An interval-arithmetic chain for the Migdal–Kadanoff flow of SU(2) plaquette weights, from $\beta = 128$ to a strong-coupling region

Kiichi Shiori Rin

2026-10-30 (version 2; version 1: 2026-10-15; revised 2026-11-14; text revised 2026-11-15)

wvbks0 [at] gmail.com · https://computoergosum.com/en/index.html

AI use is disclosed in §10; what is not claimed is in §9.

Abstract

We study the Migdal–Kadanoff recursion T in dimension $d = 4$ with scale factor 2, acting on the single-plaquette weight of SU(2) lattice gauge theory. One step consists of bond moving $f \mapsto f^4$ and decimation $a_n \mapsto a_n^4$ of the normalised character coefficients. This recursion is exact for a hierarchical lattice model. Nothing in this paper bears on the four-dimensional hypercubic lattice. Since the hierarchical model also confines for U(1), nothing here tests non-commutativity. Around the heat-kernel class we define a tube $\mathcal{U}$ in a transverse coordinate $e_n = \log a_n - (q_n/3)\log a_1$, $q_n = n(n+2)$. In this coordinate all upper bounds become ratios of sums of non-negative terms. (1) Theorem H-UV. The inclusion $T(\mathcal{U}) \subset \mathcal{U}$ holds on 6277 cells covering $\beta \in [2.46, 128]$. By Theorem 4.4, on $\beta \in [15.26, 64]$ any two orbits in $\mathcal{U}$ that enter from the top approach each other at rate $\Lambda \le 0.735$ per step, down to a tail floor of 2.3×10^{-4} in a normalised coordinate. The interval inputs of this statement are not among the distributed certificates. At the heat-kernel points, $(\beta - \beta' - \frac{1}{4})\beta \in [0.020928, 0.023690]$ for $\beta \in [2.5, 64]$. (2) On the curve of Theorem 4.4, to leading order in $1/\beta$, the shape is $\mathrm{HK}_\beta \cdot \exp(\theta^4/3600 + \theta^6/963900 + \dots)$ and the coupling flows as $\beta' = \beta - \frac{1}{4} - c^*/\beta$ with $c^* = 1/60$ (paper, leading order). On a narrower tube the flow coefficient is enclosed in $[0.0151, 0.0188]$ on 50 cells between $\beta \approx 16$ and 128. (3) Theorem MK-full. From any point of an extended tube with $\beta(w) \le 128$, at most 580 steps lead into the tube $\mathcal{D}$ of an earlier interval-arithmetic certificate, W-half(T), and at most 557 further steps lead into the region $\mathcal{K}$ of that certificate: 1137 steps in total, under five stated conditions. The certificate is stated here with its definitions as Proposition 5.2, derived in Appendix A, and distributed with its programs and logs. Items (1) and the enclosures in (2) are computer-assisted (interval arithmetic with a self-written library; not independently verified); for the statement on $[15.26, 64]$ the certificate files are not distributed. The step-count assembly, in a form with a general one-step drop, and the leading-order algebra are written in Lean 4 and pass its check. The limit $\beta \to \infty$ is not closed, and §6 explains which term of the present cell test prevents it.

Introduction

The Yang–Mills existence and mass gap problem [JW] asks for a quantum Yang–Mills theory on $\mathbb{R}^4$ with a positive mass gap. This paper does not address that problem. It studies one explicit hierarchical renormalisation-group map, the Migdal–Kadanoff recursion for SU(2) plaquette weights, as a testing ground for computer-assisted chains of renormalisation steps. Definitions, statements, derivations, certificates and Lean files used below are all contained in this paper, its appendix and its distributed bundle.

The earlier certificate. The second stage of Theorem 5.4 uses an earlier certificate for the same map, computed in the authors' working records before the computations of this paper. It shows by interval arithmetic that the orbits in a tube $\mathcal{D}$ at strong coupling enter a region $\mathcal{K}$. We call it W-half(T). Its definitions and statement are given in §5 (Definition 5.1, Proposition 5.2), and its derivation and a summary of its output in Appendix A. Its programs and logs are part of the bundle of this paper (§7). For this version, the decrease of the cell index was checked by a separate computation and part of the certificate was run again (Appendix A). To readers of the first version: in test (2) of the cell test the comparison was against the lower end of the current box rather than of the cell. We have corrected the reference point of that comparison and closed the decrease of the cell index by the additional check. The statements are unchanged.

What this paper adds. The contributions of this paper are Theorem 3.1, the results of §4, the joining computation and Theorem 5.4, Lemma 5.3 (soundness of the exit test), Lemma 6.1 (Gaussian tail of heat-kernel coefficients), and the single-cell computations at $\beta = 256$, 512, 1024 and 2048 in §6. W-half(T) is an input. It is stated and derived here so that the paper can be read and checked without other sources.

1 The hierarchical map

Scope first. The Migdal–Kadanoff (MK) recursion [Mi, Ka] replaces one step of block-spin renormalisation on the hypercubic lattice by two exactly computable operations on the single-plaquette weight. On the hypercubic lattice it is an approximation. It becomes exact on a hierarchical lattice model in which the vertical plaquettes are placed at coupling 0 or ∞ [It84]. For that model, permanent confinement for SU(N) and U(N) was announced in [It85]. Accordingly the results below are statements about one explicit map on a space of class functions on SU(2). *They do not imply anything about the four-dimensional lattice theory* (see §9). Because the hierarchical model also flows to strong coupling for U(1), *they are not a test of non-commutativity.* We have in mind a reader interested in computer-assisted proofs and hierarchical renormalisation-group maps.

Conventions. A plaquette weight is a non-negative class function w on SU(2), expanded in characters,

$$w = \sum_{n\geq 0} c_n \chi_n, \qquad c_0 = 1, \qquad a_n := \frac{c_n}{n+1}, \qquad q_n := n(n+2),$$

where $n = 2j$ and $\chi_n(\theta) = \sin((n+1)\theta)/\sin\theta$. The heat kernel at inverse coupling β is HK_β with $a_n = e^{-q_n/(2\beta)}$. For a general weight we read off an effective coupling and its inverse from the first coefficient,

$$\beta(w) := -\frac{3}{2\log a_1(w)}, \qquad u(w) := 1/\beta(w),$$

so that $\beta(\mathrm{HK}_\beta) = \beta$. The map is $T = \mathrm{D} \circ \mathrm{B}$ with

- bond moving $\mathrm{B} : w \mapsto w^4/\int w^4\, dU$ (exponent $2^{d-2} = 4$),
- decimation $\mathrm{D} : a_n \mapsto a_n^4$ (convolution of $2^2 = 4$ plaquettes).

On the heat-kernel class, decimation is exact: $\mathrm{DHK}_\beta = \mathrm{HK}_{\beta/4}$. Bond moving maps HK_β to $\mathrm{HK}_{4\beta-1}$ times a correction $\exp\rho$, $\rho = 3\log(\theta/\sin\theta) - \theta^2/2$, so the heat-kernel class is *not* invariant under T. The deviation enters at order θ^4 with coefficient $3\ell_2 = 1/60$, where ℓ_k is the coefficient of θ^{2k} in $\log(\theta/\sin\theta)$ ($\ell_2 = 1/180$, $\ell_3 = 1/2835$).

Remark 1.1 (the constant 1/4). On the heat-kernel class, bond moving sends β to $4\beta - 1$, which is the general rule $L^2\beta - (L^2-1)/3$ for scale factor $L = 2$, and decimation divides by 4. Hence one step lowers β by 1/4 at leading order. The corrections of order $1/\beta$ are the subject of Proposition 4.2. By iteration, K steps starting from β_{start} reach $\beta_{\text{start}} - K/4$ up to a logarithmic correction, so a range of β of length 64 is crossed in about 256 steps.

Relation to known results. It is known that the MK flow of SU(2) weights goes to strong coupling [It85, Mi]. We add a single chain of interval-arithmetic certificates, with explicit uniform speed and step counts, from the weak-coupling end $\beta = 128$ down to the tube $\mathcal{D}$ of W-half(T). We also add the curve near which the flow runs on $[15.26, 64]$ (Theorem 4.4; it is not shown to be unique) and the leading-order coefficient of its coupling flow.

An observation that fixes the coordinates. In double precision, the weights reaching the same coupling β' after j and after 4 steps differ in total variation by ratios 16.06 and 17.00 for $j = 1{:}2{:}3$, for starting couplings $3.5, 7, 15, 30$; a contraction 1/16 per step predicts 16.06 and 17.00. The fitted θ^4 coefficient of $\log(w/\mathrm{HK})$, also in double precision, is 2.789×10^{-4} at $\beta = 60$, against $1/3600 = 2.778 \times 10^{-4}$, and 2.615×10^{-4} after one step, against 1/3840. These observations are not part of any statement. They suggested measuring the distance from the heat-kernel class in a coordinate in which decimation is linear, and centring the tube at the heat kernel rather than at the curve.

Plan. Section 2 defines the transverse coordinate and the tube and explains why every bound is a ratio of non-negative sums. Section 3 states Theorem H-UV. Section 4 gives the leading-order shape flow, the attracting curve on a finite interval, and an enclosure of the coupling-flow coefficient. Section 5 states W-half(T), joins the tube to it and states Theorem MK-full. Section 6 explains which term of the cell test prevents $\beta \to \infty$. Sections 7, 8 and 9 list what is checked in Lean, the correspondence between statements and Lean declarations, and what is not claimed.

2 The tube and the transverse coordinate

Definition 2.1 (transverse coordinate)**.** For $n \geq 2$ put $e_n(w) := \log a_n(w) - \frac{q_n}{3} \log a_1(w)$.

On the heat-kernel class $e \equiv 0$, and e is linear in the logarithmic coefficients. Decimation acts exactly as $e \mapsto 4e$, $u \mapsto 4u$: dividing out the direction of β turns decimation into a linear map, and leaves bond moving as the only non-linear part.

Definition 2.2 (tube)**.** Fix $R > 0$, $A \geq 0$, and for each integer block $[k, k+1)$ of β a truncation index N, a radius $\rho = e^{\lambda} > 1$ and a tail size t_0 (the table `params.csv`, `params_128.csv`; N is non-decreasing in β, $N = 160$ at $\beta = 64$ and $N = 232$ at $\beta = 128$). With $W_n(u) := q_n^2 u^4 + A\, q_n u^3$,

$$\mathcal{U}_{R,A} = \Big\{ w : \; c_0 = 1, \; |e_n(w)| \leq R\, W_n(u(w)) \; (2 \leq n \leq N), \; \sum_{n>N} (n+1)\rho^n |c_n| \leq t_0 \Big\}.$$

The tube of Theorem 3.1 is $\mathcal{U} := \mathcal{U}_{10^{-3},10}$.

The weight $q_n^2 u^4$ is the size of a θ^4-type deviation, by the following lemma.

