Volume 3 Toward the mass gap — the questions that remain, and this book's results

Appendices

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Table of notation — one convention for each letter, two meanings separated by name

> This chapter is a guide to notation and terms.

What this chapter shows — The symbols of this book can be looked up by "meaning, chapter of definition, note". The three ways of writing the gauge field, the six ways of writing the coupling, and the normalisation of CP are set out with conversion formulas, tables and moving figures. Different meanings of the same letter (β, Δ, η, σ, Ω and others) are told apart in one table.

Prerequisite chapters — None. As a dictionary, it can be opened from any chapter. The "Chapter" column of each row is the chapter that defined the symbol.

Said first — This chapter is a list of conventions and contains no claims of content. The correctness of formulas is shown in each chapter; here only the arithmetic of conversions and the normalisation of generators were checked on this machine. The names of assumptions are kept separately in A-06 Index of names. This chapter shows nothing about the mass gap.

The order of this chapter

1. How to use this table — the common conventions, and three choices of convention
2. Spacetime and indices — the signature $(-,+,+,+)$ and imaginary time (Table 1)
3. Groups and Lie algebras — normalisation of generators and structure constants (Tables 2 and 3)
4. Three ways of writing the gauge field — where g and i go (Table 4). Detour: why physics likes to attach i
5. Lattice symbols — links, plaquettes, link angles (Table 5)
6. Six ways of writing the coupling — conversions and one-loop coefficients (Table 6, Figure 1, Table 7)
7. Scales, lengths, masses — lattice units and physical units, the symbol $\asymp$ (Table 8)

8. Probability, spectrum, relaxation — three CP and two η (Figure 2, Tables 9 and 10)
9. Different meanings of the same letter — a list for telling them apart (Table 11). Detour: can symbol collisions be eliminated?

01

How to use this table — one convention for each letter

This chapter is a dictionary to look things up in; it need not be read from the front. The tables have four columns, "symbol, meaning, chapter, note", and "chapter" is the chapter that defined the symbol. Check the correctness of formulas in that chapter.

There are four conventions common to all chapters. From Part 3 on, $\hbar = c = 1$. An index appearing twice in the same term is summed. Both $\ln$ and $\log$ are the natural logarithm. When the same letter has two meanings, this is declared at the start of the chapter (the list is in § 09). Beyond that, the appearance of the symbols changes with the following three choices.

$$\underbrace{\text{way of writing the gauge field}}_{\S 04\ \text{three ways}} \times \underbrace{\text{way of writing the coupling}}_{\S 06\ \text{six ways}} \times \underbrace{CP\ \text{normalisation}}_{\S 08\ \text{two ways}} \tag{1}$$

The choices in (1) do not change the physical content, only coefficients and signs. But the same condition looks like a different number. Example. The strong-coupling condition of Shen–Zhu–Zhu, $|\beta_{\mathrm{SZZ}}| < 1/(16(d-1))$, becomes $\beta_{\mathrm{SZZ}} < 1/48$ in four dimensions ($d = 4$), and in this book's Wilson β it is $\beta < 0.0833$ for SU(2) and $\beta < 0.1875$ for SU(3) (§ 06) Computed . The names of assumptions (the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption, the eight statements of construction and others) are in A-06 Index of names, and the legend of the labels is in 0-01.

02

Spacetime and indices — the signature is $(-,+,+,+)$, and in imaginary time everything becomes $+$

Physical spacetime is Minkowski spacetime, where only the time direction carries a negative sign in the squared distance (2-04, and the existing article One sign is different). Constructive field theory and lattice arguments work in Euclidean space $\mathbb{R}^4$, with time rotated to imaginary values (4-02).

$$\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1), \qquad t = -i\tau, \qquad A_0 = iA_4, \qquad F_{\mu\nu}F^{\mu\nu} \longmapsto \sum_{\mu,\nu=1}^{4} F_{\mu\nu}F_{\mu\nu} \tag{2}$$

(2) gives, from the left, the metric, the rotation (Wick rotation), and the result of rotating. In Euclidean signature the metric is the identity matrix, so upper and lower indices are not distinguished. Example. The light path $-t^2 + |\mathbf{x}|^2 = 0$ becomes $\tau^2 + |\mathbf{x}|^2$ when rotated, positive except at the origin.

Table 1 Spacetime and indices

Symbol	Meaning	Chapter	Note
$x^\mu,\ \mu = 0,\dots,3$	Minkowski coordinates ($x^0 = t$)	2-04	Some textbooks use the opposite signature $(+,-,-,-)$
$x_4 = \tau,\ \mu = 1,\dots,4$	Imaginary time and Euclidean indices	4-02	$\tau = it$
$i,j,k = 1,2,3$	Spatial indices	3-02	Do not mix with the colour indices a,b,c
$E_i,\ B_i$	Colour electric and magnetic fields	3-02	$B_i = \tfrac{1}{2}\varepsilon_{ijk}F_{jk}$; in Euclidean signature $E_{E,j} = F_{4j}$
d	Dimension of (Euclideanised) spacetime	0-03	The problem statement has $d = 4$
$\mathbb{Z}^d,\ a\mathbb{Z}^d$	The integer lattice, the lattice with spacing a	5-01	§ 05

03 Groups and Lie algebras — generators are normalised by $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$

The gauge group G is a compact simple Lie group, and in almost all examples it is $\mathrm{SU}(N)$ (1-04). For generators we take Hermitian T^a in the physics style. Unless the normalisation is fixed, both the structure constants and the coupling constant are off by constant factors.

$$[T^a, T^b] = i\,f^{abc}\,T^c, \qquad \mathrm{tr}(T^aT^b) = \tfrac{1}{2}\,\delta^{ab}, \qquad a,b,c = 1,\dots,N^2-1 \tag{3}$$

In the mathematics style one takes anti-Hermitian $X^a = -iT^a$, giving the form $[X^a, X^b] = f^{abc}X^c$ without i. It is the same f^{abc}. Example. For SU(2), $T^a = \sigma^a/2$ and $f^{abc} = \varepsilon^{abc}$; for SU(3), $T^a = \lambda^a/2$ (half the Gell-Mann matrices). The adjoint Casimir C_A ($f^{acd}f^{bcd} = C_A\delta^{ab}$) is N, the fundamental Casimir C_F ($\sum_a T^aT^a = C_F\mathbf{1}$) is $(N^2-1)/2N$, and the index of the representation $T(R)$ ($\mathrm{tr}_R\,T^aT^b = T(R)\delta^{ab}$) is $\frac{1}{2}$ for the

fundamental representation. What enters the one-loop coefficient (4-06) is C_A and $T(R)$.

Table 2 Numerical check of the normalisation of generators Computed

Group	$\max \lvert \operatorname{tr} T^a T^b - \frac{1}{2}\delta^{ab} \rvert$	C_F	C_A	Independent non-zero f^{abc} ($a<b<c$)
SU(2)	0	0.75	2	$f^{123} = 1$
SU(3)	1.1×10^{-16}	1.333333	3	$f^{123}=1$; $f^{147}=f^{246}=f^{257}=f^{345}=\frac{1}{2}$; $f^{156}=f^{367}=-\frac{1}{2}$; $f^{458}=f^{678}=\frac{\sqrt{3}}{2}$

Table 2 was obtained by writing out the matrices, computing $f^{abc} = -2i \operatorname{tr}([T^a, T^b]T^c)$, and checking the residual on substituting back into (3) (at most 1.1×10^{-16}) and that the Casimir is a constant multiple of the identity matrix.

Table 3 Symbols for groups and Lie algebras

Symbol	Meaning	Chapter	Note
G, $\mathfrak{g}$, N	Gauge group, Lie algebra, the size of $\mathrm{SU}(N)$	1-03	Large N is in 17-04
T^a, f^{abc}	Hermitian generators, structure constants	1-04, 3-02	Normalisation is (3)
σ^a, λ^a	Pauli matrices, Gell-Mann matrices	1-04	σ is also used in other meanings (§ 09)
C_A, C_F, $T(R)$	Casimirs and the index of a representation	4-06	Values in the text
χ_r, d_r	Character and dimension of the representation r	5-02	$\chi_r(U) = \operatorname{tr}_r U$
$\Omega(x)$	Gauge transformation (a group element at each point)	3-01	The letter g is kept for the coupling

04

Three ways of writing the gauge field — where to put g, and where to attach i

The same gauge field is written in three ways. The physics chapters "separate Hermitian A from the coupling g", the path-integral and dictionary chapters "absorb g into the field and put $1/g^2$ in front of the action", and the geometry chapters use "an anti-Hermitian 1-form $\mathcal{A}$". The translation is one line.

$$\hat{A} = g\,A, \qquad \mathcal{A} = -i\hat{A} = -ig\,A, \qquad \hat{F} = g\,F, \qquad \mathcal{F} = -i\hat{F} = -ig\,F \tag{4}$$

"Absorbing" is just multiplying by g, and "geometry" is just multiplying further by $-i$. But $-i$ changes the sign of $\operatorname{tr} F^2$. For Hermitian X, $\operatorname{tr} X^2 \geq 0$, while for anti-Hermitian $-iX$ it is ≤ 0, so the geometric action carries a minus sign.

Table 4 Three ways of writing the gauge field (Euclidean action)

	Physics (g placed in D)	**Physics (g absorbed)**	**Geometry (anti-Hermitian)**
Covariant derivative	$\partial_\mu - igA_\mu$	$\partial_\mu - i\hat{A}_\mu$	$d + \mathcal{A}$
Field strength	$\partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$	$\partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu, \hat{A}_\nu]$	$d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$
Gauge transformation	$\Omega A \Omega^{-1} - \frac{i}{g}(\partial\Omega)\Omega^{-1}$	$\Omega \hat{A} \Omega^{-1} - i(\partial\Omega)\Omega^{-1}$	$\Omega \mathcal{A} \Omega^{-1} - (d\Omega)\Omega^{-1}$
Action S_E	$\frac{1}{2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu} = \frac{1}{4}\int F^a_{\mu\nu}F^a_{\mu\nu}$	$\frac{1}{2g^2}\int \operatorname{tr} \hat{F}_{\mu\nu}\hat{F}_{\mu\nu}$	$-\frac{1}{g^2}\int \operatorname{tr} \mathcal{F} \wedge *\mathcal{F}$
Chapters that use it	3-02, 3-03, Part 4 (except 4-02; 4-03 uses both)	0-01, 4-02, 18-04	1-07, 3-04, 3-05

The topological charge Q is related to the second Chern number $c_2 = \frac{1}{8\pi^2}\int \operatorname{tr} \mathcal{F} \wedge \mathcal{F}$ by $Q = -c_2$ ((2) of 3-05; the difference in sign is $(-i)^2 = -1$, coming from $\mathcal{F} = -igF$), and the action has the lower bound $S_E \geq 8\pi^2|Q|/g^2$ Known . Example. For SU(2) with $g^2 = 1$ ($\beta = 4$), the action of one instanton is $8\pi^2 = 78.96$, and the weight e^{-S_E} is about 5×10^{-35} Computed .

Detour: why physics likes to attach i

You may skip this. In quantum mechanics observables are Hermitian operators, with real eigenvalues (2-05). Colour charge is an observable, so physics takes Hermitian generators and puts i in the exponent, $e^{i\theta^a T^a}$. On the geometric side, the Lie algebra is the space of tangent vectors at the identity of the group, and $\mathfrak{su}(N)$ is just the anti-Hermitian matrices. One can write e^X, and neither i nor g appears in the curvature. Precisely, the two styles differ only by changing the basis by a factor i, and represent the same connection on the same group. The differences are only the sign of $\operatorname{tr} F^2$ and the i in front of f^{abc}.

05

Lattice symbols — links, plaquettes, Wilson loops

Lattice gauge theory replaces spacetime by a lattice of spacing a and places a group element on each edge (link) (5-01). The key symbols are the Wilson action and its coefficient β.

$$S_W(U) = \beta \sum_P \Big(1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2}, \qquad U_P = U_{x,\mu}U_{x+a\hat{\mu},\nu}U^{-1}_{x+a\hat{\nu},\mu}U^{-1}_{x,\nu} \tag{5}$$

(5) sums the deviations from the identity of the product U_P around every small square (plaquette P). $\beta = 2N/g^2$ is the value for which it matches the continuum action, coefficient included, for smooth fields as $a \to 0$ (5-01 § 05) Known . Unless stated

otherwise, this book's β is this Wilson β. $\beta = 4/g^2$ in the SU(2) chapters (23-01, 23-02, 18-05) is the same formula with $N = 2$. A link sees the field only in the form $U_{x,\mu} \approx e^{-iga A_\mu}$. The parallel transport of 1-07 is its inverse, which is why the sign of the exponent of the plaquette is opposite to 1-07 (the trace does not change).

SU(2) links are also written with angles. The study in Part 23 writes $U = e^{i\vartheta\cdot\sigma}$, while the book's (3) gives $U = e^{i\theta^a T^a}$, so they are related by $\theta = 2\vartheta$.

$$U = e^{i\vartheta\cdot\sigma} = e^{i\theta^a T^a} \quad (\theta^a = 2\vartheta^a), \qquad \tfrac{1}{2}\operatorname{tr} U = \cos|\vartheta| \tag{6}$$

Example. For $\vartheta = (0.3, -0.2, 0.5)$, $\frac{1}{2}\operatorname{tr} U$ computed with the matrix exponential and $\cos|\vartheta|$ are both 0.815941 Computed. The difference in how the angle is taken acts as a constant factor on C_P in § 08.

Table 5 Lattice symbols

Symbol	Meaning	Chapter	Note
$a,\ L$	Lattice spacing, number of points on a side of the periodic lattice	5-01	$4L^4$ links, $6L^4$ plaquettes
$U_\ell = U_{x,\mu}$	Group element on the link ℓ	5-01	$U_{x+a\hat{\mu},-\mu} = U_{x,\mu}^{-1}$
$P,\ U_P$	Plaquette and the product around it	5-01	$1 - \frac{1}{N}\operatorname{Re}\operatorname{tr} U_P \in [0, 2]$
μ_β	Wilson measure $Z^{-1} e^{-S_W} \prod dU_\ell$	5-01	dU is the Haar measure
$W(C),\ W_{R\times T}$	Wilson loop $\frac{1}{N}\operatorname{tr} U_C$, rectangle	5-02	T is the length in time
σ	String tension: $\langle W_{R\times T}\rangle \approx e^{-\sigma RT}$	5-02, 5-08	Lattice units. In physical units $\sqrt{\sigma}$
$\vartheta,\ \theta$	SU(2) link angle, $\theta = 2\vartheta$	(6), 23-13	Different from θ of the θ term
$b,\ k,\ \delta(b)$	Factor per step, number of steps, the shortfall of the bookkeeping	23-02	Definitions of the study in Part 23

06

Six ways of writing the coupling — β, β_{Ch}, β_{SZZ}, λ, g^2, α

The same coupling g^2 is written with different letters and different normalisations in different literature. Comparing term by term with the action (5) fixes the conversions (repeated from (8a) of 5-01).

$$\beta = \frac{2N}{g^2}, \qquad \beta_{\mathrm{Ch}} = \frac{\beta}{N}, \qquad \beta_{\mathrm{SZZ}} = \frac{\beta}{N^2}, \qquad \lambda = g^2 N = \frac{2N^2}{\beta}, \qquad \alpha = \frac{g^2}{4\pi} \tag{7}$$

β_{Ch} is Chatterjee's form, writing the action as $\beta_{\mathrm{Ch}} \sum_P \mathrm{Re}\,\mathrm{tr}(I - U_P)$; β_{SZZ} is 't Hooft's convention, writing the inverse coupling as $N\beta_{\mathrm{SZZ}}$ (Shen–Zhu–Zhu, 5-04); λ is the 't Hooft coupling; α is what physics calls the strength of the coupling. The running coefficients also have a way of being written for each variable (Table 1 of 4-06); in this book b_0 is the coefficient for g, and κ that for the Wilson β.

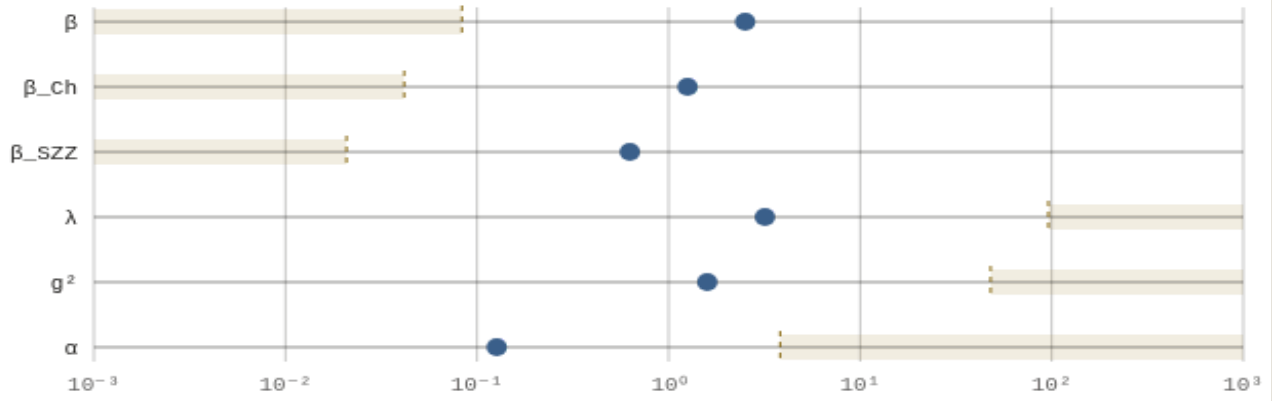
$$\mu \frac{dg}{d\mu} = -b_0 g^3 + \cdots, \qquad b_0 = \frac{11N}{48\pi^2}, \qquad \frac{d\beta}{d\ln\mu} = \kappa = 4N b_0 = \frac{11N^2}{12\pi^2} \quad (8)$$

The right of (8) is the one-loop prediction that the same physics is kept if β is raised by $\kappa \ln 2$ each time a is halved Known . $\kappa = 11/(3\pi^2)$ for SU(2) is the same number as "the slope of the step formula" of the study in Part 23, and also the same number as the value at SU(2) (C_g = 8) of the one-loop slope s = 11C_g/(24π²) of Volume 2 (10-01). Each of them is the known coefficient written in the lattice convention (23-14).

Table 6 Constants for each group (pure SU(N)) Computed

N	dim G	b_0	$2b_0$	κ	$\kappa \ln 2$	$b_0^{(\alpha)} = 11N/12\pi$ (μ² dα/dμ² = −b₀^(α) α², Table 1
2	3	0.04644	0.09288	0.3715	0.2575	
3	8	0.06966	0.13932	0.8359	0.5794	

Figure 1 Coupling converter. Moving the Wilson β with the knob moves the values of the six ways of writing (●) together on logarithmic scales. The vertical dashed line is the edge of the Shen–Zhu–Zhu condition in four dimensions ($d = 4$), $\beta_{\mathrm{SZZ}} = 1/(16(d-1)) = 1/48$, and the coloured band is the side where the condition holds. Switching SU(2) / SU(3) moves the position of the edge.



SU(2), β = 2.512: β_Ch = 1.256, β_SZZ = 0.6280, λ = 3.185, g² = 1.592, α = 0.1267. Outside the condition (edge at β = 0.08333). One-loop increment for halving a: κ ln2 = 0.2575

Computed The numbers in this figure were computed on this machine. The figure computes (7) on the spot (the same formulas as Table 7).

Table 7 Numerical examples of the conversion, and the edge of the strong-coupling condition in four dimensions ($d = 4$) Computed

Case	β	β_{Ch}	β_{SZZ}	λ	g^2	α

Case	β	β_{Ch}	β_{SZZ}	λ	g^2	α
SU(2), $\beta = 2.5$	2.5	1.25	0.625	3.2	1.6	0.12732
SU(3), $\beta = 6.0$	6	2	0.66667	3	1	0.07958
SU(2), edge of the condition	0.08333	0.04167	0.02083	96	48	3.81972
SU(3), edge of the condition	0.1875	0.0625	0.02083	96	32	2.54648

In 't Hooft's convention, β_{SZZ} and λ, the edge of the condition does not depend on N. The SU(2) $\beta = 2.5$ often used in numerical computation and the edge of the condition are about 30 times apart in α. The conditions are to be read in the normalisation of each paper; the conversion here is this book's arithmetic.

07

Scales, lengths, masses — separating lattice units from physical units

The symbols for length and mass come double, in lattice units (numbers with the lattice spacing as 1) and in physical units. If they are confused, "there is a gap on the lattice" and "there is a gap in the continuum" look like the same sentence (4-11).

$$m_{\mathrm{lat}} = \frac{1}{\xi}, \qquad \Delta = m_{\mathrm{phys}} = \frac{m_{\mathrm{lat}}}{a}, \qquad \xi_{\mathrm{phys}} = a\,\xi \tag{9}$$

ξ is the correlation length in lattice units, fixed by the rate of exponential decay of two-point correlations ((6) of 2-08). In the continuum limit, $a \to 0$ and $\xi \to \infty$ are taken together so that the product $a\xi$ stays constant. Example. If $\xi = 20$ and $a = 0.05$, then $\xi_{\mathrm{phys}} = 1$. Halving a needs $\xi = 40$, which at one loop corresponds to raising β by $\kappa \ln 2$. That two quantities are the same up to constant factors is written with the following symbol.

$$A \asymp B \quad \Longleftrightarrow \quad \exists\, 0 < c \le C : \;\; c \le A/B \le C \quad \text{(throughout the range considered)} \tag{10}$$

(10) says that the ratio goes neither to 0 nor to infinity; it is stronger than $\ln A / \ln B \to 1$, which compares only power exponents, and tells apart differences in logarithmic factors (Figure 2 of § 08).

Table 8 Symbols for scales, lengths and masses

Symbol	Meaning	Chapter	Note
H, Ω, $\mathcal{H}$	Hamiltonian, vacuum, Hilbert space	4-11	$H\Omega = 0$
Δ, m	Δ: the constant in the problem statement that bounds the energy of excitations from below. m: the largest value that can be taken as Δ (the lower end of the spectrum)	0-03, 4-11	$m = \inf \sigma(H\|_{\Omega^\perp})$. There is a gap $\Longleftrightarrow m > 0$

Symbol	Meaning	Chapter	Note
ξ, m_{lat}, m_{phys}	Correlation length (lattice units), $1/\xi$, m_{lat}/a	2-08, 18-03	(9)
μ	Renormalisation scale (momentum)	4-06	Same letter as indices and measures (§ 09)
Λ	The scale of dimensional transmutation ($\mu e^{-1/(2b_0 g^2)}$ at one loop)	4-07	Its value depends on the scheme. The ratio $m/\sqrt{\sigma}$ does not
ξ_{RG}, $a := 1/\xi_{\mathrm{RG}}$	The bookkeeping length $b^{k(\beta)}$, and the lattice spacing fixed by it	23-01	Definition of the study in Part 23. Agrees with a of Part 5 under the bookkeeping-length chain

08

Probability, spectrum, relaxation — three C_P and two η

In the second half of the book, the Poincaré constant C_P, which expresses how fast fluctuations are smoothed out, comes to the centre (1-10). It is the supremum of the ratio of the variance to the mean square of the gradient.

$$C_P = \sup_f \frac{\mathrm{Var}_\mu(f)}{\mathcal{E}(f)}, \qquad \mathcal{E}(f) = \mathbb{E}_\mu\Big[\sum_\ell |\nabla_\ell f|^2\Big], \qquad \lambda_1 = \frac{1}{C_P} \tag{11}$$

λ_1 is the gap of the Langevin dynamics that preserves μ. The same letter C_P is used for three quantities. The vacuum C_P is C_P of the vacuum measure at one time ($\Delta = 1/(2C_P^{(\alpha)})$; convention with kinetic term $\frac{1}{2}p^2$, exact for finitely many degrees of freedom; 2-05 § 08, A-06 Table 4); the path C_P is C_P of the path measure (Wilson measure) (C_P of the record of this book's study); and the path C_P measured in a graduation names the normalisation, according to whether the gradient of the path C_P is measured in the link angle ϑ or in the field $A = \sqrt{\beta}\,\vartheta$ (23-01 The line of construction and the diffusive-relaxation-time assumption).

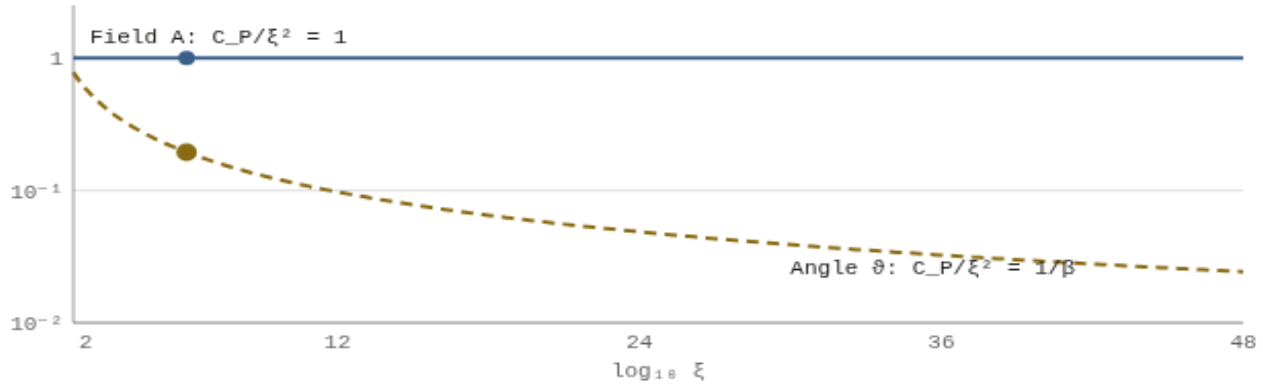
$$\mathcal{E}_A = \frac{\mathcal{E}_\vartheta}{\beta}, \qquad C_P[\mathcal{E}_A] = \beta\, C_P[\mathcal{E}_\vartheta], \qquad \text{in the free approximation } C_P^{(A)} = \xi^2,\ C_P^{(\vartheta)} = \frac{\xi^2}{\beta}$$

The free approximation is the approximation that regards the action as a Gaussian free field keeping only the quadratic terms, and gives $C_P^{(A)} = 1/m^2 = \xi^2$ (m in lattice units). A in (12) is A of § 04 measured in lattice units ($\sqrt{\beta}\,\vartheta^a = -aA^a$; the same field up to sign).

(12) shows that writing "$C_P \asymp \xi^2$" without naming the normalisation can change whether the sentence is true. A sentence comparing power exponents (the relaxation-

time assumption) moves only by the logarithm of β, and is the same sentence in both normalisations. A band statement, that the ratio stays within constant bounds, changes truth value with the normalisation. That is why this book writes band statements with the normalisation, as $(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$ (23-01 The line of construction and the diffusive-relaxation-time assumption).

Figure 2 Why the normalisation is named (SU(2), free approximation). Horizontal axis $\log_{10}\xi$, vertical axis C_P/ξ^2 (logarithmic). The solid line is the field normalisation 1, the dashed line the angle normalisation $1/\beta$. β is linked by replacing ξ with the one-loop formula for the bookkeeping length, $\xi \approx e^{3\pi^2\beta/11}$ (up to a constant factor) (asymptotic scaling assumed Physics). Moving ξ with the knob shows the local power exponent $\ln C_P/\ln\xi$ in the readout.



ξ = 10^6.0: β = 5.133, C_P^(ϑ)/ξ² = 0.1948, power exponent 1.8816 (angle) / 2 (field)

Computed The numbers in this figure were computed on this machine. The figure computes (12) and the one-loop formula on the spot (the same formulas as Table 9).

Table 9 C_P/ξ^2 in the angle normalisation and the local power exponent (SU(2), free approximation, replacing ξ by the one-loop formula for the bookkeeping length, $\beta = \frac{11}{3\pi^2}\ln\xi$. Asymptotic scaling assumed Physics)

Computed

ξ	β	$C_P^{(\vartheta)}/\xi^2$	$\ln C_P^{(\vartheta)}/\ln\xi$	$\ln C_P^{(A)}/\ln\xi$
10^2	1.7109	0.5845	1.8834	2
10^6	5.1326	0.1948	1.8816	2
10^{12}	10.2652	0.0974	1.9157	2
10^{48}	41.0609	0.0244	1.9664	2

The ratio in the angle normalisation goes to 0, so $(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$ fails in this free approximation. The power exponent $2 - \ln\beta/\ln\xi$ is still 1.97 at $\xi = 10^{48}$, but it goes to 2. This is an example showing a convention of notation, and shows nothing about C_P of the Yang–Mills measure.

There are also two exponents. The dynamical exponent z is the power in $C_P \asymp \xi^z$, and η without qualification is $\eta_{\mathrm{dyn}} = 2 - z$ (18-05 § 02, 2-10). The static $\eta_s = 2\Delta_O - (d-2)$ is the power of correlations fixed by the scaling dimension Δ_O of the operator O, and the two are equal only when the relation $z = 2 - \eta_s$ holds. Example. For Glauber

dynamics of the two-dimensional Ising model, $\eta_s = 1/4$ and $\eta_{\mathrm{dyn}} = -0.1665$ (18-03) Physics .

Table 10 Symbols for probability, spectrum and relaxation

Symbol	Meaning	Chapter	Note
$\mathbb{E}_\mu$, Var_μ, $\langle\cdot\rangle$	Expectation, variance, physical expectation	1-08	$\langle\psi,\phi\rangle$ is the inner product
$\sigma(H)$	Spectrum of an operator	1-10	Different from the string tension σ
CP, λ_1, $\mathcal{E}\vartheta$, $\mathcal{E}_A$	(11)(12)	1-10, 23-01	The three CP: the vacuum C_P, the path C_P, the path C_P measured in a graduation
τ_{exp}, τ_{int}	Exponential and integrated autocorrelation times	2-10, 5-07	For continuous-time Langevin, $\tau_{\mathrm{exp}} = CP$
z, η_{dyn}, η_s	Dynamical exponent, $2-z$, static power	2-10, 18-05	Different from the metric $\eta_{\mu\nu}$

09

Different meanings of the same letter — a list for telling them apart by context

The conventions of mathematics and physics have assigned the same letters to different things, so collisions within one book cannot be avoided. This book keeps the strong conventions as they are and declares the meaning at the start of the chapter. The guide for telling them apart is "subscripts and arguments". Example. $\beta(ba) = \beta(a) - \kappa \ln b + \cdots$ in 18-04 is the Wilson inverse coupling taking the lattice spacing as argument, not the β function $\beta(g)$.

Table 11 Different meanings of the same letter

Letter	Meaning (chapter)	How to tell apart
β	Wilson inverse coupling (5-01; the converted β_{Ch} and β_{SZZ} are in § 06) / inverse temperature $1/k_BT$ (2-08; in 2-07 the length of the loop $\beta\hbar$) / the critical exponent β (2-08 § 06, $1/8$ for the two-dimensional Ising model) / the β function (4-06) / (β) of the three CP (23-01)	Without qualification, the Wilson inverse coupling. The β function carries the argument g. The exponent is written alongside α, γ, ν
Δ	Mass gap (4-11) / Laplacian Δ_{lat} (1-08) / scaling dimension Δ_O (18-05; the scaling dimension of 17-08 is the same) / the decrease per step $\Delta(b)$ and the MCRG shift $\Delta\beta$ (18-04) / coproduct $\Delta(\Gamma)$ (4-05) / the group quantity Δ_G (Adhikari–Cao in 16-01; Δ_G in 16-05 and 16-08 is the mass gap) / difference Δx (2-04)	The scaling dimension carries an operator subscript, the Laplacian carries lat, the coproduct takes a diagram as argument. For Δ_G, see the chapter's declaration
η	Metric $\eta_{\mu\nu}$ (2-04) / $\eta_{\mathrm{dyn}}, \eta_s$ (18-05)	With two indices, the metric

Letter	Meaning (chapter)	How to tell apart
κ	One-loop slope (4-06) / decay rate of the Gaussian chain $1/\xi$ (2-08)	Decay rate only within 2-08
λ	Gell-Mann matrices (1-04) / 't Hooft coupling (17-04) / eigenvalue λ_1 (1-10)	With a colour index, a matrix; with a numeral subscript, an eigenvalue
σ	Pauli matrices (1-04) / string tension (5-02) / spin σ_x (2-08) / spectrum $\sigma(H)$ (1-10)	If the argument is an operator, the spectrum
Ω	Gauge transformation $\Omega(x)$ (3-01) / vacuum (4-11)	With an argument point x, a gauge transformation
θ	Angle of the θ term (3-05) / link angle 2ϑ (§ 05) / gac of the plaquette (5-01)	Multiplied by Q, the θ term
Λ	Lorentz transformation (2-04) / scale of dimensional transmutation (4-07)	If a matrix, the transformation
μ	Spacetime index / renormalisation scale (4-06) / measure μ_β	In subscript position, an index
a	Lattice spacing (5-01) / spacing fixed by bookkeeping $1/\xi_{\mathrm{RG}}$ (23-01) / starting point of a path (sections 01–06 of 2-07; from section 07, the step width) / colour index (3-02)	Superscript, colour. In 2-07 it switches by section
δ	Kronecker δ^{ab} (1-04) / shortfall of the bookkeeping $\delta(b)$ (23-02)	With two indices, Kronecker; with the factor b as argument, the shortfall
P	Plaquette P (5-01) / subscript of the Poincaré constant C_P (1-10)	As a subscript of C, the Poincaré constant
H	Hamiltonian (4-11) / Ising energy function (2-08)	An assumption is the name of a sentence
I	Identity matrix ($\mathrm{tr}(I-U_P)$ in 5-01) / modified Bessel function $I_k(J)=\frac{1}{2\pi}\int_0^{2\pi} e^{J\cos\theta}\cos k\theta\, d\theta$ (sections 06 and 07 of 0-04)	With a subscript k and an argument, the Bessel function
Ghost	The anticommuting Faddeev–Popov fields $c^a, \bar{c}^a$ (4-08 § 03) / states of negative weight in Pauli–Villars regularisation (4-10)	If a field, Faddeev–Popov; if a state of negative inner product, Pauli–Villars
Orientation of links	Parallel transport carries from the start point to the end point (1-07) / the lattice link $U_{x,\mu}$ is its inverse, carrying from $x+a\hat{\mu}$ to x, $U\approx e^{-igaA}$ (5-01 § 02) / numbering of chains: 3-01 uses $U_j=U(j{+}1,j)$, 3-02 uses $U_n=U(n,n{+}1)$	The value of the trace does not depend on the orientation. When comparing signs of exponents, see each chapter's declaration (the correspondence with 1-07 in 5-01 § 05)
T	Generators T^a, index $T(R)$ / length in time (5-02) / temperature (2-08) / transfer matrix T (2-08, 4-11; $T=e^{-aH}$)	Superscript, a generator. Inside a matrix power or a trace, the transfer matrix

Detour: can symbol collisions be eliminated?

You may skip this. Assigning a different letter to every quantity would eliminate collisions, but we do not, for two reasons. One is distance from the literature: writing the string tension with anything other than σ would make the tables of lattice papers unreadable as they are. The other is the number of letters: the number of quantities is far larger than the number of letters in commonly used typefaces. Precisely, a collision does harm only when two meanings stand in the same formula or the same table. For example, the three C_P in Table 3 of 23-01 stand in the same table as the Wilson β, so they are distinguished by name (the vacuum C_P, the path C_P, the path C_P measured in a graduation), rather than by a superscript such as (β) alone.

Table 12 The κ's compared (a copy of 26-05 Table 2): quantities that use the letter κ

Symbol	Meaning	Chapter
κ_{cb}	Chessboard density: the geometric mean, per block, of the product of the container biases of all blocks ((4) of 26-05)	26-05, 26-00 Table 1
κ (the one-sided bridge)	The comparison constant of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (in the form $CP \geq \kappa \xi^2$)	23-01 § 06
κ_{BE}	The lower bound on the Bakry–Émery curvature	24-09
$\hbar\kappa$, κ_f	An upper bound on the strength of the coupling to neighbouring sites; the crossing strength of Brascamp–Lieb	23-06
$\kappa = 3/1000$	The lower bound outside the defects in the Barta row	23-09
κ (Part 24)	The comparison constant between the Langevin metric and the metric of the kinetic term	24-09
$\kappa_{\mathrm{C7}}(k)$	The convexity constant of a chain of length k (the same quantity as $\kappa(k)$ in 23-11)	26-07
κ_w	Adapted stiffness: the variance of the force along the generator of the twist (width w; one of the candidate quantities for the band)	26-10

Table 13 Other meanings of θ and of "band" (a copy of 26-00 Table 0)

Word	Meaning	Chapters that use it	Relation to the band of Part 26
"Band" in Parts 23 and 24	The "band statement" of 23-01 § 05 and 23-15 (5) (the statement that a quantity lies within a width of constant multiples of the square of a length); ℓ_{band} of 23-09 (a length tied to the band at the summit); the "two-dimensional strip" (a narrow, strip-shaped lattice) of 24-02 § 03, § 05 and 24-09. The "twisted strip" of the two-dimensional model in 25-07 § 06 also has the meaning of a strip-shaped lattice	23-01, 23-09, 23-15, 24-02, 24-09, 25-07	Words with another meaning. Not the band of this part (an interval of scales)

Word	Meaning	Chapters that use it	Relation to the band of Part 26
Three meanings of θ	The threshold θ of 23-09 (θ_B, θ_σ and others; the value that divides configurations into the valley side and the summit side); the sector bias $\theta(R;T^2_L)$ of this part ((4) of 25-01); the vacuum angle θ of Tanizaki–Ünsal (set as $\theta = 0$ in 25-02)	23-09, 25-01, 25-02	Only the sector bias is a quantity of this part. The other two are different quantities

What this chapter can say

Matter	What is said	What is not said
Conventions of notation (Tables 1, 3, 5, 8, 10, 11)	A list collecting in one place the conventions used by the chapters of this book. The definitions are as in the linked chapters	Properties of the quantities each symbol refers to (existence, positivity and so on). These can be said only within the range of the labels in each chapter
Normalisation of generators (3) and structure constants (Table 2)	Checked numerically for SU(2) and SU(3) Computed . Standard convention Known	General G (the list in 1-04)
Three ways of writing the gauge field (Table 4)	That they translate into one another by the substitutions (4). $Q = -c_2$ ((2) of 3-05) Known	—
Conversion of couplings (7) and constants (Tables 6 and 7, Figure 1)	Conversions as this book's arithmetic Computed . b_0 and κ rewrite known one-loop coefficients Known	The content of each paper's conditions (conditions are read in the normalisation of the paper)
Normalisation of C_P ((12), Table 9, Figure 2)	Arithmetic under the free approximation and the one-loop formula (up to constants) Computed	The size of C_P for the lattice Yang–Mills measure. Whether the relaxation-time assumption holds
The mass gap	—	This chapter is only conventions of notation, and shows nothing, on the lattice or in the continuum

Sources and reproduction

Item	Kind	Source / tools
Definition of each symbol (the "Chapter" column of each table)	Chapters of this book	The linked chapters. Primary sources are as in the source tables of those chapters
Definitions of Δ and m in the problem statement	Primary source (text checked)	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official Clay problem description) § 4. As in the source table of 4-11
The Wilson action and $\beta = 2N/g^2$	Known	K. G. Wilson, Phys. Rev. D 10 (1974) 2445–2459. As in the source table of 5-01
The convention for β Ch	Primary source (text checked)	S. Chatterjee, “Yang–Mills for probabilists”, arXiv:1803.01950 (2018) § 3. As in the source table of 5-01
The convention for β SZZ and the condition $1/(16(d-1))$	(1.2)(1.3) of the text checked (the reading of the normalisation is in 5-04)	H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737. As in the source table of 5-04
The one-loop coefficient b_0 and its four ways of writing	Known	D. J. Gross, F. Wilczek (1973), H. D. Politzer (1973), D. Tong's lecture notes § 2.4, the QCD chapter of the PDG. As in the source table of 4-06
The three CP, $\mathcal{E}_\vartheta$, $\mathcal{E}_A$, the bookkeeping length	Definitions of the study in Part 23	Source tables of 23-01 and 23-13
η_s, η_{dyn}, z	Physics, the study in Part 23	Source tables of 18-05 and 2-10
The numbers in Tables 2, 6, 7, 9, (6) and § 04	Computed on this machine	Python 3, NumPy, SciPy: `generators.py` (normalisation, commutation relations, Casimirs), `constants.py` (the arithmetic of (7)(8)), `normalisation.py` ((12) and the one-loop formula). Figures 1 and 2 compute the same formulas in plain JavaScript

Next chapter to read: A-02 Glossary (look up the meaning of a term in one to three lines, and jump to the chapter). The names of assumptions are in A-06 Index of assumptions.

Glossary — looking up the words of this book in one line, and jumping to the chapter of definition

This chapter is a glossary; each entry points to the chapter that defines it.

What this chapter shows — The meaning of the main words in this book, in 1–3 lines each. Which word is defined in which chapter, and in what order they build up to reach "the mass gap". How to tell apart words that name different quantities under the same name (β, Δ, η, C_P, "gap").

Prerequisite chapters — None. You may look things up from anywhere, and read the details in the linked chapters. The conventions for symbols are in A-01 Table of notation, the list of assumptions in A-06 Index of names, and the literature in A-03 Guide to the literature.

Said first — The explanations here are summaries; the master copy of each definition is the linked chapter. Some conditions were dropped to fit a single line, so check the hypotheses of theorems in the linked chapters. The mass gap has not been shown (for existence in four dimensions, the committee's assessment is that it is proved in Lean in the construction published in 2026; 6-00). This book does not claim to solve the problem.

Order of this chapter

1. How to use this, and the single road of words — the form of an entry, filtering, twelve words from the group to the mass gap (Figure 1), a table of frequent words
2. Mathematical preparation (1): groups, Lie algebras, geometry — group, SU(2), compact simple group, Lie groups and Lie algebras, connection, holonomy, curvature, gauge transformation
3. Mathematical preparation (2): analysis and probability — Gaussian measure, Lebesgue measure in infinite dimensions, self-adjoint operators and spectrum, variational principle, Poincaré constant
4. Physical preparation — action, states and the Hamiltonian, path integral, imaginary time, correlation length, phase transition, transfer matrix, Langevin and z (Figure 2)

5. Yang–Mills theory: classical, quantisation, axioms — field strength, instanton, orbit space, renormalisation, asymptotic freedom, Λ, gauge fixing and BRST, Gribov, Schwinger functions, OS axioms, mass gap
6. Lattice gauge theory — link variable, plaquette, Wilson action, Elitzur's theorem, Wilson loop, area law and string tension, strong-coupling expansion, continuum limit, lattice correlation-length problem (Chatterjee's Problem 5.1)
7. Constructive field theory — constructive field theory, triviality, cluster expansion, blocking, ultraviolet stability, stochastic quantisation
8. Words for reading the published construction (Parts 6–14) — published construction, assessment, probability kernel, representation, forced activity, shooting, calendar, induction hypothesis, seed, chessboard root, phase measure and others
9. Physical pictures — dual superconductivity, centre vortex, effective string, large N, holography, one scale
10. The words of this book (Parts 16, 18, 23 and 24) — the two columns of settledness, seams, the study with the diffusive-relaxation-time assumption at its axis (the side of the four-dimensional lattice, the model of the first step, the assembly of the first step, computation and checking), the words of Part 24
11. Different quantities under the same name — β, Δ, η, b_0, three C_P, three "gaps", Detour: four uses of the word "dimension"

01

How to use this, and the single road of words

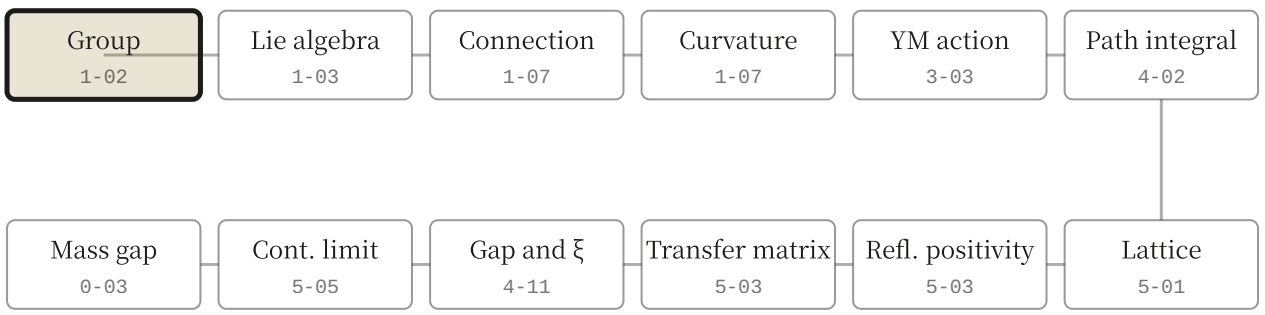
Each entry consists of "the headword", "a meaning in 1–3 lines" and "→ the chapter where it is defined". The order is not alphabetical but follows the parts of the book, so the words needed to read an entry are usually a little above it. To search by word, you can filter with the box below (without JavaScript, use in-page search).

The backbone of this book is one formula. Set the energy of the vacuum of the Hamiltonian H to 0; the distance from it to the next part of the spectrum is called the mass gap (4-11 § 01).

$$\Delta \;=\; \inf\big(\operatorname{spec} H \setminus \{0\}\big) \;>\; 0 \quad (1)$$

To read this formula you need "spectrum", "Hamiltonian" and "vacuum", and to give it meaning for four-dimensional Yang–Mills theory you need everything from groups to lattices and reflection positivity. Figure 1 is that single road.

Figure 1 The single road of words — moving the knob advances a step and shows that word and its chapter of definition below. "Play in order" advances one step at a time. The number in each box is the chapter that uses the word on this single road (the chapters of definition are the links in the entries below)



1. Group — a collection of symmetries (1-02)

Without JavaScript, the twelve boxes of the figure and their chapter numbers can still be read as they are.

Table 1 counts how many chapters the main words appear in. "First chapter" is the chapter where the name first appears, not necessarily the chapter of definition.

Table 1 Number of chapters in which the main words appear (the body text of the 105 chapters other than the appendices; counted excluding script, style, the whole-book contents and the meta line)

Word	Chapters	First chapter
Mass gap	101	0-01
Confinement	56	0-01
Correlation length	54	0-02
Gauge transformation	48	0-01
Asymptotic freedom	47	0-01
Plaquette	46	0-03
Refl. positivity	43	0-02
Wilson loop	43	0-01
String tension	40	0-02
Transfer matrix	38	0-04
diffusive-relaxation-time assumption	26	0-02
Poincaré constant	20	1-10
Dynamical exponent	14	2-08

Computed The numbers in this table were counted on this machine (a small script that counts words in the body text of the drafts; the numbers move as chapters are revised). Of the 26 words counted, we list the top 10 by number of chapters (excluding

"connection", which overlaps with ordinary Japanese) and the 3 words central to Part 23 of this book (relaxation-time assumption, Poincaré constant, dynamical exponent).

02

Mathematical preparation (1): groups, Lie algebras, geometry

Yang–Mills theory is the theory of a mechanism for comparing the orientation of a group from place to place (the connection) and of the mismatch in that comparison (the curvature). The first half of Part 1 prepares the words for this. The key formula is the one line that makes the curvature F from the connection A (in the geometric way of writing; for conversion to the physical way, see A-01).

$$F = dA + A \wedge A \quad (2)$$

Example: if the group is $U(1)$ (complex numbers of absolute value 1), then $A \wedge A = 0$ and $F = dA$, and we are back to the field strength of electromagnetism (2-03). For $SU(2)$, $A \wedge A \neq 0$, and this is where the interaction of the field with itself comes from.

Group

A set with a product, an identity and inverses, whose product is associative. An abstraction of a collection of symmetries (operations that preserve a shape). → 1-02

SU(2), double cover

The group of 2×2 unitary matrices with determinant 1. The same as the quaternions of length 1; its shape is the three-dimensional sphere. It maps two-to-one onto the rotation group SO(3). → 1-02

Compact simple group

A bounded, closed, connected matrix group (a compact connected Lie group) whose Lie algebra is non-abelian and has no ideals other than 0 and the whole (simple in the sense of Lie groups; a finite centre, as in $SU(N)$, is allowed). The problem statement lets the gauge group be chosen arbitrarily from among these. → 0-03

Lie group, Lie algebra

A group whose product can be taken smoothly, and its tangent space at the identity equipped with the commutator $[X, Y]$. The Lie algebra is the "infinitesimal directions" of the group. → 1-03

Connection

A rule for comparing vectors at neighbouring points. The gauge field A is a connection. → 1-07

Parallel transport, holonomy

Carrying a vector along a connection. The change (an element of the group) after going once round a closed path and returning is the holonomy. The prototype of the link variables of the lattice and of Wilson loops. → 1-07

Curvature

The leading part, proportional to the area, of the holonomy of a small loop. Formula (2). The field strength of gauge theory. → 1-07

Gauge transformation (geometry)

Changing the basis at each point by an element of the group. The connection changes as $A \mapsto gAg^{-1} - (dg)g^{-1}$ and the curvature as $F \mapsto gFg^{-1}$. → 1-07

03

Mathematical preparation (2): analysis and probability

Quantised Yang–Mills theory can be written, in imaginary time, as "a probability measure on configurations of the field". A configuration is infinitely many numbers, so we need probability in infinite dimensions and the language of the spectrum of operators. The inequality used most in this book is the Poincaré inequality, which bounds the variance by "how much the value changes when you shake things".

$$\mathrm{Var}_\mu(f) \;\le\; C_P\, \mathcal{E}(f), \qquad \mathcal{E}(f) = \mathbb{E}_\mu\big[|\nabla f|^2\big] \quad (3)$$

Example: if μ is the standard Gaussian distribution on the real line, then $C_P = 1$, with equality for $f(x) = x$ (variance 1, expected squared derivative 1). Known The reciprocal of the best C_P is the gap of the diffusion that has μ as its equilibrium (1-10).

Gaussian measure

A measure determined by its mean and covariance alone. The most basic measure that can be built even in infinite dimensions; constructive field theory often starts from here. → 1-08

There is no Lebesgue measure in infinite dimensions

On an infinite-dimensional separable Hilbert (Banach) space such as ℓ^2, the only measure that is invariant under translations and gives finite values to bounded sets is 0. The reason why the "$\mathcal{D}A$" of the path integral has no literal meaning. → 1-09

Self-adjoint operator, spectrum

The infinite-dimensional version of a Hermitian matrix, and the set of its "eigenvalues" (including continuous parts). The Hamiltonian is self-adjoint and its spectrum is real. → 1-10

Variational principle (Rayleigh–Ritz)

The bottom of the spectrum (the smallest eigenvalue, if there is one) is the infimum of $\langle\psi, H\psi\rangle / \langle\psi,\psi\rangle$. However many trial functions you put in, you only get upper bounds. → 1-10

Poincaré inequality, Poincaré constant C_P

Formula (3), and the best constant that goes in it. Its value depends on the normalisation of $\mathcal{E}$ (what you differentiate by), so each chapter states its normalisation. → 1-10

04

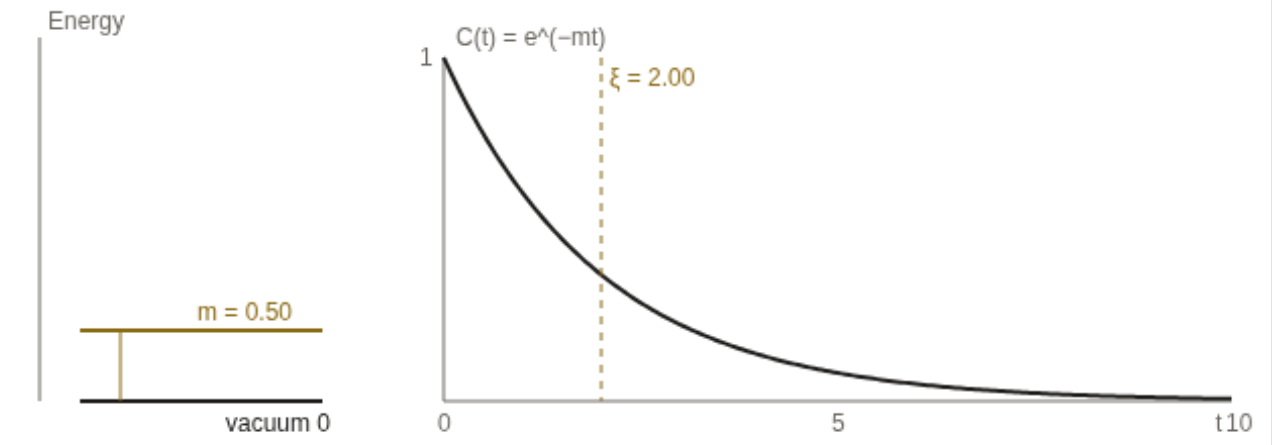
Physical preparation

The words of Part 2 have two faces, quantum mechanics and statistical mechanics. What quantum mechanics calls "the energy gap" and statistical mechanics calls "the length over which correlations decrease" become the same quantity through imaginary time. This is the rereading that runs through the whole book; as a formula, it is the following expansion using the vacuum $|0\rangle$ and the excitations $|n\rangle$ (4-11 § 02).

$$\langle O(t)\,O(0)\rangle - \langle O\rangle^2 \;=\; \sum_{n\geq 1} |\langle n|O|0\rangle|^2\, e^{-(E_n - E_0)t} \;\approx\; C\, e^{-mt}, \qquad \xi = \frac{1}{m} \quad (4)$$

Example: if $m = 0.5$ then $\xi = 2$, and the correlation at a separation of $t = 10$ decreases by a factor of $e^{-5} \approx 0.0067$. Moving the gap m with the knob in Figure 2 changes the spacing of the levels on the left and the rate of decrease on the right together.

Figure 2 Gap and correlation length — changing the gap m with the knob changes at the same time the height of the first excitation on the left and the rate at which the correlation e^{-mt} on the right decreases. The vertical dashed line is t = ξ = 1/m




```
m = 0.50, ξ = 2.00, C(5) = 0.0821, C(10) = 0.0067
```

Table 2 The gap m and the correlation $C(t) = e^{-mt}$ (with the constant C = 1)

m	ξ = 1/m	C(1)	C(5)	C(10)
2.00	0.50	0.1353	0.0000	0.00000
0.50	2.00	0.6065	0.0821	0.00674
0.10	10.00	0.9048	0.6065	0.36788

Computed The numbers in this figure and table were computed on this machine (values of the exponential function; the figure draws the same formula on the spot). The smaller the gap, the farther the correlation persists, and as $m \to 0$ the correlation length becomes infinite. This is what "critical" looks like.

Action, Euler–Lagrange equation

A functional S that assigns one number to a whole path. The classical motion is the path that makes S stationary, and that condition is the Euler–Lagrange equation. → 2-01

State, Hamiltonian

A state is a unit vector in a Hilbert space; the Hamiltonian H is the self-adjoint operator that generates the time evolution e^{-itH} and represents the energy. → 2-05

Path integral

The way of writing quantum theory in which all paths are summed with weight $e^{iS/\hbar}$ (in imaginary time, e^{-S}). → 2-07

Imaginary time, Wick rotation

Substituting $t = -i\tau$. The oscillation e^{iS} becomes the damping e^{-S}, and quantum theory becomes the probability of statistical mechanics. → 2-07

Correlation function, correlation length

The tie between the values at two distant points, and the length ξ when it decreases as $e^{-r/\xi}$. In formula (4), the reciprocal of the gap. → 2-08

Phase transition, critical point, universality

When a parameter is moved, the free energy and other quantities change in a non-smooth (non-analytic) way. If a quantity jumps, it is a first-order transition; if nothing jumps but the correlation length becomes infinite, it is a continuous transition, and that point is a critical point. At a critical point, behaviour that does not depend on the details (universality) appears. → 2-08, 16-11 § 04, 26-12

Transfer matrix

The weight of one step of imaginary time made into a matrix (an operator) $T = e^{-H}$. The ratio of the largest eigenvalue to the next determines how correlations decrease. → 2-07

Langevin dynamics, Glauber dynamics

A noisy gradient flow that has a given measure as its equilibrium (the continuous case), and a stochastic dynamics that rewrites one point at a time (the discrete case). Not the time of the theory but a provisional time for producing samples. → 2-10

Dynamical exponent z, critical slowing down

The exponent when the relaxation time grows as a power of the correlation length, $\tau \sim \xi^{z}$. The phenomenon that the dynamics becomes slower the closer one is to criticality is critical slowing down. → 2-10

05

Yang–Mills theory: classical, quantisation, axioms

These are the words of Parts 3 and 4. The classical theory is the theory of "connections that minimise the square of the curvature", and when it is quantised, one length scale that the classical theory does not have is born. That scale is expressed by Λ, and at one loop it has the following form (4-07).

$$\Lambda = \mu \exp\!\left(-\frac{1}{2b_0\, g^2(\mu)}\right), \qquad b_0 = \frac{11N}{3}\cdot\frac{1}{16\pi^2} \quad (\text{pure } SU(N)) \tag{5}$$

Example: for $SU(3)$, $11N/3 = 11$. As g^2 is made small, Λ approaches 0 in the form e^{-1/g^2}, and does not show up in any power series in g^2. Known

Classical

Field strength

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$. The last commutator is the mark of non-commutativity: the field interacts with itself. → 3-02

Yang–Mills action

$S = \frac{1}{2g^2} \int \operatorname{tr} F_{\mu\nu} F_{\mu\nu} \, d^4x$ (Euclidean, written with g absorbed into A). In four dimensions its value does not change when the scale is changed. → 3-03

Self-dual, instanton

A connection satisfying $F = \pm * F$ (+ is self-dual and − anti-self-dual; the signs of the topological charge are opposite). A finite-action solution that brings the action all the way down to a constant times the absolute value of the topological charge. The representative is the BPST solution (a solution of topological charge 1 with a size around one point). → 3-03

Second Chern number, topological charge

$\frac{1}{8\pi^2} \int \operatorname{tr} F \wedge F$ (in the anti-Hermitian way of writing). An integer that does not change when the connection is changed continuously. → 3-05

Gauge orbit space, Singer's theorem

The space in which connections related by gauge transformations are identified. In the non-abelian case, there is no global way to choose one connection from each orbit continuously. → 3-04

Quantisation

Divergence, regularisation, renormalisation

That loop integrals become infinite at short distances; the cutoff that makes them temporarily finite; and readjusting the couplings so that observables remain finite when the cutoff is removed. → 4-04

Renormalisation group, β function

The rule for how the effective coupling flows when the scale at which one looks is changed. The speed of the flow is the β function $\mu \, dg/d\mu$. → 4-04

Asymptotic freedom

That the β function is negative and the coupling becomes weaker at shorter distances. For pure $SU(N)$, the one-loop coefficient $11N/3$ is what matters. → 4-06

Dimensional transmutation, Λ

That in a theory with only a dimensionless coupling, one length scale Λ appears through quantisation. Formula (5). → 4-07

Gauge fixing, Faddeev–Popov, ghost

A condition that narrows the connections representing the same physics down to one, and the determinant that then appears in the integral. The determinant written in terms of fields is the ghost. → 4-08

BRST symmetry

A symmetry, involving the ghosts, that remains in the action after gauge fixing. It is used to check that physical quantities do not depend on the choice of gauge. → 4-08

Gribov copy, Gribov region

That one orbit contains several connections satisfying the gauge condition (a manifestation of Singer's theorem), and the region where the Faddeev–Popov operator is positive. → 4-08

Axioms and definitions

The Wightman axioms, Haag's theorem

The set of conditions that a quantum field theory on Minkowski spacetime should satisfy, and the theorem showing that an interacting field cannot be placed in the same representation as the free field (if the fields at time 0 are unitarily related to a free field and the vacuum is unique and invariant under spatial translations and rotations, then the field is a free field). → 4-09 axioms, Haag

Refl. positivity

The property that the correlation with the time-reflected function is non-negative. With it, a state space with a positive inner product can be built from the Euclidean measure. → 4-10

Schwinger functions

The correlation functions of the field in Euclidean (imaginary) time, $\langle \phi(x1) \cdots \phi(x_n) \rangle$. The Wightman functions continued to imaginary time. → 4-10

The Osterwalder–Schrader axioms, reconstruction

The set of conditions that the Schwinger functions (just above) should satisfy, and the theorem that if they are satisfied, a Hilbert space and a Hamiltonian can be built. → 4-10

Mass gap

Formula (1). That the spectrum above the vacuum is separated by $\Delta > 0$. It can be rephrased as the mass of the lightest particle, but this is a separate condition from the existence of particles. → 4-11

Non-trivial

That the constructed theory is not a free field (Gaussian). One of the requirements of the problem statement. → 0-03

06

Lattice gauge theory

These are the words of Part 5. A lattice replaces spacetime by a mesh of spacing a, and represents the gauge field by elements of the group placed on the edges (links) of the mesh. That gauge invariance is kept exactly is the strength of the lattice. The action is a sum over the small squares (plaquettes), one by one.

$$S_W = \beta \sum_P \Big(1 - \tfrac{1}{N} \operatorname{Re} \operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2} \quad (6)$$

Example: for $SU(2)$ ($N = 2$) with $\beta = 2.4$, $g^2 = 4/\beta \approx 1.67$. Making β larger makes g smaller, and by asymptotic freedom the lattice spacing a measured in physical units is expected to approach 0 (the one-loop running, 4-07) Physics . Some references use a different normalisation; for conversion, see 5-01.

