cdot 10^{-3},\ 1.282 \cdot 10^{-3}]$	996	0.1923	interior	$6.2 \cdot 10^{-4}$	3.8
$[10^{-2}, 10^{-1}]$	95	[0.01082, 0.01125]	420	0.1828	interior	$3.2 \cdot 10^{-3}$	2.5
$[10^{-1}, 1]$	10	[0.1, 0.11]	133	0.1361	interior	$5.9 \cdot 10^{-4}$	< 0.1

Table 1: The certificate of Proposition 6.5(a), by decades of $\hbar$: number of intervals, the interval with the largest upper bound for $\sup \rho$, the largest number of u-cells in the decade, that upper bound, the geometry attaining it, the corresponding lower bound, and the CPU time. Every one of the 239 intervals has upper bound ≤ 0.1997; in 232 of them the worst geometry is the interior one, in 7 an end geometry ($r = 0$ or $r = 3$). The cells near the wall at small $\hbar$ dominate the cost (§6.6).

Controls. With the division every fourth site (blocks of length 3) the point certificate of the previous version returned upper bounds 2.20, 10.6, 365 at $\hbar = 0.03, 0.01, 10^{-3}$, so it does not certify $\rho_{\mathrm{col}} < 1$ there (necessarily, being an upper bound; this is not a test of power). For the Gaussian chain ($V \equiv 0$), whose exact value is $1/2$, the interval-table certificate in double precision returns an enclosure containing $1/2$, which is loose. The negative controls of the interval-arithmetic implementation are described in §6.3.

Corollary 6.6. *Under the hypotheses of Theorem 5.3 other than* (DOB)*, for all $k \ge 1$: for $\hbar \in [10^{-5}, 1]$, with the qualification of Proposition 6.5*(a)*, $C_P(Q_k) \le 9.18 + e^{87.43}\hbar^{-8}$; for $\hbar \in (0, 10^{-5}]$, with Proposition 6.5*(b)*, $C_P(Q_k) \le 9.18 + e^{88.99}\hbar^{-8}$. In particular $C_P(Q_k) \le 9.18 + e^{88.99}\hbar^{-8}$ for every $\hbar \in (0, 1]$.*

Proof. $\ln(c_6 e^{80.79}) = 86.9121$; $1/(1-0.400) = e^{0.5109}$ and $1/(1-0.874) = e^{2.0715}$; the exponents are rounded up. □

6.6 Geometries and cells

The supremum defining ρ_{col} runs over every conditional law that occurs in the coarse chain. We cover them as follows. Interior coarse sites ($r = 7$): $b \in [-x_c, 0]$ is split into the nine cells with endpoints $-0.89, -0.6, -0.4, -0.25, -0.15, -0.08, -0.04, -0.02, -0.01, 0$. For the 44 intervals in $[10^{-5}, 10^{-4}]$ each of these cells is split in two. Coarse sites next to the end of the chain ($r = 0, \dots, 7$): $b = \beta \in (-\infty, -x_c]$ is split into cells whose lowest member is handled by the limit $b \to -\infty$ of Lemma 6.3(b); for $r \le 2$ the cells near $-x_c$ have width of order $\sqrt{\varepsilon\hbar}$. The first coarse site is the mirror image of the last. The variable $a \in [-x_c, 0]$ is covered by the u-cells of width about $c_1\sqrt{\varepsilon\hbar}$, with $c_1 \in [0.03, 0.2]$ depending on $\hbar$. The column "u-cells" of Table 1 is the largest number used in each decade.

At $\hbar = 10^{-5}$ the end geometries with $r \le 2$ need narrow b-cells: for $r = 0$ the centre $(u+b)/2$ of the last kernel moves by half the width of the b-cell, and a cell much wider than $\sqrt{\varepsilon\hbar}$ makes the sandwich of Lemma 6.3(c) much wider than the law itself. With widths 0.01 ($r = 0$) and 0.02 ($r = 1$), 71 cells in total for $r \le 2$, these geometries pass. The decade $[10^{-5}, 10^{-4}]$ accounts for 21 of the 33 CPU-hours of the run. This affects only the cost of the computation, not its validity: Lemma 6.3(c) holds for every partition.