Lemma 2.3 (transverse shape of a θ^4 deformation; paper, leading order)**.** *Let $w = \mathrm{HK}_\beta e^{\sigma\theta^4}$ and $u = 1/\beta$. To first order in σ and up to images of order $e^{-\beta\pi^2/2}$,*

$$\Delta \log a_n = \sigma(q_n(q_n+2)u^4 - 10 q_n u^3) + O(\sigma u^5), \qquad e_n = \sigma\, q_n(q_n - 3)\, u^4 + O(\sigma u^5).$$

Proof. By Poisson summation $\mathrm{HK}_\beta(\theta) \propto (\theta/\sin\theta)e^{-\beta\theta^2/2}$ up to images. With the Haar weight $\sin^2\theta$ and $\chi_n/(n+1) = \sin(k\theta)/(k\sin\theta)$, $k = n+1$, the integrand of a_n becomes $\theta^2 e^{-\beta\theta^2/2}\sin(k\theta)/(k\theta)$. This is the plane-wave average $\mathbb{E}[e^{i\,k\cdot x}]$ of a centred Gaussian x on $\mathbb{R}^3$ with covariance $u\,I$ and $r = |x|$. Since $c_0 = 1$ and $n = 0$ corresponds to $k = 1$, $\Delta\log a_n = \sigma(\mathbb{E}_k[r^4] - \mathbb{E}_1[r^4])$ to first order, where $\mathbb{E}_k$ is the expectation tilted by the plane wave. The complex shift $x \mapsto x + iuk$ gives $\mathbb{E}_k[r^4] = 15u^2 - 10k^2u^3 + k^4u^4$. With $k^2 = q_n + 1$ the difference is $\sigma\big((q_n+1)^2 - 1\big)u^4 - 10\sigma q_n u^3 = \sigma\big(q_n(q_n+2)u^4 - 10q_nu^3\big)$. For $n = 1$ ($q = 3$) this is $\sigma(15u^4 - 30u^3)$, and

$$e_n = \Delta\log a_n - \tfrac{q_n}{3}\Delta\log a_1 = \sigma(q_n^2 + 2q_n - 5q_n)u^4 + \sigma(-10q_n + 10q_n)u^3 = \sigma q_n(q_n - 3)u^4.$$

The u^3 terms cancel exactly, which is the point of dividing out the direction of β. □

On the curve of §4, $e_3/(q_3^2u^4) = 2.23\times10^{-4} = 0.80/3600$ in double precision, in agreement with $(q_3 - 3)/q_3 = 0.8$. The term Aq_nu^3 gives room to the low rows.

Why the bounds are ratios of non-negative sums. Write $\ell' = \log a(Tw)$. Since a_i' is the fourth power of a normalised coefficient of w^4, $\log P_i$ with $P = c^{\star4}$ (four-fold Clebsch–Gordan convolution) determines ℓ'. The derivatives of $\log P_i$ in the logarithmic coordinates are the mean, variance and higher cumulants of a probability distribution on the quadruples of input indices that contribute to the output index i. These weights are non-negative. Hence the first derivative is a ratio of non-negative sums, the second derivative in the transverse direction is a variance $\mathrm{Var}_i(X_e) \in [0, \mathbb{E}_i[(\sum|e|)^2]]$, and the third-order remainder is bounded by $8\,\mathbb{E}_i[(\sum_k D_{j_k})^3]$ with $D_j = RW_j + q_j|\delta u|/2$. Each bound is a ratio of non-negative sums, monotone in each summand. It can therefore be evaluated with the numerator at the upper end of a box and the denominator at the lower end, and no cancellation of large terms occurs. This is why the tube survives interval evaluation at all: on boxes of coefficients the same computation loses control already at widths of order 10^{-12}.

The cumulant form. Let Q_i be the set of ordered quadruples $J = (j_1, \dots, j_4)$ for which χ_i occurs in $\chi_{j_1}\chi_{j_2}\chi_{j_3}\chi_{j_4}$, with multiplicity $m_J \geq 0$. Then the coefficient of χ_i in w^4 is

$$P_i = \sum_{J\in Q_i} m_J \prod_{k=1}^{4}(j_k+1)\,e^{\ell_{j_k}}, \qquad \ell_j = \log a_j,$$

and $\ell_i' = 4\big(\log P_i - \log P_0 - \log(i+1)\big)$. Put $\pi_i(J) \propto m_J\prod_k(j_k+1)e^{\ell_{j_k}}$, a probability on Q_i, and $X_\Delta(J) = \sum_k \Delta_{j_k}$. Then

$$D\log P_i[\Delta] = \mathbb{E}_{\pi_i}[X_\Delta], \qquad D^2\log P_i[\Delta,\Delta] = \mathrm{Var}_{\pi_i}(X_\Delta),$$
$$D^3\log P_i[\Delta,\Delta,\Delta] = \kappa_3^{\pi_i}(X_\Delta) \quad \text{(third cumulant)},$$

and $|\kappa_3| \leq 8\,\mathbb{E}_{\pi_i}[|X_\Delta|^3]$. Each row of the transverse linearisation is $4(\mathbb{E}_{\pi_i} - \mathbb{E}_{\pi_0})$ applied to X_e. The matrix of π_i-occupation numbers is non-negative with row sums 4. In double precision the spectral radius of the absolute value of the projected linearisation is 0.136–0.17. This is why a one-step weighted box suffices and no adapted multi-step norm is needed.

The tail. If the input tail satisfies $\sum_{n>N}(n+1)\rho^n|c_n| \leq t_0$ and S bounds $\|c^{\leq N}\|_\rho$, the relative change of P_i caused by the tail is at most $\delta_i \leq \big((S+t_0)^4 - S^4\big)/\big((i+1)\rho^iP_i\big)$, by submultiplicativity of $\|\cdot\|_\rho$. The image tail is bounded by the computed part for $N_t < n \leq N$ plus $(\Pi/P_0)^4\sum_{n>N}(n+1)^{-6}(\rho_t/\rho^4)^n$. On $[2, 64]$ the largest ratio of image tail to t_0 is 3.5×10^{-3} and the largest δ_i is 3.0×10^{-13}. On $[64, 128]$ the tail ratio is at most 5.9×10^{-4}.

The cell test. A cell is an interval $[u_{\rm lo}, u_{\rm hi}]$ of width 0.02 in β, with centre $h = \mathrm{HK}_{1/u_c}$ truncated at N. A point of the tube in the cell is $\ell(h) + \Delta$ with $\Delta = e + v\,\delta u$, $v = -q/2$. We use

$$\ell'(h+\Delta) = \ell'(h) + Je + g_1\delta u + \tfrac{1}{2}g_2\delta u^2 + He\,\delta u + \tfrac{1}{2}D^2\ell'(h)[e,e] + R_3, \tag{1}$$

with the point quantities $\ell'(h)$, J, g_1, g_2, H evaluated at the centre in interval arithmetic, and the second order in e and the remainder R_3 bounded as above. The tail is handled in the norm $\|f\|_\rho = \sum(n+1)\rho^n|c_n|$, which is submultiplicative since $\chi_m\chi_k = \sum_{j=|m-k|}^{m+k}\chi_j$. The cell *passes* if, after the projection $(Py)_i = y_i - (q_i/3)y_1$, the resulting bound on $|e'_i|$ is at most $RW_i(u'_{\rm lo})$ for all $2 \le i \le N$, the image tail is at most the target t_0, and the image coupling β' lies in blocks whose parameters were used. We write ratio for the maximum over rows of bound/requirement.

Arithmetic. Arithmetic operations assume correctly rounded IEEE 754 double precision and are widened by one ulp outward per operation. Sums of non-negative terms are bounded with Higham's γ_{2N+6} [Hi]. The functions exp and log are evaluated in 34-digit decimal arithmetic and then widened by one ulp. The library is self-written and has not been checked against an established interval package (in the sense of [Mo, Tu]). The rounding model and the derivation of the remainder and cumulant bounds are on paper and are not formalised.

3 Theorem H-UV: invariance and one-step drop

Theorem 3.1 (H-UV: invariance of the tube and one-step drop of the coupling; computer-assisted (interval arithmetic with a self-written library; not independently verified)). *Let T be the MK map of §1 on SU(2) plaquette weights, and $\mathcal{U} = \mathcal{U}_{10^{-3},10}$ the tube of Definition 2.2 with the tabulated tail parameters. Assume correctly rounded IEEE 754 arithmetic and the paper bounds of §2 for the remainder and cumulants. Then:*

(i) (invariance) for every $w \in \mathcal{U}$ with $\beta(w) \in [2.46, 128]$, $Tw \in \mathcal{U}$; this is verified on 6277 *consecutive cells of width* 0.02*;*

(ii) (heat-kernel points) for the 3075 *cell centres $\beta \in [2.5, 64]$, $(\beta - \beta(T\mathrm{HK}_\beta) - \frac{1}{4})\beta \in [0.020928, 0.023690]$, each enclosure of width at most 10^{-6}, decreasing towards $1/48 = 0.020833$;*

(iii) (one-step drop) for every $w \in \mathcal{U}$ with $\beta(w) \in [2.46, 128]$, $\beta(w) - \beta(Tw) \ge 0.2195$, and $\beta(w) - \beta(Tw) \le 0.2990$ for $\beta(w) \le 64$.

The 23 cells in $[2.00, 2.46)$ fail with the same parameters (ratio up to 9.7). Table 1 lists representative cells.

Controls. (a) Decimation alone, in interval arithmetic: $\max|e(\mathrm{DHK}_\beta)| \le 1.3 \times 10^{-11}$ and $u'/u - 4 \in \pm 6 \times 10^{-14}$ at $\beta = 4.02, 8.02, 32.02, 64$. (b) An independent implementation in position space (trapezoidal rule, 40001 nodes, no interval arithmetic) was applied to 20 random points of the tube per tested cell. It gives $|e'|/\text{bound} \le 0.81$ and u' inside the certified range; the bounds are 1.2 to 1.9 times the true values. (c) Negative controls fail as they should. With $R = 2 \times 10^{-4}$ the ratio is 1.45–1.47 (at $\beta = 8, 64, 96, 128$). Adding $1.0 \cdot RW_2$ to the image coefficient ℓ'_2 gives 1.09–1.28, while $0.3 \cdot RW_2$ passes. (d) The 200 cells in $[60, 64]$ were recomputed on a second machine; all columns except the timing agree as strings. These controls are evidence that the implementation does what is described. They are not a substitute for independent verification of the library.

β	verdict	ratio	centre defect	remainder + tail	λ_{tube}
$[2.00, 2.46)$	23 cells fail	up to 9.7			$\gg 1$
2.50	pass	0.889	0.348	0.777	15.1
8.00	pass	0.516	0.268	0.171	0.811
16.00	pass	0.501	0.257	0.120	0.461
32.00	pass	0.496	0.253	0.098	0.282
63.98	pass	0.494	0.253	0.088	0.269
$[64, 128]$ (3200 cells)	all pass	≤ 0.494			≤ 0.269

Table 1: The cell test for the tube $\mathcal{U}_{10^{-3},10}$, cell width 0.02. The columns "centre defect" and "remainder + tail" are row-wise maxima divided by $R\,W_i(u_c)$; they are attained in different rows and do not add up to the ratio. λ_{tube} is the interval bound on the norm of the transverse linearisation over the whole cell (§4). Certificates: `main_lo.csv`, `main_hi.csv` ($\beta \in [2, 64]$, 3100 cells), `main_128.csv` ($[64, 128]$, 3200 cells).

The coupling flow across the whole tube. At the heat-kernel points the enclosure of $(\beta - \beta' - \frac{1}{4})\beta$ is 0.023690 at $\beta = 2.5$, 0.022506 at 4, 0.021629 at 8, 0.021222 at 16, 0.021025 at 32 and $[0.0209283, 0.0209293]$ at 64. Over the whole tube, $\beta - \beta' - \frac{1}{4}$ lies in $[-0.0106, 0.0295]$ at $\beta \approx 2.5$, $[-0.0031, 0.0085]$ at 8, $[-0.0007, 0.0021]$ at 32 and $[-0.0004, 0.0010]$ at 64. This width comes mainly from the term $Aq_n u^3$ of the tube weight, and it is what prevents $\mathcal{U}$ from separating $1/60$ from $1/48$ (§4).