Link variable, lattice spacing

The element of the group U_ℓ placed on the edge joining neighbouring lattice points (the lattice version of the parallel transport of a connection), and the width of the mesh a. → 5-01

Plaquette, staple

The product U_P of the four edges of a unit square (the holonomy of a small loop), and the product of the three edges with one edge removed. → 5-01

Wilson action

Formula (6). The lattice action that adds $1 - \frac{1}{N} \operatorname{Re} \operatorname{tr} U_P$ for each plaquette; in the naive limit $a \to 0$ it returns to the Yang–Mills action. → 5-01

Elitzur's theorem

On the lattice, the expectation value of a quantity that changes under gauge transformations (such as a component of a single link) is 0 in every volume and every limit. Local gauge symmetry does not break spontaneously. → 5-04

Wilson loop

The trace of the product of links along a closed path. Gauge invariant; it represents the weight for carrying a pair of heavy charges. → 5-02

Area law, perimeter law, string tension

The area law is that the expectation value of a Wilson loop decreases as the exponential of the enclosed area, $e^{-\sigma \cdot \mathrm{area}}$; the perimeter law is that it decreases with the length of the perimeter. The coefficient σ is the string tension. → 5-02

Confinement

That a colour charge cannot be taken out on its own. On the lattice it is often judged by the area law (the static potential growing in proportion to the distance), but there are several formulations. → 5-02

Strong-coupling expansion, character expansion

A method of expanding the weight in characters of the group when β is small. Here the area law and a positive gap can be shown Known (Osterwalder–Seiler 1978). → 5-02

Continuum limit, scaling

The limit $\beta \to \infty$ ($a \to 0$) that keeps quantities in physical units fixed. Scaling is when dimensionless ratios stop depending on a. → 5-05

Glueball

The lightest excitation of pure Yang–Mills regarded as a particle. On the lattice its mass is measured from how correlations decrease. → 5-08

The lattice correlation-length problem, the direction of the lattice version of the mass gap

Chatterjee's (2018) problem. For four-dimensional lattice gauge theory, show that for each β the correlation length $\xi(\beta)$ exists and is finite, and that $\xi(\beta) \to \infty$ as $\beta \to \infty$. If the gap in lattice units stayed uniformly positive, the mass would become infinite in the continuum limit, so the question is posed in this direction. That the gap in physical units is kept is the step beyond (the physical-unit gap statement). → 5-09

07

Constructive field theory

These are the words of Part 7. Constructive field theory is the field that builds interacting field theories as probability measures and proves that they satisfy the axioms. Which dimensions are easy, and where things stop, can be estimated from the "dimension" of the coupling. The Yang–Mills coupling has a power of length.

$$g^2 \sim (\mathrm{length})^{d-4} \quad (7)$$

Example: for $d = 2$, g^2 has the dimension of mass squared, and for $d = 3$ of mass to the first power, and it stops mattering at short distances (super-renormalisable). For $d = 4$ it is dimensionless; it does not weaken by a power of the scale, and moves only logarithmically under the renormalisation group (marginal). In Yang–Mills, by asymptotic freedom, it weakens logarithmically at short distances (4-06). Known Where things stand in each dimension is in 16-01.

Constructive field theory

The field that builds field theories as mathematical objects (measures, operators) and checks the axioms. This book reads it divided into six stages. → 7-01

Triviality

That only a Gaussian (free field) remains when the continuum limit is taken. A result of the type shown for ϕ^4 in four dimensions. → 7-08

Cluster expansion

A technique that splits the measure into a sum of contributions from small lumps far from each other, and shows convergence. Effective at strong coupling and large mass. → 7-04

Blocking, renormalisation group step

One step that coarsens the lattice by a factor b and integrates out the fine fluctuations to get a new action. Repeating it controls the continuum limit. → 7-05

Ultraviolet stability

That, in finite volume, the upper and lower bounds of the final action after repeated blocking do not depend on the lattice spacing. A theorem in three dimensions, a theorem with a hypothesis in four. There are three readings of different strength. → 7-06, 7-07 three readings

Stochastic quantisation, regularity structures

A method of building the measure as the equilibrium of a Langevin equation, and the theory that gives meaning to singular stochastic partial differential equations. → 7-09

08

Words for reading the published construction (Parts 6–14)

These are the words of Volume 2. Volume 2 reads the construction of the existence of four-dimensional Yang–Mills theory published on 2026-11-10 (the Lean code and the committee's assessment). The objects of the published construction are called by

names that carry their meaning, and the identifiers in the code are added as sources. The list of names and symbols is in A-06 § 02, and the conventions for the labels are in 6-00.

Statements and checks

Published construction

The Lean 4 code submitted on 2026-10-28 (yang-mills-group/OpenAI-YM-20261028), and the assessment of the committee that examined it (2026-11-10). This book cites its theorem at the level of "the committee's assessment is that it is proved in Lean". → 6-00

Assessment

The committee's assessment (docs/assessment-1.pdf in yang-mills-group/YM). Part I compares the result of the construction with the problem statement; Part II is the record of the checks. → 6-00

Public theorem, construction theorem, the committee's combined target

The statement in the form of the published question (whose only assumption is that the Lie algebra is simple), a stronger statement that contains it as a projection, and a statement on top of that in which the committee bundled its independent checks into one family. They differ in how much they say. → 6-02

Headline theorems, outside the dependencies

The 17 headline theorems listed by the verification record, and declarations not reached from any of them. Being "inside the dependencies" does not mean being used by the public theorem. → 6-00

Re-checking in the kernel, Comparator

Re-checking the proof in a fresh Lean kernel initialised only with the external libraries, and a tool that matches the statements and definitions against independently written definitions. Neither guarantees that the definitions express the intent of the problem. → 6-02

One step of renormalisation, and the coupling

Probability kernel, predictor, penalty coefficient

One step of renormalisation in the published construction is not a deterministic map but a probability kernel that draws the coarse link with Gaussian-type fluctuations around a central value (the predictor) built from the fine configuration. The strength of the pull is the penalty coefficient c_{pen}. → 9-01, 9-04

Representation

The fixed form in which the density at each step is written: the vacuum term, the main action, regular activities, and the gas of forced activities. The representation is not unique. → 9-06

Regular activity

The sum of the local correction terms in the representation. It vanishes at flat configurations. → 9-06

Forced activity

A correction function with a support that exactly covers the large-field flags and a non-empty "inventory" on it. The whole of it is carried into the representation as a gas. → 9-06, 11-01

Truncated action, flag

The action in which the plaquette terms are cut off linearly above a step-dependent threshold, and the mark of a large field exceeding the threshold. The threshold goes to 0 with the step. → 9-07

Coupling at a step

The coefficient multiplying the term of the normalised curvature squared in the representation at step m. A number that is not determined by the measure alone. → 10-01

Rereading

After one step, rereading the coupling of the representation from the log-ratio of the density at two points, a flat configuration and a test configuration. Strong monotonicity fixes the root uniquely. → 10-03

Shooting

Choosing the bare coupling that brings the coupling at step 0 to the reference value $\bar{\beta}$, by nested intervals of the bands at each step and the intermediate value theorem (the shooting method). Uniqueness is not claimed. → 10-04

Calendar

The bundle of conditions for choosing, in a fixed order, the constants used in the one-step estimates (block side, smoothing exponent, constant of the energy lower bound, rate, entry of the coupling). The conditions uniform in the step are paid here. → 10-05, 11-08

Calibration

Reading the coefficients of insertions of local fields at a reference step and matching them. The inverse of the renormalisation transport is used as input. → 8-04

Multistep estimates

Induction hypothesis, one-step certificate

That the density at every step has a representation of the same form, with a per-step ledger (estimates) attached to its ingredients; and building the representation at step m from that at step $m+1$, with the push-forward identity and the ledger. → 11-00

Seed, region

Each large fine plaquette, each edge far from its prediction and each flag (a seed), and the condition that sorts configurations by the type of their seeds (a region; with hysteresis of some width). → 11-03

Price, mark

The factor per seed (the price), and the part of its allocation left at the exit (the mark; half of the guaranteed lower bound). → 11-05

Strong estimate, weak ledger

The estimate $e^{-\mathrm{rate}\cdot P^2}$ at the exit of one step, and the estimate $e^{-P^{1/5}}$ carried by the induction hypothesis. The strong estimate is converted into the weak ledger and carried. → 11-07

Infinite volume and reconstruction

Slice, thread

The configuration of links within one layer perpendicular to the direction of reflection (a slice), and the set of links in the direction joining adjacent layers (a thread). The split used to build the kernel of the transfer operator. → 12-01

Inside function

A function that is gauge-invariant, measurable and bounded, and depends only on links with both ends inside the strip. The class of functions on which reflection positivity and the chessboard root act. → 12-01

Chessboard root

The size of the expectation of the product of one function copied to all boxes, raised to the power one over the number of boxes. A seminorm satisfying the triangle inequality (not necessarily a norm). → 12-02

Centred tiling

A tiling by boxes whose side is the lattice spacing of step m, with the reference box placed at the centre of the observation point. The size of renormalised observables is measured by this root. → 12-02, 12-03

Column

The arrangement in which a block and its mirror image alternate so as to fill the time circle. A chessboard copy in the single direction of time. → 12-04

Extraction sequence

A pair of a subsequence of cutoffs and a subsequence of volumes. A refinement composes each of the two subsequences. The limit is read as two ordinary limits along this sequence. → 12-05

Window (window function)

In a Ward identity, a smooth function of compact support that equals 1 near one point and whose support contains none of the other points. → 13-01, 12-06

Zero-energy space

The subspace $\ker H$ fixed by the whole time semigroup. That it is one-dimensional is the uniqueness of the vacuum in that phase. → 12-07

Phase measure

The probability measure μ (ν in the assessment) that splits a mixed limiting theory into phases (components). It is built from the Gelfand representation of the commutative algebra of operators compressed to the zero-energy space. → 12-07

Growth condition

That when all pairwise distances of a configuration of points are at least r, the correlations and their derivatives are bounded by a constant times r^{-A}. One form of the bundle of reconstruction axioms. → 12-08

09 Physical pictures

The words of Part 17 are the pictures by which physics, assuming that "the theory exists", explains the mass gap and confinement; most are at the level of Physics. The representative one is the picture in which the colour electric flux is squeezed into a thin tube and behaves as a vibrating string; the potential between heavy charges then takes the following form at long distances R (17-03).

$$V(R) \;=\; \sigma R \;-\; \frac{\pi(d-2)}{24R} \;+\; \cdots \tag{8}$$

Example: for $d = 4$ the second term is $-\pi/(12R)$. The first term is the area law (the string tension), and the second is the zero-point energy of the transverse vibrations of the tube. Physics That the coefficient of the second term does not depend on the details is a result under the assumption that a long tube can be regarded as an effective string.

Dual superconductivity, abelian projection

The picture of confinement in which, when magnetic monopoles condense, electric flux is squeezed into a tube, dually to the way a superconductor squeezes magnetic flux into a tube. Abelian projection is the procedure that partially fixes the gauge, shrinks the group to its abelian part (the diagonal components), and sees monopoles at the singularities of the fixing. → 17-02

Centre vortex

A sheet-like defect across which the phase shifts by an element of the centre of the group. The picture is that if there are many of them in disorder, Wilson loops show an area law. → 17-02

Effective string, Nambu–Goto, Lüscher term

The theory that regards the flux tube as a string weighted by its area, and the $1/R$ term of formula (8) produced by its fluctuations. → 17-03

Large N, 't Hooft coupling, planar diagrams

The limit $N \to \infty$ for $SU(N)$ with $\lambda = g^2 N$ held fixed. Only the Feynman diagrams that can be drawn on a plane remain. → 17-04

Holography, AdS/CFT

The conjecture that the gravitational theory of a certain spacetime is the same thing as a gauge theory on its boundary. The radial direction corresponds to scale. → 17-08

One scale ("a single scale")

The physical premise that every dimensionful quantity of pure Yang–Mills is a constant times a power of Λ. On the lattice it can be written in the form $m^2 \asymp \sigma$. → 17-12

10

The words of this book (Parts 16, 18, 23 and 24)

From Part 18 on, words named within this book appear. The exhaustive list of assumptions is in A-06; here we list only the central set. The quantity $a^2 C_P$ that the study in Part 23 sets up splits into a product of three ratios (17-12; $a = 1/\xi_{\mathrm{RG}}$, $\lambda_1 = 1/C_P$).

$$a^2 C_P \;=\; \underbrace{\frac{a^2}{\sigma}}_{\text{(iii)}} \cdot \underbrace{\frac{\sigma}{m^2}}_{\text{(ii)}} \cdot \underbrace{\frac{m^2}{\lambda_1}}_{\text{(i)}} \tag{9}$$

Example: with the lattice values for SU(2), $\beta = 2.60$ (the table in 17-12), $m/\sqrt{\sigma} = 3.837$, so factor (ii) is $1/3.837^2 \approx 0.068$. Computed Whether (ii) can be bounded above and below by positive constants independent of β is the question of one scale; the lattice values show the tendency, but are not a proof.

The dictionary and where things stand (Parts 16 and 18)

The two columns of settledness

A way of reading that attaches two grades separately to the correspondence between the physical and the mathematical vocabulary: (a) whether they correspond as definitions, and (b) whether the quantity has been shown to exist in four-dimensional Yang–Mills. → 18-01

Seam

A place where the physical way of digging and the mathematical way of digging do not connect. This book lines up seven of them by type. → 16-05

The study with the relaxation-time assumption at its axis (Part 23)

The side of the four-dimensional lattice

Eight statements of construction

The eight statements into which the problem statement is divided along the road that enters from the lattice (infinite volume, reflection positivity, Wilson loops, physical gap, slowness of fluctuations, recovery of rotations, OS reconstruction, non-triviality). The relaxation-time band at the bookkeeping length is not among the requirements of the problem statement; it is a restatement of the physical-unit gap statement. → 23-01

The bookkeeping length ξ_{RG}

A length fixed by keeping the books on how the coupling decreases at each step that coarsens the lattice by a factor b, and taking the number of steps it takes to fall to a fixed value. We set $a := 1/\xi_{\mathrm{RG}}$ (the definition of the four-dimensional study in Part 23). → 23-01

Relaxation-time assumption

The power statement that the Poincaré constant C_P (the relaxation time of Langevin dynamics) grows with the same power as the square of the correlation length: $\ln C_P / \ln \xi \to 2$. In four dimensions, for the Wilson measure as $\beta \to \infty$. There is no proof. → 23-01

The band statement

The statement "C_P lies within a band of constant multiples of the square of the ruler": $a^2 C_P \asymp 1$. It is moved by logarithmic factors, so in this part it is not called the relaxation-time assumption. → 23-01

Relaxation-time upper-bound assumption, relaxation-time lower-bound assumption

The upper and lower halves of the band statement: the upper-bound assumption is $a^2 C_P \le C$ (an upper bound over all test functions), and the lower-bound assumption is $a^2 C_P \ge c$ (a lower bound for which one test function suffices). The hard one is the upper-bound assumption. Names distinct from the relaxation-time assumption. → 23-01

Band form of the diffusive-relaxation-time assumption $(\xi^2)_{\mathrm{band}}[\mathcal{E}]$

An assumption that compares within a band of constant multiples rather than by powers. Whether the band statement holds depends on the graduation of $\mathcal{E}$ (link angle or field), so it is written with the graduation named. An assumption distinct from the relaxation-time assumption. → 23-01

Relaxation chain, one-scale chain, bookkeeping-length chain

That each of the three factors into which $a^2 C_P$ splits is $\asymp 1$: relaxation and correlation length (the relaxation chain), correlation length and string length (the one-scale chain), string length and bookkeeping length (the bookkeeping-length chain). If all three hold, $a^2 C_P \asymp 1$ follows (the converse does not). → 23-01, 17-12

The one-sided bridge

A proposition that, for the path measure with continuous time, a finite spatial lattice and $T = \infty$, under assumptions such as on the ground state, bounds C_P from below

by a constant times the square of the correlation length, using test functions local in time. It spans only the side of the lower-bound assumption. → 23-01

Topological sectors and the order of volume

In a periodic box, the topological charge Q, which moves between sectors with difficulty, also enters the test functions. Take the reading of the original sources for $SU(3)$ and above that the autocorrelation of Q grows faster than ξ^2 (for $SU(2)$, a single primary source, close to a^{-2}), and apply to it, extended, the conjecture attached to the observation with open boundaries that HMC and Langevin are in the same universality class: then the upper-bound assumption reads as false in a periodic box. In this book's order, where the volume is made infinite first, open boundaries are not expected to be a rescue (an assumption row). → the note to 23-01 § 05

Seventeen roads tried on the four-dimensional lattice

Each is a pair of one question and a set of tools applied to it. The road of comparison with the defect of a submission and the road of a one-dimensional chain of scales are restatements of the original questions. → 23-02

Twenty-one walls of the four-dimensional lattice

The reason, in one sentence, why a road stopped with a tool short of an arrow. The twenty-one walls are divided into five types: the wall of principle, the wall of structure, the wall of tools, the wall of assumptions, and the wall of measurement and literature. → 23-02

Complete family

A set of links that removes exactly one link from every plaquette. If the remaining links are held fixed, the weights of the links of the family separate one by one and can be integrated independently of each other. → 23-02

The shortfall δ(b), the slope κ

The amount by which the window of one lattice step falls short of the continuum $\ln b$ (a number measured in the model of the one-loop kernel), and the slope of the one-loop running of the coupling $\kappa = 11/(3\pi^2)$ (SU(2), $\beta = 4/g^2$; a rewriting of known coefficients). → 23-02

The form of the answer

The statement "under the relaxation-time assumption ($\eta = 0$), the physical-unit gap statement $\Longleftrightarrow \sum_k \delta(b_k) < \infty$". Besides the relaxation-time assumption, it stands on the power form, on a model in which the errors of the steps are additive and

bounded, and on δ being the value of the model of the one-loop kernel (assumption rows). → 23-02

Lower bound on the overlap

The input that, in the representation of the correlation by non-negative weights, the weight of the lightest state is at least q times C(0). With it, a lower bound on the gap follows from correlations at finite k. Without it, spectroscopy ends with an upper bound on the gap (the spectroscopic upper-bound wall). → 23-02

Assumption of projection onto a one-dimensional chain of scales

The assumption that the measure can be projected onto a one-dimensional chain in the scale $s = \ln \ell$. Under it, the two-weight Hardy inequality (a weighted inequality in one dimension whose best constant is bounded above and below by integrals of the two weights) can be used. It cannot be said to be weaker than the relaxation-time assumption. → 23-02

Tube transfer-operator inequality

An inequality for the transfer operator saying that the weight of a tube of perimeter ℓ is at least $\sigma\ell + c/\ell$. If it holds, $m^2 \geq 4c\sigma$. There is no proof. → 23-02

The model of the first step

The staircase (four steps)

Four steps placed from an easy place to a hard place: the path measure of the double well in 0+1 dimensions ($\hbar \to 0$), the Wilson measure of the four-dimensional lattice ($\beta \to \infty$), the continuum limit and the OS axioms, and the mass gap of the limiting theory. This part deals with the first and second steps. → 23-03

Double-well relaxation-time power statement

The statement of the first step, $\ln C_P / \ln \xi \to 2$ ($\hbar \to 0$). It is a statement of the same form as the four-dimensional relaxation-time assumption, differing only in the family of measures, so we call it by the same name on both steps. Distinct from the band statement, which speaks of a band of constant multiples. → 23-03, 23-03 § 01

The five grades (proved here / cited / computer-assisted / reading / assumption)

The labels for the strength of claims in this part: elementary (the proof is placed in the text), a known theorem cited from the original source, computer-assisted, reading (floating point, Monte Carlo, fitting), assumption. One label per sentence. → 23-00

The double well

The potential energy with two valleys, $V(x) = (x^2 - 1)^2/8$. The valleys are at $x = \pm 1$ (curvature 1 at the bottom), and the top at $x = 0$ (curvature −1/2). → 23-04

The path measure

The probability measure that gives paths in imaginary time the weight $e^{-S/\hbar}$. $T \to \infty$ is taken first, then $\hbar \to 0$. The numbers are taken on the lattice with step $\varepsilon = \frac{1}{2}$. There is no gauge field in it. → 23-04

Tunnelling splitting

The difference between the two lowest levels of the Hamiltonian, $\Delta = E_1 - E_0$. In terms of paths, it is determined by the weight $e^{-S_0/\hbar}$ of inserting one kink. → 23-04

Correlation length (of the double well)

The length in imaginary time over which the correlation between two points of a path decreases, $\xi = \hbar/\Delta$. At the leading semiclassical order it grows exponentially, $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. → 23-04, § 03

Kink

A path that moves from one valley to the other; after the paths that stay in a valley, the path with the smallest action. Its action is $S_0 = 2/3$ (the action of the instanton), and its width is about 2 in imaginary time. → 23-04

Zero mode

The direction $\dot{x}_{\mathrm{cl}}$ with eigenvalue 0 of the fluctuation operator of the kink. It comes from the fact that shifting the kink in the direction of imaginary time does not change the action: one slow degree of freedom, the position of the kink. → 23-04

Diffusion coefficient of the kink

$D = \hbar/S_0$, obtained by projecting the Langevin noise onto the direction of the position of the kink. A statement about the position of one kink, not about the gap $1/CP$ of the whole path. → 23-04

Herring's formula

A formula that writes the difference between the even and odd eigenvalues of the transfer operator directly as a ratio of inner products of eigenfunctions, without finding the two separately and subtracting. It avoids the loss of digits when $\hbar$ is small. → 23-04

Plateau

A flat direction, along which the action hardly changes, made by the stretch between a pair of kinks where the path stays near the top. The curvature of one site

at the top falls short by $\frac{1}{4}$ of the value needed for convexity ($\varepsilon = \frac{1}{2}$). → 23-04

The convex core

The set deep in a valley, $\{x_i \geq a\}$ ($a > 1/\sqrt{3}$). If the path is confined here, $C_P \leq 2/(3a^2 - 1)$ follows, of Bakry–Émery type, uniformly in the length and in $\hbar$. → 23-04

Phase (the phase without kinks)

The measure restricted to configurations in which the path stays on the side of one valley. For example $\mu^{(+)}$, restricted to $x_i \geq 0$ at all sites (reflected at the wall $x_i = 0$). The measure of the in-phase local Poincaré condition. → 23-05

The assembly of the first step

The splitting of the variance

An identity that splits the configuration space into the two valleys and divides the variance into "the variance within each valley" and "the difference of the means of the two valleys". It is the first tool for assembling an upper bound on C_P from the constants of local Poincaré within the valleys and of the transport of the mean difference, but for long paths a single coarse variable is not enough. → 23-05

Local Poincaré

The Poincaré inequality for the measure restricted to one of the parts (a valley, or the phase without kinks). Its constant C_{loc} becomes the in-phase local Poincaré condition. → 23-05

Transport of the mean difference

An inequality that bounds the square of the difference of the means over the two valleys by the Dirichlet form. Its constant C_{md} gives the type of the saddle-tube condition. → 23-05

Four double-well conditions

The four conditions into which the upper side of the first step is divided: the in-phase local Poincaré condition (local Poincaré in the phase without kinks), the saddle-tube condition (transport along the family of flat saddle points; inside the box, the rows of the tube and the core), the box-reduction condition (reduction to the box dynamics), and the gap-inside-the-box condition. Each leaves assumption rows. They are distinct from the relaxation-time assumption, which is a statement on the side of the conclusion. → 23-05

Double-Well Relaxation-Time Upper-Bound Proposition

The conditional proposition that, for the path measure of the double well, assuming the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$ holds, and through $1 \le R \le K'/\hbar$ the double-well power statement follows. It is algebra on top of the four conditions, and does not say that the first step is closed. → 23-05, 23-03

Scrap

A stretch of the phase without kinks, sandwiched between kink and kink inside the box (of length at most $C\xi$, with values at both ends). The version of the in-phase local Poincaré condition on it is the segment local Poincaré assumption, which requires uniformity also in the end values. → 23-05

Tube

The collection of configurations around a kink, or a kink–antikink pair. The row that bounds the Poincaré inequality of the tube left after the configuration of kinks is held fixed is the saddle-tube Poincaré assumption. The one on a stretch with both ends held in the two valleys is called "the bridge tube". → 23-05, 23-07

Core

The stretch inside the tube where the path stays near the top ($Q_k = (-x_c, 0)^k$, $x_c = 0.89$). The Hessian of the action has negative directions there; bounding its Poincaré constant uniformly in the length k of the core is the row of the core Poincaré uniformity assumption. → 23-07, 23-11

The box dynamics (reduction to the box)

The step that bounds the Poincaré constant of an infinitely long path by the Poincaré constant of the conditional measure on a box of length $C\xi$ with the values on both neighbours held fixed (the box-reduction condition). The Box-Reduction Proposition has the form $C_P \le (1 + 1/n)\, C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$ and does not use the logarithmic Sobolev inequality. → 23-05, 23-11

Even and odd boxes

The line covered by boxes of length L, neighbours overlapping by l sites, divided into two families by the parity of their numbers. Boxes of the same family are apart from each other, and become independent of each other when the values outside the family are held fixed. → 23-11

Averaging over shifts

Shifting the arrangement of boxes by l sites at a time in n ways, and averaging the inequality for each shift. Since the overlap sites are disjoint across the shifts, the coefficient is only 1+1/n. → 23-05

Maximal correlation

For two σ-algebras, the supremum of the correlation between functions measurable with respect to each. For variables at two points, it can be expressed as the second singular value of the normalised matrix of the joint distribution. → 23-05, 23-11

Alternating maximal correlation

The maximal correlation c_{alt} of the two σ-algebras generated by the values outside the even family and the values outside the odd family. That this is below 1 uniformly in $\hbar$ is the assumption row that remains in the Box-Reduction Proposition (the alternating maximal-correlation assumption for boxes). → 23-05

The lemma of two projections

The Two-Projection Lemma. If the maximal correlation c of two σ-algebras is below 1, the variance is at most $1/(1-c)$ times the sum of the expectations of the two conditional variances. An elementary proof is placed in the text. → 23-05

The Efron–Stein inequality

The inequality that the variance of a function on a product measure is at most the sum of the expected variances when the coordinates are moved one at a time. Used for the inequality on the product of the even boxes. → 23-11

Reduced kink gas

A model that keeps only the slow variables inside the box (the number and positions of the kinks). If it is put into the same type as the Glauber dynamics of the one-dimensional Ising model (the row of the kink-gas gap assumption), the gap takes the form of a sum of the creation and annihilation of pairs and the diffusion of kinks. → 23-05

Cross term

The row that bounds the Dirichlet form of the slow side inside the box (the configuration of kinks Y) by $(1+\chi)$ times the original Dirichlet form (the cross-term uniformity assumption). That χ is uniform in $\hbar$ and in the boundary values is an assumption. → 23-05

Kept site

In the two-scale assembly of the in-phase local Poincaré condition, the site kept every 12 sites (the coarse side). When these are held fixed, the stretches between them become independent of each other, and the marginal on the kept sites becomes a nearest-neighbour chain. → 23-06

Bridge (a stretch with fixed ends)

A stretch with the values at both ends fixed. In the in-phase local Poincaré condition, the 11 sites between kept sites (the fine side); in the saddle-tube condition, a stretch with both ends held in the two valleys and one kink. → 23-06, 23-07

Nesting

The assembly that keeps sites 4 and 8 inside the non-convex 11-site bridge as one more level, splitting it into three small bridges (log-concave) and two sites. → 23-06

Helffer's criterion

The Poincaré version of a tool that judges the gap of a chain from the gap of one site and the strength of the coupling between neighbours. Used in Gershgorin form on the ring of kept sites. → 23-06

Margin

The key quantity of the criterion on the coarse side, $\hbar\rho - 2\hbar\kappa$ (the gap of one site minus twice the coupling to a neighbour). That it is positive is confirmed by interval arithmetic for every pair of end values. → 23-06

Node (of $\hbar$)

A point of $\hbar$ confirmed with computer assistance. The Six-Node In-Phase Poincaré Proposition is a statement at each of the six nodes $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, with a margin of at least 0.067. Distinct from the nodes of the dependency graph. → 23-06

Subinterval

A small interval into which the range of $\hbar$ is cut in order to fill in the $\hbar$ between nodes. Within each subinterval, the movement of the expectation values with $\hbar$ is enclosed in an interval. → 23-06, 23-08

Allowance

In a subinterval, the amount of swelling permitted above the pointwise upper bound (the remainder after subtracting the pointwise upper bound from the upper limit fixed by the margin). If the same amount is allowed for every box, the worst box decides the allowance. → 23-06

Route

The seven lines of attack set up to close the row of the core of the saddle-tube condition (the chain outside the window, the remaining numbers, the sector framework, small assumptions, the one remaining proposition, the seam of the ring, the range of $\hbar$). The seven are called by name, and their order does not express sequence or weight. → 23-07

Site window

The range W of 17 sites around one site inside the core. The logarithmic Sobolev constant A_W of the marginal of its mark is central to the route of the window-constant numerics, the route of the sector framework and the route of the window log-Sobolev inequality. Distinct from the window of $\hbar$. → 23-07

The window of $\hbar$

The range $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ of $\hbar$ handled with computer assistance. The "window" in "the whole window" and "the lower end of the window" is this one; where they could be confused, "site" is added to the site window. → 23-07

The chain outside the window

The chain extending outside the site window. Its contraction ratio $q < 1$ is the condition of the route of the chain outside the window and the route of the seam of the ring; for the core of the bridge, $q_{\mathrm{free}} \le 0.998854904$ became a computer-assisted inequality over the whole window of $\hbar$. Nothing is said outside the window or about $\hbar \to 0$. → 23-07, 23-08

Ring

A short periodic path of length $T \ll \xi$ with a configuration containing one kink–antikink pair. → 23-07

Seam

The place on the ring where the separation ℓ of the pair shrinks and the pair disappears. The configuration there is the plateau at the top itself, and the same difficulty as in the core of the bridge tube appears. → 23-07

Lump

A run of sites close to the top ($|z| < \theta$, $\theta = \frac{1}{2}$), taken as the unit of a sector that does not depend on the sign at the seam. With the rest held fixed, the action of a lump of k sites is convex for $k \le 7$. → 23-07

Outer chain and inner chain

The two kinds of chain that enter the logarithmic Sobolev constant of the site window of the ring. The outer chain is the chain on the side of fixed sign extending outside the window; its contraction is a computer-assisted inequality under conditions such as inside the window and pin ≤ 0. The inner chain is the stage inside the window; its ratio reads 1.24–1.28 and exceeds 1 (a finite coefficient). → 23-07

Boundary stage

A finite stage that appears only for configurations in which the lump touches the edge of the window (pin in $(0, \frac{1}{2})$). Its ratio reads 1.004–1.045, and the computer-assisted row of the outer chain does not cover this pin. → 23-07

Sector framework

The procedure for bounding the gap of the core uniformly in k by paying with the rarity of the places of negative curvature, divided into five stages (sector sum, Gibbs decomposition, componentwise localisation, volume locality, sector composition). What remains as an assumption is not the structure but the size of the constants. → 23-09

Sector

The direct product of boxes for each type of the marks attached to the sites (on the side of the top or not). The variance, the Dirichlet form and the entropy split into sums over sectors, and the number of sectors does not enter the constants. → 23-09

Defect set

The set N_j of configurations in which site j or one of its neighbours is on the side of the top beyond the threshold. Negative curvature exists only inside it. → 23-09

Rarity exponent

The minimum $\ell_{\min}$ of the logarithm of the measure of sets containing a defect. Not a length; it does not depend on the length k of the core. → 23-09

Closing condition

The condition $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$ under which the right-hand side of the inequality assembled from the five stages of the sector framework is positive. If it is satisfied, the gap of the core is bounded below by a positive number independent of k. → 23-09

Needed value

The upper limit $A_{\max}$ obtained by rewriting the closing condition for A_W. That A_W is smaller than this is the one remaining proposition, the Window Log-Sobolev Threshold Proposition (the route of the window log-Sobolev inequality). The present upper bound lies outside the needed value, and the lower bound from test functions does not exclude the needed value. → 23-09

The three strands

Attacks on the one remaining proposition in three forms that differ in the functional form of the condition: the componentwise form, the scalar form (the Scalar-Strand Proposition), and the Feynman–Kac form. None of them is closed. → 23-09

Scalar-Strand Proposition

The proposition of the scalar strand, with a table of assumptions. With settings such as core length $k = 20$, it bounds the remainder quantities at each of 8 nodes in $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$. The interval-arithmetic rows have not yet been run. → 23-09

Lifting assumption

The assumption lifting the proxy ratio, in the Scalar-Strand Proposition. The only assumption, which lifts the proxy ratio linearly outside the measured range (the values for lumps of two sites). → 23-09

The common wall

The same place where the three strands stop: at the top, the diagonal of one site, $4 - \frac{1}{4} = 15/4$, does not reach the sum 4 of the couplings to its neighbours. A tool that has convexity as its entrance must pay for this ¼ with the rarity of the plateau, and $\hbar$ enters into the way it is paid. → 23-09

Diagonal dominance

That in each row of a matrix, the diagonal is at least the sum of the absolute values of the off-diagonal entries. For the Hessian of the core it is broken by just ¼ at the top, and on the face where every site is at the top, the smallest eigenvalue of a lump with $k \geq 8$ is negative. → 23-09

Detour plan

A plan that moves one of the two sides of the closing condition, or the form of the inequality itself. The list of detour plans was judged Alive, Conditional, On hold or Dead. → 23-10

The line that killed it

The fact that stopped a detour plan, shrunk to one line. The numbers are readings from a floating-point grid or a transfer matrix. → 23-10

Needed side, form, insufficient side (floor)

The three sides into which the detour plans are divided by direction. The needed side holds plans that make the needed value $A_{\max}$ larger; form, plans that change the functional form of the condition; the insufficient side (floor), plans that reduce the structural loss in the upper bound on A_W. → 23-10

Choice of structure and of tools

The two sources of the constants in the closing condition. The structural quantities (κ, b, $\ell_{\min}$) are decided by the model and by how the defect set is defined; the tool quantities (c_H, ln 2) depend on the choice of the steps of the proof and can be

changed with the same set. What moved the needed value a great deal was the tool side. → 23-09, 23-10

Types of breakage

The reasons that appeared repeatedly in the lines that killed the dead detour plans, gathered into seven (one source, ceiling, paying twice, mixing up the currency, the address of $1/\hbar$, misdirection, transfer). Each type comes with a question to test against a plan. → 23-10

Paying twice

The double-payment type. Paying the same amount again on a set that an earlier step has already paid for. The original Bochner step paid for the negative curvature over the whole core, but that was inside the defect set. A type where fixing it has a large effect. → 23-10

Currency (speed of exit and entropy)

A word of the currency-mix-up type. What the Poincaré inequality measures is the speed of exit; what can pay for the rarity $\mu(B) \le e^{-\ell}$ is the currency of entropy. With only the ratio of the variance to the Dirichlet form, the rarity cancels between numerator and denominator. → 23-10

The address of $1/\hbar$

The type of the address of 1/ħ. If one misjudges which quantity $1/\hbar$ multiplies, a plan merely moves $1/\hbar$ into sectors, marks, comparison constants or the cost of a perturbation. On the plateau at the top, $1/\hbar$ lives not in the direction of the sum but in the radius from the corner of the top. → 23-10

Threshold of convexity

The boundary: the lower bound $\kappa(k) = 8\sin^2\!\big(\pi/(2(k+1))\big) - \frac{1}{4}$ on the Hessian of the action of the core is positive only up to $k \le 7$. A core with $k \le 7$ is bounded by convexity (the Short-Core Convexity Lemma), independently of the end values and of $\hbar$. → 23-11

Three-Range Core-Constant Proposition

The proposition that bounds the Poincaré constant of the core in three ranges by length k: $k \le 7$ by convexity, $8 \le k \le 14$ by the Two-Block Lemma and the maximal correlation of two points, and $k \ge 15$ by applying the reduction to the box to the core. The inputs of the medium-core step and the long-core step are measured values. → 23-11

The wall of conditioning

A wall standing in the direction in which the zero wants to move, because the stretch of the core was conditioned to be "exactly this stretch". In the core on the side of the box whose ends cross the top, this creates bistability. Not physical metastability, but something made by the choice of conditioning. → 23-11

The action barrier

The row of the core action-barrier assumption. The assumption that the Poincaré constant of the core on the box side is bounded in the form $e^{\Delta S_*/\hbar}$ (with a prefactor a power of $\hbar$), and that the barrier is smaller than $2S_0$ ($\Delta S_* < 2S_0$). Under it, the local term of the Gap-Inside-the-Box Proposition vanishes as $\hbar \to 0$. → 23-11

Computation and checking

Computer-assisted

What has been confirmed as an inequality by interval arithmetic (the grade "computer-assisted"). Premises such as the rounding convention are written on the same row. Nothing is said to hold outside the computed range. → 23-08, 23-00

Interval arithmetic

Computation that holds every number as "an interval that is sure to contain the true value". If the upper end of the final interval is smaller than 1 ($q^+ < 1$), then $q < 1$ follows as an inequality. → 23-08

Outward rounding

When computing the ends of an interval, rounding the lower end down and the upper end up. In this part, the result of each operation is moved outward by one to the neighbouring floating-point number. Correct rounding in IEEE 754 is placed in a premise row. → 23-08

Loss of dependence

That intervals forget the ties between quantities coming from the same variable, and widen (evaluating $x - x^2$ naively on [0,1] gives [−1,1]). Fixed by subdividing finely or expanding about the centre. → 23-08, § 04

Positive control, negative control

A positive control asks whether the machine reproduces the answer in a case where it is known in closed form. A negative control asks whether the machine rejects a claim that should not hold. → 23-08, 23-14

Pre-registration

Writing down, before running a computation, the expected range of values, the rate-limiting place and the pass line for the controls, and recording after the result whether they were right. It becomes a record that the pass line was not moved after seeing the result. → 23-08

The integrated table

A table that puts the rows of the five grades for the four conditions of the first step on one sheet, row by row. A map of where there are proofs and where assumptions remain; the proportion of rows does not express progress. → 23-12

The dependency graph

A list that turns the rows of the table into numbered nodes and lines up the premises of each row to the right of "⟸". Which rows the conclusion of the First-Step Upper-Bound Proposition depends on is read by following the premises. → 23-12, 23-05

Node, leaf

A node is one row of the dependency graph. A leaf is a node with no "⟸": a row of assumption, known, computer-assisted or reading. → 23-12

The rule of numbering

The rule that the numbers of the premises to the right of "⟸" are always smaller than the number of the conclusion on the left. If it is kept on every row, the number goes down each time a premise is followed until a leaf is reached, and there is no circle. → 23-12

The required set of leaves

All the leaves that can be picked up by following the premises from the conclusion [111]. The only fork is the single "or" at [103], so there are two sets. → 23-12

The two roads (the core on the box side, the core on the sector side)

The two ways of supporting the fork [103] (the row that the Poincaré constant of the core is uniform). The road of the core on the box side supports it with the Three-Range Core-Constant Proposition and the vanishing of the local term as $\hbar \to 0$; the road of the core on the sector side with a uniform lower bound on the gap of the core in the sector framework and the regularity of the seam. → 23-15

Kinds of work

Six ways of classifying the work needed to close rows of assumption or reading: interval arithmetic, paper (writing a proof), bibliography (checking in the original

source and raising it to "cited"), measurement, another method (out of reach of the present tools), method undecided. → 23-15, § 03

A $\hbar$ that fits all

A $\hbar$ at which every row of the four conditions holds at the same time. Overlaying the ranges of the rows, there is none at present. The double-well power statement ultimately needs every row stated in a form uniform as $\hbar \to 0$. → 23-12

The criterion of cleanness

That the output of `#print axioms` for a Lean theorem is entirely a subset of the three axioms `propext`, `Classical.choice` and `Quot.sound`. It shows that the proof has no holes and that no arbitrary axioms have been added. → 23-13

Putting the conclusion into a definition

The form in which the desired property is written into a definition and that definition is taken as an assumption. The conclusion becomes a copy of the assumption, and the proof closes in one line. It is invisible to the criterion of cleanness. → 23-13

Circle of dependence

The form in which theorem A takes the conclusion of B as an assumption and B takes the conclusion of A. Theorems are checked one at a time, so it is invisible to machine checking; it is checked with the rule of numbering of the dependency graph. → 23-13, 23-12

The four verdicts of reconciliation

The verdicts when the numbers measured in this book's study are compared with the known ones: Agreement, Conditional agreement (how they agree depends on the choice of step, box, convention or form of extrapolation), Disagreement, Not comparable. → 23-14

The words of Part 24

24-00 Entrance

Further considerations

The title of Part 24. A part that considers, in order, the shape of what hinders the relaxation-time assumption, two candidates for restating while keeping the relaxation-time assumption as a premise, two views that do not go through the relaxation-time assumption, and what is at work in four dimensions. Many of its sentences are estimates. → 24-00

The trade of the restatement

Restating the quantity of the problem statement (the gap in Euclidean time) as the quantity of the relaxation-time assumption (the relaxation C_P in Langevin time: a quantity defined by a supremum, for which one test function gives a lower bound, and whose upper bound can be assembled from parts). The two are different quantities. → 24-00, 23-01

The estimate label Estimate

A sixth label added to the five labels of Part 23 (proved here, cited, computer-assisted, reading, assumption). Attached to sentences that are opinions offered as material for judgement, with neither proof nor numbers. → 24-00

24-01 A nearly conserved global quantity

A nearly conserved global quantity (one type)

A global quantity Q whose variance is large and whose Dirichlet form is small (it is passed along through fluxes, or changes only in rare configurations), and which does not show up in the gap. Its relaxation time is decided not by the correlation length but by the scale of the events that break Q, and it hinders the upper-bound assumption. That the five examples fall into the same type is a classification (an estimate). → 24-01

The coefficient of breaking of conservation h

The density of the expectation of the square of the sum of the gradients of the charges of subregions (the gradient of the total charge). In the continuum it is 0 by topological invariance; only on the lattice is it non-zero, on configurations with dislocations. What the upper-bound assumption as a whole asks about is this coefficient. This book has no value for the four-dimensional lattice. → 24-01

The transport coefficient d

The density of the expectation of the square of the first moment (the dipole) of the field of gradients of the charge. In the regime where h is small, what the local diffusive-relaxation-time assumption asks about is this coefficient. There is no value for the four-dimensional lattice. → 24-01, 24-03

The boundary-layer coefficient g

The quantity per unit area obtained by subtracting the volume term from the Dirichlet form of the charge cut by a half-space. It enters the boundary correction of the ratio for a sharp box. → 24-01

The crossing length R_*

A length determined by the ratio of the transport coefficient to the coefficient of breaking of conservation (the square of R_* is dλ$_W$/h). The boundary at which the ratio of the charge of a smooth window is of transport type, proportional to R^2, in boxes shorter than this, and of barrier type, saturating at χ_t/h, in longer boxes (an estimate resting on the assumption of cluster decay). → 24-01

Flux through the box

A field running uniformly through the box, which enters the space of tests if the gauge transformations at the boundary sites of the box are left without dividing out. It moves the reference value by a factor of 2.2–2.5 (a reading of the free-field reference numbers). An example where the same type arises not from the dynamics but from the choice of family. → 24-01, 24-03

24-02 The length at which conservation breaks

Three rulers

The candidates to compare CP against, divided into three: the static correlation length ξ, the length at which conservation breaks ℓ_*, and the time for the global quantity to move by one τ_{glob}. They can take different values within the same model. → 24-02

The length at which conservation breaks ℓ_*

The length at which the charge of a subregion begins to relax not by transport through the boundary but by breaking of conservation inside the region. The candidate definition is (2): the smallest R at which the ratio Var/☒ reaches half its saturation value (an estimate). For two-dimensional $U(1)$ on a strip of width 2 with the outer boundary frozen, β = 2–6 and strip length $M \le 256$, the length read from the reading of τ_{int} (an upper bound on the gap) using the fit form $A/M^2 + B$ is not proportional to the correlation length. → 24-02, 24-02 § 03

Tree coordinates and link coordinates

In the Gaussian approximation on the hierarchical lattice, taking the values of the plaquettes as variables gives $CP = 1/\beta$ independent of the number of steps, while taking the link variables makes the gap shrink in inverse proportion to the square of the side. An example where the type of growth of CP changes with the choice of coordinates (the four-dimensional hierarchical lattice is an extension of this book). → 24-02

Conservation of flux

The property that when a link is moved, the values of the plaquettes (the flux) are only passed between neighbours. The source of the slowness of link coordinates. → 24-02

Flux dipole

The quantity X obtained by summing the angles of the plaquettes of the strip with cosine weights. The slowest mode of the Gaussian approximation; the length at which conservation breaks is fixed from the bend in its relaxation. → 24-02, § 04

Wrapping

The event in which the angle of one plaquette crosses $\pm\pi$. A rare event that crosses a barrier of height 2β; it breaks conservation because the group is compact. → 24-02

Topological charge of the strip

The integer obtained by reducing the angles of the plaquettes of the strip to $(-\pi, \pi]$, summing and dividing by 2π. It changes by one at each wrapping. We do not claim it is the same thing as the four-dimensional topological charge. → 24-02

Raw rate and static fluctuation

The rate of relaxation of the topological charge of the strip, split into the ratio of the rate of wrapping events (a dynamical quantity) and the static fluctuation E (a quantity of the measure), which measures how much the charge moves in one event (an estimate of the chain). → 24-02

τ_{glob}

The time for the global quantity to move by one. The form that compares CP with $\max(\xi^2, \tau_{\mathrm{glob}})$ is a candidate estimate set up from readings of the numbers of the two-dimensional strip, and is not shown in any model. → 24-02

Two-dimensional test bed

For the two-dimensional CP^{N-1} model, a comparison of freezing in a periodic box with the local ratio of a box as N is varied. The ratio is a ratio of autocorrelations with the lattice side fixed at $L = 64$, a different quantity from the static ratio of 24-03. The two-dimensional results imply nothing about four dimensions. → 24-02

24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption

The algebra of time-averaged functionals $\mathcal{D}$

The collection of test functions built with smooth functions from functions at one time averaged over time with weights of bounded support. The CP of Part 24 is

defined as a supremum over this. → 24-03

Window

The time interval $[0, c\ell]$ of c times the ruler of length ℓ (in the version local also in space, a spacetime box of side $c\ell$). ℓ is not fixed by the gap but given from outside (in four dimensions, the bookkeeping length). → 24-03

The local Poincaré constant $C_P^{\mathrm{loc}}(c)$

The supremum of $\mathbf{Var}/\mathcal{E}$ taken only over test functions whose weights have support inside the window. It increases in c, and returns to C_P without assumption as $c \to \infty$. → 24-03

Local relaxation-time assumption ($(\xi^2)\uparrow_c^{\mathrm{loc}}$)

The statement that $C_P^{\mathrm{loc}}(c)$ grows with the same power as ℓ^2, and its upper half $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ ($C_P^{\mathrm{loc}}(c)$ is at most K times ℓ^2). It has content different from the relaxation-time assumption as a whole only when stated with one c held fixed. The definition stands on continuous time, a finite spatial lattice, $T = \infty$ and the assumptions of the one-sided bridge. The version for the four-dimensional Wilson measure is a copy of it, with the existence of the infinite-volume measure placed as a premise. → 24-03

The ratio form K_{rel}

The ratio of the local Poincaré constant of a box for the target measure to the value of the massless reference on the same lattice, the same box and the same graduation. The form "bounded by some $K(c)$" is satisfied by the free-field computation for boxes in four spacetime dimensions that are local also in space, and so is empty; hence it is stated with this ratio. → 24-03

The massless reference and Γ

The reference is the local Poincaré constant $C_P^{\mathrm{loc},0}(R)$ of a box for the infinite-volume massless free field, and Γ is its coefficient $C_P^{\mathrm{loc},0}(R)/R^2$. The numbers are readings of the free-field reference, not numbers for the target measure. → 24-03

Gauge quotient reference

The reference for the gauge field measured on functions invariant under gauge transformations at all sites of the box (for a contractible box, functions of the angles of the plaquettes). It does not depend on the parameter of gauge fixing, so this one is used. → 24-03

Sector relaxation-time assumption (upper-bound assumption^{sec})

The statement that, in a box with its physical side held fixed, the Poincaré constant within a sector of topological charge is bounded by the square of the bookkeeping length. It only pushes the obstacle to the relaxation-time assumption into the quantity between sectors; it does not remove it. → 24-03

Sectors and the reflecting form

The sectors cut from the smoothed topological charge at regular values, and the Dirichlet form on them with no boundary condition imposed (the form of Langevin reflecting at the walls). The Poincaré constant of a sector is finite only when the sector is connected. → 24-03

The quantity between sectors C_{trans}

The supremum of the ratio of the variance of the conditional expectation onto the sector number to the Dirichlet form. It gives the sandwich $C_{\mathrm{trans}} \le C_P \le \sup C_P^{\mathrm{sec}} + C_{\mathrm{trans}}$. Its physical reading is the rate of dislocations. → 24-03

Open box and closed box

A box of the conditional measure with the boundary values held fixed (the box of the Box-Reduction Proposition in Part 23) is a closed box; a box of the marginal with the boundary not held fixed (the box of the local relaxation-time assumption) is an open box. → 24-03

24-04 Which form of the relaxation-time assumption the one-sided bridge needs

The family local in time $\mathcal{F}_1$

The family of functionals obtained by integrating functions at one time over time with bounded weights. Under the assumptions of the one-sided bridge and the canonical graduation, the supremum of $\mathrm{Var}\,/\mathcal{E}$ over this family is exactly ξ^2. The bridge uses only this part. → 24-04

The window factor $\varphi(x)$

$1-(1-e^{-x})/x$. The bracket in writing the ratio for the window T as $\kappa\xi^2\varphi(T/\xi) = \kappa T^2 g(mT)$; $\varphi(x)/x^2$ is strictly decreasing (the Window-Factor Monotonicity Proposition). Distinct from the angle of a plaquette φ_p. → 24-04

In-Window Supremum Proposition

The proposition that writes the supremum over the family local in time with the support of the weights restricted to the window $[0,T]$ in closed form, via the root of a transcendental equation (under the assumptions of the one-sided bridge and the canonical graduation). → 24-04

Two forms of the local relaxation-time assumption

The absolute form $(\xi^2)\uparrow_c^{\mathrm{loc}}$ ($C_P^{\mathrm{loc}}(c)$ is at most $K(c)\ell^2$) and the ratio form $(\xi^2)\uparrow_c^{\mathrm{rel}}$ ($K_{\mathrm{rel}}(c)$ is at most K). The constants correspond by $K(c) = K_{\mathrm{rel}}\,\Gamma\,c^2$. → 24-04

Threshold K_*

$K_* = \kappa/(\lambda_W \Gamma)$. Under the local-excitation moment assumption and the infinite-volume ground-state transformation identity, if the ratio K_{rel} is smaller than this, a lower bound on the gap follows (algebra on top of assumptions). Its value changes with the choice of reference. → 24-04

Local-excitation moment assumption

The assumption that there is a local observable supported in the box such that the moment ratio of the spectral measure of the excitation it creates is at most the square of the infinite-volume gap m_∞ plus a box-size correction $\gamma/(c\ell)^2$. It holds for the free field, but is not shown for any interacting model. → 24-04

The band $[K_*, 1)$

For fields whose ratio lies in this range, the local bridge says nothing. The massive free field also lies in this band. Whether there is another argument that sees the mass inside the band is undecided. → 24-04

Colour singlet and the second chaos

Gauge invariants of $SU(2)$ (colour singlets) have no first-order term around the identity, and at the leading order ($\beta \to \infty$), with three independent Gaussian colours, they lie in the second chaos or higher. For this reason the colour-singlet reference is exactly half the value of the gauge quotient. → 24-04

Functionals of winding-number type

Functionals outside the family local in time, such as the functional that integrates with weights the increment of the lifted angle of a free particle on a circle. All of the amount by which the upper-bound assumption is stronger than the gap lies outside this family. → 24-04

24-05 The Hamiltonian lattice and the vacuum measure

The vacuum measure ν

The distribution of the vacuum at one time (the distribution with density the square of the ground state, ψ_0^2). In gauge theory the functions are restricted to gauge-invariant ones. → 24-05

Ground-state transformation

The isometry that sends a function F to $F\psi_0$. With it, the quadratic form of $(H - E_0)$ is carried to $\hbar^2/2$ times the Dirichlet form of ν. → 24-05

Gap and Vacuum Measure Proposition

Under the ground-state assumption ((A1) in the paper), the gap can be written with the Poincaré constant of the vacuum measure as $\Delta = \hbar^2/(2C_P(\nu))$, and the supremum is attained by ψ_1/ψ_0. → 24-05

Three Poincaré constants

The C_P of the path measure (a one-sided relation to the gap), the supremum over the family local in time ($\xi^2 = \hbar^2/\Delta^2$), and the constant of the vacuum measure ($C_P(\nu) = \hbar^2/(2\Delta)$). → 24-05

The Kogut–Susskind lattice

The Hamiltonian of a gauge theory with time taken continuous and only space made a lattice (the sum of an electric term and a magnetic term). The quantity of the problem statement is exactly the gap on its gauge-invariant subspace. → 24-05

frustration-free

That, when the Hamiltonian is viewed as a sum of local terms, there is a state that realises the ground states of all the local terms at once. Local-to-global gap theorems need this. → 24-05

Electric–Magnetic Splitting Frustration Proposition

The sum into which the Kogut–Susskind lattice is split as an electric term and a magnetic term is not frustration-free ($g > 0$, finite spatial lattice). For this reason local-to-global gap theorems do not apply to this splitting. → 24-05

The Hamiltonian limit

The order of limits in which the temporal lattice spacing is sent to 0 first. That it leads to the same continuum limit as Wilson's isotropic lattice is an assumption. → 24-05

24-06 Paying with rarity instead of convexity

The ceiling of convexity

On a compact manifold without boundary, the oscillation of a potential satisfying $\operatorname{Hess} V \geq -K\, g$ is at most $KD^2/2$ (the Convexity Ceiling on Groups Lemma). On $SU(2)$ a convex measure cannot leave the range of constant multiples of Haar, and cannot concentrate at the identity at weak coupling. → 24-06

The Bakry–Émery lower bound on curvature κ_{BE}

The constant in $\mathrm{Ric} + \mathrm{Hess}\, V \geq \kappa_{\mathrm{BE}} > 0$. If this holds, log-Sobolev and $C_P \leq 1/\kappa_{\mathrm{BE}}$ follow (a copy of the statement in the original source). A different quantity from the comparison constant κ of the one-sided bridge. → 24-06

Fibre

Where the conditional measure lives on the fine fluctuations (products of many links) when the coarse field is held fixed. Reading the inductive convexity assumption of hierarchical models as Bakry–Émery on the fibres over the group is an estimate. → 24-06

The reversal of convex and non-convex places

For flat ϕ^4, what is non-convex is small fields; on the group, it is the hemisphere far from the identity (large fields). On a closed manifold there is always a non-convex place, but the probability of being there is small. → 24-06

The rate of rarity s

The exponent when a bad set B satisfies $\mu(B) \leq e^{-s}$. The quantity written ℓ in 23-09 and 23-10 of Part 23; in Part 24 the letter was changed because ℓ is used for the ruler. → 24-06

Currency (entropy)

The distinction that the Poincaré inequality is a currency that measures the speed of exit, and that what can pay for rarity is the currency of entropy. A step that pays with rarity goes through an inequality of log-Sobolev type. → 24-06

The Holley–Stroock perturbation

The tool that says that if a measure is changed by a factor e^{-U} with bounded U, the Poincaré constant changes by at most a factor $e^{\operatorname{osc} U}$. The cost is not rarity but the exponential of the oscillation (in this book, a standard form from memory, not checked against the original source). → 24-06

Lyapunov condition

The type that bounds the global Poincaré constant from a drift $LW \leq -aW + b\mathbf{1}_G$ returning from the bad set to the good set and a local Poincaré inequality on the good set. Convexity is used only on the good set. → 24-06

Field-dependent constants

The direction of letting the constants of the convexity assumption at each scale depend on the coarse field (but uniformly below 1). As long as uniformly below 1 is required, it does not get past the ceiling of convexity. → 24-06

The Polchinski flow on the group

The multiscale Bakry–Émery condition along the flow of covariances, carried over to convolution with the heat kernel on the group. A formulation on the group was not found within the range searched. → 24-06

The class of measures that pay with rarity

A candidate class of measures with the rules: locally convex on the good set, the bad set rare, a drift back from the bad set, and rarity paid only through entropy inequalities (an estimate). → 24-06

Subcriticality

The key condition of the solution theory of stochastic quantisation (regularity structures); it holds only when $d < 4$. Four dimensions is exactly critical. → 24-06

24-07 Because it is four dimensions

Critical dimension

The dimension in which the coupling g^2 is exactly dimensionless. For Yang–Mills it is four, and the correction at each change of scale does not shrink by a power but remains. → 24-07

marginal

A term whose coupling is dimensionless and which, by power counting, neither strengthens nor weakens. Four-dimensional ϕ^4 and Yang–Mills are both marginal, but the directions of their flows are opposite. → 24-07

Infrared asymptotic freedom

That the coupling approaches 0 at long distances (four-dimensional ϕ^4). The asymptotic freedom of Yang–Mills is at short distances, the opposite direction. → 24-07

Dislocation

A small instanton, the size of the lattice spacing, passing through the lattice. The smoothed topological charge on the lattice changes only at such moments. → 24-07, 24-01

The four hurdles

A table that lines up, for each of the four steps of Part 23 (the 0+1 double well, the four-dimensional lattice, the continuum limit and the OS axioms, the mass gap of the limit), the hurdle, the known point reached, whether it is in reach, and the kind of cost. Whether it is in reach and the cost are estimates. → 24-07

Three sources of difficulty

The sources of the difficulty of the third step: the source of a dimensionless coupling (all scales act with the same weight), the lack of an expansion linking scales (there is no expansion linking the ultraviolet and the infrared), and the lack of global gauge fixing (non-abelian gauge invariance; there is no global gauge fixing, and large fields cannot be controlled by convexity). → 24-07

Ultraviolet stability (as used in Part 24)

That on a finite-volume lattice, the effective action on the unit lattice stays bounded after arbitrarily many repetitions of the block-spin transformation. Cited as a known point reached in the third step. → 24-07, the entry above

The window of coupling

The range that remains between the currency of rarity on the ultraviolet side (the rate of rarity is proportional to the effective coupling and runs out at the confinement scale) and the currency of curvature on the strong-coupling side, where neither is enough. A restatement of the second source, the lack of an expansion linking scales (an estimate). → 24-07, 24-06

Dual superconductivity (as treated in Part 24)

The mechanism that explains confinement by the condensation of monopoles. The condensation has not been shown and the projection depends on the choice of gauge, so it is not used as a tool of proof. → 24-07, the entry above

Centre vortex (as treated in Part 24)

The mechanism by which vortices taking values in the centre carry the string tension. The evidence is lattice numerics, and there are neither estimate inequalities nor statements about the limit, so it is not used as a tool of proof. → 24-07, the entry above

24-08 Questions, and what is not claimed

Candidate rules (six, and three combinations)

Candidate systems that remove one assumption of an existing framework and put an alternative rule in its place: the axiomatisation of scale, the algebra of loops, conserved quantities and sectors, the class of measures that pay with rarity, the axiom of asymptotic freedom, the rule of curvature; and the rule of axiomatising scale + the rule of the rarity class, the rule of conserved quantities and sectors + the rule of the rarity class, the rule of the loop algebra + the rule of conserved quantities

and sectors. All are candidates, and not even their consistency has been checked. → 24-08

Status of the questions (done, partly done, not yet, new)

The status column of the table of questions in Part 24. "Done" means that the text has answered the question conditionally; "new" is a question added in the writing of this part. → 24-08

What does not move when settled

A column attached to each row of questions and candidate rules, stating what does not move even if that question is settled. Whichever question is settled, the mass gap problem does not move. → 24-08

11

Different quantities under the same name

The same letter or the same word can refer to different quantities in different chapters. This is because the conventions of each field are carried over; the meaning is declared at the start of each chapter, but it is missed when skimming. The heavy one is "gap": the gaps of three operators are called by the same word. The conversion between physical units and lattice units is the following line.

$$\Delta = \frac{m_{\mathrm{lat}}}{a}, \qquad \xi_{\mathrm{phys}} = a\,\xi_{\mathrm{lat}} \tag{10}$$

Example: if the gap in lattice units m_{lat} is held at 0.5 while a goes from 1 to 1/16, Δ increases from 0.5 to 8. If it is made proportional to a, as $m_{\mathrm{lat}} = 0.5a$, then $\Delta = 0.5$ is kept. Computed What the continuum limit asks for is the latter (23-01).

