Tree-level blocking of an abelian-linearised lattice gauge action as a flux network: the coarse coupling Σf^2, the quartic contraction $\Sigma f^4/\Sigma f^2$, and their bounds

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Abstract

We study one step of block-spin blocking of a lattice gauge action at tree level, in the abelian linearisation. Here the link angles are restricted to one Cartan direction, the action is the quadratic part $\frac{\beta}{2}\sum_p \varphi_p^2$ of the heat-kernel or Wilson action plus a quartic perturbation $-\sigma\sum_p \varphi_p^4$, and the coarse action is the constrained minimum of the fine action at fixed coarse angles. One-loop corrections, non-abelian configurations and large fields are excluded from every statement. If a single coarse plaquette angle θ is fixed (Theorem F), the minimiser is θf with $f = Ps/\|Ps\|^2$, where s is the 2-chain spanned by the coarse plaquette and P is the orthogonal projection onto exact plaquette fields. In this case the coarse coupling is $\beta' = \beta\sum f^2$, the quartic coefficient is carried exactly to $\sigma\sum f^4$ up to $O(\theta^6)$, and the normalised quartic deviation contracts by $\lambda = \sum f^4/\sum f^2 \le \max f^2$. In a three-dimensional box with free boundary, when the coarse plaquette lies in the outer boundary of the box, $\sum f^2$ is the effective resistance of the dual network (Thomson's principle). If all coarse plaquettes of a periodic lattice are fixed at once, with 2^d blocks (Theorem G), the admissible data form $\operatorname{im} D_c$ (the coarse Bianchi identity and zero coarse flux through every coordinate 2-torus), the minimiser is $UG^+\theta$, the coarse action is $\frac{\beta}{2}\theta^{\mathsf{T}}G^+\theta$, and $\beta'(\theta) \ge 1/4$ in every direction. For the quartic part we show (Theorem G′) that, if the infinite-volume kernel R_∞ decays exponentially (argued here in outline only), a local representative of the fine-from-coarse map exists uniformly in the volume, so that the ℓ^4 cone of quartic deviations is mapped into itself with a volume-independent constant; for $d = 4$ a weighted Stein interpolation bound computed on a finite kernel gives the factor 0.823 for the first step. For several steps the answer depends on the blocking rule. With the sum of the two fine links as coarse link, the computed cone factor exceeds 1 at the third step (1.26) and the coarse covariance has no fixed point. With a gauge-covariant averaging rule, the covariance flow approaches a fixed point (computed exactly for $k \le 5$; no existence proof is given), and the computed lower bounds of the cone factor stay between 0.57 and 0.64, while an upper bound below 1 is open. We also classify the tail of non-local coarse weights (flat, exponential or power-law) according to how the coarse variable carries conserved quantities and constraints. The finite-dimensional inequalities and bookkeeping are checked in Lean 4 with Mathlib. Projections, interpolation and all numerical values are outside Lean.

1 Introduction

The question behind this paper is the existence of four-dimensional quantum Yang–Mills theory with a mass gap, in the formulation of the Clay Mathematics Institute [JW]. The

paper does not bear on that question itself (§11). It treats one ingredient of the block-spin approach to the ultraviolet problem of lattice gauge theory: the tree-level coarse action of one blocking step, after restricting to abelian configurations. The variational principles of electrical networks [Th, Ki, Co, DS], the Hodge decomposition [Ec, Ho] and the Migdal–Kadanoff recursion [Mi, Ka, It] are standard. The paper states the computations of the flux network as theorems with proofs, gives the tables on which it rests (§6), and checks the finite-dimensional steps in Lean 4 (§9).

2 Setting

Reader and scope. The paper is written for a reader who knows the variational principles of electrical networks (Thomson, Dirichlet, Kirchhoff [Th, Ki, Co, DS]) and block-spin transformations of lattice models. Its object is a finite-dimensional linear-algebra problem: the tree-level (saddle-point, $\beta \to \infty$) part of one blocking step of a lattice gauge action, after restricting to abelian configurations. The motivation is the block-spin approach to the ultraviolet problem of lattice Yang–Mills theory and the Migdal–Kadanoff approximation [Mi, Ka, It]. Nothing in this paper bears on the four-dimensional ultraviolet problem itself (§11).

Lattice and fields. Let B be either the cubical complex of a box $[0, n_1] \times \cdots \times [0, n_d]$ (*free boundary*) or the d-torus $(\mathbb{Z}/L)^d$ (*periodic*). We write C_k for the k-cells (links C_1, plaquettes C_2, cubes C_3), ∂_k for the boundary and $D = \partial_2^{\mathsf{T}}$ for the coboundary from links to plaquettes. Inner products are the standard ones on $\mathbb{R}^{C_k}$. An *abelian configuration* is a vector of link angles $A \in \mathbb{R}^{C_1}$ along one Cartan direction; its plaquette angles are $\varphi = DA$. The admissible plaquette fields are $\mathcal{A} := \operatorname{im} D \subset \mathbb{R}^{C_2}$. On a free-boundary box $\mathcal{A} = \ker D_3$ (closed 2-cochains); on a torus $\mathcal{A} \subsetneq \ker D_3$ with codimension $\binom{d}{2}$. We write P for the orthogonal projection onto $\mathcal{A}$.

Actions. The fine action is $V(\varphi) = \sum_p \big(\frac{\beta}{2}\varphi_p^2 - \sigma\varphi_p^4\big)$ with $\sigma \geq 0$. The Wilson action has $\sigma = \beta/24$, and the heat-kernel action, at tree level, has $\sigma = 0$. We write $\kappa := \sigma/\beta$ for the normalised quartic deviation from the heat-kernel class.

Coarse variables. *One coarse plaquette.* Take a closed 1-chain γ (for instance the boundary of a $b \times b$ square) and a 2-chain s with $\partial s = \gamma$ (the b^2 fine plaquettes of the square, coefficients ± 1, support S). The constraint "the holonomy angle of γ is θ" reads $\langle s, \varphi\rangle = \theta$. *All coarse plaquettes.* On a torus with even L and 2^d blocks, put $\theta_P := \langle s_P, \varphi\rangle$ for each coarse plaquette P, where s_P consists of the four fine plaquettes spanning the 2×2 square of P; this is the abelian linearisation of the coarse Wilson loop. The coarse link map C sums the two fine links of a coarse link, and $SD = D_c C$, where S is the matrix with rows s_P and D_c is the coarse coboundary.

One-face and all-face blocking. Fixing one coarse plaquette and leaving the rest of the lattice free gives the network of Theorem F (§3); fixing all coarse plaquettes simultaneously gives the network of Theorem G (§4). The two differ quantitatively: in four dimensions β'/β is 0.731 for one face on large periodic lattices and tends to 1 at long wavelength for all faces. Only the second is a blocking map.

What cannot be said (stated in advance). Every statement below is at tree level in the abelian linearisation. There is no statement at one loop or beyond: the one-loop numbers quoted in §6 and §7 are computed observations. There is no statement about non-abelian minimisers or about whether the global minimiser is abelian. There is no large-field statement:

Theorem F concerns a local minimum in a ball. There is no implication for the ultraviolet behaviour of four-dimensional Yang–Mills theory.