Cost. The cost of the four-fold convolution is of order N^3 per cell ($N = 232$ at $\beta = 128$), and the convolution table has $(N+1)^3$ entries. How far the upper end can be raised is limited by the size of this table rather than by the bounds, up to the rounding effect described in §6.

4 The attracting curve and the coupling flow

4.1 Leading order in $1/\beta$

Write $w = \mathrm{HK}_\beta\, e^S$ with $S = \sum_{k\geq 2} \sigma_k \theta^{2k}$.

Proposition 4.1 (shape flow; paper, leading order)**.** *To leading order in $1/\beta$ (saddle-point order; error constants are not derived), one step of T acts on the shape coefficients by*

$$\sigma_k \longmapsto \frac{16\,\sigma_k + 12\,\ell_k}{4^{2k}}, \qquad k \geq 2.$$

Hence the θ^{2k} direction contracts by 16^{1-k} ($1/16$, $1/256$, ...), the fixed point is $\sigma_k^* = 12\ell_k/(4^{2k} - 16)$, in particular $\sigma_2^* = 1/3600$ and $\sigma_3^* = 1/963900$, and one step from the heat kernel gives $\sigma_2 = 1/3840$.

Sketch. Bond moving: $w^4 = \mathrm{HK}_{4\beta}(\theta/\sin\theta)^3 e^{4S} = \mathrm{HK}_{4\beta-1} e^{\rho+4S}$ up to images, with $\rho = \sum_{k\geq 2} 3\ell_k\theta^{2k}$. Decimation: the heat-kernel part is a semigroup, $\mathrm{HK}_{(4\beta-1)/4} = \mathrm{HK}_{\beta-1/4}$. For the factor e^S the saddle point places the four group elements equally spaced along the geodesic from the identity, giving $4S(\theta/4)$, while fluctuation corrections of order S''/β move to lower powers of θ (a triangular structure). Collecting the coefficient of θ^{2k} gives the map. □

Proposition 4.2 (coupling flow; paper, leading order)**.** *For $w = \mathrm{HK}_\beta e^{\sigma\theta^4}$,*

$$\beta(Tw) - \beta(w) = -\frac{1}{4} - \frac{c(\sigma)}{\beta} + O(\beta^{-2}), \qquad c(\sigma) = \frac{1}{48} - 15\sigma.$$

On the heat-kernel class $c = 1/48$; at $\sigma = \sigma_2^* = 1/3600$, $c^* = 1/48 - 1/240 = 1/60$.

Sketch. In $\mathbb{R}^3$-Gaussian form, $\mathrm{HK}_\beta \chi_n/(n+1)\sin^2\theta \propto \theta^2 e^{-\beta\theta^2/2}\sin(k\theta)/(k\theta)$ with $k = n+1$, which is the plane-wave average of a centred Gaussian on $\mathbb{R}^3$ with variance $1/\beta$. A complex shift gives $\mathbb{E}_k[r^4] = 15/\beta^2 - 10k^2/\beta^3 + k^4/\beta^4$. For $n = 1$ this yields $\log a_1 = -\frac{3}{2}u - 30\sigma u^3 + O(u^4)$, i.e. $\beta(w) = \beta - 20\sigma/\beta + \dots$. After bond moving the weight is $\mathrm{HK}_{\tilde\beta} e^{\rho_2\theta^4 + \cdots}$ with $\tilde\beta = 4\beta - 1$, $\rho_2 = 1/60 + 4\sigma$, and decimation raises a_1 to the fourth power exactly, so $\beta' = \beta - \frac{1}{4} - 5\rho_2/(4\beta) + O(\beta^{-2})$. Subtracting gives $c(\sigma)$. □

The arithmetic in both propositions (fixed points, contraction ratios, $1/48 - 15/3600 = 1/60$) is checked in Lean (§7); the saddle-point steps are not. In double precision, after eight steps from the heat kernel the flow coefficient reads $1/60 + 0.0078/\beta$; the largest eigenvalue of the transverse linearisation is $(1/16)(1 + 1.0/\beta)$ at $\beta = 4, \dots, 32$ and the second approaches $1/256$. These are observations, not part of any statement.

4.2 The attracting curve on a finite interval

We work on the cells with $\beta \in [5, 64]$ (width 0.02, 2950 cells, all of which also pass the invariance test). On these cells we compute the following in interval arithmetic, over the whole tube in each cell and in the normalised coordinate $\tilde y = e/(RW(u))$: the Lipschitz bound λ of $\tilde y \mapsto \tilde y'$, the bound $\tilde\mu$ of $|\partial\tilde y'/\partial\beta|$, the range $[\kappa_{\mathrm{lo}}, \kappa_{\mathrm{hi}}]$ of $\partial\beta'/\partial\beta$, and the bound $\tilde\delta$ of $|\partial\beta'/\partial\tilde y|$.

Lemma 4.3 (graph transform on a finite interval; paper)**.** *Suppose that on every cell of $I = [\beta_a, 64]$, for some $\tilde M > 0$, (a) $\kappa_{\mathrm{eff}} := \kappa_{\mathrm{lo}} - \tilde\delta(\tilde M + 4/\beta) > 0$, (b) $\tilde\mu + \lambda\tilde M \le \tilde M\kappa_{\mathrm{eff}}$, and (c) $\Lambda := \lambda + (\tilde\mu + \lambda\tilde M)\tilde\delta/\kappa_{\mathrm{eff}} < 1$. Then the image under T of a graph $\tilde y = \varphi(\beta)$ with $|\varphi| \le 1$ and $\mathrm{Lip}(\varphi) \le \tilde M$ is again such a graph, and for two such graphs $\|\Gamma\varphi_1 - \Gamma\varphi_2\| \le \Lambda\|\varphi_1 - \varphi_2\| + \varepsilon_{\mathrm{tail}}$, where $\varepsilon_{\mathrm{tail}} \le 6 \times 10^{-5}$ is the contribution of the tail beyond N.*

Proof. By (a), $\beta \mapsto \beta'$ is strictly increasing on the graph, so the image is a graph. By (b), its Lipschitz constant is at most $\tilde M$. By Theorem 3.1, it stays in $|\tilde y| \le 1$. When the images are compared at the same β', the preimages satisfy $|\beta_1 - \beta_2| \le \tilde\delta\|\varphi_1 - \varphi_2\|/\kappa_{\mathrm{eff}}$, and the mean-value theorem row by row gives the contraction. This is the finite-interval form of Hadamard's graph transform. □

Theorem 4.4 (orbits approach each other on a finite interval; computer-assisted (interval arithmetic with a self-written library; not independently verified) for the inputs, certificate files not distributed; paper for Lemma 4.3)**.** *With $\tilde M = 80$, conditions (a)–(c) hold on every cell of $[15.26, 64]$ with $\Lambda \le 0.735$; with $\tilde M = 30$ they hold on $[23.62, 64]$ with $\Lambda \le 0.384$. Consequently, for $\beta \in [15.26, 64]$ the images of any two orbits in $\mathcal{U}$ that enter from the top agree after k steps up to $2\Lambda^k + \varepsilon_{\mathrm{tail}}/(1-\Lambda)$ in the normalised coordinate; since one step lowers β by at most 0.299, at $\beta = 32$ this is at most 1.6×10^{-14} plus the tail floor 2.3×10^{-4}.*

We call the curve so determined $\mathcal{C}$; it depends on the input at the upper end and is not shown to be unique. Representative inputs (normalised coordinate, whole tube per cell) are: at $\beta = 8$, $\lambda = 0.811$, $\tilde\mu = 22.3$, $[\kappa_{\mathrm{lo}}, \kappa_{\mathrm{hi}}] = [0.949, 1.052]$, $\tilde\delta = 6.7 \times 10^{-3}$; at $\beta = 16$, 0.461, 19.2, $[0.979, 1.021]$, 3.0×10^{-3}; at $\beta = 32$, 0.282, 17.5, $[0.991, 1.010]$, 1.4×10^{-3}; at $\beta = 64$, 0.269, 16.6, $[0.996, 1.005]$, 6.8×10^{-4}. The bound $\tilde\mu$ is dominated by the third-order remainder. The point part of $|\partial e'/\partial\beta|$ is about $1.2 \times 10^{-4}/R$, consistent with the β-derivative $4\sigma u$ of the θ^4-type defect. Other choices $\tilde M = 50, 60, 100, 120$ give the lower ends 16.42, 15.62, 15.54, 16.06. Of the 2950 cells, 374 were computed on two machines with bitwise identical output.

Uniformity in the number of steps. If d_k is the transverse distance to $\mathcal{C}$ after k steps, then $d_{k+1} \le \Lambda d_k$ and $d_K \le \Lambda^K d_0$. If instead the distance is measured to the heat-kernel class, the class is not invariant and $d_{k+1} \le \Lambda d_k + \varepsilon$ with ε the size of the $\theta^4/3600$ shape, so $d_k \le \Lambda^k d_0 + \varepsilon/(1-\Lambda)$, uniformly in k (`affine_contraction`). On the coupling side, Proposition 4.2 gives, to leading order, $\beta_{\mathrm{end}} = \beta_{\mathrm{start}} - K/4 - \sum_k c^*/\beta_k + O(\sum_k \beta_k^{-2})$ with $\sum_k c^*/\beta_k \approx 4c^* \log(\beta_{\mathrm{start}}/\beta_{\mathrm{end}})$. If the coupling read backwards grows by at least $c > 0$ per step, then $\sum_k \beta_k^{-2} \le \beta_{\mathrm{end}}^{-2} + 1/(c\,\beta_{\mathrm{end}})$ (`inv_sq_sum_le`); with $c = 0.2195$ this is about $4.6/\beta_{\mathrm{end}}$. Within $[15.26, 64]$ the weight after K steps is thus a point of $\mathcal{C}$ determined by β_{end} up to a term Λ^K. The statement stops at the finite upper end. Pointwise transverse contraction $\lambda_{\mathrm{tube}} < 1$ holds on $[5.88, 64]$ (Table 1). This bound is far from the true rate $1/16$ because it uses a weighted sup norm and covariance bounds. For general k the leading-order contraction of the θ^{2k} direction is 16^{1-k}. At leading order the curve is therefore a graph over β whose higher shape coefficients $\sigma_k^* = 12\ell_k/(4^{2k} - 16)$ decrease geometrically ($\sigma_3^*/\sigma_2^* = 3600/963900 \approx 1/268$). Theorem 4.4 does not use this and works with the full tube. Since the upper end 64 is finite, uniqueness of $\mathcal{C}$ as $\beta \to \infty$ is *not* established. The inputs of Lemma 4.3 were not computed on $[64, 128]$.

4.3 Enclosing the flow coefficient

The tube $\mathcal{U}$ is too wide to separate $c^* = 1/60$ from $1/48$: across $\mathcal{U}$, $\beta - \beta' - \frac{1}{4}$ varies by about ± 0.0014 at $\beta = 32$. One can re-centre the tube at the leading transverse shape of $\mathrm{HK} \cdot e^{\theta^4/3600}$, i.e. $e_n = \sigma r_n u^4$ with $r_n = q_n(q_n - 3)$ and $\sigma = \mathrm{fl}(1/3600)$. This does not help by itself while the heat kernel ($e \equiv 0$) remains inside the tube: the flow coefficient is continuous on the tube, so its enclosure must contain the heat-kernel value. Excluding the heat kernel requires $RW_n < \sigma r_n u^4$ for some n; with $A = 0$ this means $R \lesssim 2.7 \times 10^{-4}$. We therefore use the tube $\mathcal{U}_\sigma = \{|e_n - \sigma r_n u^4| \le RW_n(u)\}$ with $R = 10^{-4}$, $A = 0$ and cell width 0.002.