Table 3 Different quantities under the same name

Name	Meaning	Where
β	The inverse of the lattice coupling $2N/g^2$ / the inverse temperature of statistical mechanics $1/k_BT$ / the β function $\mu\,dg/d\mu$	5-01 / 2-08 / 4-04
Δ	Mass gap / Laplacian (on the lattice, Δ_{lat})	4-11 / 1-11
η	Static anomalous dimension η_s (the power of correlations) / dynamical $\eta_{\mathrm{dyn}} = 2 - z$. Always attach the subscript	18-05
b_0	One-loop coefficient. With $\mu\,dg/d\mu = -b_0 g^3$ it is $11N/(48\pi^2)$ (formula (5)) / in the table of 0-03 it is $11C_A/(12\pi)$ (0.5836 for $SU(2)$). The normalisations of the coupling differ	4-06 / 0-03

Name	Meaning	Where
CP	Vacuum C_P: the Poincaré constant of the distribution of the vacuum at one time (equivalent to the gap of the Hamiltonian) / path C_P: the Poincaré constant of the path measure (the reciprocal of the Langevin gap; what the relaxation-time assumption refers to) / path C_P measured in a graduation: the path C_P with its normalisation named	23-01
Gap	The gap of the Hamiltonian (the mass gap) / the gap of the transfer matrix (the mass m in lattice units) / the gap of Langevin dynamics λ_1	4-11 / 5-03 / 2-10
Mass	The mass gap / the mass of a particle (an isolated eigenvalue) / the coefficient of the mass term of the action (the bare mass)	4-11 / 2-02
Critical	The point where the correlation length becomes infinite / "critical" for stochastic partial differential equations (counting of scales) / the critical dimension of string theory	2-08 / 7-09 / 17-05

Of the three gaps, the problem statement asks only for the first. The second passes to the first if the continuum limit exists and the ratio (10) converges (the physical-unit gap statement, 23-01), and the third is a property of the dynamics that produces samples. The relaxation-time assumption is a statement comparing the third with the second (23-01).

The words of the second road, the band and the ultraviolet (Part 15 and Parts 25–27)

The crossover band (the band)

The interval around the scale $L \sim 1/\Lambda$ at which weak coupling gives way to strong coupling. The subject of Part 26, and the place where the two roads reach the same wall. It is different from the "windows" of Parts 23 and 24 (the window of A_W, the window of $\hbar$, the time window), and the "band statement" of 23-01 § 05, ℓ_band of 23-09, the "two-dimensional strip" of 24-02 and the "twisted strip" of 25-07 also have different meanings (compared in 26-00 Table 0). → 26-00

The twisted T^2

A small torus with the 't Hooft twist $n_{12} = 1$ put in. The container in which the second road shrinks two directions. → 25-01

't Hooft flux

The sector n_{34} and the flux free energy. Uniqueness of the vacuum, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector. → 25-01

Fractional instanton

A self-dual lump of topological charge ½. Action $4\pi^2/g^2$. Its dilute gas gives the string tension. → 25-02

Container

The region $D \times T^2$ with the frame held fixed. By the container identity (Mack–Petkova), conditioning on the frame makes things independent. → 25-03

The ensemble summed over sectors, the bias θ, the insertion ratio ρ

The ensemble defined by the sum over sectors, the sector bias $\theta(R;T^2_L)$ ((4) of 25-01), and the vortex insertion ratio ρ. Driven by an involution oriented to the sectors. θ is a different quantity from the thresholds θ_B, θ_σ of 23-09 and from the vacuum angle θ (26-00 Table 0). → 25-05

Slab Confinement Theorem

The theorem that, with the lattice spacing and the T^2 held fixed, the area law holds at every β. In the continuum limit the lower bound on the tension goes to 0. → 25-04

Frame decorrelation

How correlations decrease once the frame is fixed. Together with the lateral-widening inequality (strong form, weak form), it becomes the condition for the reduction to a single inequality. → 26-02

The lateral-widening inequality (strong form, weak form)

An inequality that controls the bias when widening sideways. The conclusion of the reduction theorem differs between the strong form and the weak form. → 26-02

The uniform slab tension, the uniform slab gap

The two quantities that remain after the band, with L held fixed, is collected into the density κ_cb. The band is collected into these two. → 26-05

The renormalisation comparison, β-monotonicity

Band = β-monotonicity + renormalisation comparison. A comparison that distinguishes groups; only 26-06 uses the abbreviations. → 26-06

κ_cb (the chessboard density)

The density of frame decorrelation with L held fixed. Different from the κ of Part 23, the κ of the one-sided bridge, κ_BE and others (comparison table in A-01 § 09). → 26-05

Diffusive-relaxation-time assumption

The assumption that the relaxation time grows as a power of the square of the correlation length. The short name is the relaxation-time assumption; the upper side is the upper-bound assumption, and the lower side the lower-bound assumption. The Lean identifiers and the names inside the papers have not been changed. → A-06

A class that closes in one step

A class of effective actions that returns to the same form (a family with the same parameters) after one renormalisation step. Ultraviolet control is the question of whether this class closes across the steps. In 15-03 this is written as the class-closure condition and placed as the first half of the decomposition of the ultraviolet wall. The one-link heat-kernel class is the smallest example that closes with one parameter. → 15-03, 15-04

The telescoping sum of losses

When a class closes with a single parameter, the losses of the steps do not pile up as a product but cancel as a sum of differences of the parameter (a telescoping sum). The one-link heat-kernel class is the smallest example. → 26-11

The properties of the weight (four)

The four properties asked of the weight of one step in multiscale analysis (an arrangement of how the original sources put it). For a group-valued field, the property of closing in one step is missing. The group version of submultiplicativity in sums of fields ((ii) of BBS) is the Background–Fluctuation Decomposition Lemma. → 26-11

Decomposition of the ultraviolet wall (class-closure condition, coupling-flow remainder condition, shape remainder condition, definition and continuity of the effective coupling)

A statement [Assumption] that splits ultraviolet control into four parts: a class that closes with finitely many parameters, the remainder of the flow, the remainder of the shape, and continuity within the domain. The contrast with the decomposition of the band (β-monotonicity × renormalisation comparison, 26-06) is in 15-03 § 06. The questions of this part are in Table 1 of 15-09. → 15-03

Group version of closure under the fluctuation expectation (group version of the exponential moment of fluctuations) and the background dependence of the measure η_k

A statement [Assumption] that restates closure under the fluctuation expectation ((iii) of BBS) and the fourth property, among the four properties of the Balaban–Brydges–Slawny weight, for group-valued fields. For a product of one-link heat kernels it holds by the invariance of Haar measure. The difficulty is the background dependence η of the measure at each step: under a naive estimate Σ η_k diverges, and absorption into the parameters is needed (15-07). → 15-07, 15-01

The invariant curve, the transverse contraction λ, the source σ

The invariant curve: a family of weights indexed by one parameter that the one-step renormalisation map sends to itself. The heat-kernel family is an approximation to it (15-04, 15-06). **The transverse contraction** (λ): the factor by which a deviation of the weight from the invariant curve is multiplied in one step. If $\lambda<1$ the deviation is bounded uniformly in the number of steps (15-03, 15-05). **The source** (σ): the terms outside the class newly fed in at each step (for example, the coefficient of θ^4 of the single coarse plaquette) (15-04, 15-05). → 15-04, 15-05, 15-03

The flux network

A network that finds the commuting configuration of minimal action, with the angle of the single coarse plaquette fixed, as the ratios f of the angles of the plaquettes. In three dimensions with a free boundary it is the current in a resistor network. $\beta'/\beta = \Sigma f^2$, the transverse contraction $\lambda = \Sigma f^4/\Sigma f^2$, and the inequality $1/(64\Sigma f^2) \leq \lambda \leq \max f^2$. → 15-05

Couplings by loop shape

The coefficients when the weight with the fluctuations integrated out is written as a quadratic form in the angles of classes such as single loops and the composite loop. The coordinates for counting the parameters. For four plaquettes in a row, two are needed at the quadratic level, β and the ratio κ_ab/κ_a, and for the heat kernel a part $2/(3\beta)$ due to the curvature of the group appears. → 15-06

The attracting class

When the one-step map contracts the shape coordinates, the class "one parameter + a fixed shape of $O(1/B)$" that the family approaches. It is determined by the fixed point of the flow of the power of the prefactor. → 15-06

The shift of local couplings

A view (a proposal) that reads the part η by which the measure of the fluctuations depends on the background as a change in local couplings $b_p(W)$ determined by the background. The multi-step induction has not yet been written. → 15-07, 15-08

How the small-field contribution to the coupling is split

The view that splits the one-step remainder for small fields into the local part, the Taylor truncation, the small-field cut and the explicit quadratic term between blocks. The first three have the same form as in the sources for flat fields. → 15-08 § 06

The zero-sum system

A system with only one block, in which the explicit quadratic term between blocks vanishes (four plaquettes in a row, the hierarchical model). → 15-08 § 06

Shape-Remainder Theorem for Four Plaquettes in a Row

In the projected model of four plaquettes in a row, the one-step constant of the shape remainder is given as a number, in the form of an upper bound independent of the number of steps. → 15-06 § 09

The local representative

A representative that agrees with F on im D_c and is bounded uniformly in the volume. Built from the cone of axes whose base point moves with the row (Bundle Network Theorem (local form)). → 15-05

One-Link Heat-Kernel Comparison Lemma

A comparison that bounds the one-link Wilson law by a constant multiple of the heat kernel inside the angle cutoff and by a rare tail outside it. → 15-07 § 05

The group-geometry part (the cubic BCH term)

The part of the derivative in the background that remains only for SU(2) and does not depend on the shape coordinate. Its coefficient is 1/9 in the Gaussian main term of two links. → 15-07 § 07

The budget of background change

The sum over steps of the width of the background change times the coupling. It diverges however the threshold is chosen. → 15-07 § 07

The non-locality of the background map

The fact that a background built from the block constraint moves the fine variables non-locally. A one-loop formula that is local in the fine background looks, in the coarse variables, like a non-local O(1) term. → 15-08 § 06

The tail type

How the weight of (1−Loc)W decays with distance (flat, exponential, power). It is fixed by the definition of the coarse variables (how the fine variables straddle the coarse ones). → 15-08 § 06

Coarse links plus global tags

Coordinates made from the coordinates of coarse links, with the Bianchi constraint promoted to coarse variables, plus finitely many global quantities attached as tags, such as the flux threading the torus and the magnetic flux of the centre Z_2. → 15-08 § 06

The agreement of the two gauges

The fact that, for one step, at one loop and in the background of the critical point, the one-loop effective actions in the tree gauge and in the background-covariant gauge are equal (eq. (5)). → 15-08 § 05

The cost of recentring, and the recentring condition

The loss in the Taylor radius (about a factor of 4) and the shift of the coupling when the conditional maximiser of one link is re-taken at the background. → 15-07 § 05

Proposition on a Class with the Background as a Parameter

The proposal that, in a class that carries the logarithm a of the background holonomy as a parameter of the class, the remainder of the background dependence drops by one order and the budget becomes uniform in the number of steps. → 15-07 § 06

The one-face step

One step in which the coarse variable is a single loop. The update of a is exact as an ordered product in the group. → 15-07 § 07

The corner triple product

$\kappa \, \Sigma \, a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of a cube. The first non-commutative coordinate of the multi-loop class. → 15-07 § 07

Row-weighted interpolation

Riesz–Thorin with the rows weighted by their ℓ^1 norms (Stein's analytic family). It brought the factor of the four-dimensional ℓ^4 cone down to 0.823 or less. → 15-05 § 07

The ratio coordinates

The coefficients of the neighbour, the skew neighbour, the cubic and the quartic terms divided by the class coupling B. Their values are fixed at the fixed point of the flow, and they are not independent parameters. → 15-03 § 02, 15-07 § 07

The skew neighbour

The quadratic coupling of a pair of faces shifted by one edge that share no vertex. It is needed in the class in four dimensions; its sign and its being non-zero do not depend on the rule, while its value does. → 15-07 § 07

Normalisation (a)

The coordinate in which the cone constant is divided by the long-wavelength coupling β'_0. The direction-wise normalisation λ is not an amount of contraction. → 15-05 § 07

Blocking by edge sums / by transverse averaging

The rule that takes a coarse edge to be the sum of two fine ones, and the rule that takes it to be the gauge-covariant average of the transverse staples. With the latter, the flow of the covariance heads to a fixed point. → 15-05 § 07

The recursion of W

The one-step recursion of the non-local term. In a row it is "a contracting image + a new term independent of the step + an exponential three-body tail". → 15-08 § 06

Proposal of a one-step inequality with the comparison partner fixed

The one-step inequality that the string tension of the one-step MK image does not exceed that of the genuine coarse theory. On finite boxes the threshold is 1. → 15-04 § 05

Home (of a tool)

The chapter that first places the definition of a tool and the known results about it. The home and label of each tool used by this book's study are in 16-10 Table 1. → 16-10

Where it stops (of a tool)

How far the tool reaches in this book's study and where it stops (the third column of 16-10 Table 1). → 16-10

Estimate (label)

A label used only in Part 24. At the same level as "Reading", it corresponds to a physical estimate ("Physics" among the five labels). The legend is in 0-01 § 03. → 0-01 § 03

Splitting the core in two

Viewing the core of the band as two jobs: "extending the ultraviolet" and "the statement of the last interval to strong coupling". → 16-11 § 01

Extending the ultraviolet

The job of extending the control of the published construction from the threshold to a coupling β_* of order one, while keeping the sum of the per-step errors finite. → 16-11 § 01

Statement of the last interval to strong coupling

A non-perturbative statement over a small number of steps from β_* to the strong-coupling window (distinct from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06). → 16-11 § 01

Lossy chain

Renormalisation comparisons that allow a loss of a factor K at each step, chained over the number of steps. The rate is uniform, but the prefactor grows as $\xi^{\log_2 K}$. → 16-11 § 02

Prefactor

The factor in front of the exponential on the right-hand side of an estimate. In the lossy chain it is $K^{k_0}=\xi^{\log_2 K}$. → 16-11 § 02

Facts on the energy side

Facts about the action of configurations (existence of twisted vacua, the fixed group ± 1, positivity of the action of a box with two twists, upper and lower bounds on S_N). → 16-11 § 03

Shortfall on the entropy side

That this book has no estimate, needed for the band, that counts the fluctuations and the number of arrangements of vortices from below, uniformly in the scale. → 16-11 § 03

Mixing statement for boxes of physical size

That if the side of a box is taken to be the physical length C/Λ, the total variation of the law of gauge-invariant observables inside, when the frame is changed, is smaller than a threshold (uniformly in β). A restatement of the mass gap in a strong form. → 16-11 § 04

Seven test models

The models against which arguments for the band are tested: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups, the limit of the adjoint Higgs, the line and endpoint of the mixed action for SU(2), the bulk transition of SU(N), and G_2. → 16-11 § 04, 26-12 § 02

Three classes of value level

The weak-coupling behaviour of the quantity for a symmetric box with one twist: proportional to β for U(1), constant for a frozen finite group, of the type $(3/2)\ln\beta$ for SU(2). → 16-11 § 04

Mixed action

A lattice action that adds a term in the adjoint representation to the Wilson action (the term in the fundamental representation). → 16-11 § 04, 26-12

Critical endpoint

The point where a line of first-order transitions ends. → 16-11 § 04, 26-12

Frozen phase

In a gauge theory with a finite group, the phase on the side of large β that sticks close to flat connections. → 16-11 § 04, 26-12

Control of the continuum limit of twisted boxes

A comparison across scales, with a finite sum of errors, of quantities fixed by the measure in twisted (framed) boxes. → 16-11 § 06

Coupling of the laws of gauge-invariant observables

A construction that couples the conditional laws of gauge-invariant functions, and a finite-box criterion on top of it. → 16-11 § 06

Sieve

Testing whether an argument for the band would go through equally for other models, and the collection of those models. → 26-12 § 01

Confinement quantity / gap quantity

The two directions of the sieve: a quantity that should be false in a non-confining phase / a quantity that should be true for a model that does not confine but has a gap. → 26-12 § 01

Maximal torus

The commutative subgroup $\{\mathrm{diag}(e^{i\phi}, e^{-i\phi})\}$ of SU(2). It has the same form as U(1). → 26-12

Adjoint Higgs

A model that adds to the SU(2) Wilson action a term coupling a unit vector at each site (in the adjoint representation) with the links. The limit in which the hopping coupling J goes to infinity is formally the U(1) Wilson action. → 26-12 § 02

Bulk transition

A temperature-independent (zero-temperature) transition coming from the lattice action. Not physics of the continuum limit, but a phenomenon on the lattice. → 26-12

Centreless group

A group with trivial centre (G_2 and others). Quantities written with centre twists and vortices are not defined for such a group. → 26-12 § 02

Hysteresis

A disagreement in the value of a quantity between the path that raises the coupling and the path that lowers it (it can be a trace of a first-order transition). → 26-12 § 05

Aspect ratio

For a twisted box, the ratio r = M/N of the side M of the vortex plane to the side N of the transverse plane. It separates comparisons at fixed ratio from the quantification that fixes the transverse side and extends the longitudinal one. → 26-12 § 05

Sign term by term

A test that places a separate coupling β_p on each plaquette and checks whether ΔF increases in each β_p (the sign of $\partial \Delta F / \partial \beta_p$). → 26-12 § 05

Flat connection (flat configuration)

A configuration in which the holonomy of every plaquette equals the identity (or, when there is a twist, the sign of the twist). In 21-07 it is called a "flat configuration". → 16-11 § 04, 26-12

The words of objects, firm theorems and premises (Parts 19–22, Part 28 and the second half of Part 15)

Flux free energy ΔF

$\Delta F = \ln(Z_0/Z_1)$. The logarithm of the ratio of partition functions when one twist n_{34} is inserted. A word of 25-01; the same quantity as the vortex free energy F_v of 26-03. → 19-02

Vortex ratio θ

$\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1) = \tanh(\Delta F/2)$. $|\theta_M| < 1$ follows from the definition. $\theta \geq 0$ does not follow from the definition; it is a theorem when M is even (21-06). → 19-02

Base time

For a link $\ell = (x, \mu)$, the start point x is called the base point of the link; for a link in the time direction, the time of its base point x (the base time) is taken as the time of the link. A different word from the "roots" of the tree of premises. → 19-01

Unnormalised Wilson measure $\tilde{\mu}$, normalised Wilson measure μ

$\mu\tilde{}_{N,\beta}$ is the measure given by the weight $w_\beta(U)$ times the product of Haar measures; $\mu_{N,\beta}$ is the probability measure obtained by dividing it by the partition function. $\mu\tilde{}_{N,\beta} = Z_N(\beta)\ \mu_{N,\beta}$. → 19-01

Bridge between the two Wilson measures

That the measure on the strong-coupling side (Part 20) and the measure on the vortex side (Part 21) are equal, for M = N and no twist, when carried over by a one-to-one correspondence given by definition. An identification shown in Lean. → 19-06

Twist signs n_{12}, n_{34}

Two indices that put in 't Hooft's twisted boundary conditions through the signs $z_p \in \{+1, -1\}$ of the plaquette weights. → 19-02

Twisted vacuum

The configuration V of action 0 built from the unit quaternions i and j. → 19-02

Polyakov loop $P_b(t)$

With the transverse position b and the time t fixed, the real part of the product of the variables of the M links that go once around direction 3. → 19-02

Minimal action of a vortex $S_V(N, m)$, S_N

The number that appears in the exponent of the thick-vortex floor: the minimal action, measured by dividing a plane of height M = 2m into four blocks of side m. $0 \leq S_N \leq 2N^2$. $S_N > 0$ is the theorem of 21-05. → 19-02

Positive side, reflection pairing

For functions F, G that depend only on the links on one side of the mirror (the positive side), $\langle F, G\rangle = \int F(\Theta\omega)\, G(\omega)\, d\mu$. → 19-03

Vessel of reconstruction

The tuple that takes out of the Osterwalder–Schrader reconstruction the parts that do not depend on the measure (pairing, translation, constants). From it one builds the Hilbert space, the vacuum Ω and the transfer operator T. → 19-03

Mass gap (definition on the vessel)

$\| T - P_\Omega \| \leq e^{-m}$. A definition at a fixed lattice spacing, not the definition of the mass gap in the continuum limit. → 19-03

Non-degenerate (two uses)

Non-degeneracy of the vessel means that there are states outside the vacuum (20-04). Non-degeneracy in the statements to be shown means that the coarse field stays in small fields (19-07). → 20-04

Block map B

The map that builds a configuration of the coarse lattice from a configuration of the fine lattice by products along a family of paths, weights, averaging and a return to the group. → 19-04

Path kernel C

The first-order part of the block map B when U is close to the identity. → 19-04

Push-forward measure

$(B{*}\mu)(E) = \mu(B^{-1}(E))$. The measure on the fine lattice carried to the coarse lattice by the block map. → 19-04

Wilson–Dirac operator, measure with fermions

The measure on gauge fields whose weight is the determinant of the Wilson–Dirac operator. A detour outside the main object of the book (pure gauge theory). → 19-05

Cylinder function

A function determined only by the values on a finite set S of links (support S). → 20-03

Dobrushin's condition

The influence that one link receives from the surrounding links is measured by coefficients, and their sum is less than 1. A condition that fixes the infinite-volume limit uniquely. → 20-03

Strong-coupling window

The range of β in which the theorems of Part 20 hold ($18 \tanh|\beta| < 1$ and $24(1 - e^{-4|\beta|}) < 1$). The opposite end from the weak coupling that the continuum limit needs. → 20-02

Sign system, sign character

An abstract system, with the lattice forgotten, written with multiplicative functions (sign characters) on a finite group G (on the lattice, the field of signs on the links). → 21-01

Covering inequality

An abstract inequality about sign systems. A part of the thin-vortex floor. → 21-01

Z_2 monotonicity

With non-negative couplings, increasing the couplings term by term does not increase the twisted ratio and does not decrease the flux free energy. → 21-01

Floor

An upper bound on the vortex ratio θ; equivalently a lower bound on the flux free energy ΔF, an estimate of the slab tension from below. → 21-02

Thin-vortex floor

$\theta \le \tanh(N^2|\beta|)^{M^2}$ (all β, N, M and background twists; the SU(2) Wilson measure). Not uniform in N. → 21-02

Thick-vortex floor

An upper bound on θ whose exponent contains the minimal action of a vortex S_N. The thresholds β_0 and $M_0(\beta)$ are not explicit. → 21-03

One-step contraction

The inequality ☒P ☒ (3/5)P for the symbol of one step of the free field. → 15-12

Cap h(r)

The open cap $\{U \in SU(2) : |U - 1| < r\}$ around the identity, and its Haar volume $h(r)$. → 22-01

Statements to be shown (the root)

Six statements written only with the objects defined in Part 19: matching, convergence of the full sequence, non-degeneracy, equicontinuity, mass gap (uniform decay), and the band. → 19-07

Matching

The way the sequence of bare couplings β_n is chosen: the adjustment that sets the scale of the lattice spacing, written as a rule using only defined objects. → 19-07

Convergence of the full sequence

For the sequence fixed by matching, the push-forward measures converge without passing to a subsequence. → 19-07

Equicontinuity

Observables are continuous uniformly in the physical position inside the box. → 19-07

Sufficient condition, representation

A sufficient condition is a premise placed in order to derive the root along the present road. Many of them are statements made through effective couplings, activities of large fields, background fields and classes (a representation), and many of the quantities of a representation are not yet defined from the lattice. → 28-00

Labels of premises

The column of the list of premises: theorem, theorem (weak form), paper, form fixed, unproved, form not fixed, depends on design, literature. Separate from the labels of the text (Proved here, Cited, Computer-assisted, Reading, Assumption). → 28-00

Reason marks (a)–(f)

The reasons attached to a premise whose form is not fixed: too light, too strong, the object is undefined, a numerical hint, no form that crosses steps, the order of quantifiers is not fixed. → 28-00

Design choices

Things that the mathematics does not fix uniquely and that we choose and fix: the weights, scale factor and thresholds of the block map, norms, backgrounds and so on. → 28-05

Implication (as used in Part 28)

A conditional theorem with all of its assumptions shown. → 28-01

Skew product

A one-step map of the form in which the coupling direction does not contract, only the irrelevant directions contract, and the two affect each other. → 28-02

Endpoint matching

Choosing the bare value for each number of steps so that the value at the coarse end hits the target. Convergence of the full sequence does not follow from contraction alone, and this becomes part of the statement. → 28-02

Tension-doubling inequality

A premise of the form $4\sigma_{2N} \geq e^{\delta_0}\sigma_N$ ($N \geq N_1$) for some $\delta_0 > 0$ and N_1. → 28-04

Detour: four uses of the word "dimension"

You may skip this. "Dimension" has four meanings. (1) The dimension of spacetime d (for space alone, D; in the chapters on string theory, D is used for the dimension of spacetime to match the literature, 17-05). (2) The physical dimension of a quantity (which power of mass). "Dimensional transmutation" is this one (4-07). (3) The dimension of a linear space ($\mathfrak{su}(N)$ has $N^2 - 1$). (4) The anomalous dimension η_s. "Dimensional regularisation" is the technique of extending (1) to complex numbers (4-05). The four of formula (7) and of the article on four dimensions are both (1).

What this chapter can say

What is said	What is not said
Each entry is a shortened version of the definition in the linked chapter.	Some entries omit the hypotheses of theorems (dimension, group, normalisation). The exact statements are only in the linked chapters.
The numbers in Table 1 were counted on this machine from the drafts.	They move as chapters are revised. They are not an indicator of importance.
The entries of Part 17 are written as physical pictures and conjectures.	We do not say that they hold as theorems for four-dimensional Yang–Mills.
The entries of Part 23 are the names and contents of the assumptions that this book's study set up.	None of them is proved. This book shows nothing about the mass gap of the continuum theory (the mass gap in this book's Lean is a statement about the lattice at strong coupling).

Sources and reproduction

Item	Kind	Source / tools
The meaning of each entry	Summary	The chapters of this book (the links). Bibliographic data for primary sources are in the source table of each chapter and in A-03
Conventions for symbols (normalisation of β and tr, the two ways of writing the gauge field)	Conventions of this book	A-01 Table of notation
Table 1 (number of chapters in which words appear)	Computed	python3: from the drafts of the 105 chapters other than the appendices, script, style, the whole-book contents, the meta line and head were removed, tags stripped, and the occurrences of words counted (`term_counts.py`)
Table 2, Figure 2 (e^{-mt} and $\xi = 1/m$), the example for formula (10)	Computed	python3 (`gap_corr.py`). Formula (4) is the known relation in 4-11
The b_0 and $11N/3$ of formula (5)	Known	Sources for 4-06: D. J. Gross, F. Wilczek, PRL 30 (1973) 1343. H. D. Politzer, PRL 30 (1973) 1346 (both bibliographic data only; from the source table of each chapter)
The entry on the strong-coupling expansion (area law and positive gap)	Known	Source for 5-02: K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only; from the source table of each chapter)
The Lüscher term of formula (8)	Physics (under the assumption of an effective string)	Source for 17-03: M. Lüscher, NPB 180 (1981) 317 (bibliographic data only; from the source table of each chapter)
The $m/\sqrt{\sigma} = 3.837$ in the example for formula (9)	Computed (a ratio of values from a primary source)	17-12: B. Lucini, M. Teper, JHEP 0106 (2001) 050, arXiv:hep-lat/0103027, Table 1 ($\beta = 2.60$, 24^4, $a\sqrt{\sigma} = 0.13395(62)$, $am = 0.514(8)$). 3.837 = 0.514/0.13395

Next chapter to read: A-03 Guide to the literature — the primary sources, reviews and textbooks of each chapter, with their availability.

Guide to the literature — what to read first, where to get it, and how far this book read it

This chapter is a guide to the literature.

What this chapter shows — The 52 main works cited by the chapters of this book, arranged by lineage (reviews, textbooks, primary sources) and year (in addition, the three items of the construction published in 2026 are in § 07, and 18 works whose bibliographic data for individual chapters' sources were checked against primary sources are in Table 3). A few to read first. Where each can be obtained (arXiv, the publisher, public release by an institute or the authors, books), and whether the chapters of this book read each work in the text, only the abstract, or only the bibliographic data.

Prerequisite chapters — None. It can be consulted from anywhere, and the content of each work is read in the linked chapters. Terms are in A-02 Glossary, symbols in A-01 Table of notation.

Said first — This is not an exhaustive bibliography. The chapters of this book cite several hundred works, and here we select those that matter across chapters. "Extent read" is a value read mechanically from how each chapter's source table is written, not an assessment of the value of the work. Existence in four dimensions is given by the construction published in 2026 (§ 07); the committee's assessment is that it is proved in Lean. The mass gap has not been shown. This book does not claim to solve the problem.

The order of this chapter

1. How to use this guide — three kinds and two statuses. Detour: how to read bibliographic data
2. Map of the literature — seventy years arranged in seven lineages (a moving figure)
3. What to read first: the problem statement and reviews — the text of the problem, and reviews describing where things stand
4. Textbooks and lecture notes — for going deeper into the tools of Parts 1 and 2

5. Primary sources in physics — from the original paper of 1954 to the skeleton of the 1970s (and later mathematical and stochastic-quantisation viewpoints)
6. Lattice and numerics — Wilson's lattice, and measurements of glueballs
7. Constructive field theory and probability — axioms, the four-dimensional ultraviolet, probabilistic constructions in two and three dimensions
8. Strings and holography — models at the level of physics
9. Counting the source tables of this book — which works appear in how many chapters, and how far they were read (computed)
10. Notes on obtaining and reading — differences between versions, unrefereed work, finding old journals. Bibliographic data checked in primary sources for chapters' citations (Table 3)

01

How to use this guide

Works are divided into three kinds. A **primary source** is the paper that first showed a result; the exact hypotheses and statements of theorems are checked here. A **review** is an overall picture of results up to some point, and an entrance for deciding which primary sources to read. **Textbooks and lecture notes** are for acquiring the tools in order.

We attach two statuses. The first is **availability**: arXiv (the public repository for preprints; free), the publisher (only the journal version; often needs a library subscription; those made freely available are marked "Publisher (open)"), PDFs made public by an institute or the authors, and books. Preprints whose journal publication had not been confirmed at the time of this book are marked "not refereed".

The second is **the extent to which this book read it**. The "Sources and reproduction" table of each chapter writes whether a work was read in the text, only the abstract, or only the bibliographic data. We collect this over all chapters and take the maximum extent over chapters.

$$\mathrm{best}(r) = \max_{c \in C(r)} s(c, r), \qquad s \in \{\text{bibliographic data only} < \text{abstract} < \text{text}\} \tag{1}$$

Here r is the work, $C(r)$ is the set of chapters citing it in their source tables, and $s(c, r)$ is the extent read from the source table of chapter c. "Text" includes cases where only the introduction or one section was checked. For example, Osterwalder–Seiler (1978) is cited by 40 chapters, of which 25 write "bibliographic data only" (§ 09).

Claims of this book that rest on works marked "bibliographic data only" go through other works whose text was checked. Where they go through is in each chapter's source table.

Detour: how to read bibliographic data

You may skip this. Bibliographic data for papers in physics and mathematics are usually written in the order "authors, journal volume (year) first page". For example, `Comm. Math. Phys. 102 (1985) 255` means Communications in Mathematical Physics, volume 102, 1985, starting on page 255. Journal names are abbreviated; in this book `CMP` (Comm. Math. Phys.), `PRL` (Phys. Rev. Lett.) and `NPB` (Nucl. Phys. B) appear often.
arXiv numbers come in two forms. Until March 2007 they had the form "field/year-month and serial number", and from April 2007 the form "year-month.serial number". The serial number has 4 digits until December 2014 and 5 digits from January 2015.



The "Year" in the tables of this chapter is the year of the arXiv submission if there is one, otherwise the year of the journal (Shen–Zhu–Zhu: arXiv 2022, journal 2023). The Clay problem statement is given the year of its announcement (2000).

02

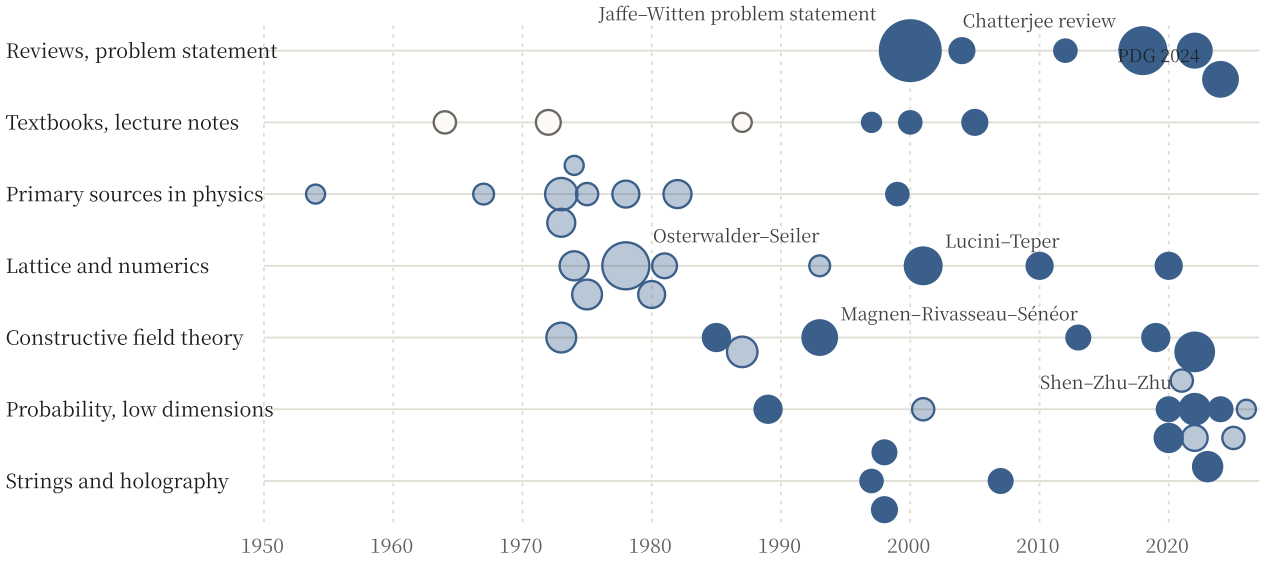
Map of the literature

First we look at the whole on one sheet. The horizontal axis is the year, the vertical the lineage. The size of a dot is the number of chapters citing the work. The fill of a dot is the extent read in this book: solid for "text", light fill for "abstract", hollow for "bibliographic data only".

The primary sources in physics cluster from 1954 to the 1970s. The work aimed at four-dimensional construction (Bałaban, Magnen–Rivasseau–Sénéor) is concentrated in the 1980s and the early 1990s. This line of work did not complete the four-dimensional construction (7-07). For existence in four dimensions, the committee's assessment is that it is proved in Lean in the construction published in 2026 (the table of the published construction in § 07, 6-00). Work from the side of probability theory has been increasing since around 2018.

Moving the year knob leaves dark only the works published up to that year. Moving the part knob makes dark only the works cited by the chapters of that part. Pressing a dot shows its lineage, extent read and the parts citing it below, and lets you jump to its table row (which has links to chapters).

Figure 1 Map of the literature — the year knob brings up "works up to that year", the part knob "works cited by that part". Pressing a dot shows its lineage, extent read and the parts citing it.



Computed The numbers in this figure (positions, sizes and fills of dots, citations by part) were counted on this machine from the source tables of this book (Sources and reproduction). The same content is in the tables by lineage in § 03– § 08, readable even where the figure does not run.

03

What to read first: the problem statement and reviews

If you read only one, read the official problem statement of the Clay Mathematics Institute (Jaffe–Witten). It writes, in the words of those who set it, what the problem asks for and why it is hard in four dimensions. This book reads it word by word in 0-03. It is also the work cited most often among the chapters of this book.

The core of the property the problem statement asks for is the shape of the spectrum of the Hamiltonian H. With the vacuum energy set to 0, it asks that a positive gap Δ open above it.

$$\operatorname{spec}(H) \subset \{0\} \cup [\Delta, \infty), \qquad \Delta > 0 \quad (2)$$

The exact reading of this formula (uniqueness of the vacuum, relation to the decay of correlations) is in 4-11.

The second is Chatterjee's review "Yang–Mills for probabilists". It defines Yang–Mills theory on the lattice in the language of probability theory, and lists what has been shown and what is open in the form of questions (the lattice correlation-length problem and the continuum string-tension problem; Chatterjee's Problems 5.1 and 5.2). Many chapters of Parts 5 and 7 of this book use it as a guide. For a third, Chevyrev

if you follow the probabilistic constructions in two and three dimensions, and the review by Summers for the history of axioms and construction.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2000	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute)	The official text of the problem, and an exposition of its background	0-01 0-02 and 74 more	The institute's public PDF	Text checked
2004	M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem (report to the Clay Mathematics Institute, 2004)	A status report a few years after the problem was announced	0-02 4-07 and 4 more	Public PDF of the Clay Mathematics Institute (archive of annual reports)	Text checked
2012	S. J. Summers, arXiv:1203.3991	A review of the history of constructive field theory	4-10 7-03 and 2 more	arXiv	Text checked
2018	S. Chatterjee, arXiv:1803.01950	A review for readers in probability. Lattice definitions, the area law at strong coupling, unsolved questions	0-01 0-02 and 37 more	arXiv	Text checked
2022	I. Chevyrev, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359	A review of the road approaching Yang–Mills from stochastic quantisation	0-02 0-04 and 14 more	arXiv	Text checked
2024	Particle Data Group (S. Navas et al.), Phys. Rev. D 110 (2024) 030001 (the QCD chapter and others)	The reference for standard physical numbers, such as the running of the coupling constant	0-02 0-03 and 15 more	Publisher (open)	Text checked

> 04

Textbooks and lecture notes

Parts 1 and 2 of this book prepare, from second-year knowledge, the minimum of tools needed. This table is where to go to learn one step deeper. For groups and representations, Hall; for manifolds and connections, Carroll; for operators and spectra, Reed–Simon; for calculations in quantum field theory, Tong's lecture notes; for axioms and construction, Streater–Wightman and Glimm–Jaffe.

One example. The Baker–Campbell–Hausdorff formula used in 1-03 says that the product of matrix exponentials receives a correction that can be written with commutators alone.

$$e^{X}e^{Y} = \exp\Big(X + Y + \tfrac{1}{2}[X,Y] + \tfrac{1}{12}\big([X,[X,Y]] + [Y,[Y,X]]\big) + \cdots\Big) \tag{3}$$

This book checked it up to the third-order terms with Hall's notes. Near the identity, the product of the group is determined by the commutators of the Lie algebra alone. This formula is the reason (the group itself is not determined by the Lie algebra alone; 1-03). Many lecture notes have public versions by the authors, and the arXiv numbers are given in the table.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1964	R. F. Streater, A. S. Wightman, PCT, Spin and Statistics and All That, Benjamin (1964)	The Wightman axioms and their consequences	0-03 0-04 and 2 more	Book	Bibliographic data only
1972	M. Reed, B. Simon, Methods of Modern Mathematical Physics I, Academic Press (1972)	Textbook of operator theory and functional analysis (4 volumes)	1-03 1-10 and 4 more	Book	Bibliographic data only
1987	J. Glimm, A. Jaffe, Quantum Physics: A Functional Integral Point of View, 2nd ed., Springer (1987)	Textbook of constructive field theory	4-10 7-01	Book	Bibliographic data only
1997	S. M. Carroll, arXiv:gr-qc/9712019	Lecture notes on general relativity. An introduction to manifolds, connections and curvature	1-05 1-07	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2000	B. C. Hall, arXiv:math-ph/0005032 (expanded version Springer GTM 222, 2003)	An introduction to Lie groups, Lie algebras and representations	1-02 1-03 and 2 more	arXiv	Text checked
2005	D. Tong's lecture notes (Gauge Theory, Statistical Field Theory, TASI Lectures on Solitons, arXiv:hep-th/0509216)	Lecture notes on gauge theory, statistical field theory and solitons	3-05 4-04 and 4 more	Author's public version	Text checked

05

Primary sources in physics

Yang–Mills theory appeared in the 1954 paper of Yang and Mills, in the form of building a field from the local symmetry of isospin. In the 1960s the rules of gauge fixing and ghosts (Faddeev–Popov) were set in order, and in 1973 asymptotic freedom was found: the discovery of the sign that makes the coupling weaker at short distances.

In this book's notation (formula (8) of A-01), the one-loop running has the following form.

$$\mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11N}{48\pi^2} \quad (4)$$

Since $b_0 > 0$, raising the energy μ makes g smaller. Known This is a one-loop perturbative calculation, not a theorem about the four-dimensional theory. Physics This sign is the ground for expecting Yang–Mills theory to make sense at short distances. The account of its content ($11/3 = 4 - 1/3$) is in 4-06.

Many in this lineage are "abstract" or "bibliographic data only" in the chapters of this book. This is because they are in old journals with no public version; formulas and numbers are checked against reviews whose text was checked (PDG, Tong, Chatterjee).

The table also includes three that come after the skeleton of the 1970s. Gribov (1978) showed that gauge fixing is not uniquely determined. Uhlenbeck (1982) is a mathematical paper in geometric analysis showing that singularities of classical Yang–Mills fields can be removed. Baulieu–Zwanziger (1999) is the viewpoint that sees stochastic quantisation as a five-dimensional field theory.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1954	C. N. Yang, R. L. Mills, Phys. Rev. 96 (1954) 191	The original paper on non-abelian gauge fields	3-01 3-02	Publisher	Abstract checked
1967	L. D. Faddeev, V. N. Popov, Phys. Lett. B 25 (1967) 29	The rules of gauge fixing and ghosts	4-03 4-08 and 1 more	Publisher	Abstract checked
1973	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343	Asymptotic freedom: the sign of the one-loop β function being negative	0-02 0-03 and 12 more	Publisher	Abstract checked
1973	H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346	Asymptotic freedom (an independent calculation in the same issue as Gross–Wilczek)	0-03 4-06 and 7 more	Publisher	Abstract checked
1974	G. ’t Hooft, Nucl. Phys. B 72 (1974) 461	The expansion in a large number of colours N. That planar diagrams dominate	17-04 18-04	Publisher	Abstract checked
1975	A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. B 59 (1975) 85	The instanton solution (a self-dual solution with finite action)	1-12 3-03 and 2 more	Publisher	Abstract checked
1978	V. N. Gribov, Nucl. Phys. B 139 (1978) 1	That gauge fixing is not uniquely determined (Gribov copies)	3-04 4-01 and 6 more	Publisher	Abstract checked
1982	K. Uhlenbeck, Commun. Math. Phys. 83 (1982) 11	Removable singularities of Yang–Mills fields	1-05 1-07 and 7 more	Publisher	Abstract checked
1999	L. Baulieu, D. Zwanziger, Nucl. Phys. B 581 (2000) 604, arXiv:hep-th/9909006	Seeing stochastic quantisation as a five-dimensional field theory	23-00 23-01 and 2 more	arXiv	Text checked

06

Lattice and numerics

In 1974 Wilson put forward the formulation that replaces spacetime by a lattice and puts group elements on the links. With it Yang–Mills theory becomes an integral over finitely many variables, and can be measured by computer. In this book's notation the action has the following form (5-01).

$$S_W(U) = \beta \sum_P \Big(1 - \frac{1}{N} \operatorname{Re}\operatorname{tr} U_P\Big), \qquad \beta = \frac{2N}{g^2} \tag{5}$$

The sum is over all unit squares (plaquettes) P, and U_P is the product of the links around it. Osterwalder–Seiler (1978) gave the mathematical foundation of this formulation (reflection positivity and the transfer matrix). Among the chapters of this book, it is cited most often, together with Jaffe–Witten and Chatterjee.

On the numerical side, starting with the Monte Carlo calculations of Creutz (1980), Lucini–Teper (2001) and Athenodorou–Teper (2020, 2021) have measured glueball masses and their ratios to the string tension. Physics These are values supported by numerical computation, not theorems. The correction coming from fluctuations of the string (Lüscher 1981) has the following form (17-03).

$$V(R) = \sigma R + c_0 - \frac{\pi (d-2)}{24 R} + \cdots \tag{6}$$

In four dimensions ($d = 4$) the last term is $-\pi/(12R)$.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1974	K. G. Wilson, Phys. Rev. D 10 (1974) 2445	Lattice gauge theory and Wilson loops. Confinement at strong coupling	5-01 5-02 and 8 more	Publisher	Abstract checked
1975	S. Elitzur, Phys. Rev. D 12 (1975) 3978	Local gauge symmetry does not break spontaneously (Elitzur's theorem)	3-01 3-07 and 9 more	Publisher	Abstract checked
1978	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440	Reflection positivity, the transfer matrix and strong coupling for lattice gauge theory	0-01 0-02 and 38 more	Publisher	Abstract checked
1980	M. Creutz, Phys. Rev. D 21 (1980) 2308	Monte Carlo calculation of $SU(2)$ lattice gauge theory	4-06 5-02 and 6 more	Publisher	Abstract checked
1981	M. Lüscher, Nucl. Phys. B 180 (1981) 317 (the string fluctuation term)	The universal correction term coming from fluctuations of the string (effective string)	2-02 5-08 and 4 more	Publisher	Abstract checked
1993	R. Sommer, Nucl. Phys. B 411 (1994) 839, arXiv:hep-lat/9310022	Introduction of the length r_0 that sets the lattice scale	5-05 5-08 and 1 more	arXiv	Abstract checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2001	B. Lucini, M. Teper, JHEP 0106 (2001) 050, arXiv:hep-lat/0103027	The string tension of $SU(N)$ (the strength of the force between separated charges) and glueballs, extrapolation to large N	5-02 5-08 and 18 more	arXiv	Text checked
2010	S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228	Critical slowing down of lattice computations (updates of the computation becoming slower closer to the continuum limit), and the autocorrelation of the topological charge (an integer counting the twisting of the field)	5-07 23-01 and 5 more	arXiv	Text checked
2020	A. Athenodorou, M. Teper, JHEP 11 (2020) 172, arXiv:2007.06422; JHEP 12 (2021) 082, arXiv:2106.00364	Lattice computation of the mass spectrum of glueballs	5-08 5-09 and 5 more	arXiv	Text checked

07

Constructive field theory and probability

The mass gap problem consists of two stages, "existence" and "gap". The works approaching it from the mathematical side fall into three groups: the axioms; the four-dimensional ultraviolet; and probability theory in low dimensions and on the lattice.

Axioms: returning from probability to quantum mechanics

The Osterwalder–Schrader axioms are conditions for rebuilding a Hilbert space and a Hamiltonian from Euclidean (imaginary-time) correlation functions. At the centre is reflection positivity, the condition that the expectation of the product of an observable reflected in time, ΘF, and the original F is not negative (4-10).

$$\langle F, F\rangle := \mathbb{E}\big[\,\Theta F\, F\,\big] \;\geq\; 0 \tag{7}$$

If this holds, a positive inner product can be made, and translation in time appears as e^{-sH} ($H \geq 0$). H in formula (2) comes from here.

The four-dimensional ultraviolet: Bałaban and MRS

The work that came closest to the four-dimensional continuum limit is Bałaban's series (the 1985 paper in three dimensions and the 1987–1989 papers in four dimensions; ultraviolet stability of lattice Yang–Mills by the renormalisation group) and the construction with an infrared cutoff by Magnen–Rivasseau–Sénéor (1993). Neither reaches the stage that this book can read as "complete" in the sense of the problem statement. What is shown and what remains is read in 7-07.

Existence in four dimensions: the published construction (2026)

On 2026-11-10, code written in Lean 4 for a construction of the existence of four-dimensional Yang–Mills theory was published, together with the assessment of the committee that examined it. The assessment's verdict is that the submitted code proves, for every simple compact group, that a theory satisfying the Wightman axioms can be built as the scaling limit of four-dimensional Wilson lattice gauge theory. It also states that the uniqueness of the infinite-volume limit and the mass gap are not shown. This book cites that theorem at the level of "the committee's assessment is that it is proved in Lean". Parts 6–14 of Volume 2 read these three items as primary sources. The way they are cited follows the conventions of 6-00 (for the code, the path relative to the repository root, the declaration name and the line; for the assessment, the section number).

Year	Work	What it contains	Chapters of this book	Availability	Extent read
2026	yang-mills-group/OpenAI-YM-20261028 (https://github.com/yang-mills-group/OpenAI-YM-20261028, Apache-2.0). The Lean 4 source submitted on 2026-10-28. The version this book read is 3456e5e	A construction of the existence of four-dimensional pure Yang–Mills theory (lattice, one renormalisation step, multi-step estimates, infinite volume, reconstruction)	6-00–6-04, Parts 8–14	GitHub (public)	Definitions and statements (part of the proof contents)

Year	Work	What it contains	Chapters of this book	Availability	Extent read
2026	The committee's assessment and checks: docs/assessment-1.pdf (2026-11-10, 35 pages) in yang-mills-group/YM (https://github.com/yang-mills-group/YM), and the checking code in submissions/openai-20261028/. The version this book read is 55ba9fc	The result of the construction compared with the problem statement (Part I); the record of the rebuild, re-checking in the kernel, the audit of axioms, and the matching by Comparator (Part II); five open questions	6-00, 6-02, 16-02 and others	GitHub (public)	Text checked
2026	The committee's third statement (page of the Kavli Institute for Theoretical Physics, 2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov	A short statement of the conclusion of the assessment. The committee members' names are not in the text of the assessment; they come from the signatures on this statement	6-00	Public page of the institute	Text checked

The three items of the published construction (table in § 07). They are not included in the count (52 works) of Table 1 and Figure 1.

Two and three dimensions, and probability theory on the lattice

Known In two dimensions the Yang–Mills measure has been constructed (Driver 1989, Lévy 2001 and others). In the 2020s, the road that makes the measure as the equilibrium of a stochastic dynamics (stochastic quantisation) is advancing in two and three dimensions. On the lattice, exponential decay of correlations at strong coupling follows from the cluster expansion of Osterwalder–Seiler (1978), and Shen–Zhu–Zhu showed Poincaré-type inequalities and exponential decay by methods of stochastic analysis in an explicit range of β Known . The reach and where it stops in four dimensions are in 7-10 and 16-01. Unrefereed preprints are marked "not refereed" in the table.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1973	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281	Euclidean axioms and the reconstruction theorem (including the second part of 1975)	0-03 0-04 and 9 more	Publisher	Abstract checked
1985	T. Bałaban, Comm. Math. Phys. 102 (1985) 255 (three dimensions)	Ultraviolet stability of three-dimensional lattice Yang–Mills (controlling short-distance contributions independently of the lattice spacing)	0-02 7-01 and 6 more	Publisher	Text checked
1987	T. Bałaban, Comm. Math. Phys. 109 (1987) 249; 122 (1989) 175; 122 (1989) 355 (the four-dimensional series, 1987–1989)	The series on ultraviolet stability of four-dimensional lattice Yang–Mills by the renormalisation group	0-02 4-04 and 10 more	Publisher	Abstract checked
1993	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325	Construction of four-dimensional Yang–Mills with an infrared cutoff (an operation that artificially suppresses long-distance contributions)	0-02 0-03 and 15 more	Publisher	Text checked
2013	M. Hairer, Invent. Math. 198 (2014) 269, arXiv:1303.5113	The theory of regularity structures (a framework giving meaning to stochastic partial differential equations that have no solutions in the ordinary sense)	1-11 7-03 and 3 more	arXiv	Text checked
2019	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163, arXiv:1912.07973	Triviality of the scaling limit (the limit as the lattice spacing goes to 0) of four-dimensional φ^4 and Ising	0-03 1-08 and 6 more	arXiv	Text checked
1989	B. K. Driver, Comm. Math. Phys. 123 (1989) 575	Two-dimensional Yang–Mills measure on the plane	0-02 0-04 and 6 more	Publisher	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2001	T. Lévy, Mem. AMS 166 no. 790 (2003), arXiv:math/0101239	Yang–Mills measure on compact surfaces	0-02 7-01 and 2 more	arXiv	Abstract checked
2020	A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Publ. Math. IHÉS 136 (2022) 1, arXiv:2006.04987	Langevin dynamics and the state space of the two-dimensional Yang–Mills measure	3-04 7-09 and 3 more	arXiv	Text checked
2020	S. Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229	A probabilistic mechanism of confinement (centre symmetry)	5-10 17-01 and 7 more	arXiv	Text checked
2021	S. Cao, S. Chatterjee, Comm. Math. Phys. 405 (2024), arXiv:2111.12813	The state space of three-dimensional Euclidean Yang–Mills theory	7-06 7-11 and 2 more	arXiv	Abstract checked
2022	A. Adhikari, S. Cao, Ann. Probab. 53 (2025) 140–174, arXiv:2202.10375	Decay of correlations at weak coupling for lattice gauge theory with finite groups	5-06 5-09 and 10 more	arXiv	Text checked
2022	A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Invent. Math. 237 (2024) 541, arXiv:2201.03487	Stochastic quantisation of three-dimensional Yang–Mills–Higgs	4-05 7-06 and 5 more	arXiv	Abstract checked
2022	H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805, arXiv:2204.12737	Stochastic analysis of lattice Yang–Mills at strong coupling (Poincaré, exponential decay)	4-11 5-01 and 21 more	arXiv	Text checked
2023	I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026) (volume and pages not confirmed), arXiv:2302.12160	The invariant measure and universality of two-dimensional Langevin dynamics	0-02 4-08 and 8 more	arXiv	Text checked
2024	S. Chatterjee, Prob. Math. Phys. 7 (2026) 339, arXiv:2401.10507	The scaling limit (the limit as the lattice spacing goes to 0) of $SU(2)$ lattice Yang–Mills–Higgs	7-08 7-10 and 3 more	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
2025	S. Cao, R. Nissim, S. Sheffield, arXiv:2505.16585, arXiv:2509.04688	Enlarging the region of the area law for lattice Yang–Mills (2505.16585), and its proof by a dynamical method (2509.04688) (not refereed)	5-02 5-11 and 2 more	arXiv (not refereed)	Abstract checked
2026	I. Chevyrev, T. Klose, A. Mohamed, arXiv:2607.22236 / R. Chhaibi et al., arXiv:2607.24640	New methods of constructing the two-dimensional Yang–Mills measure (not refereed)	7-02 16-01	arXiv (not refereed)	Abstract checked

08

Strings and holography

On the physics side, there is a view that reads confinement and glueball masses of Yang–Mills theory in the geometry of gravity in one more dimension. The starting points are the correspondence proposed by Maldacena (submitted 1997) and Witten (1998), who used it for a model of confinement. Csáki–Ooguri–Oz–Terning derived ratios of glueball masses from this model as solutions of an eigenvalue problem.