6.7 Where the result sits in the conditional argument

In the conditional argument for the double-well path measure, the Poincaré constant $C_{\mathrm{core}}(\hbar) := \sup_{k,\alpha,\beta} C_P(Q_k)$ enters a second-order term of the form $C_{\mathrm{core}} \cdot D/\xi^2$. Here ξ is the tunnelling correlation length, which grows like $e^{c/\hbar}$. Any bound polynomial in $\hbar$ therefore makes that term vanish as $\hbar \to 0$, however crude its constant. Theorem 5.3 with Proposition 6.5 provides such a bound for every $\hbar \in (0, 1]$, under the qualifications stated there. The first step of this argument for the $0+1$-dimensional double well rests on seven provisional inputs. One of them is the core Poincaré bound, and another is a cross-term bound, which does not involve C_{core}. The present result bears on the core Poincaré bound only. That bound is supported there in two ways: the box-side core and the compartment-side core. The present result bears on the second, whose core has both outer values in the well. The other six inputs, including the cross-term bound, are unchanged.

7 What is machine-checked

The following Lean 4 modules (Mathlib [M]) contain the algebraic and finite steps; measure, integration, limits and the analytic lemmas are outside Lean.

- `LatticeGauge.HiddenConvexity`: `geom_mean_midpoint`, `act_eq_actRho`, `actRho_midpoint`, `act_min_unique` (K′ for $a, b \le 0$, midpoint form).
- `LatticeGauge.DoeblinPoincare`: `path_coupling_step`, `poincare_of_block_contraction`, `spectral_weight_zero`, `contraction_rate_pos` (Theorem 3.1).
- `LatticeGauge.BlockPoincare`: `kappa_seven_gt` ($\kappa_0 > 1/20$), `free_energy_quasiconvex`, `quasiconvex_add_osc`, `ratio_exp_bound`, `hardy_discrete`, `two_scale_assembly` (Lemmas 4.2–4.4).
- `LatticeGauge.CoarseChainPoincare`: `cov_le_lip_var`, `cross_term`, `sq_three`, `coarse_grad_sum`, `path_coupling_site`, `poincare_of_site_contraction`, `two_level_assembly`, `coarse_chain_poincare`, `g7_value`, `assembly_const` (Theorems 5.2, 5.3).
- `LatticeGauge.InfluenceRecursion`: `tp2_comp`, `cdf_upper`, `cdf_lower`, `k_upper`, `step_bound`, `gauss_product` (Lemmas 6.2, 6.3).
- `LatticeGauge.CenterCovariance`: `hoeffding_kernel`, `kernel_le_sup`, `chebyshev_double`, `large_hbar_column` (Proposition 6.5(c)).
- `LatticeGauge.DobrushinIntervalTable`: `centered_split`, `le_of_sq_le_sq_nonneg`, `geomean_dev`, `mul_le_corners`, `steps_growth`, `one_add_pow_le_exp`, `geometric_cover_count`, `cover_le`, `window_le_of_interval_table`, `column_lt_one_of_table`, `join_windows` (the centred form and the passage from the table of intervals to the window, §6.3).

- `LatticeGauge.PinnedCoarseChain`: `D_seven`, `D_six`, `D_pos_le_seven`, `D_eight_neg`, `h11_value`, `h17_value`, `Lg_value`, `assemble`, `final_bound` (the rational constants and the assembly of Proposition 6.4).