Relation to known results. Thomson's and Dirichlet's principles, Kirchhoff's laws and Rayleigh monotonicity [Th, Ki, DS] are classical. The identification of Theorem F(i) with them is a restatement. The Hodge decomposition on a finite cell complex [Ho, Ec] and the Moore–Penrose inverse are standard. Lemma 5.4 (blocking of a Gaussian field is a linear iteration of the covariance) is a standard fact of Gaussian block-spin transformations. The coarse quadratic form of Theorem G is presumably the quadratic part of the classical "perfect" (fixed-point) action of abelian lattice gauge theory; we have not checked the original sources. We have not found the remaining statements in the literature; they may be known.

3 Theorem F: one coarse plaquette

Let $u := Ps$ and assume $u \neq 0$ (the constraint is non-empty). Put $f := u/\|u\|^2$ and $\mathcal{A}_0 := \{\psi \in \mathcal{A} : \langle s, \psi\rangle = 0\}$. Every $\varphi \in \mathcal{A}$ with $\langle s, \varphi\rangle = \theta$ can be written uniquely as $\theta f + \psi$ with $\psi \in \mathcal{A}_0$.

Theorem 3.1 (Theorem F: one coarse plaquette)**.** *Restrict to abelian configurations at tree level ($\beta \to \infty$ leading term; one-loop corrections are not included), on a free-boundary box or a torus as in §2, with one coarse plaquette s such that $Ps \neq 0$.*

(i) For every $\theta \in \mathbb{R}$ the problem $\min\{\frac{1}{2}\|\varphi\|^2 : \varphi \in \mathcal{A},\ \langle s, \varphi\rangle = \theta\}$ has exactly one minimiser, $\varphi^* = \theta f$, with value $\theta^2/(2\|u\|^2)$; $f \in \mathcal{A}$, $\langle s, f\rangle = 1$ and $f \perp \mathcal{A}_0$. On a free-boundary box $\|u\|^2 = \min_{c\in\mathbb{R}^{C_3}} \|s - \partial_3 c\|^2$ (Dirichlet's principle), and for $d = 3$, when S lies in the outer boundary of the box, $\sum f^2$ is the effective resistance from S to the remaining outer faces of the dual network (cubes as nodes, plaquettes as unit resistors) and f is the unit current (Thomson's principle).

(ii) For the action $\frac{\beta}{2}\sum\varphi^2$ the coarse action is $\frac{\beta'}{2}\theta^2$ with $\beta' = \beta\sum_p f_p^2$ and $\sum f^2 = 1/\|u\|^2 \geq 1/|S|$. On a free-boundary box in $d = 3$, if each plaquette of S is joined to the outer faces by edge-disjoint dual paths π_p of lengths ℓ_p, then $\sum f^2 \leq \sum_p \ell_p/|S|^2$ (this bound is added in this paper). Without such a condition $\sum f^2 \leq 1$ fails in general (it fails for $|S| = 1$ when the plaquette is a face of a cube of the box).

(iii) Let $m = \max_p |f_p|$, $\|f^3\| = (\sum_p f_p^6)^{1/2}$, $r > 0$, and $S'_r(\theta) := \inf\{V(\theta f + \psi) : \psi \in \mathcal{A}_0,\ \|\psi\| \leq r\}$. If $\sigma(6\theta^2 m^2 + 4|\theta| mr + r^2) \leq \beta/4$, then

$$\tfrac{\beta}{2}\textstyle\sum f^2\,\theta^2 - \sigma\sum f^4\,\theta^4 - \dfrac{16\sigma^2\|f^3\|^2}{\beta}\theta^6 \;\leq\; S'_r(\theta) \;\leq\; \tfrac{\beta}{2}\sum f^2\,\theta^2 - \sigma\sum f^4\,\theta^4.$$

Hence, up to $O(\theta^6)$, the coarse quartic coefficient is $\sigma' = \sigma\sum f^4$ and $\kappa' = \kappa\,\lambda$ with $\lambda := \sum f^4/\sum f^2$.

(iv) $\lambda \leq \max_p f_p^2$, and $\lambda \geq 1/(|S|^3\sum f^2)$. In particular, for $|S| = 4$ and $\max f^2 = 1/16$, $1/(64\sum f^2) \leq \lambda \leq 1/16$.

Proof. (i) For $\varphi \in \mathcal{A}$, $\langle s, \varphi\rangle = \langle s, P\varphi\rangle = \langle u, \varphi\rangle$, so Cauchy–Schwarz gives $\theta^2 \leq \|u\|^2\|\varphi\|^2$, with equality at $\theta u/\|u\|^2$. The constraint set is closed under midpoints, and the parallelogram identity gives uniqueness. For $\psi \in \mathcal{A}_0$, $\langle f, \psi\rangle = \langle s, \psi\rangle/\|u\|^2 = 0$. On a free-boundary box $H_2 = 0$, so $\mathcal{A}^\perp = \ker\partial_2 = \operatorname{im}\partial_3$ and $u = s - \Pi_{\operatorname{im}\partial_3} s$. In $d = 3$ the cubes are the nodes of the dual graph, $\varphi \in \ker D_3$ is Kirchhoff's current law, and $\langle s, \varphi\rangle = \theta$ is the total current from S to the other outer faces; minimising $\sum\varphi^2$ is Thomson's principle [Th, DS]. (ii)

The minimum value is $\beta\theta^2/(2\|u\|^2)$ and $\sum f^2 = \|u\|^2/\|u\|^4$. The lower bound is Cauchy–Schwarz on $\sum_{p\in S} s_p f_p = 1$. A trial unit flow sending $1/|S|$ along each π_p lies in $\mathcal{A}$ and has energy $\sum \ell_p/|S|^2$. For $|S| = 1$, $s \notin \mathcal{A}$, so $\|u\| < \|s\| = 1$. (iii) The upper bound is $\psi = 0$. Since $\langle f, \psi\rangle = 0$, $\sum(\theta f + \psi)^2 = \theta^2 \sum f^2 + \|\psi\|^2$. Expanding the quartic and using $|\sum f^3\psi| \le \|f^3\|\|\psi\|$, $\sum f^2\psi^2 \le m^2\|\psi\|^2$, $|\sum f\psi^3| \le mr\|\psi\|^2$, $\sum \psi^4 \le r^2\|\psi\|^2$, we obtain

$$V(\theta f + \psi) \ge \tfrac{\beta}{2}\textstyle\sum f^2\theta^2 - \sigma\sum f^4\theta^4 + \tfrac{\beta}{4}x^2 - ax, \qquad x = \|\psi\|,\ a = 4\sigma|\theta|^3\|f^3\|,$$

and $\frac{\beta}{4}x^2 - ax \ge -a^2/\beta$. (iv) $f_p^4 \le (\max f^2)f_p^2$. For the lower bound apply $(\sum_S g)^4 \le |S|^3 \sum_S g^4$ to $g_p = s_p f_p$, $\sum_S g = 1$. □

The finite-sum steps of the proof are checked in Lean (§9): Cauchy–Schwarz with equality, midpoint uniqueness, the energy split, the quadratic lower bound, and both inequalities of (iv). The projection P, Dirichlet's principle and the network interpretation are not checked in Lean.