Physics These are correspondences and models at the level of physics, not mathematical theorems. Nor have they been shown in a form that applies to pure four-dimensional Yang–Mills theory itself. What the model assumes, and what happens in the limit back to four dimensions, are read in 17-08, 17-09 and 17-10.

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1997	J. M. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231, arXiv:hep-th/9711200	Proposal of the AdS/CFT correspondence (holography)	17-05 17-08 and 2 more	arXiv	Text checked
1998	C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, arXiv:hep-th/9806021	Deriving ratios of glueball masses from supergravity (the low-energy theory of gravity) calculations	17-06 17-08 and 4 more	arXiv	Text checked

Year	Reference	What it contains	Chapters of this book	Availability	Extent read
1998	E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505, arXiv:hep-th/9803131	A model, in geometry of one more dimension, of the thermal phase transition and confinement	17-05 17-07 and 3 more	arXiv	Text checked
2007	U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 0802 (2008) 019, arXiv:0707.1349	An improved holographic model of QCD	5-08 17-01 and 3 more	arXiv	Text checked

09

Counting the source tables of this book

The source tables of the 110 chapters of this book (excluding this one) have 1035 rows in total. The arXiv numbers appearing in them are of 206 kinds, and bibliographic data in the form "volume (year) page" of about 602 kinds (the latter may pick up sequences of numbers inside formulas, so it is approximate). Computed

For each work, we counted the number of chapters citing it and the extent read in each chapter. Table 1 is the top 10 in order of the number of citing chapters. "Undetermined" counts chapters whose source-table row had no extent written, or where matching of words could not decide.

Table 1 Works cited often by the chapters of this book (breakdown of the extent in each chapter's source table)

Rank	Reference	Citing chapters	Text checked	Abstract checked	Bibliographic data
1	Jaffe–Witten problem statement	76	61	4	
2	Osterwalder–Seiler	40	0	15	
3	Chatterjee review	39	27	4	
4	Shen–Zhu–Zhu	23	6	12	
5	Lucini–Teper	20	12	0	
6	Magnen–Rivasseau–Sénéor	17	2	7	

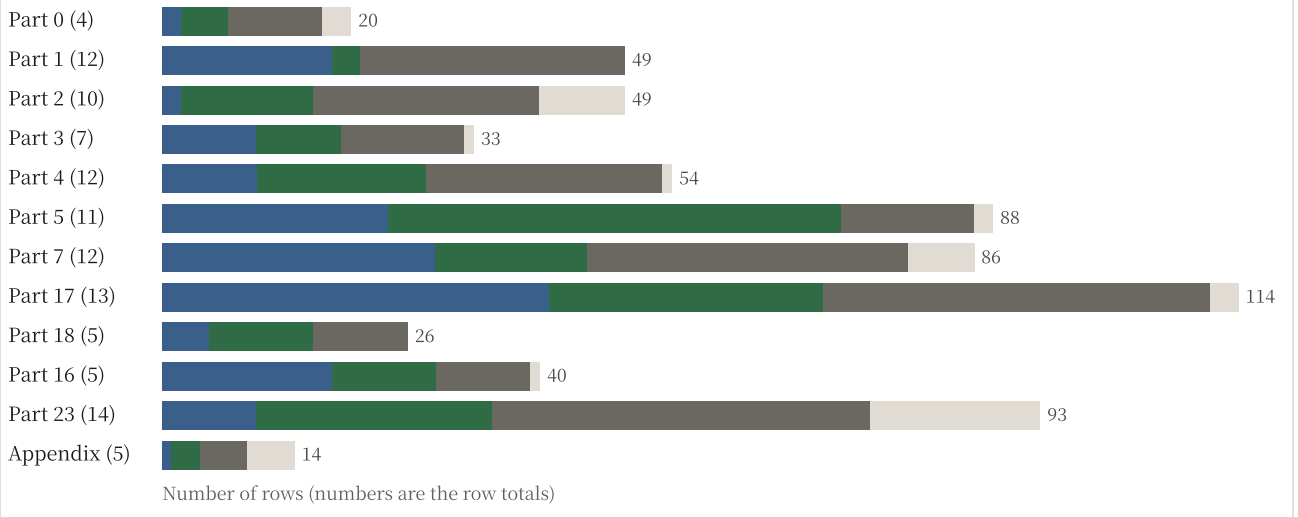
Rank	Reference	Citing chapters	Text checked	Abstract checked	Bibliographic data
7	PDG 2024	17	8	1	
8	Chevyrev review	16	11	1	
9	Gross–Wilczek	14	0	2	
10	Bałaban four-dimensional series	12	0	7	

Osterwalder–Seiler is cited by 40 chapters, but many chapters go through the abstract and reviews whose text was checked (Chatterjee § 4 and others). For Lucini–Teper, 12 of 20 chapters write that they checked as far as the tables of the text.

We also counted by part. Among the source-table rows, for those containing bibliographic data of primary sources, we count the extent in that row. The fraction of text is the following ratio.

$$\text{fraction of text} = \frac{\#\{\text{text}\}}{\#\{\text{text}\} + \#\{\text{abstract}\} + \#\{\text{bibliographic data only}\} + \#\{\text{undetermined}\}}$$

Figure 2 Extent read by part — the button switches between "number of rows" and "fraction".



Computed The numbers in this figure were counted on this machine from the source tables of this book. Table 2 below has the same numbers.

Table 2 Extent of source-table rows containing primary sources, by part

Part	Chapters	Rows	Text checked	Abstract checked	Bibliographic data only	Unde
Part 0	4	20	2	5	10	
Part 1	12	49	18	3	28	

Part	Chapters	Rows	Text checked	Abstract checked	Bibliographic data only	Unde
Part 2	10	49	2	14	24	
Part 3	7	33	10	9	13	
Part 4	12	54	10	18	25	
Part 5	11	88	24	48	14	
Part 7	12	86	29	16	34	
Part 17	13	114	41	29	41	
Part 18	5	26	5	11	10	
Part 16	5	40	18	11	10	
Part 23	14	93	10	25	40	
Appendix	5	14	1	3	5	
Total	110	666	170	192	254	

The extent was decided by the words in the source tables ("text checked", "abstract", "bibliographic data only" and so on), so it is affected by variations in wording. Table 1 reads with priority the words inside the separators (the full-width slash and semicolon) in which the name of the work appears.

10

Notes on obtaining and reading

Differences between versions. The arXiv version and the journal version may differ in page and theorem numbers. The chapters of this book write in their source tables which version was checked.

Publication status. The "journal-ref" in the arXiv catalogue is a field filled in by the authors, and volume and pages are sometimes empty. For example, for Chevyrev–Shen the catalogue gives only "Comm. Pure Appl. Math. (2026)", and volume and pages are not confirmed in this book. For the preprints of 2025 and 2026 (Cao–Nissim–Sheffield, and the two of 2026), publication had not been confirmed at the time of this book. Unrefereed results are not treated with the same weight as refereed results.

Finding old journals. Many papers of the 1970s and 80s exist only in journal versions. Bibliographic data can be checked in INSPIRE (the catalogue of high-energy physics literature) or Crossref. Some old volumes of Comm. Math. Phys. have public versions on Project Euclid (for example Driver 1989, CMP 123). Otherwise, going through a university library is the sure way.

Bibliographic data checked in primary sources for chapters' citations

The tables of § 03– § 08 are the 52 works that matter across chapters. Separately, among the works supporting claims of one or two chapters, those whose bibliographic data or text were rechecked in primary catalogues (Crossref, INSPIRE, DML-CZ, Math-Net.Ru, publishers' pages) are listed in Table 3. "Level of checking" is how far each work was checked. They are not included in the counts of Figure 1, Table 1 or Table 2.

Table 3 Bibliographic data for individual chapters' sources, with the level of checking confirmed

Year	Reference	What it is used for	Chapters of this book	Level of checking
2006	J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, Clay Mathematics Institute / American Mathematical Society (2006). Jaffe–Witten is pp. 129–152	The book containing the problem statement (Jaffe–Witten). The preface gives the date and place of the announcement (24 May 2000, Collège de France)	0-02	Contents and preface checked in the PDF published by CMI, year of publication on the AMS book page
1907	O. Perron, Math. Ann. 64 (1907) 248–263, DOI 10.1007/BF01449896	The largest eigenvalue of a positive matrix is simple	5-03 17-13	Bibliographic data checked in Crossref. Text: bibliographic data only
1912	R. Jentzsch, "Über Integralgleichungen mit positivem Kern", J. reine angew. Math. 141 (1912) 235–244, DOI 10.1515/crll.1912.141.235	The largest eigenvalue of an integral operator with a positive kernel is simple (uniqueness of the ground state of the transfer matrix)	2-07 5-03 and 3 more	Bibliographic data checked in Crossref. Text not read. The claim checked through a citation in a secondary source (arXiv:1910.01120)
1948	M. G. Krein, M. A. Rutman, Uspekhi Mat. Nauk 3:1(23) (1948) 3–95 (English translation AMS Translation 26, 1950)	Generalisation to operators preserving a cone	5-03 7-01	Bibliographic data checked in the Math-Net.Ru index. Text not read
1974	H. H. Schaefer, *Banach Lattices and Positive Operators*, Springer (1974), the theorem on p. 337	Generalisation of Jentzsch's theorem to positive kernels on compact spaces (continuous positive kernels on a finite box fall under this hypothesis)	5-03 7-01 17-13	Bibliographic data only. The claim checked through citations in arXiv:1910.01120 and arXiv:1501.06011

Year	Reference	What it is used for	Chapters of this book	Level of checking
1958	J. Feldman, Pacific J. Math. 8 (1958) 699–708	Two Gaussian measures are either equivalent or mutually singular	1-09	Text checked (introduction and theorem)
1958	J. Hájek, Czechoslovak Math. J. 8 (1958) 610–618, DOI 10.21136/CMJ.1958.100333	The same dichotomy (independently of Feldman)	1-09	Bibliographic data checked in DML-CZ. In Russian; text not read
2023	M. Schröfl, S. Floerchinger, Ann. Henri Poincaré 26 (2025) 3233–3319, arXiv:2307.15548	Free fields of different masses are mutually singular for $d \geq 4$ (Dirichlet, Neumann, periodic and Robin boundaries), and the relative entropy is finite $\Leftrightarrow d < 4$	1-09	Text checked (§ 4.1, Theorem 3)
2002	H. Boschi-Filho, N. R. F. Braga, JHEP 0305 (2003) 009, arXiv:hep-th/0212207	Deriving glueball masses by placing a Dirichlet condition on a hard wall cutting off AdS	17-09	Abstract checked
2003	M. Kruczenski, D. Mateos, R. C. Myers, D. J. Winters, JHEP 0405 (2004) 041, arXiv:hep-th/0311270	The string tension (2.7) of the D4 model, $\sigma \propto g^2_{\rm YM} N_c M^2_{\rm KK}$	17-09	Text checked (formula (2.7))
2005	H. Boschi-Filho, N. R. F. Braga, C. N. Ferreira, Phys. Rev. D 73 (2006) 106006, arXiv:hep-th/0512295	A linear term appears in the static potential in the hard-wall model	17-09	Text checked ((12)–(16))
2006	A. Karch, E. Katz, D. T. Son, M. A. Stephanov, Phys. Rev. D 74 (2006) 015005, arXiv:hep-ph/0602229	The soft-wall model. The squared meson masses are proportional to c ((15))	17-09	Text checked ((15))
2006	O. Andreev, V. I. Zakharov, Phys. Rev. D 74 (2006) 025023, arXiv:hep-ph/0604204	The string tension of the soft-wall model ((19), $g = R^2/\alpha'$)	17-09	Text checked ((19))

Year	Reference	What it is used for	Chapters of this book	Level of checking
1981	D. Zwanziger, "Covariant quantization of gauge fields without Gribov ambiguity", Nucl. Phys. B 192 (1981) 259–269	Stochastic gauge fixing (fixing the equilibrium distribution of Parisi–Wu stochastic quantisation without imposing a gauge condition)	23-02	Title, volume, pages and abstract checked in INSPIRE. Text not read
1990	C. Parrinello, G. Jona-Lasinio, Phys. Lett. B 251 (1990) 175–180	A modified Faddeev–Popov formula that handles Gribov copies by a weighted average over gauge orbits	23-02 23-13 and 1 more	Title, volume, pages and abstract checked in INSPIRE. Text not read
1990	D. Zwanziger, "Quantization of gauge fields, classical gauge invariance and gluon confinement", Nucl. Phys. B 345 (1990) 461–471	A globally correct gauge fixing (cited by 23-02 alongside Parrinello–Jona-Lasinio)	23-02 23-13 and 1 more	Title, volume, pages and abstract checked in INSPIRE. Text not read
2023	C. Chattopadhyay, J. Ott, T. Schäfer, V. Skokov, Phys. Rev. D 108 (2023) 074004, arXiv:2304.07279	The dynamical exponent of conserving dynamics $z = 3.972(2)$ and the prediction $z = 4 - \eta$	2-10	Text checked. Volume and article number checked in Crossref
2025	R. Masaoka, T. Soejima, H. Watanabe, J. Stat. Phys. 192 (2025) 76, arXiv:2502.09908	$z \geq 2$ for Ising in two or more dimensions (periodic boundary, $\beta \geq \beta_c$)	2-10	Abstract checked

The two papers by Zwanziger are different papers. The 1981 paper is gauge fixing within stochastic quantisation (stochastic gauge fixing), and the 1990 paper, from the same year as Parrinello–Jona-Lasinio, is gauge fixing by a weighted average over orbits. What Part 23 of this book calls "the scheme of Parrinello–Jona-Lasinio and Zwanziger" is the latter pair of 1990, and the source table of 23-02 lists both. For Cao–Nissim–Sheffield in the table of § 07, 5-11 checked arXiv:2509.04688 as far as Definition 1.4 and Remark 1.5.

Chapter authors and overall authors. The PDG review as a whole is written as Navas et al., and when a chapter within it (for example the QCD chapter) is cited, it is written with the names of the chapter's authors. The difference in how 0-03 and 4-06 write it is this distinction, and both point to the same edition (2024).

The meaning of the labels is in A-04, the verification procedure in A-05, and the papers of this site in the list of papers.

What this chapter can say

What is said	What is not said
The bibliographic data of the 52 works listed (those with arXiv numbers checked against the arXiv catalogue)	A list of all works cited by this book (see each chapter's source table)
The chapters citing each work, and the extent read written in each chapter's source table (read by matching words)	Complete correctness of the reading (mix-ups due to variations in wording may remain)
One-line descriptions (summaries within the range of the chapters of this book and the arXiv titles and abstracts)	The exact hypotheses and statements of the theorems of each work (checked in the primary sources and the linked chapters)
The distinction of where to obtain each work. That the preprints of 2025–2026 were not refereed at the time of this book	Whether the journal version can be read for free (this changes with time and subscription). The correctness of unrefereed results
The bibliographic data of the 18 works of Table 3, and the level of checking written in each row (Crossref, INSPIRE, DML-CZ, Math-Net.Ru, publishers' pages)	The content of the text of the works in Table 3 marked "bibliographic data only" or "abstract"
That the mass gap problem is unsolved	A quantitative assessment of how close the works listed, taken together, come to a solution

Sources and reproduction

Item	Kind	Source / tools
Titles, authors, dates of first appearance and journal-ref of works with arXiv numbers	Known	Checked against the arXiv catalogue (API queries). In this chapter only as far as the abstracts. Checking of the texts is as in each chapter's source table
The format of arXiv numbers and when it switched	Known	The arXiv help page "Understanding the arXiv identifier" (text checked)
Where the Clay problem statement is	Known	The Yang–Mills page of the Clay Mathematics Institute (the link to the official problem statement checked). Its content is in 0-03
Where Douglas's report (2004) is	Known	Public PDF in the archive of annual reports of the Clay Mathematics Institute (title, author and date checked)

Item	Kind	Source / tools
Bibliographic data and level of checking of Table 3	Known	Checked against Crossref, INSPIRE (titles, volumes, pages and abstracts of Zwanziger 1981 and 1990 and Parrinello–Jona-Lasinio 1990), DML-CZ, Math-Net.Ru, the PDF published by CMI, and the AMS book page. Rows whose text was checked are the same as in the source tables of the chapters (1-09, 2-10, 17-09)
Bibliographic data of works without arXiv numbers	Known	Taken from the bibliographic data in each chapter's source table. In this chapter, bibliographic data only
Formulas (2)–(7)	Known	Formulas repeated from 4-11, 1-03, A-01, 5-01, 17-03 and 4-10. The sources are in each chapter's source table
Tables 1 and 2, Figures 1 and 2, the counts of § 09, the "Chapters of this book" and "Extent read" columns of the tables	Computed	python3: read the source table of each chapter of this book, collected citing chapters with a regular expression per work, and decided the extent by matching words (`refs.py`, `figdata.py`, `gen.py`; each under 1 second)

Next chapter to read: A-04 Lean and machine checking — what "Lean" means among the labels of this book.

Lean and machine checking — what the label "Lean" guarantees is only the arrow from premises to conclusion

> This chapter is a guide. § 02, § 03, § 04, § 05 and § 08 are established knowledge.

What this chapter shows — What has been confirmed when something is "checked" in Lean (propositions are types, proofs are terms, and what is trusted is the kernel). How to read the goals in the middle of a proof, and how to find `sorry`, `native_decide` and axioms (with the results of checking seven small files on this machine). Four types in which a statement passes but does not say what was meant, how this book decides its labels, and the procedure by which readers can check things again themselves.

Prerequisite chapters — None. You can read this even if you have never used Lean. Detailed usage is in this site's guide to Lean (thirteen pages); this appendix is an entrance to it, and gathers the conventions for reading the labels in this book. The list of Lean theorems in this book is in 23-13, and the legend of the labels in 0-01.

Said first — The Lean label guarantees only that "the conclusion follows logically from the premises"; it does not guarantee that the premises are true, nor that the statement says the mathematics that was intended. The Lean theorems in this book are mainly statements about inequalities of real numbers and finite sums; nothing in this book shows the existence of four-dimensional Yang–Mills theory or the mass gap in Lean. Four-dimensional existence was shown in Lean by a different construction published in 2026, and the committee's assessment is that it is proved (6-00). The mass gap has not been shown.

Order of this chapter

1. The role of this appendix — what the Lean label refers to among the five labels of this book, and the division of work with the guide pages
2. Lean on one sheet — propositions are types, proofs are terms of those types, and only the kernel is trusted; Detour: the name of the correspondence between propositions and types

3. One step at a time — the "goal" in the middle of a proof, and tactics (Figure 1, Table 1)
4. Holes and axioms — `sorry` passes with only a warning; finding it with `#print axioms` (Table 2); Detour: using real numbers brings out the three axioms
5. It passes, but is it what was meant? — the four gaps: definition, name, vacuous truth, assumption
6. What the Lean statements in this book talk about — named real numbers, the "if assumption then conclusion" type, mathematics out of stock
7. How the labels are decided — one label per sentence; the strength of a chain is its weakest step (Figure 2, Table 5)
8. A procedure to check for yourself — the four stages of checking, the record of versions, seven questions (Table 7)

01

The role of this appendix — the label "Lean" and the guide pages

Sentences in this book carry five kinds of label (the legend in 0-01): Lean , Paper , Computed , Known and Physics . Of these, the Lean label looks the strongest and is the most easily misread.

The typical misreading is to read "checked in Lean" as "settled as true". What Lean confirms is that the written conclusion follows logically from the written premises. Whether the premises hold, and whether the statement says the intended mathematics, lie outside Lean. This appendix shows that boundary with small examples that were actually checked.

Written as a formula, a sentence with the Lean label has the following form.

$$\underbrace{P_1,\ P_2,\ \ldots,\ P_k}_{\text{premises: outside Lean}} \quad \underbrace{\Longrightarrow}_{\text{the Lean label}} \quad \underbrace{Q}_{\text{conclusion}} \qquad (1)$$

What the label guarantees is the arrow in the middle. If the premises P_i contain unproved assumptions (for example 23-01 The line of construction and the diffusive-relaxation-time assumption) or measured numbers, the conclusion Q is conditional to the same extent. When this book cites a Lean theorem name, it writes the premises on the same line.

C_P (the Poincaré constant, Part 23), m (the mass) and ξ (the correlation length) are quantities of Part 23, but in this appendix it is enough to read them as plain real numbers.

Example. "If $m > 0$, $C_P = 1/m^2$ and $C_P \le K\xi^2$, then $1 \le K(m\xi)^2$" is a statement that only multiplies both sides by m^2, and it can be confirmed in Lean (in § 03 this machine actually checks it). But whether $C_P \le K\xi^2$ holds in four-dimensional Yang–Mills theory has nothing to do with this arrow. That is an assumption.

The division of work between this appendix and this site's guide to Lean (thirteen pages) is as follows. The guide is an introduction to Lean itself, covering installation, how to assemble concrete proofs, and common pitfalls. This appendix restricts itself to the minimum needed to read this book, and to how this book decides its labels. The correspondence of each theorem's premises with the known results is listed in 23-13.

02

Lean on one sheet — propositions are types, proofs are terms, and the kernel is what is trusted

Lean is both a programming language and a tool for checking proofs. It can do both with one mechanism because it **treats propositions as types, and proofs as values (terms) of those types**.

In a program we write "`3` is a value of the type of natural numbers `Nat`". In Lean, with the same notation, we write "`p` is a proof of the proposition P".

$$\texttt{3} : \texttt{Nat} \qquad\qquad p : P \ \text{(2)}$$

The formula on the right is read "p has type P", and that means "p is a proof of P". Type checking can be done mechanically, so checking proofs becomes mechanical too.

A proposition with premises becomes a function type. A proof of "if P_1 and P_2 then Q" is a function that takes a proof of P_1 and a proof of P_2 and returns a proof of Q.

$$(h_1 : P_1) \to (h_2 : P_2) \to Q \ \text{(3)}$$

This is what the arrow of formula (1) really is. In the declaration of a Lean theorem, the arguments lined up to the left of the colon (brackets like `(hm : 0 < m)`) are the premises, and the right of the colon is the conclusion. **When reading, the most important thing is this row of brackets** ("Make a list of the hypotheses" in the guide).

Example. The smallest example checked on this machine is the following one line.

```
theorem pos_nat (n : Nat) (h : 0 < n) : 0 < n := h
```

The premise `h` is itself the proof of the conclusion. It is a theorem with no content, but the form "a function that takes a premise and returns a conclusion" is clearly

visible.

The other pillar is the **kernel**. The proof a person writes is a script lined up from tactics (commands that direct one step of a proof; § 03), and Lean expands it into a term of the form of formula (2). Finally that term is type-checked by the kernel, a checker kept small. Even if a tactic has an error, a term the kernel does not pass does not become a theorem. Narrowing what is trusted down to the one kernel is where the trust in machine checking comes from ("Kernel and tactics" in the guide, How checking works).

Detour: the name of the correspondence between propositions and types

You may skip this. The correspondence that sees propositions as types and proofs as terms is generally called the "Curry–Howard correspondence". In this view, the logical "implies" corresponds to a function type, "and" to a pair type, and "or" to a type with two kinds of value. Lean's logic is built on this correspondence, as a type theory with dependent types (types that may depend on values, as in "vectors of length n"). For the history and the relation to other proof assistants, see "Where the idea came from" in the guide.

03

One step at a time — goals and tactics

When writing a proof in Lean, "what must be shown now" is always on the screen. This is called the **goal**. The goal is written to the right of the symbol $\vdash$, and the hypotheses that may be used are lined up above it. Each time one tactic is entered, a hypothesis or the goal is rewritten, and when no goals remain the proof is finished.

As an example we use the statement of § 01 written in Lean. There are three premises (excluding the types of the variables). It is the same statement as `hup_is_gap_positivity` in 23-13, with only the names of the variables different.

$$m > 0, \quad C_P = \frac{1}{m^2}, \quad C_P \le K\xi^2 \implies 1 \le K(m\xi)^2 \quad (4)$$

On paper it is three lines: "substitute C_P, multiply both sides by $m^2 > 0$, and tidy up". In Lean it takes four steps, and the second step, `have`, is preparation for multiplying by m^2. The correspondence is: substitution is step 1, multiplication is steps 2 and 3, tidying is step 4. In Lean, what each step rewrote is recorded mechanically. The numbers in Figure 1 (the text of the goals) are copied as they are from the states that Lean 4.33.1 on this machine output in the middle of the proof.

Figure 1 Advancing one tactic with "Next" shows the goal after it. Lines that changed from the previous state are shown in colour

Proof

```
theorem gap_from_bound (CP m K ξ : ℝ) (hm : 0 < m)
    (hCP : CP = 1 / m ^ 2) (hH : CP ≤ K * ξ ^ 2) :
    1 ≤ K * (m * ξ) ^ 2 := by
  rw [hCP] at hH
  have hm2 : 0 < m ^ 2 := by positivity
  rw [div_le_iff₀ hm2] at hH
  calc (1 : ℝ) ≤ K * ξ ^ 2 * m ^ 2 := hH
    _ = K * (m * ξ) ^ 2 := by ring
```

State after step 0

```
CP m K ξ : ℝ
hm : 0 < m
hCP : CP = 1 / m ^ 2
hH : CP ≤ K * ξ ^ 2
⊢ 1 ≤ K * (m * ξ) ^ 2
```

Step 0/4: Right after the declaration. The premises are lined up as hypotheses, and the goal is the conclusion itself.

Computed The numbers in this figure were computed on this machine (checked with Lean 4.33.1 with `trace_state` inserted after each step, and the output copied as it is; the figure omits the `trace_state` lines. The "No goals" of step 4 is not a copy of output; it was confirmed by finishing without errors). On a narrow screen the same content can be read in Table 1.

Table 1 has the same content as Figure 1.

Table 1 Each step of the proof of formula (4), and the state after it Computed

Stage	Tactic entered	What changed
0	(declaration)	Hypotheses `hm : 0 < m`, `hCP : CP = 1 / m ^ 2`, `hH : CP ≤ K * ξ ^ 2`; goal `⊢ 1 ≤ K * (m * ξ) ^ 2`
1	`rw [hCP] at hH`	The `CP` in `hH` is replaced by `1 / m ^ 2`
2	`have hm2 : 0 < m ^ 2 := by positivity`	The hypothesis `hm2 : 0 < m ^ 2` is added (`positivity` is an automatic step that shows something is positive)
3	`rw [div_le_iff₀ hm2] at hH`	`hH` changes to `1 ≤ K * ξ ^ 2 * m ^ 2` (a Mathlib lemma that clears the denominator)
4	`calc … := hH`, `_ = … := by ring`	The goal closes (`ring` is an automatic step that shows ring identities)

How to read it. At step 3, `hH` has almost the same form as the goal; the only difference is the order of the product. The final `ring` confirms that reordering as an identity. You can see that the three lines on paper correspond to Lean's four steps with just one preparatory step added. Detailed examples of choosing steps in Lean are in "The skeleton of the proof" in the guide.

What Lean confirmed in this example is only algebra. Inside Lean, CP is just a named real number, and that it is a Poincaré constant is not checked. We look at this point for the theorems of the whole book in § 06.

04

Holes and axioms — `sorry` passes with only a warning

Lean provides two ways of skipping part of a proof. One is `sorry`, a mark meaning "fill this in later". The other is an axiom, a statement declared true without proof. With either, the theorem passes. So the fact that it "passed" alone does not tell you there are no holes.

The tool for finding holes is `#print axioms`. Given a theorem name, it lists which axioms that theorem ultimately depends on (following the statement and the proof). `sorry` is internally replaced by an axiom called `sorryAx`, so it shows up here. The Lean reference manual writes, as a checking procedure, "confirm that only the three standard axioms `propext`, `Classical.choice` and `Quot.sound` appear". The condition for the Lean label in this book has this form.

$$\mathrm{axioms}(\mathrm{theorem}) \subseteq \{\texttt{propext},\ \texttt{Classical.choice},\ \texttt{Quot.sound}\} \quad (5)$$

The meanings of the three are, in order, "equivalent propositions are equal", "an element can be extracted from a proof of existence (the axiom of choice)" and "equivalent elements are equal in the quotient". A quotient is the set obtained by lumping together the elements regarded as the same under an equivalence relation (the set of equivalence classes), for example the classes of remainders of integers divided by 7. The everyday reasoning of mathematics (including the law of excluded middle) needs nothing more than these three. For details, see 23-13 Lean statements and the limits of machine checking and "Axioms" in the guide.

On this machine we checked seven small files. Table 2 is the result.

Table 2 Results of checking seven small Lean files, eight theorems (Lean 4.33.1) Computed

File (theorem)	Content	Exit code	Warnings, errors	Output of `#print axioms`
`pos_nat`	For natural numbers, "if $0 < n$ then $0 < n$"	0 (success)	None	Does not depend on any axioms

File (theorem)	Content	Exit code	Warnings, errors	Output of `#print axioms`
`pow_mod_decide`	$2^{10} \bmod 7 = 2$ by `decide` (computation inside the kernel)	0	None	Does not depend on any axioms
`pow_mod_native`	The same statement by `native_decide` (computation with compiled code)	0	None	`pow_mod_native._native.native_deci`
`pow_mod_wrong`	The false statement $2^{10} \bmod 7 = 3$ by `decide`	1 (failure)	Error: showed that the proposition is false	—
`gap_from_bound`	Formula (4) (Mathlib's real numbers)	0	None	The standard three
`gap_from_bound'`	The proof of formula (4) replaced by `sorry`	**0 (success)**	Warning: `declaration uses `sorry``	The standard three + `sorryAx`
`vacuous`	$x = 42$ from contradictory assumptions (§ 05)	0	None	The standard three
`mass_gap_positive`	An empty theorem with only a big name (§ 05)	0	None	The standard three

Seconds are wall-clock time including loading `import Mathlib` (a 4-core machine, nice 19). Rows that load Mathlib take about 5 seconds for loading alone. The values are from one run in `results.json` and vary by about 10% from run to run (the exit codes, warnings and axiom rows are the same on re-running).

There are three things to read. First, a file containing `sorry` also passes with exit code 0. All that appears is a warning. If a checking record only says "no errors", the absence of `sorry` is not guaranteed. Second, `native_decide` creates one dedicated axiom just for that call. The Lean reference manual explains that this is so that what each axiom asserted can be audited individually. It differs by version: up to 4.28.0, every call appeared as depending on a common axiom `Lean.ofReduceBool` (with `Lean.trustCompiler` behind it). From 4.29.0, it changed to a dedicated axiom per call (of the form `…ax_1_1` in Table 2). Axiom names change with the version, so this book judges by "is it a subset of the standard three?". Third, a false statement does not pass with `decide`.

Detour: using real numbers brings out the three axioms

You may skip this. `mass_gap_positive` in Table 2 is "if $0 < m$ then $0 < m$", and its proof just returns the premise. Even so, the standard three appear. For `pos_nat` (natural numbers), of the same form, nothing appears. The only difference is the type, so the three come from the side of the real numbers. `#print axioms` follows not only the proof but also the definitions appearing in the statement (including the type $\mathbb{R}$). Mathlib's type of real numbers is built as a quotient of Cauchy sequences, and that definition itself depends on the three. In fact, even the empty theorem "True for an element m of $\mathbb{R}$" brings out the three. So theorems that talk about real numbers almost always bring out the three. That the three appear is not a sign that the proof is heavy.

For how this book uses this, see 23-13 § 02 (the #print axioms at the end of each bundle as the standard of cleanliness).

05

It passes, but is it what was meant? — the four gaps

Even with no holes and only the standard three axioms, something is still unchecked: **whether the statement says what it means to say**. At the start of its checking procedure, the Lean reference manual writes that it is important to distinguish the question "does the theorem have a correct proof?" from the question "what does the statement of the theorem mean?". The latter cannot be checked by machine.

This site's page 8 of the guide divides this gap into four types (Table 3).

Table 3 The four gaps (the guide's classification, with examples from this book)

Type	What happens	How to find it
① Definition	The definition is weaker or stronger than intended (e.g. if "gap" is defined by ≥ 0, it passes even with 0)	Follow every definition appearing in the statement
② Name	The name of the theorem is bigger than the statement	Read the statement, not the name
③ Vacuous truth	The premises cannot hold at the same time, so anything can be concluded	Build one concrete example that satisfies the premises
④ Assumption	What should be proved is placed in an assumption, and when it fails is not written	Make a list of assumptions and look at the label of each

② and ③ actually occur in the last two rows of Table 2. `mass_gap_positive` carries the name mass gap, but the statement is "a positive number is positive". To Lean a name is just a string and guarantees nothing about the content. This is why, when this book cites a theorem name, it writes the statement and the premises on the same line.

An example of ③ is the following statement.

$$(x < 0) \ \wedge \ (0 < x) \implies x = 42 \ \text{(6)}$$

The premises contradict each other, so the conclusion can be anything, and Lean passes it with a single `linarith` (an automatic step that combines linear inequalities). The more premises a theorem has, the harder this danger is to see. The way to check is to give one example that satisfies all the premises (a case where the answer is known, such as a free field). In Part 23 of this book, the role of an example satisfying all the premises is played by checking against the free field (23-16). Negative controls (23-13 Lean statements and the limits of machine checking) go the other way: they show that removing one assumption breaks the conclusion, confirming that the assumption is not decoration.

④ (assumptions) is the most important in this book. In a theorem that draws a conclusion from an unproved assumption such as the diffusive-relaxation-time assumption, the content is on the side of the assumption, and the theorem does not prove a single step of that assumption (23-13 Lean statements and the limits of machine checking).

06

What the Lean statements in this book talk about

A Lean theorem is a statement about typed objects. So looking at the types of a theorem's arguments tells you what the statement actually talks about. 23-13 Lean statements and the limits of machine checking divides the forty-six theorems that this book cites as conclusions into five layers by the types appearing in their statements. Exactly half, 23, are inequalities or identities of real numbers, and the theorems that have the lattice gauge field as a type are the 3 of 5-11.

Written as a formula, a sentence in this book with the Lean label consists of two parts.

$$\text{a sentence of this book} \ = \ \underbrace{(P_1 \wedge \cdots \wedge P_k \Rightarrow Q)}_{\text{checked by Lean}} \ + \ \underbrace{\text{reading: "}C_P \text{ is the Poincaré constant of}}_{\text{attached by the text; not ch}}$$

Just as C_P was a named real number in the example of § 03, in many theorems of this book C_P, ξ and m are real variables. Reading them as physical quantities is done in the text, and whether that reading is right lies outside Lean. The conventions for the reading (the definition of β, the normalisation of tr, the metric of the lattice) are in 23-13 Lean statements and the limits of machine checking and A-01 Table of notation.

Why is there no theorem about the Wilson measure itself? One reason is stock. Counting words in the source of Mathlib, Lean's standard library of mathematics, in the version on this machine (the version is in Table 7 of § 08) gave Table 4.

Table 4 Number of files, out of the 8,311 source files of Mathlib, that contain the word Computed

Word	Files	Word	Files
`Haar` (Haar measure)	70	`plaquette`	0
`Wilson loop`	0	`lattice gauge`	0
`Yang-Mills` (three spellings)	0	`spectral gap`, `SpectralGap`	0
`Poincare inequality`, `PoincareInequality`	0		

There are foundations for the uniform measure on a group (Haar measure), but the words of lattice gauge theory do not appear. The absence of a word is not a proof of the absence of a definition, but at least as named machinery, whoever uses it must build it from scratch. The guide's "Mathematics that mathlib has no stock of" divides the sources of such difficulty into three (quantity, way of stating, stock). The continuum limit and the construction of four-dimensional field theory lie beyond that (7-10).

Example. This site's paper (5-11) is one of the few examples that have the lattice gauge field as a type. Even there the Lean label is split by step: the chain from the action to a certain lower bound is Lean , and the step from that lower bound to the lattice mass gap is Paper (and only a sketch) (0-01).

Table 4b lists the bundles used in the assembly of the first step of Part 23, and Table 4c the bundles cited by Part 24. The single files outside Tables 4b, 4c and 4f whose declarations are cited by Part 23 and other chapters are in Table 4h. All of them are collected in the bundle `principia-ym-lattice-gauge-2026-11-13` (143 modules, about 30,000 lines; every module imports only Mathlib). The rows are the same as Table 1 of 23-13; what a bundle guarantees is only the algebra, arithmetic and one-variable analysis from the list of assumptions to the conclusion. Every bundle follows the convention of writing the assumptions next to the theorem name (23-13 § 04).

Table 4b The bundles of the first step (Part 23). The number of theorems is the number of `#print axioms` lines bundles were rebuilt on another computer, and every line was confirmed to be a subset of the three standa source is the bundle `principia-ym-lattice-gauge-2026-11-13` (the same bundle as Tables 4c and 4f). On 2026-1 built again from this bundle, one at a time, and the kernel re-check and `#print axioms` for the declarations cit were confirmed Lean

Bundle	Theorems	Algebra of what	Representative theorem (assumptic

Bundle	Theorems	Algebra of what	Representative theorem (assumptio
LatticeGaugeOneScale	12	The relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption, and rereading as mass (the equality version)	hup_is_gap_positivity (0<m, C_P= C_P≤Kξ²)
LatticeGaugeHupBridge	18	The one-sided bridge, the lower half of the power, negative control	hup_is_gap_positivity_of_lower (0<m_lat, κ/m_lat²≤C_P, C_P≤Kξ
LatticeGaugeDoubleWellTestbed	7	The skeleton of the finite sums of the One-Dimensional One-Sided Bridge Proposition	local_rayleigh_ratio_le_one (0<m a_k≥m, c_k≥0, Σc_ka_k>0)
LatticeGaugeTwoScalePoincare	8	Splitting of the variance, the convex core, two states, the Glauber band	split_poincare_le (the splitting formula, two local Poincaré inequalities, the mean difference, negativity)
LatticeGaugeDoubleWellKramers	10	From the sandwich of the ratio to the double-well relaxation-time power statement	power_H_of_kink_ratio (0<K,s,δ, 1≤R≤K/ħ, s/ħ≤ln ξ)

Bundle	Theorems	Algebra of what	Representative theorem (assumptic
LatticeGaugeH1Assembly	17	The algebra of the rates of the Six-Node In-Phase Poincaré Proposition, the arithmetic of the margins at the six nodes	H1_assembly (seven named ones)
LatticeGaugeCoarseSideUniform	20	The arithmetic of the crossing row, both sides of Muckenhoupt	T4_paper_numeric (λ_c≥1.3×10
LatticeGaugeIntervalQfreeWindow	150	The Linear-Fractional Lemma, from the cover to the window, recomputation of the rate-limiting cell	win_qfree_window_le_of_interval (the table of interval-arithmetic o
LatticeGaugeRingUnconditionedWindow	20	The assembly of the window of the chain outside the ring	unc_window_lt_one_of_table (the of interval-arithmetic output)
LatticeGaugeRingInnerChain	22	The ratio at the corner at the wall, the S_W of the ring	sw_inner_le (q_in≤13/10, q_out≤0.998888530)
LatticeGaugePartitionFramework	14	The sector sum, the Gibbs stage, the closing condition	comp_close (X>0, b,A≥0, ℓ>0, Bar componentwise localisation, Bocl
LatticeGaugeRemVProxy	23	The assembly of the Scalar-Strand Proposition	path5_conditional (the remainder proxy ratio, the rarity and the ma named)
LatticeGaugeDetourKills, DetourKills2	17, 16	The arithmetic of the lines that killed the detour plans	Lemmas by type

Bundle	Theorems	Algebra of what	Representative theorem (assumptio
LatticeGaugeBoxReduction	23	The Two-Projection Lemma, even and odd boxes, averaging over shifts, controls	box_shift_average (the two-family inequality for each shift, C_box≥ sum of overlaps≤$\mathcal{E}$)
LatticeGaugeBoxGap	18	The Gap-Inside-the-Box Proposition, c_kg, composition of H3 and H4	H3_H4_compose (product of boxe of the box, D=ħ/S_0, n,ħ,S_0>0, c_alt<
LatticeGaugeCoreMixing, CoreMaxCorr, CoreThreeStage	15, 20, 18	The three stages of the core	three_stage (the upper bound of ϵ stage)
LatticeGaugeCoreActionGap	15	The vanishing of the local term, drawing the line at the ends	cloc_term_vanishes (the core acti barrier assumption, the reading c ΔS_*<2S_0)
LatticeGaugeGlobalLyapunov	21	The algebra of the Lyapunov criterion, and a positive control	gauss1d_bbcg, gauss1d_equality (assumptions)

Table 4c The bundles cited by Part 24. The source is the bundle `principia-ym-lattice-gauge-2026-11-13`. The l paper, EuclideanPath.GapBridge, is in `principia-gap-bridge-2026-10-09.tar.gz` Lean

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeLocalH (LatticeGauge.LocalH)	31	Properties of the supremum defining the local Poincaré constant, the algebra of the charges of subregions, monotonicity of the window factor, the local bridge	local_le_global (bounded above, non-empty, subset), wr_strictAntiOn (no assumptions), gap_lower_of_local_window (κ/(m^2+w^2)≤C_P, w^2≤2m/(cℓ), C_P≤K$ℓ^2$)
LatticeGaugeLocalHReference (LatticeGauge.LocalHReference)	11	The bridge in ratio form, the algebra of the threshold K_*	ratio_bridge, kstar_le_kappa (the assumptions of each are in the classification table of 24-04)

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeLocalHDoubleWell (LatticeGauge.LocalHDoubleWell)	14	The one-site ratio in 0+1, the three regimes	one_slice_ratio (0<ε, 0<m, 0<ω)
LatticeGaugeGaugeReference (LatticeGauge.GaugeReference)	16	The gauge quotient reference, and that the reference vanishes when gauge fixing is removed	kstar_gauge_measured (measured values), kstar_vanishes_without_gauge_fixing
LatticeGaugeWilsonLoopReference (LatticeGauge.WilsonLoopReference)	14	That the family of loops spans the quotient, that colour singlets have no first-order term, the threshold of the second chaos	no_linear_singlet, chaos2_bracket
LatticeGaugeConservationLength (ConsLen)	14	The sandwich of the Gaussian gap of the strip	neumann_gap_lower / upper (1≤M)
LatticeGaugeConservationExponent (ConsExp)	16	The interval arithmetic of the slope of the strip, the relation for ℓ_c, the decomposition of the rate	lc_eq_iff (0<E, 0<λ, 0<ℓ), lamQ_log_split (0<κ, positivity)
LatticeGaugeHierarchicalLink (HierLink)	14	The arithmetic of coordinates and gaps on the hierarchical lattice	lam_le (no assumptions), tree_no_decay (0<β)

Bundle (namespace)	Theorems	Algebra of what	Representative theorem (assumptions)
LatticeGaugeFourDTrial (LatticeGauge.FourDTrial)	8	The finite sums of $\tau_int \le C_P$, the one line in which the upper-bound assumption breaks, and a negative control	not_hup_of_fast_tau ($\tau_n \le C_n$, $c\xi_n^p \le \tau_n$, $p>2$, $\xi_n \to \infty$), z_two_compatible_with_hup (no assumptions)
LatticeGaugeHupBridge (LatticeGauge.HupBridge)	18	The one-sided bridge, the rotor ratio	hup_is_gap_positivity_of_lower ($0<m$, $\kappa/m^2 \le C_P$, $C_P \le K\xi^2$), rotor_ratio_unbounded ($0<r$)
LatticeGaugeCoreMaxCorr (LatticeGauge.CoreMaxCorr)	20	The algebra of the maximal correlation of the core	Not cited by name in the text of Part 24 (the core bundle of 23-11; background to the reduction to the box in 24-03)
AlternatingMaxCorrBloch	33	The Bloch quadratic formula for the alternating maximal correlation	Not cited by name in the text of Part 24 (background to the alternating maximal correlation of the reduction to the box)
The bundle of the paper, EuclideanPath.GapBridge (`principia-gap-bridge-2026-10-09.tar.gz`)	17	The paper's version of the one-sided bridge (optimal over the family local in time, the rotor)	local_rayleigh_le (the assumptions of each are in the classification table of 24-04)

Table 4h Single files outside Tables 4b, 4c and 4f that are named by Part 23 and other chapters (40). The numb of theorem and lemma declarations in each file (excluding private ones), 725 in all. The source is the bundle `2026-11-13` (the same bundle as the files of Tables 4b, 4c and 4f). On 2026-11-13 each file was built one at a ti and `#print axioms` for the declarations cited by the chapters (all the three standard axioms) were

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugePoincareNormalisation (LatticeGauge.PoincareNormalisation)	9	Normalisation of the Dirichlet form and invariance of the exponent	`exponent_invariant_of_slowly_vary`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeAutocorrelationBound (LatticeGauge.AutocorrelationBound)	10	Comparison of the integrated autocorrelation time of a reversible chain with the Poincaré constant	`rayleigh_le_tau`, `tau_int_le`
LatticeGaugeBlockKernel (LatticeGauge)	26	A one-parameter family of one-step block kernels, and the condition for a fixed step to be bounded	`fixed_step_bounded_iff`
LatticeGaugeCoarseGraining (LatticeGauge)	24	The finite part of one coarse-graining step (quaternion identities, upper and lower bounds on one-link quantities)	`log_F0_ge`, `log_F0_le`
LatticeGaugeFiberAverage (LatticeGauge)	37	Counting for one packing step, and that blocking alone does not lower it	`no_blocking`
LatticeGaugeFlowedPoincare (LatticeGauge.FlowedPoincare)	17	Families of flowed functions and the chain rule for the Dirichlet form, the price of one smearing step	`comp_class_eq_of_equivariant_bije`
LatticeGaugeGapVersusCost (GapVersusCost)	13	That the gap bounds the target but does not bound the cost of local updates	`cost_ge`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeGaugeFixing (LatticeGauge.GaugeFixing)	10	The cost of gauge-fixed long-wavelength test functions	`jump_energy_ge`, `prop_at_gap_scal`
LatticeGaugeHardyLadder (LatticeGauge.HardyLadder)	19	A weighted Hardy inequality on a finite chain, and the ladder of scales	`conditional_scale_bound`, `const_defect_scale_unbounded`
LatticeGaugeMarginalAccumulation (LatticeGauge.Marginal)	22	Accumulation of one-step deficits along the renormalisation ladder, and marginal dimension	`dev_eq_of_marginal`, `dev_le_of_contracting`
LatticeGaugeMultiStep (LatticeGauge)	19	Slope observables of multi-step blocking, the functional equation for the correlation length, anomalous exponents	`S4_two_sided`, `kappaDisj_lt_one`, `xi_eq_exp`
LatticeGaugeOverlapExponent (LatticeGauge.Overlap)	20	Zero-momentum overlap, operator dimensions, what the renewal series does not see	`eta_ge_two_of_bounded_susc`, `fisherEta_eq_zero_iff_canonical`, `overlap_le_susc_mul_gap`, `renewal_chi_mul_massSq`, `renewal_residue_pos`
LatticeGaugeProfileFreedom (LatticeGauge.ProfileFreedom)	9	The freedom in the shape of translations in the variational lower bound	`profileQuotient_isGreatest`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeRunning (LatticeGauge)	19	What one blocking step does to the coupling constant (the strong-coupling end and the weak-coupling end)	`discStep_semigroup`, `step_lowers_coupling`
LatticeGaugeSlicePositivity (SlicePositivity)	17	Smearing of time slices, positivity of the transfer matrix, monotonicity of the effective mass	`meff_antitone`
LatticeGaugeSmearedSquare (LatticeGauge.SmearedSquare)	10	The variational quotient of the squared smeared field, and what smearing gains	`sqQuotient_between`
LatticeGaugeSmoothGaugeFixing (LatticeGauge.SmoothGaugeFixing)	11	Orbit averages for smooth gauge fixing, and that they do not act on the hinge	`orbitAvg_continuous`, `orbitAvg_gauge_invariant`
LatticeGaugeSpectralHinge (LatticeGauge.Hinge)	22	Decomposition of the hinge between the Poincaré constant and the square of the correlation length (the Gaussian optimum, the drain exponent)	`c2_ge_of_nonneg_weights`, `drainExp_eq_zero_iff`, `drain_expo`
LatticeGaugeStepSequence (LatticeGauge)	19	The sequence of steps (decomposition of the packing integral and the one-step Hessian lower bound)	`blocking_shortfall`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeSubspacePoincare (LatticeGauge.SubspacePoincare)	16	Poincaré constants on a restricted space of functions	`autocov_sum_le_on_invariant`
LatticeGaugeSusceptibilityRatio (SusceptibilityRatio)	8	What the ratio of the susceptibility to the equal-time correlation can and cannot tell	`gap_ge_of_overlap`, `light_admixture_invisible`
LatticeGaugeTubeDefect	14	The sign of the tube deficit in the model of surfaces with positive weights	`linear_deficit_breaks_inv`
LatticeGaugeTubeEntropy (LatticeGaugeTube)	18	Counting surfaces (the entropy of the tube) and the lower half at one scale	`mass_sq_ge_of_tube`, `linear_regime_breaks_onescale`
LatticeGaugeTwoLink (LatticeGauge.Character)	28	The finite part of the two-link integral	`intact_frac_le`
LatticeGaugeVariationalBound (VariationalBound)	9	That the variational effective mass is an upper bound on the gap	`meff_ge_of_corr`
LatticeGaugeWulffCorrection	13	The tension of a tilted wall and the Wulff subtraction (the deficit of a closed tube)	`casimir_only_no_bound`, `mass_bound_localized`
LatticeGaugeH1TaylorStage (LatticeGauge.H1TaylorStage)	19	Three-point interpolation in the β direction and the cubic remainder, endpoints of the linear fraction on a small piece	`cube_le_exp_split`, `lebesgue_le`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeH1Tilt (LatticeGauge.H1Tilt)	22	Upper bound on the chord in the β direction and the Hölder loss, the number of subintervals	`chord_beta_upper`, `holder_r117_exceeds`, `nodes_reach`
LatticeGaugeRingMirror (LatticeGauge.RingMirror)	20	Algebra of the mirror image at the seam of the ring	`chain_action_reflect`
LatticeGaugeAdjointSandwich (LatticeGauge.AdjointSandwich)	19	Arithmetic that sandwiches the remaining assumption (MC-sup) between the reading from below and path-by-path upper bounds	`mbar_bound_false`
LatticeGaugeDriftDual (LatticeGauge.DriftDual)	20	Algebra of the weak dual measured by the dual of the conditional drift, and arithmetic on the measured values	`tilt_g1`
LatticeGaugeHbarZero (HbarZero)	23	Algebra of the form that fixes the range of ħ by re-choosing the window	`p5_assembly`
LatticeGaugeLocalizedBochner (LatticeGauge.LocalizedBochner)	29	Composition that localises the negative curvature of the Bochner step to the defect set, and the cost of the tilt	`bochner_localized`, `judge_i_insid` `negw_le_localized`, `tilt_cost_le`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeBTDressed (BTDressed)	17	The form of one-dimensional sequences counted in the dressed direction (the margin does not depend on the direction, scale invariance)	`threshold_gap_invariant`
LatticeGaugeBartaRefine (LatticeGauge.BartaRefine)	19	How to split the tilt between the Barta and Bochner defects, the ceiling lemma, negative controls	`ceiling_same_set`, `ceiling_split`
LatticeGaugeEigenGradient (LatticeGauge.EigenGradient)	16	A form that uses only the gradient of the eigenfunction, one step with the Agmon weight, negative controls	`agmon_no_gain_on_negative_curvatu`
LatticeGaugeExtendedChain (LatticeGauge.ExtendedChain)	19	Composition of the LSI constant of the extended chain, and the condition for that constant to be bounded	`S_ext_01_iff`
LatticeGaugePlateauLSI (LatticeGauge.PlateauLSI)	26	Decomposition of the LSI on the plateau at the top (collective coordinate and transverse directions), transverse eigenvalues	`transverse_fails_k17`

File (namespace)	Theorems	Content	Theorems cited by the chapters
LatticeGaugeWallCut (LatticeGauge.WallCut)	21	Composition of the multi-step decomposition cut at the position of the wall	`chain_rule_cells`
LatticeGaugeKinkPosition (LatticeGauge.KinkPosition)	16	Bistability at the negative end of the core, and conditioning on the binary of the bump	`pn_vs_bistab`, `y_side_k9_3em4`

Table 4d The bundle of Parts 25 and 26 (Y7, 11 files). The number of theorems is the number of theorem and lemma declarations in the files (32). Rechecked on another computer on 2026-10-15: all use the three standard axioms, `sorry` 0. `Lean`

File (namespace)	Theorems	Content	Chapters that cite it
InsertionIdentity (Insertion)	5	The identity for the sum of insertion ratios, mean_one, mean_rho_sigma, sector_bias_le	25-05
BoundedInsertion (Bounded)	5	single, single_mul, multi, averaged, averaged_bound	25-05
SecondMoment	2	sqrt_identity, pz_bound	25-05
MultiCell	3	pushforward, expand_identity, bound ("odd under insertion" is an assumption)	25-05
FlipLemma	2	flip_le, cut_formula	25-06
FluxLemma	1	flip_le_one_sub_abs	25-06, 26-05
Containers	2	factorize, container_bound	25-03
FluxDuality	3	split, twisted_le, bias_mem (positivity is assumed for $\theta \leq 1$)	26-04
Doubling	2	iterate, decay	26-02, 27-01
ReflPos	2	refl_pos, refl_cs	26-04, 26-05, 27-02
ColumnChessboard	5	signed_sum, unsigned_sum, prod_expand, binomial_bound, exp_bound	25-02 (paper on lattice fractional instantons)

Only two files have a file name different from the namespace, InsertionIdentity (Insertion) and BoundedInsertion (Bounded); the text cites them in the form "file (namespace.theorem)". The bundle is `principia-ym-insertion-identities-2026-10-15.tar.gz` (package `yminsertion`, lib `Y7`, sha256 `58d60c433cb23cb9cfab1f2299732385f79a362b96a1d49ac109334b4832bdf8`, record of axioms axioms-ym-insertion-identities-2026-10-15.txt).

