An interval-arithmetic certificate for the (T1) margin of the one-dimensional double well for $\hbar \in [0.00195, 0.005]$

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Abstract

We consider the lattice path measure of the symmetric double well $V(x) = (x^2-1)^2/8$ with time step $\varepsilon = \frac{1}{2}$, restricted to the phase $x_i \geq 0$. For this measure a two-scale construction reduces a Poincaré inequality uniform in the ring length to a margin inequality on the coarse chain of kept sites, $\lambda = \min \hbar\rho - 2\max \hbar\kappa \geq 0.067$, where $\hbar\rho$ is the scaled one-site gap of the coarse chain and $\hbar\kappa$ the scaled coupling of a bridge of 11 sites. The extrema are taken over neighbour and end values in $[0, 2.5]^2$. Earlier computations established this margin at six isolated values of $\hbar$ down to 0.005. Here we certify it, by interval arithmetic in float64 with outward widths, for every $\hbar$ in the continuum $[0.005/1.17^6, 0.005] \supset [0.00195, 0.005]$ ($0.005/1.17^6 \approx 0.0019492$). We split this range into six subintervals of ratio 1.17, with certified margins at least 0.076, 0.154, 0.240, 0.321, 0.352, 0.366. On the $\hbar\rho$ side the main step is an interval version of the banded Muckenhoupt estimate (cellwise comparison of the exponents at the two ends of the $\hbar$-interval). On the $\hbar\kappa$ side we use a three-point interpolation in $\beta = 1/\hbar$ with an explicit third-order remainder. The elementary inequalities of the $\hbar$-interval step and the final subtractions for all six subintervals are written in Lean 4 and pass its check. The interval outputs, the analytic lemmas (Muckenhoupt, FKG, the tail lemma, the interpolation remainder) and the floating-point premise are inputs. The cost of this scheme grows like $\hbar^{-3.5}$. The remaining range, down to the floating-point estimate $\hbar_0 \approx 5.5 \times 10^{-4}$ of a Laplace-method lemma, is not covered; we explain why this needs a different machine. The status is computer-assisted and conditional.

1 Introduction

This note belongs to the study of a Poincaré inequality for a one-dimensional lattice path measure of a double well, uniform in $\hbar$ and in the system size. This inequality is used as a model step in a programme on the Yang–Mills mass gap; the problem statement is [6]. Nothing here addresses gauge theory directly. Numbers quoted without a citation are taken from the authors' working records (programs, outputs and reports), which are not published.

The inequality in question is condition (H1) of §2: a bound $\operatorname{Var} f \leq C\,\mathcal{E}(f)$ for the measure restricted to one phase, with C independent of $\hbar$ and of the ring length. Section 2.2 states a two-scale construction. It keeps one site in every twelve, treats the eleven sites between two kept sites as a bridge with fixed ends, and bounds the Poincaré constant of the whole measure by an explicit algebraic expression in a few inputs. On the coarse chain it uses the Poincaré version of Helffer's criterion ([4]; Proposition 3.1 of [10]). The inputs other than the margin inequality (T1) of Definition 2.1 — the gap of the two-site marginal inside a bridge (T3), the cross term (T4) and the resulting bridge gap λ_S — hold for every $\hbar \in (0, 0.2]$.

They are stated in §2.2 and graded in Table 2. The margin (T1) was checked by interval arithmetic only at the six nodes $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$ (margins 0.7039, 0.5812, 0.5677, 0.3574, 0.0670, 0.0845). Because the rate condition of the construction is monotone in the margin, any lower bound $\lambda \geq 0.067$ at other values of $\hbar$ gives the same constant there as at the nodes (Proposition 2.2). We call the following *the question (H1-b)*: for $\hbar < 0.005$, does the margin stay ≥ 0.067? Within this construction it is the only input not available below $\hbar = 0.005$, apart from the rows for large neighbour and end values that are left open at every $\hbar$ (Table 2).

Two routes exist. The first route is semiclassical: as $\hbar \to 0$ the margin tends to a limit determined by a zero-temperature landscape. We bounded this limit by interval arithmetic from below by 0.758. A quantitative Laplace-method lemma (QL) would give an explicit $\hbar_0$ below which the margin is safe. With the constants we can currently write, a floating-point evaluation gives $\hbar_0 \approx 5.5 \times 10^{-4}$. The second route is computational: extend the interval machine of the six nodes from points to intervals of $\hbar$. This note follows the second route, starting from the top of the window. It covers $[0.005/1.17^6,\ 0.005]$ and stops there.

Status of the results. Computer-assisted (interval arithmetic in float64 with outward widths, presupposing correctly rounded IEEE arithmetic), with Lean 4 for the elementary inequalities of the $\hbar$-interval step and the final arithmetic of the six subintervals; conditional on the reference and assumption rows of Table 2.

2 The model, the construction and the margin

2.1 The model and condition (H1)

On a ring of $N = 12n$ sites ($n \geq 3$) consider

$$S(x) = \sum_i \frac{(x_{i+1} - x_i)^2}{2\varepsilon} + \varepsilon \sum_i V(x_i), \qquad V(x) = \frac{(x^2-1)^2}{8}, \qquad \varepsilon = \frac{1}{2},$$

and the measure $\mu^{(+)} \propto e^{-S/\hbar}\mathbf{1}\{x_i \geq 0\ \forall i\}$, with Dirichlet form $\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$. The paths stay on the side of the well $x = 1$ and carry no kink. *Condition (H1)* (ring version) asks for $\hbar_0 > 0$ and $C < \infty$ such that $\mathrm{Var}_{\mu^{(+)}} f \leq C\,\mathcal{E}(f)$ for all $\hbar \in (0, \hbar_0]$ and all n. The measure is not log-concave, and the simple arguments (a convex core, a global or a site-by-site perturbation) lose factors exponential in $1/\hbar$. This is why the construction below has two scales. Keep the sites $y_j = x_{12j}$.

Definition 2.1 (bridge, coupling, one-site measure)**.** (i) For ends $a, b \geq 0$ the bridge measure on $[0,\infty)^{11}$ is $\mu_{a,b} \propto e^{-S_{a,b}/\hbar}$ with $S_{a,b}(x) = \sum_{k=0}^{11}(z_{k+1} - z_k)^2/(2\varepsilon) + \varepsilon\sum_{i=1}^{11} V(x_i)$, $z = (a, x_1, \ldots, x_{11}, b)$. Write $e(a,b) = \mathbb{E}_{a,b}x_{11}$.

(ii) The coupling: $\hbar\kappa(a,b) = \mathrm{Cov}_{a,b}(x_1, x_{11})/(\varepsilon^2\hbar) = \varepsilon^{-1}\partial_a e(a,b) \geq 0$.

(iii) The one-site measure of a kept site with neighbours c, d is $\nu_{c,d}(dt) \propto e^{-U(t)/\hbar}dt$ on $[0,\infty)$, $U'(t) = \varepsilon V'(t) + (2t - e(c,t) - e(d,t))/\varepsilon$. We write $\hbar\rho(c,d)$ for $\hbar$ times its Poincaré gap for the flat form $\int |f'|^2 d\nu$.