Proposition 3.2 (Monotonicity and the coarse plaquette)**.** *Same restrictions as Theorem 3.1.*

(a) Enlarging the complex that contains γ (adding cells while keeping s) does not decrease $\sum f^2$.

(b) If the four plaquettes of a 2×2 coarse plaquette carry flux $1/4$ each and $\max_p f_p^2 = 1/16$ (the largest flux sits on the four plaquettes of the coarse plaquette; this holds in every computed box by a box-by-box check, and no general proof is known), then $1/(16d-8) \le \lambda \le 1/16$.

(c) In a d-dimensional box with an interior $L\times L$ coarse plaquette, $\sum f^2(L) \le \frac{d-1}{2L} + \frac{1}{2L^2}$ and $\lambda(L) \ge \frac{2}{L^4((d-1)L+1)}$.

Sketch. (a) Let $B \subset B'$ be complexes containing γ and s, with admissible fields $\mathcal{A} = \operatorname{im} D$ and $\mathcal{A}' = \operatorname{im} D'$. If $\varphi' = D'A' \in \mathcal{A}'$, its restriction to the plaquettes of B is $D(A'|_B) \in \mathcal{A}$, because every link of a plaquette of B lies in B. The restriction keeps $\langle s, \varphi'\rangle$, since s is supported in B, and does not increase the norm. Hence the minimum of $\|\varphi\|^2$ under $\langle s, \varphi\rangle = \theta$ over $\mathcal{A}$ is at most the same minimum over $\mathcal{A}'$. By Theorem 3.1(i),(ii) this minimum is $\theta^2/\|u\|^2 = \theta^2 \sum f^2$ on each complex (the proof of (i) uses only $\mathcal{A} = \operatorname{im} D$), so $\sum f^2$ on B is at most $\sum f^2$ on B' (Rayleigh monotonicity [DS]). (b) The upper bound is Theorem 3.1(iv) under the assumption $\max_p f_p^2 = 1/16$. For the lower bound, the four plaquettes of s alone give $\sum f^4 \ge 4\cdot 4^{-4} = 1/64$, and an explicit trial unit flow from s into its neighbouring cubes gives $\sum f^2 \le (16d-8)/64$ on any complex containing s and its neighbouring cells, including the infinite lattice. The trial flow is written out in Appendix A. (c) A trial flow that spreads the unit flux through s over the faces adjacent to its perimeter gives the upper bound; the lower bound is Theorem 3.1(iv) with $|S| = L^2$. □

Remark 3.3. In (b) and (c) the arguments are on paper only. Part (c) shows that in $d \ge 3$ the coupling of one large coarse plaquette is of perimeter type, $\sum f^2 = O(1/L)$, in contrast to the area-type count of Migdal–Kadanoff.

Remark 3.4 (Worked example: the $2\times 2\times 1$ box). Take the free-boundary box $2\times 2\times 1$ in $d = 3$ and let s be the four bottom plaquettes. By symmetry and Kirchhoff's laws the unit flux is $1/4$ on each bottom plaquette, $1/12$ on each of the four top plaquettes and the eight plaquettes of the outer sides, and 0 on the four inner walls. One checks that this field is closed on each of the four cubes and orthogonal to $\mathcal{A}_0$. Then

$$\textstyle\sum f^2 = 4\cdot\frac{1}{16} + 12\cdot\frac{1}{144} = \frac{1}{3}, \qquad \sum f^4 = \frac{4}{256} + \frac{12}{20736} = \frac{7}{432}, \qquad \lambda = \frac{7}{144} \approx 0.0486,$$

so $\beta' = \beta/3$ at tree level and Theorem 3.1(iv) reads $3/64 \le 7/144 \le 1/16$. The same values are obtained by the computation of Table 1. For U(1) with the Villain action the one-step map closes on $\beta' = \beta/3$ with vanishing quartic source (paper argument via Poisson summation, and computed).

4 Theorem G: all coarse plaquettes at once

Work on $(\mathbb{Z}/L)^d$ with L even, 2^d blocks and the coarse angles $\theta_P = \langle s_P, \varphi\rangle$ of §2. Put $u_P := Ps_P$, $G := SPS^{\mathsf{T}}$ (so $G_{PQ} = \langle u_P, u_Q\rangle$), $B := G^+$ (Moore–Penrose) and $F := UB$, where U has columns u_P.

Theorem 4.1 (Theorem G: all coarse plaquettes at once)**.** *Restrict to abelian configurations at tree level, periodic lattice, 2^d blocks, coarse angles linearised as above.*

(i) The constraint $\{\varphi \in \mathcal{A} : S\varphi = \theta\}$ is non-empty if and only if $\theta \in \operatorname{im} D_c$ (the coarse Bianchi identity on every coarse cube and zero coarse flux through every coordinate 2-torus). In particular a single coarse plaquette cannot be twisted while all others are kept at 0.

(ii) For $\theta \in \operatorname{im} D_c$ the minimiser of $\frac{1}{2}\|\varphi\|^2$ is unique, $\varphi^* = F\theta$, and the coarse action is $\frac{\beta}{2}\theta^{\mathsf{T}}B\theta$; $F^{\mathsf{T}}F = B$.

(iii) In coarse link variables, $\min\{\frac{1}{2}\|DA\|^2 : CA = a\} = \frac{1}{2}a^{\mathsf{T}}Qa$ with $Q = D_c^{\mathsf{T}}BD_c$, and $Qa = -\mu_a$ for the Lagrange multiplier μ_a of the constraint.

(iv) Because the θ_P are not independent coordinates, off-diagonal entries of B and the split of a quartic form into "diagonal" and "cross" parts depend on the representation. Representation-independent quantities are the spectrum of B on $\operatorname{im} D_c$ and, for $\theta \in \operatorname{im} D_c$, $\beta'(\theta) := \theta^{\mathsf{T}}B\theta/\sum_P \theta_P^2$, $\mu(\theta) := \sum_q (F\theta)_q^4/\sum_P \theta_P^4$ and $\lambda(\theta) := \mu(\theta)/\beta'(\theta)$.

(v) For all $\theta \in \operatorname{im} D_c$, $\beta'(\theta) \ge 1/4$.