Proposition 4.5 (enclosure of the flow coefficient; computer-assisted (interval arithmetic with a self-written library; not independently verified), 50 cells only)**.** *On each of the* 50 *cells of width* 0.002 *listed in Table 2 (ten consecutive cells starting at* $\beta = 16, 32, 64, 96$ *and ending at* 128*),* $T(\mathcal{U}_\sigma) \subset \mathcal{U}_\sigma$ *and, for every* $w \in \mathcal{U}_\sigma$ *in the cell,* $(\beta(w) - \beta(Tw) - \frac{1}{4})\beta(w) \in [0.0151, 0.0188]$*. This interval contains* $1/60 = 0.01667$ *and contains neither* $1/48 = 0.02083$ *nor the heat-kernel value at* $\beta = 64$ *(Theorem 3.1(ii)).*

cells	passed	ratio (row)	centre value of $(\beta - \beta' - \frac{1}{4})\beta$	enclosure over $\mathcal{U}_\sigma$
$[16.00, 16.02]$	10/10	0.288 ($n = 77$)	$[0.017239, 0.017240]$	$[0.01570, 0.01878]$
$[32.00, 32.02]$	10/10	0.200 ($n = 112$)	$[0.016951, 0.016951]$	$[0.01539, 0.01851]$
$[64.00, 64.02]$	10/10	0.207 ($n = 2$)	$[0.016808, 0.016809]$	$[0.01524, 0.01838]$
$[96.00, 96.02]$	10/10	0.360 ($n = 2$)	$[0.016759, 0.016763]$	$[0.01519, 0.01833]$
$[127.98, 128.00]$	10/10	0.781 ($n = 2$)	$[0.016732, 0.016742]$	$[0.01516, 0.01831]$

Table 2: The re-centred tube $\mathcal{U}_\sigma$ ($R = 10^{-4}$, $A = 0$, width 0.002). Certificate: `cstar_A0.csv`. The centre values lie within 10^{-4} of the double-precision reading $1/60 + 0.0078/\beta$.

Remark 4.6. Proposition 4.5 is a statement about $\mathcal{U}_\sigma$, not about $\mathcal{C}$. It becomes a statement about the coefficient on the curve only under the additional hypothesis $\mathcal{C} \subset \mathcal{U}_\sigma$, which is not shown. Showing it would need invariance of $\mathcal{U}_\sigma$ on a connected range from the top and the inputs of Lemma 4.3 for $\mathcal{U}_\sigma$. With cell width 0.02 the same tube fails: the remainder exceeds the requirement 62-fold at $\beta = 64$. Covering $[64, 128]$ would therefore need 32000 cells.

5 Theorem MK-full: from weak to strong coupling

The earlier certificate W-half(T)**.** W-half(T) works in the coordinate $t := a_1$ of the normalised coefficients a_n of §1. Hence

$$t = a_1 = \exp(-3/(2\beta)).$$

For $\rho > 1$ write $\chi_n(\rho) := \sum_{k=0}^{n} \rho^{n-2k}$; this is the character χ_n continued to the point $\mathrm{diag}(\rho, \rho^{-1})$ of the complexified torus. Put $p_1(n) := n(n+1)(n+2)/3$ and $p_2(n) := (n-1)n(n+1)(n+2)(n+3)/15$.

Definition 5.1 (the tube $\mathcal{D}$ and the region $\mathcal{K}$)**.** Let $J = 20$.

(a) *Reference curve.* For each of the 33 values $\beta_0 \in \{2, 2.125, \dots, 6\}$, start from the Wilson weight $e^{\beta_0 \cos\theta}$ ($a_n = I_{n+1}(\beta_0)/I_1(\beta_0)$, $n < 60$) and iterate T in double precision with 60 characters, the integrals over θ by a Riemann sum on the 7999 interior points of a uniform grid of 8001 points on $[0, \pi]$ and the weight rescaled to maximum 1 before the fourth power; keep the coefficient vectors $(a_0, \dots, a_{20})$ of the iterates from the third on with $0.004 < a_1 < 0.95$, and stop once $a_1 < 0.004$. Sorted by a_1, this gives 405 nodes (S_j, P_j). For $0 \le n \le 20$, $\gamma_n(t)$ is the piecewise-linear interpolation of the values $P_{j,n}$ at the nodes S_j, constant outside $[S_1, S_{405}]$.

(b) *Tube.* $\mathcal{D}$ is the set of weights with $a_0 = 1$, $a_n \ge 0$ for all n, $a_n \le \gamma_n(t) + 0.005\, t^n$ for $2 \le n \le 20$, and $\sum_{n>20}(n+1)a_n\chi_n(2) \le 10^{-15}$. (Images under T have $a_n \ge 0$, since decimation takes fourth powers.)

(c) *Region.* For a weight with $F(w) := 1 - \sum_{n\ge1}(n+1)^2 a_n > 0$ put $D_i(w) := \sum_{n\ge1}(n+1)a_n\, p_i(n)$ $(i = 1, 2)$ and

$$\kappa(w) := 1 - 12\Big(\frac{D_1}{F} + \frac{D_2}{F} + \Big(\frac{D_1}{F}\Big)^2\Big).$$

$\mathcal{K}$ is the set of weights with $a_0 = 1$, $t(w) \le 0.01658$, $F(w) > 0$ and $\kappa(w) > 0$.

We use κ only through its formula and $\mathcal{K}$ only as a region. We do not state what positivity of κ implies for the hierarchical measure, and we do not use the words confinement or mass gap for it.

Proposition 5.2 (W-half(T): from the strong-coupling tube $\mathcal{D}$ into the region $\mathcal{K}$; computer-assisted, Appendix A)**.** *Cover $[0.0144, 0.21)$ by cells $[s_k, s_{k+1})$ with $s_{k+1} = s_k(1+r)$, $r = 0.005$ on $[0.0144, 0.198)$ and $r = 0.0025$ on $[0.198, 0.21)$ (556 cells, overlapping where the computation was restarted). Precisely, the cells are the 556 cells recorded in the logs of the bundle (Table 6); each is a cell of the grid of one of the ten ranges listed in Appendix A (*Output*), where in a range $[\ell, h]$ the grid starts at $s = \ell$ and continues by $s \mapsto \min(s(1+r), h)$; four ranges start at the restart points 0.14836, 0.16711, 0.18639, 0.20452, and a run that was stopped before the end of its range contributes only the cells it logged. The logged cells cover $[0.0144, 0.21)$ without gaps. The cells are numbered in increasing order of their lower ends s_k.*

(i) For every cell and every $w \in \mathcal{D}$ with $t(w) \in [s_k, s_{k+1}]$: $t(Tw) < s_k$, and either $Tw \in \mathcal{D}$, or $t(Tw) \le 0.0165$ and $Tw \in \mathcal{K}$.

(ii) Every $w \in \mathcal{D}$ with $t(w) \le 0.01658$ lies in $\mathcal{K}$; on this set $\kappa \ge 0.0618$.

Consequently the cell index of an orbit in $\mathcal{D}$ with $t < 0.21$ decreases strictly at each step until the orbit is in $\mathcal{K}$, so $\mathcal{K}$ is reached within a number of steps bounded by the number of cells.

The consequence is immediate from (i) and (ii). A point of $\mathcal{D}$ with $t \le 0.01658$ is in $\mathcal{K}$ by (ii). Otherwise it lies in a cell, and by (i) its image is either in $\mathcal{K}$ or in $\mathcal{D}$ in a cell of smaller index.

The tube $\mathcal{U}$ of Theorem 3.1 covers $t \in [0.5435, 0.988]$ and W-half(T) covers $t < 0.21$; the gap is $\beta \in [0.961, 2.46)$. In the transverse coordinate, the upper boundary of $\mathcal{D}$ is $e_n \le E_n(t) := \log(\gamma_n(t) + 0.005t^n) - \frac{q_n}{3}\log t$. In row $n = 2$ it lies only 0.014–0.03 above γ itself, so no tube centred at the heat kernel is contained in $\mathcal{D}$. We join the two by inclusion of *images*, not of tubes. For orientation: the region $\mathcal{K}$ corresponds to $\beta \le 0.366$, and the range of W-half(T) to $\beta \in [0.354, 0.961)$. The tube $\hat{\mathcal{U}}$ below covers $t \in [0.2062, 0.988]$ and therefore overlaps the range of W-half(T) in $t \in [0.2062, 0.21)$, but the overlap is not used as an inclusion of tubes.

The extended tube $\hat{\mathcal{U}}$. For $\beta > 2.46$, $\hat{\mathcal{U}}$ is the tube $\mathcal{U}$. For $\beta \le 2.46$, on pieces $(2.46 - 0.05(m+1),\ 2.46 - 0.05m]$ with $x = 0.05(m+1)$, put

$$R_m = \max\bigl(4 \times 10^{-4},\ 10^{-3} - 6 \times 10^{-4}\max(0, x - 0.3)\bigr),$$
$$A_m = \max\bigl(0,\ 10 - 10\max(0, x - 0.3)\bigr),$$

with tail parameters on blocks of width 0.1 ($N = 24$ at $\beta = 2.46$ down to $N = 13$ at 0.96); both are non-increasing in m, and at a boundary point the smaller tube applies. The cell test is that of §2 with cell width 0.002, except that the requirement on the image is replaced. The part of the image with $\beta' \ge 0.95$ must lie in $\hat{\mathcal{U}}$; it is checked against every piece and block it meets. The part with $\beta' < 0.95$ must lie in $\mathcal{D}$. For the latter, the range of a_1' is split into 64 subintervals. On each, rows $2 \le n \le \min(N, 20)$ are checked as $\exp(\frac{q_n}{3}\log v + \mathrm{bound}_n) \le \inf_{[u,v]}(\gamma_n + 0.005t^n)$ with relative margin 10^{-9}, rows up to 20 beyond N through the image tail norm, and the tail beyond 20 through $\chi_n(2) \le \frac{4}{3}2^n$ and $\rho \ge 2$. Non-negativity $a_n' \ge 0$ is automatic since decimation takes fourth powers.