(iv) The margin: $\lambda := \min_{c,d\in[0,2.5]} \hbar\rho(c,d) - 2\max_{a,b\in[0,2.5]} \hbar\kappa(a,b)$.

(T1) is the inequality $\lambda \geq 0.067$. Muckenhoupt's estimate [11, 2] squeezes the Poincaré constant of $\nu_{c,d}$ between B and $4B$, where B is a Hardy-type functional of the density; hence $\hbar\rho \geq \hbar/(4B)$. The monotonicity of $e(c,t)$ in c (an MTP_2 correlation inequality [7]) allows a finite set of neighbour values to control the continuum. A tail lemma (Holley's comparison [5] and a tilted Gaussian bridge) handles t beyond the computed range.

2.2 The two-scale construction

The following statements are the steps of the construction. (C1)–(C6) number the steps, and the labels (T1)–(T5) name the inputs on which they rest; (T1) is the margin inequality above. Their derivations are given in Appendix A, and their grades are the rows of Table 2. Here λ, λ_S, λ_c and ρ_f are $\hbar$ times spectral gaps for the flat Dirichlet form $\int |\nabla f|^2$, and T_k denotes the Dirichlet Laplacian tridiag$(-1, 2, -1)$ of a path of k sites.

(C1) (T5) $\mathrm{Var}_{\mu^{(+)}} f = \mathbb{E}[\mathrm{Var}(f \mid y)] + \mathrm{Var}(\mathbb{E}[f \mid y])$, for test functions in a class whose regularity and density are assumed (Appendix A.1).

(C2) (T2) Given y, the bridges between consecutive kept sites are independent, the one between y_j and y_{j+1} distributed as $\mu_{y_j, y_{j+1}}$, and the marginal of y is a nearest-neighbour chain on a ring of n sites whose one-site conditional law with neighbours c, d is $\nu_{c,d}$ (Appendix A.2).

(C3) (Fine side; nested estimate.) The 11-site bridge is not log-concave (the smallest eigenvalue of the lower bound $T_{11}/\varepsilon - (\varepsilon/2)I$ of its Hessian is ≈ -0.11). Inside it keep the sites 4 and 8. Each of the three 3-site sub-bridges has Hessian $\geq A_3 := T_3/\varepsilon - (\varepsilon/2)I$, whose smallest eigenvalue is $\rho_f = 15/4 - 2\sqrt{2} \approx 0.9216$; the coupling between the sub-bridges and the two kept sites is at most $\kappa_f = 1.98$; and (T3) the marginal of (x_4, x_8) has gap $\lambda_c \geq 1.3 \times 10^{-4}$ for all end values in $[0, 4]^2$ and all $\hbar \in (0, 0.2]$. The gap of the bridge is then at least the smaller root of $t^2 - (\rho_f + \lambda_c + \kappa_f^2/\rho_f)t + \rho_f \lambda_c = 0$, which gives $\lambda_S \geq 2.3 \times 10^{-5}$ (Appendix A.4).

(C4) (Coarse side.) The Poincaré version of Helffer's criterion, applied on the ring in Gershgorin form and using $\hbar\kappa \geq 0$, bounds the gap of the chain of kept sites from below by the margin λ whenever $\lambda > 0$ (Appendix A.5).

(C5) (T4) The cross term between the two scales is controlled by the constant $\kappa_\times$ with $\kappa_\times^2 = 2(v/\hbar)\lambda_S/\varepsilon^2$, where $v/\hbar \leq 2\big(644/1455 + (60/97)^2/\lambda_c\big) \leq 5888$ (Appendix A.6).

(C6) (Rate condition.) If $0 \leq t < \lambda$ and $t\,\kappa_\times^2 \leq \lambda_S(\lambda_S - t)(\lambda - t)$, then $C_P(\mu^{(+)}) \leq \varepsilon/t$ (Appendix A.7).

Proposition 2.2 (reduction to (T1); conditional)**.** *Assume the rows of Table 2 for (C1)–(C6) other than the open row, and assume moreover that (C3)–(C5) and the margin hold, with the stated constants, for all neighbour and end values in* $[0, \infty)$ *(this note establishes them for* $[0, 4]^2$ *and* $[0, 2.5]^2$ *only; the rest is the open row). Then, at every* $\hbar \in (0, 0.2]$ *at which* $\lambda \geq 0.067$*, one has* $\mathrm{Var}_{\mu^{(+)}} f \leq C\,\mathcal{E}(f)$ *with* $C \leq 1.6 \times 10^{10}$*, independently of* n*.*

The proof is in Appendix A.8. At the six nodes this gives $C \leq 1.6 \times 10^{10}$. The extrema in λ are taken over $[0, 2.5]^2$ and (T3) is stated for end values in $[0, 4]^2$. The passage to larger neighbour and end values is a separate row (Table 2) that this note does not treat.

3 Extending the machine from points of $\hbar$ to intervals

We take subintervals $I_k = [0.005/1.17^k,\ 0.005/1.17^{k-1}]$, $k = 1, \ldots, 6$, and certify on each

$$\inf_{\hbar \in I_k} \min_{c,d} \hbar\rho \ \geq\ r_k, \qquad \sup_{\hbar \in I_k} \max_{a,b} \hbar\kappa \ \leq\ k_k, \qquad r_k - 2k_k \geq 0.067.$$

Consecutive subintervals share their endpoints exactly. The ratio 1.17 was chosen from the size of the fluctuation of the action.

3.1 The $\hbar\kappa$ side

At a single $\hbar$ the computation used at the six nodes (the *box bound*) tilts the bridge measure at the centre of a box of end values and controls the whole box by the second-order Taylor expansion of the cumulant generating function, with a Hölder remainder. The integrals in this bound are computed by transfer operators on a quadrature grid, in interval arithmetic. To make $\beta = 1/\hbar$ a third variable we use the following scheme.

- *Three-point interpolation (Lemma TI).* For each non-negative weight X (products of end observables) the unnormalised integral $J_X(\beta) = \int X e^{-\beta H} dx$ is computed at $\beta_{\mathrm{lo}}, \beta_c, \beta_{\mathrm{hi}}$; the quadratic interpolant p_X satisfies $|J_X - p_X| \le (|\omega(\tau)|/6) \sup |J_X'''|$, $\omega = (\tau+\Delta)\tau(\tau-\Delta)$, $\max|\omega| = 2\Delta^3/(3\sqrt{3})$, by the standard Lagrange remainder.

- *Third-order remainder (Lemma EM).* $|y|^3 \le c_3(\lambda_+)e^{\lambda_+ y} + c_3(\lambda_-)e^{-\lambda_- y}$ with $c_3(\lambda) = (3/(e\lambda))^3$, so $\sup|J_X'''|$ is bounded by J_X at four shifted values of β; log-convexity of J_X puts the supremum at an endpoint.