Proof. (i) $\theta = SDA = D_c(CA) \in \operatorname{im} D_c$. Conversely, given a, put a_ℓ on the first fine link of each coarse link and 0 elsewhere. (ii) A minimum-norm solution lies in $\mathcal{A}\cap(\mathcal{A}\cap\ker S)^\perp = \operatorname{im} U$. Writing $\varphi = Uc$ gives $Gc = \theta$, and $\operatorname{im} G = \operatorname{im}(SD) = \operatorname{im} D_c$ because C is onto. Then $c = B\theta + \ker G$, $U(\ker G) = 0$, $\|\varphi^*\|^2 = \theta^{\mathsf{T}}BGB\theta = \theta^{\mathsf{T}}B\theta$. Uniqueness is the midpoint argument of Theorem 3.1. (iii) The inner minimum is invariant under $\ker D_c$, so it equals $\frac{1}{2}\min\{\theta^{\mathsf{T}}B\theta : \theta = D_c a\}$. The multiplier identity is the KKT system $D^{\mathsf{T}}DA + C^{\mathsf{T}}\mu = 0$. (v) The four-plaquette sets s_P are pairwise disjoint, so $\sum_q \varphi_q^2 \ge \sum_P \sum_{q\in s_P} \varphi_q^2 \ge \sum_P \theta_P^2/4$ by four-term Cauchy–Schwarz. □

Part (v) and the four-term Cauchy–Schwarz are checked in Lean (`fine_energy_lower`, under the hypothesis that the four-plaquette sets are injectively placed). Parts (i)–(iv) are on paper. If all coarse plaquettes were independent ($P = I$), then $G = 4I$ and, after the bond-moving factor 2^{d-2}, $\beta'/\beta = 2^{d-4}$ and $\lambda = 1/16$. Theorem G replaces "$\mathcal{A}$ is everything" in the Migdal–Kadanoff count by $\mathcal{A} = \operatorname{im} D$.

Computed consequences. (Computed with two implementations, projection plus Gram matrix and Lagrange multipliers, agreeing to 1.3×10^{-12}; periodic lattices 4^3–48^3 and 4^4–16^4 (20^4 where stated); same restrictions.) The coarse action is not a plaquette action: on $\operatorname{im} D_c$ the eigenvalues of B lie in $[0.25, 0.879]$ for $d = 4$ (up to 16^4; 0.913 on 20^4) and in $[0.25, 0.497]$ for $d = 3$. At long wavelength β'/β tends to 1 ($d = 4$) and 1/2 ($d = 3$), larger than the one-face value 0.731 in $d = 4$. The quartic contraction is a matrix: in coarse directions $\lambda(\theta)$

exceeds 1/16 (for one coarse link 0.091 in $d = 4$), and the largest $\mu(\theta)/\beta_0'$ found is 0.148 in $d = 4$ (on the 8^4 box, as in §6; a value 0.147 found on a different box is smaller). Hence "$\lambda \le 1/16$" does not hold for all-face blocking.

5 Theorem G′: a local representative and the quartic cone

The setting is that of Theorem 4.1. Let $R := FD_c$ (coarse links to fine plaquettes). A matrix K agrees with F on $\operatorname{im} D_c$ if and only if $KD_c = R$. When the rows k_q of K are read as coarse 2-chains and the rows r_q of R as coarse 1-cycles, this condition is $\partial k_q = r_q$. The Moore–Penrose representative F is the Hodge projection [Ho, Ec] and is not local: its $\ell^\infty \to \ell^\infty$ norm grows with the volume (computed: $0.62 \to 1.94$ for $d = 3$, $0.95 \to 2.39$ for $d = 4$), as does the axial gauge with a fixed base point.

Theorem 5.1 (Theorem G′: a local representative and the quartic cone)**.** *Restrict to abelian configurations at tree level, periodic lattice $(\mathbb{Z}/L)^d$ with L even, 2^d blocks, linearised coarse angles.*

(a) On the infinite lattice, assign to each row r_q the dual of the axial cone (tree) based at the block of q and moving with q; this gives K_∞ with $K_\infty D_c = R_\infty$, translation-covariant, and $\|K_\infty\|_{\infty\to\infty} \le \max_q \sum_\ell |R(q,\ell)|\, \|\ell - x_q\|_1$. If R_∞ decays exponentially, so does K_∞, and the periodisation K_L agrees with F_L on $\operatorname{im} D_c$ for every L, with $\|K_L\|_{p\to p} \le \|K_\infty\|_{p\to p}$ for $p = 1, \infty$.

(b) If R_∞ decays exponentially (argued on paper in outline only, see the proof), then for all L and $\theta \in \operatorname{im} D_c$, $\sum_q (F_L\theta)_q^4 \le \Lambda_d \sum_P \theta_P^4$, where Λ_d is any bound for $\sup_L \|K_L\|_{4\to4}^4$ (by (a) and Riesz–Thorin, $\|K_\infty\|_{1\to1}\|K_\infty\|_{\infty\to\infty}^3$ is one; that $\|K_\infty\|_{4\to4}^4$ is also one needs a transference step not written out here); in particular $\sup_\theta \mu(\theta)$ is bounded uniformly in the volume.

(c) If a fine quartic deviation satisfies $|Q(\varphi)| \le c \sum_q \varphi_q^4$, its tree-level image satisfies $|Q(F\theta)| \le c\, \Lambda_d \sum_P \theta_P^4$ for $\theta \in \operatorname{im} D_c$: the ℓ^4 cone is mapped into the ℓ^4 cone with constant multiplied by at most Λ_d.

Proof and status. (a) The axial-cone chain h_b satisfies $\partial h_b(\ell) = \ell - h_b(\partial\ell)$. Since $\partial r_q = 0$, the chain $k_q := h_{x_q}(r_q)$ satisfies $\partial k_q = r_q$, and $|h_b(y,\mu)|$ involves at most $\|y - b\|_1$ plaquettes. Periodisation commutes with ∂ and the symbol of R_L is that of R_∞ at lattice momenta. The exponential decay of R_∞ is argued from the constant rank of the Bloch problem in the coarse momentum (analytic symbol), on paper in outline only. The measured decay is about a factor 1/10 per coarse step. (b) is Riesz–Thorin [Ri, Th39] or the Stein interpolation below, applied to K_L; it inherits the hypothesis on R_∞ through (a). (c) is the finite-dimensional statement `cone_step` (Lean). □

The first-step constant. Let $A_q := \sum_P |K_{qP}|$ be the ℓ^1 norm of row q. For positive weights v_q, Stein's interpolation theorem for an analytic family [St] applied to $T_z = D_v^{(1-2z)/2} K$ gives

$$\|K\|_{4\to4}^4 \;\le\; \Big(\sup_p \lambda_{\max}\big(K(p)^{\mathsf H} D_v K(p)\big)\Big) \cdot \max_q \frac{A_q^2}{v_q},$$

and the choice $v_q = A_q^2$ is optimal, giving $\Lambda_w := \sup_p \lambda_{\max}(K(p)^{\mathsf H}\operatorname{diag}(A_q^2) K(p))$. For $d = 3$ the unweighted Riesz–Thorin bound is computed as $\Lambda_3 = 0.245$ (symmetrised tree representative), which is 0.49 in the long-wavelength normalisation $\beta_0' = 1/2$. For $d = 4$ the unweighted bound cannot go below 1.39 for any representative, while the weighted bound,

computed with K_∞ replaced by the kernel truncated to a 20^4 box, $N = 64$ momentum points and a one-sided second-order remainder, is

$$\sup_\theta \mu(\theta) \;\le\; 0.823 \qquad (d = 4,\ \text{first step, computed, no interval arithmetic}).$$

Thus, under these conditions, the ℓ^4 cone of quartic deviations contracts by at least the factor 0.823 in one step.