Lemma 5.3 (soundness of the exit test; paper)**.** *Let $[u_0, v_0]$ enclose a_1' over the image of a cell, with $v_0 < 0.21$, and let $[u_0, v_0] = \bigcup_m [u_m, v_m]$. Suppose that for every m and every $2 \le n \le \min(N, 20)$, $\exp(\frac{q_n}{3}\log v_m + b_n) \le \inf_{t\in[u_m,v_m]}(\gamma_n(t) + 0.005\,t^n)$, where $|e_n'| \le b_n$ on the image, and that the rows $N < n \le 20$ and the tail beyond 20 satisfy the bounds stated before this lemma. Then the image of the cell lies in $\mathcal{D}$.*

Proof. Take a point of the image with $a_1' = t \in [u_m, v_m]$. Then $a_n' = \exp(\frac{q_n}{3}\log t + e_n') \le \exp(\frac{q_n}{3}\log v_m + b_n)$ because log is increasing and $q_n > 0$, and the right-hand side is at most $\gamma_n(t) + 0.005t^n$ by hypothesis. Since $\mathcal{D}$ is stated for the normalised coefficients a_n (Definition 5.1), the row condition holds. The lower bound $a_n' \ge 0$ holds since a_n' is a fourth power. The remaining rows and the tail are bounded through the image norm. The infimum of γ_n over $[u_m, v_m]$ is attained at an end point or at a break point of the piecewise-linear γ_n, and is evaluated with a relative margin 10^{-9}. □

Results of the joining computation (computer-assisted (interval arithmetic with a self-written library; not independently verified)). All 755 cells of $\beta \in [0.95, 2.46]$ and the 27 seam cells in $[2.46, 3.0]$ pass (782 cells, certificate `fprof_F1.csv`). The largest image ratio is 0.934 and the largest ratio against $\mathcal{D}$ is 0.9888. For the 139 exit cells ($\beta \in [0.95, 1.228]$), the parts of the images with $\beta' < 0.95$ lie in $\beta' \in [0.664, 0.95)$ and $a_1' \in [0.1046, 0.2062)$, all inside the range of W-half(T). For a few cells near $\beta = 1.22$ the image also reaches above 0.95; that part is checked against $\hat{\mathcal{U}}$. Other profiles of (R_m, A_m) also reach 0.95: the largest image ratios are 0.878, 0.9999, 0.995 for three variants, and 0.929 with cell width 0.001 on 1537 cells. By contrast, keeping $R = 10^{-3}$, $A = 10$ and integer tail blocks leaves the lower end at 2.46 or, with partial narrowing, between 2.17 and 2.50, limited by the last row $n = N$. Two details matter. First, the term Aq_nu^3, which helps at weak coupling, enlarges the remainder at strong coupling: for one cell at $\beta = 1.0$ and $R = 10^{-3}$ the ratio is 17.4 with $A = 10$ and 0.586 with $A = 0$, $R = 6 \times 10^{-4}$. Second, row $n = 2$ of $\mathcal{D}$ has a margin of only 1–7% over

the image of the heat kernel itself (ratio 0.93–0.99 for $\beta \in [0.65, 1.3]$ with $R \to 10^{-5}$), which is why R is lowered to 4×10^{-4}. Without the subdivision of the a_1'-range, the correlation between the upper bound of a_n' and the lower bound of γ_n is lost and the exit cells fail with ratio 1.03–1.06. In double precision, Wilson-action orbits started at $\beta = 3, 6, 12$ stay inside a tube of radius $R \approx 2.6 \times 10^{-4}$ (with $A = 10$) down to $\beta \approx 0.4$ after a transient of two or three steps. The failure of the unmodified tube below 2.46 therefore reflects the bounds, not the orbits.

On $\beta \in [0.95, 64]$ the one-step drop satisfies $\beta(w) - \beta(Tw) \in [0.2195, 0.2990]$; on $[64, 128]$ it is at least 0.2296. The smallest guaranteed drop, $2.46 - 2.24045 = 0.21955$, occurs on the cell $[2.46, 2.48]$.

Theorem 5.4 (MK-full: step count from $\beta \le 128$ into $\mathcal{K}$; computer-assisted (interval arithmetic with a self-written library; not independently verified) inputs, Lean assembly with a general drop)**.** *Let T be the MK map on SU(2) plaquette weights, $\hat{\mathcal{U}}$ the extended tube, $\mathcal{D}$ and $\mathcal{K}$ the tube and region of Definition 5.1. Assume:*

(a) the interval arithmetic of §2 is sound (self-written library; IEEE 754 correct rounding and Higham's bound assumed; the derivation of the remainder and cumulant bounds is on paper and not formalised);

(b) Proposition 5.2 holds, i.e. the certificate W-half(T) is correct (Appendix A: interval arithmetic of mpmath, a double-precision reference curve evaluated with relative margin 10^{-9});

(c) the reference curve used in the joining computation, recomputed by the construction of Definition 5.1(a), is the curve γ of $\mathcal{D}$ (the two computations agree at all 405 nodes to within 9×10^{-14}, less than 2×10^{-10} of the margin $0.005\, t^n$; $\gamma_2(0.3)/0.3^2 = 0.451$);

(d) the bound 557 for the second stage counts cells of W-half(T) and is crude;

(e) at the boundary between two pieces of $\hat{\mathcal{U}}$ the intersection of the two tubes is used.

Then every $w \in \hat{\mathcal{U}}$ with $0.95 \le \beta(w) \le 128$ satisfies $T^j w \in \mathcal{D}$ for some

$$j \le \left\lceil \frac{\beta(w) - 0.95}{0.2195} \right\rceil + 1 \le 580,$$

and then $T^{j+i} w \in \mathcal{K}$ for some $i \le 557$; in total at most 1137 steps. For $\beta(w) \le 64$ the bounds are 289 and 846.

Proof. Above 0.95 each step either stays in $\hat{\mathcal{U}}$ with a drop of at least 0.2195 or lands in $\mathcal{D}$ (Theorem 3.1(i),(iii) on $[2.46, 128]$, the joining computation below 2.46). Hence after at most $\lceil (\beta(w) - 0.95)/0.2195 \rceil$ steps the orbit has left $\hat{\mathcal{U}}$ through its lower end, and the exit cells map into $\mathcal{D}$. Inside $\mathcal{D}$ the cell index decreases strictly at each step until $\mathcal{K}$ is reached (assumption (b)). The composition with a general drop $\delta > 0$ is checked in Lean (`mk_full_chain` and `chain_top`, which take δ as a parameter; §7), and we apply it with $\delta = 0.2195$. The specialised Lean statements `mk_full_chain_64`, `mk_full_chain_128`, `mk_full_chain_general` and the numerical lines `steps_numeric`, `steps_numeric_128` are written with $\delta = 0.2196$. Their hypothesis on the drop is not met by the certificate on the cell $[2.46, 2.48]$, so they are not used here. With $\delta = 0.2195$ the ceilings $\lceil 63.05/0.2195 \rceil = 288$ and $\lceil 127.05/0.2195 \rceil = 579$, hence the totals 846 and 1137, are the same as with 0.2196; this arithmetic is on paper. □

Remark 5.5 (where each condition enters). Condition (a) enters every computer-assisted step: Theorem 3.1(i),(iii) on $[2.46, 128]$, the joining computation on $[0.95, 3.0]$, and the exit test against $\mathcal{D}$. Conditions (b) and (d) enter only the second stage, inside $\mathcal{D}$. Condition (c) enters the exit test, through the upper boundary $\gamma_n + 0.005 t^n$ of $\mathcal{D}$. Condition (e) is a convention that makes $\hat{\mathcal{U}}$ a well-defined set. With the other convention the boundary cells would have to be tested against both pieces, which the computation does in any case for images that

$\beta(w) \le$	steps to $\mathcal{D}$ (upper)	steps in $\hat{\mathcal{U}}$ (lower)	then to $\mathcal{K}$ (upper)	total (upper)
16	70	51	557	627
32	143	104	557	700
64	289	211	557	846
128	580	—	557	1137

Table 3: Step counts of Theorem 5.4. The upper bounds use the drop ≥ 0.2195; the lower bounds use the drop ≤ 0.2990, certified on $[0.95, 64]$ only. A double-precision count, $4\beta - 1 = 255$ steps from the heat kernel at $\beta = 64$, lies between the bounds.

meet several pieces. The Lean statements take as hypotheses the one-step alternatives (stay with a drop, or exit into D; decrease of the cell index, or entry into K) and the bounds $0.95 \le \beta \le 128$ and $r \le 556$. They do not know what U, D, K are. To apply them, U is $\hat{\mathcal{U}} \cap \{0.95 \le \beta \le 128\}$, D is $\mathcal{D} \cap \{0.0144 \le t < 0.21\}$, K is $\mathcal{K}$, and r is the smallest number of a logged cell containing t (on D; any value elsewhere).

6 Why $\beta \to \infty$ is open

The tube is *easier* at large β (Table 1): the ratio decreases from 0.889 to 0.492, and λ_{tube} from above 1 to 0.26. It is therefore natural to try to cover $[128, \infty)$ by one asymptotic cell. With the present cell test this fails, for three reasons, of which the first is structural.

Scaling variables. Put $\xi_n = q_n u$. The heat kernel is $a_n = e^{-\xi_n/2}$ and the tube weight is $W_n = u^2(\xi_n^2 + A\xi_n)$. The tail parameters chosen by our rule scale as $N \approx \nu\sqrt{\beta}$, $\lambda \approx \ell_0/\sqrt{\beta}$ ($N/\sqrt{\beta} = 20.5$–22.0 and $\lambda\sqrt{\beta} = 2.70$–$2.93$ for $\beta = 128, \dots, 2048$). When each contribution is normalised by RW_i, the four columns of the cell test behave as follows.

- Centre defect: tends to $(\sigma_1/R)\,\xi_N/(\xi_N + A) \le \sigma_1/R = 0.2604$, where $\sigma_1 = 1/3840$ is the one-step θ^4 coefficient from the heat kernel [paper, leading order].
- Linear part: approximately constant ($0.2566 \to 0.2553$ measured).
- Second order in e: exactly proportional to u (measured at five points).
- Third-order remainder: of order u^0 and proportional to the cube of the cell width $\Delta\beta$.

Since the image satisfies $u' > u_c$, each row ratio is at most the sum of the four column maxima.

The cubic-remainder wall (paper). In the low rows the remainder is dominated by the δu-direction: $D_j \approx q_j|\delta u|/2 \le \xi_j \Delta\beta\, u/4$, so $\mathbb{E}_i[(\sum_k D_{j_k})^3] \approx (\Delta\beta\, u/4)^3\, \mathbb{E}_i[(\sum \xi_{j_k})^3]$. The requirement in row 2 is $RAq_2u^3 \propto u^3$. The ratio is $\Delta\beta^3/(RA)$ times a constant independent of β. Hence a test that expands around the centre to second order in δu and bounds the third order in absolute terms needs cells of bounded *absolute* width in β. Covering $u \in [0, 1/128]$ by one cell makes δu comparable to u, and the remainder exceeds the requirement by orders of magnitude. Measured: at $\beta = 128$ and 256 the remainder column is 7.4–7.9 times larger for width 0.08 than for 0.01, and cells pass only up to width about 0.04.

Measured columns. For single cells of width 0.02 ending at $\beta = 128, 256, 512, 1024$ ($N = 232, 335, 483, 693$) the four columns are: centre defect 0.2538, 0.2543, 0.2551, 0.3691; linear 0.2566, 0.2559, 0.2556, 0.2554; second order 0.0149, 0.0075, 0.0037, 0.0019; remainder 0.0834, 0.0817, 0.0856, 0.1261 (computer-assisted (interval arithmetic with a self-written library; not independently verified)). The jump of the defect and the remainder at $\beta = 1024$ comes from rows $n = 2$ (rounding) and from the tail floor. Dividing t_0 by 1000 lowers the remainder at 1024 to 0.0793 but leaves the rounding term unchanged.

The rounding floor. The absolute error of $\ell' = 4(\log P_i - \log P_0)$ is of order $4\gamma_{2N+6}$, and the smallest requirement is $RAq_2u^3 = 0.08\,u^3$. With $N \propto \sqrt{\beta}$ the ratio grows like $u^{-7/2}$. Single cells $[\beta - 0.02, \beta]$ of the tube $\mathcal{U}$ pass at $\beta = 256$, 512, 1024 (ratio 0.4916, 0.4942, 0.5313; computer-assisted (interval arithmetic with a self-written library; not independently verified)) and fail at $\beta = 2048$ (ratio 4.69, limited by rounding in row 2). A third, removable, term is the floor of the tail parameters, of order u^{-3}.