- *Linear-fractional endpoints (Lemma TF).* On each of 32 pieces of the β-interval, the ratio of two such interpolants is a linear-fractional function of the position, whose values lie between those at the two ends of the piece when the denominator is positive there; the interval hull of the endpoint values is taken.

This gives an upper bound on $\hbar\kappa$ for every end value in a box and every $\hbar$ in I_k. The square $[0, 2.5]^2$ of end values is covered by $N \times N$ boxes, with $N = 720$ on I_1 and $N \propto 1/\hbar$ (to keep the box width relative to the thermal width fixed). The quadrature step is $h_q \propto \hbar^{3/4}$ (to keep the relative width of the partition function fixed). Boxes above the threshold after the main pass are recomputed with finer settings: the corner near $(0, 0)$ with twice the boxes, and the band at the large end with a finer quadrature step. The final map takes, box by box, the minimum of all upper bounds obtained, and the symmetry $\hbar\kappa(a, b) = \hbar\kappa(b, a)$ fills the other half.

3.2 The $\hbar\rho$ side: an interval version of the banded Muckenhoupt estimate

1. *The band of e.* $e(c, t)$ is enclosed for all $\hbar \in I_k$ by the same three-point scheme in β as in §3.1 (three nodes, Lemmas TI and EM with the four shifted values of β, 32 pieces), applied to the raw integrals of the weights 1, P_+, P_-. The centring constant is fixed on the grid of β_c and used for all β. As a positive control, at $\hbar = 0.0046$ (a point inside I_1, computed with a different step) all 315126 point bands lie inside the interval band.

2. *Exponents compared at the two ends.* Every exponent appearing in the upper bounds $B_\pm$ of Muckenhoupt's functional is a difference of forward sums $\sum_{\mathrm{cells}} U' \cdot h_t/\hbar$, that is, a sum over the cells lying between two points, so it can be bounded cell by cell. For each cell we replace $U_{\mathrm{lo}}/\hbar$ by $\min(U_{\mathrm{lo}}/\hbar_1, U_{\mathrm{lo}}/\hbar_2)$ in B_+ and $U_{\mathrm{hi}}/\hbar$ by $\max(U_{\mathrm{hi}}/\hbar_1, U_{\mathrm{hi}}/\hbar_2)$ in B_-. This is correct whatever the sign of U (Lemma 3.1). The within-cell corrections are taken at $\hbar_1$ and the tail term $\log(\hbar/g)$ at $\hbar_2$, the ends at which each is largest. The conditions of the tail lemma are monotone in the thermal width $\propto \sqrt{\hbar}$ and are checked once at $\hbar_2$.

3. *Pieces.* The Muckenhoupt step alone is repeated on $K = 8$ subpieces of I_k, which share the band of item 1. The bound for I_k is the minimum over pieces (Lemma 3.1(iv)). This reduces the loss from the ratio 1.17 to $1.17^{1/8}$: on I_1, $\hbar\rho \ge 0.74627$ with $K = 1$ and ≥ 0.84797 with $K = 8$.

4. $\hbar\rho \ge \hbar_1/(4 \sup B)$ on the piece $[\hbar_1, \hbar_2]$ (Lemma 3.1(iii)).

Lemma 3.1 (elementary steps; Lean)**.** *Let* $0 < \hbar_1 \le \hbar \le \hbar_2$.

(i) For every real U*:* $\min(U/\hbar_1, U/\hbar_2) \le U/\hbar \le \max(U/\hbar_1, U/\hbar_2)$*, and the same for finite sums of such terms.*

(ii) If $\hbar \mapsto r(\hbar)$ *and* $\hbar \mapsto k(\hbar)$ *satisfy* $r \le r(\hbar)$ *and* $k(\hbar) \le k$ *on* $[\hbar_1, \hbar_2]$ *and* $0.067 \le r - 2k$*, then* $0.067 \le r(\hbar) - 2k(\hbar)$ *there.*

(iii) If $0 < B \le \bar{B}$ *then* $\hbar_1/(4\bar{B}) \le \hbar/(4B)$.

(iv) If P *is a finite nonempty family of pieces and each* $\hbar$ *lies in a piece* $p \in P$ *on which* $r_p \le f(\hbar)$*, then* $\min_{p\in P} r_p \le f(\hbar)$.

3.3 The floating-point premise and the weight scale

The interval arithmetic is the one used at the six nodes: float64 with outward widths added (relative 10^{-10} on sums of non-negative terms, 10^{-12} on interval operations), presupposing correctly rounded IEEE arithmetic. It is not arithmetic with a directed rounding mode for each operation, nor a multiprecision interval library. Weights below e^{-700} are replaced by an upper bound, and the region where the weight would underflow is covered by a tail upper bound. Two changes to the settings were needed below $\hbar = 0.005$, neither of which changes a formula:

- The weight cut-off was an absolute value, so at large β and large end values the tail bound exceeded the integral itself. A uniform factor e^C multiplied into all site weights moves the cut-off outward. At first $C = 650$; from I_4 on $C = 686$, set by a formula with a small safety margin below the float64 limit: $C = 687$ still gives finite forward recursions, $C = 694$ overflows.
- From I_4 on, the shift λ_- of Lemma EM is lowered automatically so that the tail at the corner $(2.5, 2.5)$ stays e^{-25} below the integral at the shifted β ($\lambda_-/\beta_c = 0.721$, 0.460, 0.236 for I_4, I_5, I_6). This enlarges the constant $(3/(e\lambda_-))^3$ of the remainder, by a factor about 4.4 on I_5 and about 30 on I_6.

Where the stored kernels would exceed the memory limit, we used a lazily evaluated version of the transfer kernels, built block by block instead of stored. On a recomputed run of I_3 it reproduces the stored version bit for bit.

4 Results

Proposition 4.1 (conditional, computer-assisted)**.** *Under the premises and rows of Table 2, for every* $\hbar \in [0.005/1.17^6, 0.005]$ *(which contains* $[0.00195, 0.005]$*) and all neighbour and end values in* $[0, 2.5]^2$*, the margin satisfies* $\lambda \ge 0.067$*; on* I_k *it is at least the value in the last column of Table 1.*

Combined with the six nodes, (T1) then holds, under the same premises, on $\{0.2, 0.1, 0.05, 0.02, 0.01\} \cup [0.005/1.17^6, 0.005]$. At the five nodes above 0.005 this rests on the earlier interval computation at the nodes, which is not reproduced in this note (Table 2). By Proposition 2.2, the constant $C \le 1.6 \times 10^{10}$ then holds at every such $\hbar$, under the assumptions of that proposition, which include the open row for large neighbour and end values. The gaps between the nodes above 0.005 are a separate matter not treated here.