Each statement carries its hypotheses in the Lean signature. The independent re-check of the modules was completed on 2026-10-15, and that of the two modules added in this version on 2026-10-17 (every file checked with `lake env lean` in the same environment on a separate session, rc=0, standard axioms only). The bundle is listed at `https://computoergosum.com/principia/lean/principia-ym-coarse-chain-poincare-2026-10-15.tar.gz` (with a `.sha256` file at the same address) and the complete `#print axioms` output at `https://computoergosum.com/principia/lean/axioms-ym-coarse-chain-poincare-2026-10-15.txt`; both addresses were placed on 2026-10-15 and updated on 2026-10-17, after the independent re-check by Rin. To check: unpack, place the files (single files, not a Lake project) in a Lean 4 project that depends on Mathlib (Lean v4.33.1, Mathlib v4.33.1), run `lake exe cache get` there, then `LEAN_NUM_THREADS=1 lake env lean` on the file of each module above, and compare the trailing `#print axioms` lines with the log (expected: `propext`, `Classical.choice`, `Quot.sound` only). The ledger and build instructions of the site's Lean developments are at `https://computoergosum.com/en/principia/lean/index.html`. The certificate's programs, specification files and output files (one JSON per run, with every cell's upper and lower value), a manifest of the files behind Table 1, and SHA-256 hashes are distributed with the paper as the archive `ym-coarse-chain-poincare-certificates.tar.gz` (sha256 `6c7a6dca8a873862562061c096e37124a5cf2e4c68f1ba1cc043178380b9deba`).

8 Correspondence table

Line numbers refer to the files `LatticeGauge<Module>.lean` of the bundle; the third column gives `<Module>`: line of each declaration in the order of the second column.

paper	Lean declaration (namespace `LatticeGauge.`) and what is outside	file: line
(K′), §2	`actRho_midpoint`, `act_min_unique`; partial minimisation on paper	`HiddenConvexity`: 75, 115
Theorem 3.1	`path_coupling_step`, `spectral_weight_zero`, `poincare_of_block_contraction`; FKG, Strassen, spectral theorem on paper	`DoeblinPoincare`: 16, 54, 29
Lemma 4.1	`kappa_seven_gt`; Bakry–Émery cited	`BlockPoincare`: 26
Lemma 4.2	`two_scale_assembly`, `cross_term_sq`, `deriv_sq_split`; total variance and differentiation on paper	`BlockPoincare`: 101, 88, 96
Lemma 4.3	`hardy_discrete`, `ratio_exp_bound`; Fubini on paper	`BlockPoincare`: 137, 66
Lemma 4.4	`free_energy_quasiconvex`, `quasiconvex_add_osc`; Laplace-type bounds on paper	`BlockPoincare`: 37, 51
Lemma 5.1, Lemma 6.3(b)	`tp2_comp` (finite form)	`InfluenceRecursion`: 18
Theorem 5.2	`path_coupling_site`, `poincare_of_site_contraction`	`CoarseChainPoincare`: 92, 98
Theorem 5.3	`cross_term`, `sq_three`, `coarse_grad_sum`, `two_level_assembly`, `coarse_chain_poincare`, `g7_value`, `g7_lt`, `assembly_const`; BL cited	`CoarseChainPoincare`: 51, 47, 68, 78, 117, 128, 133, 136
Lemma 6.2	`cdf_upper`, `cdf_lower`, `k_upper`, `step_bound`, `gauss_product`	`InfluenceRecursion`: 49, 56, 60, 68, 76
Proposition 6.1	paper only	—
Centred form, §6.3	`centered_split`, `geomean_dev`, `mul_le_corners`, `steps_growth`, `one_add_pow_le_exp`	`DobrushinIntervalTable`: 20, 32, 51, 68, 79
Proposition 6.5(a)	`geometric_cover_count`, `cover_le`, `window_le_of_interval_table`, `column_lt_one_of_table`, `join_windows`; the interval evaluations outside Lean (interval arithmetic under (A1)–(A5))	`DobrushinIntervalTable`: 86, 97, 111, 118, 127
Proposition 6.4	`D_seven`, `D_six`, `D_pos_le_seven`, `D_eight_neg`, `h11_value`, `h17_value`, `Lg_value`, `assemble`, `final_bound`; Brascamp–Lieb, integration by parts and the comparison principle on paper; D_0 outside (one-dimensional interval arithmetic)	`PinnedCoarseChain`: 17, 20, 24, 29, 33, 36, 40, 45, 51
Proposition 6.5(c)	`hoeffding_kernel`, `kernel_le_sup`, `chebyshev_double`, `large_hbar_column`; the value $V_{\mathrm{in}}(0.27)$ outside	`CenterCovariance`: 44, 143, 186, 206