Table 4e The bundle of slabs and containers (lean-y6, 30 files + Lemma5Core). The number of theorems is the number of lines in the `#print axioms` record (150: of these, 137 are declarations written in this book's study

for Parts 25 and 26, and 13 are declarations added on the verification branch). On 2026-10-15 it passed `lake build` on another computer: all use the three standard axioms, `sorry` 0. The bundle is `principia-ym-slab-containers-2026-10-15.tar.gz` (package `ymslabcontainers`, lib `Y6`, sha256 `4892aca93897af4940828fd14d5fc45e e72bdcfca0914a2dcd20dd2d77229341`, record of axioms axioms-ym-slab-containers-2026-10-15.txt). `Lemma5Core.lean` is outside the lean_lib and is not checked by `lake build` alone (it is checked separately with `lake env lean`; its record of axioms is `axioms-Lemma5Core.txt`). `Lean`

Main files	Content	Chapters that cite it
SlabConfine, SlabAverage	The Slab Confinement Theorem and the slab average	25-04
FrameLemma, CommutingFlips	The frame lemma of the container identity, commuting flips	25-03, 25-04
GapFromDecay, LevyArea, TwistLocality and others	Gap from decay, Lévy area, locality of the twist	25-03–26-05

Table 4f The bundles of Part 15 (ultraviolet control) (72 single files; the number of theorems is the number of lemma declarations in each file, 704 in all; the last twenty-five files were added on 2026-10-17, 2026-10-22 and 2 have been re-verified on this book's side; the four from DominantMatrixContraction to ConeCalculus bel systematic chapter 15-10, the seven after them were added on 2026-10-22, and the last seven on 2026-10-29). R file at a time on the main machine (a different computer) on 2026-10-15: all rc=0, `sorry` 0, `native_decide` standard axioms. The assumptions of each theorem and "what is not included" are in 15-09 Table 3. An inde verification (same environment, one file at a time, the 62-file inventory) also gave rc=0 and the three standard (evening of 2026-10-15). `Lean`

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.ClassResidualTelescoping	6	`telescoping_residual`, `telescoping_residual_multi`, `inv_sq_sum_le`, `uniform_deviation`	15-03
LatticeGauge.HierarchicalShapeFlow	11	`quarticStep_fixed`, `shapeStep_contract`, `flow_coefficient`, `affine_contraction`, …	15-04
LatticeGauge.FluxNetwork	11	`ratio_le_max`, `ratio_ge_of_unit_flux`, `energy_split`, `thomson_lower`, …	15-05 Lemr Step Back Indep Fluct
LatticeGauge.AllFacesBlocking	6	`fine_energy_lower`, `block_energy_lower`, `smooth_ratio`, `equal_split_ratio`	15-05
LatticeGauge.RatioTwoCriterion	9	`ratio_two_iff`, `one_loop_D`, `gamma_family_one_loop_lower`, `fund_adj_one_loop_root`, …	15-06
LatticeGauge.FundAdjointRefit	8	`refit_solution`, `tree_quadratic_split`, `tree_value_at_saddle`, `image_rho`, …	15-06

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.PrefactorFlowFixedPoint	12	`step_k2`, `step_k2_fixed_iff`, `step_k2_contract`, `fixed_point_general`, …	15-06
LatticeGauge.HeatKernelFourPlaquette	8	`conv_assoc`, `convPow_add`, `four_plaquette_of_semigroup`, `four_plaquette_convPow`	15-06
QuaternionReImSplit	9	`re_mul_split`, `commutator_re`, `J_imI`, `two_link_split`	15-07
LatticeGauge.ShapeRemainderUniform	9	`uv_uniform`, `stays_in_box`, `pow_sub_pow_norm_le`, `keff_half_le`, …	15-06 Rema Theo Plaqu Row
LatticeGauge.HierarchicalTubeChain	8	`tube_exit`, `cell_descent`, `chain_compose`, `tube_weight_mono`, …	15-04
LatticeGauge.ShapeResidualGeometric	4	`forced_uniform`, `geom_partial_le`, `residual_sum_uniform`, `harmonic_part_unbounded`	15-07
LatticeGauge.LargeFieldSource	4	`weighted_tail_le_exp_moment`, `union_tail_le_sum`, `coarse_source_commute`, `coarse_source_commute_comp`	15-08
LatticeGauge.TreeGaugeFixing	29	`exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, …	15-08
LatticeGauge.WilsonWindowComparison	4	`one_sub_cos_window`, `wilson_weight_window`, `window_constant_chain`, `window_split_bound`	15-07
LatticeGauge.QuarticRowColumnBound	7	`weighted_quartic`, `row_quartic`, `schur_quartic`, `representative_bound`, …	15-05 Netw (loca
LatticeGauge.BCHCubicResidual	6	`nT_sum_diff`, `shape_split`, `score_generator_split`, `bch3_coefficient`, …	15-07
LatticeGauge.SmallFieldLocSplit	3	`loc_split_unique`, `geom_tail_le`, `taylor_tail_bound`	15-08
LatticeGauge.BackgroundBudget	4	`budget_term_sq`, `budget_unbounded_of_threshold`, `inv_sq_partial_le`, `inv_sq_budget_bounded`	15-07

File (namespace)	Theorems	Main theorems	Chap
LatticeGaugeInterBlockLocAbsorption	7	`loc_split`, `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_split`, …	15-08
LatticeGaugeBlockCouplingCancellation	28	`locM_idem`, `locLin_idem`, `split_unique`, `bil_split`, …	15-08
LatticeGaugeSmallFieldRemainderAssembly	16	`one_step_remainder`, `gamma_ge_one`, `cut_sum_le`, `taylor_step_le`, …	15-08
LatticeGaugeToeplitzTailSplit	10	`split`, `split_unique`, `qform_split`, `cform_split`, …	15-08
LatticeGaugeWeightedTailBound	5	`weighted_geom_le`, `weighted_tail_le`, `weighted_sum_uniform_linear`	15-08
LatticeGaugeTailTypeClassification	6	`flat_weighted_sum`, `flat_weighted_unbounded`, `exp_weighted_le`, `power_sum_uniform`, …	15-08 § 07
LatticeGaugeClassParameterBudget	4	`ghat3_forms`, `ghat3_zero_iff`, `inv_sq_partial_le`, `class_parameter_budget_bounded`	15-07
LatticeGaugeStokesSecondOrder	7	`log_quadratic_part`, `log_quadratic_part_five`, `commSum_eq_zero_of_comm`, `pure_quaternion_commutator`	15-07
LatticeGauge.CubeFaceCoupling	9	`total_sq`, `opp_eliminate`, `general_to_diag_adj`, `diag_adj_unique`, …	15-07
LatticeGaugeStapleSumLowerBound	5	`sum_norm_ge_of_inner_ge`, `card_cos_ge_quadratic`, `first_order_ratio`, `quadratic_margin`, …	15-07
LatticeGauge.QuarticSplit	4	`titu_two`, `quartic_two`, `quartic_triangle`, `class_split`	15-05
LatticeGauge.GhostDeterminantMatching	8	`det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks`, `coeff_tree_eq_coeff_cov`, `coeff_drop_ghost_shift`, …	15-08
LatticeGauge.BondMovingSandwich	6	`bond_moving_finite`, `iterate_le_of_le_of_monotone`, `comparison_fails_at_fixed_point`, `shrink_shape_shift`, …	15-04

File (namespace)	Theorems	Main theorems	Chap
LatticeGauge.InfluenceRecursion	6	`tp2_comp`, `cdf_upper`, `cdf_lower`, `k_upper`, …	The s row c Poinc unifo assur Part 2 text i 15-00
LatticeGauge.CornerCouplingAffineFlow	5	`fixpt_is_fixed`, `iterate_dev`, `abs_iterate_dev`, `ratio_gt_one_below_fixpt`, …	15-07 § 02
LatticeGauge.HypercubeFaceCoupling	5	`sigma_is_relation`, `relation_is_multiple_of_sigma`, `opposite_in_cell_reduces`, `diag_adj_not_enough`, …	15-07
LatticeGauge.SkewCouplingSpectrum	5	`coeffs_of_spectrum`, `skew_zero_iff`, `one_step_skew`, `increment_pow`, …	15-07
LatticeGauge.HypercubeCochainCount	6	`faceCount_four`, `euler_cube`, `hypercube_cochain_ranks`, `edge_face_dof_agree`, …	15-03 § 07
LatticeGaugeNonlocalImageRecursion	12	`affine_iter`, `fixed_point`, `affine_bound_sum`, `affine_bound_uniform`, …	15-08 § 03
LatticeGaugeThreeBodyTailRecursion	8	`weighted_geom_le`, `three_body_sum_le`, `affine_uniform`, `pair_tail_uniform`, …	15-08
CoarseFaceHodgeSplit	7	`range_sup_tags`, `finrank_ker_eq`, `disjoint_range_tags`, `decomp_unique`, …	15-08 § 07
LatticeGauge.QuarticConeNorm	5	`ratio_sandwich`, `ledger`, `transfer`, `cone_b_to_a`, …	15-05
LatticeGauge.BlockCovariance	7	`cone_chain`, `cone_chain_uniform`, `growth_ledger`, `inverse_decay`, …	15-05
LatticeGauge.AveragedBlockFlow	7	`affine_iter`, `affine_dist`, `tail_sum`, `tail_bound`, …	15-05
LatticeGaugeHierarchicalUpperTube	8	`tube_exit_top`, `chain_top`, `steps_numeric_128`, `steps_numeric_64`, …	15-04
LatticeGauge.TensionDomination	6	`iterate_bound_var`, `iterate_bound`, `lower_bound_four`, `area_law_transfer`, …	15-04
LatticeGauge.BlockStepBetaShift	6	`beta_shift_of_inv_sq`, `su2_coeff`, `log_steps`, `shift_bounds`, …	15-01 § 02

File (namespace)	Theorems	Main theorems	Chap
LatticeGaugeTwoBlockRowSumLoc	14	`sRow_eq_sComp_add`, `nRow_eq_nComp`, `row_sum_cancels`, `row_split_unique`, …	15-08
LatticeGauge.MultiStepCone	5	`step_pair`, `contraction`, `cone_quartic`, `block_three`, `cone_transfer`	15-08 boun re-fix
LatticeGauge.MultiStepPerturb	3	`three_product_perturb`, `three_product_perturb_grouped`, `block_three_affine`	15-08 boun re-fix
LatticeGauge.QuarticCone	2	`four_amgm`, `quartic_sum_le`	15-08
LatticeGauge.ColorAverage	10	`colorAverage`, `colorAverage_of_design`, `design_identity`, `isQuarticDesign`, …	15-08 boun re-fix
LatticeGauge.CubicCone	7	`three_amgm`, `cubic_sum_le`, `triple_amgm`, `cubic_cone_pullback`, `block_three_uniform`, …	15-08
SmallSetRowSum	20	`rowSum_total`, `row_cancel`, `total_split`, `piece_loc_zero`, `theta_ledger`, …	15-08
ThetaSquareOneStep	6	`one_step_affine`, `theta_ledger_half`, `sup_monotone_in_mu`, …	15-08
DominantMatrixContraction (DominantMatrix)	21	`liftWeight_step`, `weighted_step`, `weighted_iter`, `weight_change`, `norm_pow_le_const_mul`, `sum_norm_pow_le`, `aggregate_step`, `two_by_two_weight`, `signedEx_sq`, …	15-10
SignLossBound (SignLoss)	16	`stein_le_inv_sq_mul_test`, `stein_form_le`, `schur_four`, `loss_le_of_coherent`, `comp_coherent`, `row_l1_comp_le`, …	15-10
ConstrainedExtension (ConstrainedExtension)	20	`isExtension_iff_mem_coset`, `finrank_freedom`, `exists_extension_with_conditions`, `finrank_freedom_with_conditions`, `eq_of_eqOn_iSup`, `comp_isExtension`, `comp_depends_of_not_le`, …	15-10
ConeCalculus (ConeCalculus)	20	`cone_pullback`, `contracts_comp`, `block_decay`, `lp_schur`, `lp_schur_sharp`, `delta_dilution`, `weighted_rpow_midpoint`, `weighted_rpow_shift`, `midpoint_below_one`, `affine_varying`, `affine_block`, …	15-10

File (namespace)	Theorems	Main theorems	Chap
ColorAverageN (LatticeGauge.ColorAverageN)	13	`colorAverage_of_design`, `design_identity`, `colorAverage`, `oneColour_of_colorAverage`, `design_identity_two`, `cube_moment`, …	15-08 boun re-fix § 07
OneStepWindow (LatticeGauge.OneStepWindow)	20	`holder_dual_ratio`, `exists_lpRatio_ge_row`, `lamMax_mono`, `steinWeighted_ge`, `stein_row_weight_saturation`, `window_ordConnected_of_logConvex`, …	15-10
LargeSetWeight (LatticeGauge.LargeSetWeight)	16	`summable_shell_iff`, `bbs_weight_vs_decay`, `bsv_weight_vs_decay`, `card_shell`, `manhattanShell_three`, …	15-08
LatticeGaugeAcceptanceSandwich (LatticeGauge.AcceptanceSandwich)	15	`half_metro_le_fermi`, `fermi_le_metro`, `barker_le_of_crooks`, `overlap_sandwich`, `mixture_sandwich`, `jarzynski_of_crooks`, …	25-06 § 03
LatticeGaugeAbelianTwistAction (LatticeGauge.AbelianTwistAction)	13	`abelianAction_lt_pi_sq`, `totalAction_eq`, `plaqTerm_mixed`, …	27-02 small
LatticeGaugeNestedBlocks (LatticeGauge.NestedBlocks)	7	`nested_supermultiplicative`, `doubling_iterate`, `doubling_bound`, …	26-05 § 03
CenterFluxPolymerCertificate (PolymerCert)	6	`tree_k`, `kp_bound`, `kp_lt_one`, `rate_rational`, `exp_le_inv_one_sub`, `final_real`	26-06
ClassFixedPoint (LatticeGauge.ClassFixedPoint)	24	`exists_unique_fixed`, `exists_unique_fixed_in_ball`, `norm_iterate_sub_le`, `exists_unique_fixed_of_iterate`, `dist_fixed_fixed_le`, `tracking`, `exists_unique_entire_orbit`, `entire_tracking`, …	15-03
LocalCancellationOneRow (LatticeGauge.LocalCancellationOneRow)	8	`pairs_sum_to_total`, `loc_remainder0_eq_zero`, `pairAB_rank_one`, `pairAB_local_plus_gradient`, `mirror`, …	15-08
RankOneRemainder (LatticeGauge.RankOneRemainder)	12	`pair_split`, `edge_sum_split`, `chain_remainder_eq`, `chain_remainder_zero_of_match`, `deflation_split`, …	15-08
BlockDeflation (LatticeGauge.BlockDeflation)	9	`blockDeflate_split`, `blockDeflate_left_zero`, `blockDeflate_right_zero`, `twoStep_zero`, `twoStep_split`, …	15-08

File (namespace)	Theorems	Main theorems	Chap
ConeCalculusSOS (LatticeGauge.ConeCalculusSOS)	27	`cert_sound`, `certValue_anti`, `floor_of_coeff`, `classSum_pushforward`, `classSum_pullback`, `shift_rigid`, `tensor_ratio_product`, `schur_weighted_bound`, …	15-10
LatticeGaugeH1HbarInterval (LatticeGauge.H1HbarInterval)	15	`cell_lower`, `cell_upper`, `rho_lower`, `pieces_min`, `margin_on_interval`, `margin_I1`, `margin_I2`, …	23-15 unifo doub
StrongCouplingOneSidedTerm (StrongCouplingOneSided)	9	`area_le_strip`, `strip_coeff_lower`, `tube_coeff_neg`, `h_main_term`, `h_neg`, `h_neg_on_interval`, …	27-02 langu coup

The labels in the text of Part 15 stay the five grades (paper proofs are [Proved here]); Lean is attached in the text of each chapter as the theorem name and its assumptions on the same line (15-09 §01). The source is the bundle `principia-ym-lattice-gauge-2026-11-13` (the tar.gz under `principia/lean/`, and the axiom log `axioms-ym-lattice-gauge-2026-11-13.txt`). Of these, 23 files are also in the bundles published in October 2026 (flux-network-blocking, hierarchical-chain, one-row-renormalisation, coarse-chain-poincare, h1b-margin-interval, boundary-mixed-derivative), as the same files.

07

How the labels are decided — one label per sentence

This book attaches labels sentence by sentence, not chapter by chapter or paper by paper, because even within one paper the label differs from step to step (the example in § 06). Which label is attached is decided by the kind of guarantee the sentence has. Figure 2 puts that decision in the form of questions asked from the top down.

There is a reason for the order of the questions. The Lean label guarantees the arrow from premises to conclusion; the Known label refers to a proof in the literature, the Paper label to a proof on this book's side, the Computed label to a check over a finite range, and the Physics label to acceptance at the level of physics. The strong guarantees are asked first, and the first label that applies is attached. A sentence that matches none gets no label and is written as an "assumption" or a "question" (the questions of 23-17 carry no label).

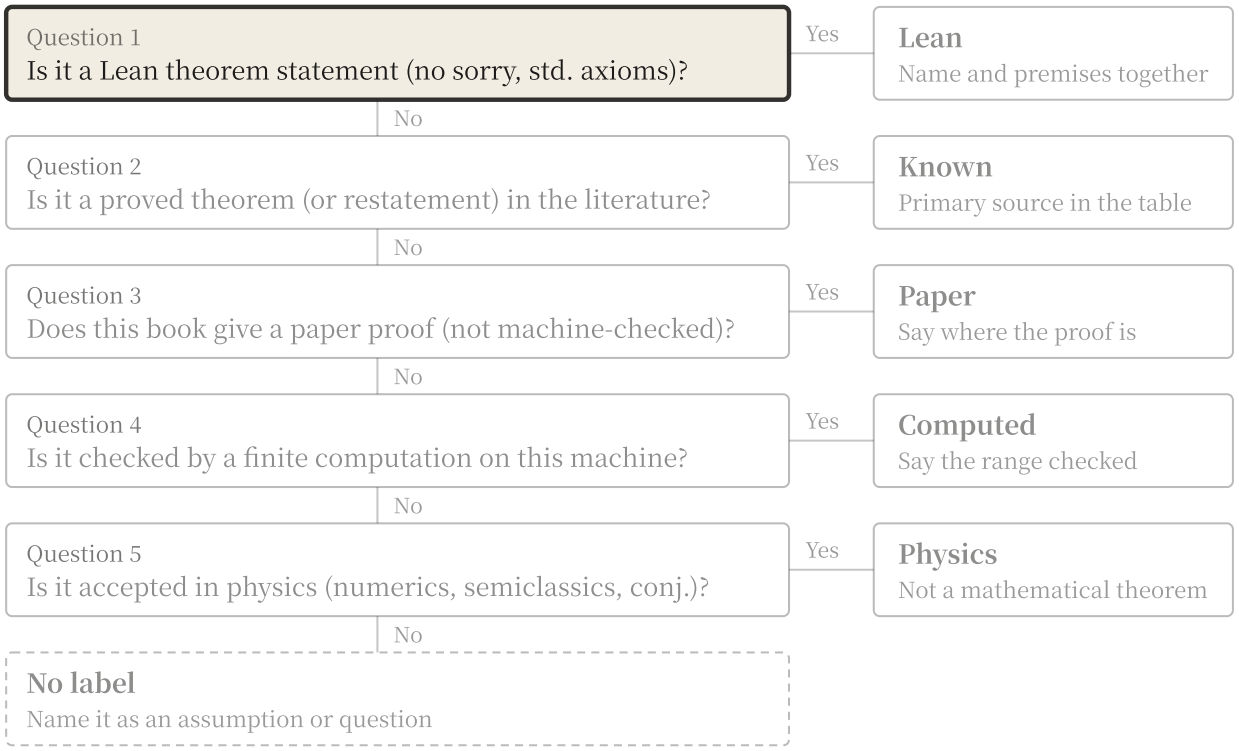
When the sentences form a chain, read it as follows.

$$\text{for the chain } A_0 \Rightarrow A_1 \Rightarrow \cdots \Rightarrow A_n\text{, the certainty of the conclusion } = \text{ the label of the weal}$$

If just one step is Paper , then even if all the other steps are Lean , the conclusion is at the level of Paper. If the starting point A_0 is an assumption, the conclusion is also conditional on it. The premises P_i of formula (1) correspond to this starting point.

Figure 2 Answering the questions from the top with "Yes / No" shows the path taken and the label attached. The

example buttons pick one sentence of this book and show its path



`Question 1: Is it a Lean theorem statement (no sorry, std. axioms)?`

This figure puts the conventions for labels in 0-01 in the form of questions. It contains no numbers.

Table 5 Examples of sentences in this book, and the path they take in Figure 2

Sentence	Path taken (answers to questions 1–5)	Label	Chapter
Formula (4): $m > 0$, $C_P = 1/m^2$, $C_P \le K\xi^2 \Rightarrow 1 \le K(m\xi)^2$ (the same statement as `hup_is_gap_positivity` in 23-13)	Yes	Lean (with premises stated alongside)	§ 03, 23-13
The one-loop coefficient b_0 (including $11N/3$)	No → Yes	Known	4-06
The step from the one-link lower bound to the lattice mass gap	No → No → Yes	Paper (sketch)	5-11
The behaviour of C_P/L^2 in small boxes ($L = 3, 4, 5$) for the two-dimensional Ising model	No → No → No → Yes	Computed	2-10
The ratio of the glueball mass to the string tension from lattice numerical simulations	No ×4 → Yes	Physics	5-08

Sentence	Path taken (answers to questions 1–5)	Label	Chapter
Relaxation-time assumption (at the level of powers, $\ln C_P / \ln \xi \to 2$; the band statement is named the band form of the diffusive-relaxation-time assumption, and its upper half is the upper-bound assumption: $C_P \le K\xi^2$)	No ×5	No label (assumption)	23-01

How to read it. The Lean premise $CP \le K\xi^2$ in the first row of the table is the upper half of the band statement of the assumption in the last row (the band form of the relaxation-time assumption), that is, the upper-bound assumption (A-06 Index of names). In the first row it is a premise; in the last row, the claim of an assumption. The labels differ because the roles in the sentence differ.

08

A procedure to check for yourself

The Lean label presupposes that readers can check again on their own machines. The Lean reference manual describes checking as a staircase of four stages (Table 6). The lower stages are easier; the higher stages widen the range of doubt.

Table 6 The four stages of checking (from the structure of "Validating a Lean Proof" in the Lean reference manual)

Stage	What to do	What it guards against
1	See that the editor's check mark appears, or that `lake build` finishes with no warnings or errors	Unfinished proofs, `sorry` inside that theorem, honest mistakes in tactics
2	Write `#print axioms theorem-name` after the theorem, and confirm formula (5)	`sorry` inside theorems it depends on, extra axioms, `native_decide`
3	Use `lean4checker` to pass the proofs saved by the build through the kernel again	Bugs in the core of Lean that handles the kernel's state, metaprograms or tactics that deliberately bypass checking
4	Write only the statements in a trusted place, assemble the proofs in isolation with `lake comparator`, pass them again through both Lean's kernel and an external checker written independently of it, and also confirm that the statements match	Malicious proofs, implementation bugs present in only some of the checkers used

The manual itself positions the isolated-comparison step for high-risk settings, such as markets for proofs, competitions with large prizes, and AI whose intentions are not aligned. About the build-check step, the manual warns that the check mark appears even if a dependency contains `sorry`. This is the same situation as `gap_from_bound'` in Table 2 having exit code 0, and it is why the axiom-check step (a checking step of the Lean reference manual) is needed. The Lean labels in this book are attached to theorems that have been through at least the axiom-check step.

The concrete procedure has the following form. How to install and match versions is in "Installing" in the guide and "Checking a distributed proof locally".

```
lake build                      # Stage 1: check the whole project
lake env lean File.lean         # check a single file only
#print axioms gap_from_bound    # Stage 2: write this inside the file
```

This site's Lean record (the list of theorems and the record of checking) is at the Lean record. Even the same proof may stop passing when the versions of Lean and Mathlib differ, because lemma names change. For this reason this book records the versions used for checking.

Table 7 The environment in which the examples of this appendix were checked Computed

Item	Version
Lean	4.33.1 (commit `819816b2`)
Mathlib	rev `0df444a3`
Machine	x86_64, 4 cores, Linux. Checking was run at low priority (nice 19)

Finally, we gather into seven the things to ask when you receive a sentence with the Lean label. This is the checklist of "The procedure for reading a statement" in the guide, adapted to the way this book is read.

1. Is the theorem name given, so that you can check it again yourself (build-check step)?
2. Is there no `sorry`, and do the axioms fit within the three of formula (5) (axiom-check step)?
3. Have you followed every definition that appears in the statement (① of § 05)?
4. Did you read the statement rather than the name (② of § 05)?
5. Is one example satisfying all the premises given (③ of § 05)?
6. Are the list of premises and the label of each premise written (④ of § 05, formula (8))?
7. Where in the text is the reading of the real variables as physical quantities declared (§ 06, formula (7))?

The seventh is the question specific to this book. Many of the Lean theorems in this book connect to the mass gap problem at this point, and that connection lies outside the Lean label.

```
09
```

> **Declarations of Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 — list, and the checking procedure**

The Lean declarations cited by Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 (Parts 9, 10, 12 and 14) are listed chapter by chapter. The detailed form of the scope and of what is not claimed is in the scope chapter of each part (20-07, 21-08, 15-14), in "What this chapter can say" of 14-02, and in the correspondence table of definitions (19-06).

The checking procedure. Every declaration has been checked again in the core (kernel) of Lean 4 (kernel recheck), confirming that there is no `sorry` and that the axioms used are only the three standard ones. In addition, with eyes other than those that wrote the proofs, we have compared whether the Lean definitions and statements point to the same things as the definitions and statements of the text, and whether the assumption fields can be satisfied by the actual objects (cross-checking by a reader other than the writer). What machine checking guarantees goes only as far as the written statement following from the written assumptions (the four gaps of § 05).

The bundle. The declarations of Table 4g, and all the files on which they depend through `import`, are collected in the bundle `principia-ym-three-volumes-2026-11-13`, a single lake package. It uses Lean 4 v4.33.1 and Mathlib v4.33.1, has 400 modules and about 115,000 lines, and the declarations cited by the chapters number 677 (of which 455 are theorems). A lexical search finds 0 occurrences of `sorry`, `admit`, `native_decide` and custom `axiom`. How to check it (`lake build`, and `#print axioms` for each declaration), and the pairs of modules that cannot be `import`ed together because they have declarations of the same name, are written in the README of the bundle. The earlier bundle of the same day, `principia-ym-premise-first-2026-11-13` (317 modules), is entirely contained in this bundle (the 317 module files are the same).

Table 4g Lean declarations cited by Parts 19–22 and 28, the second half of Part 15, and the sections "What this book's study could see" of Volume 2 (Parts 9, 10, 12 and 14) (bundle `principia-ym-three-volumes-2026-11-13`, 400 modules)

Chapter	Theorem or definition	Scope	Lean declaration
19-06	Strong-coupling side ↔ vortex side (measure): $\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure N β`) equals the measure on the vortex side, a Haar product multiplied by a weight, transported by a one-to-one correspondence given by definition. No constant factor	Only for $M = N$ and no twist. There is no bridge for the twisted measure (background $n_{12} = 1$)	`WilsonBridge.wilsonMeasure_e`
19-06	Same (expectations, total mass): the expectations of measurable observables and the total mass $Z_N(\beta)$ agree on the two sides	As above	`WilsonBridge.integral_wilson` `WilsonBridge.wilsonMeasure_u`
19-06	Same (correspondence with aligned time): the measures also agree under the correspondence that puts the time of the strong-coupling side (index 0) onto index 3 of the vortex side, and the reflections in planes through lattice sites on the two sides become the same map	As above	`WilsonBridgeT.wilsonMeasure_` `WilsonBridgeT.refl_toBallT`
19-06	Same (vortex ratio): the vortex ratio with no background twist ($n_{12} = 0$) is written as an expectation of the probability measure on the strong-coupling side. It is not the book's θ_M ($n_{12} = 1$)	As above	`WilsonBridgeV.vortex_ratio_e`
19-06	Strong-coupling side ↔ fermion side: $\tilde{\mu}_{2h,\beta}$ is $e^{6(2h)^4\beta}$ times the pure-gauge measure `WilsonGauge.gaugeMeasure` transported by a relabelling of indices	Spatial side $N = 2h$, $2h > 1$	`WilsonGauge.wilsonMeasure_ec`

Chapter	Theorem or definition	Scope	Lean declaration
20-01	Reflection positivity in planes through lattice sites	$(\mathbb{Z}/2M)^4$, all β, bounded measurable f depending only on the closed positive half	`TorusRP.wilson_rp`
20-01	Reflection positivity in mid-planes	$(\mathbb{Z}/2M)^4$, $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F	`MidRP.wilson_rp_mid`
20-01	Reflection positivity in planes through lattice sites, in four directions	$(\mathbb{Z}/2M)^4$, all β, for each direction ρ	`PeierlsPerm.wilson_rp_dir`
20-01	Symmetry of the reflection pairing	$(\mathbb{Z}/2M)^4$, all β, measurable f, g	`TorusRP.wilson_refl_symm`
20-02	Decay of correlations at strong coupling in finite volume	Narrow window, for each finite N, no N in the constants, distance in time	`TorusRP.wilson_strong_coupli`
20-02	Decay of covariances in the Dobrushin distance	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, finite N, no N in the constants	`LinkSpec.wilson_cov_dist`
20-03	Theorem on the limit without subsequences	$\beta \not= 0$, $q(\beta) < 1$	`InfVol.wilson_infvol_unique`
20-03	Theorem on the limit independent of the size	$\beta \not= 0$, $q(\beta) < 1$, all sizes $m + 1$, any $s(k) \to \infty$	`AllN.wilson_infvol_allN`
20-03	Translation invariance of the limit	$\beta \not= 0$, $q(\beta) < 1$, unit translations in four directions	`TransI.wilson_infvol_transla`
20-03	Clustering of limit points	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, at every limit point, four directions	`WilsonLimit.cluster_limit`
20-04	Mass gap with non-degeneracy (Dobrushin window)	$\beta \not= 0$, $q(\beta) < 1$, the statement chooses one limit point (the same estimate at every limit point: `WilsonLimit.wilson_massGap_dob`), $\|T - P_\Omega\| \leq q(\beta)^{1/2}$, $\mathrm{Var}(F_0) \geq \frac{1}{4} e^{-12\|\beta\|}$	`WilsonLimit.wilson_massGap_d`
20-04	Mass gap with non-degeneracy (all limit points)	$\beta \not= 0$, narrow window, all limit points, rate $-\log 24(1 - e^{-4\|\beta\|})$ (not included in the combined statement)	`WilsonLimit.wilson_massGap_r`
20-04	Mass gap in the gauge-invariant sector	$\beta \not= 0$, $q(\beta) < 1$, the statement chooses one limit point, witness $F_p = \mathrm{Re}\, U_p$	`GaugeZ.wilson_massGap_gauge_`
20-04	$T \geq 0$ (on the whole of W)	$\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$, all limit points	`FullT.wilson_Tpos_full`
20-04	The image of T lies in the gauge-invariant sector	$\lambda > 0$, $e^{\lambda} q(\beta) < 1$, all limit points	`Bind.T_range_gauge_all`

Chapter	Theorem or definition	Scope	Lean declaration
20-04	The combined statement	$0 < \beta$, $18 \tanh \beta < 1$. Convergence, translation, reflection positivity, gap and non-degeneracy in the two sectors, $T[F_0] = 0$, $T \geq 0$, clustering	`InfMain2.wilson_infvol_main2`
20-05	Reflection positivity of the Wilson–fermion weight	$m > -1$, all β and j, time planes through lattice sites	`WilsonGauge.site_reflection_`
20-05	The partition function is positive	All β, m, j	`WilsonGauge.partition_pos_wi`
20-05	Global gauge invariance of the determinant	Unconditional, constant g	`GaugeDet.physDirac_det_conj`
20-05	Invariance under the generators of the lattice symmetries	All N, j, m, β: global gauge, translations in four directions, spatial 90° rotations, reflections of spatial axes, time reflection	`CubicSym.wfProb_lattice_gene`
20-05	Invariance under the hypercubic generators	Isotropic lattice $N = 2h$; the above with 90° (space, time) rotations added	`CubicSym.wfProb_hypercubic_g`
20-05	Invariance of the determinant under time reflection	Unconditional (all N, h, m, ω), reversal $t \mapsto 2h - 1 - t$	`CubicSym.physDirac_det_trefl`
21-01	The three results of the sign systems (covering inequality, monotonicity for $\mathbb{Z}_2$, comparison with $\mathbb{Z}$ 2)	Abstract groups and sign characters. Implications with assumption fields	`CoveringInequality.covering_` `dominated_covering_ineq`, `in`
21-02	Thin-vortex floor (ratio form)	All real β, $N, M \geq 1$, any background twist: `twisted_vortex_bound` is the signed form $(Z_0 - Z_1)/(Z_0 + Z_1) \leq t^{M^2}$, `_abs` is the absolute-value form $\lvert Z_0 - Z_1\rvert/(Z_0 + Z_1) \leq t^{M^2}$	`TwistedWilson.twisted_vortex` `twisted_vortex_bound_abs`
21-02	Thin-vortex floor (exponential form)	$\beta \neq 0$: $\sigma_0 = -\ln t > 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$	`TwistedWilson.thin_vortex_fl`
21-04	Block estimate	$mk \leq M$, all real β: $\lvert\theta_M\rvert \leq \tanh(K_{\text{eff}})^{k^2}$	`TwistBlocks.block_bound_abs`
21-03	Thick-vortex floor	All $N \geq 1$, any $\varepsilon > 0$, for each sufficiently large β a sufficiently large M: $\lvert\theta_{M+1}\rvert \leq \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	`TwistBlocks.thick_vortex_flo`

Chapter	Theorem or definition	Scope	Lean declaration
21-05	Positivity of the minimal vortex action	All $N \geq 1$: $\delta \leq S_{\mathrm{v}}(N, m)$ with $\delta > 0$ uniform in m, and $S_N > 0$	`VortexPos.Svortex_ge_unif`,
21-05	Certificates of upper bounds Computer-assisted	$S_{\mathrm{v}}(2,8) \leq 6.67401$, $S_{\mathrm{v}}(3,8) \leq 9.09321$, $S_{\mathrm{v}}(4,4) \leq 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	`VortexCert.Svortex_two_eight` `Svortex_three_eight_le`, `Svo` `SN_two_lt`, `SN_three_lt`, `SN_` `VortexCert.Svortex_le_of_cer`
21-05	Thick-vortex floor with explicit exponents	$N = 2,3,4$, for each sufficiently large β a sufficiently large M: exponents 6.675, 9.094, 12.021	`TwistBlocks.thick_vortex_flo`
21-06	Sign of the vortex ratio	M even, all real β: $Z_1 \leq Z_0$, $\theta_M \geq 0$	`HalfSplit.Z1_le_Z0`, `TwistBl`
21-06	Signed thick-vortex floor	Within the range of the thick-vortex floor, with $M+1$ even: $0 \leq \theta_{M+1} \leq \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	`TwistBlocks.thick_vortex_flo`
21-06	Estimate of Polyakov loop pairs by θ	M even, all real β, all b, b': $\lvert\langle P_b(0) P_{b'}(M/2)\rangle_0\rvert \leq 2\sqrt{\theta}/(1+\theta)$	`HalfSplit.polyakov_pair_thet`
21-06	Polyakov loop pairs at general time separation	$M \equiv 0 \bmod 4$, all real β, $0 \leq T \leq M$: $\lvert\langle P_b(0) P_{b'}(T)\rangle_0\rvert \leq 2t^{M \min(T, M-T)}$	`PolyakovT.polyakov_pair_all`
21-06	Same (other parities)	M even ≥ 4 with $\beta \geq 0$ / N, M even $(M \geq 4)$ with all β	`PolyakovT.polyakov_pair_ever`
21-06	Pairs at distance $M/2$ and the thin-vortex floor	M even (including $M = 2$), all real β: $\lvert\langle P_b(0) P_{b'}(M/2)\rangle_0\rvert \leq 2t^{M \cdot M/2}$	`PolyakovT.half_pair_tanh`
21-06	Polyakov loop pairs combined with the thick-vortex floor	For each sufficiently large β a sufficiently large M, $M+1$ even: pair at distance $(M+1)/2 \leq 2\exp(-\frac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	`HalfSplit.polyakov_pair_floo`
15-12	Upper side	All steps k, all $q \in \mathbb{R}^4$ with $\omega(q) > 0$	`OSUpQpos.os_up_Qpos`, `OSUpQp`
15-12	Lower side	As above (a family closed under a different name)	`HsSplit.lower_side_c`
15-12	Bracketing of the propagator	As above; constants uniform in k and q	`LatticeRec.lattice_bracket`, `OneStepLower.bracket_Qpos_fr` `NearAliasB.bracket_Qpos_spli`
15-12	One-step contraction	All q	`OneStepBridge.Mfam_P_le_row`
15-12	Contraction uniform in the step	All steps k, all q	`TorusLam.stage_tau_17`
15-12	Convergence of Γ_k	All steps; constants uniform in the step and the momentum	`AliasPoinc.coupling_converge`

Chapter	Theorem or definition	Scope	Lean declaration
15-13	Locality, gauge covariance	$r0 > 0$, $\lvert g\rvert = 1$	`BoxWalk.boxMap_local`, `BoxWa`
15-13	Linearisation	$0 < r0 \le 1$, $\varepsilon M \le 1/32$	`BlockLinearKernel.blockMap_e`
15-13	Covariance of the pushforward	Any measure (meaningful when second moments exist), one colour	`PushCov.lattice_pushCov`
15-13	Covariance of plaquettes	All steps m, $N \ge 2$. The field obtained by pushing the free field forward with the linearised blocking	`LatticeFace.lattice_face_T1h`
22-01	Chessboard Estimate	$N = 2m$ ($m \ge 1$), all real β, functions on a unit cube	`FoldWilson.wilson_chessboard`
22-01	Rarity uniform in the volume	$\beta \ge 81$, $m \ge 2$, any set X of plaquettes. $q1$ does not depend on N or X	`FoldCap.rarity_explicit`
22-01	Rarity at rate $1/4$	$0 < y \le 2$, at or above some $\beta0$, $m \ge 1$	`FoldQuarter.rarity_uniform4`
28-01	Subsequential limits, and the properties that pass to the limit (implications with assumption fields)	The assumption fields are as in the table of 28-01	`WeakUltravioletExistence.exi` `limit_consistent`, `BlockConc` `ChainBlockMap.chain_end_to_e` `continuous_blockMap`; `BoxWal` `chain_end_to_end_box_wilson2`
28-02	Convergence of skew-product orbits with matched endpoints (implications with assumption fields)	The one-step map is an abstract skew product and is not defined from the lattice (28-02 § 08)	The 13 declarations of `SkewOr` `Admissible`, `two_point_bound` `orbit_compare`, `coarse_step`, `coarse_tendsto`, `coarse_rate` `observable_tendsto` and the r `SkewOrbitProbe.theta_exists_` `one_step_after` and others; `WeakUltravioletExistence.ter` `weak_uv_observable_full`
28-03	From decay to the mass gap; reduction of the band (implications with assumption fields)	The assumption fields are as in the table of 28-03. Inside the strong-coupling window all fields can be discharged	`OSGap.massGap_of_measures_su` `OSGap.Datum.hasMassGap_of_de` `BandReduction.band_reductior`

Chapter	Theorem or definition	Scope	Lean declaration
20-06	Area law at strong coupling (finite volume): the expectation of the Wilson loop on the boundary of a rectangle in directions 1 and 2 with a corner at the origin is exponentially small in the area of the rectangle (the formula is (3) of 20-06)	Torus of equal size in the four directions, β at least 0, inside the strong-coupling window (the 4096 window), both sides of the rectangle at least 1 and smaller than the side of the lattice. The rate and the prefactor depend neither on the size of the lattice nor on the size of the rectangle. The form whose exponent is exactly the area, when twice a side is at most the side of the lattice, is area_law_wilson_area	`AreaLaw.area_law_wilson`, `ar`
20-06	Same (rectangles with any corner and orientation)	As above. From the invariance of the Wilson measure under translations and permutations of directions	`AreaLaw.area_law_wilson_gene` `wilson_cfgShift_invariant`,
20-06	Same (version with tightened constants, and version with the exponent taken over the larger of the two directions)	β at least 0, inside the tightened window (the 441 window). Only the constants of the prefactor and the rate differ. Nothing is said outside the window, about general loops, or about lower bounds on Wilson loops	`AreaLaw.area_law_wilson_shar` `area_law_wilson_general_shar` `area_law_wilson_max_sharp`
20-06	Area law at strong coupling (infinite volume): in the limiting measure, the expectation of the Wilson loop on the boundary of a rectangle with a corner at the origin is exponentially small in the area	β positive, the intersection of the window of the limit without subsequences of 20-03 with the 4096 window, all side lengths. Also at every limit point (β at least 0, the 4096 window). There is no infinite-volume version with tightened constants or with any corner and orientation	`AreaLaw.area_law_infvol`, `ar` `area_law_limit_sub`, `zRectWo`
20-06	Parts of the proof of the area law: positive mixtures, the domination lemma, terms that vanish by sign flips and mod 2 duality, connectedness, lower bound on the area and counting of minimal sets of plaquettes, definition of the rectangular path	As in the statement of each declaration (20-06 § 05– § 08). The family of spanning sets of plaquettes is placed concretely inside the definition and is not an assumption	`AreaLaw.rectWord`, `rectWordG` `rectWord_one_one`, `abs_wordE` `exp_eq_mix`, `mixVar_mem`, `pro` `integral_mul_prod_one_add`, `sum_superset_le`, `q_eq`, `flip` `hol_eq_wordHol`, `integral_word_mul_prod_eq_ze` `not_mem_span_iff_exists_sep` `rect_closed_cases`, `minSpan_` `card_ge_of_spans_rect`, `word` `card_rootP_le`, `card_nbQ_le`

Chapter	Theorem or definition	Scope	Lean declaration
20-06	Upper bound on the vortex free energy (isotropic torus): the difference between the partition function with one central twist inserted and the ordinary partition function is exponentially small in the area of the twisted plane	Torus of equal size in the four directions, β at least 0, the 4096 window (the 441 window for the tightened version). The rate per unit area does not depend on the size of the lattice. A prefactor (the transverse area) remains. Nothing is said about the sign of the difference or outside the window	`VortexStrong.vortex_floor`, `vortex_floor_rel`, `vortex_fl`
20-06	Same (torus with transverse size N and longitudinal size M)	β at least 0, the 441 window (the 9216 window for the untightened version vortex_floor_MN), all backgrounds. The difference is exponentially small in the area of the longitudinal plane, and the rate per unit area depends neither on the transverse nor on the longitudinal size. A prefactor (the transverse area) remains	`VortexStrongMN.vortex_floor_`
20-06	Parts of the proof of the vortex upper bound: good sets and bad sets, independence of where the twist is placed, lower bound on the size of bad sets and their connectedness	As in the statement of each declaration (20-06 § 07, § 08)	`VortexStrong.Good`, `ZM_gauge` `good_twist_iff`, `card_bad_ge` `VortexStrongMN.card_bad_ge`,
21-07	Continuity of the vortex ratio, and its value at coupling 0: the twisted partition function and the vortex ratio are continuous in the coupling; at coupling 0 the partition function is 1 and the vortex ratio is 0	Transverse N and longitudinal M fixed at 1 or more, all real couplings	`ThetaTuning.ZW_continuous`, `thetaM_zero`
21-07	Weak form of tuning: if the target value is at least 0 and at most the vortex ratio at some coupling, then between 0 and that coupling there is a solution, and a largest solution among them	With the size of the lattice fixed. The largest is the largest within that interval	`ThetaTuning.tuning_exists`,

Chapter	Theorem or definition	Scope	Lean declaration
21-07	A box with one twist on each of two orthogonal planes has no flat configuration (from the discrete Stokes theorem and the algebra of quaternions)	All N, M (at least 1, including degenerate lattices), no assumptions. In a background with only one of the twists, flat configurations exist	`TwistFlat.no_flat_twist11`, `no_twisted_quadruple`, `Theta`
21-07	The minimal defect is positive and is attained at some configuration	All N, M (at least 1), no assumptions. No explicit value is stated	`TwistFlat.defect11`, `defect1` `ThetaLimit.Smin`
21-07	Uniform lower bound on the defect: in a symmetric box, the defect is at least 1/2 independently of the transverse size	All N, M (at least 1) and all configurations, no assumptions. In a background with only one twist there is no positive lower bound. The true value of the minimal defect is not stated	`TwistFloor.defect11_ge_two`, `defect11_uniform_half`, `Smin` `QuadFloor.quad_floor_two`, `FlatStokes.approx_torus_comm`
21-07	With the lattice fixed, the vortex ratio approaches 1 as the coupling grows. Its values lie above −1 and at most 1	With the size of the lattice fixed. The constant of the lower bound depends on the size of the lattice and has no explicit value. Nothing is said about the speed being uniform in the size of the lattice	`ThetaLimit.tendsto_theta_one` `Z10_ge`, `volV_pos`, `ratio_ter` `theta_gt_neg_one`, `theta_le_`
21-07	Slope of the flux free energy: the log ratio of the two twisted partition functions, divided by the coupling, approaches the minimal defect	The limit of large coupling with the size of the lattice fixed. In a symmetric box the limit is at least 1/2. No uniform estimate at finite coupling is stated	`ThetaLimit.tendsto_logratio_` `TwistFloor.logratio_limit_ha`
21-07	Tuning the vortex ratio: for every target value at least 0 and less than 1 there is a coupling that makes the vortex ratio equal to it, and the set of solutions has a largest element	With the size of the lattice fixed. If the target is positive, the largest solution is positive too. Monotonicity, uniqueness of solutions, and sequences moving the transverse size and the coupling together are not stated	`ThetaLimit.tuning_all`, `tuni` `exists_theta_gt`, `sanity_hal` `envelope_max_sol`
14-02	Block map of the side-3, centred design: a coarse link is made as the uniform average over a family of paths through the centre of the block (definition)	1944 path indices, weights summing to 1. Not the design of the main line of this book's study (side 2)	`Block3.paths`, `wts`

Chapter	Theorem or definition	Scope	Lean declaration
14-02	Same: the map commutes with reflections in planes through lattice sites without a gauge transformation, and flipping the signs of the links crossing a section flips only one coarse link	One step and all steps, with values in the sphere	`Block3.blockMap_refl_cfg`, `d` `down_flipSet`, `down_local`
14-02	Same: reflection positivity, in planes through lattice sites, of the pushed-forward Wilson measure	The SU(2) Wilson measure (normalised or not) pushed forward by a composite of several steps of the side-3 map, all real β, bounded measurable functions depending only on coarse links strictly between the two planes through the coarse lattice sites at the ends (gauge invariance not imposed). The version with the plane through the middle of the links, infinite volume and the mass gap are not stated	`Block3.rp_push`, `rp_push_pro`
14-02	Same: the map commutes with coarse translations and with reflections in boundary planes	One step and all steps. Within the lattice symmetries	`Block3.blockMap_trans`, `down`
14-02	Same: existence of a tuned sequence. The expectation of the coarse plaquette average of the pushed-forward measure is 0 at coupling 0 and approaches 1 as the coupling grows, and at each step there is a non-negative coupling tuned to a target value in between	On the side-3 design, retract threshold at most 1, side of the coarse lattice at the tuned step at least 2, target value at least 0 and less than 1 (the assumptions are only the values of the design and arithmetic). Monotonicity, divergence and uniqueness of the coupling are not stated	`Block3.Y3_zero`, `tendsto_Y3_`
10-01	In the book's current design (side 2), the expectation of the coarse plaquette average of the pushed-forward measure is continuous in the coupling, and for a target between its values at two couplings there is a solution	For each step and each coarse plaquette. That a tuned sequence exists at all steps is not stated by these two	`TunedCoupling.continuous_Yc`

Chapter	Theorem or definition	Scope	Lean declaration
9-05	In the book's current design (side 2), the pushed-forward Wilson measure is invariant under coarse translations	No assumption on the sequence of couplings. For each step, when the fine-side step is taken sufficiently large, for bounded continuous functions	`TunedCoupling.hsym_translati`

What this chapter can say

What is said	What is not said
The output of Table 1, Table 2 and Figure 1 is exactly the result of checking seven small files with Lean 4.33.1 and Mathlib (the version in Table 7) on this machine.	We do not say that other versions give the same output. In particular the names of the axioms for `native_decide` change with the version.
The Lean label guarantees the arrow from premises to conclusion (formula (1)). `sorry` passes with only a warning, and is found by `#print axioms` (Table 2).	This appendix does not confirm the soundness of Lean's logic itself, or that the kernel has no bugs (the kernel-recheck step and the isolated-comparison step in the sense of the reference manual were not carried out for the examples of this appendix).
Of the four gaps, name and vacuous truth were shown with examples that actually pass (the last two rows of Table 2).	This appendix does not confirm that the individual theorems of this book have no gaps. Read the premises and readings in each row of 23-13.
The words of lattice gauge theory (Wilson loop, plaquette, Yang–Mills and so on) do not appear in the source of Mathlib on this machine.	The absence of a word is not a proof that the same content is not defined under another name.
How the labels are decided (Figure 2, Table 5) is a restatement of this book's conventions.	A label shows the kind of guarantee; it is not a ranking of the importance or plausibility of sentences.
The Lean theorems of this book show neither the existence of four-dimensional Yang–Mills theory nor the mass gap (23-16). Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (6-00). The mass gap has not been shown.	

Sources and reproduction

Item	Kind	Source / tools
Table 1, Figure 1 (the states of the goal)	Computed	Lean 4.33.1 + Mathlib: `e1_gap.lean` (with `trace_state` after each step) checked with `run.sh`, and the output copied

Item	Kind	Source / tools
Table 2 (exit codes, warnings, axioms and times of the seven files)	Computed	python3: `table.py` checks the seven files `e1` – `e7` in turn with `run.sh` and keeps them in `results.json`. The seconds column is from that one run
The observation in the detour "Using real numbers brings out the three axioms"	Computed	Lean 4.33.1 + Mathlib: `e8_real_probe.lean` (an empty theorem, and `#print axioms` on `Real`, `CauSeq.Completion.Cauchy`, `Real.instZero` and so on)
Word counts in the source of Mathlib	Computed	`mathlib_grep.sh` (a case-insensitive search of 8,311 files)
Table 7 (versions)	Computed	`lean --version` and the Mathlib rev in `lake-manifest.json` (`versions.txt`)
The four steps of checking; distinguish "is there a correct proof?" from "what does the statement mean?"; the mark of the build-check step appears even with `sorry` in a dependency; the three standard axioms; the common axiom `Lean.ofReduceBool` up to 4.28.0 (with `Lean.trustCompiler` behind it) and the dedicated axiom per call from 4.29.0	Known	Lean Reference Manual, “Validating a Lean Proof” (lean-lang.org/doc/reference/latest/ValidatingProofs/). Text read
`native_decide` creates a dedicated axiom for each call	Known	Lean Reference Manual, “Axioms” (lean-lang.org/doc/reference/latest/Axioms/). Text read. Agrees with the output of Table 2
Lean 4	Known	L. de Moura, S. Ullrich, “The Lean 4 Theorem Prover and Programming Language”, CADE-28, LNCS 12699 (2021), doi:10.1007/978-3-030-79876-5_37. Bibliographic data only
Lean (the first system description)	Known	L. de Moura, S. Kong, J. Avigad, F. van Doorn, J. von Raumer, “The Lean Theorem Prover (System Description)”, CADE-25, LNCS 9195 (2015), doi:10.1007/978-3-319-21401-6_26. Bibliographic data only
Mathlib	Known	The mathlib Community, “The Lean Mathematical Library”, CPP 2020, doi:10.1145/3372885.3373824, arXiv:1910.09336. Bibliographic data only
The classification of the four gaps, the procedure for reading a statement, the three-way classification of stock	Article on this site	Guide 08, 09, 10

Item	Kind	Source / tools
The layers of types of the theorems in this book (46 theorems; 23 only real numbers; 3 gauge fields)	Computed (another chapter)	23-13 Lean statements and the limits of machine checking
The legend of the labels, and how the labels of the paper in 5-11 are split	Conventions of this book	0-01, 5-11
The name "Curry–Howard correspondence"	Known	As a general name. The history is left to Guide 13; years and attributions are not stated here

Next chapter to read: A-05 How this book was made — how sentences and numbers were checked before labels were attached.

How this book was made — labels and source tables record how far a sentence was checked

This chapter is a guide (how this book was made and checked).

What this chapter shows — The seven stages a sentence passes through before it appears in the text, and what each stage catches and does not catch (Figure 1).
The meaning of "text", "abstract" and "bibliographic data only" in the source tables, and that about half of the verdicts in the refutation check were corrections (with tables counted on this machine).
The convention for the label "Computed", the stage that aligns symbols and names across chapters, the mechanical checks of assembly, and what this procedure does not guarantee.

Prerequisite chapters — None. It is easier to read after looking at the legend of the labels (0-01). The procedure of machine checking with Lean is not dealt with in this appendix; it is in A-04.

Said first — What this appendix explains is a procedure for reducing errors, not a guarantee of correctness. This book has not been through expert peer review. Also, no step of this book shows the existence or the mass gap of four-dimensional Yang–Mills theory. Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (6-00). The mass gap has not been shown.

The order of this chapter

1. The role of this appendix — labels and source tables are a graduation for measuring how far one may believe; the procedure that marked that graduation
2. The road a sentence travels — seven stages, and what each catches and does not catch (Figure 1, Table 1)
3. Cross-checking against primary sources — three levels: text, abstract, bibliographic data only (Table 2); Detour: writing in the form of hearsay
4. Looking for refutations — another reader checks "is this too strong?"; the breakdown of verdicts and the types of correction (Tables 3 and 4)

5. Making computations reproducible — the label "Computed" covers a finite range; write the conventions; a scale that finishes in minutes
6. Attaching labels — a record of how far a sentence passed; the composition of labels by part (Figure 2, Table 5)
7. Aligning across chapters — five things: symbols, conventions, names, duplication, degree of certainty
8. Assembling and checking mechanically — pre-rendering of formulas, links, forbidden words (Table 6)
9. What this procedure does not guarantee — not peer review; texts not obtained; finite computations; problems that remain open

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01

The role of this appendix — so that readers can measure for themselves how far to believe

Sentences in textbooks tend to be written on the premise that readers will believe them as they are. This book does not do that. Each sentence carries a label (the legend in 0-01), the source tables write "whether the text was read", "only the abstract" or "only the bibliographic data", and computations are kept in a form that can be reproduced at hand. Labels and source tables are a graduation with which readers can measure for themselves "how far this sentence may be believed". Precisely, a label is a record of how far a sentence was checked, not a number expressing a degree of correctness.

A graduation cannot be read unless the procedure that marked it is known. For example, even for a sentence labelled "Known", the weight differs according to whether it was checked word for word against the text of a primary source or copied from the description in a review. This appendix explains that procedure from the reader's side. Stage by stage, it writes what was checked and what was not.

The flow in one line:

research → cross-check against primary sources → look for refutations → reproduce com

Each of the seven stages is placed to catch what the previous stage overlooks. But no stage guarantees "correctness". What is not guaranteed is collected in § 09.

Example. The sentence of Part 5 that "the Wilson action has reflection positivity" carries the label "Known", and the source table lists the paper of Osterwalder–Seiler (5-03). The label "Known" shows that it is a known result checked in a primary source. The depth to which it was checked (text, abstract, bibliographic data only) is written in

the source table, and the source table for this example says "abstract checked, text not checked". Whether it passed the refutation check (§ 04) is not distinguished by the label. If the source table says "bibliographic data only", the cross-checking stopped at the level of bibliographic data.

The procedure of machine checking with Lean is not dealt with in this appendix. It is in A-04. How labels are decided (one label per sentence; the strength of a chain is its weakest stage) also has its master copy in A-04 § 07.

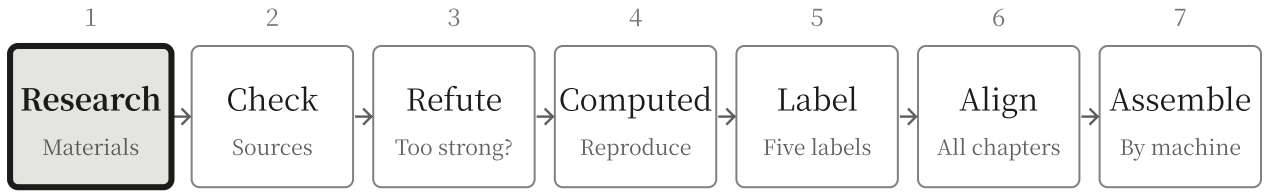
02

The road a sentence travels — seven stages

Figure 1 lines up the stages a sentence passes through before it appears in the text. Moving the knob shows below what that stage catches and what it does not.

The order of the steps has reasons. If the refutation step (step 3) were placed before the cross-checking step (step 2), the refuting side would not know what to compare against. The label step (step 5) comes after the cross-checking step and the reproduction step because a label is a record of "which step it passed up to". The consistency step (step 6) and the assembly step (step 7) are steps for errors that cannot be caught by looking at one chapter at a time, and are run each time a part is complete (and finally for the whole book).

Figure 1 Choosing a stage with the knob shows below what that stage catches and does not catch. The frame of the chosen stage becomes thicker



What the previous stage overlooks, the next stage catches

```
Step 1/7 Research. Catches: Literature on each topic, known results, candidate numbers. Misses:
Whether the candidates are correct (here candidates are only collected).
```

Table 1 has the same content as Figure 1 (the figure is schematic and contains no numbers).

Table 1 What the seven stages catch and do not catch

Stage	Name of the stage	What it catches	What it does not catch
1	Research (survey step)	Literature on each topic, known results, candidate numbers	Whether the candidates are correct (here candidates are only collected)

Stage	Name of the stage	What it catches	What it does not catch
2	Cross-check against primary sources (cross-checking step)	Errors in bibliographic data, over-paraphrased citations, misattribution	Errors in the primary sources themselves, the content of texts that could not be obtained
3	Look for refutations (refutation step)	Wording stronger than the source, promotion of the degree of certainty, omitted conditions	Cases where the cross-checking reader and the refuting reader share the same misunderstanding
4	Reproduce computations (reproduction step)	Miscopied numbers, mixed-up conventions, counterexamples within a finite range	Whether it holds outside the range computed
5	Attach labels (label step)	Mixing of "proved", "conjecture" and "numerically supported"	Errors in the judgement of the label itself
6	Align across chapters (consistency step)	Discrepancies in symbols, names and conventions; the same name pointing to different sentences; duplication	All chapters being wrong in the same direction
7	Assemble and check mechanically (assembly step)	Formulas that cannot be rendered, broken links, forbidden words, unfilled placeholders	Errors of content (the machine does not read meaning)

03

Cross-checking against primary sources — three levels: text, abstract, bibliographic data

What this book calls a "primary source" is the paper that first stated a result, or the problem statement itself. Reviews and textbooks are used as guides to the primary sources, not as substitutes for them. Years and attributions (who showed what, when) are not written from memory but checked in the sources.

Cross-checking has three levels of depth, and the source table writes which level was reached.

Table 2 Depth of cross-checking, and how it is written in the source tables

Depth	What was read	How it is written in the source table	What can be written in the text
Text checked	The text of the paper (a public PDF or the like) was read, and the relevant passage checked word for word	Authors, year, journal or arXiv number, section	Exactly the strength stated by the source

Depth	What was read	How it is written in the source table	What can be written in the text
Abstract checked	Only the abstract (the abstract in a literature database) was read	As above + "checked in the abstract", "abstract only" and so on (the wording differs slightly by chapter)	As far as is written in the abstract
Bibliographic data only	Only that the authors, title, journal and year exist was checked	"Bibliographic data only"	As far as the form in which another source reports that "... did ..."

Word-for-word checking is the work of confirming by search that the original text enclosed in quotation marks exists literally in the source. Even when a translated sentence is placed in the text, the cross-checking is done on the original.

Example. The problem statement of the Clay Institute (Jaffe–Witten) was cross-checked in the text of the public PDF. The requirement of the problem, in the original, is as follows.

> “Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.”

0-03 reads this sentence word by word. Because it was cross-checked in the text, 0-03 can use each of the words "existence", "non-trivial" and "mass gap" at the strength of the problem statement.

On the other hand, for papers of which only the abstract could be read, claims in the abstract are written as "checked in the abstract", and claims not in the abstract are not brought into the text. The same goes for section numbers: if the abstract does not mention sections, which section something is written in is not known until the text is read.

Example. The abstract of the review by Wilson–Kogut (Phys. Rep. 12 (1974) 75) is a single sentence and does not mention sections. So this book does not write that some topic "is discussed in § 10 of Wilson–Kogut", and writes " § 10 not checked" in the source table.

Detour: writing in the form of hearsay

You may skip this. That the upper limit on the spacetime dimension in which supersymmetry can be realised with fields of spin at most 2 (supergravity, including the graviton) is 11 is widely known as a consequence of Nahm's 1978 classification. Yet the number 11 does not appear in the abstract of that paper. While the text cannot be obtained, this book writes "known as a

consequence of Nahm's classification", and does not attribute the conclusion directly to Nahm's paper (17-05). It is not that the fact is doubtful. It is to show honestly, in the form of the sentence, how far this book checked. Precisely, the form of hearsay is not a mark that "it may be wrong" but a mark that "the cross-checking did not reach the text, and the abstract does not contain the number".

04

Looking for refutations — another reader checks "is this too strong?"

Material that has been cross-checked is handed to another reader. That reader's job is not confirmation but refutation. Going back to the primary sources once more, they look for "is this sentence stronger than what the source says?", "it was written as settled, but does it not need a condition?", "does this bibliographic entry exist?". There are three verdicts.

- **Confirmed** (confirmed): against the source, the written strength is fine as it is.
- **Corrected** (corrected): it needs to be weakened, given a condition, or have its bibliographic data fixed. The wording of the correction is attached.
- **Unverifiable** (unverifiable): the primary source could not be reached, and neither can be said. It is lowered to the form of hearsay.

At the stage of writing chapters, the rule is to reflect every correction. Material that received a correction is not cited at its strength before the correction.