Table 1: The six subintervals $I_k = [0.005/1.17^k,\ 0.005/1.17^{k-1}]$ (lower end shown to seven digits; consecutive subintervals share endpoints). Boxes: number of boxes of end values up to the symmetry $(a, b) \leftrightarrow (b, a)$, i.e. $N(N+1)/2$. r_k: lower bound on $\hbar\rho$ (Muckenhoupt on 8 pieces); k_k: upper bound on $\hbar\kappa$ over all boxes; threshold $(r_k - 0.067)/2$; margin $r_k - 2k_k$ (computed from unrounded values). Neighbour and end values $[0, 2.5]^2$.

	lower end of I_k	boxes	r_k ($\hbar\rho \geq$)	k_k ($\hbar\kappa \leq$)	threshold	margin $\geq$
I_1	0.0042735	259560	0.84797	0.38560	0.39049	0.07677
I_2	0.0036526	354903	0.85245	0.34907	0.39272	0.15431
I_3	0.0031219	486591	0.87360	0.31665	0.40330	0.24030
I_4	0.0026683	665281	0.87539	0.27705	0.40419	0.32129
I_5	0.0022806	910575	0.86653	0.25695	0.39977	0.35262
I_6	0.0019492	1247410	0.85199	0.24263	0.39250	0.36672

Remarks on the numbers.

- The margin grows as $\hbar$ decreases because $h_q \propto \hbar^{3/4}$ shrinks the quadrature width faster than the scaling requires. The maxima of $\hbar\kappa$ on I_1 lay just outside the refined corner. For comparison, floating-point values of the true margin are 1.67 to 1.79 for $\hbar \in [0.001, 0.02]$, so the certified values are limited by the machine, not by the model.

- On I_1 the main pass alone, without the refinement of the large-end band, gives a maximum of $\hbar\kappa$ of 4.33884 and a negative margin; refinement is needed. This negative control is one of the Lean statements.

- Where an endpoint is rounded in the text, the lower end of the covered range is rounded up ($0.00195 > 0.0019492$).

5 Cost, and why the range below $0.005/1.17^6$ is not covered

The number of boxes per side grows like $1/\hbar$ and the number n of quadrature points like $\hbar^{-3/4} \cdot \hbar^{-1/2}$. The cost of a run is close to (number of rows) $\times n^2$ and nearly independent of the number of columns, because the forward chain of each row passes through all weights and grids. The resulting model is cost $\propto \hbar^{-3.5}$, i.e. about 1.7 to 1.8 times per subinterval. The measured ratios scatter around this: I_2 cost about 1.8 times I_1, and I_6 about 2.8 times I_5, the latter including a slowdown of about 2.4 from the lazily evaluated kernels. Peak memory reached about 2.6 GB on I_5.

Below $0.005/1.17^6$ three things fail together:

1. *Float64 range.* The single weight scale e^C cannot be raised beyond $C \approx 690$ without overflow, and the automatic choice of λ_- falls to 0.045 on I_7. The remainder constant then grows beyond what the margin can absorb, and the tail at the corner $(2.5, 2.5)$ cannot be closed with one weight scale. Per-node or per-row scales do not help, since the required range is set by the depth of the well relative to the cut-off, independently of β.

2. *Cost.* Down to the floating-point $\hbar_0 \approx 5.5 \times 10^{-4}$ of (QL), about nine further subintervals would be needed, at a cost growing by 1.7 to 1.8 per step.

3. *The lower end is not explicit.* $\hbar_0 \approx 5.5 \times 10^{-4}$ is a floating-point evaluation of constants in a paper lemma, some of whose rows (the action gap outside a convex neighbourhood, uniformity over the end values, the tail) are not certified.

Possible remedies (not tried): a quadrature centred at the minimiser of the bridge action on the thermal scale $\sqrt{\hbar}$ (Laplace-type coordinates), so that n no longer grows with $1/\hbar$ and the cost would drop to $\propto 1/\hbar$; weights in logarithmic form; or sharper constants in (QL).

6 Status of the rows

Table 2 lists, for each step of Proposition 4.1, whether it rests on interval arithmetic, on a derivation by the authors, on Lean, on the literature, or on an assumption.

7 What is machine-checked

The file `LatticeGaugeH1HbarInterval.lean` (namespace `LatticeGauge.H1HbarInterval`, 15 theorems, sha256 `4231182160d4e4721de2aa299f819e24634121d7b2293e4cf0fb0b2a c5561530`) is a single file beginning with `import Mathlib`. It is distributed as a bundle of its own (index of the Lean files and build instructions: `https://computoergosum.com/en/principia/lean/index.html`; bundle: `https://computoergosum.com/principia/lean/principia-ym-h1b-margin-interval-2026-10-29.tar.gz`, checksum file `https://computoergosum.com/principia/lean/principia-ym-h1b-margin-interval-2026-10-29.tar.gz.sha256`; axiom log: `https://computoergosum.com/principia/lean/axioms-ym-h1b-margin-interval-2026-10-29.txt`). The file was checked with Lean v4.33.1 and Mathlib v4.33.1 (commit `0df444a3`); the bundle holds the file, a README and the axiom log, and is not a Lake project. To verify: extract the bundle, place the file in a Lean 4 project that depends on Mathlib at that version, run `lake exe cache get` there, then run `LEAN_NUM_THREADS=1 lake env lean LatticeGaugeH1HbarInterval.lean`; each of the fifteen `#print axioms` lines at the end must print `[propext, Classical.choice, Quot.sound]`. There is no `sorry`, no `axiom` declaration and no `native_decide`.

Lean checks Lemma 3.1, with item (iv) in a form whose covering hypothesis is required for every real $\hbar$, stronger than the form stated in the lemma; its content is that a minimum is at most each member. Lean also checks the threshold arithmetic and the margins $r_k - 2k_k \geq 0.067$ of $I_1, \ldots, I_6$. These margins are the theorems `margin_I1`–`margin_I6`; each is a `norm_num` check of the literal numbers of Table 1, which does not certify those numbers themselves. Finally, Lean checks that the unrefined value 4.33884 would not give a margin. Lean does not check the interval outputs r_k, k_k (they enter as hypotheses), the transfer-operator integrals, Muckenhoupt, FKG, the tail lemma, Lemmas TI, EM (sharp constant) and TF, the box bound at a single $\hbar$, or the floating-point premise. Some elementary steps of Lemmas TI, EM and TF (the Lagrange basis, its Lebesgue constant, $|y|^3$ against exponentials with the constant $6/\lambda^3$, the linear-fractional endpoint lemma) are in a separate earlier file, `LatticeGaugeH1TaylorStage.lean`, and are not used by this note's Lean file.

8 What is not claimed

- $[\hbar_0, 0.005/1.17^6]$ **is not covered**, and (H1) is not established as $\hbar \to 0$: the computed range and the range of (QL) do not meet, and $\hbar_0$ itself is a floating-point estimate. Here $0.005/1.17^6 \approx 0.0019492$; where the covered range is written with a rounded end, it is rounded up to 0.00195.

Table 2: What each step rests on. Grades: *interval arithmetic* = computed under the float64 premise; *paper* = derived by the authors in the section or appendix named, not machine-checked; *Lean* = passes the Lean check of §7; *not reproduced* = derived by the authors but not written out in this note; *reference* = theorem from the literature, with the level of checking stated; *assumption* = taken as given.