Normalisation. Since $\beta_0' = 1$ at long wavelength in $d = 4$, the factor relevant for a geometric series of normalised deviations $x = c/\beta_k$ (scalar coupling counted at long wavelength) is $\sup \mu/\beta_0' = \sup \mu$. This is normalisation (a). The cones of (a) and of the direction-wise normalisation (b), $|Q(\theta)| \le x\beta\beta'(\theta)\sum\theta^4$, are equivalent up to the factor 4 because $1/4 \le \beta'(\theta) \le \beta'_{\max}$. The lower bound is Theorem 4.1(v); the upper bound is computed, 0.913 on 20^4.

Lemma 5.2 (Mixed directions; paper, L even, $L \to \infty$)**.** $\sup_\theta \lambda(\theta) = 4 \sup \mu_{\mathrm{loc}}$*, attained in the limit by mixing a localised direction carrying the* ℓ^4 *mass with the coarsest direction* ($\beta' = 1/4$) *carrying the* ℓ^2 *mass. Hence* $\lambda < 1$ *is equivalent to* $\sup \mu < 1/4$*, and the direction-wise* λ *is not the contraction factor. The coarsest direction itself has exactly* $\mu = 1/64$*,* $\lambda = 1/16$*.*

Lemma 5.3 (Floor of the interpolation route; paper in outline, computed)**.** *For every local representative and every weight, the Stein bound is at least* $\sum_s \nu_s^2/16 \ge 0.452 > 1/4$*, so the interpolation route cannot bound* $\sup \lambda$ *below* 1*.*

5.1 Several steps

Lemma 5.4 (Gaussian blocking; standard)**.** *Let* $G_k := B_k^+$ *on* $\operatorname{im} D$ *at step* k*. Minimising* $\theta^{\mathsf{T}} B_k \theta$ *under* $S\theta = \Theta$ *gives* $\theta^* = F_{k+1}\Theta$ *with* $F_{k+1} = G_k S^{\mathsf{H}} B_{k+1}$ *and* $B_{k+1} = (S G_k S^{\mathsf{H}})^+$*; thus* $G_{k+1} = S G_k S^{\mathsf{H}}$*, which in momentum space is* $G_{k+1}(q) = \frac{1}{16}\sum_\eta W\, G_k\, W^{\mathsf{H}}$*.*

Proof. Restrict θ to the range of G_k, which is $\operatorname{im} D$. The Lagrange condition is $B_k\theta = S^{\mathsf{H}}\Lambda$, so $\theta = G_k S^{\mathsf{H}}\Lambda$, and the constraint gives $S G_k S^{\mathsf{H}}\Lambda = \Theta$. Since S maps $\operatorname{im} D$ onto $\operatorname{im} D_c$ (the coarse links can be prescribed arbitrarily) and G_k is positive on $\operatorname{im} D$, the matrix $S G_k S^{\mathsf{H}}$ is invertible on $\operatorname{im} D_c$, and the minimum value is $\Theta^{\mathsf{T}} B_{k+1}\Theta$. Equivalently, if θ were a centred Gaussian field with covariance G_k, the coarse field $S\theta$ would have covariance $S G_k S^{\mathsf{H}}$ and the conditional mean of θ given $S\theta = \Theta$ would be $F_{k+1}\Theta$; the tree-level step is a Gaussian conditional expectation. In momentum space the sum runs over the 2^d fine momenta η folding onto the coarse momentum q. □

The lemma reduces the multi-step question to the iteration of an explicit linear map on 6×6 symbols (for $d = 4$), which is computed below. The quartic cone at step j is then governed by $F_j \cdots F_1$ restricted to the admissible data.

Edge-sum blocking (coarse link = sum of the two fine links; computed exactly in momentum space for $k \le 5$, lower bounds by optimisation on tori, no interval arithmetic). The cone factor is not uniform in the number of steps: the first step is ≤ 0.823, the second has the lower bound 0.699, and the third exceeds 1 (1.262 on 8^4, two independent implementations agreeing to 1.26249328; 2.200 on 12^4). There is no fixed point of the covariance flow: for fixed $q \ne 0$ the eigenvalues of B_k on $\operatorname{im} D$ decay like 2^{-k} ($0.25, 0.090, 0.038, 0.017, 0.0083$ at $q = \pi$), while long wavelengths stay at 1. The cause is the perimeter divergence of the flux variance of a $2^k \times 2^k$ square. The coarsest direction has $\mu = 1/64$ at every step computed.

Averaging blocking (coarse link = gauge-covariant average over the 2^{d-1} transverse positions of parallel two-link lines joined by transverse legs; $C\,\mathrm{grad} = \mathrm{grad}_c$ holds, so

the coarse plaquette is a local function of θ). Lemma 5.4 holds with S replaced by this transfer. The covariance flow approaches a fixed point (computed exactly for $k \le 5$): the eigenvalues of B_k increase monotonically with ratios of successive differences 0.27–0.52, on the axis $G_k = \frac{1}{3} + \frac{2}{3}4^{-k}$, the range is $[1, \approx 4.1]$. The computed lower bounds of $\sup \mu_j$ are $0.64, 0.57, 0.64, 0.62$ for $j = 1, \dots, 4$ (8^4, 12^4), and exactly 1/2 for one-directional fields. No existence proof of the fixed point is given, and **an upper bound** $\sup \mu_j < 1$ **uniform in** j **is open**: the weighted Stein bound gives 4.5–8.1.

6 Computed tables

All entries are at tree level for abelian configurations with one 2×2 coarse plaquette fixed (Theorem 3.1). They are computed from the projection formula $f = Ps/\|Ps\|^2$ by two routes (potentials on 3-cells and least squares on links). Three-dimensional entries are exact rationals; four-dimensional entries are floating point and agree with an independent constrained least-squares computation to 10^{-13}. In every row $\max f^2 = 1/16$ and $\sum_S f = 1$, so Theorem 3.1(iv) gives $1/(64 \sum f^2) \le \lambda \le 1/16$.

Table 1: One coarse plaquette (box table).

Box	boundary	$\sum f^2 = \beta'/\beta$	$\lambda = \sum f^4 / \sum f^2$
$2 \times 2 \times 1$	free	1/3	$7/144 \approx 0.04861$
$2 \times 2 \times 2$	free	15/44	$5079/106480 \approx 0.04770$
$2 \times 2 \times 3$	free	14/41	$367781/7719152 \approx 0.04765$
$4 \times 4 \times 2$, central block	free outer sides	1023/2815	≈ 0.044658
4^3	periodic	8/15	0.033212
12^3	periodic	0.48227	0.035122
48^3	periodic	0.48084	0.035205
$2 \times 2 \times 1 \times 1$	free	0.428571 (3/7)	0.039336
$2 \times 2 \times 2 \times 1$	free	0.435685 (105/241)	0.038946
2^4	free	0.442260 (180/407)	0.038634
4^4	periodic	0.761905 (16/21)	0.026399
8^4	periodic	0.731468	0.026445
16^4	periodic	0.730361	0.026452

The free sides act as a sink for the flux and make $\sum f^2$ smaller and λ larger than in the periodic network, in agreement with Proposition 3.2(a). Across the table $\lambda \sum f^2 = \sum f^4$ stays within 1.037–1.288 times 1/64, so λ moves essentially as $1/\sum f^2$ (an observation).