Directions. The obstruction is in the cell test, not in the bounded quantities. At the measured points $\beta = 128$ and 256 the four columns stay below $0.2604 + 0.2570 + 2u + 0.0835 \le 0.62$ for $u \le 1/128$. At $\beta = 512$ the remainder 0.0856, and at $\beta = 1024$ the centre defect 0.3691 and the remainder 0.1261, exceed these constants. These measurements are not a proof over $[128, \infty)$. What would be needed:

1. bounding the third-order term *after* the projection P, so that the δu direction, which P removes at leading order (Lemma 2.3: the u^3 terms cancel), does not enter the remainder; alternatively, adding the third derivative g_3 in the δu direction to the point quantities, so that only the e-part remains in R_3;
2. computing the projected quantities e_i' directly, instead of as differences of logarithms of size $q_i u$, so that the rounding error scales with the requirement;
3. taking the tail size t_0 proportional to u^3, which removes the tail floor.

With (1)–(3) the cell test would need to verify four column bounds as functions of u on $[0, 1/128]$. This is the content of the bookkeeping statement `row_lt_req_of_columns`: if each row bound is a sum of four columns, each column is bounded by a constant (or by a multiple of u), and the requirement is at least the normalising weight, then the row passes. The statement `uniform_columns_128` instantiates this with these constants. That the columns lie below them is not among its hypotheses and is not established. The tail itself can be controlled in closed form.

Lemma 6.1 (Gaussian tail of heat-kernel coefficients; paper)**.** *Let $p \ge 0$, $\lambda > 0$, $u > 0$, $M \ge 2$, and $r := (1 + 1/M)^p e^{\lambda - (2M+1)u/2} < 1$. Then*

$$\sum_{n \ge M-1} (n+1)^p e^{\lambda n} e^{-q_n u/2} \le e^{u/2-\lambda}\, \frac{M^p e^{\lambda M - M^2 u/2}}{1 - r}.$$

Proof. Since $q_n = (n+1)^2 - 1$, the term with $m = n + 1$ equals $e^{u/2-\lambda} m^p e^{\lambda m - m^2 u/2}$. The ratio of consecutive terms is $((m+1)/m)^p e^{\lambda - (2m+1)u/2}$, which decreases in m and is hence at most r for $m \ge M$. Now sum the geometric series. □

With $N = \nu\sqrt{\beta}$, $\lambda = \ell_0/\sqrt{\beta}$ and $M = N + 2$, the exponential factor is independent of β while $1/(1-r)$ grows like $\sqrt{\beta}$: if $2p \le \nu(\nu - \ell_0)$ and $(\nu - \ell_0)\sqrt{u} \le 1$ then $1/(1-r) \le 4/((\nu - \ell_0)\sqrt{u})$. Hence ν growing like $\sqrt{\log \beta}$ suffices for the tail, in line with the slow growth of $N/\sqrt{\beta}$ chosen by our rule. The Lean statement `chain_top` already takes the upper end β_0 and the drop δ as parameters (step bound $\lceil (\beta_0 - 0.95)/\delta \rceil + 558$). Its hypotheses are verified, with $\delta = 0.2195$, only for $\beta_0 \le 128$.

7 What is machine-checked

What is in Lean. Five Lean 4 modules with Mathlib [M] check the *finite algebra* of the paper. The MK map, the measure, the saddle-point approximations and all interval computations are outside Lean. In each statement the computer-assisted inputs enter as hypotheses.

- `LatticeGaugeHierarchicalShapeFlow` (10 declarations): `quarticStep_fixed`, `quarticStep_iterate` (the map $\sigma \mapsto \sigma/16 + 1/3840$, its fixed point $1/3600$ and k-step contraction; no hypotheses), `shapeStep_fixed`, `shapeStep_contract`, `shapeStep_ratio_le` (hypothesis $k \geq 2$: fixed point $12\ell/(4^{2k} - 16)$, ratio $16/4^{2k} \leq 1/16$), `shapeStep_quartic`, `sextic_fixed` ($1/963900$), `flow_coefficient` ($1/48 - 15/3600 = 1/60$), `affine_contraction` (hypotheses $0 \leq \lambda < 1$, $0 \leq \varepsilon$, $d_{k+1} \leq \lambda d_k + \varepsilon$: $d_k \leq \lambda^k d_0 + \varepsilon/(1-\lambda)$), `tube_iterate`.
- `LatticeGaugeHierarchicalTubeChain` (8): `tube_exit`, `cell_descent`, `chain_compose`, `mk_full_chain`, `steps_numeric` ($\lceil (64 - 0.95)/0.2196 \rceil = 288$), `mk_full_chain_64` (hypotheses: $0.95 \leq \beta \leq 64$ on U; each step stays in U with drop ≥ 0.2196 or lands in D — a constant the certificate supports only as 0.2195, see the proof of Theorem 5.4; in D the cell index $r \leq 556$ decreases or the orbit lands in K; conclusion: K within 846 steps).
- `LatticeGaugeHierarchicalUpperTube` (8): `steps_numeric_128` (579), `mk_full_chain_128` (the same hypotheses with $\beta \leq 128$; K within 1137 steps), `union_tube_step`.
- `LatticeGaugeHierarchicalChainGeneral` (12): `chain_top` (drop δ and upper end β_0 as parameters), `mk_full_chain_general` (upper end β_0 as a parameter; bound $\lceil (\beta_0 - 0.95)/0.2196 \rceil + 558$, with the constant 0.2196), `steps_general_at_128`, `steps_general_mono`, `row_lt_req_of_columns`, `uniform_columns_128` (the numerical inequality for the column constants of §6; that the columns lie below these constants is not a hypothesis of it and is not shown).
- `LatticeGaugeClassResidualTelescoping` (6, shared with a companion paper): `telescoping_residual`, `geometric_part_le`, `inv_sq_sum_le`, `uniform_deviation` (hypotheses $a, c > 0$, $C, D \geq 0$, $0 \leq q < 1$ and a one-step residual at most $C/(a + ck)^2 + Dq^k$; conclusion: the total deviation after K steps is at most $C(1/a^2 + 1/(ca)) + D/(1-q)$, uniformly in K).

Trust base. Each module ends with one `#print axioms` line per listed declaration (44 in all). Each line reads a subset of `[propext, Classical.choice, Quot.sound]` (one of them, `chain_compose`, reads `[propext, Quot.sound]`), and the outputs are included with the Lean files as `axioms-<module>.txt`. `sorry`, `axiom` declarations and `native_decide` do not occur. An independent re-check of these modules was completed on 2026-10-15 (every file checked with `lake env lean` in the same environment on a separate session, rc=0, standard axioms only).

Reproduction. The archive of the modules, `https://computoergosum.com/principia/lean/principia-ym-hierarchical-chain-2026-10-15.tar.gz` (with a `.sha256` file at the same address with that suffix), and the complete `#print axioms` output, `https://computoergosum.com/principia/lean/axioms-ym-hierarchical-chain-2026-10-15.txt`, were placed on 2026-10-15, after the independent re-check by Rin; the ledger and build instructions are at `https://computoergosum.com/en/principia/lean/index.html`. To verify: unpack, place the files (single files, not a Lake project) in a Lean 4 project that depends on Mathlib (Lean v4.33.1, Mathlib v4.33.1), run `lake exe cache get` there, then `LEAN_NUM_THREADS=1 lake env lean` on the file of each module above; the `#print axioms` lines at the end of each module should agree with the report above.

Certificates. The interval certificates are distributed with the paper as CSV files, one row per cell, with a `SHA256SUMS` file: `main_lo.csv` and `main_hi.csv` (3100 cells, $\beta \in [2, 64]$; Theorem 3.1, Table 1), `main_128.csv` (3200 cells, $[64, 128]$), `fprof_F1.csv` (782 cells of $\hat{\mathcal{U}}$,

[0.95, 3.0]; Theorem 5.4), `cstar_A0.csv` (50 cells; Proposition 4.5), and the tail parameters `params.csv`, `params_ext.csv`, `params_128.csv`. The interval inputs of Theorem 4.4 ($\tilde{\mu}$, κ, $\tilde{\delta}$ per cell) and the single cells of §6 are not among these files and are not distributed. Until they are, those statements cannot be checked from the bundle. In `main_lo.csv`, `main_hi.csv` and `main_128.csv` each row contains the integer block, the cell endpoints in β, the truncation N, the pass flags of the individual requirements, the ratio and the row where it is attained, the four column maxima of §6 (centre defect, linear, second order, remainder), λ_{tube}, the tail ratio, the enclosure of the image coupling, and the enclosures of the coupling flow at the centre and over the tube. For example, the last cell [127.98, 128.00] has $N = 232$, ratio 0.49196 in row 98, columns 0.2538, 0.2566, 0.0149, 0.0834 and $\lambda_{\mathrm{tube}} = 0.2616$. The program that produced them is not part of the Lean development; its rounding model is described in §2.

The bundle of this paper. All files needed to check the paper are collected in one archive, `ym-hierarchical-chain-bundle-v2.tar.gz`, distributed with the paper (sha256 `0b0ca2c92638f20b575649e5ede851df6bdb7ca2bfaddb20c0bc786d2e56d0bc`); Table 4 lists its parts. The Lean archive inside it is the archive of §7 above, unchanged (sha256 `d549fd8969bd4ec348e09020ceff836a250ca45355deaaf00525f56dcfd33da5`). The file `SHA256SUMS` at the top of the bundle lists every file. Re-running W-half(T) needs Python 3 with mpmath, NumPy and SciPy. The script `whalf/run.sh` runs the programs, which read each other from their own directory, on a range of cells, e.g. `bash whalf/run.sh lift 0.0144 0.06 0.005`. The command `bash whalf/run.sh kappa 0.01658` runs part (ii), and `whalf/cover_all.py` checks the coverage of [0.0144, 0.21) by the logged cells.

path in the bundle	content
`lean/`	the Lean archive and its axiom log (§7)
`certificates/`	the CSV certificates listed above, with `SHA256SUMS`
`whalf/programs/`	the programs of Proposition 5.2: enclosure of T (`mk_tail_cert.py`), reference curve (`tube_cap4.py`), envelope (`envelope.py`), cell test (`tube_lift.py`; `tube_lift_strong.py` with the decrease test against the lower end of the cell), decrease of the cell index, one box per cell (`drop_check.py`), part (ii) (`tube_be_lift.py`), coverage (`lift_cover.py`)
`whalf/run.sh`	runs the programs above
`whalf/cover_all.py`	coverage of [0.0144, 0.21) by the logged cells
`whalf/logs/`	the original output: one line per cell, restarts included, and the coverage and part (ii) summaries
`whalf/rerun/`	the output of the re-run for this version (Appendix A)
`whalf/curve/`	the reference curve recomputed for the joining computation (`gamma_S.txt`, `gamma_P.txt`, the program that writes them) and the comparison with the curve of the certificate

Table 4: Parts of the bundle of this paper.

8 Correspondence table

Table 5 lists, for each statement with a Lean counterpart, the declarations and their place in the distributed files. Abbreviations of the files:

- SF: `LatticeGaugeHierarchicalShapeFlow.lean`;
- TC: `LatticeGaugeHierarchicalTubeChain.lean`;

- CG: `LatticeGaugeHierarchicalChainGeneral.lean`;
- RT: `LatticeGaugeClassResidualTelescoping.lean`.