Row	Grade
The margin on $[0.005/1.17^6, 0.005]$ *(§§3–4)*	
r_k, k_k and the covering of all boxes, I_1–I_6	interval arithmetic
float64 with outward widths, correct IEEE rounding	assumption (premise of the implementation)
Lemma 3.1 (cellwise comparison, pieces, $\hbar_1/(4\bar{B})$, margin on an interval)	paper (§3.2) and Lean
substitution into $B_\pm$ (exponents as sums over intermediate cells; within-cell corrections at $\hbar_1$, tail term at $\hbar_2$)	paper (§3.2, items 2–4)
final subtraction $r_k - 2k_k \geq 0.067$	Lean (`margin_I1`–`margin_I6`)
Lemma TI (Lagrange interpolation remainder)	reference (standard; not re-checked against a source)
Lemma EM with the sharp constant $(3/(e\lambda))^3$	paper (§3.1; Lean has the weaker constant $6/\lambda^3$)
Lemma TF (linear-fractional endpoints)	paper (§3.1)
box bound at a single $\hbar$ (tilt at the centre of a box, second-order cumulant expansion, Hölder remainder; §3.1)	not reproduced (the same bound as at the six nodes)
Muckenhoupt $B \leq C_P \leq 4B$	reference [11, 2] (statement checked in the restatement [2])
monotonicity of e (MTP_2/FKG)	reference [7] (statement checked in a restatement)
tail lemma (continuous Holley comparison)	assumption (theorem number not checked [5])
uniform weight scale e^C, symmetry $(a,b) \leftrightarrow (b,a)$	paper (§3.3, §3.1)
(T1) at the six nodes	interval arithmetic in earlier computations; not reproduced in this note (unpublished records)
$\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$ (margins 0.7039, …, 0.0670, 0.0845; §1)	
The construction of §2.2 (valid for $\hbar \in (0, 0.2]$*)*	
(C1) variance decomposition	paper (Appendix A.1)
(C1) regularity and density of the test functions, differentiation under the integral	assumption
(C2) nearest-neighbour chain of kept sites, formulas for U' and $\hbar\kappa$	paper (Appendix A.2)
two-scale lemma (Lemma A.1)	paper (Appendix A.3)
(C3) curvature bound, $\rho_f = 15/4 - 2\sqrt{2}$, $\kappa_f \leq 1.98$, root $\lambda_S \geq 2.3 \times 10^{-5}$	paper (Appendix A.4)
Bakry–Émery and Brascamp–Lieb on convex domains	reference (convex-domain versions from Theorem 1.2 of [8]; [1, 3] bibliographic entries only)
(C3) (T3) $\lambda_c \geq 1.3 \times 10^{-4}$, end values $[0,4]^2$, $\hbar \in (0, 0.2]$	interval arithmetic (multiprecision and float64 with outward widths), with a derivation not reproduced here (Lyapunov-type test, monotonicity in the end values, corner boxes; Bakry–Émery directly for $\hbar \geq 0.0355$); its references not re-checked (Appendix A.4)
Poincaré–Miranda theorem in the box covering of (T3)	assumption (statement not checked in [9])
(C4) Poincaré version of Helffer's criterion, Gershgorin form	paper (Appendix A.5, after [4] and Proposition 3.1 of [10])
(C5) cross term and $v/\hbar \leq 5888$	paper (Appendix A.6)
(C6) rate condition $\Rightarrow C_P \leq \varepsilon/t$	paper (Appendix A.7)
Proposition 2.2 (monotonicity, $C \leq 1.6 \times 10^{10}$)	paper (Appendix A.8)
neighbour and end values above 2.5 in (T1), above 4 in (T3)	not treated (open row)
$\hbar_0 \approx 5.5 \times 10^{-4}$ of (QL)	floating-point estimate; rows of (QL) partly unverified
semiclassical lower bound 0.758 for the limit of the margin (§1; not used in Proposition 4.1)	interval arithmetic in earlier computations; not reproduced in this note (unpublished records)

- This note does not treat the gaps between the nodes above 0.005, neighbour and end values above 2.5 in (T1) and above 4 in (T3), or the version of condition (H1) for a segment of the chain with given end values. Only the ring version of §2 is considered.
- The interval arithmetic is float64 with added outward widths and presupposes correct IEEE rounding. It is not directed rounding and not a multiprecision interval library. Underflowed terms are treated as in the node computations.
- The relative cost figures were measured on one shared machine and are indicative; the extrapolation to further subintervals is a model.
- The model is $0+1$-dimensional; any bearing on gauge theory is by analogy only.

9 Correspondence table

Table 3: Statements and Lean declarations (namespace `LatticeGauge.H1HbarInterval`, file `LatticeGaugeH1HbarInterval.lean`).

Statement	Lean declaration	Line
Lemma 3.1(i), lower	`cell_lower`	10
Lemma 3.1(i), upper	`cell_upper`	21
Lemma 3.1(i), sums	`sum_lower`, `sum_upper`	28, 33
Lemma 3.1(iii)	`rho_lower`	38
Lemma 3.1(iv)	`pieces_min`	45
Lemma 3.1(ii)	`margin_on_interval`	51
margin of I_1	`margin_I1`	59
margin of I_2	`margin_I2`	62
margin of I_3	`margin_I3`	65
margin of I_4	`margin_I4`	68
margin of I_5	`margin_I5`	71
margin of I_6	`margin_I6`	74
threshold arithmetic	`threshold`	77
negative control on I_1	`margin_I1_coarse_fails`	80

10 Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic), run as the coding agent Claude Code. The reduction, the interval scheme, its implementation and runs, the Lean 4 file including the formalised statements and this text were produced by Shiori over many sessions, which share no memory and rely on the written record; further sessions of Shiori reviewed the text against the Lean statements. Rin re-checked the Lean development from the distributed archive, confirmed that the declarations cited here are in it, and prepared the web edition. Kiichi, the human author, set the task and the working rules and decides what is published; he is responsible for this note.

This note has not been refereed. All review was done by further instances of the same model family, which share the authors' blind spots. What the Lean kernel checks is checked; everything else is the authors' word. In particular, the interval outputs r_k, k_k enter the Lean file as hypotheses, and the other steps listed in §7 as not checked by Lean rest on interval arithmetic, on paper, on derivations not reproduced in this note, on the literature or on assumptions, as graded in Table 2. The interval arithmetic presupposes correctly rounded IEEE arithmetic (§8).

A The two-scale construction

Throughout, $H = S/\hbar$, all measures live on orthants $[0,\infty)^m$, and *gap* means the largest g with $g\,\mathrm{Var}(f) \le \hbar \int |\nabla f|^2 d\mu$ (so that λ, λ_S, λ_c, ρ_f, $\hbar\rho$ are all in these units). For test functions we take a class of smooth functions satisfying the Neumann condition on the faces of the orthant, assumed dense in the form domain. Differentiation under the integral sign is used without further comment. These two points are the regularity assumption of row (C1) in Table 2. Since $\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$, a gap t for the whole measure gives $C_P(\mu^{(+)}) \le \varepsilon/t$ in the units of $\mathcal{E}$.