Table 3 counts the verdicts for the six materials used mainly by Parts 7 to 23 (and partly by chapters of Parts 0, 4 and 5). Many chapters of Parts 1 to 5 have not been through the materials of this refutation check.

Table 3 Verdicts of the refutation check by material (counting the materials as of 2026-10-01) Computed

Subject of the material	Confirmed	Corrected	Unverifiable	Fraction corrected
Dictionary of the physical and the mathematical vocabulary	25	20	0	44%
Extra dimensions and where gauge fields come from	16	12	1	41%
The physical view (confinement, lattice numerics, holography)	25	17	0	40%
Arrangement of this book's study	7	14	0	67%
Where things stand, dimension by dimension	25	8	0	24%
Where constructive field theory stands, and candidates for new concepts	16	29	0	64%

Subject of the material	Confirmed	Corrected	Unverifiable	Fraction corrected
Total	114	100	1	47%

How to read it. About half of the verdicts are corrections. The highest fraction of corrections is for "arrangement of this book's study": in this material corrections were many, 14 of 21, and those that weakened wording stronger than the record, or added conditions that had been dropped, stood out. Conversely, materials close to a list of theorems, such as where things stand dimension by dimension, have fewer corrections.

Table 4 sorts the content of the corrections into four types by words. The sorting is mechanical, by words appearing in the wording of the correction, and one correction may fall into several types.

Table 4 Types of the 100 corrections (mechanical sorting by words; with overlaps) Computed

Type	Cue words	Count	Typical
Strength	too strong, promoted, showed, restricted	39	Lowering "showed" to "proposed"; "settled" to "conditional"
Level of the source	abstract, text, not obtained, not checked	34	Rewriting a claim not in the abstract as "text not obtained"
Numbers and signs	number, coefficient, sign, digit	15	Adding the convention for the coefficient (which β)
Bibliographic data	bibliographic data, volume, page, author, journal	12	Corrections to the list of authors, such as a missing co-author
None of the above	—	30	Supplements, suggestions for rewording

Example. What § 6.5 of the Clay problem statement says is that "with many regularisations, such as dimensional regularisation and Pauli–Villars, reflection positivity is lost, and there are few ways to recover it"; it does not reject continuum regularisations across the board. This book cites it at this strength, and does not write "continuum regularisations do not preserve reflection positivity". What the refutation stage catches is summaries that have become one step stronger than the source in this way (regularisation and reflection positivity are in 4-10 and 5-03).

05

Making computations reproducible — the content of the label "Computed"

Of the numbers in the text, those that can be recomputed on this machine are recomputed and given the label "Computed". All computations are short programs (Python), kept chapter by chapter. Their names are written in the "tools" column of the source tables, so running the same programs gives the same tables.

What the label "Computed" means is that it was **checked on this machine, over the range written**. As a formula, having checked a property $P(n)$ for finitely many n is the distinction

$$P(n_1) \wedge P(n_2) \wedge \cdots \wedge P(n_k) \qquad \text{and not} \qquad \forall n\, P(n)$$

The left is a computation, the right a theorem. A sentence with the computed label is not cited as a theorem.

Computations have three conventions. The first is not to make numbers inside figures. The numbers in moving figures are limited to tables computed in advance and embedded, or simple formulas that can be computed on the spot (rotations, partial sums of series and so on). The second is to write the conventions. If the definition of the lattice coupling β or the normalisation of the trace tr differs, the same quantity is off by several times, so the conventions are stated in the premises of a computation (conversion between conventions is in 5-01 and A-01). The third is to keep to a scale that finishes within a few minutes on an ordinary computer. A computation that readers cannot reproduce loses the meaning of the label.

Example. Tables 3, 4 and 5 of this appendix are also computations. We wrote one program that reads the materials and chapter pages and counts, and copied its output. Over the whole book, the places where computations are kept, chapter by chapter, number over 100, and there are several hundred programs.

06

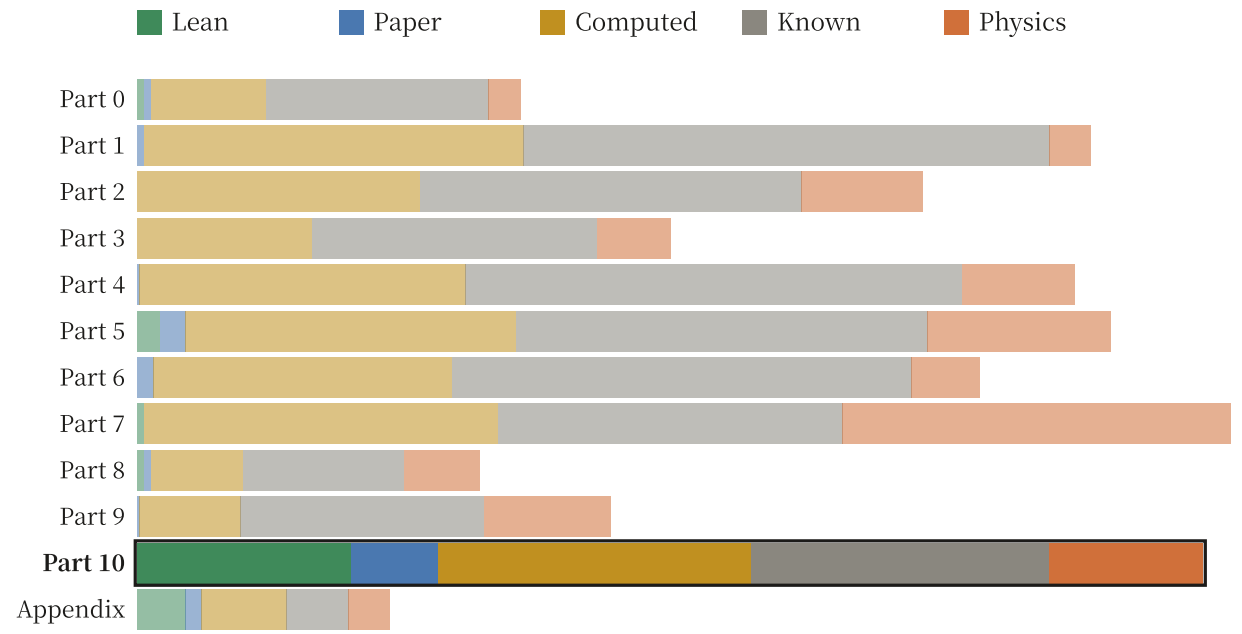
Attaching labels — a record of how far a sentence passed

A label is a record that tells readers how far a sentence passed. A sentence that stopped at cross-checking gets "Known", a number reproduced at hand gets "Computed", something accepted in physics but without proof gets "Physics", and so on. The meaning of the five labels, and what the labels do not guarantee, are in the table of 0-01 § 03.

There are two rules for attaching labels. One sentence gets one label. And for a claim connected through several stages, the label of the weakest stage is attached to the whole (A-04 § 07). For example, if numbers are put as assumptions into an inequality checked in Lean and a conclusion is drawn, the label of the conclusion is "Computed" or "Physics". Labels are not pushed towards the strong side.

Figure 2 counts, by part, the labels appearing on the chapter pages of this book. The composition of labels differs clearly from part to part.

Figure 2 Number of labels by part. "View by fraction" scales each part to 100%. Choosing a part with the knob shows its breakdown below



`Part 10 (total 464) : Lean 93 (20%) , Paper 38 (8%) , Computed 136 (29%) , Known 130 (28%) , Physics 67 (14%)`

Computed The numbers in this figure were computed on this machine (counting the chapter pages of 110 chapters as of 2026-10-01, excluding this appendix itself. Labels placed in legend tables are also counted. The numbers change as chapters are added).

Table 5 Number of labels, number of figures, and occurrences of "bibliographic data only" by part (as of 20 Computed)

Part	Chapters	Lean	Paper	Computed	Known	Physics	Figures	Bibliographic
Part 0 Entrance	4	3	3	50	97	14	8	
Part 1 Mathematical preparation	12	0	3	165	229	18	40	
Part 2 Physical preparation	10	0	0	123	166	53	28	
Part 3 Classical theory	7	0	0	76	124	32	13	
Part 4 Quantum theory	12	0	1	142	216	49	23	
Part 5 Lattice	11	10	11	144	179	80	22	

Part	Chapters	Lean	Paper	Computed	Known	Physics	Figures	Bibliographic
Part 7 Constructive field theory	12	0	7	130	200	30	24	
Part 17 From the side of physics	13	3	0	154	150	169	28	
Part 18 A dictionary of two languages	5	3	3	40	70	33	7	
Part 16 Where things stand, and open tasks	5	0	1	44	106	55	10	
Part 23 The record of this book's study	14	93	38	136	130	67	25	
Appendices (A-01, A-02, A-03, A-04, A-06)	5	21	7	37	27	18	10	
Total	**110**	**133**	**74**	**1241**	**1694**	**618**	**238**	

How to read it. Most labels are "Known" and "Computed". This is because much of this book is an exposition of known mathematics and physics, with numbers reproduced at hand added to it. "Physics" is most frequent in Part 17, because its centre is material accepted in physics but not theorems, such as mechanisms of confinement and the string picture. The "Lean" labels are concentrated in Part 23. However, many of those Lean theorems are statements about inequalities of real numbers or finite sums, and do not show the existence or the mass gap of four-dimensional Yang–Mills theory (23-16, A-04 § 06).

07

Aligning across chapters — symbols, names, conventions

Checking chapters one at a time cannot catch discrepancies between chapters. For example, if one chapter writes $\beta = 2N/g^2$ and another $\beta = 1/g^2$, each chapter may be correct on its own, but readers take them to be the same β and get lost when the numbers do not match. So each time a part is complete, a stage of reading across parts is run.

This stage looks at five things.

1. **Symbols**: whether the same symbol is used for the same quantity. The standard is the table of notation A-01.
2. **Conventions**: when the coupling β, the normalisation of the trace, or the lattice metric differs between chapters, whether a conversion is written (5-01).
3. **Names**: whether one name points to different sentences in different chapters. For example, the diffusive-relaxation-time assumption of Part 23 can be read in eight ways, according to which length it is measured in, which normalisation it is measured in, and whether it is stated as a power or a band. A name is assigned to each reading and fixed (calling the power level the diffusive-relaxation-time assumption and the band statement the band form of the diffusive-relaxation-time assumption; A-06), and other chapters link there and use the same name. The list of assumptions is in A-06.
4. **Duplication**: when the same explanation is in two chapters, make one the master copy, and reduce the other to a one-line summary and a link.
5. **Degree of certainty**: whether one chapter writes the same correspondence as "settled" and another as "conditional". In the dictionary part, correspondence as a definition (which can be settled) and actually holding in four-dimensional Yang–Mills theory (unsettled) are written in two separate columns (18-01).

When a discrepancy is found, it is aligned to one side or the other. Discrepancies that cannot be aligned (for example when conventions genuinely differ between papers) are written in the text as discrepancies.

Example. CP in Part 23 is a number that serves as a measure of the time it takes for fluctuations to be smoothed out (defined in 1-10 and 23-01). With the same symbol CP, it can point to different numbers according to which fluctuations are meant (of the field at one time, or of the whole path including the time direction). For the free field one is 4 times the square of the other, so mixing them up doubles the power. This is an example distinguished at this stage, and the whole book writes the two as different quantities (23-01).

08

Assembling and checking mechanically — what can be left to the machine

The last stage is done by machine. The tool that assembles the chapter pages does the following mechanically for all chapters, and reports what gets caught.

Table 6 Mechanical checks at the assembly stage

Check	What it looks at	What it does not look at

Check	What it looks at	What it does not look at
Rendering of formulas	Renders formulas written in TeX to static HTML with KaTeX, and reports formulas that cannot be rendered	Mathematical correctness of the formulas
Links and headings	Whether the files linked between chapters and the heading `id`s exist	Whether the link target fits the flow of the argument
Forbidden words	Whether real names, paths, internal numbers or unfilled-placeholder marks remain	The strength of wording (this is looked at by people and by the refutation stage)
Templates	Whether a class name not to be used for figure frames (`fig`) remains	The content of figures
PDF	Makes a PDF for printing for each part (moving figures go in as still images of their initial state)	Whether it is easy to read in print

Formulas are rendered at the assembly stage rather than in the reader's browser so that no JavaScript is needed on the reader's side, and so that they do not break when printed or made into PDF. Only moving figures use JavaScript, and even when JavaScript is disabled, the same content can be read in the table below the figure.

Mechanical checks do not read meaning. Passing this stage guarantees that the page is not broken, not its content. Judgements of the strength of wording, the choice of labels and the depth of sources are not left to the machine; people and the writers read them at the stages of cross-checking, refutation and consistency.

09

What this procedure does not guarantee

Errors can remain even after the seven stages. We write the limits of this book from the side of the procedure.

It has not been through expert peer review. The reader at the refutation step is a different reader from the writer, but not a referee of an academic journal. Much of the research, drafting and refutation for this book was done by the AI writers of this site; people decide the policy and read before publication. Readers of the same kind can share the same misunderstanding. What the refutation step cannot catch is exactly such errors (the "Look for refutations" row of Table 1).

The content of texts that could not be obtained has not been checked. For works marked "bibliographic data only" or "checked in the abstract" in the source tables, this book does not know what is written in the text. The count of "bibliographic data only" in Table 5 is a measure of the size of that range.

Computations cover finite ranges. Numbers with the label "Computed" are what the computations at hand gave over the range written, and say nothing outside the range or about the continuum limit (§ 05).

The mass gap problem remains open. No step of this book shows the existence or the mass gap of four-dimensional Yang–Mills theory. Four-dimensional existence was shown by a construction published in 2026, and the committee's assessment is that it is proved in Lean (Part 6). A list of what this book does not claim is in 23-16.

If you notice an error, please let us know through the board or the contact page of this site. Corrections are recorded with dates in the "Revised" field at the end of each chapter.

What this chapter can say

What is said	What is not said
That the sentences of this book appear in the text after passing through the seven stages (research, cross-checking, refutation, computation, labels, consistency, assembly). What each stage catches (Table 1)	That sentences that passed the seven stages are correct. No stage guarantees correctness
That "text", "checked in the abstract" and "bibliographic data only" in the source tables are records of the depth of cross-checking (Table 2)	What is written in the texts of works marked "bibliographic data only" or "abstract"
That in the refutation check of the materials as of 2026-10-01, of 215 verdicts, 100 were corrections and 1 was unverifiable Computed (Table 3)	That the 114 judged confirmed contain no errors. If the refuting reader and the cross-checking reader share the same misunderstanding, this stage cannot catch it
The counts from sorting the wording of corrections by words Computed (Table 4)	That the sorting classifies the content of corrections correctly (it is a mechanical classification by words, with overlaps)
The numbers of labels, figures and "bibliographic data only" in the chapter pages of 110 chapters as of 2026-10-01 (excluding this appendix itself) Computed (Table 5, Figure 2)	That the number of labels represents the weight of claims. Labels placed in legends are also counted, and the numbers change as chapters are added
What the assembly stage looks at mechanically (Table 6)	That the content of pages that passed the mechanical checks is correct
That this book has not been through expert peer review, and that much of the research and refutation was done by AI writers	—

What is said	What is not said
—	The existence and mass gap of four-dimensional Yang–Mills theory. No step of this book shows them (existence was shown by the published construction, and the committee assessed it so; the mass gap is open; 23-16)

Sources and reproduction

Item	Kind	Source / tools
Tables 3 and 4 (verdicts of the refutation check and types of correction)	Computed	`tally.py` (counts the verdicts of this book's materials and sorts the wording of corrections by words; output `tally.json`)
Table 5 and Figure 2 (numbers of labels, figures and "bibliographic data only" by part)	Computed	`tally.py` (counts `class="label …"`, figure frames and the words "bibliographic data only" on the chapter pages, excluding the inside of figures' JavaScript)
Quotation of the Clay problem statement (§ 03)	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement of the Clay Mathematics Institute, public PDF). Checked in the text
What the problem statement says about regularisation and reflection positivity (the example in § 04)	Known	Same, § 6.5. Checked in the text (details in 4-10)
Reflection positivity of the Wilson action (the example in § 01)	Known	K. Osterwalder, E. Seiler, “Gauge Field Theories on the Lattice”, Ann. Phys. 110 (1978) 440–471. Abstract checked, text not checked (how the text is handled is in the source table of 5-03)
An example where the abstract does not mention sections (§ 03)	Known	K. G. Wilson, J. Kogut, Phys. Rep. 12 (1974) 75–199. Abstract only (the abstract is a single sentence; § 10 not checked)
The upper limit 11 on the dimension for supersymmetry (detour)	Known	W. Nahm, Nucl. Phys. B135 (1978) 149. Abstract only. "11 is the upper limit" is not in the abstract and not checked. The upper limit carries the condition of fields of spin at most 2 (17-05)
Pre-rendering of formulas	Tool	KaTeX (a JavaScript typesetting library that renders TeX formulas to static HTML. Used only at the assembly stage; it does not run on the reader's side)
Figure 1, Tables 1, 2 and 6 (description of the procedure)	—	A summary of this book's writing conventions and assembly procedure

Next chapter to read: A-06 Index of names.

Index of names

> This chapter is guide material (an index).

What this chapter shows — What the names in this book (assumptions, conditions, propositions, lemmas, theorems, statements, routes, table rows, questions) and its symbols refer to, and in which chapter each was introduced. The names and symbols given in Volume 2 (the volume that reads the published construction) are gathered in § 02; the rest are listed in § 03 in the order of the parts of their introducing chapters.

Prerequisite chapters — None. The list of symbols is A-01, and explanations of words are in A-02.

Said first — This book does not use names made only of letters or numbers; it uses names that show their content. Standard labels from the literature are added at the first occurrence of a name, in the form "(author label)". "What it refers to" is a one-line guide; the exact statement is in the introducing chapter.

Order of this chapter

1. How to use this index
2. Names and symbols of Volume 2 — conventions for names and symbols when reading the published construction (Tables 1 and 2)
3. Index of names — in the order of the parts of the introducing chapters (Table 3)
4. Eight readings of the diffusive-relaxation-time assumption — why the normalisation is named
5. Bookkeeping and exponent quantities — the one-step slope, the shortfall, the exponents
6. Assumptions of the second road and the band — name, content, status

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01

How to use this index

1. Look things up by name. The quickest way is to search for part of a name ("relaxation time", "widening", and so on) with the browser's in-page search.

2. The "introducing chapter" in Table 3 is the chapter where the statement of that name was first placed. Later chapters may use the same object in another form, but there is one name for it across the whole book.
3. Long names are sometimes called by a short form from their second occurrence on. The short form is added in the name column of Table 3 as "(short: ...)".
4. "Kind" is one of: assumption (a premise that has not been shown), condition (a condition imposed within a statement), proposition or theorem (something shown), statement (a sentence to be shown, or a sentence in the form of a question), table row (a row of a table referred to by name), question, and so on. What has been shown and what has not is read from the labels in the introducing chapter.

02

Names and symbols of Volume 2

Volume 2 is the volume that reads the construction of four-dimensional Yang–Mills existence published on 2026-11-10 (the Lean code and the committee's assessment report). Objects of the published construction are called by names that carry their meaning, and the identifiers of the code are added as sources (6-00). There are three labels, Source (we read the source of the published construction), Inferred (this book's inference) and Assessment (from the committee's assessment report); the labels of the whole book (Proved here, Cited, Computer-assisted, Reading, Assumption, Lean) have the same meaning in all volumes. Theorems of the published construction are written with Source , and their level (the committee's assessment is that they are proved in Lean) with Assessment ("Labels" in 6-00).

Name	What it refers to	Introducing chapter
Published theorem	The statement in the form of the published question (its only assumption is that the Lie algebra of the group is simple)	6-02
Construction theorem	A stronger statement, for each penalty coefficient, that contains the published theorem as a projection	6-02
Committee's combined target	A statement in which the committee bundles independent checks on a single family, on top of the construction theorem	6-02
Headline theorems, outside the dependencies	The 17 headline theorems listed in the verification record, and the declarations not reached from any of them	6-00
Family certificate	The bundle of conditions that the construction theorem imposes on a family (renormalisation relations at all steps, local cross-sections, label templates, calibration of insertion coefficients)	6-02
Run	A construction satisfying the conditions, chosen once for each penalty coefficient. All later objects are built on this single run	6-03
Template	A finite set of offsets of lattice sites, used to make lattice representatives of local fields	8-03

Name	What it refers to	Introducing chapter
Probability kernel, predictor	One step of renormalisation: a probability kernel that draws a coarse link with Gaussian-type fluctuations around a central value (the predictor) made from the fine configuration	9-01
Representation	The fixed form in which the density of each step is written (vacuum term, main action, regular activity, gas of forced activities). Not unique	9-06
Regular activity, forced activity	The local correction terms in the representation, and the gas of corrections that exactly covers the large-field flags	9-06
Cut action, flag	The action in which the plaquette terms are cut off linearly above a step-dependent threshold, and the mark of a large field exceeding the threshold	9-07
Step coupling	The coefficient multiplying the term of the squared normalised curvature in the representation at step m. Not determined by the measure alone	10-01
Rereading	After one step, reading the coupling of the representation again from the log ratio of the densities at two configurations	10-03
Shooting method	Choosing, by nested intervals and the intermediate value theorem, the bare coupling that makes the coupling at step 0 equal to the reference value	10-04
Calendar	The bundle of conditions that chooses, in a fixed order, the constants used in the one-step estimate (sides, smoothing exponents, entry coupling and so on). Conditions uniform in the step are discharged here	10-05, 11-08
Induction hypothesis, one-step certificate	That the density at every step has a representation of the same form and carries the ledger; and making the representation at step m from the representation at step $m+1$ together with the pushforward identity	11-00
Seed, seed type, region	Each large fine plaquette, each edge deviating strongly from the prediction, and each flag (seeds); the triple of these three sets (seed type); and the condition that divides configurations by seed type (region)	11-03
Price, mark	The factor per seed, and the part of its allocation left for the exit (half of the guaranteed lower bound)	11-05
Strong estimate, weak ledger	The exit estimate of one step, $e^{-\mathrm{rate}\cdot P^2}$, and the estimate carried by the induction hypothesis, $e^{-P^{1/5}}$	11-07
Polymers with and without flags, hard and soft clusters	The way of sorting polymers, after the integration of one step, into the exit gas and the regular activity	11-06
Early representation	In the small-field step, the way of writing the expansion of the corrections into the form of the representation first	11-09
Convergence of ingredients, anchor, forgetting the top end, the step-removing step	The parts that give convergence of the cutoff limit at fixed step (uniform convergence of the ingredients, an anchor of uniqueness, vanishing of the influence of the fine-side end, and one step that passes from step 0 to the limit)	11-10

Name	What it refers to	Introducing chapter
Slice, thread, inner function	In reflection positivity: the configuration of links in one layer (slice), the set of links joining layers (thread), and a bounded gauge-invariant function depending only on the inside of a band (inner function)	12-01
Chessboard root, centred tiling	The expectation of the product of a function copied to all boxes, raised to the power one over the number of boxes (a seminorm); and the tiling that puts the reference box in the middle of the observation points	12-02
Column	An arrangement placing a block and its mirror image alternately around the time circle (the chessboard transcribed in the time direction)	12-04
Zero-energy space, phase measure	The subspace $\ker H$ fixed at all times, and the probability measure that splits a mixed theory into phases	12-07
Order bundle	The input bundling the choice of kernel, the step constants of fine and coarse steps, the number of convolutions of the block weights, and so on (`Teller` in the source)	11-04
Extraction sequence	The pair of a subsequence of cutoffs and a subsequence of volumes (`Extraction.Order` in the source). Used in the two-stage Arzelà–Ascoli extraction	12-03
Minimal defect of the twisted box	The lower bound $D_{\min}(N, M)$, over the whole twisted box, of the loss from the maximum of the action to the actual action. A different quantity from the minimal vortex action	21-07

Table 1 The main names given in Volume 2 (and one name from a new chapter of Volume 3). Explanations of the words are also in A-02.

Symbol	What it refers to	Introducing chapter
s	The one-loop slope $s = 11C_g/(24\pi^2)$ ($2b_0$ in the assessment report, `slopeConstant` in the source). Each time the lattice is made one step coarser, the coupling decreases by about $s \log b$	10-01
κ	The one-step slope of Volume 3, $\kappa = 11/(3\pi^2)$ (SU(2)). The coefficient in the one-step bookkeeping $\beta \mapsto \beta - \kappa(\ln b - \delta(b))$ for a step coarsening by a factor b. For SU(2), $C_g = 8$ and $s = 11 \cdot 8/(24\pi^2) = 11/(3\pi^2)$, so κ and s are the same number, and both are the coefficient of the logarithm in the decrease of the coupling over one step Proved here (this book's $\beta = 4/g^2$ and the published construction's $\beta = 1/g^2$ are the same number for SU(2), 8-01)	23-02, 10-01
C_g	The Casimir constant ($2N^2$ in the defining representation of SU(N))	8-01
$\bar\beta$, $\hat\beta_m$	The reference value of the coupling at step 0, and the one-loop running $\hat\beta_m = \bar\beta + s \log D_m$	10-01
c_{pen}	The penalty coefficient of the probability kernel (`kernel.penalty` in the source text)	9-01
cE	The constant of the energy per seed	11-04
c_{sup}	The covering exponent	11-01

Symbol	What it refers to	Introducing chapter
σ	The chordal distance (given a different letter so as not to be confused with the slope s)	11-01
$\kappa 0$	The tuning target value (the value to which the vortex ratio or the expectation of the coarse plaquette average is tuned, $0 \le \kappa 0 < 1$)	19-07, 14-03
$\mu_{N,p}^{\mathrm{W}}$, μ_{β}^{W}	The Wilson lattice measures of the published construction	8-01
μ	The phase measure (`phaseMeasure` in the source; ν in the assessment report). This book's infinite-volume limiting measure at strong coupling stays ν_β	12-07
σ_A	The spectral measure of the state A	12-08
τ	The Euclidean time of the heat estimate	6-04

Table 2 Symbols of Volume 2. The full list of symbols is in A-01.

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Index of names

The entries are listed in the order of the parts of their introducing chapters, and within a part in the order in which they appear in the text. A row whose "What it refers to" is empty is one for which the introducing chapter does not yet have a one-line statement of its content.

Part 0 Entrance — how to read this book

Name	Kind	What it refers to	Introducing chapter
local-field footnote · weak-existence footnote (footnotes of Jaffe–Witten) (short: local-field footnote · weak-existence footnote)	Name from the literature	Two footnotes of the problem statement (Jaffe–Witten): asking for a correspondence between local fields and quantum operators / not accepting as a solution a weak existence obtained from compactness alone	0-03

Part 2 Physical preparation

Name	Kind	What it refers to	Introducing chapter
Maxwell equations with sources	Name	The two equations by which charges and currents produce fields (Gauss's law and the Ampère–Maxwell law)	2-03
source-free Maxwell equations	Name	The two equations constraining the form of the fields (absence of magnetic charge and Faraday's law of induction). They hold automatically from the potential representation	2-03
potential representation	Name	The formulas writing E and B in terms of potentials	2-03
dynamical exponent (symbol kept)	Name	The exponent z when the relaxation time grows as a power of the correlation length, C_P ≍ ξ^z (together with η_dyn = 2 − z)	2-10

Name	Kind	What it refers to	Introducing chapter
gauge-transformation formula	Formula	A relabelling of the potentials that leaves the electric and magnetic fields unchanged (a transformation changing ϕ and A simultaneously by a single function χ)	2-03
Euler–Lagrange equation	Formula	The equation of motion obtained from the condition that the action is stationary (obtained by a variation with fixed ends and one integration by parts)	2-01

Part 4 Quantisation

Name	Kind	What it refers to	Introducing chapter
states-and-symmetry axiom (Wightman axioms) (short: states-and-symmetry axiom)	Name from the literature	States are elements of a separable Hilbert space, on which the Poincaré group acts as a strongly continuous unitary representation	4-09
spectral condition (Wightman axioms) (short: spectral condition)	Name from the literature	The joint spectrum of $P^\mu = (H, \boldsymbol{P})$ lies in the forward light cone V_+ (energy is never negative)	4-09
uniqueness of the vacuum (Wightman axioms) (short: uniqueness of the vacuum)	Name from the literature	There is only one unit vector Ω (the vacuum), up to phase, invariant under the spacetime transformations U	4-09
field axiom (Wightman axioms) (short: field axiom)	Name from the literature	Fields are operator-valued tempered distributions (for each test function $f \in \mathcal{S}(\mathbb{R}^4)$, $\phi(f)$ is defined on a dense domain containing the vacuum)	4-09
covariance (Wightman axioms) (short: covariance)	Name from the literature	Moving a field by the unitary operator of a Poincaré transformation gives the field with the test function moved by the same transformation (there is no special coordinate system)	4-09
local commutativity (Wightman axioms) (short: local commutativity)	Name from the literature	If the supports of f, g are spacelike separated, then $[\phi(f), \phi(g)] = 0$ (no influence faster than light)	4-09
cyclicity (Wightman axioms) (short: cyclicity)	Name from the literature	Linear combinations of products of fields applied to the vacuum, $\phi(f_1)\cdots\phi(f_n)\Omega$, are dense in $\mathcal{H}$ (there are no states unrelated to the fields)	4-09
regularity (Osterwalder–Schrader axiom E0) (short: regularity)	Name from the literature	Schwinger functions are tempered distributions on test functions vanishing where points coincide (used together with the linear-growth condition)	4-10
Euclidean covariance (Osterwalder–Schrader axiom E1) (short: Euclidean covariance)	Name from the literature	Schwinger functions are unchanged by Euclidean rotations and translations	4-10

Name	Kind	What it refers to	Introducing chapter
reflection positivity (Osterwalder–Schrader axiom E2) (short: reflection positivity)	Name from the literature	The quadratic form of correlations between past and future, with a time reversal in between, is never negative (the condition for the theory to have a Hilbert space)	4-10
permutation symmetry (Osterwalder–Schrader axiom E3) (short: permutation symmetry)	Name from the literature	Schwinger functions are unchanged by permutations of the points	4-10
clustering (Osterwalder–Schrader axiom E4) (short: clustering)	Name from the literature	The correlation of two groups of points pulled apart splits, at large distance, into the product of the separate correlations (corresponds to uniqueness of the vacuum)	4-10
Counterterm Uniqueness Theorem (Theorem B of Costello) (short: Counterterm Uniqueness Theorem)	Name from the literature	Once the renormalisation scheme is fixed, local counterterms can be chosen uniquely and the limit of the effective action exists	4-05
Classification Theorem for Families of Effective Actions (Theorem C of Costello) (short: Classification Theorem for Families of Effective Actions)	Name from the literature	Local actions correspond one-to-one to families of effective actions (with the scheme fixed)	4-05

Part 5 Lattice gauge theory

Name	Kind	What it refers to	Introducing chapter
lattice area-law problem (Chatterjee's Problem 4.1) (short: lattice area-law problem)	Name from the literature	The problem of showing the area law for Wilson loops at all β in four-dimensional lattice gauge theory with a non-abelian compact group	5-02, 16-07
Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021) (short: Theorem from Uniform Decay to the Area Law)	Name from the literature	If a lattice gauge theory has uniform exponential decay under arbitrary boundary conditions, then centre symmetry is unbroken, and Wilson loops in representations on which the centre acts non-trivially obey the area law	5-10
definition of unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) (short: definition of unbroken centre symmetry)	Name from the literature	The theory on a slab is invariant under centre transformations under every boundary condition (centre symmetry is not broken)	5-10
definition of strong decay (Definition 2.3 of Chatterjee 2021) (short: definition of strong decay)	Name from the literature	Correlations decay exponentially, uniformly over every cube and every boundary condition	5-10

Name	Kind	What it refers to	Introducing chapter
Strong-Coupling Uniqueness Theorem · Exponential-Decay Corollary (labels of Shen–Zhu–Zhu) (short: Strong-Coupling Uniqueness Theorem · Exponential-Decay Corollary)	Name from the literature	Under the strong-coupling condition, the invariant measure of stochastic quantisation (the Langevin equation) is unique and has an infinite-volume limit, and correlations decay exponentially (a set of assumption, theorem and corollary)	5-04

Part 7 Constructive field theory

Name	Kind	What it refers to	Introducing chapter
six steps of construction	Table row	The six steps of the work of building a quantum field theory (regularisation, finite volume, ultraviolet limit, infinite volume, axioms, uniform gap)	7-01
regularisation step	Table row	Cutting off short distances, by a lattice or otherwise	7-01
finite-volume step	Table row	Building the finite-volume measure $\mu_{a,L}$	7-01
ultraviolet-limit step	Table row	The ultraviolet limit $a \to 0$ and uniform estimates	7-01
infinite-volume step	Table row	The infinite-volume limit $L \to \infty$ and clustering	7-01
axiom step	Table row	The limit satisfies the OS axioms (reflection positivity)	7-01
uniform-gap step	Table row	A mass gap uniform in the volume	7-01
level of perturbation theory	Table row	Computation as a power series in the coupling (perturbation theory). The premise of the levels of control	7-07
level of uniform estimates	Table row	The level at which estimates uniform in the lattice spacing are obtained (Bałaban's main four-dimensional theorem reaches this far, with assumptions)	7-07
level of subsequential limits	Table row	The level at which, from uniform estimates, a limit can be taken along a subsequence of a sequence of finer lattice spacings	7-07
level of uniqueness	Table row	The level at which the limit is determined uniquely, independently of the choice of subsequence	7-07
level of symmetry and reflection positivity	Table row	The level at which the limiting theory has Euclidean symmetry and reflection positivity	7-07
level of non-triviality	Table row	The level at which the limit is not a free (Gaussian) field. The problem statement further asks that the short-distance behaviour agree with the predictions of asymptotic freedom	7-07

Name	Kind	What it refers to	Introducing chapter
level of infinite volume	Table row	The level at which the limiting theory with infinite volume exists	7-07
level of a uniform gap	Table row	The level at which there is a mass gap uniform in the volume	7-07
obstacle of the infrared parameter	Table row	The obstacle that at long distances the coupling is strong and there is no small parameter to expand in	7-11
obstacle invisible to perturbation theory	Table row	The obstacle that there are contributions (of the form e^{−c/α}) that appear at no order of the power series in the coupling	7-11
obstacle of non-triviality	Table row	The obstacle that it must be shown separately that the limit does not collapse to a free field (non-triviality)	7-11
obstacle of the curved state space	Table row	The obstacle that the space of fields (the space of gauge orbits) is curved, and the signed sum of gauge fixing becomes 0 (0/0 remains)	7-11
obstacle of positive regularisation	Table row	The obstacle that, apart from the lattice, there are few regularisations that preserve reflection positivity	7-11
obstacle of infinitely many degrees of freedom	Table row	The structure of infinitely many degrees of freedom: the obstacle that as the volume grows, the overlap of different vacua tends to 0	7-11
Marginal-Triviality Theorem and others (labels of Aizenman–Duminil-Copin) (short: Marginal-Triviality Theorem and others)	Name from the literature	A set of results of Aizenman–Duminil-Copin: the scaling limits of four-dimensional ϕ^4 and Ising are Gaussian (Theorem 1.2), an improved tree-diagram upper bound (Theorem 1.3), an error estimate for the characteristic function (Proposition 1.4)	7-08

Part 15 Record of the attempts at ultraviolet control

Name	Kind	What it refers to	Introducing chapter
ultraviolet control	Condition	Keeping, at every step from the lattice spacing a to the scale L_0 where the coupling is small, the effective density in the same step-independent form, with the three conditions of small-field analyticity, large-field rarity and the coupling staying small	15-00
small-field analyticity condition	Condition	The localised correction E_k(X,·) of each step is analytic in a complex neighbourhood of radius r of the background-field variables, with $\lvert E_k\rvert \le C\, e^{-\kappa d}$ (constants independent of the step k and the lattice spacing a)	15-00

Name	Kind	What it refers to	Introducing chapter
large-field rarity condition	Condition	The large-field factor satisfies $\|R_k(\Omega^c)\| \le e^{-c\, p(g_k)^2\, \|\Omega^c\|}$ (constants independent of the step and the lattice spacing)	15-00
condition that the coupling stays small (short: the coupling stays small)	Condition	The sequence of effective couplings $g_0,\dots,g_K$ all lies in $(0,\gamma]$ (γ independent of the step and the lattice spacing)	15-00
three ultraviolet conditions (twisted-box version)	Condition	Ultraviolet control stated up to the last step of the twisted M^4 lattice, with constants depending only on M	15-02
class-closure condition	Condition	The number of parameters is finite independently of the step, and the main part of the one-step map returns into the family. The class is taken to be the one-parameter class into which the one-step map pulls (a fixed form of coupling $B + O(1/B)$)	15-03
coupling-flow remainder condition	Condition	The β component of Φ satisfies $\beta_{k+1} = \beta_k - 8\, b_0 \ln L + O(1/\beta_k)$ (the net one-step bracket of the coupling written in the variable β)	15-03
shape remainder condition	Condition	$\| r_{\theta_k}[\rho_k] \|_{G_{k+1}} \le (1+\eta_k)\, 2^{\|B\|}\, \| \rho_k \|_{G_k}$ with $\sum \eta_k < \infty$, and the deviation from the class ρ_k stays uniformly bounded in the step	15-03
definition and continuity of the effective coupling (short: continuity of the coupling)	Condition	On $D_k = \{g_0 : g_0,\dots,g_k \in (0,\gamma_1]\}$, the map $g_0 \mapsto g_{k+1}(g_0)$ is continuous	15-03
net one-step bracket of the coupling	Condition	$1/g_{k+1}^2 = 1/g_k^2 - b + r_k$, $\|r_k\| \le C\, g_k^2$ (the change of the coupling at each step deviates from the one-loop value b only by $O(g_k^2)$)	15-03
small-field contribution to the coupling	Condition	The part of the remainder of the net one-step bracket of the coupling conditioned on small fields, $r^{small} = b(\rho_S) - \beta_{pred}$ (local part + Taylor truncation + small-field cut + explicit quadratic inter-block term)	15-03
large-field contribution to the coupling	Condition	The large-field part of the remainder of the net one-step bracket of the coupling, $r^{large} = b(\rho) - b(\rho_S)$	15-08
Coupling-Smallness Induction Lemma	Lemma	Under the net one-step bracket of the coupling and the definition and continuity of the effective coupling, for every $\gamma_0 \in (0,\gamma_1)$ there is a bare coupling g_0 with $g_0 < g_1 < \dots < g_n = \gamma_0$ (monotonicity and intermediate values; the condition that the coupling stays small is split into these two)	15-03

Name	Kind	What it refers to	Introducing chapter
Background–Fluctuation Decomposition Lemma	Lemma	In SU, $\| U_{\partial p} - 1 \| \le \| W_{\partial p} - 1 \| + \sum \| A_{l_i} \|$, and its squared version (decomposition into background × fluctuation, U = W e^A)	15-07
Splitting of the Weight	Corollary	$G_c(X, W e^A) \le G_c(X, W)^2 \exp(48c \sum \| A_l \|^2)$	15-07
Lemma on One Step with Background-Independent Fluctuations	Lemma	If the law of the fluctuations ν does not depend on the background and satisfies the background-independence condition for fluctuations, the scale-contraction condition and the exponential-moment condition for fluctuations, then $\| \theta_\nu K \|_{G_{h+}} \le 2^{|B|} \| K \|_{G_h}$ (one step only; no loss in exchanging the expectation and the seminorm)	15-07
background-independence condition for fluctuations	Condition	ν does not depend on the background W	15-07
scale-contraction condition	Condition	$h_+^2 \le h^2/2$ (the scale of the next step is at most half)	15-07
exponential-moment condition for fluctuations	Condition	$M_\nu(8qc/h^2, X) \le 2^{|B|}$ (exponential moment of the fluctuations)	15-07
One-Link Heat-Kernel Comparison Lemma	Lemma	Bounds the single-link Wilson law μ^b of SU(2) by $K(b,\theta_0)$ times the heat kernel $p_{1/b'}$ inside the angle bound θ_0, and by a rare tail outside it	15-07
One-Link One-Step Fluctuation Lemma	Lemma	The chain of the Lemma on One Step with Background-Independent Fluctuations also goes through for a single Wilson link, and the loss splits into K (on the side of the closure under the fluctuation expectation) and 1+η (the movement of the coupling)	15-07
recentring condition	Condition	A condition that assumes the angle of the background plaquette is at most $\theta_0 < \pi/2$, together with the geometry of recentring (the Taylor radius of the map to the direction of the sum of staples)	15-07
group version of the exponential moment of fluctuations	Condition	For all small-field backgrounds W and $0<t\le t_0$, $E_{\mu_{k,W}} \exp((48tc/h_k^2) \sum \| A_l \|^2) \le 2^{|B|}$ (for the true fluctuation measure)	15-07
background-regularity condition	Condition	For W in the image of the background map, $G_k(X,W)^{2t} \le G_{k+1}(X,W)$ (fine curvature is bounded by coarse curvature)	15-07
group version of exchanging the expectation and the seminorm	Condition	$\| E \theta_W F \|_{T_W(h_{k+1})} \le (1+\eta_k) E \| F \|_{T_{We^A}(h_k)}$, $\sum \eta_k < \infty$ (the group version of exchanging the expectation and the seminorm)	15-07

Name	Kind	What it refers to	Introducing chapter
product factorisation ((i) of BBS) (short: product factorisation)	Name from the literature	The first of the four properties of the weight (regulator) of BBS Appendix A.6.2: on disjoint sets the weight splits into a product	15-01
submultiplicativity in sums of fields ((ii) of BBS) (short: submultiplicativity in sums of fields)	Name from the literature	The second of the four properties of the weight: for sums of fields, $\tilde{G}(X,\phi+\phi') \leq \tilde{G}^2(X,\phi)\tilde{G}^2(X,\phi')$	15-01
closure under the fluctuation expectation ((iii) of BBS) (short: closure under the fluctuation expectation)	Name from the literature	The third of the four properties of the weight: taking the expectation over fluctuations, the weight closes with the weight of the next step	15-01
monotonicity to the next step ((iv) of BBS) (short: monotonicity to the next step)	Name from the literature	The fourth of the four properties of the weight: the weight of the current step is bounded by the weight of the next step	15-01
One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) (short: One-Step Bound on the Activity Norm)	Name from the literature	If the weight satisfies submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step, then $\| E_+\theta K \|_{\tilde{G}_+} \leq 2^{\|B\|} \| K \|_{\tilde{G}}$ (for flat fields the activity norm stays within a factor $2^{\|B\|}$ over one step)	15-01
inequality exchanging the expectation and the seminorm ((A.6.5) and others of BBS) (short: inequality exchanging the expectation and the seminorm)	Name from the literature	Formulas of BBS Appendix A.6: the inequality exchanging the fluctuation expectation and the seminorm (together with the definition of the norm and how the properties of the weight are applied)	15-02
Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989) (short: Conditional Ultraviolet Stability Theorem)	Name from the literature	If one assumes that the effective coupling stays small, then the form of the effective density and the estimates of small-field analyticity and large-field rarity hold at all steps (four dimensions, finite volume)	15-02
basic inequality (unpublished) (Theorem 2 of Bałaban 1989) (short: basic inequality (unpublished))	Name from the literature	The inequality for removing the assumption of the Conditional Ultraviolet Stability Theorem (that the coupling stays small). Unpublished	15-02
Orbit-Direction Flatness Proposition	Proposition	At a minimum point of a gauge-invariant smooth function, $\mathrm{Hess}\, V(x_0)(\xi,\xi) = 0$ for tangent vectors $\xi \neq 0$ of the orbit	15-01
Proposition on a Class with the Background as a Parameter	Proposition	In the class $C_2 = (B,q,a)$ that has the logarithm a of the background holonomy as a parameter, the background-dependent remainder drops to $\theta^0\beta^{-3/2}$, $\theta^1\beta^{-1}$, and the budget is uniform in the number of steps	15-07

Name	Kind	What it refers to	Introducing chapter
Shape-Remainder Theorem for Four Plaquettes in a Row (short: Four-in-a-Row Theorem)	Theorem	In the projected model of four plaquettes in a row (SU(2), diagonal definition, ψ^8 projection, $B \geq 32$, O(1) box), $r_n \leq \lambda^n r_0 + C/((5-\lambda)B_n)$ with $C = 0.0108$ (independent of the number of steps)	15-06
One-Step Contraction of the ℓ^4 Cone	Corollary	The four-dimensional ℓ^4 cone contracts by a factor of at most 0.823 in one step (normalisation $\beta'_0 = 1$, row-weighted interpolation)	15-05
Mixing-Limit Lemma	Lemma	In the volume limit, $\sup \lambda = 4 \sup \mu_{loc}$ (the limit of mixing between localised directions and the coarsest direction)	15-05
Floor Lemma for the Stein Route	Lemma	The row-weighted Stein route cannot go below $\Sigma\, s\, \nu_s^2/16 \geq 0.452 > 1/4$, for any local representative and any weights	15-05
Conditional-Mean Covariance Lemma	Lemma	One step at tree level is a Gaussian conditional mean, and the inverse of the coarse action (the covariance of the blocked flux) iterates linearly	15-05
Heat-Kernel Gaussian Upper-Bound Lemma	Lemma	For the SU(2) heat kernel p_t and non-decreasing $g \geq 0$, $E\, g(\theta) \leq e^{t/2}\, E\, g(\lvert Z\rvert)$, $Z \sim N(0, tI_3)$	15-08
Heat-Kernel Fluctuation Rarity Lemma	Lemma	Under the edgewise independent heat-kernel condition through the background-independence condition for fluctuations, the probability that a link exceeds the threshold h and the exponential moment are bounded in a closed form independent of the background (with heat-kernel fluctuations, the group version of the exponential moment of fluctuations holds)	15-08
Large-Field Remainder Bound Lemma	Lemma	Splitting the remainder as $r = r^{small} + r^{large}$, the Lipschitz constant of the read-out b with respect to total variation gives $\lvert r^{large}\rvert \leq (4/3)(b_S^2/E_S)\, P_L$	15-08
edgewise independent heat-kernel condition	Condition	The condition that the law of the fluctuations is an edgewise independent heat kernel (a premise of the Heat-Kernel Fluctuation Rarity Lemma and others)	15-08
single-step condition	Condition	The condition restricting the claim to a single step of renormalisation (multistep induction is not included)	15-08
analyticity condition at complex backgrounds	Condition	Analyticity at complex backgrounds and an upper bound independent of β (a Cauchy-type estimate of the Taylor coefficients of the small-field effective action)	15-08

Name	Kind	What it refers to	Introducing chapter
definition of the inter-block term	Condition	The definition of the explicit quadratic inter-block term W^gauge (algebra)	15-08
local cancellation	Condition	Local cancellation (the Loc identity)	15-08
cubic-remainder estimate	Condition	The cubic estimate (the remainder is $O(g^3)$, uniformly in complex backgrounds and in the volume)	15-08
Gaussian comparison condition for the partition function	Condition	e^{−CM⁴} ≤ Z_n/Z_n^G ≤ e^{CM⁴} (the ratio to the sum of Gaussian integrals around the isolated flat connections)	15-02
observable-source condition	Condition	The statement holds with the same constants even when a source of a smooth observable (strength ≤ s_0) with support larger than the scale L_0/M is inserted	15-02
ultraviolet statement for a small twisted four-dimensional torus (short: twisted-torus ultraviolet statement)	Statement	On a small twisted M^4 lattice, with a suitable choice of bare coupling, the three ultraviolet conditions (twisted-box version), the Gaussian comparison condition for the partition function and the observable-source condition hold (proposal, unproved)	15-02
row of the remainder including large fields	Table row	One of the four rows of 15-08 that do not close: the one-step remainder of the coupling including large fields (showing the net one-step bracket of the coupling including large fields)	15-08
row of background-dependent absorption and many steps	Table row	The core of the four rows that do not close: the row that absorbs the background-dependent loss η into a parameter of the class and passes it through multistep induction	15-08
row of the boundary of gauge re-fixing	Table row	One of the four rows that do not close: the handling of the boundary when the gauge fixing is re-fixed at each step	15-08
covariance-flow fixed-point condition	Condition	The flow of the covariance of the blocked field has a scale-invariant fixed point	15-03
cone-contraction condition at the fixed point	Condition	At the fixed point of the covariance flow, the contraction rate of the cone is less than 1 (sup μ_∞ < 1)	15-03
ultraviolet questions	Question	The thirty-one questions left by the chapters of Part 15 (Table 1 of 15-09). All touch on parts of the row of ultraviolet control	15-09
questions of the systematic attempts	Question	The sixteen questions left by 15-10 (the systematic attempts) (Table 15 of 15-10)	15-10

Name	Kind	What it refers to	Introducing chapter
contraction-map window for the fixed point of the class	Name	When the fixed point of the class-closure condition is stated in the form of a contraction map, the range of β in which the contraction map opens (from β of about 28–54, with $\tau = 0.695$, $a = 4.335$, $c_0 = 0.02$–0.04)	15-03
question "transverse contraction on a one-parameter class" (short: transverse contraction on a one-parameter class)	Question	Whether the one-step map contracts transversally on the one-parameter class it pulls into (whether λ and the sources stay finite, whether the constants are independent of the step, whether a weight on the sign of the centre is needed)	15-09
question "minimal model of the remainder including large fields" (short: minimal model of the remainder including large fields)	Question	Whether, in the minimal model (hierarchical SU(2)) in which the net one-step bracket of the coupling can be shown including large fields, the constrained absorption of one plaquette can be restated at the true critical point	15-09
question "smallest adjoint eigenvalue on the twisted lattice" (short: smallest adjoint eigenvalue on the twisted lattice)	Question	Whether, on the twisted M^4 lattice, the smallest eigenvalue of the adjoint fluctuation operator around the isolated flat connections is at least c/M^2	15-09
question "extending the tube of the hierarchical model" (short: extending the tube of the hierarchical model)	Question	Whether the tube and the attracting curve of the hierarchical model can be extended to $\beta > 64$ (the seam between the asymptotic estimates and interval arithmetic)	15-09
question "interval arithmetic for curvature outside the layer" (short: interval arithmetic for curvature outside the layer)	Question	Whether, by confirming the lower bound on the curvature outside the layer and the weight of the layer with interval arithmetic, the flow coefficient on the curve $c^* = 1/60$ can be separated from 1/48	15-09
question "lateral constancy of the coarse-plaquette source" (short: lateral constancy of the coarse-plaquette source)	Question	Whether, along a sequence of coarse-plaquette sides L, source $\times L^4$ approaches a constant value even when the lateral extent is widened	15-09
question "local representative of the tree-level network" (short: local representative of the tree-level network)	Question	Whether, in the tree-level network with the coarse plaquette fixed, a local representative can be made and an upper bound uniform in the volume obtained	15-09
question "one-step correction for four plaquettes in a row" (short: one-step correction for four plaquettes in a row)	Question	Whether the O(1/B) one-step correction for four plaquettes in a row can be derived on paper, giving the constant C of the one-step inequality of the shape remainder condition	15-09

Name	Kind	What it refers to	Introducing chapter
question "uniqueness of the class for other loop shapes" (short: uniqueness of the class for other loop shapes)	Question	Whether the class pulled into is determined uniquely even when loop shapes other than four in a row (2×2, L-shaped, mixed quartic) are included	15-09
question "splitting the density of the joint law" (short: splitting the density of the joint law)	Question	Whether, for four in a row, the joint law of two plaquettes can be split into a product of local couplings × a density, giving the lowest order in θ of the density and the isotropy of the quadratic form	15-09
question "minimal multistep statement" (short: minimal multistep statement)	Question	Whether a minimal multistep statement can be written in a hierarchical model or a two-dimensional model (the background-regularity condition and contraction of many-body norms)	15-09
question "Lipschitz constant of the read-out" (short: Lipschitz constant of the read-out)	Question	Deriving on paper the Lipschitz constant of the read-out used in the large-field remainder estimate, and measuring the difference in total variation of the densities with and without large fields	15-09
question "shift of the coupling when a source is added" (short: shift of the coupling when a source is added)	Question	Whether, when a source s Re W is added to four plaquettes in a row, the shift of the coupling is linear in s	15-09
question "parity of critical points" (short: parity of critical points)	Question	Whether the parity of critical points holds for four in a row (paired with the question of splitting the density of the joint law)	15-09
question "one-loop Hessian and ghosts" (short: one-loop Hessian and ghosts)	Question	Whether the one-loop Hessian of a block with factor 2 can be split and the F^2 coefficient of the ghosts measured (together with formalising the direct-product decomposition of the tree gauge)	15-09
question "smallest system that is not zero-sum" (short: smallest system that is not zero-sum)	Question	Whether, in the smallest system that is not zero-sum (two sets of four in a row side by side, sharing the link between them), the breaking of the local cancellation of the second-order contribution to the marginal coupling can be turned into a number	15-09
question "odd BCH orders on a 2×2 slab" (short: odd BCH orders on a 2×2 slab)	Question	Deriving, for the central plaquette of a 2×2 slab, the cyclically symmetrised BCH coefficients of odd order, and comparing them with the two-link values	15-09
question "class with the holonomy as a parameter" (short: class with the holonomy as a parameter)	Question	Whether, in a class having the logarithm of the background holonomy as a parameter, the background-dependent part moves exactly into the flow of that parameter	15-09

Name	Kind	What it refers to	Introducing chapter
question "radius of recentring" (short: radius of recentring)	Question	Whether, in the recentring of the One-Link One-Step Fluctuation Lemma, the Taylor radius can be made explicit and the loss coefficient made a single number	15-09
question "direct upper bound for the four-dimensional ℓ^{4}" (short: direct upper bound for the four-dimensional ℓ^{4})	Question	Bounding the contraction of the four-dimensional ℓ^4 cone directly (a finite network for the near part, Riesz–Thorin only for the tail), and seeing whether sup $\mu < 1$	15-09
question "two definitions of Loc" (short: two definitions of Loc)	Question	Whether, when the localisation Loc is defined in two ways, by row sums (zero momentum) and by matrix components, the quadratic inter-block terms agree (and which cut the original source uses)	15-09
question "three-dimensional non-local weights" (short: three-dimensional non-local weights)	Question	Whether, in three dimensions without boundary, the one-loop non-local weights in the coordinates of coarse links + global labels decay exponentially	15-09
question "second step of the multi-loop class" (short: second step of the multi-loop class)	Question	The second step of the multi-loop class: whether these are coordinates in which the triple product of angles κ is preserved at each step, and whether the same coordinates suffice in four dimensions with 2^4	15-09
question "gauge agreement on non-abelian backgrounds" (short: gauge agreement on non-abelian backgrounds)	Question	Measuring, on non-abelian backgrounds (curvature including commutators), the agreement of the two gauges and the coefficient of the ghosts	15-09
question "normalisation factors by direction" (short: normalisation factors by direction)	Question	Whether the normalisation factor by direction, sup $\lambda \leq 3.29$, can be brought below 1 with symbols weighted by momentum band	15-09
question "class with the coupling as a parameter" (short: class with the coupling as a parameter)	Question	Whether, giving the class the coupling as a parameter, the movement of the coupling of a single Wilson link moves into the update of the parameter	15-09
question "fixed point of the four-dimensional class ratio" (short: fixed point of the four-dimensional class ratio)	Question	Whether the value at the fixed point of the ratio coordinate of the four-dimensional class can be derived directly from a semi-infinite problem near the corner	15-09
question "upper bound on the fixed point of the lateral average" (short: upper bound on the fixed point of the lateral average)	Question	Whether, at the fixed point of lateral-average blocking, an upper bound on sup μ_∞ below 1 can be obtained by a route other than Stein	15-09

Name	Kind	What it refers to	Introducing chapter
question "quartic width of the three-body tree" (short: quartic width of the three-body tree)	Question	Measuring, with four-dimensional coarse links + global labels, the rate of dependence of the three-body tree term on the quartic width	15-09
question "one-loop coefficient of the long-wavelength network" (short: one-loop coefficient of the long-wavelength network)	Question	Whether, with lateral-average blocking or a large factor, the dependence on the factor of the one-loop coefficient of the long-wavelength network approaches the one-loop value	15-09
question "checking the whole range of the recentred tube" (short: checking the whole range of the recentred tube)	Question	Whether, passing the recentred tube through the whole of $\beta \in [15.26, 128]$ in intervals of width 0.02, the curve entering the tube and the enclosure of the flow coefficient c* can be made unconditional	15-09
question "alignment of rows in the worst direction" (short: alignment of rows in the worst direction)	Question	Whether, by estimating the lower bound of the alignment of rows in the worst direction with the kernel in the limit of steps, the loss from discarding signs can be bounded uniformly at all steps	15-10
question "periodic shift of the Stein maximiser" (short: periodic shift of the Stein maximiser)	Question	Where the amount by which the maximiser of the Stein upper bound on a large torus shifts away from period 2 comes from (upper bound 10.2 against a loss of 6.4)	15-10
question "cancellation on the side of the true operator" (short: cancellation on the side of the true operator)	Question	Once a quantity measuring the cancellation on the side of the true operator is fixed (for example, the ratio of $\|T^{k}\|$ to $\||T|^{k}\|$ divided by the minimum within the similarity class of signs), what can be said to cause the regularity of the loss that grows only slowly with the step	15-10
question "finite types and aggregation of returns" (short: finite types and aggregation of returns)	Question	Whether, aggregating the finite types and the remaining returns into two-by-two, $bc < (1-a)(1-d)$ holds for the recorded matrix	15-10
question "rereading with Schur weights" (short: rereading with Schur weights)	Question	What becomes of the one-step contraction constant when the record of the W coordinates (measured in ℓ^2) is reread with the weights of Schur's test	15-10
question "supremum of constrained sum rules" (short: supremum of constrained sum rules)	Question	Whether, for all $k \le \operatorname{ord}^{V}(F)$, a local representative with $\operatorname{ord}(K) \ge k$ can be taken with finite range	15-10
question "canonical choice of degrees of freedom" (short: canonical choice of degrees of freedom)	Question	Whether a representative minimising a Stein-type norm exists, is unique, and is local	15-10