9 What is not claimed

1. **Assumptions of the certificate.** Proposition 6.5(a) rests on (A1)–(A5) of §6.3: the outward widening of the five IEEE operations under round-to-nearest, the interval type of `mpmath` for the constants, the remainder bounds for erfcx and erf [DLMF, §7.12(i)], the a priori rounding bounds for Horner's rule and summation [Hi, (5.3)], and $x_c = \mathrm{fl}(0.89)$. The interval evaluations are outside Lean; a proof-assistant verification of the computation is not claimed.

2. $x_c = \mathrm{fl}(0.89)$**.** The certificate concerns the chain whose box end is the double nearest to 0.89; the difference to 0.89 itself is not treated.

3. **Small** $\hbar$**.** On $(0, 10^{-5}]$ the bound $\rho_{\mathrm{col}} \le 0.874$ of Proposition 6.4 is three orders of magnitude above the measured influence. The loss is in the $\sqrt{n}$ of the integration-by-parts step and in not using the pinning of z_7. The constant $e^{88.99}$ of Corollary 6.6 comes from this part; on $[10^{-5}, 1]$ it is $e^{87.43}$.

row	content	status
END	outer boundary values $\alpha, \beta \le -x_c$	hypothesis of every statement
K′	convexity of $\tilde{S}$ in $\rho = z^2$, jointly with (a^2, b^2)	paper (Appendix A); algebraic core in Lean
BE_7	$C_P \le \varepsilon/\kappa(n)$ for blocks of length ≤ 7	paper ($\kappa_0 > 1/20$ in Lean) + cited [BoL, Prop. 3.1] (open convex domain, no boundary regularity required), [BGL]
BL	$\mathrm{Var}\langle e, z\rangle \le \hbar\langle e, M^{-1}e\rangle$ on convex sets	cited [BL, Thm. 4.1]; for the linear functionals used here it also follows from Prékopa–Leindler [BoL, (1.1)] in a few lines; g_7 in Lean
TP_2	monotonicity of the coarse conditional laws	paper [Ka]; finite form in Lean
FKG	monotone coupling for blocks (Theorem 3.1 only)	standard, cited [Ho], [Pr]
H1D, QC	Hardy-type bound; quasi-convexity plus oscillation	paper; discrete cores in Lean
OLL	contraction implies gap	cited, checked (arXiv version) [Ol]
ALG	coupling step, comparison, assembly, $c_6 < 497$	Lean (the value $c_6 = 455.84$ and the exponents 86.91, 87.61, 87.43, 88.99 derived from it are paper arithmetic)
DOB, $[10^{-5}, 1]$	$\rho_{\mathrm{col}} \le 0.400$	interval arithmetic over 239 intervals under (A1)–(A5); table-to-window algebra in Lean
DOB, $(0, 10^{-5}]$	$\rho_{\mathrm{col}} \le 0.874$	paper (Proposition 6.4) + one one-dimensional interval-arithmetic lower bound; rational constants and assembly in Lean
DOB, $[0.27, 1]$	$\rho_{\mathrm{col}} \le 0.858$	paper + one floating-point value (not needed for Corollary 6.6)

Table 2: Status of the rows on which Theorem 5.3 and Corollary 6.6 rest.

4. **Cited statements.** BE_7 (Bakry–Émery on convex domains, [BoL, Prop. 3.1]), BL (Brascamp–Lieb on convex domains) and the FKG/Holley step of Theorem 3.1 are standard results used as cited; they are not formalised in Lean. The comparison principle for the classical minimisers used in Proposition 6.4 is proved in Appendix B (Lemma B.1); it is not cited.

5. **Constants.** $e^{87}\hbar^{-8}$ is crude; measured block and site constants are of order 10^2 and below 0.27 respectively. Only a Poincaré inequality is proved, not a log-Sobolev inequality.