Theorem 3.1 has no Lean statement; it rests on the certificates `main_lo.csv`, `main_hi.csv`, `main_128.csv`. Proposition 4.5 rests on `cstar_A0.csv`. Lemmas 4.3, 5.3 and 6.1 are on paper only, and so are Lemmas A.1–A.4 of Appendix A. Proposition 5.2 has no Lean statement; it rests on the logs `whalf/logs/`.

Statement	Lean declaration	File:line
Prop. 4.1 (algebra only)	`quarticStep_fixed`	SF:20
	`quarticStep_iterate`	SF:24
	`shapeStep_fixed`	SF:45
	`shapeStep_contract`	SF:55
	`shapeStep_ratio_le`	SF:65
	`shapeStep_quartic`	SF:71
	`sextic_fixed`	SF:75
Prop. 4.2 (algebra only)	`flow_coefficient`	SF:78
Thm. 4.4 (iteration only)	`affine_contraction`	SF:82
	`tube_iterate`	SF:99
Thm. 5.4 (assembly;	`tube_exit`	TC:35
inputs `fprof_F1.csv` and the	`cell_descent`	TC:71
certificates above as	`mk_full_chain`	TC:92
hypotheses, $\delta = 0.2195$)	`chain_top`	CG:71
§6 (bookkeeping only)	`row_lt_req_of_columns`	CG:127
	`uniform_columns_128`	CG:152
Uniform deviation (shared module)	`uniform_deviation`	RT:83

Table 5: Statements of this paper and the Lean declarations that check their finite algebra. Hypotheses supplied by interval computations are not checked in Lean.

9 What is not claimed

- **No implication for four dimensions.** The map T is exact only on a hierarchical lattice, where one plaquette weight returns to one plaquette weight after a step: multi-link couplings, gauge re-fixing and large-field remainders are absent by construction. Nothing here says that the same tube argument works for one step of the hypercubic lattice. A comparison of the MK partition function with the hypercubic one exists [To]: an upper bound by translation invariance and convexity, a lower bound by reflection positivity. Its use for confinement, however, requires a common interpolation parameter for twisted and untwisted boundary conditions that is not established [IS1, IS2]. The comparison concerns numbers (partition functions), not effective actions, so it does not transport the chain of this paper to four dimensions.

- **No transfer of the tube.** In a one-step comparison of the type of [To], a representative weight αc^{U} raises the transverse coordinate e_2 by at least $(5/3)(1-\alpha)$. Staying inside the tubes of this paper would force $1-\alpha$ to be very small (of order 10^{-2} in $\mathcal{D}$ and far smaller in $\hat{\mathcal{U}}$ at $\beta = 64$), in tension with the requirement $\alpha \le 1-\delta$ of that approach. We record this only as a reason why no transfer is attempted.

- **No test of non-commutativity.** The hierarchical model flows to strong coupling and confines also for U(1) [It85]; the same chain would be expected there.

- **No statement about a given initial action.** Theorem 5.4 starts from points of $\hat{\mathcal{U}}$. Whether the Wilson-action weight at a given β lies in $\hat{\mathcal{U}}$, or enters it after finitely many

steps, is not certified. The double-precision orbits of §5 suggest it, after a transient of two or three steps, but this is an observation.

- **No meaning attached to $\mathcal{K}$.** We do not state what positivity of the lower bound defining $\mathcal{K}$ implies for the hierarchical measure.
- **Self-written interval arithmetic.** All interval statements except Proposition 5.2 are computer-assisted (interval arithmetic with a self-written library; not independently verified). Proposition 5.2 uses the interval arithmetic of mpmath [Mp] with a double-precision reference curve. The rounding model is assumed, the remainder and cumulant bounds are derived on paper only, and no established interval package was used for cross-checking.
- **W-half(T) is an input.** Proposition 5.2 is an earlier computation of the authors, stated and derived here and distributed with its programs and logs. For this version the decrease of the cell index was checked separately, and part (ii) and the cells with $t < 0.11$ were run again (Appendix A). Its reference curve is a double-precision object, and the certificate was not cross-checked with a second interval package. The curve used in the joining computation was recomputed from Definition 5.1(a) and agrees with the curve of the certificate at all nodes to within 9×10^{-14}.
- $\beta \to \infty$**.** Invariance is certified on $[2.46, 128]$, the attracting curve on $[15.26, 64]$, and the enclosure of the flow coefficient on 50 cells. The shape coefficients and $c^* = 1/60$ are leading-order statements without error constants.
- **No external review.** The computations, the Lean development and this text were produced within the authors' group (§10).

10 Disclosure of AI use

Two of the three authors are AI agents. *Shiori* and *Rin* are instances of Claude (Anthropic), run as the coding agent Claude Code. The observation, the informal proofs, the interval computations and their programs, the Lean 4 development including the formalised statements, the literature search and this text were produced by Shiori over many sessions, which share no memory and rely on the written record; further sessions of Shiori reviewed the text against the Lean statements. Rin re-checked the Lean development from the distributed archive, confirmed that the declarations cited here are in it, and prepared the web edition. Kiichi, the human author, set the task and the working rules and decides what is published; he is responsible for this paper.

This paper has not been refereed. All review was done by further instances of the same model family, which share the authors' blind spots. What the Lean kernel checks is checked; everything else is the authors' word. In particular the analytic steps listed in §7 are checked on paper only.

A The certificate W-half(T)

This appendix derives Proposition 5.2. It gives the enclosure of the untruncated map T used by the certificate (Lemmas A.1 and A.2), the lower bound for κ (Lemma A.3), the evaluation of the tube boundary (Lemma A.4), the cell test, and a summary of the output. The programs and logs are in the bundle (§7, file list in Table 4).

Notation. Write a weight as $w = \sum_n c_n \chi_n$ with $c_n = (n+1)a_n$, $c_0 = 1$, and write $c \geq 0$ for coefficientwise non-negativity. The Clebsch–Gordan product is $(c \star d)_m := \sum c_i d_j$ over the pairs (i, j) with $m \in \{|i-j|, |i-j|+2, \ldots, i+j\}$, so that the product of the class functions with coefficients c and d has coefficients $c \star d$. In coefficients the map is: $g := c^{\star 4}$ (bond moving $w \mapsto w^4$), $a'_m := g_m/((m+1)g_0)$, and $T(w)$ has normalised coefficients $(a'_m)^4$ (decimation). For $\rho \geq 1$ put $M_\rho(c) := \sum_n c_n \chi_n(\rho)$, the value of w at $\mathrm{diag}(\rho, \rho^{-1})$; note $\chi_n(1) = n+1$. We write P_K for the projection onto the modes $n \leq K$. In the programs of the bundle the two letters are the other way round: the array `c` holds the normalised coefficients a_n of this paper and the array `a` holds $c_n = (n+1)a_n$. The comments of the programs use c_n in the sense of a_n.

Lemma A.1 (paper)**.** *Let $c, d \geq 0$ and $1 \leq \rho < \rho'$.*

(i) $c \star d \geq 0$, and $M_\rho(c \star d) = M_\rho(c) M_\rho(d)$ whenever the sums converge.

(ii) $c_n \leq M_\rho(c)/\chi_n(\rho)$ for every n.

(iii) $\chi_n(\rho)/\chi_n(\rho')$ is non-increasing in n; $\chi_{n+1}(\rho)/\chi_n(\rho) \geq \rho$; and $\chi_n(\rho) \geq n+1$.

(iv) If $c \geq 0$ and $w = \sum c_n \chi_n \geq 0$ as a function, then $0 \leq a'_m \leq 1$ for the coefficients a'_m of w^4 normalised as above.

Proof. (i) The multiplicities of the Clebsch–Gordan rule are 0 or 1; the identity $\chi_i \chi_j = \sum_m \chi_m$ holds for Laurent polynomials in z with $\chi_n = \sum_k z^{n-2k}$, and evaluation at $z = \rho$ is a ring homomorphism. (ii) One term of a sum of non-negative terms is at most the sum. (iii) With $\rho = e^\sigma$, $\chi_n(\rho) = \sinh((n+1)\sigma)/\sinh\sigma$, and $\frac{d}{dx} \log \frac{\sinh(x\sigma)}{\sinh(x\sigma')} = \sigma \coth(x\sigma) - \sigma' \coth(x\sigma') < 0$ because $u \mapsto u \coth(xu)$ is increasing; $\rho = 1$ is the limit. Further $\chi_{n+1}(\rho) - \rho\chi_n(\rho) = \rho^{-n-1} \geq 0$, and $\chi_n(\rho) \geq \chi_n(1) = n+1$ term by term. (iv) $|\chi_m| \leq m+1$ on SU(2), so $|\int w^4 \chi_m \, dU| \leq (m+1) \int w^4 \, dU$; and $a'_m \geq 0$ because $c \geq 0$ gives $c^{\star 4} \geq 0$ by (i). □

Lemma A.2 (one-step enclosure with a tail; paper)**.** *Let $K \geq 1$, $\rho' > 1$, $c \geq 0$ with $c_0 = 1$, and $\tau \geq \sum_{n>K} c_n \chi_n(\rho')$. Put $\hat{c} := P_K c$, $h := P_K(\hat{c} \star \hat{c})$, $G := P_K(h \star h)$, $F_1 := \sum_{n \leq K} (n+1) c_n$, $F' := \sum_{n \leq K} c_n \chi_n(\rho')$, $M_1(h) = \sum_{n \leq K} (n+1) h_n$, $\tau_h := \sum_{K < n \leq 2K} (n+1)(\hat{c} \star \hat{c})_n$ and*

$$\Delta := 4(F_1 + \tau)^3 \tau + 2 M_1(h) \tau_h + \tau_h^2.$$

Then for $0 \leq m \leq K$,

$$G_m \leq g_m \leq G_m + \frac{\Delta}{m+1},$$

and the image tail satisfies

$$\sum_{n>K} (n+1)(a'_n)^4 \chi_n(\rho') \leq \Big(\frac{(F' + \tau)^4}{G_0} \Big)^4 \frac{(K+2)^{-3} \rho'^{-3(K+1)}}{1 - \rho'^{-3}}.$$

Proof. Since $\hat{c} \leq c$, Lemma A.1(i) gives $h \leq \hat{c} \star \hat{c} \leq c \star c$, hence $g - h \star h = (c \star c - h) \star (c \star c + h) \geq 0$, and $g_m \geq (h \star h)_m = G_m$ for $m \leq K$. For the upper bound, Lemma A.1(i) at $\rho = 1$ gives $\sum_{m \leq K} (m+1)(g - h \star h)_m \leq M_1(g - h \star h) = M_1(c)^4 - M_1(h)^2$. Since $\hat{c} \star \hat{c}$ has no modes above $2K$, $M_1(\hat{c})^2 = M_1(h) + \tau_h$, so

$$M_1(c)^4 - M_1(h)^2 = \big[M_1(c)^4 - M_1(\hat{c})^4 \big] + \big[(M_1(h) + \tau_h)^2 - M_1(h)^2 \big].$$

By Lemma A.1(iii), $0 \leq M_1(c) - M_1(\hat{c}) = \sum_{n>K} (n+1) c_n \leq \tau$, and $M_1(c) \leq F_1 + \tau$, so the first bracket is at most $4(F_1 + \tau)^3 \tau$; the second equals $2 M_1(h) \tau_h + \tau_h^2$. Each term $(m+1)(g - h \star h)_m$ is non-negative, so each is at most Δ. For the tail, Lemma A.1(ii) at ρ' and (i) give $g_n \leq M_{\rho'}(c)^4 / \chi_n(\rho')$ with $M_{\rho'}(c) \leq F' + \tau$, and $g_0 \geq G_0$; hence $a'_n \leq (F' + \tau)^4 / ((n+1)\chi_n(\rho') G_0)$ and $\sum_{n>K} (n+1)(a'_n)^4 \chi_n(\rho') \leq ((F' + \tau)^4 / G_0)^4 \sum_{n>K} (n+1)^{-3} \chi_n(\rho')^{-3}$. Finally $\chi_n(\rho') \geq \rho'^n$ and $(n+1)^{-3} \leq (K+2)^{-3}$ for $n > K$, and the geometric series gives the stated factor. □

The bound Δ is written as a sum of products of a small and a large quantity on purpose. Evaluating $M_1(c)^4 - M_1(h)^2$ as a difference of two enclosures lets the width of the enclosures enter Δ and grow from step to step.