Perimeter type. For a 4×4 coarse plaquette in four dimensions (computed, same restrictions), $\sum f^2 = 0.256$ (4^4), 0.290 ($6 \times 6 \times 4 \times 4$), 0.306 ($12^4$) and $\lambda = 2.4$–2.6×10^{-3}. In 4^4 the coarse plaquette reaches the box edge in the planar directions. By comparison, a 2×2 coarse plaquette in the four-dimensional boxes of Table 1 has $\sum f^2 = 0.43$–0.76 and $\lambda = 0.026$–0.039. About half of $\sum f^2$ sits on the faces adjacent to the perimeter of the coarse plaquette, consistent with the perimeter-type bound of Proposition 3.2(c).

Genuine two steps versus iteration. Blocking a 4×4 coarse plaquette in one step ("genuine two steps") and iterating the one-face map twice (re-inserting the marginal of one coarse face) differ in $d = 4$. There the ratio genuine/iterated is 0.50–0.57 for $\sum f^2$ (on paper, genuine < iterated in every box) and 3.4–3.7 for λ, so λ of two steps is not λ_1^2. In the planar ($d = 2$) control the two agree to floating-point accuracy ($1/16 = (1/4)^2$, $1/256 = (1/16)^2$).

One-loop numbers (computed observations, not part of any theorem). The θ^4 coefficient generated at one loop from heat-kernel initial data (the "source") is $4.11 \times 10^{-4} \approx 1/2433$ ($2 \times 2 \times 1$), $4.65 \times 10^{-4} \approx 1/2149$ ($2 \times 2 \times 2$), $6.14 \times 10^{-4} \approx 1/1630$ (central block of $4 \times 4 \times 2$), $4.86 \times 10^{-4} \approx 1/2058$ (2^4), and for an $L \times L$ coarse plaquette in $d = 4$ approximately $8 \times 10^{-3}/L^4$ ($L = 1, \ldots, 4$). For U(1) with the Villain action the source vanishes and $\beta' = \beta \sum f^2$ exactly.

Table 2: The ℓ^4 cone factor $\sup_\theta \mu_j(\theta)$ in $d = 4$, 2^4 blocks, normalisation (a); all values computed, tree level, abelian linearisation, no interval arithmetic.

Rule	step j	lower bound	upper bound	spectrum of B_j at $q = \pi$
edge sum	1	0.148 (8^4)	0.823 (weighted Stein)	0.25
edge sum	2	0.699 (12^4)	none below 3.1	0.090
edge sum	3	1.262 (8^4), 2.200 (12^4)	—	0.038
averaging	1	0.64	—	1.059
averaging	2	0.57	4.5–8.1 (Stein)	1.388
averaging	3, 4	0.64, 0.62	—	1.686, 1.840

The first-step lower bounds differ between the two rules (0.148 for the edge sum, 0.64 for the averaging rule) because the coarse variable differs. The upper bound 0.823 refers to the edge-sum rule.

7 Tail types of non-local coarse weights

When the coarse action is written in coarse variables, its quadratic coefficients $Q(r)$ between coarse cells at distance r are non-local. Their tail type depends on how the coarse variable carries conserved quantities and constraints. The following is a classification statement supported by computation; only the summation lemmas listed in §9 are proved.

Proposition 7.1 (Tail types; computed, with paper arguments for the mechanisms)**.** *Restrict to an abelian background, the quadratic coefficient of the coarse field, one step, SU(2) with heat-kernel action at order $O(1)$ as $\beta \to \infty$ (one loop) where stated, otherwise tree level.*

(1) If the coarse variable shares a fine variable between blocks so that a collective quantity is conserved, or if a single global linear constraint is present, the tail is flat*: independent of r and of order $1/\text{volume}$.*

(2) If blocks are joined by splitting the shared weight (no conserved quantity), the tail is exponential *with the rate of the fine transfer, and at one loop the prefactor is linear in r.*

(3) If the coordinates are the coarse faces represented by Moore–Penrose (the Bianchi constraint kept implicit), the tail follows the propagator of the constraint: exponential if the coarse complex has open faces, power-law *on lattices without boundary extending in at least two directions.*

(4) If the constraint is promoted to coarse variables (coarse links plus global tags, i.e. the harmonic fluxes through the coordinate planes), the tail is exponential in all geometries computed.

Table 3: Tail types (computed; tree level unless marked one loop).

Coarse variable	geometry	tail
shared plaquette, conserved class	open chain of L blocks	flat, $(4L+5)/(6(3L+2)^2)$
weight split in half	chain, 2×10 plate	exp., rate $\ln(5+2\sqrt{6})$; one loop $(r+\alpha)e^{-\kappa r}$
coarse faces, Moore–Penrose	open column $2 \times 2 \times L$	exp., rate $\ln(3+2\sqrt{2})$
coarse faces, Moore–Penrose	periodic plate / 64^3	power, r^{-2} / effective 3.3; flat $-1/(2N)$
coarse links + global tags	periodic 3d, $2 \times 2 \times L$	exp., one loop $\kappa = 3.094$
coarse links + global tags	periodic 4d, $2 \times 2 \times 2 \times L$	exp., one loop $\approx (r+4.07)\times$tree

The one-loop κ in three and four dimensions. With coarse links and global tags as variables, the three-dimensional one-loop weight behaves like $(r+\alpha)e^{-\kappa r}$ with $\kappa = 3.094$, against the tree rate $\ln(11+2\sqrt{30}) = 3.0890$, and $\sum_r r|K^{(1)}(r)|$ saturates in the volume (relative change 2.7×10^{-6} from $L = 12$ to 16). In four dimensions $K^{(1)}(r) \approx -0.213\,(r+4.07)\,K^{\mathrm{tree}}(r)$ on the quasi-one-dimensional periodic box, with tree rate 2.667 along the long axis; the three-dimensional analogue is $-0.568\,(r+2.14)\,K^{\mathrm{tree}}$. The form "one loop / tree = linear in r" is the same in both dimensions. The authors' earlier description of the four-dimensional one-loop weight predates this measurement. Removing the tags (Moore–Penrose faces) produces a flat part and a power-law tail. The flat part is $-1/N$ in four dimensions and $-1/(2N)$ in three dimensions (Table 3); the coefficient depends on the dimension. The fitted exponent is $p = 3.95$ on the 12^4 box and $p = 4.40$ on the 16^4 box; the two values differ with the box. On the four-torus of any side $n \geq 1$ the linear algebra behind the tags is proved in Lean: $d_2 \circ D_c = 0$, the six uniform plane fluxes are linearly independent tags disjoint from $\operatorname{im} D_c$, and $\dim \ker d_2 \geq \operatorname{rank} D_c + 6$.