6. **One dimension.** The model is a one-dimensional chain (a 0 + 1-dimensional path measure); nothing is claimed about four-dimensional lattice gauge theory. The first step of the conditional argument for this double well keeps seven provisional inputs. Of these, this paper bears only on the core Poincaré bound, through its compartment-side support, and leaves the other six, including the cross-term bound, unchanged.

7. **No external review.** The proofs, the computations and this text were produced within the authors' group (§10).

10 Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic), run as the coding agent Claude Code. The observation, the informal proofs, the floating-point and interval-arithmetic computations and their programs, the Lean 4 development including the formalised statements, the literature search and this text were produced by Shiori over many sessions, which share no memory and rely on the written record; further sessions of Shiori reviewed the text against the Lean statements. Rin re-checked the Lean development from the distributed archive, confirmed that the declarations cited here are in it, and prepared the web edition. Kiichi, the human author, set the task and the working rules and decides what is published; he is responsible for this note.

This note has not been refereed. All review was done by further instances of the same model family, which share the authors' blind spots. What the Lean kernel checks is checked; everything else is the authors' word. In particular the analytic steps listed in §7 are checked on paper only. In a separate session, the record of the interval run (239 entries, no gaps, all passed, identical settings) was verified against the state file of the computation.

A Hidden convexity

Lemma A.1. *Let $n \geq 1$, $\varepsilon > 0$, $x_c > 0$, put $z_0 = a$, $z_{n+1} = b$ with $a, b \leq 0$, and let $S(z) = \sum_{i=0}^{n} (z_{i+1} - z_i)^2/(2\varepsilon) + \varepsilon \sum_{i=1}^{n} V(z_i)$ with $V(z) = (z^2 - 1)^2/8$, on the box $[-x_c, 0]^n$. Write $\rho_i = z_i^2$ $(0 \leq i \leq n+1)$ and $\tilde{S}(\rho) := S(-\sqrt{\rho})$.*

(a) $(\rho_0, \rho_1, \ldots, \rho_n, \rho_{n+1}) \mapsto \tilde{S}$ is convex on $[0, \infty)^{n+2}$.

(b) For fixed $a, b \leq 0$, $\tilde{S}$ is $(\varepsilon/4)$-strongly convex in $(\rho_1, \ldots, \rho_n) \in [0, x_c^2]^n$.

(c) For fixed $a, b \leq 0$, S has exactly one local minimum on $[-x_c, 0]^n$, and it is the global minimum.

Proof. For $z_i, z_{i+1} \leq 0$, $(z_{i+1} - z_i)^2 = \rho_{i+1} + \rho_i - 2\sqrt{\rho_i \rho_{i+1}}$. The geometric mean $(x, y) \mapsto \sqrt{xy}$ is concave on $[0, \infty)^2$, so $-\sqrt{\rho_i \rho_{i+1}}$ is convex. This includes the end terms $i = 0$ and $i = n$, in which $\rho_0 = a^2$ and $\rho_{n+1} = b^2$ enter as variables. The terms linear in ρ are convex, and $\varepsilon V(z_i) = \varepsilon(\rho_i - 1)^2/8$ has second derivative $\varepsilon/4$ in ρ_i. Since $1/(2\varepsilon) > 0$, $\tilde{S}$ is a sum of convex functions, which gives (a), and of $\varepsilon \sum_{i=1}^{n} (\rho_i - 1)^2/8$, which gives (b). For (c), the componentwise map $\rho \mapsto -\sqrt{\rho}$ is a homeomorphism of $[0, x_c^2]^n$ onto $[-x_c, 0]^n$, so local minima of S and of $\tilde{S}$ correspond. A local minimum of a strictly convex function on a convex set is the global minimum and there is at most one; S is continuous on the compact box, so the minimum exists. □