Lemma A.3 (lower bound for κ; paper)**.** *Let $a_n \ge 0$ for all n, $K \ge 6$ and $\tau \ge \sum_{n>K}(n+1)a_n\chi_n(2)$. Put $w_i := p_i(K+1)/\chi_{K+1}(2)$, $\underline{F} := 1 - \sum_{n=1}^{K}(n+1)^2 a_n - \tau$ and $\overline{D}_i := \sum_{n=1}^{K}(n+1)a_n p_i(n) + \tau w_i$. If $\underline{F} > 0$, then $F(w) \ge \underline{F}$, $0 \le D_i(w) \le \overline{D}_i$ and*

$$\kappa(w) \ge 1 - 12\Big(\frac{\overline{D}_1}{\underline{F}} + \frac{\overline{D}_2}{\underline{F}} + \Big(\frac{\overline{D}_1}{\underline{F}}\Big)^2\Big).$$

Proof. For $n > K$, $(n+1)^2 a_n \le (n+1)a_n\chi_n(2)$ by Lemma A.1(iii), which gives $F \ge \underline{F}$. For $n \ge K+1 \ge 7$, $p_1(n+1)/p_1(n) = (n+3)/n \le 2$ and $p_2(n+1)/p_2(n) = (n+4)/(n-1) \le 2$, while $\chi_{n+1}(2)/\chi_n(2) \ge 2$; so $p_i(n)/\chi_n(2) \le w_i$ for $n > K$, which gives $D_i \le \overline{D}_i$. The right-hand side of the definition of κ is decreasing in $D_1, D_2 \ge 0$ and increasing in $F > 0$. □

Lemma A.4 (envelope of the tube boundary; paper)**.** *Let γ_n be piecewise linear with nodes S_j and constant outside $[S_1, S_{405}]$. For $0 \le a < b$, $\sup_{[a,b]}(\gamma_n(t) + 0.005t^n) \le \max_{x\in X}\gamma_n(x) + 0.005\,b^n$ and $\inf_{[a,b]}(\gamma_n(t) + 0.005t^n) \ge \min_{x\in X}\gamma_n(x) + 0.005\,a^n$, where $X = \{a,b\} \cup \{S_j : a < S_j < b\}$.*

Proof. A piecewise-linear function attains its extrema on $[a,b]$ at the end points or at nodes inside, and t^n is increasing. □

The program evaluates γ_n at the points of X in double precision and multiplies the maximum by $1 + 10^{-9}$ and the minimum by $1 - 10^{-9}$. That this margin covers the rounding of the interpolation is part of assumption (b) of Theorem 5.4. A box with $a_1 \in [s, s']$ (no restriction on a_1 beyond this) and $0 \le a_n \le \sup_{[s,s']}(\gamma_n + 0.005t^n)$ therefore contains every point of $\mathcal{D}$ with $t \in [s, s']$, up to the tail, which is carried separately as $\tau \le 10^{-15}$.

The cell test. All enclosures are computed in the interval arithmetic of mpmath [Mp] at 25 significant digits, with $K = 20$ and $\rho' = 2$. For a cell $[s_k, s_{k+1}]$ the program starts from the box $a_1 \in [s_k, s_{k+1}]$, $a_n \in [0, \overline{\gamma}_n]$ ($2 \le n \le 20$, $\overline{\gamma}_n$ the upper envelope of Lemma A.4), $\tau = 10^{-15}$. On a box it computes the image by Lemma A.2: a'_m is enclosed through $g_m \in [G_m, G_m + \Delta/(m+1)]$ and $g_0 \in [G_0, G_0 + \Delta]$, the image coefficients are $(a'_m)^4$, and the image tail is τ'. The box *passes* if

(1) $\tau' \le 10^{-15}$;
(2) the upper end of the image a'_1 is below the lower end of the a_1-range of the box (which is at least s_k);
(3) either the upper end of the image a'_1 is at most 0.0165 and the lower bound of Lemma A.3 for the image is positive (with $\underline{F} > 0$), or, with $[u, v]$ the enclosure of the image a'_1, the upper end of every image a'_n ($2 \le n \le 20$) is at most the lower envelope $\min_X \gamma_n \cdot (1 - 10^{-9}) + 0.005u^n$ of Lemma A.4 over $[u, v]$.

A box that does not pass is bisected in the coordinate among $a_1, \dots, a_5$ whose freezing at its midpoint shrinks the enclosure of the image a'_1 most. A cell passes if all boxes of a finite bisection pass within a budget of 3000 boxes. If a cell passes, every $w \in \mathcal{D}$ with $t(w)$ in the cell lies in a passing box, and (1)–(3) give the alternative of Proposition 5.2(i): $Tw \in \mathcal{D}$ by (1), non-negativity of fourth powers and the envelope, or $Tw \in \mathcal{K}$ by Lemma A.3. In both cases $t(Tw) < t(w)$ by (2), and $t(Tw) < s_k$ by the separate check described under *Output*. Part (ii) evaluates the lower bound of Lemma A.3 on the boxes $a_1 \in [x_i, x_{i+1}]$, $a_n \in [0, \overline{\gamma}_n]$, $\tau = 10^{-15}$, where $x_0 = 0$, $x_1 = 10^{-3}$ and $x_1 < \cdots < x_{80} = 0.01658$ are geometrically spaced (80 boxes).

Output. The original logs (Table 6) record that every cell passes, and they cover $[0.0144, 0.21)$ without gaps. Test (2) compares the image with the lower end of the current box, not of the cell. By itself this gives $t(Tw) < t(w)$ but not a decrease of the cell index. In the first version of this paper the decrease of the cell index was stated on the basis of these logs; test (2) as run does not give it, and the check below closes this. For this version two computations were added (`whalf/rerun/`; mpmath 1.3.0, NumPy 2.5.2, SciPy 1.18.0). First, `drop_check.py` encloses the image a_1' over the whole box of each cell (one box per cell, the enclosure of Lemma A.2, tail 10^{-15}). It does so for every cell of the logged grids, that is, of the ten ranges $[0.0144, 0.06]$, $[0.06, 0.11]$, $[0.11, 0.163]$, $[0.14836, 0.163]$, $[0.163, 0.18]$, $[0.16711, 0.18]$, $[0.18, 0.198]$, $[0.18639, 0.198]$ with $r = 0.005$ and $[0.198, 0.21]$, $[0.20452, 0.21]$ with $r = 0.0025$ (610 cells in all, counted per range). The upper end of the enclosure is below s_k in every case, and at most $0.67\, s_k$. Together with the logs this gives Proposition 5.2(i) as stated. Second, the cell test with (2) taken against s_k (`tube_lift_strong.py`) was run on the 409 cells with $t < 0.11$: all pass, each with the same number of boxes as in the logs. Part (ii) was also run again with the original program and returns the same smallest lower bound of κ, 0.0618141797083238, to 15 significant digits.

range of t	relative cell width	cells	boxes	largest boxes per cell
$[0.0144, 0.06]$	0.5%	287	287	1
$[0.06, 0.11]$	0.5%	122	612	33
$[0.11, 0.163]$	0.5%	80	6072	225
$[0.163, 0.198]$	0.5%	42	18926	1491
$[0.198, 0.21)$	0.25%	25	24065	1289
total		556	49962	1491

Table 6: Output of the certificate of Proposition 5.2(i), from the logs `whalf/logs/` (cells counted as logged, including the cells repeated at restarts). Part (ii): 80 boxes pass, smallest lower bound of κ equal to $0.061814\ldots$

Revision history. Revised 2026-11-14: the one-step drop is corrected from 0.2196 to 0.2195 (the certificate gives 0.21955 on the cell $[2.46, 2.48]$; the step counts are unchanged, and the Lean assembly is now cited in its form with a general drop); the cells of W-half(T) are defined as the logged cells; the abstract and the statement on $[15.26, 64]$ no longer speak of attraction to one curve, and their undistributed inputs are marked; several figures and one hypothesis of Lemma A.1 are corrected; a reference to background material outside the note is removed, so that the note stands alone. Revised 2026-11-15: the text was rewritten for readability. In addition, the predicted ratios 16.06 and 17.00 in §1, the variation ± 0.0014 at $\beta = 32$, the tail floor 2.3×10^{-4} (abstract and Theorem 4.4), the caption of Table 2 and the wording of Lemma 5.3 were corrected, and the abstract and §1 now say which curve is meant; the other statements, the proofs and the numerical tables are unchanged. The description of how the Lean files are distributed and checked, and the row of Theorem 5.4 in the correspondence table, were corrected.

References

[Hi] N. J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, 2002 (not consulted directly).

[IS1] K. R. Ito and E. Seiler, arXiv:0711.4930; cited for Eq. (3.5).

[IS2] K. R. Ito and E. Seiler, arXiv:0803.3019; cited for §3.3.

[It84] K. R. Ito, Commun. Math. Phys. 95 (1984), 247; cited for the hierarchical lattice models in which the Migdal–Kadanoff recursion is exact (Eqs. (1)–(3)).

[It85] K. R. Ito, *Permanent quark confinement in four-dimensional hierarchical lattice gauge theories of Migdal–Kadanoff type*, Phys. Rev. Lett. 55 (1985), 558–561 (abstract only consulted).

[JW] A. Jaffe and E. Witten, *Quantum Yang–Mills theory*, official problem description, Clay Mathematics Institute, `https://www.claymath.org/millennium/yang-mills-the-maths-gap/`.

[Ka] L. P. Kadanoff, Ann. Phys. 100 (1976), 359 (not consulted directly).

[M] Mathlib, `https://github.com/leanprover-community/mathlib4`.

[Mi] A. A. Migdal, *Recursion equations in gauge field theories*, Zh. Eksp. Teor. Fiz. 69 (1975), 810–822; Sov. Phys. JETP 42 (1975), 413–418; and Zh. Eksp. Teor. Fiz. 69 (1975), 1457 [Sov. Phys. JETP 42 (1975), 743] (not consulted directly).

[Mp] F. Johansson and others, *mpmath: a Python library for arbitrary-precision floating-point arithmetic*, `https://mpmath.org` (interval arithmetic `mpmath.iv`; version 1.3.0 for the re-run).

[Mo] R. E. Moore, *Interval Analysis*, Prentice-Hall, 1966 (not consulted directly).

[To] E. T. Tomboulis, *Confinement for all values of the coupling in four-dimensional SU(2) gauge theory*, arXiv:0707.2179.

[Tu] W. Tucker, *Validated Numerics: A Short Introduction to Rigorous Computations*, Princeton University Press, 2011 (not consulted directly).