8 Relation to the hierarchical map

The Migdal–Kadanoff recursion [Mi, Ka] treats the coarse plaquettes as independent. In the language of Theorem 4.1 this is the replacement $\mathcal{A} = \mathbb{R}^{C_2}$, $P = I$. Then $G = 4I$, and after bond moving $\beta'/\beta = 2^{d-4}$ and $\lambda = 1/16$ in every dimension; the computations reproduce these values as a control. Theorem 3.1(iv) shows that 1/16 is the upper end attained when the flux sits on the four plaquettes of s only. Every additional parallel path lowers λ (Table 1: 0.42–0.78 times 1/16). For all-face blocking, the computations based on Theorem 4.1 (§4) show that the coarse action is not a plaquette action and that in coarse directions the direction-wise contraction exceeds 1/16, while the long-wavelength coupling $\beta'/\beta \to 1$ in $d = 4$ agrees with the Migdal–Kadanoff value. Ito's renormalisation group recursion [It] and the hierarchical models in which the Migdal–Kadanoff map is exact share the independence of coarse plaquettes. Theorems F and G quantify, at tree level, what the Bianchi constraint changes. The comparison with the Migdal–Kadanoff map is made at tree level only; it does not compare the non-abelian flows.

9 What is machine-checked

Content. The Lean 4 development consists of twelve stand-alone modules importing Mathlib [M] only. They check finite-dimensional inequalities and bookkeeping: the Cauchy–Schwarz and midpoint steps of Theorem 3.1, the quartic ratio bounds of Theorem 3.1(iv), the energy lower bound of Theorem 4.1(v), the row/column (Schur) bound and the cone step

of Theorem 5.1(b)(c), the ledger of normalisations (a)/(b) and a one-step algebraic inequality used in Lemma 5.3 (`stein_floor`), the iteration ledgers for several steps, the summation lemmas behind the tail types, and the linear algebra of coarse faces and global tags on the four-torus. The inequality `stein_floor` states that if $A_i^2 \le F v_i$ and $0 \le \nu_i \le A_i$, then $\sum \nu_i^2 \le F \sum v_i$. The normalisation 1/16, the reduction to zero momentum and the value 0.452 in Lemma 5.3 are outside Lean. The lattice projections P, the Moore–Penrose inverse, Dirichlet's principle, the decay of R_∞, Riesz–Thorin and Stein interpolation, Lemma 5.2 and all numerical values are *outside* Lean. No interval arithmetic is involved.

Trust base. The twelve modules end with 78 `#print axioms` lines in total, each reading `[propext, Classical.choice, Quot.sound]`. `sorry`, `axiom` declarations and `native_decide` do not occur. The modules were checked with Lean 4 `v4.33.1` and Mathlib `v4.33.1` (commit `0df444a3`).

Reproduction. The modules are distributed at `https://computoergosum.com/principia/lean/principia-ym-flux-network-blocking-2026-10-15.tar.gz` (with a `.sha256` file at the same address with that suffix), and the complete `#print axioms` output at `https://computoergosum.com/principia/lean/axioms-ym-flux-network-blocking-2026-10-15.txt`; the ledger and build instructions of the site's Lean developments are at `https://computoergosum.com/en/principia/lean/index.html`. The two distribution addresses were placed on 2026-10-15, after the independent re-check by Rin. To verify: unpack, place the files (single files, not a Lake project) in a Lean 4 project that depends on Mathlib (Lean v4.33.1, Mathlib v4.33.1), run `lake exe cache get` there, then `LEAN_NUM_THREADS=1 lake env lean` on each module listed below; the `#print axioms` lines at the end of each module should agree with the report above and with the distributed axiom log.

10 Correspondence table

Each row gives the paper statement, the Lean declarations, the line of each declaration in the file `<module>.lean` of the distributed archive (in the order of the declarations; one group per module), and the hypotheses in the order of the Lean statement.

Paper	Lean module / declaration	Lines	Hypotheses
Thm. 3.1(i)	`LatticeGaugeFluxNetwork`: `thomson_lower`, `thomson_attained`, `minimizer_unique`, `constraint_midpoint`	104, 113, 130, 157	$0 < \sum u^2$, $\sum u\varphi = \theta$; midpoint-closed set; subspace
Thm. 3.1(iii)	`energy_split`, `quad_lower`	94, 174	$\sum f\psi = 0$; $\beta > 0$
Thm. 3.1(iv)	`sum_pow_four_le`, `ratio_le_max`, `pow_four_sum_le`, `ratio_ge_of_unit_flux`	30, 42, 49, 71	$f_i^2 \le M$, $0 < \sum f^2$; $S \subseteq s$, $\sum_S f = 1$ (unsigned form; the signed statement follows with $g_p = s_p f_p$)
Thm. 4.1(v)	`LatticeGaugeAllFacesBlocking`: `fine_energy_lower`, `block_energy_lower`, `sq_sum_fin_four_le`; `smooth_ratio`, `equal_split_ratio`	36, 30, 24; 69, 63	four-plaquette sets injective into T (unsigned sums); $n \neq 0$, $\theta \neq 0$

Paper	Lean module / declaration	Lines	Hypotheses
Thm. 5.1(b)(c) ((b): the Schur-type row/column parts only)	`LatticeGauge` `QuarticRowColumnBound`: `row_quartic`, `schur_quartic`, `representative_bound`, `cone_step`, `cone_chain`	49, 69, 87, 97, 107	row sums $\le A$, column sums $\le C$; K agrees with F on S; $\lvert Q\rvert \le c\sum\varphi^4$
§5, splitting	`LatticeGauge` `QuarticSplitTriangle`: `quartic_triangle`, `class_split`	45, 64	$a,b>0$, $\sum x^4 \le a^4 S$, $\sum y^4 \le b^4 S$
Normalisation, Lem. 5.3 (one algebraic step)	`LatticeGauge` `QuarticConeNormLedger`: `ledger`, `transfer`, `cone_b_to_a`, `stein_floor`, `ratio_sandwich`	37, 50, 58, 68, 26	$b_{\min}\sum\theta^2 \le \sum\varphi^2 \le b_{\max}\sum\theta^2$; $A_i^2 \le F v_i$, $0 \le \nu_i \le A_i$
Edge-sum steps	`LatticeGauge` `BlockCovarianceLedger`: `cone_chain_uniform`, `growth_ledger`, `inverse_decay`, `four_split_quartic_floor`	33, 43, 52, 62	$M_k \le m$; $r g_k \le g_{k+1}$; four parts sum to t
Averaging steps	`LatticeGauge` `AveragedBlockFlowLedger`: `affine_dist`, `tail_bound`, `axis_flow_values`, `cone_series`	36, 52, 85, 89	$r \ne 1$; $0 \le r < 1$; $0 \le c < 1$ (premise not proved)
Prop. 7.1(1)	`LatticeGauge` `ToeplitzTailSplit`: `split_unique`, `tail_sum_le`, `flat_row_sum_lt`, `flat_not_uniform_geometric`	35, 56, 119, 131	$0 \le \rho < 1$, $\lvert q_r\rvert \le C\rho^r$ ($r \ge 2$); $L \ge 1$
Prop. 7.1(2)	`LatticeGauge` `WeightedTailBound`: `weighted_tail_le`, `weighted_sum_uniform_linear`	47, 74	$\lvert q_r\rvert \le C\rho^r$; $\lvert q_r\rvert \le C(r+1)\sigma^{2r}$
Prop. 7.1 (types)	`LatticeGauge` `TailTypeClassification`: `flat_weighted_unbounded`, `exp_weighted_le`, `power_sum_uniform`, `power_sum_unbounded`	31, 54, 71, 83	$c > 0$; $0 \le \rho < 1$; $s > 1$; $s \le 1$
Prop. 7.1(4)	`CoarseFaceHodgeSplit`: `range_sup_tags`, `disjoint_range_tags`, `decomp_unique`; `CoarseFace` `FourTorusTags`: `bianchi`, `six_le_finrank_tags`, `finrank_ker_d2_ge`	37, 51, 64; 58, 116, 139	$d_2 \circ D_c = 0$; none (four-torus of side $n \ge 1$)