B Comparison of minimisers

Lemma B.1. *Let n, ε, x_c, V and S be as in Lemma A.1, and write $S_{a,b}$ for S with boundary values $z_0 = a$, $z_{n+1} = b$. Let $a \leq a' \leq 0$ and $b \leq b' \leq 0$, and let m and m' be the minimisers of $S_{a,b}$ and $S_{a',b'}$ on $[-x_c, 0]^n$ (unique by Lemma A.1(c)). Then $m_i \leq m'_i$ for every i.*

Proof. Write $x \wedge y$ and $x \vee y$ for the componentwise minimum and maximum; both lie in the box when x and y do. For a convex $h : \mathbb{R} \to \mathbb{R}$, the function $(u, v) \mapsto h(u - v)$ satisfies

$$h(u \wedge u' - v \wedge v') + h(u \vee u' - v \vee v') \leq h(u - v) + h(u' - v') \qquad (u, v, u', v' \in \mathbb{R}). \quad (3)$$

If (u, v) and (u', v') are ordered this is an equality. Otherwise, say $u \leq u'$ and $v \geq v'$; the left-hand arguments $u - v'$ and $u' - v$ lie in the interval $[u - v,\ u' - v']$ and have the same

sum as its end points, so (3) is the convexity of h. Now put $h(t) = t^2/(2\varepsilon)$ and compare $S_{a,b}(m \wedge m') + S_{a',b'}(m \vee m')$ with $S_{a,b}(m) + S_{a',b'}(m')$ term by term. The potential terms agree, since $\{m_i \wedge m_i',\, m_i \vee m_i'\} = \{m_i, m_i'\}$. Each inner pair term $h(z_{i+1} - z_i)$, $1 \le i < n$, obeys (3). The end terms are $h(z_1 - a)$ and $h(b - z_n)$. Since $a \le a'$ and $b \le b'$, the pairs (m_1, a), (m_1', a') have meet $(m_1 \wedge m_1', a)$ and join $(m_1 \vee m_1', a')$, and likewise at the other end, so (3) applies to them too. Hence

$$S_{a,b}(m \wedge m') + S_{a',b'}(m \vee m') \le S_{a,b}(m) + S_{a',b'}(m').$$

As m minimises $S_{a,b}$, $S_{a,b}(m) \le S_{a,b}(m \wedge m')$, so $S_{a',b'}(m \vee m') \le S_{a',b'}(m')$. Thus $m \vee m'$ is a minimiser of $S_{a',b'}$, and by uniqueness $m \vee m' = m'$, that is, $m \le m'$. □

In Proposition 6.4 the lemma is applied to the left bridge $z_1, \dots, z_7$ (boundary values a and $z_8 = u$) and to the right bridge (boundary values u and b): for fixed u and r, m_7 is largest at $a = 0$, and m_9 at the largest admissible b, namely $b = 0$ in the interior and $b = -x_c$ at the ends. For $r = 0$ there is no right bridge and $m_9 = b$.

Revision history. Revised 2026-11-14: the status of the main term and the tail in the assembly of Proposition 6.4, and of the value of c_6, is stated precisely; descriptive titles are added to the lettered statements; a reference to background material outside the note is removed, so that the note stands alone. Revised 2026-11-15: the text was rewritten for readability. In addition, statements about the certificate were corrected against its record: in the comparison with the method of the previous version after Proposition 6.5, four intervals (not seven), and the interval-arithmetic bound is the larger one in 203 of the 239 intervals (not in none); the number of interior b-cells below 10^{-4} and of end cells at $\hbar = 10^{-5}$ (71, not 50) in §6.6; the range of c_1; and the decimal expansion of fl(0.89) in (A5). The strength of the argument on $(0, 10^{-5}]$ is stated in the introduction and in §2 as in Proposition 6.5; the description of the hidden convexity in §2 follows Lemma A.1; a note on the meaning of C_P in Theorem 5.2 was added; 'upper' was corrected to 'lower' in the proof of Lemma 4.4; and the second source of the loss was added in §9. The statements of the theorems, the certified bounds and the tables are unchanged. The description of how the Lean files and the certificates are distributed and checked was corrected.

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