Name	Kind	What it refers to	Introducing chapter
question "consistency of composing representatives across steps" (short: consistency of composing representatives across steps)	Question	Whether the output of the representative at step k lies in the constrained subspace of step $k+1$ (if not, the composition of representatives depends on the choice)	15-10
question "decay of the Moore–Penrose kernel" (short: decay of the Moore–Penrose kernel)	Question	How fast the kernel of the Moore–Penrose representative (the minimal-norm representative) decays	15-10
question "zero modes of the symmetric representative" (short: zero modes of the symmetric representative)	Question	Why the zero modes of the symmetric representative on a finite torus appear to approach $U/4$ faster than exponentially	15-10
question "band of triple valuations" (short: band of triple valuations)	Question	Whether, with triple valuations adding a third valuation (the exponent of the transseries), the difference between the bands of U(1) and SU(2) can be distinguished	15-10
question "ties in the four-dimensional series" (short: ties in the four-dimensional series)	Question	When the ties of the errors are found for the genuine four-dimensional series (higher orders at strong and weak coupling), how they overlap with the crossover band	15-10
question "skeleton of the space of two couplings" (short: skeleton of the space of two couplings)	Question	Whether, in the space of the two couplings, fundamental and adjoint, the skeleton of the tie set becomes a non-trivial polyhedral complex	15-10
question "motion of the lower edge of the window" (short: motion of the lower edge of the window)	Question	How the lower edge of the window moves with the number of steps along a sequence of blocking maps (whether the true value goes below 1 for one step with $3.5 \le p \le 5$)	15-10
question "divergence of the number of bundles" (short: divergence of the number of bundles)	Question	How the number of bundles diverges as p $\downarrow$ 2 (of the c/(p−2) type, or with a critical exponent fixed by the geometry of the lattice)	15-10
question "Schur threshold and Stein window" (short: Schur threshold and Stein window)	Question	Where the gap between the exponent of the Schur threshold and the lower edge of the Stein window comes from, and how far p^* can be lowered with weighted Schur (positive test functions)	15-10

Part 16 Where things stand, and the questions that remain

Name	Kind	What it refers to	Introducing chapter
seven stages a proof must contain	Table row	The seven stages a proof of the mass gap must pass through (vessel, ultraviolet, small-T^2 gap, small-T^2 confinement, band, R^4, general groups)	16-09

Name	Kind	What it refers to	Introducing chapter
vessel stage	Table row	Lattice, reflection positivity, block observables, compactness. The foundational stage that derives a gap of the transfer matrix from decay on a dense set	16-09
ultraviolet-control stage	Table row	The stage of multiscale analysis uniform in a (locality of the effective action, rarity of large fields, flow of the coupling), from the lattice spacing a to the scale L	16-09
small-T^2 gap stage	Table row	The stage that removes zero modes with twisted boundary conditions and shows, uniformly in a, a gap of the transfer matrix $\geq c/L$ on a small T^2	16-09
small-T^2 confinement stage	Table row	The stage that reduces confinement on a small T^2, by container estimates and insertion estimates, to "insertion of one vortex" and "independence between regions"	16-09
band stage	Table row	That the gap holds with the same constant even when T^2 is widened. The stage that crosses the intermediate scale $L \sim 1/\Lambda$ non-perturbatively	16-09
R^4 stage	Table row	The stage of the infinite-volume limit, the OS axioms, recovery of rotation invariance and non-triviality	16-09
general-group stage	Table row	The stage that extends to SU(N) and to groups without centre (such as G_2)	16-09
seam of the premise of existence	Table row	The seam where physics posits that the theory exists and computes quantities, while mathematics must build a limit (not a subsequence) satisfying the axioms	16-05
seam of "there is one scale"	Table row	The seam where physics regards both the mass and the string tension as constant multiples of a single scale Λ, while mathematics must show this holds uniformly in β	16-05
seam of the different shores	Table row	The seam where physics computes on shores where computation works (small coupling, small β, large N), while mathematics must control all scales at finite N	16-05
seam of finite and infinite	Table row	The seam where physics measures in finite boxes and extrapolates, while mathematics must show a gap uniform in the volume	16-05
seam of spacetime and space	Table row	The seam where physical intuition lives on spacetime (Minkowski), while mathematics returns from the Euclidean measure by reflection positivity	16-05
seam of the word "particle"	Table row	The seam between the physics statement "the gap is the mass of the lightest glueball" and the mathematical spectral gap (an isolated one-particle state is an extension problem)	16-05
seam of extra dimensions	Table row	The seam where physics produces the gap from the geometry of extra dimensions, while mathematics needs the limit shrinking the extra dimensions to 0 and the exchange of limits	16-05

Name	Kind	What it refers to	Introducing chapter
type in which a premise becomes a goal	Table row	The type of seam in which a premise of physics becomes a goal of mathematics (premise of existence; there is one scale)	16-05
type of limits and uniformity	Table row	The type of seam in which physics computes at particular values of parameters (shores), while mathematics needs limits to the values of the problem statement and uniform estimates (different shores; finite and infinite; extra dimensions)	16-05
type in which the same word points to different things	Table row	The type of seam in which the same word points to different things in physics and mathematics (spacetime and space; the word "particle")	16-05
task of the T^4 construction	Task	On T^4, the expectations of gauge-invariant quantities converge as $a \to 0$ along the whole sequence	16-07
task toward infinite volume	Task	The limit $L \to \infty$ exists along the whole sequence	16-07
task of a uniform gap	Task	$\inf_L \Delta_L > 0$	16-07
task of the remaining axioms	Task	Recovery of rotation symmetry, and the regularity needed for reconstruction	16-07
task of non-triviality	Task	The limit is not Gaussian	16-07
task of local fields and short distances	Task	Local fields, and agreement with asymptotic freedom at short distances	16-07
task of strong decay	Task	Uniform exponential decay under arbitrary boundary conditions	16-07
task of one scale	Task	$c\sigma \le m^2 \le C\sigma$ for $\beta \ge \beta_0$	16-07
task of the three-dimensional measure	Task	Construction of the measure on T^3	16-07
rule of existence	Table row	The rule that the existence of the theory on R^4 (the requirement of existence) follows from the task of the T^4 construction and the task toward infinite volume (a definition)	16-07
rule of the axioms	Table row	The rule that the requirement of the axioms follows, by OS reconstruction, from the existence of the theory and the task of the remaining axioms (a theorem)	16-07
rule of the gap	Table row	The rule that the requirement of the gap follows from the existence of the theory and the task of a uniform gap (a theorem: a uniform gap survives in the limit)	16-07
rule of non-triviality	Table row	The rule that the requirement of non-triviality follows from the task of non-triviality and the task of local fields and short distances (a definition)	16-07
rule of the problem statement	Table row	The rule that the whole problem statement for the group G follows from the three requirements of the axioms, non-triviality and the gap (a definition)	16-07

Name	Kind	What it refers to	Introducing chapter
rule from decay to the area law	Table row	The rule that the lattice area law follows from the task of strong decay (the Theorem from Uniform Decay to the Area Law of Chatterjee 2021)	16-07
rule of the string tension	Table row	The rule that the requirement of the string tension follows from the continuum string-tension problem (a definition)	16-07
rule of the problem setters' route	Table row	The route indicated by the problem setters (Jaffe–Witten), in which the task of a uniform gap opens the task toward infinite volume	16-07
rule from the string tension to the gap	Table row	The physical expectation that the requirement of the gap follows from the existence of the theory, the requirement of the string tension and the task of one scale (there is one scale)	16-07
rule from the gap to the string tension	Table row	The physical expectation that the requirement of the string tension follows from the existence of the theory, the requirement of the gap and the task of one scale (the reverse of the rule from the string tension to the gap)	16-07
requirement of existence	Name	That the theory on R^4 exists (one of the requirements of the problem statement)	16-07
requirement of the axioms	Name	That the theory satisfies axioms at least as strong as the Wightman and Osterwalder–Schrader axioms (one of the requirements of the problem statement)	16-07
requirement of non-triviality	Name	That the theory is non-trivial, has local fields, and agrees with asymptotic freedom at short distances (one of the requirements of the problem statement)	16-07
requirement of the gap	Name	That there is a mass gap $\Delta > 0$, and $m < \infty$ (one of the requirements of the problem statement)	16-07
requirement of the string tension	Name	That the continuum string tension d is not 0 (confinement; outside the requirements of the problem statement)	16-07
requirement of the problem statement	Name	The whole problem statement for the group G: the four requirements of existence, axioms, non-triviality and the gap taken together	16-07
two-way split of the core	Name	Viewing the core of the band as two jobs: "the ultraviolet extension" and "the statement of the last interval to strong coupling"	16-11 § 01
ultraviolet extension	Name	The job of extending the control of the published construction from the threshold to a coupling β_* of order one, while keeping the sum of the errors over the steps finite	16-11 § 01
statement of the last interval to strong coupling	Statement	A non-perturbative statement over a small number of steps, from β_* to the strong-coupling window (different from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06)	16-11 § 01

Name	Kind	What it refers to	Introducing chapter
lossy chain	Name	Renormalisation comparisons allowing a loss of a factor K at each step, joined over the number of steps. The rate is uniform, but the prefactor grows as $\xi^{\log_2 K}$	16-11 § 02
facts on the energy side	Name	Facts about the action of configurations (existence of twisted vacua, stabiliser group ± 1, positivity of the action of the box with two twists, upper and lower bounds on S_N)	16-11 § 03
shortfall on the entropy side	Name	That this book lacks the estimate the band needs, which would count vortex fluctuations and the number of placements from below, uniformly in the scale	16-11 § 03
mixing statement for boxes of physical size	Statement	If the side of the box is taken to be the physical length C/Λ, the total variation of the law of the gauge-invariant observables inside, when the frame is changed, is smaller than a threshold (uniformly in β). A restatement of the mass gap in a strong form	16-11 § 04
seven test models	Name	The models against which arguments for the band are tested: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups, the adjoint Higgs limit, the line and endpoint of the SU(2) mixed action, the bulk transition of SU(N), and G_2	16-11 § 04, 26-12 § 02
three classes of value levels	Name	The weak-coupling behaviour of the quantity for a symmetric box with one twist: proportional to β for U(1), constant for frozen finite groups, of the $(3/2) \ln \beta$ type for SU(2)	16-11 § 04
control of the continuum limit of twisted boxes	Task	A comparison of scales with a finite sum of errors, for quantities determined by the measure, in twisted (framed) boxes	16-11 § 06
coupling of the laws of gauge-invariant observables	Task	A way of coupling the conditional laws of gauge-invariant functions, and a finite-box criterion on top of it	16-11 § 06
method for computing lower bounds on the entropy side	Task	Insertions adapted to the configuration, with relative entropy tending to 0	16-11 § 06
ordering of actions in the twisted sector	Task	An inequality saying that ΔF increases in the plaquette couplings, or an ordering of actions replacing it	16-11 § 06
substitute for the vortex free energy for groups without centre	Task	A quantity that plays the role of the vortex free energy for groups without centre, such as G_2	16-11 § 06
termwise sign inequality	Question	$\partial \Delta F / \partial \beta_p \geq 0$ for the plaquette-wise couplings β_p	16-11 § 04, 26-12 § 05

Part 18 A dictionary of two languages

Name	Kind	What it refers to	Introducing chapter
entry "path integral (quantum mechanics)"	Dictionary entry	Path integral (quantum mechanics) ⟷ Feynman–Kac formula	18-02
entry "functional integral"	Dictionary entry	Functional integral ⟷ Euclidean probability measure	18-02
entry "space of states"	Dictionary entry	Space of states ⟷ reconstructed Hilbert space	18-02
entry "Wick rotation"	Dictionary entry	Wick rotation ⟷ OS reconstruction theorem	18-02
entry "Green functions"	Dictionary entry	Green functions ⟷ Wightman and Schwinger functions	18-02
entry "Källén–Lehmann representation"	Dictionary entry	Källén–Lehmann representation ⟷ spectral measure	18-02
entry "vacuum and uniqueness"	Dictionary entry	Vacuum and uniqueness ⟷ Ω, clustering	18-02
entry "Hamiltonian"	Dictionary entry	Hamiltonian ⟷ logarithm of the transfer matrix	18-02
entry "one-particle state"	Dictionary entry	One-particle state ⟷ isolated eigenvalue of the mass operator	18-02
entry "S-matrix"	Dictionary entry	S-matrix ⟷ Haag–Ruelle scattering theory	18-02
entry "positive norm"	Dictionary entry	Positive norm ⟷ reflection positivity	18-02
entry "unitarity of the S-matrix"	Dictionary entry	Unitarity of the S-matrix ⟷ positive norm + asymptotic completeness	18-02
entry "spacetime and lattice"	Dictionary entry	Spacetime and lattice ⟷ $a\mathbb{Z}^4 \to \mathbb{R}^4$	18-02
entry "regularisation"	Dictionary entry	Regularisation ⟷ family of measures with lattice spacing a	18-02
entry "renormalisability"	Dictionary entry	Renormalisability ⟷ BPHZ, Epstein–Glaser	18-02
entry "field φ(x)"	Dictionary entry	Field $\phi(x)$ ⟷ operator-valued distribution	18-02
entry "gauge fixing"	Dictionary entry	Gauge fixing ⟷ section into the orbit space	18-02
entry "physical states"	Dictionary entry	Physical states ⟷ reconstruction from gauge-invariant quantities	18-02
entry "local symmetry does not break"	Dictionary entry	Local symmetry does not break ⟷ Elitzur's theorem	18-02
entry "mass gap"	Dictionary entry	Mass gap ⟷ spectral gap, uniform decay	18-03

Name	Kind	What it refers to	Introducing chapter
entry "correlation length ξ"	Dictionary entry	Correlation length ξ ⟷ inverse of the gap of the transfer matrix	18-03
entry "gap in physical units"	Dictionary entry	Gap in physical units ⟷ uniform lower bound in lattice units	18-03
entry "V(R)"	Dictionary entry	$V(R)$ ⟷ decay rate of Wilson loops	18-03
entry "string tension σ"	Dictionary entry	String tension σ ⟷ coefficient of the area law, d of Chatterjee's Problem 5.2 (the continuum string-tension problem)	18-03
entry "confinement"	Dictionary entry	Confinement ⟷ one-directional theorems	18-03
entry "centre symmetry"	Dictionary entry	Centre symmetry ⟷ centre invariance on slabs	18-03
entry "Λ, one scale"	Dictionary entry	Λ, one scale (all masses are constant multiples of a single scale) ⟷ no corresponding term on the mathematical side	18-03
entry "m/√σ"	Dictionary entry	$m/\sqrt{\sigma}$ ⟷ the limit combining Chatterjee's Problems 5.1 and 5.2 (the lattice correlation-length problem and the continuum string-tension problem)	18-03
entry "glueball"	Dictionary entry	Glueball ⟷ isolated eigenvalue of the mass operator	18-03
entry "Langevin relaxation"	Dictionary entry	Langevin relaxation ⟷ generator L	18-03
entry "z=2"	Dictionary entry	$z = 2 \longleftrightarrow \mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$ (the Langevin gap is the square of the gap of the transfer matrix)	18-03
entry "exponential autocorrelation time"	Dictionary entry	Exponential autocorrelation time ⟷ Poincaré constant C_P	18-03
entry "thermalisation time"	Dictionary entry	Thermalisation time ⟷ mixing time	18-03
entry "dynamical exponent z"	Dictionary entry	Dynamical exponent $z \longleftrightarrow C_P \asymp \xi^z$	18-03
entry "z=2-η_s"	Dictionary entry	$z = 2 - \eta_s$ ⟷ variational lower bound $C_P \geq \chi$	18-03
entry "no slowing down away from criticality"	Dictionary entry	No slowing down away from criticality ⟷ uniform mixing ⟺ uniform log-Sobolev	18-03
entry "Lüscher term"	Dictionary entry	Lüscher term ⟷ Casimir energy of the strip	18-03

Name	Kind	What it refers to	Introducing chapter
entry "roughening, shape of the tube"	Dictionary entry	Roughening, shape of the tube ⟷ Wulff construction	18-03
entry "topological freezing"	Dictionary entry	Topological freezing ⟷ bottleneck of the state space	18-03
entry "renormalisation group"	Dictionary entry	Renormalisation group ⟷ sequence of steps, UV stability	18-04
entry "β function"	Dictionary entry	β function ⟷ recursion for the effective coupling	18-04
entry "Polchinski's equation"	Dictionary entry	Polchinski's equation ⟷ Polchinski equation of measure decomposition (heat flow)	18-04
entry "Δβ of MCRG"	Dictionary entry	Δβ of MCRG ⟷ one-step map	18-04
entry "change of scheme for Λ"	Dictionary entry	Change of scheme for Λ ⟷ reparametrisation of the coupling	18-04
entry "Migdal–Kadanoff"	Dictionary entry	Migdal–Kadanoff ⟷ recursion that closes on hierarchical lattices	18-04
entry "continuum limit"	Dictionary entry	Continuum limit ⟷ scaling limit	18-04
entry "a(β), setting the scale"	Dictionary entry	a(β), setting the scale ⟷ ε(β) (the scale function of Chatterjee's Problem 5.2)	18-04
entry "scaling"	Dictionary entry	Scaling ⟷ limits of dimensionless ratios	18-04
entry "asymptotic scaling"	Dictionary entry	Asymptotic scaling ⟷ asymptotic form of ε(β)	18-04
entry "universality"	Dictionary entry	Universality ⟷ uniqueness of the limit	18-04
entry "Symanzik's $O(a^2)$"	Dictionary entry	Symanzik's $O(a^2)$ ⟷ the shortfall of a step δ(b)	18-04
entry "improved action"	Dictionary entry	Improved action ⟷ loss of reflection positivity (contributions with negative weights)	18-04
entry "marginal coupling"	Dictionary entry	Marginal coupling ⟷ Sobolev critical exponent	18-04
entry "large N, planar diagrams"	Dictionary entry	Large N, planar diagrams ⟷ planar maps, sums over strings	18-04
entry "master field"	Dictionary entry	Master field ⟷ limit of non-commutative distributions	18-04
entry "radial direction of holography"	Dictionary entry	Radial direction of holography ⟷ Fefferman–Graham expansion	18-04

Name	Kind	What it refers to	Introducing chapter
entry "KK tower"	Dictionary entry	KK tower ⟷ λ_1 of the internal Laplacian	18-04
entry "topological charge"	Dictionary entry	Topological charge ⟷ second Chern number	18-04
entry "instanton"	Dictionary entry	Instanton ⟷ anti-self-dual connection	18-04
entry "Wilson flow"	Dictionary entry	Wilson flow ⟷ Yang–Mills heat flow	18-04
entry "Gribov region"	Dictionary entry	Gribov region ⟷ region where the FP operator is positive	18-04
entry "trace anomaly"	Dictionary entry	Trace anomaly ⟷ no corresponding term on the mathematical side	18-04
Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) (short: Theorem from Centre Symmetry to Confinement)	Name from the literature	If centre symmetry is not broken, the static pair potential V(R) tends to infinity as $R \to \infty$ (confinement)	18-01
definition column	Table row	The first column of the dictionary's grade: whether the correspondence as a definition is settled, conditional or unsettled	18-01
existence column	Table row	The second column of the dictionary's grade: whether the object exists in four dimensions (constructed, conditional, none)	18-01
measurement column	Table row	The third column of the dictionary's grade: in which direction finite measurements can reach	18-01
static exponent (symbol kept)	Name	The static correlation exponent η_s: the exponent determined from the scaling dimension Δ of an operator by $\eta_s = 2\Delta - (d - 2)$ (distinct from the dynamical exponent)	18-05

Part 19 Objects, and the statements to be shown

Name	Kind	What it refers to	Introducing chapter
uniform-decay statement	Statement	One of the statements to be shown: along a tuned sequence of couplings, correlations decay exponentially, uniformly in the lattice spacing and the box (its consequence is the mass gap)	19-07
statements to be shown	Statement	The set of statements to be shown toward the continuum limit and the mass gap, written only with the objects defined in Part 19 (tuning, convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band)	19-07

Name	Kind	What it refers to	Introducing chapter
base time	Statement	The time of the starting point x of the link (x,μ). For a link in the time direction, this is taken as the time of that link	19-01

Part 21 Firm theorems — the vortex floor and Polyakov loops

Name	Kind	What it refers to	Introducing chapter
upper bound on the flipping cost	Statement	For some S, for every $\varepsilon > 0$ there is an m such that $\kappa_c(N,M,m,\lfloor M/m \rfloor) \le S + \varepsilon$ for all $M \ge m$ (an upper bound on the cost of flipping the class of the window)	21-04

Part 23 The first road, first half — from the direction of time

Name	Kind	What it refers to	Introducing chapter
diffusive-relaxation-time assumption (short: relaxation-time assumption)	Name	For the family of Wilson measures, $\lim_{\beta\to\infty} \ln C_P(\beta)/\ln \xi(\beta) = 2$ (the relaxation time grows with the same power as the square of the correlation length; a statement at the level of powers)	23-01
relaxation-time upper-bound assumption (short: upper-bound assumption)	Name	$a^2 C_P \le C$, that is, $C_P \lesssim \xi_{RG}^2$ (ξ_{RG} is the bookkeeping length; the upper half of the band statement, a Poincaré inequality for all test functions f)	23-01
relaxation-time lower-bound assumption (short: lower-bound assumption)	Name	$a^2 C_P \ge c$, that is, $C_P \gtrsim \xi_{RG}^2$ (ξ_{RG} is the bookkeeping length; the lower half of the band statement, for which one test function suffices)	23-01
band form of the diffusive-relaxation-time assumption (short: band form of the relaxation-time assumption)	Name	For some $0<c\le C$ and β_0, $c\,\xi^2 \le C_P[E] \le C\,\xi^2$ (a band statement agreeing up to a constant factor; it names the graduation E)	23-01
double-well relaxation-time power statement (short: double-well power statement)	Name	For the double-well path measure, $\lim_{\hbar\to 0} \ln C_P(\hbar)/\ln \xi(\hbar) = 2$ (a statement of the same form as the diffusive-relaxation-time assumption, stated for the family of measures of the first step)	23-03
relaxation-time band at the bookkeeping length	Assumption	The band statement $a^2 C_P \asymp 1$ ($a = 1/\xi_{RG}$) that follows from the band assumption (angle) together with the one-scale chain and the bookkeeping-length chain	23-01

Name	Kind	What it refers to	Introducing chapter
eight statements of construction	Statement	The eight component statements into which the requirements of the problem statement are split along the road entering from the lattice (infinite-volume uniqueness, reflection-positivity preservation, the Wilson-loop limit, the physical-unit gap, rotation recovery, OS reconstruction, non-triviality and so on)	23-01
infinite-volume uniqueness statement	Statement	At each β the infinite-volume limiting measure is determined uniquely	23-01
reflection-positivity preservation statement	Statement	A positive self-adjoint transfer operator, preserved under weak limits	23-01
Wilson-loop limit statement	Statement	Renormalised Wilson loops have a non-trivial limit as $a \to 0$	23-01
physical-unit gap statement	Statement	The gap in physical units $\Delta(\beta)$ converges as $\beta \to \infty$ to $\Delta_\infty \in (0, \infty)$	23-01
rotation-recovery statement	Statement	Rotation and translation invariance return in the limit	23-01
OS reconstruction statement	Statement	A Hilbert space and a Hamiltonian are built from the OS axioms	23-01
non-triviality statement	Statement	The limit is not Gaussian and matches asymptotic freedom at short distances	23-01
relaxation chain	Assumption	$\lambda_1 \asymp m^2$ (the Langevin gap and the square of the mass agree)	23-01
one-scale chain	Assumption	$m^2 \asymp \sigma$ (the square of the mass and the string tension agree)	23-01
bookkeeping-length chain	Assumption	$\sigma \asymp a^2$ (the string tension and the bookkeeping length agree)	23-01
tube transfer-operator inequality (short: tube inequality)	Assumption	The tube transfer operator satisfies $T \leq \exp(-(\sigma \ell + c/\ell))$ (a tube of circumference ℓ is heavier by at least c/ℓ in addition to the tension)	23-02
localised tube inequality	Assumption	T for short loops, a Casimir floor for large loops, and the junction between them. The three give $m^2 \geq 4c\sigma$	23-02
assumption of projection onto a one-dimensional chain of scales (short: projection assumption)	Assumption	The state space can be projected onto a one-dimensional chain in the scale $s = \ln \ell$ (four parts, from the scale decomposition of the variance to the single-line property)	23-02
summability-of-shortfalls assumption	Assumption	The sum of the one-step shortfalls $\delta(b_k)$ is finite	23-02

Name	Kind	What it refers to	Introducing chapter
time-step comparison assumption ((D) in the paper "one-sided bridge") (short: time-step comparison assumption)	Assumption	On a lattice with discrete time, $E\|dF\|^2_G \le c_D\langle\phi,(1-\hat{T})\phi\rangle$ (a comparison of the increment between neighbouring times with the gradient, with c_D uniform in β)	23-01
ground-state assumption ((A1) in the paper) (short: ground-state assumption)	Assumption	The spectrum is discrete and the ground state is simple and positive	23-01
stationary path-measure assumption ((A2) in the paper) (short: stationary path-measure assumption)	Assumption	A stationary path measure with $T = \infty$ (infinitely long in the time direction) exists	23-01
metric comparison assumption ((A3) in the paper) (short: metric comparison assumption)	Assumption	Comparison of the Langevin metric, $G \le \lambda_G M^{-1}$	23-01
vacuum C_P	Name	The Poincaré constant C_P of the vacuum measure at a single time (one of the three C_P)	23-01
path C_P	Name	The Poincaré constant C_P of the distribution of all fields in spacetime (the path measure = the Wilson measure). The relaxation-time assumption is a statement about this	23-01
path C_P measured in a graduation	Name	The C_P for the same measure as the path C_P, with the graduation of the Dirichlet form named (measured in angles or in fields). Multiplying the graduation by c multiplies C_P by 1/c	23-01
One-Sided Bridge Proposition (Theorem 3.1 of the paper) (short: one-sided bridge)	Proposition	With continuous time, a finite spatial lattice and $T=\infty$, under the ground-state assumption, the stationary path-measure assumption and the metric comparison assumption, $C_P \ge \kappa\xi^2 = \kappa\hbar^2/\Delta^2$ (best possible within the family of test functions local in time)	23-01
Discrete-Time One-Sided Bridge Proposition	Proposition	On a time lattice with step 1, under the time-step comparison assumption, $C_P \ge 2/(c_D \Delta^2)$	23-01
One-Dimensional One-Sided Bridge Proposition	Proposition	For any one-dimensional well with continuous time and $T\to\infty$, the ceiling of the family of test functions local in time is ξ^2, hence $C_P \ge \xi^2$ ($R \ge 1$)	23-04
Free-Particle-on-a-Circle Counterexample	Proposition	For a free particle on a circle, the gap $\Delta = \hbar^2/2 > 0$, yet the C_P of the path measure is ∞ (a negation of the converse direction)	23-01

Name	Kind	What it refers to	Introducing chapter
Double-Well Relaxation-Time Upper-Bound Proposition (short: First-Step Upper-Bound Proposition)	Proposition	For the double-well path measure, under the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$, so with $1 \le R \le K'/\hbar$ the power statement follows	23-03
Six-Node In-Phase Poincaré Proposition	Proposition	For the phase measure $\mu^{(+)}$ with $V=(x^2-1)^2/8$, $\varepsilon=1/2$ and ring $N=12n$, at each of the six nodes in $\hbar$, Var $f \le C\,E(f)$ with $C \le 1.6\times 10^{10}$ (658 with numerical input), independent of n	23-06
Box-Reduction Proposition	Proposition	If $c_{alt} < 1$, then $C_P \le (1+1/n)\,C_{box}/(1-c_{alt})$ (two families, even and odd, of overlapping boxes; Poincaré inequalities only)	23-05
Gap-Inside-the-Box Proposition	Proposition	Splitting the inside of the box by kink configurations, $C_{box} \le C_{loc} + (1+\chi)C_{kg} \le K_4(C\xi)^2/D$	23-05
Three-Range Core-Constant Proposition	Proposition	Bounds the Poincaré constant of the core by splitting according to the core length k into three ranges: convexity for $k\le 7$, two blocks for $8\le k\le 14$, and box reduction applied to the core for $k\ge 15$ (the Box-Reduction Inequality for Long Cores)	23-11
Two-Projection Lemma	Lemma	If the maximal correlation of two σ-algebras is $c < 1$, then Var $f \le (1-c)^{-1}(E\,\mathrm{Var}(f\|G_1) + E\,\mathrm{Var}(f\|G_2))$	23-05
Short-Core Convexity Lemma	Lemma	If $\kappa(k) > 0$ ($k \le 7$), then $C_P(\mu_{k}^{\{\alpha,\beta\}}) \le \varepsilon/\kappa(k)$ for all end values and $\hbar$	23-11
End-Reduction Lemma	Lemma	For a Markov chain, the maximal correlation of two separated blocks equals the maximal correlation of the two facing end points	23-11
Two-Block Lemma	Lemma	Bounds C_P for cores with $8 \le k \le 14$ by splitting them into two blocks	23-11
Closed Formula for the Alternating Maximal Correlation	Proposition	For a stationary Gaussian chain with correlation t within pieces and correlation s across seams, the maximal correlation between the even pieces and the odd pieces is $\phi(\|s\|,\|t\|)$ (a closed formula)	23-11
Box-Reduction Inequality for Long Cores	Proposition	The inequality for the Poincaré constant of the core obtained by applying the box reduction of the Box-Reduction Proposition to cores of length $k \ge 15$	23-11
Tilted Second-Order Taylor Lemma	Lemma	Tilts the measure at the centre of the box with given end values, and bounds the whole box by the second-order Taylor expansion of the cumulant generating function with a Hölder remainder	23-06

Name	Kind	What it refers to	Introducing chapter
Gap Non-Monotonicity Proposition	Proposition	FKG gives an ordering of likelihood ratios, but the single-site gap is not monotone in the neighbouring values	23-06
Linear-Fractional Lemma	Lemma	Setting $h = g - g^*$ and $\beta = \beta_0 + \tau$ makes the expectation a linear-fractional function of τ	23-08
Tilted-Enclosure Lemma	Lemma	Keeps the enclosure of the fixed point of the step map tilted in τ, in the form $m_k - \tau\mu_k - \tau^2\nu_k$	23-08
Moving-Centre Lemma	Lemma	Moves the centres of the two one-sided parts of the variance bound with τ, as $c_\tau = c + \tau\kappa$	23-08
Scalar-Strand Proposition	Proposition	At 8 nodes with $\hbar \in [3\cdot10^{-4}, 10^{-3}]$, bounds the Poincaré constant of the core using Bakry–Émery for the convex bottom and paying for the band at the top with the Gibbs step (the condition is $\ln k < \ell_{band}$)	23-09
Product Bound for the Window Log-Sobolev Constant	Proposition	An assembly bounding the log-Sobolev constant of the site window from above by a product of factors, $A_W \le c_W C_{1D}^W S_W T_W$	23-07
Window Log-Sobolev Threshold Proposition	Proposition	The log-Sobolev constant of the site window is within the required value: $A_W < A_{max}(\hbar)$	23-09
marginal log-Sobolev input for the window	Assumption	The marginal log-Sobolev inequality for the window, $\mathrm{Ent}(E[g^2\|\sigma_W]) \le A_W E(g) + D_W \int g^2$	23-09
Coarse-Chain Dobrushin Theorem (Theorem CC of the paper) (short: Coarse-Chain Dobrushin Theorem)	Name from the paper	For a core with both ends inside the wells, the Poincaré constant of the core is bounded uniformly by the Dobrushin criterion for the coarse chain (for $\hbar \in [10^{-5},1]$, $\rho_{col} \le 0.400$ by interval arithmetic over 239 intervals)	23-05
Small-ħ Coarse-Chain Proposition (Proposition PIN of the paper) (short: Small-ħ Coarse-Chain Proposition)	Name from the paper	For small $\hbar$ ($\hbar \le 10^{-5}$), the Dobrushin coefficient of the coarse chain stays within $\rho_{col} \le 0.874$ (the small-ħ side of the Coarse-Chain Dobrushin Theorem)	23-05
Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper) (short: Coarse-Chain Brascamp–Lieb Proposition)	Name from the paper	The coarse-chain version of the Brascamp–Lieb inequality (a part of the Coarse-Chain Dobrushin Theorem)	23-05
Closed-Core KKT-Point Proposition	Proposition	When the end values are below the box ($a,b \le -x_c$), the only KKT point on the closed core is the bottom corner	23-15

Name	Kind	What it refers to	Introducing chapter
four double-well conditions	Condition	The four conditions into which the upper bound $C_P \le K\xi^2 S_0/\hbar$ of the first step (the double well) is split (in-phase local Poincaré, saddle tube, box reduction, gap inside the box)	23-03
in-phase local Poincaré condition	Condition	The Poincaré constant C_loc of the kink-free phase measure $\mu^{(+)}$ is O(1) uniformly in ħ and the length of the path (inside the box, the segment local Poincaré assumption)	23-05
saddle-tube condition	Condition	In the tube around the family of kink–antikink pairs, the transport constant and the Poincaré constant of the tube are uniform in the dimension and ħ (inside the box, the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption)	23-05
box-reduction condition	Condition	$C_P(\mu) \le C_{box}(L)/c_M$, with c_M depending only on the length in units of ξ (reducing the overall relaxation time to the relaxation time of boxes of length L)	23-05
gap-inside-the-box condition	Condition	For the conditional measure on a box of length Cξ, $1/C_P \ge cD/(C\xi)^2$, with c uniform in ħ and the boundary values	23-05
segment local Poincaré assumption	Condition	The local Poincaré inequality for segments with end values (length $\le C\xi$) is uniform also in the end values	23-05
saddle-tube Poincaré assumption	Condition	The Poincaré constant of the tube is uniform in tube lengths $\le C\xi$	23-05
core Poincaré uniformity assumption	Condition	The Poincaré constant of the core within the tube (the interval that stays near the top) is uniform in the core length k and the end values	23-05
alternating maximal-correlation assumption for boxes	Assumption	The alternating maximal correlation between boxes, $c_{alt} < 1$, is uniform in ħ and the boundary values	23-05
alternating maximal-correlation assumption for the core	Assumption	An upper bound $\bar{c} < 1$ on the alternating maximal correlation of the core ($k \ge 15$)	23-11
core action-barrier assumption	Assumption	C_P of the core and the window $\le c\,\hbar^{-p}\, e^{(\Delta S/\hbar)}$, with barrier $\Delta S < 2S_0$	23-11
kink-gas gap assumption	Assumption	The box gap of the reduced kink gas is of the Glauber type of the one-dimensional Ising model ($C_{kg} \le (C\xi)^2/(c_{kg}(1-\eta_{KG})D)$)	23-05
cross-term uniformity assumption	Assumption	The cross term satisfies $E_Y(E[f\|Y]) \le (1+\chi)E(f)$, with χ uniform in ħ and the boundary values	23-05

Name	Kind	What it refers to	Introducing chapter
kink diffusion-coefficient row	Table row	The mobility of a single kink, $D = \hbar/S_0$ (at the level of the process)	23-05
bridge volume-locality input	Assumption	Volume locality on the bridge	23-09
seam limit assumption	Assumption	The limit of stacking steps without end lies, at each $\hbar$, in the enclosure of the fixed point	23-07
random-end-value assumption	Assumption	The assumption that the seam estimate holds also when the end values of the outer chain are given by a distribution rather than a single point	23-07
ratio conditioned on the plateau	Assumption	The ratio measured at the seam of the ring (the plateau at the top), conditioned on the length of the plateau	23-07
row of the contraction of the chain outside the window	Table row	$q_{\mathrm{free}} \le 0.998854904$ (contraction of the chain outside the window of the bridge core, over the whole window of $\hbar$)	23-07
small assumptions of the sector input	Assumption	The list of eleven small assumptions needed as input to the sector framework (the bridge volume-locality input, the core log-Sobolev uniformity assumption, the longitudinal term, regularity and others; seven of them do not yet have a one-line statement of content)	23-07
margin row	Table row	The first row of the two-scale assembly: the coarse-side margin $\hbar\rho - 2\hbar\kappa \ge 0.067$ (for neighbouring and end values in $[0, 2.5]^2$)	23-06
nearest-neighbour chain row	Table row	A row of the two-scale assembly: the marginal measure of the retained sites is a nearest-neighbour chain	23-06
two-site gap row	Table row	A row of the two-scale assembly: the gap of the marginal measure of the two sites (x_4, x_8) is $\lambda_c \ge 1.3 \times 10^{-4}$ (end values in $[0, 4]^2$, $\hbar \in (0, 0.2]$)	23-06
crossing row	Table row	A row of the two-scale assembly: the strength of the crossing between the two scales, $v/\hbar \le 5888$	23-06
variance-decomposition row	Table row	The last row of the two-scale assembly: the formula that splits the variance into the retained sites and the rest, and assembles the other rows into a single Poincaré constant	23-06
lattice target level	Condition	The first level of the goal: at each β the lattice correlation length $\xi(\beta)$ is finite, and it diverges as $\beta \to \infty$	23-15
physical-unit target level	Condition	The lattice mass in physical units, m_{lat}/a, tends to a positive finite value	23-15

Name	Kind	What it refers to	Introducing chapter
continuum target level	Condition	The third level of the goal: a non-trivial continuum theory on R^4 with a positive gap	23-15
remaining item between nodes	Condition	A row remaining for closing the in-phase local Poincaré condition: ħ between the six nodes	23-06
small-ħ remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the side $ħ < 0.005$ (computer assistance that subdivides into intervals would need infinitely many intervals)	23-06
large-end-value remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the case where neighbouring and end values exceed 2.5	23-06
sharp-constant remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: sharpening the constant $C \leq 1.6 \times 10^{10}$	23-06
segment-version remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: the version for segments with end values (the segment local Poincaré assumption)	23-06
bibliographic remaining item	Condition	A row remaining for closing the in-phase local Poincaré condition: checking the bibliographic data of the cited theorems	23-06
sector sum	Step	The first step of the sector framework: even when the configuration space is split into a product of boxes, the Dirichlet form, the variance and the entropy split into sums over sectors	23-09
Gibbs decomposition	Step	The second step of the sector framework: by the Gibbs variational inequality, the weight on rare sets is paid for by the entropy of the marginal of the coarse variables	23-09
componentwise localisation	Step	The third step of the sector framework: for each component of the gradient, the marginal log-Sobolev constant of the site window is applied, so that the core length k does not enter the sum	23-09
volume locality	Step	The fourth step of the sector framework: the rarity exponent ℓ_min of the defect set does not depend on the core length k	23-09
sector composition	Step	The fifth step of the sector framework: combines the three inequalities of Barta, componentwise localisation and Bochner, and bounds the gap of the core independently of k under the closing condition	23-09

Name	Kind	What it refers to	Introducing chapter
quantitative uniform Laplace method	Condition	Local Poincaré inside the wells of the first step, and the reduction of the margin as $\hbar \to 0$	23-15
step aligning the assumptions	Step	The first step of the proof of the First-Step Upper-Bound Proposition: aligning the premise rows	23-05
box-reduction step	Step	A step of the proof of the First-Step Upper-Bound Proposition: by the Box-Reduction Proposition, the whole is reduced to boxes of length $C\xi$	23-05
step inside the box	Step	A step of the proof of the First-Step Upper-Bound Proposition: the gap inside the box is bounded by splitting according to kink configurations	23-05
composition step	Step	A step of the proof of the First-Step Upper-Bound Proposition: combining the box reduction and the gap inside the box gives $C_P \le K\xi^2 S_0/\hbar$	23-05
step bracketing the ratio	Step	A step of the proof of the First-Step Upper-Bound Proposition: the ratio $R = C_P/\xi^2$ is bracketed as $1 \le R \le K'/\hbar$	23-05
step reading the power	Step	The last step of the proof of the First-Step Upper-Bound Proposition: the power statement $\ln C_P/\ln \xi \to 2$ is read from the bracketed ratio	23-05
short-core step	Step	The first of the three ranges of the core constant: cores of length $k \le 7$ are bounded by convexity	23-11
medium-core step	Step	The second of the three ranges of the core constant: cores of length $8 \le k \le 14$ are bounded by splitting into two blocks	23-11
long-core step	Step	The third of the three ranges of the core constant: box reduction is applied to cores of length $k \ge 15$	23-11
seven routes of the saddle tube	Route	The seven routes set up to close the core row of the saddle-tube condition	23-07
route of the chain outside the window	Route	Contraction of the chain outside the window ($q_free < 1$ for the bridge core, computer-assisted)	23-07
route of the window-constant numerics	Route	The remaining numerics (putting the site-window constants into intervals)	23-07
route of the sector framework	Route	The sector framework (five steps and the closing condition)	23-07
route of the sector input	Route	The small assumptions (the list of sector inputs)	23-07

Name	Kind	What it refers to	Introducing chapter
route of the window log-Sobolev inequality	Route	The route that attacks the one remaining proposition (the Window Log-Sobolev Threshold Proposition) with three strands	23-07
route of the seam of the ring	Route	The seam of the ring (outer chain and inner chain)	23-07
route of the range of ħ	Route	The range of ħ (outside the window and ħ→0, outside the scope)	23-07
componentwise strand	Route	The first strand of the route of the window log-Sobolev inequality: the componentwise form of the gradient (stops at 1.4–5.5 times the required value)	23-09
scalar strand	Route	The second strand of the route of the window log-Sobolev inequality: the scalar form (the Scalar-Strand Proposition)	23-09
Feynman–Kac strand	Route	The third strand of the route of the window log-Sobolev inequality: the Feynman–Kac form (Monte Carlo readings)	23-09
road of the core on the box side	Route	The road that supports the fork (the core Poincaré uniformity assumption) with the core on the box side: the Three-Range Core-Constant Proposition, and the local term vanishing as ħ→0	23-15
road of the core on the sector side	Route	The road that supports the fork (the core Poincaré uniformity assumption) with the core on the sector side: a uniform lower bound on the gap of the core in the sector framework, and the seam limit assumption	23-15
road of blocks and mixing	Route	A third branch that supports the fork (the core Poincaré uniformity assumption) with log-Sobolev inequalities for blocks ($\ell_B \geq 12$) and strong mixing (the fourteen rows of Table C1 of 23-15)	23-15
summit-curvature type	Type	A type of breakage of detour plans: there is a single source — the curvature of the single site at the summit	23-10
type of the ceiling on structural factors	Type	A type of breakage of detour plans: structural factors that enter as ratios have a ceiling	23-10
double-payment type	Type	A type of breakage of detour plans: the same place was being paid for twice	23-10
currency-mix-up type	Type	A type of breakage of detour plans: a mix-up of currencies — exit speed and rarity	23-10
type of the address of 1/ħ	Type	A type of breakage of detour plans: the address of 1/ħ (a mistake about where the factor 1/ħ enters)	23-10

Name	Kind	What it refers to	Introducing chapter
type of the misaligned fluctuation direction	Type	A type of breakage of detour plans: a mismatch between the direction of fluctuation and the soft direction	23-10
transfer type	Type	A type of breakage of detour plans: transfer — the weight only moves to another column	23-10
seven types of breakage	Table row	The types classifying into seven the reasons why detour plans broke. Used as questions to put to new candidates	23-10
detour plan "localising negative curvature" (short: localising negative curvature)	Detour plan	Localising Bochner's negative curvature on the defect set	23-10
detour plan "optimising the Gibbs tilt" (short: optimising the Gibbs tilt)	Detour plan	Setting the Gibbs tilt to $t^{*}=\ell-5.10$	23-10
detour plan "raising the threshold" (short: raising the threshold)	Detour plan	Raising the threshold to $1/\sqrt{3}$ and making the localisation remainder 0	23-10
detour plan "layering defects by depth" (short: layering defects by depth)	Detour plan	Dividing the defect depth b into layers	23-10
detour plan "narrowing the defect set" (short: narrowing the defect set)	Detour plan	Narrowing the defect set to the set where the row falls below κ	23-10
detour plan "defects as long clusters" (short: defects as long clusters)	Detour plan	Defining defects by clusters of length 8 or more	23-10
detour plan "paying for the summit band" (short: paying for the summit band)	Detour plan	Charging the summit band to the defects	23-10
detour plan "scalar payment" (short: scalar payment)	Detour plan	Paying for the sector defects in scalar form	23-10
detour plan "non-linear Barta weight" (short: non-linear Barta weight)	Detour plan	Making the Barta weight non-linear	23-10
detour plan "Agmon-type payment" (short: Agmon-type payment)	Detour plan	Paying only with the gradient of the true eigenfunction (Agmon type)	23-10
detour plan "re-examining downstream needs" (short: re-examining downstream needs)	Detour plan	Re-examining what is needed downstream	23-10

Name	Kind	What it refers to	Introducing chapter
detour plan "coarse and propagation steps at once" (short: coarse and propagation steps at once)	Detour plan	Doing the coarse step and the propagation step at once	23-10
detour plan "replacing the functional inequality" (short: replacing the functional inequality)	Detour plan	Poincaré / modified LSI / Φ-entropy instead of log-Sobolev	23-10
detour plan "log-Sobolev with clusters fixed" (short: log-Sobolev with clusters fixed)	Detour plan	Log-Sobolev for the conditional measure with the cluster state fixed	23-10
detour plan "perturbing only the summit band" (short: perturbing only the summit band)	Detour plan	Bakry–Émery outside the window, Holley–Stroock only for the summit band	23-10
detour plan "known uniform log-Sobolev" (short: known uniform log-Sobolev)	Detour plan	Known theorems on uniform log-Sobolev in one dimension	23-10
detour plan "finite-volume criteria" (short: finite-volume criteria)	Detour plan	Finite-volume criteria (Martinelli–Olivieri, Lu–Yau)	23-10
detour plan "Glauber log-Sobolev" (short: Glauber log-Sobolev)	Detour plan	Known log-Sobolev for Glauber dynamics	23-10
detour plan "two-step cluster expansion" (short: two-step cluster expansion)	Detour plan	Two-step cluster expansion, block spins	23-10
detour plan "extension to the classical limit" (short: extension to the classical limit)	Detour plan	Taking $\hbar$ below the window (the classical limit)	23-10
detour plan "one- or two-site windows" (short: one- or two-site windows)	Detour plan	Making the window one or two sites	23-10
detour plan "localising at the cluster core" (short: localising at the cluster core)	Detour plan	Localising at the core of the cluster rather than at the window	23-10
detour plan "Hölder and hypercontractive steps" (short: Hölder and hypercontractive steps)	Detour plan	Replacing the Gibbs step with Hölder and hypercontractivity	23-10

Name	Kind	What it refers to	Introducing chapter
detour plan "paying with a height function" (short: paying with a height function)	Detour plan	Paying for the negative curvature with a height function	23-10
list of detour plans	Table row	The twenty-four detour plans that moved either side, or the form, of the one remaining proposition of the route of the window log-Sobolev inequality	23-10
road of strong-coupling curvature	Table row	Widening the range of β where the curvature method works (the strong-coupling side)	23-02
road of one-step bookkeeping	Table row	Bookkeeping the continuum limit as a sequence of single steps	23-02
road of the sum of shortfalls	Table row	Whether the sum of the shortfalls δ(b) is finite	23-02
road of the strong-coupling series	Table row	Whether the dynamical exponent η can be read from the strong-coupling series	23-02
road of the dimension of complete families	Table row	In which dimensions complete families of edges and plaquettes exist	23-02
road of the dimensional threshold	Table row	Why four dimensions is the threshold	23-02
road of reflection positivity	Table row	Using reflection positivity for estimates from below	23-02
road of an area law in the bookkeeping	Table row	Whether the form of an area law appears in the bookkeeping	23-02
road of multi-link integrals	Table row	Whether the closed forms of k-link integrals are analytic	23-02
road of comparison with the defect of a submission	Table row	Whether the defect of a submission claiming a construction is the same quantity as δ	23-02
road of multiscale curvature	Table row	Stacking the curvature method scale by scale	23-02
road of volume and boundary	Table row	Whether the estimate of C_P is uniform in the volume and the boundary conditions	23-02
road of lower-bound test functions	Table row	Whether there is a test function giving the lower-bound assumption	23-02
road of gauge-fixed long wavelengths	Table row	Whether gauge-fixed long-wavelength modes can be made single-valued	23-02
road of spectroscopy	Table row	Bounding the gap with smoothed operators and spectroscopy	23-02
road of the lower half at one scale	Table row	Making the lower half of one scale, $m^2 \gtrsim \sigma$	23-02
road of a one-dimensional chain of scales	Table row	Bracketing the upper-bound assumption with a one-dimensional chain of scales	23-02

Name	Kind	What it refers to	Introducing chapter
seventeen roads tried on the four-dimensional lattice	Table row	The seventeen roads tried on the four-dimensional lattice (each a pair of one question and one set of tools)	23-02
one-sided test-function wall	Wall	Test functions bound C_P only from below	23-02
spectroscopic upper-bound wall	Wall	Spectroscopy ends with an upper bound on the gap	23-02
local-operator dimension wall	Wall	Local gauge-invariant operators have dimension 4 or more, so η_s=2	23-02
gauge-fixing single-valuedness wall	Wall	Gauge fixing: if sharp, it is multivalued; if blunt, the long wavelengths disappear	23-02
measurement-cost wall	Wall	The cost of measuring the numerator of smoothed operators is $\geq cL^{d+2}$	23-02
positive-weight kernel wall	Wall	A kernel with $\delta(2)=0$ has negative neighbouring weights	23-02
renewal-series wall	Wall	The strong-coupling series has renewal form and forces $\eta=0$	23-02
single-step integral wall	Wall	The one-link integral of a complete family works for one step only	23-02
limit-property wall	Wall	The Wilson-loop limit statement and the rotation-recovery statement do not follow from compositions of one-step maps	23-02
upper-bound tool wall	Wall	There are no tools of the type of the upper-bound assumption (they stop at the cross term)	23-02
tube-inequality wall	Wall	$m^2 \gtrsim \sigma$ has moved to the tube inequality, but there is no pattern of proof for it	23-02
projection-assumption wall	Wall	The projection assumption is not weaker than the relaxation-time assumption	23-02
checking-tool wall	Wall	There are few tools for machine checking	23-02
wall of the assumption itself	Wall	The relaxation-time assumption and the three chains have not been proved	23-02
equivalence-scaffold wall	Wall	The scaffold of equivalences for the physical-unit gap statement (power form, $\eta \geq 0$, additive step errors, δ of the one-loop kernel)	23-02
periodic-boundary conjecture wall	Wall	The breaking of the upper-bound assumption with periodic boundary conditions depends on a conjecture	23-02
fitting wall	Wall	That the violation of $m/\sqrt{\sigma}$ is $O(a^2)$ is a fit	23-02
box-size wall	Wall	The boxes are small	23-02
gauge dependence of the shortfall wall	Wall	Differences in c_2 depending on the choice of gauge	23-02

Name	Kind	What it refers to	Introducing chapter
tube-model sign wall	Wall	The sign of the 1/P coefficient of the tube model is undecided	23-02
primary-source wall	Wall	Missing primary sources	23-02
wall of principle	Type	One of the five types of walls: a wall that comes from the direction of an inequality and does not depend on the details of the tools	23-02
wall of structure	Type	One of the five types of walls: a wall where the requirements are incompatible within the chosen family, bookkeeping or series (outside the family it is undecided)	23-02
wall of tools	Type	One of the five types of walls: a wall where tools of the needed kind do not yet exist	23-02
wall of assumptions	Type	One of the five types of walls: a wall where the assumption relied on has itself not been proved, or depends on a conjecture	23-02
wall of measurement and literature	Type	One of the five types of walls: a wall coming from the sizes that can be measured and from missing primary sources	23-02
question "complete families of k=3 cells" (short: complete families of k=3 cells)	Question	Whether complete families of $k = 3$ cells exist in $d = 9$	23-17
question "largest dimension with a complete family" (short: largest dimension with a complete family)	Question	The largest dimension $d_{\max}(k)$ in which complete families exist	23-17
question "constant of the Hermite family" (short: constant of the Hermite family)	Question	The question of finding the constant in 1 − λ_k for the Hermite family	23-17
question "SU(N) version of the ratio on forests" (short: SU(N) version of the ratio on forests)	Question	The $SU(N)$ version, and a Lean statement, of the fact that the ratio of functions on forests saturates at $1/\lambda_1(G)$	23-17
question "C_P(β) on the two-dimensional lattice" (short: C_P(β) on the two-dimensional lattice)	Question	$C_P(\beta)$ in the link variables of two-dimensional lattice YM	23-17
question "curvature after one-link integration" (short: curvature after one-link integration)	Question	The best constant $c^* \in [21, 27]$ for the curvature after one-link integration	23-17

Name	Kind	What it refers to	Introducing chapter
question "the N^{−3} coefficient of the column model" (short: the N^{−3} coefficient of the column model)	Question	Deriving analytically the N^{-3} coefficient $C \approx 320$ of the column model (finite-size expansion of the XX chain)	23-17
question "sign of 1/P in the column model" (short: sign of 1/P in the column model)	Question	In the column model, the sign of $1/P$ at finite u	23-17
question "gauge dependence of the shortfall" (short: gauge dependence of the shortfall)	Question	How much the shortfall c_2 changes with the choice of gauge of the block transformation	23-17
question "C_P and ξ² of the O(3) σ model" (short: C_P and ξ² of the O(3) σ model)	Question	C_P and ξ^2 of the two-dimensional $O(3)$ σ model	23-17
question "C_P/ξ² of three-dimensional Z_2 gauge theory" (short: C_P/ξ² of three-dimensional Z_2 gauge theory)	Question	C_P/ξ^2 of three-dimensional $\mathbb{Z}_2$ gauge theory	23-17
question "volume monotonicity of C_P(V)" (short: volume monotonicity of C_P(V))	Question	Volume monotonicity of $C_P(V)$ and boundary conditions (computing all eigenvalues of the generator on small lattices)	23-17
question "dynamical exponent of four-dimensional SU(2)" (short: dynamical exponent of four-dimensional SU(2))	Question	Collecting primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	23-17
question "interval arithmetic for the maximal correlation of the core" (short: interval arithmetic for the maximal correlation of the core)	Question	The question of bounding from above, by interval arithmetic, the two-point maximal correlation of medium cores ($8 \le k \le 14$)	23-17
question "power of the prefactor for the core and window" (short: power of the prefactor for the core and window)	Question	Reading the power of the prefactor $\hbar^{-p}$ of C_P for the core and the window from the capacity of degenerate points	23-17
question "comparison for the alternating maximal correlation of boxes" (short: comparison for the alternating maximal correlation of boxes)	Question	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$ while avoiding examples that do not reduce to two-point quantities	23-17

Name	Kind	What it refers to	Introducing chapter
question "length set by a conservation-breaking term" (short: length set by a conservation-breaking term)	Question	Asks, when a small conservation-breaking term is added to a conservative dynamics, at what length the gap departs from the behaviour as the square of the side	23-17
row of common premises	Table row	Uniqueness of the T=∞ measure, the Markov property, extension from local functions, the limiting step, regularity (the common premises)	23-11
row of interval-arithmetic premises	Table row	Premises of the interval arithmetic: correct rounding of the IEEE 754 arithmetic operations, errors of sums, errors of the self-written exponential function	23-07
summit-curvature shortfall wall	Table row	A row of Table 4 of 23-07: the curvature of the single site at the summit is short by 1/4 (a wall common to the three strands)	23-07
core lower-bound row	Table row	A row of Table 4 of 23-07: a uniform lower bound on the gap of the core	23-07
sector-framework row	Table row	A row of Table 4 of 23-07: the sector framework (five steps and the closing condition)	23-07
row of small assumptions	Table row	A row of Table 4 of 23-07: the small assumptions of the route of the sector input	23-07
window log-Sobolev threshold row	Table row	A row of Table 4 of 23-07: the one remaining proposition (the Window Log-Sobolev Threshold Proposition)	23-07
assumption lifting the proxy ratio	Table row	The lifting assumption of the Scalar-Strand Proposition (lifting the proxy ratio linearly beyond the measured range)	23-09
box-product inequality row	Table row	The box-product inequality (of Efron–Stein type)	23-05
row of the growth of the correlation length	Table row	The reading $\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{S_0/\hbar}$)	23-05
kink-action row	Table row	A row of Table 3 of 23-04: the action S_0 of a single kink (paired with the zero-mode row)	23-04
zero-mode row	Table row	A row of Table 3 of 23-04: the translational zero mode of the kink (paired with the kink-action row)	23-04
next-eigenvalue row	Table row	A row of Table 3 of 23-04: the next eigenvalue 3/4 of the fluctuation operator around the kink	23-04

Name	Kind	What it refers to	Introducing chapter
Box-Dynamics Proposition (Propositions 3.4 and 3.5 of Martinelli's lectures) (short: Box-Dynamics Proposition)	Name from the literature	Box dynamics (the pattern that, for discrete spins, bounds the overall gap by the box dynamics and the gap inside the box)	23-05
Variance-Splitting Lemma (Lemma 2.4 of Menz–Schlichting) (short: Variance-Splitting Lemma)	Name from the literature	The formula for splitting the variance (the skeleton of the Poincaré version of Eyring–Kramers)	23-05
Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) (short: Poincaré Version of Helffer's Criterion)	Name from the literature	The Poincaré version of Helffer's criterion (Proposition 3.1 of Bobkov–Ledoux 2000 is cited with the same number)	23-06
Integrated Bochner Formula (Proposition 1.3 of Ledoux) (short: Integrated Bochner Formula)	Name from the literature	The identity obtained by integrating Bochner's formula (used in sector composition to bound the Dirichlet form of the gradient by the curvature)	23-09
Holley–Stroock Perturbation Lemma (Proposition 6.1 of Ledoux) (short: Holley–Stroock Perturbation Lemma)	Name from the literature	A lemma that perturbs the Poincaré constant by the oscillation of the difference from a convex action (of Holley–Stroock type)	23-06
Single-Interface Diffusion Theorem (Theorem 2.1 of Bertini–Brassesco–Buttà) (short: Single-Interface Diffusion Theorem)	Name from the literature	The diffusion coefficient of a single interface	23-04
Two-Block Gibbs-Sampler Rate Theorem (Theorem 5 of Roberts–Sahu) (short: Two-Block Gibbs-Sampler Rate Theorem)	Name from the literature	The rate of convergence of the two-block Gibbs sampler	23-11
lattice correlation-length problem (Chatterjee's Problem 5.1) (short: lattice correlation-length problem)	Name from the literature	Chatterjee's Problem 5.1: at each β the lattice correlation length $\xi(\beta)$ is finite, and $\xi(\beta) \to \infty$ as $\beta \to \infty$	23-01, 16-07
continuum string-tension problem (Chatterjee's Problem 5.2) (short: continuum string-tension problem)	Name from the literature	Chatterjee's Problem 5.2: in the continuum limit of Wilson loops with a chosen lattice spacing, the string tension d is not 0	23-01, 16-07
angle graduation	Name	The Dirichlet form measured in the angles of the links	23-01
field graduation	Name	The Dirichlet form measured in the field $A = \sqrt{\beta}\,\vartheta$	23-01