A.1 Variance decomposition (C1)

For $f \in L^2(\mu^{(+)})$ and the σ-algebra generated by $y = (y_j)$, $\mathbb{E}[(f - \mathbb{E}f)^2] = \mathbb{E}[(f - \mathbb{E}[f \mid y])^2] + \mathbb{E}[(\mathbb{E}[f \mid y] - \mathbb{E}f)^2]$, the cross term vanishing because $f - \mathbb{E}[f \mid y]$ is orthogonal to functions of y. This is (C1).

A.2 Markov structure and the coarse chain (C2)

S is a sum of terms each of which involves one site or two neighbouring sites, and the orthant is a product. Hence, given y, the density of the remaining sites factorises over the n bridges, the bridge between y_j and y_{j+1} having density $\propto e^{-S_{y_j,y_{j+1}}/\hbar}$, i.e. law $\mu_{y_j,y_{j+1}}$. Integrating out the bridges, the marginal density of y is

$$p(y) \propto \prod_j e^{-\varepsilon V(y_j)/\hbar}\, Z(y_j, y_{j+1}), \qquad Z(a,b) = \int_{[0,\infty)^{11}} e^{-S_{a,b}/\hbar}\, dx,$$

a nearest-neighbour chain on the ring. Its one-site conditional law with neighbours c, d is $\propto e^{-U(t)/\hbar}$ with $U(t) = \varepsilon V(t) - \hbar \log Z(c,t) - \hbar \log Z(t,d)$. Since $\partial_b S_{a,b} = (b - x_{11})/\varepsilon$, we have $-\hbar\,\partial_b \log Z(a,b) = (b - e(a,b))/\varepsilon$. By the reflection $i \mapsto 12 - i$ of the bridge, $\mathbb{E}_{t,d} x_1 = e(d,t)$, so $-\hbar\,\partial_t \log Z(t,d) = (t - e(d,t))/\varepsilon$. Adding gives the formula for U' in Definition 2.1, so the conditional law is $\nu_{c,d}$. Differentiating once more gives $\partial_a\partial_b(-\log Z(a,b)) = -\hbar^{-2}\,\mathrm{Cov}_{a,b}(\partial_a S, \partial_b S) = -\,\mathrm{Cov}_{a,b}(x_1, x_{11})/(\varepsilon^2\hbar^2) = -\hbar\kappa(a,b)/\hbar$, and similarly $\partial_a e(a,b) = \mathrm{Cov}_{a,b}(x_1, x_{11})/(\varepsilon\hbar)$. These are the two expressions for $\hbar\kappa$ in Definition 2.1(ii). This is (C2).

A.3 A two-scale lemma

Let $\mu \propto e^{-H}$ live on a product of orthants in coarse variables $y \in \mathbb{R}^m$ and fine variables x, and let $\partial_u H$ (u a coarse coordinate) depend on x only through a function $X_u(x)$ that does not involve y, i.e. $\partial_u H(y,x) = \phi_u(y) + X_u(x)/\hbar$.

Lemma A.1. *Suppose that (a) the conditional laws $\mu(\cdot \mid y)$ have gap $\ge \rho > 0$; (b) the marginal of y has gap $\ge \lambda > 0$; (c) for every test function f, $\hbar^{-2}\sum_u \mathrm{Cov}(f, X_u \mid y)^2 \le (\kappa/\rho)^2\,\mathbb{E}[|\nabla_x f|^2 \mid y]$ with $\kappa \ge 0$. Put $a^2 = \hbar\mathbb{E}|\nabla_x f|^2$, $b^2 = \hbar\mathbb{E}|\nabla_y f|^2$. Then*

$$\mathrm{Var}\, f \le \frac{a^2}{\rho} + \frac{(b + \kappa a/\rho)^2}{\lambda},$$

and if $0 \le t < \lambda$ and $t\kappa^2 \le \rho(\rho - t)(\lambda - t)$, then $t\,\mathrm{Var}\, f \le a^2 + b^2$, i.e. μ has gap $\ge t$.

Proof. By the decomposition of §A.1 and (a), $\mathbb{E}\,\mathrm{Var}(f \mid y) \le (\hbar/\rho)\mathbb{E}|\nabla_x f|^2 = a^2/\rho$. For $g(y) = \mathbb{E}[f \mid y]$, differentiating the conditional density gives $\partial_u g = \mathbb{E}[\partial_u f \mid y] - \hbar^{-1}\,\mathrm{Cov}(f, X_u \mid y)$ (the term $\phi_u(y)$ has zero covariance, and the fine domain does not depend on y). By Minkowski's inequality in $L^2(\mu; \mathbb{R}^m)$, Jensen's inequality for the first term and (c) for the second, $(\mathbb{E}|\nabla_y g|^2)^{1/2} \le (\mathbb{E}|\nabla_y f|^2)^{1/2} + (\kappa/\rho)(\mathbb{E}|\nabla_x f|^2)^{1/2}$. Then (b) gives $\mathrm{Var}\, g \le$

$(\hbar/\lambda)\mathbb{E}|\nabla_y g|^2 \le (b+\kappa a/\rho)^2/\lambda$. Thus $\operatorname{Var} f \le Q/(\rho^2\lambda)$ with $Q = \rho\lambda a^2 + (\rho b + \kappa a)^2$. The polynomial identity

$$\rho^2(\lambda - t)[\rho^2\lambda(a^2+b^2) - tQ] = \big(\rho^2(\lambda-t)b - t\rho\kappa a\big)^2 + \rho^2\lambda(\rho(\rho-t)(\lambda-t) - t\kappa^2)a^2$$

has a non-negative right side under the rate condition, and $\rho^2(\lambda - t) > 0$; hence $tQ \le \rho^2\lambda(a^2+b^2)$ and $t \operatorname{Var} f \le a^2 + b^2$. □

A.4 The fine side (C3)

Fix ends a, b of the bridge and put $z = (x_4, x_8)$; the fine variables are the three sub-bridges $\{1,2,3\}$, $\{5,6,7\}$, $\{9,10,11\}$.