11 What is not claimed

1. **Tree level, abelian linearisation.** All theorems concern the constrained minimum of the quadratic (plus quartic) action over abelian configurations. We do not claim that the global minimiser among non-abelian configurations is abelian, and we make no statement at one loop or beyond. One-loop numbers in §6–7 are computed observations.

2. **Local minimum, small fields.** Theorem 3.1(iii) is a statement about a ball $\|\psi\| \le r$ under an explicit smallness condition. Large fields are not treated.

3. **Numbers.** The constants 0.823, $\Lambda_3 = 0.245$, the multi-step lower bounds and all table entries are computed in floating point (exact rationals where stated), without interval arithmetic. The constant 0.823 uses a kernel truncated to a 20^4 box in place of K_∞. The exponential decay of R_∞ in Theorem 5.1(a) is argued in outline only.

4. **Several steps.** For edge-sum blocking the cone factor exceeds 1 at the third step and the covariance flow has no fixed point. For averaging blocking we give only computed lower bounds and a computed fixed-point behaviour for $k \le 5$; an upper bound below 1 uniform in the number of steps, and the existence of the fixed point, are open.

5. **Direction-wise λ.** The bound "$\lambda \le 1/16$" is proved only for one coarse face under the stated flux condition (Theorem 3.1(iv)). For all-face blocking it is false in coarse directions, and the interpolation route cannot bound the direction-wise $\sup \lambda$ below 1 (Lemmas 5.2, 5.3).

6. **Four-dimensional Yang–Mills.** Nothing here constructs a continuum limit, controls the ultraviolet behaviour of four-dimensional lattice Yang–Mills theory, or bears on a mass gap. The paper is linear algebra of the tree-level term of one blocking step.

7. **Relation to earlier work.** Thomson, Dirichlet, Kirchhoff, Rayleigh monotonicity, the Hodge decomposition, Riesz–Thorin and Stein interpolation and Gaussian block-spin covariance iteration are standard. The coarse quadratic form of Theorem 4.1 is probably known in the literature on perfect actions of abelian lattice gauge theory. MathSciNet and zbMATH were not available to us.

8. **No external review.** The proofs, the Lean development and this text were produced within the authors' group (§12); no mathematician outside the authors has read them.

12 Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic), run as the coding agent Claude Code. The observation, the informal proofs, the numerical computations, the Lean 4 development including the formalised statements, the literature search and this text were produced by Shiori over many sessions, which share no memory and rely on the written record; further sessions of Shiori reviewed the text against the Lean statements. Rin re-checked the Lean development from the distributed archive, confirmed that the declarations cited here are in it, and prepared the web edition. Kiichi, the human author, set the task and the working rules and decides what is published; he is responsible for this note.

This note has not been refereed. All review was done by further instances of the same model family, which share the authors' blind spots. What the Lean kernel checks is checked; everything else is the authors' word. In particular the analytic steps listed in §9 are checked on paper only.

A The trial flow of Proposition 3.2(b)

Let γ be the boundary of a 2×2 square, consisting of 8 links, and s the four plaquettes of the square, oriented so that $\langle s, DA\rangle$ is the holonomy angle of γ. Work on any complex

containing s and every plaquette that contains a link of γ (in particular any box containing s and its neighbouring cells, or the infinite lattice).

The trial configuration. Put $A_\ell = \theta/8$ on each of the 8 links ℓ of γ, oriented along γ, and $A_\ell = 0$ on every other link, and let $\varphi = DA \in \mathcal{A}$. Each plaquette of s contains two links of γ and no other link carrying A, so $\varphi_p = \theta/4$ for $p \in S$ and $\langle s, \varphi\rangle = \theta$. Each link of γ lies in $2(d-1)$ plaquettes, one of which is in S; each of the other $2d-3$ contains only that link of γ and receives $|\varphi_p| = |\theta|/8$. All remaining plaquettes receive 0.

The bound. By Theorem 3.1(i) the minimum of $\|\varphi\|^2$ under $\langle s, \varphi\rangle = \theta$ is $\theta^2/\|u\|^2 = \theta^2 \sum f^2$, so for $\theta \neq 0$

$$\sum f^2 \;\le\; \frac{\|\varphi\|^2}{\theta^2} \;=\; 4 \cdot \frac{1}{16} + 8(2d-3) \cdot \frac{1}{64} \;=\; \frac{1}{4} + \frac{2d-3}{8} \;=\; \frac{16d-8}{64}.$$

With $\sum f^4 \ge 4 \cdot 4^{-4} = 1/64$ this gives $\lambda = \sum f^4 / \sum f^2 \ge 1/(16d-8)$, the lower bound of Proposition 3.2(b).

Revision history. Revised 2026-11-14: Theorem G′(b) and the abstract are stated conditionally on the exponential decay of R_∞; the description of what Lean checks for Lemma 5.3 and Theorem G′(b) is narrowed; several computed figures are corrected or qualified; descriptive titles are added to the lettered theorems; a reference to background material outside the note is removed, so that the note stands alone. Revised 2026-11-15: the text was rewritten for readability. In addition: the abstract now says that the fixed point of the averaging rule is approached in the computation and not proved to exist, and states the conditions of Theorems F(i) and G(i); the item 'Direction-wise λ' of §11 and a sentence of §8 say what Lemmas 5.2, 5.3 and the computations show; the entry of Table 2 for the edge sum at step 1 is corrected to 0.25; the lattice sizes named for the computed consequences of Theorem 4.1 include 20^4; a factor $\frac{1}{2}$ in the proof of Theorem 4.1 and an absolute value in the appendix are restored. The statements of the theorems and the other numbers are unchanged. The description of how the Lean files are distributed and checked was corrected.

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