Convexity. Since $V''(x) = (3x^2-1)/2 \ge -1/2$, the Hessian of S restricted to a sub-bridge with fixed ends is $T_3/\varepsilon + \varepsilon \operatorname{diag} V''(x_i) \ge A_3$,

$$A_3 = \begin{pmatrix} 15/4 & -2 & 0 \\ -2 & 15/4 & -2 \\ 0 & -2 & 15/4 \end{pmatrix},$$

with $\lambda_{\min}(A_3) = 15/4 - 2\sqrt{2} = \rho_f$, $\det A_3 = 1455/64$, and $(A_3^{-1})_{11} = (A_3^{-1})_{33} = 644/1455$, $(A_3^{-1})_{13} = 256/1455$, $(A_3^{-1})_{22} = 900/1455$. The same bound for the 11-site bridge is $T_{11}/\varepsilon - (\varepsilon/2)I$, with smallest eigenvalue $2(2 - 2\cos(\pi/12)) - 1/4 \approx -0.114$, so the bridge itself is not covered by a convexity argument. Given z, the three sub-bridges are independent with log-concave laws on the convex set $[0,\infty)^3$. Hence the Bakry–Émery criterion on convex domains [1, 8] gives (a) of Lemma A.1 with $\rho = \rho_f$, and the Brascamp–Lieb inequality on convex domains [3, 8] gives $\operatorname{Var}(\langle w, x_F\rangle \mid z) \le \hbar\langle A_3^{-1}w, w\rangle$ blockwise.

Coupling. $\partial_{x_4} S$ depends on the fine variables through $-(x_3+x_5)/\varepsilon$ and $\partial_{x_8} S$ through $-(x_7+x_9)/\varepsilon$. For a unit vector $\theta \in \mathbb{R}^2$, $\sum_u \operatorname{Cov}(f, X_u \mid z)^2 = \sup_\theta \operatorname{Cov}(f, \theta\cdot X \mid z)^2 \le \operatorname{Var}(f \mid z) \sup_\theta \operatorname{Var}(\theta\cdot X \mid z)$, and by Brascamp–Lieb $\sup_\theta \operatorname{Var}(\theta\cdot X \mid z) \le \hbar q/\varepsilon^2$ with $q = \lambda_{\max}(M)/1455 = 1544/1455$, $M = \left(\begin{smallmatrix} 1288 & 256 \\ 256 & 1288 \end{smallmatrix}\right)$ (the diagonal entries are $(A_3^{-1})_{33} + (A_3^{-1})_{11}$ and the off-diagonal entry is $(A_3^{-1})_{13}$ of the middle sub-bridge). With $\operatorname{Var}(f \mid z) \le (\hbar/\rho_f)\mathbb{E}[|\nabla_x f|^2 \mid z]$ this is (c) of Lemma A.1 with $\kappa_f = \sqrt{q\rho_f}/\varepsilon \approx 1.978 \le 1.98$.

Two-site marginal (T3). (b) of Lemma A.1 with $\lambda = \lambda_c \ge 1.3 \times 10^{-4}$, for end values in $[0,4]^2$ and $\hbar \in (0, 0.2]$, is the input of this row. It is obtained by interval arithmetic together with a derivation by the authors that is not reproduced in this note: a Lyapunov-type argument with Bakry–Émery on a convex ball, a monotonicity in the end values, and an interval check of boxes near the corner $(0,0)$ that uses the Poincaré–Miranda theorem [9]; for $\hbar \in [0.0355, 0.2]$ the Bakry–Émery criterion is applied directly, with the Hessian of the marginal bounded by interval arithmetic. The references used in that derivation (Brascamp–Lieb, a Lyapunov criterion, FKG/Holley) were not re-checked against their sources (Table 2).

The root. Let t be the smaller root of $t^2 - (\rho + \lambda + \kappa^2/\rho)t + \rho\lambda = 0$ with $\rho = \rho_f$, $\lambda = \lambda_c$, $\kappa = \kappa_f$. Then $(\rho - t)(\lambda - t) = \kappa^2 t/\rho$, so the rate condition of Lemma A.1 holds with equality; at $t = 0$ the left side exceeds the right and at $t = \min(\rho, \lambda)$ it is smaller, so $0 < t < \min(\rho, \lambda)$. Replacing λ_c by its lower bound and κ_f by 1.98 only weakens (b) and (c). The root with $\rho_f = 15/4 - 2\sqrt{2}$, $\lambda_c = 1.3 \times 10^{-4}$, $\kappa_f = 1.98$ is 2.31×10^{-5}. Hence the bridge has gap $\lambda_S \ge 2.3 \times 10^{-5}$ for all ends in $[0,4]^2$ and all $\hbar \in (0, 0.2]$.

A.5 The coarse side (C4)

Let $H_c = -\log p$ on $[0,\infty)^n$, $L = \Delta - \nabla H_c \cdot \nabla$ with Neumann condition, so that $\int f(-Lf)\,dp = \int |\nabla f|^2 dp$. We show $\int (Lf)^2 dp \ge (\lambda/\hbar)\int |\nabla f|^2 dp$ for test functions. By the spectral theorem this gives the gap $\ge \lambda$ (in our units) for the coarse marginal. This is the Poincaré version of Helffer's criterion ([4]; Proposition 3.1 of [10]); we write the argument for the case at hand.

Integration by parts (Bochner's formula) gives

$$\int (Lf)^2 dp = \int \Big(\sum_{i,k} (\partial_{ik} f)^2 + \langle \nabla f, \operatorname{Hess} H_c \nabla f \rangle \Big) dp;$$

on the flat faces of the orthant the Neumann condition makes the boundary term vanish. Drop the terms $(\partial_{ik} f)^2$ with $i \neq k$. For fixed i and fixed other coordinates, $h = f(\dots, y_i, \dots)$ as a function of y_i satisfies the one-dimensional formula $\int (L_i h)^2 d\nu = \int \big((h'')^2 + \partial_{ii} H_c \, (h')^2 \big) d\nu$ for the conditional law $\nu = \nu_{c,d}$ and its generator L_i; since $\nu_{c,d}$ has gap $\hbar\rho(c,d)$, we have $\int (L_i h)^2 d\nu \geq (\hbar\rho(c,d)/\hbar) \int (h')^2 d\nu$. Integrating over the other coordinates,

$$\int (Lf)^2 dp \;\geq\; \frac{1}{\hbar} \int \Big(\sum_j \hbar \rho_j (\partial_j f)^2 - 2 \sum_j \hbar |\partial_{j,j+1} H_c| \, |\partial_j f \, \partial_{j+1} f| \Big) dp.$$

By §A.2, $\hbar |\partial_{j,j+1} H_c| = \hbar \kappa(y_j, y_{j+1})$, which is non-negative by the MTP_2 inequality [7] and at most $\max \hbar\kappa$. On a ring with $n \geq 3$ sites, $|x_j x_{j+1}| \leq (x_j^2 + x_{j+1}^2)/2$ and a shift of the index give $\sum_j \rho x_j^2 - 2\kappa \sum_j |x_j x_{j+1}| \geq (\rho - 2\kappa) \sum_j x_j^2$ for $\kappa \geq 0$ (Gershgorin form). With $\rho = \min \hbar\rho$, $\kappa = \max \hbar\kappa$ this is $\int (Lf)^2 dp \geq (\lambda/\hbar) \int |\nabla f|^2 dp$. This is (C4), with the extrema over $[0, 2.5]^2$ as in Definition 2.1. Neighbour and end values above 2.5 are the open row of Table 2.

A.6 The cross term (C5)

On the ring, $\partial_{y_j} S$ depends on the fine variables through $X_j = -(x_{12j-1} + x_{12j+1})/\varepsilon$, which involves only the two bridges B_{j-1}, B_j adjacent to y_j. Let $f_j = \mathbb{E}[f \mid y, B_{j-1}, B_j]$. Since the bridges are independent given y, $\operatorname{Cov}(f, X_j \mid y) = \operatorname{Cov}(f_j, X_j \mid y)$, and the law of (B_{j-1}, B_j) given y is a product with gap $\geq \lambda_S$, so with Jensen's inequality $\operatorname{Var}(f_j \mid y) \leq (\hbar/\lambda_S) \mathbb{E}[|\nabla_{B_{j-1}} f|^2 + |\nabla_{B_j} f|^2 \mid y]$. Also $\operatorname{Var}(X_j \mid y) = (\operatorname{Var} x_{12j-1} + \operatorname{Var} x_{12j+1})/\varepsilon^2 \leq v/\varepsilon^2$ with $v = \sup_{a,b} \operatorname{Var}_{a,b} x_{11} + \sup_{a,b} \operatorname{Var}_{a,b} x_1$, the suprema over $(a,b) \in [0,4]^2$ (larger end values are the open row of Table 2). By Cauchy–Schwarz and since each bridge is adjacent to two kept sites,

$$\hbar^{-2} \sum_j \operatorname{Cov}(f, X_j \mid y)^2 \leq \frac{v}{\hbar^2 \varepsilon^2} \cdot \frac{2\hbar}{\lambda_S} \mathbb{E}[|\nabla_x f|^2 \mid y] = \Big(\frac{\kappa_\times}{\lambda_S} \Big)^2 \mathbb{E}[|\nabla_x f|^2 \mid y], \qquad \kappa_\times^2 = \frac{2(v/\hbar)\lambda_S}{\varepsilon^2},$$

which is (c) of Lemma A.1 on the ring.

The bound on $v/\hbar$. By reflection, $\sup \operatorname{Var} x_{11} = \sup \operatorname{Var} x_1$. In a bridge, x_1 lies in the sub-bridge $\{1,2,3\}$ with ends a and x_4. By §A.1, $\operatorname{Var} x_1 = \mathbb{E} \operatorname{Var}(x_1 \mid z) + \operatorname{Var} g$ with $g = \mathbb{E}[x_1 \mid z]$, a function of x_4 only. Brascamp–Lieb gives $\operatorname{Var}(x_1 \mid z) \leq \hbar (A_3^{-1})_{11} = \hbar \cdot 644/1455$. Next, $g' = \operatorname{Cov}(x_1, x_3 \mid z)/(\varepsilon\hbar)$, because $\partial_{x_4} S$ involves this sub-bridge only through $-x_3/\varepsilon$. From $4 \operatorname{Cov}(X,Y) = \operatorname{Var}(X+Y) - \operatorname{Var}(X-Y)$ and Brascamp–Lieb, $|\operatorname{Cov}(x_1, x_3 \mid z)| \leq \frac{\hbar}{4} \langle A_3^{-1}(e_1 \pm e_3), e_1 \pm e_3 \rangle \leq \frac{\hbar}{4} \cdot \frac{644+644+2\cdot 256}{1455}$, so $|g'| \leq 1800/(4 \cdot 1455\, \varepsilon) = 60/97$. The gap λ_c of the marginal of z gives $\operatorname{Var} g \leq (\hbar/\lambda_c) \mathbb{E} g'^2 \leq \hbar (60/97)^2/\lambda_c$. Hence $v/\hbar \leq 2\big(644/1455 + (60/97)^2/\lambda_c\big)$, and with $\lambda_c \geq 1.3 \times 10^{-4}$, $v/\hbar \leq 5887.3 \leq 5888$, for end values in $[0,4]^2$ and $\hbar \in (0, 0.2]$.

A.7 The rate condition (C6)

On the ring, take the kept sites as coarse variables. By §A.2 and §A.4, (a) of Lemma A.1 holds with $\rho = \lambda_S$ (a product of bridges); by §A.5, (b) holds with the margin λ; by §A.6, (c) holds with $\kappa = \kappa_\times$. Lemma A.1 gives gap $\geq t$ for every t with $0 \leq t < \lambda$ and $t\kappa_\times^2 \leq \lambda_S(\lambda_S - t)(\lambda - t)$, hence $C_P(\mu^{(+)}) \leq \varepsilon/t$. Nothing in this depends on n.

A.8 Proof of Proposition 2.2

Divide the rate condition of (C6) by $\lambda_S > 0$ and insert $\kappa_\times^2 \le 2(v/\hbar)\lambda_S/\varepsilon^2$: it suffices that $2t(v/\hbar)/\varepsilon^2 \le (\lambda_S - t)(\lambda - t)$. The left side is nondecreasing in $v/\hbar$; for $t < \min(\lambda_S, \lambda)$ the right side is nondecreasing in λ_S and in λ. Hence it suffices to check it with the bounds $v/\hbar \le 5888$, $\lambda_S \ge 2.3 \times 10^{-5}$ (both independent of $\hbar \in (0, 0.2]$) and $\lambda \ge 0.067$. For $t = 3.2 \times 10^{-11}$: $2t \cdot 5888/\varepsilon^2 \approx 1.507 \times 10^{-6}$ and $(2.3 \times 10^{-5} - t)(0.067 - t) \approx 1.541 \times 10^{-6}$, so the condition holds, $t < \min(\lambda_S, \lambda)$, and $C_P(\mu^{(+)}) \le \varepsilon/t = 1.5625 \times 10^{10} \le 1.6 \times 10^{10}$. □

Revision history. Revised 2026-11-14: the dependence of (T1) at the six nodes and of the semiclassical bound on computations not reproduced in this note is stated in Table 2; the form of the covering lemma in Lean is described; a reference to background material outside the note is removed, so that the note stands alone. Revised 2026-11-15: the text was rewritten for readability. In addition, two cross-references were corrected: (T1) is the inequality stated before Proposition 4.1, and Lemma EM, to which the four shifted values of β belong, is now cited together with Lemma TI; the statements, proofs, numbers and tables are otherwise unchanged. The description of how the Lean files are distributed and checked was corrected.

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[1] D. Bakry, M. Émery, *Diffusions hypercontractives*, Sém. Probab. XIX, Lecture Notes in Math. 1123 (1985) 177–206 (used through the convex-domain version of [8]; bibliographic entry checked, statement not re-checked).

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[4] B. Helffer, *Remarks on decay of correlations and Witten Laplacians III. Application to logarithmic Sobolev inequalities*, Ann. Inst. H. Poincaré Probab. Statist. 35 (1999) 483–508 (bibliographic entry only; the criterion is used in the form of [10]).

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[10] M. Ledoux, *Logarithmic Sobolev inequalities for unbounded spin systems revisited*, Sém. Probab. XXXV, Lecture Notes in Math. 1755 (2001) 167–194 (Proposition 3.1, checked in the text).

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