Name	Kind	What it refers to	Introducing chapter
exponent of accumulation of the shortfall	Name	The exponent θ of the speed at which the shortfall accumulates (aξ ∝ a^θ)	23-01
one-step slope (symbol kept)	Name	The one-step slope $\kappa = 4Nb_0 = 11/(3\pi^2)$	23-02
one-step shortfall (symbol kept)	Name	The one-step shortfall. It vanishes as c_2/b^2	23-02
bookkeeping length	Name	The length ξ_RG = b^{k(β)} determined by the number k(β) of coarsening steps of the lattice. The lattice spacing is set to a := 1/ξ_RG	23-01
band exponent	Name	The exponent p when a^2C_P behaves as a power a^p of a (p = 0 is the band statement)	23-01
core log-Sobolev uniformity assumption	Condition	The log-Sobolev constant of the core is bounded uniformly	23-09
upper-side target row	Table row	A row of Table 3 of 23-04: the upper-side target $C_P \le K\xi^2 S_0/\hbar$ (the form of the conclusion of the First-Step Upper-Bound Proposition)	23-04
row of the numerical kink diffusion coefficient	Table row	A row of Table 3 of 23-04: the numerical kink diffusion coefficient $D/(\hbar/S_0) = 1.00–1.14$	23-04
row of the semiclassical leading term of the correlation length	Table row	A row of Table 3 of 23-04: the semiclassical leading term of the correlation length ξ (ξ ∝ ħ^{1/2} e^{S_0/ħ})	23-04
row of the lattice ln ξ	Table row	A row of Table 3 of 23-04: numerical values of the lattice ln ξ and the lattice kink action S_0^lat	23-04
row of the growth of the lower bound on R	Table row	A row of Table 3 of 23-04: the lower bound on the ratio $R = C_P/\xi^2$ grows from 1.06 to 6.5	23-04
dilute-gas accounting row	Table row	A row of Table 3 of 23-04: the accounting $C_P \approx \xi^2 S_0/\hbar$ obtained by counting kinks as a dilute gas	23-04
band-ratio row	Table row	A row of Table 3 of 23-04: the ratio to the component orthogonal to the magnetisation approaches 1	23-04
Glauber-eigenvalue row	Table row	A row of Table 3 of 23-04: the Glauber eigenvalue appears in the spectrum of the full state	23-04
row of the curvature of a convex core	Table row	A row of Table 3 of 23-04: for a convex core, $C_P \le 2/(3a^2-1)$	23-04
assumption that the maximum over searched starting points suffices	Assumption	The assumption that the maximum over the starting points searched by Monte Carlo suffices (it is not an upper bound over the whole fibre)	23-09

Name	Kind	What it refers to	Introducing chapter
assumption of the Barta row	Table row	The assumption of the inequality of the Barta row used in sector composition (lower bound $\kappa = 3/1000$ outside the defects, defect depth $b = 493/1000$)	23-09
eigenfunction-regularity assumption	Table row	The assumption, used in sector composition, that the Bochner boundary term vanishes, and of the regularity of the eigenfunction	23-09

Part 24 The first road, second half

Name	Kind	What it refers to	Introducing chapter
local diffusive-relaxation-time assumption (short: local relaxation-time assumption)	Name	The local Poincaré constant C_P^loc(c), with support restricted to a window c times the yardstick ℓ, has the same power as ℓ^2 (the upper side is C_P^loc(c) $\leq$ K(c)ℓ^2). Stated in the ratio form K_rel	24-03
sector diffusive-relaxation-time assumption (short: sector relaxation-time assumption)	Name	A statement of the diffusive-relaxation-time assumption for the measure conditioned on each topological sector (value of the charge) (the sector Poincaré constant C_P^sec(n))	24-03
local-excitation moment assumption	Assumption	The moment ratio of the spectral measure of local observables supported in a spatial box satisfies $M_3/M_1 \leq m_\infty^2 + \gamma/(c\ell)^2$	24-04
alternating maximal-correlation assumption for four-dimensional boxes	Assumption	The alternating maximal correlation between families of closed four-dimensional boxes is less than 1, uniformly in β	24-03
gap assumption for closed four-dimensional boxes	Assumption	For closed four-dimensional boxes, the relaxation time of a box satisfies C_box(cξ_RG) $\leq$ K'ξ_RG^2 (ξ_RG is the bookkeeping length)	24-03
infinite-volume ground-state transformation identity	Assumption	The assumption that the ground-state transformation identity of the lattice with continuous time (the relation between the gap and the vacuum measure) holds also in infinite volume	24-04
Window-Factor Monotonicity Proposition	Proposition	g(x) = ϕ(x)/x^2 (ϕ(x) = 1 − (1−e^{−x})/x) is strictly decreasing for x > 0 (the window factor decreases)	24-04
In-Window Supremum Proposition	Proposition	Under the assumptions of the one-sided bridge and the canonical graduation, writes in closed form, via the root of a transcendental equation, the supremum over the family local in time with weights supported in the window [0,T]	24-04

Name	Kind	What it refers to	Introducing chapter
Gap and Vacuum Measure Proposition	Proposition	On a lattice with continuous time, the gap is exactly the reciprocal of the Poincaré constant of the vacuum measure at a single time (ground-state transformation)	24-05
Electric–Magnetic Splitting Frustration Proposition	Proposition	With the splitting into electric and magnetic parts, the Kogut–Susskind Hamiltonian is not frustration-free	24-05
Local Poincaré Constant Properties Proposition	Proposition	The local Poincaré constant C_P^loc(c) is determined only by the marginal law of the window; D_c is not dense; it is monotone in c and returns to C_P as c→∞	24-03
Sector Poincaré Constant Proposition	Proposition	The sector Poincaré constant C_P^sec(n) is finite if the sector is connected; the variance splits into within-sector and between-sector parts, and can be bracketed with C_trans	24-03
Convexity Ceiling on Groups Lemma (short: Convexity Ceiling Lemma)	Lemma	For measures on closed manifolds, the lower bound of the Bakry–Émery curvature has a ceiling	24-06
No Concentration at the Identity Corollary	Corollary	On SU(2), a convex measure cannot concentrate at the identity	24-06
One-Link Wilson Weight Proposition	Proposition	The range of convexity of the one-link Wilson weight (at weak coupling, the non-convex region is pushed into a region of small probability)	24-06
rule of local convexity	Rule	One of the candidate rules for a class of measures that pay with rarity: the measure is locally convex on the good set	24-06
rule of rarity of bad sets	Rule	One of the candidate rules for a class of measures that pay with rarity: the non-convex bad set is rare	24-06
rule of return drift	Rule	One of the candidate rules for a class of measures that pay with rarity: there is a drift returning from the bad set to the good set	24-06
rule of paying for entropy	Rule	One of the candidate rules for a class of measures that pay with rarity: the rarity of the bad set is paid for with an entropy inequality	24-06
rule of closing under coarse-graining	Rule	One of the candidate rules for a class of measures that pay with rarity: the estimates close on the coarse-grained variables	24-06

Name	Kind	What it refers to	Introducing chapter
source: dimensionless coupling	Question	One of the sources of difficulty: the coupling is dimensionless, all scales act with the same weight, and infinitely many steps must be controlled uniformly in the number of steps	24-07
source: no expansion linking scales	Question	One of the sources of difficulty: there is no expansion linking the ultraviolet and the infrared	24-07
source: no global gauge fixing	Question	One of the sources of difficulty: because of non-abelian gauge invariance, there is no continuous global gauge fixing	24-07
three sources of difficulty	Question	The three sources of the difficulty of four-dimensional Yang–Mills (dimensionless coupling, no expansion linking scales, no global gauge fixing)	24-07
rule of axiomatising scale	Table row	A candidate rule in Table 2 of 24-08: treating scale as an axiom (axiomatising scale)	24-08
rule of the loop algebra	Table row	A candidate rule in Table 2 of 24-08: rules of the algebra of Wilson loops	24-08
rule of conserved quantities and sectors	Table row	A candidate rule in Table 2 of 24-08: rules for conserved quantities and the sectors divided by them	24-08
rule of the rarity class	Table row	A candidate rule in Table 2 of 24-08: the class of measures that pay with rarity	24-08
rule of an asymptotic-freedom axiom	Table row	A candidate rule in Table 2 of 24-08: a rule that sets asymptotic freedom as an axiom	24-08
curvature rule	Table row	A candidate rule in Table 2 of 24-08: a rule on the Bakry–Émery curvature on the space of gauge orbits	24-08
question "comparing local excitations of the free field" (short: comparing local excitations of the free field)	Question	The question of comparing energies of local excitations in the free field	24-08
question "monotonicity of the window factor" (short: monotonicity of the window factor)	Question	A question about the monotonicity of the window factor $\phi(x)/x^2$	24-08
question "sign of the Polyakov loop" (short: sign of the Polyakov loop)	Question	Asks whether, with the sign of the Polyakov loop in boxes of small period, a static ratio captures the slowness of relaxation	24-08
question "U(1) gap on two-dimensional layers" (short: U(1) gap on two-dimensional layers)	Question	Asks about the gap of U(1) on two-dimensional layers when the strip is lengthened	24-08

Name	Kind	What it refers to	Introducing chapter
question "data on topological-charge autocorrelation" (short: data on topological-charge autocorrelation)	Question	The question of collecting primary sources on the autocorrelation of the topological charge of SU(2)	24-08
question "many-family version of the Two-Projection Lemma" (short: many-family version of the Two-Projection Lemma)	Question	Asks for a many-family version of the Two-Projection Lemma (families of 2^d colours of d-dimensional boxes) and its condition on the maximal correlation	24-08
question "Hessian of the one-link weight" (short: Hessian of the one-link weight)	Question	Asks about the Hessian of the one-link Wilson weight convolved once with the heat kernel, and how the parameters of the class are mapped	24-08
question "time-step comparison for the transfer operator" (short: time-step comparison for the transfer operator)	Question	Asks whether the time-step comparison assumption holds for the Wilson transfer operator	24-08
question "Lean for τ_int and C_P" (short: Lean for τ_int and C_P)	Question	The question of writing τ_int ≤ C_P and the ground-state transformation identity in Lean	24-08
question "rereading the modes of CP^{N−1}" (short: rereading the modes of CP^{N−1})	Question	The question of rereading the local modes and topological modes of the CP^{N−1} model	24-08
question "loop families of two-dimensional SU(2)" (short: loop families of two-dimensional SU(2))	Question	The question of finding λ_W and Γ for loop families of two-dimensional SU(2)	24-08
question "values of large four-dimensional loops" (short: values of large four-dimensional loops)	Question	The question of finding the values for families of large loops in four-dimensional boxes	24-08
question "reduction to kink positions" (short: reduction to kink positions)	Question	The question of reducing the double-well problem to the dynamics of kink positions	24-08
question "the single point β=7 of the band" (short: the single point β=7 of the band)	Question	The question of measuring the single point β = 7 of the band	24-08
question "mass in the band between the threshold and 1" (short: mass in the band between the threshold and 1)	Question	The question of building another argument that sees the mass in the band between the threshold and 1	24-08

Name	Kind	What it refers to	Introducing chapter
question "window bridge on a time-step lattice" (short: window bridge on a time-step lattice)	Question	The question of making a version of the In-Window Supremum Proposition (the window bridge) on a lattice with discrete time	24-08
question "frustration-free sector splitting" (short: frustration-free sector splitting)	Question	Asks whether a sector splitting of the Kogut–Susskind Hamiltonian can be frustration-free	24-08
question "extending the CP^{N−1} test bed" (short: extending the CP^{N−1} test bed)	Question	The question of extending the CP^{N−1} test bed by adding seeds and values of N	24-08
question "conservation length of subregion charge" (short: conservation length of subregion charge)	Question	Asks at what length conservation breaks when measured with the charge of a four-dimensional subregion	24-08
type of a nearly conserved global quantity	Type	The type in which a global quantity not reflected in the gap slows relaxation at a length different from the correlation length (three conditions: the variance of the quantity is large, its Dirichlet form is small, and it is not reflected in the gap)	24-01
Infrared Asymptotic-Freedom Proposition (Proposition 6.1.7 of BBS) (short: Infrared Asymptotic-Freedom Proposition)	Name from the literature	The one-step estimate of the large-field weight norm; infrared asymptotic freedom	24-06
Covariance-Compression Lemma (Lemma 2.2 of Hairer–Stuart–Voss–Wiberg) (short: Covariance-Compression Lemma)	Name from the literature	An expression for the operator norm of a covariance compressed to a subspace	24-03

Part 25 The second road — from the direction of space

Name	Kind	What it refers to	Introducing chapter
five moves of the proof	Name	The five steps of the proof of the Slab Confinement Theorem	25-04
Moving-Surface Lemma	Lemma	Flipping the signs of all plaquettes containing a link leaves the partition function unchanged, the surface Π_q can be moved, and the partition function is determined only by the parity of the number of surfaces	25-01
definition of the sector and flux free energies	Name	The definitions $\Delta F = \ln(Z(0)/Z(1))$ and $\theta = \tanh(\Delta F/2)$	25-01

Name	Kind	What it refers to	Introducing chapter
Parity Representation of the Bias	Proposition	$\theta = E[(-1)^{\wedge}\lvert Q\rvert]$ (the expectation of the parity of the number of surfaces in the ensemble summed over twists)	25-01
Zero-Action Configuration Lemma	Lemma	$S^{\wedge}z=0 \iff U_p = z_p \cdot 1$ for all plaquettes	25-01
Lemma on the Sign of the Holonomy Commutator	Lemma	If all plaquettes take central values, the commutator of the holonomies around the two directions is the product of the signs of that plane	25-01
Determination-by-Holonomy Lemma	Lemma	A configuration with central values is determined, up to gauge, by the four holonomies at the base point	25-01
Lemma on the Configuration that Absorbs the Twist	Lemma	U^0 has action 0 in the sector (1,0) ($\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -1$)	25-01
Rigidity Lemma for Anticommuting Pairs	Lemma	A pair with $ABA^{-1}B^{-1} = -1$ is conjugate to $i\sigma_1$, $i\sigma_2$, and the only elements commuting with both are ± 1	25-01
Uniqueness of the Twisted Vacuum	Proposition	The configuration of action 0 in the sector (1,0) is U^0 alone, up to gauge and centre transformations in the plane directions	25-01
Twisted Adjoint Gap Lemma	Proposition	The smallest eigenvalue of the adjoint Laplacian around U^0 is $4\sin^2(\pi/2N) > 0$ (there are no zero modes)	25-01
Positivity of the Minimal Action in the Twisted Sector	Proposition	There is no configuration of action 0 in the sector (1,1), and $0 < S_{min} \leq 2N^2$	25-01
Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) (short: Confinement Lower Bound by Fractional Instantons)	Theorem	For fixed $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $-\ln \theta \geq c\, \beta^{\{-a\}}\, e^{\{-\beta(S_*+\varepsilon(\beta))\}}\, M^2$ (bounding the flux free energy from below by the action of fractional instantons)	25-02
Route to Existence by the Nahm Transform	Proposition	A route that, assuming equivariance of the Nahm transform, gives the existence of self-dual solutions of charge 1/2 on twisted $T^2 \times R^2$	25-02
Container Identity	Lemma	With the frame fixed, the containers become independent, and $\lvert\langle W\rangle\rvert \leq E_\Phi[\prod \theta_j]$, $\theta_j = \tanh(\lvert\Delta F_j\rvert/2)$	25-03
$\sigma_{avg}(\ell)$	Name	The estimate of the string tension of two-dimensional SU(2) obtained from the Container Identity (frame average) when tiling with containers of side ℓ	25-03

Name	Kind	What it refers to	Introducing chapter
Inner and Outer Container Proposition	Proposition	If containers are placed inside and outside the loop, the Wilson loop in a fixed sector is bounded by the product of the container biases and an outer term	25-03
Nesting Inequality in the Plane Directions	Proposition	The bias of a large container is bounded by the average of the product of the biases of the smaller containers into which it is split by the inner frame	25-03
assumption on the form of the thick-container free energy ((23) of Kovács–Tomboulis) (short: assumption on the form of the thick-container free energy)	Name from the literature	The free energy of a thick container has the form $\lvert V\rvert\, e^{-cd^2}$, uniformly in the boundary values	25-03
Slab Confinement Theorem	Theorem	With N and the lattice spacing fixed, for all β, $\lvert\langle \tfrac{1}{2}\mathrm{tr}\, U_C\rangle\rvert \le \tanh(\beta N^2)^A$ and $\sigma \ge -\ln\tanh(\beta N^2)$	25-04
finite configuration-space condition	Condition	One of the three conditions of the insertion identities: the configuration space X is a finite set	25-05
positive-weight condition	Condition	One of the three conditions of the insertion identities: the weight $w : X \to (0,\infty)$ is positive	25-05
sector-flipping bijection condition	Condition	One of the three conditions of the insertion identities: there is a bijection $T : X \to X$ that flips the sector sign $\sigma : X \to \{\pm 1\}$ ($\sigma(Tx) = -\sigma(x)$)	25-05
kernel moving only inside the container	Condition	A condition of driven insertion: the kernel updates only the variables inside the container and does not change the frame	25-06
Mean Insertion-Ratio Identity	Lemma	$E[\rho] = 1$, $E[\rho\sigma] = -E[\sigma]$	25-05
Bias Estimate by Insertion-Ratio Fluctuation	Theorem	$4E[\sigma]^2 \le \mathrm{Var}(\vec{\rho})$	25-05
Bounded Insertion Identity	Lemma	For an involution τ and an odd H, $E[H] = E[H\tanh(\Delta S/2)]$	25-05
Insertion Fluctuation Relation Lemma	Lemma	$P(\Delta S{=}s) = e^{s}\, P(\Delta S{=}{-}s)$, $E[t] \ge 0$, and $1-E[t]$ is twice the acceptance rate	25-05
Position-Averaged Insertion Estimate	Theorem	Bounds $\lvert E[H]\rvert$ by the product of the tanh factors averaged over positions	25-05
Square-Root Insertion Identity	Lemma	$E[H\sqrt{\rho}] = 0$	25-05
Paley–Zygmund-Type Bias Estimate	Theorem	$E[H]^2 \le 1 - E[Y]^2/E[Y^2]$, $E[Y^2] \le 1$	25-05

Name	Kind	What it refers to	Introducing chapter
Many-Cell Second-Moment Estimate	Theorem	Under "odd under insertion", $4^{\lvert C\rvert}\,\mathbb{E}[H]^2 \le \mathbb{E}[\prod(1-r_c)^2]$	25-05
assumption that composed insertions are odd	Assumption	For all $S \subseteq C$, $\mathbb{E}[H \prod_{c\in S} r_c] = (-1)^{\lvert S\rvert}\,\mathbb{E}[H]$	25-05
Crooks Relation	Lemma	Under the path involution ι, the ratio of weights is e^{-W}, and $W\circ\iota = -W$	25-06
Driven Insertion Identity	Theorem	$\mathbb{E}[H] = \mathbb{E}_{\Omega}[H \tanh(W/2)]$	25-06
Jarzynski Equality	Corollary	$\mathbb{E}[e^{-W} \mid \Phi, \sigma=+] = Z_-/Z_+$	25-06
Driven Insertion Acceptance-Rate Corollary	Corollary	$1 - \mathbb{E}[\tanh(W/2)] = 2 \times$ the Barker-type acceptance rate	25-06
Many-Cell Driven Estimate	Theorem	Bounds by the product of the position averages of $\lvert\tanh(W/2)\rvert$ of driven insertions over many cells	25-06
Cut-Measure Formula	Proposition	$Z_{\pm}/Z_{\mathrm{cut}} = \mathbb{E}_{\mathrm{cut}}[e^{\pm\beta X/2}]$, $\theta = \lvert E\sinh\rvert/E\cosh$	25-06
Flip-Probability Lemma	Theorem	For a stationary kernel, the probability of a flip in one step is $\le 1-\theta^2$	25-06
Flip-Probability Corollary for Positive Kernels	Corollary	For a reversible positive kernel, $(1-\theta^2)/2$	25-06
Flow Lemma	Theorem	For the one-step flip probability a of a stationary kernel, $\theta \le 1-a$	25-06
Simultaneous-Flip Estimate	Theorem	$\mathbb{E}\prod\theta_t \le$ P(no container flips)	25-06
Transport Identity	Proposition	$\Delta F = \mathbb{E}_+[S\circ T - S - \ln J_T] - \mathrm{KL}$ (the transport identity and the Gibbs variational principle)	25-07
Isometric-Map Loss Proposition	Proposition	The loss of a map isometric in the fluctuations is $\tfrac{1}{2}\Sigma(\lambda-1-\ln\lambda)$, which is not diluted by the degrees of freedom	25-07
Fluctuation-Shaping Proposition	Proposition	The optimal value for maps of determinant 1 is $(n/2)(r^{1/n}-1)$, with loss $\le (\Delta F^2/n)e^{2\lvert\Delta F\rvert/n}$	25-07
First-Order Vanishing of the Spin Term	Proposition	The spin term does not appear in the first-order trace (at one-loop level)	25-07
Isometric versus Shaping Comparison Proposition	Proposition	(a) The isometric comparison gives $\beta' \ge \beta+0.023$; (b) the shaping comparison gives $\beta-\beta' = 0.258$	25-07
Relative-Entropy Obstruction to the Area Law	Proposition	If the remaining $\mathrm{KL} \ge c'A$, the area law does not follow from the variational argument	25-07
Additivity of Relative Entropy over Containers	Proposition	For a map that is a product over containers, the remaining KL is the sum over containers	25-07

Part 26 The crossover band

Name	Kind	What it refers to	Introducing chapter
Hidden-Convexity Proposition	Proposition	On the box [−c,0]^l the action has exactly one local minimum, and it is global	26-07
jump step	Name	The first step of the proof of the Conditional Reduction Theorem: from frame decorrelation and the base point at small T^2, jump to the first bias by the lateral-widening inequality	26-02
doubling step	Name	The second step of the proof of the Conditional Reduction Theorem: doubling the container and the transverse T^2 at each step, show that the bias satisfies $t_{k+1} \le K\, t_k^4$	26-02
iteration step	Name	The third step of the proof of the Conditional Reduction Theorem: iterating $t_{k+1} \le K\, t_k^4$ gives that the bias becomes small with an area exponent	26-02
Tension Lower Bound from the Naive Form	Proposition	The lower bound on the tension $\ln(1/\theta_0)/L^2$ obtained by container estimates from the naive form of the band $\theta \le \theta_0$ goes to 0	26-01
Area Law from the Area Exponent	Proposition	If $\theta \le e^{-cL^2}$, then $\lvert\langle W\rangle\rvert \le C_D\, e^{-cA}$	26-01
Tension Lower Bound from Tomboulis–Yaffe	Corollary	Under Tomboulis–Yaffe, if $F_\nu \le e^{-\rho\, L_\mu\, L_\nu}$, then the tension is $\ge \rho$	26-01
naive form of the band	Name	The naive form of the band: the sector bias stays away from 1 uniformly in the scale ($\sup \theta(L,L) \le \theta_0 < 1$). From this, container estimates give no positive lower bound on the tension	26-01
correct form of the band	Name	The correct form of the band: the sector bias becomes small with an area exponent ($\ln(1/\theta(L,L)) \ge cL^2$)	26-01
exclusion of a deconfined phase	Condition	One of the intermediate milestones toward the band: the vortex free energy $F_\nu(L,L)$ tends to 0 (excluding a non-confining phase)	26-01
almost area law	Condition	One of the intermediate milestones toward the band: $\ln(1/\theta) \ge cL^2/\ln L$ (the area exponent up to a logarithmic loss)	26-01
base point at small T^2	Assumption	One of the assumptions of the reduction theorem: at some physical scale L_0, the sector bias on a small T^2 satisfies $\theta(L_0;T^2_{L_0}) \le 1-\varepsilon_0$	26-02

Name	Kind	What it refers to	Introducing chapter
frame decorrelation	Assumption	One of the assumptions of the reduction theorem: multiplying the container by s raises the bias to the power s^2, up to a constant ($\theta(sR;T^2_L) \le C_D\, \theta(R;T^2_L)^{s^2}$)	26-02
lateral-widening inequality (strong form)	Assumption	One of the assumptions of the reduction theorem: with the container fixed, widening the transverse T^2 by a factor L'/L increases the bias by at most a factor $C(L'/L)^2$	26-02
lateral-widening inequality (weak form)	Assumption	The lateral-widening inequality weakened to a form that may lose, at each step doubling the transverse size, a fraction ε_k of $\ln(1/\theta)$ (with finite sum) and a constant	26-02
Conditional Reduction Theorem	Theorem	Under the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k,R_k)) \ge \gamma R_k^2$ (the bias becomes small with an area exponent)	26-02
Reduction Theorem in the Weak Form	Theorem	Under frame decorrelation, the weak form and a condition on the size of the base point, $u_k \ge \gamma' R_k^2$	26-02
Slab-Tension Monotonicity Proposition	Proposition	If the lateral-widening inequality (strong form) holds, the effective tension σ_eff does not decrease in L (the reason the strong form is stronger than confinement)	26-02
Reduction Read in the Chessboard Density	Corollary	Read in the chessboard density κ_cb, the same reduction goes through with nesting constant 1 (under the chessboard-density base point and the lateral comparison)	26-02
chessboard-density base point	Assumption	The assumption of the base point at small T^2 read in the chessboard density: $\kappa_cb(L_0;L_0) \le 1 - \varepsilon_0$	26-02
lateral comparison of the chessboard density	Assumption	The assumption of the lateral-widening inequality (strong form) read in the chessboard density: $\kappa_cb(R;L') \le C(L'/L)^2\, \kappa_cb(R;L)$	26-02
base-point translation inequality	Condition	The single line linking the bias at the base point and the chessboard density: $1 - \kappa_cb(L_0;L_0) \ge \eta(1 - \theta(L_0;L_0))$	26-02
Flux-Equalisation Lemma	Lemma	$\int_\Omega f^2 \ge \Phi^2/\lvert\Omega\rvert$, with equality for constants (Cauchy–Schwarz)	26-03

Name	Kind	What it refers to	Introducing chapter
Contrapositive of the Reduction	Proposition	If the conclusion of the reduction theorem is false, one of the three assumptions is false (contrapositive)	26-03
Electric-Flux Decomposition Lemma	Lemma	$Z(1,0) = Y(0)+Y(1)$, $Z(1,1) = Y(0)-Y(1)$, $\theta = Y(1)/Y(0) \geq 0$	26-04
positivity assumption for the twisted partition function	Assumption	The twisted partition function $Z(1,1)$ is at least 0 (positivity of the Boltzmann weight)	26-04
tanh Comparison Lemma	Lemma	If $F' \leq KF$ ($K \geq 1$), then $\tanh(F'/2) \leq K \tanh(F/2)$	26-04
Reflection-Positivity Theorem for z (short: reflection positivity of z)	Theorem	The z ensemble with the twist fixed is positive under reflections through lattice sites in directions 3 and 4 (the row "reflection positivity of z" in the ledger of conditions of 27-02 also refers to this theorem)	26-04, 27-02
Lemma on the Non-Positivity of Lateral Reflections	Lemma	The weight of a twisted plaquette has negative coefficients at half-integer j, so lateral reflections are not of positive type	26-04
Lemma that Structural Inputs Alone Do Not Suffice	Lemma	The set of structural inputs ☒ also holds for Z_2 and U(1), so the lateral-widening inequality does not follow from it alone	26-04
set of structural inputs	Name	The set consisting of positivity of the transfer matrix, the centre and duality identities, reflection positivity, the lattice symmetries and finite lemmas	26-04
thin-vortex floor	Name	The strong form reaching down to the floor of slab confinement: $\theta \leq \tanh(N^2\|\beta\|)^{M^2}$	26-04
Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) (short: Chessboard Estimate)	Name from the literature	For a reflection-positive measure, $\|\langle\prod f_t\rangle\| \leq \prod\langle\prod f_t^{(s)}\rangle^{1/\|T\|}$	26-05
Chessboard Estimate with the Lateral Size Fixed	Corollary	$E\prod_{t\in S}\theta_t \leq \kappa_{cb}^{\|S\|}$, $\|\langle W\rangle\| \leq \kappa_{cb}^{n}$	26-05
Nested Supermultiplicativity Lemma	Lemma	$\kappa_{cb}(sR;L) \leq \kappa_{cb}(R;L)^{s^2}$ (constant 1, all L)	26-05
$\kappa_{cvx}(k)$	Name	The convexity constant at the top of a chain of length k ($\kappa(k)$ of the Short-Core Convexity Lemma of 23-11)	26-07
β-monotonicity	Condition	With the lattice geometry fixed, $\partial_\beta \Delta F \geq 0$	26-06
renormalisation comparison	Condition	$\Delta F(2N,2M,\beta) \leq K\cdot\Delta F(N,M,\beta')$, $\beta' < \beta$	26-06

Name	Kind	What it refers to	Introducing chapter
rate form of the renormalisation comparison	Name	$\sigma_N(\beta)$, $s(N,\beta) = N^2\sigma_N(\beta)$, and the rate form $s(N,\beta') \le s(2N,\beta)$	26-06
Strong-Coupling-End Expansion Proposition	Proposition	For $u \le u_0 \approx 10^{-80}$, $\Delta F = 2N^2 e^{-\sigma_N M^2}(1+\varepsilon)$	26-06
Comparison at the Strong-Coupling End	Corollary	At the strong-coupling end, choosing $\sigma_N(\beta') = 4\sigma_{2N}(\beta)$ gives $\Delta F(2N,2M,\beta)/\Delta F(N,M,\beta') = 4(1+O(u))$	26-06
assumption of two-sided area laws inside the band	Assumption	An assumption combining, on the band I, the two-sided area law at fixed β (with errors of the order of the perimeter) and the uniform positivity and continuity of the rate	26-06
two-sided area law at fixed β	Assumption	The assumption that at each β in the band I the vortex free energy is bounded above and below by the area rate $\sigma_N(\beta)$ (with errors of the order of the perimeter)	26-06
uniform positivity and continuity of the rate	Assumption	The assumption that the rate $\sigma_N(\beta)$ is uniformly positive on the band and uniformly continuous in β (for nearby β, $\sigma_N(\beta') \le \sigma_{2N}(\beta) + \sigma_*$)	26-06
Lemma Reducing the Comparison inside the Band	Lemma	Under the assumption of two-sided area laws inside the band, the renormalisation comparison inside the band holds with a constant K	26-06
Corollary Crossing the Band in Finitely Many Steps	Corollary	The band can be crossed in $\lceil(\beta_b - \beta_a)/\delta\rceil$ steps, with a loss independent of N and M	26-06
criterion "distinguish from the U(1) transition"	Condition	Show that SU(2) has no transition, distinguishing it from the transition of U(1) inside the band	26-06
tension-doubling inequality	Condition	$4\sigma_{2N}(\beta) \ge e^{\delta_0}\sigma_N(\beta)$ (for all $N \ge N_1$): the ratio of the tensions when the box is doubled exceeds 1 uniformly	26-06
softness in the twist direction	Name	One of the four faces of the middle band: the variance of the force along the generator of the twist within a band of width w (adapted stiffness)	26-06
mixing of box magnetic flux	Name	One of the four faces of the middle band: the total variation distance of the law of the inner cube when the central magnetic flux on the boundary of the box is changed	26-06

Name	Kind	What it refers to	Introducing chapter
twist cost of a column	Name	One of the four faces of the middle band: the twist-conditioned free energy and bias of the column $\mathrm{T}^2_\mathrm{N} \times \ell \times \ell$ with the outer frame fixed	26-06
pressure margin	Name	One of the four faces of the middle band: the difference in pressure $p_{\infty} - p_{G,L}$ between systems with and without the twist pattern inserted (the exponent of the chessboard density κ_{cb})	26-06
interpolation-parameter commonality inequality ((3.5) of Ito–Seiler) (short: interpolation-parameter commonality inequality)	Name from the literature	The inequality that reduces to A(α) > A⁺(α) the claim that the Migdal–Kadanoff interpolation parameter t* can be taken in common for two sectors	26-06
candidate "reduction by compactness of the band" (short: reduction by compactness of the band)	Name	One of the candidates for the band: since the band is a bounded closed interval of β, reduce the comparison inside the band to the two-sided area law at fixed β at each β, and cross in finitely many steps	26-10
candidate "log-concavity of the Weyl measure" (short: log-concavity of the Weyl measure)	Name	One of the candidates for the band: using the log-concavity of the density obtained by projecting the uniform measure on the sphere onto one coordinate, and a height–azimuth decomposition, bring it to a conditional ferromagnetic model	26-10
candidate "local step function of a column" (short: local step function of a column)	Name	One of the candidates for the band: replace the global renormalisation comparison by a comparison of the twist free energies of columns with the outer frame fixed (the local step function of a column)	26-10
candidate "confinement as sign cancellation" (short: confinement as sign cancellation)	Name	One of the candidates for the band: write θ = E[tanh(X/2)] in the measure summed over twists, and view the bias as becoming small through cancellation of the sign of X	26-10
candidate "centre-only influence coefficients" (short: centre-only influence coefficients)	Name	One of the candidates for the band: the influence coefficient $\delta_{\mathcal{C}}(L_0, \beta)$ when the outside is conditioned only on its SO(3) part and only the central flux on the boundary is flipped	26-10

Name	Kind	What it refers to	Introducing chapter
candidate "flux diffusion by monopole flow" (short: flux diffusion by monopole flow)	Name	One of the candidates for the band: since the sign flux of each plane changes with the parity of the closed $\mathbb{Z}_2$ monopole curves crossing between them, view the flux as being mixed by the winding density of monopoles	26-10
candidate "determinant in self-dual coordinates" (short: determinant in self-dual coordinates)	Name	One of the candidates for the band: in each sector, look at the difference between sectors of the Jacobian when the measure is viewed as a pushforward to its anti-self-dual part	26-10
candidate "quantitative 1-form symmetry of the slab" (short: quantitative 1-form symmetry of the slab)	Name	One of the candidates for the band: read slab confinement as the lattice version of "a discrete 1-form symmetry is not broken in two dimensions", and make its quantitative version uniform in N	26-10
stopped road "isometric comparison" (short: isometric comparison)	Table row	A road stopped in the band chapters: an isometric comparison that does not change the shape of the fluctuations (the one-loop spin term becomes 0, and only the anti-asymptotic-freedom direction comes out)	26-08
stopped road "structural inputs only" (short: structural inputs only)	Table row	A road stopped in the band chapters: showing it with structural inputs only — reflection positivity, duality, the transfer matrix and the centre identities (these also hold for $\mathbb{Z}_2$ and U(1), so they cannot distinguish groups)	26-08
stopped road "deterministic squeeze" (short: deterministic squeeze)	Table row	A road stopped in the band chapters: obtaining the area law by a squeeze with deterministic maps (the remaining relative entropy grows with the area)	26-08
stopped road "glassy squeeze" (short: glassy squeeze)	Table row	A road stopped in the band chapters: a squeeze with the $\mathbb{Z}_2$ glass in the upper half of the band (for $\beta \gtrsim 2$ the glass is on the ordered side and the squeeze is loose)	26-08
stopped road "flip certificate" (short: flip certificate)	Table row	A road stopped in the band chapters: using container flip certificates and simultaneous flips to test the band (the flip probability saturates at about 1/2, and only the weak form comes out)	26-08
stopped road "supremum estimate over frames" (short: supremum estimate over frames)	Table row	A road stopped in the band chapters: estimates using the supremum over frames (they get super-exponentially worse in L, and for uniformity only the frame-averaged form can be used)	26-08

Name	Kind	What it refers to	Introducing chapter
stopped road "Dobrushin-type criterion" (short: Dobrushin-type criterion)	Table row	A road stopped in the band chapters: using the criteria of Dobrushin, Dobrushin–Shlosman and van den Berg–Maes with frames (continuous fields follow the boundary and the influence coefficients are nearly 1)	26-08
stopped road "β-monotonicity from easy parts" (short: β-monotonicity from easy parts)	Table row	A road stopped in the band chapters: treating β-monotonicity as an easy part and splitting the band into β-monotonicity and the renormalisation comparison (β-monotonicity turned out to be as heavy as monotonicity of the tension)	26-08
stopped road "one-directional lateral widening" (short: one-directional lateral widening)	Table row	A road stopped in the band chapters: restricting lateral widening to one direction (doubtful given the behaviour of the high-temperature spatial string tension; restricted to square shapes only)	26-08
stopped road "transport by Wilson flow" (short: transport by Wilson flow)	Table row	A road stopped in the band chapters: using Wilson flow to transport vortex insertions (numerically negative; a form recast as forward transport remains open)	26-08
stopped road "conformal frame" (short: conformal frame)	Table row	A road stopped in the band chapters: turning scale into position with the conformal frame $H^2 \times S^2$ (with hyperbolic nesting the number of containers does not grow, and the crux remains in the same place)	26-08
stopped road "strong-coupling expansion only" (short: strong-coupling expansion only)	Table row	A road stopped in the band chapters: covering the band with the strong-coupling expansion alone (the flat-surface expansion stops at roughening $\beta_R \approx 1.9$)	26-08
approach of the holonomy to Haar	Table row	An idea that turned out to be only a restatement: the law of the holonomy of large loops approaches the Haar measure (the decay of the odd part is the Wilson area law itself)	26-08
anomaly of the centre and the θ term	Table row	An idea that turned out to be only a restatement: consistency of the mixed anomaly of the centre and the θ term (a kinematic constraint; at θ = 0 it forces nothing)	26-08

Name	Kind	What it refers to	Introducing chapter
success in the hierarchical model	Table row	An idea that turned out to be only a restatement: taking success in the hierarchical model (the model where Migdal–Kadanoff decimation is exact) as evidence for the band (U(N) also confines there, so it is not a test)	26-08
criterion "stands on the lattice alone" (short: stands on the lattice alone)	Condition	A criterion a tool for crossing the band must meet: it holds on the lattice alone and does not rely on rotation invariance	26-08
criterion "distinguishes phases" (short: distinguishes phases)	Condition	A criterion a tool for crossing the band must meet: it distinguishes phases (it breaks down in a massless phase, or the limiting values differ)	26-08
criterion "does not rely on positivity" (short: does not rely on positivity)	Condition	A criterion a tool for crossing the band must meet: it does not rely on positivity or FKG	26-08
criterion "uses non-commutativity essentially" (short: uses non-commutativity essentially)	Condition	A criterion a tool for crossing the band must meet: it uses non-commutativity essentially (the same argument does not go through for U(1) or $\mathbb{Z}_2$)	26-08
criterion "compatible with asymptotic freedom" (short: compatible with asymptotic freedom)	Condition	A criterion a tool for crossing the band must meet: it is compatible with asymptotic freedom	26-08
criterion "handles topological sectors" (short: handles topological sectors)	Condition	A criterion a tool for crossing the band must meet: it can handle topological sectors (twists)	26-08
criterion "not a rewording of one loop" (short: not a rewording of one loop)	Condition	A criterion a tool for crossing the band must meet: it is a non-perturbative estimate, not a rewording of one loop	26-08
criterion "dynamical" (short: dynamical)	Condition	A criterion a tool for crossing the band must meet: it is a dynamical argument, not a kinematic constraint	26-08
criterion "passes through non-convex regions" (short: passes through non-convex regions)	Condition	A criterion a tool for crossing the band must meet: it can pass through non-convex (negatively curved) regions. Tools based only on convexity, log-concavity or curvature lower bounds cannot be used in the band	26-08
criterion "crosses the smallest band" (short: crosses the smallest band)	Condition	A criterion a tool for crossing the band must meet: it can first cross the smallest band of two-dimensional asymptotically free models (such as the O(N) model at large N) with the ultraviolet cutoff removed	26-08

Name	Kind	What it refers to	Introducing chapter
criterion "not broken by small loops" (short: not broken by small loops)	Condition	A criterion a tool for crossing the band must meet: it does not break down for small loops (short scales), or it is in a form that claims only large loops	26-08
criterion "non-isometric linearisation" (short: non-isometric linearisation)	Condition	A criterion a tool for crossing the band must meet: the linearisation of the comparison map is not isometric in the fluctuations	26-08

Part 27 Closing — conditional statements

Name	Kind	What it refers to	Introducing chapter
Reverse Estimate at a Fixed Plane (Theorem B of Paper 3) (short: Reverse Estimate at a Fixed Plane)	Theorem	The reverse estimate at fixed M (an upper estimate on the tension) of the paper on lattice fractional instantons	27-01
Doubling-Iteration Theorem	Theorem	Under θ(2R_k;T²_{R_k}) ≤ K₁θ^4 and lateral widening (s=2), θ(R_k;T²_{R_k}) ≤ K^{−1/3}exp(−cR_k²)	27-01
uniform acceptance of insertion	Table row	One of the assumptions of the conditional statement: under small-field frames, the average of tanh(W/2) for vortex insertions driven by dynamics is at most 1 − δ uniformly in the lattice spacing	27-01
frame decorrelation (form with L fixed)	Table row	One of the assumptions of the conditional statement: correlations between container frames decay, and E∏θ can be split into the product of the individual θ with a constant loss (form with L fixed)	27-01
decay of block observables (form with L fixed)	Table row	One of the assumptions of the conditional statement: correlations of block observables decay exponentially at a rate $m \geq c/L$ uniform in a (the form assumed by the theory on a small T^2)	27-01
lateral-widening inequality	Table row	One of the assumptions of the conditional statement: with the container fixed, widening the transverse T^2 by a factor s multiplies θ by at most Cs^2 (strong form), or a form with a finite total loss (weak form)	27-01
uniform slab tension	Table row	One of the assumptions of the conditional statement: the string tension $\sigma_{\text{eff}}(L)$ of the slab $T^2_L \times \mathbb{R}^2$ is positive uniformly in L	27-01
uniform slab gap	Table row	One of the assumptions of the conditional statement: correlations of block observables decay exponentially, uniformly in the lattice spacing and the slab size L (a premise of the arrow of the R^4 gap)	27-01

Name	Kind	What it refers to	Introducing chapter
limit to R^4	Table row	One of the assumptions of the conditional statement: the infinite-volume limit, the OS axioms, recovery of rotation invariance, uniqueness of the limit, non-triviality (outside the ledger)	27-01
tying the two roads together	Table row	One of the assumptions of the conditional statement: the road of the gap and the road of confinement give both conclusions for the same theory on R^4 (outside the ledger)	27-01
theory on a small T^2	Name	The theory on $R^2 \times$ a small twisted T^2 with the lattice spacing taken to 0	27-01
theory on R^4	Name	The theory on R^4 (where the T^2 of the theory on a small T^2 has been widened)	27-01
arrow of the small-T^2 gap	Condition	The arrow that gives, from ultraviolet control and the decay of block observables (form with L fixed), the existence of the theory on a small T^2 and a gap $m \geq c/L$	27-01
arrow of the R^4 gap	Condition	The arrow that gives the R^4 gap from the arrow of the small-T^2 gap, the uniform slab gap and the limit to R^4	27-01
arrow of slab confinement	Condition	The arrow of slab confinement with the lattice spacing and N fixed	27-01
arrow of the fractional-instanton rate	Condition	The arrow that gives the rate of the flux free energy from the Confinement Lower Bound by Fractional Instantons	27-01
arrow of the small-T^2 area law	Condition	The arrow that gives the area law of the theory on a small T^2 from ultraviolet control and uniform acceptance of insertion	27-01
arrow of the reduction of the band	Condition	From the base point, frame decorrelation and lateral widening, $\theta(R;T^2_R) \leq e\text{^}\{-cR^2\}$	27-01
arrow from the container estimate to the area law	Condition	From the reduced form and frame decorrelation, an area law uniform in L (the container estimate)	27-01
chessboard arrow	Condition	The arrow of the Chessboard Estimate with T^2 fixed	27-01
arrow from the two uniformities to the area law	Condition	The arrow that gives the area law from the uniform slab tension and the uniform slab gap	27-01
arrow of the R^4 area law	Condition	The arrow that gives the R^4 area law from the arrow from the container estimate to the area law, or the arrow from the two uniformities to the area law, together with the limit to R^4	27-01
arrow tying the two roads together	Condition	Both the mass gap and confinement for the same theory on R^4	27-01

Name	Kind	What it refers to	Introducing chapter
arrow of the road from the direction of time	Condition	From the upper-bound assumption and the one-sided bridge, m_phys $\geq$ (κ/K)^{1/2} at each β	27-01
small-T^2 gap	Table row	With the twist, zero modes disappear and the gap of the transfer matrix $\geq$ c/L uniformly in a	27-02
band comparison (in the language of the gap)	Table row	The lowest mass of the neutral sector of the box measured in units of the box size, x(L) = L m_0(L), satisfies x(2L) $\geq$ x(L) + δ over the interval of the band	27-02
uniform decay of block observables	Table row	Correlations of block observables with vacuum expectation 0 decay at a rate independent of a and L	27-02
positivity of the 't Hooft tension	Table row	$\ln(1/\theta(L,L)) \geq cL^2$ (c independent of L)	27-02
uniform gap of the double well	Table row	A gap, uniform in k and ħ, of the path measure of the 0+1-dimensional double well	27-02, A-08
uniformity of the relaxation-time upper bound and the one-sided bridge	Table row	The upper side of the diffusive-relaxation-time assumption and the one-sided bridge hold uniformly as $\beta \to \infty$	27-02, A-08
ledger of conditions	Table row	The ledger listing, one row each, the assumptions set by the conditional statement toward the mass gap and confinement, with what is to be shown and its status (Table 2 of 27-02)	27-02
lower end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the lower end, which pushes confinement and the comparison up to $\beta \approx 0.3$–1	27-02
upper end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the semiclassical upper end at fixed N and large β	27-02
middle of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the middle, where one inequality using non-commutativity is needed (one of the four faces, uniformly in the scale)	27-02
strong-coupling end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: the strong-coupling end $\beta \lesssim 10^{-79}$ (the record has a proof, with crude constants)	27-02

Name	Kind	What it refers to	Introducing chapter
one-loop end of the renormalisation comparison	Table row	A breakdown of the renormalisation comparison: at one-loop level, the comparison that shapes the fluctuations gives $\beta - \beta' = 0.258$, and the comparison that does not change their shape gives $\beta' \geq \beta + 0.023$	27-02
band in the language of the coupling	Table row	The row that states the band in the language of the coupling: written as a mixed derivative of a box with uniform background holonomies placed at both ends in time (a Schrödinger-functional box)	27-02
ultraviolet control in arbitrary volume	Condition	Ultraviolet control extended to arbitrary volume with extensive constants, with continuity in the sources made uniform in the lattice spacing and the volume	27-02
limiting measure on R^4 and its positivity	Table row	A gap outside the ledger: within the limit to R^4, the limiting measure, translations, preservation of reflection positivity, the Hilbert space and $H \geq 0$	27-02
recovery of rotation invariance	Table row	A gap outside the ledger: recovery of rotation invariance in the limit	27-02
local fields and OS regularity	Table row	A gap outside the ledger: local fields, OS regularity and the linear-growth condition (Wightman fields)	27-02
uniqueness and non-triviality of the limit	Table row	A gap outside the ledger: uniqueness of the limit, non-triviality, clustering	27-02
all compact simple groups (short: general groups)	Table row	A gap outside the ledger: extending to all compact simple groups	27-02
verification and peer review	Table row	A gap outside the ledger: writing out the whole, Lean, expert peer review, publication	27-02
boundedness of the insertion free energy	Condition	One of the premises of uniform acceptance of insertion: the insertion free energy ΔF is bounded uniformly in the lattice spacing	27-02
uniform lower bound on the path overlap	Condition	One of the premises of uniform acceptance of insertion: the overlap of the forward and reverse paths is bounded from below uniformly in the lattice spacing	27-02
ultraviolet control around a vortex background	Condition	One of the premises of the base point at small T^2: ultraviolet control around a background of constant curvature	27-02
insertion acceptance in a symmetric container	Condition	One of the premises of the base point at small T^2: uniform acceptance of insertion in a symmetric container of size L_0	27-02

Name	Kind	What it refers to	Introducing chapter
Lemma on the Upper Bound for the Minimal Action of a Symmetric Box	Lemma	For all N, the minimal action of the symmetric box satisfies $S_min(N,N) \leq 2N^4(1-\cos(\pi/N^2)) < \pi^2$	27-02
twelve tasks	Table row	The list dividing the continuum tasks into twelve (T^4 construction, infinite volume, uniform gap, remaining axioms, non-triviality, local fields and short distances, and so on; Table 1 of 16-07)	27-03
linear-growth condition (Osterwalder–Schrader axiom E0′) (short: linear-growth condition)	Name from the literature	A condition controlling the growth of Schwinger functions in n (added in 1975 to go back from the OS axioms to the Wightman axioms)	27-02
question "reverse direction of the fractional-instanton lower bound" (short: reverse direction of the fractional-instanton lower bound)	Table row	The reverse direction of the Confinement Lower Bound by Fractional Instantons: the question of showing, at fixed N, a dilute-gas upper bound uniform in M (an upper estimate on the tension)	27-03
question "agreement of the minimal actions" (short: agreement of the minimal actions)	Table row	The question of showing that the minimal action of a column with the frame frozen to the vacuum agrees with the minimal action of a free fractional instanton	27-03
question "existence of fractional instantons" (short: existence of fractional instantons)	Table row	The question of showing, on twisted $T^2 \times R^2$, the existence of a solution of topological charge 1/2 attaining the lower bound $4\pi^2/g^2$ of the action	27-03
question "SU(N) versions of the three facts" (short: SU(N) versions of the three facts)	Table row	The question of showing SU(N) versions of the uniqueness of the vacuum that absorbs the twist, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector	27-03
question "bias of the outer container" (short: bias of the outer container)	Table row	The second term when moving to a fixed sector: the question of showing that the bias of the outer container is typically small (0 as $M \to \infty$)	27-03
question "estimating the container free energy" (short: estimating the container free energy)	Table row	The question of estimating the container free energy uniformly over good frames and uniformly in the lattice spacing	27-03
question "cost of inserting a thick vortex" (short: cost of inserting a thick vortex)	Table row	The question of bounding the cost of insertion independently of N with the thick-vortex map (an improvement of the Slab Confinement Theorem toward the continuum limit)	27-03

Name	Kind	What it refers to	Introducing chapter
question "gap from insertion to the area law" (short: gap from insertion to the area law)	Table row	The question of showing the three things missing between the insertion estimates and the area law (rarity of simultaneous rejections, uniformity of the lower bound on the acceptance rate, an upper bound on the variance of the insertion ratio)	27-03
question "U(1) negative control" (short: U(1) negative control)	Table row	A negative-control question: applying the insertion estimates to four-dimensional U(1) to see where they stop working	27-03
question "oddness of composed insertions" (short: oddness of composed insertions)	Table row	The question of showing that the map composing insertions in separate cells satisfies the identity "odd under insertion"	27-03
question "literature check for slab confinement" (short: literature check for slab confinement)	Table row	The question of checking the Slab Confinement Theorem against arguments of Tomboulis–Yaffe type and the area law of high-temperature spatial Wilson loops (Borgs 1985)	27-03
question "kernels with flip probability near 1" (short: kernels with flip probability near 1)	Table row	The question of building a kernel with flip probability near 1 (for one-step kernels, the flip probability saturates at about 1/2)	27-03
question "quasi-static limit" (short: quasi-static limit)	Table row	The question of showing that, as the number of driving steps goes to infinity, the work W approaches the conditional free energy of the container	27-03
question "error of the shaping map" (short: error of the shaping map)	Table row	The question of bounding the error of the map that shapes the fluctuations (the relative entropy near the vortex surface and higher loops) at intermediate coupling and for large fields	27-03
question "ultraviolet cost of a fixed map" (short: ultraviolet cost of a fixed map)	Table row	Asks whether the extra ultraviolet cost of a fixed map grows in the same way in four-dimensional gauge theory	27-03
question "extending the measurements and their errors" (short: extending the measurements and their errors)	Table row	The question asking for measurements extended beyond the measured sizes of T^2, and for error estimates of driven insertion and replica exchange	27-03
question "non-perturbative proof of the lateral-widening inequality" (short: non-perturbative proof of the lateral-widening inequality)	Table row	The question of showing the lateral-widening inequality non-perturbatively, through $L \sim 1/\Lambda$ between the dilute region and the confining region	27-03
question "uniformity of frame decorrelation" (short: uniformity of frame decorrelation)	Table row	The question of showing frame decorrelation uniformly in L (with L fixed, it becomes a comparison of the chessboard density across scales)	27-03

Name	Kind	What it refers to	Introducing chapter
question "proof of the base point at small T^{2}" (short: proof of the base point at small T^{2})	Table row	The question of showing the base point at small T^{2} under ultraviolet control and uniform acceptance of insertion	27-03
question "excluding a deconfined phase" (short: excluding a deconfined phase)	Table row	The question of showing the exclusion of a non-confining phase, $\Delta F(L,L) \to 0$ (at any speed)	27-03
question "almost area law" (short: almost-area-law question)	Table row	The question of showing the almost area law $\ln(1/\theta(L,L)) \geq cL^{2}/\ln L$	27-03
question "β-monotonicity" (short: β-monotonicity question)	Table row	The question of showing $\partial_{\beta} \Delta F \geq 0$ with the lattice geometry fixed (β-monotonicity)	27-03
question "Z_{2} and U(1) on twisted slabs" (short: Z_{2} and U(1) on twisted slabs)	Table row	The question of checking that, on twisted slabs, β-monotonicity for $\mathbb{Z}_{2}$ and U(1) does not follow from Ginibre's inequality	27-03
question "criterion at the lower end" (short: criterion at the lower end)	Table row	The question of pushing up the lower end with a Dobrushin–Shlosman-type finite-size criterion based on the mixing of box magnetic flux, and with a computer-assisted count of the growth constant of surfaces	27-03
question "bookkeeping at the strong-coupling end" (short: bookkeeping at the strong-coupling end)	Table row	The question of writing out the bookkeeping of the Strong-Coupling-End Expansion Proposition and enlarging the constant $u_{0} \approx 10^{-80}$	27-03
question "four faces of the middle" (short: four faces of the middle)	Table row	The question of showing any one of the four faces of the middle, uniformly in the scale, in a form that breaks down for U(1)	27-03
question "two-sided area laws inside the band" (short: two-sided area laws inside the band)	Table row	The question of showing, at each β inside the band, the two-sided area law at fixed β (if it can be said, the comparison inside the band shrinks to finitely many steps)	27-03
question "what remains of the road of blocks and mixing" (short: what remains of the road of blocks and mixing)	Table row	The question of clearing up what remains of the road of blocks and mixing for the first step (the second and third stages of strong mixing, the check against a Varadhan-type limit and Witten's Morse theory)	27-03
question "convexity threshold and mixing length" (short: convexity threshold and mixing length)	Table row	Asks why, at the first step, the convexity threshold (the first conjugate point at the top ≈ 7.84) and the mixing length (about 9) are numerically close	27-03
question "touchstone of the smallest band" (short: touchstone of the smallest band)	Table row	The question of first testing a tool claimed to cross the band on two-dimensional asymptotically free models with the ultraviolet cutoff removed	27-03

Name	Kind	What it refers to	Introducing chapter
question "comparison principle for large loops" (short: comparison principle for large loops)	Table row	The question of showing the asymptotic form of the comparison principle for large loops only, σ_tw(L) ≤ σ_∞	27-03
question "testing the three new candidates" (short: testing the three new candidates)	Table row	The question of putting the three new candidates (reduction by compactness of the band, log-concavity of the Weyl measure, local step function of a column) through the tests of the selection criteria	27-03

Part 28 The tree of premises

Name	Kind	What it refers to	Introducing chapter
premise of the box continuum limit	Step	One of the groups of the tree of premises: the premises for the chain of block maps on a box of fixed size to give subsequential limits and the properties that pass to the limit (Table 2 of 28-04)	28-04
premise of ultraviolet stability	Step	One of the groups of the tree of premises: the premises for deriving the representation-level rows of the premise of the box continuum limit from one-step estimates (Table 3 of 28-04)	28-04
free-field premise	Step	One of the groups of the tree of premises: the absolute conditions on the free field. Given by reference to the free-field rows of the premise of ultraviolet stability (bracketing of the propagator and others), without separate rows in the table	28-04
premise of full-sequence convergence	Step	One of the groups of the tree of premises: the premises for the full sequence, along a tuned sequence of couplings, to converge without taking a subsequence (Table 4 of 28-04)	28-04
observable premise	Step	One of the groups of the tree of premises: the premises for handling fine observables (flowed observables and others) (Table 4 of 28-04)	28-04
infinite-volume premise	Step	One of the groups of the tree of premises: the premises for taking the infinite-volume limit (Table 4 of 28-04)	28-04
rotation premise	Step	One of the groups of the tree of premises: the premises for recovering rotation invariance in the limit (Table 4 of 28-04)	28-04
non-triviality premise	Step	One of the groups of the tree of premises: the premises that the limit is not a free field (Table 4 of 28-04)	28-04
fermion premise	Step	One of the groups of the tree of premises: the premises for theories containing fermions (outside the scope of the list of 28-04)	28-04

Name	Kind	What it refers to	Introducing chapter
implication of subsequential limits	Route	A conditional implication that, for the pushed-forward measures, if the premise fields hold then subsequential limits exist, and the limit has reflection positivity, symmetry and a lower bound of non-degeneracy	28-01
implication of full-sequence convergence	Route	A conditional implication that gives convergence of the full sequence, without taking a subsequence, as a statement about orbits of an abstract skew-product map	28-02
implication of the gap and the band	Route	A conditional implication that derives the mass gap from the uniform decay of block observables and reduces the band	28-03
continuity, composition and push-forward consistency of the block map	Table row	The block map B is continuous, the map from step to step is the composition of single steps, and the pushed-forward measures are consistent ($(B_m)_*\nu_{n,m+1} = \nu_{n,m}$)	28-04
gauge covariance and locality	Table row	The block map is covariant under gauge transformations ($B(U^g) = B(U)^{g\vert_{\text{coarse points}}}$), and the value of a coarse link depends only on fine links within a finite distance	28-04
commuting with the lattice symmetries	Table row	The block map commutes with the lattice translations, reflections and the hypercubic group	28-04
reflection positivity of the fine lattice	Table row	Reflection positivity of the Wilson measure on the fine lattice: for bounded measurable f depending only on links on the positive side, $\int f(\Theta U)\, f(U)\, d\mu_W \geq 0$	28-04
reflection positivity at the coarse level	Table row	Reflection positivity of the pushed-forward measure: for F in the positive-time class, the integral of the product of $F\circ\Theta$ and F with respect to the coarse-level measure is at least 0	28-04
symmetries of the measure	Table row	The pushed-forward measure $\nu_{n,m}$ is invariant under the lattice symmetries, and the symmetry breaking of observables tends to 0	28-04
family of observables and small-field set	Table row	The family of observables D is countable, dense and uniformly bounded, and the small-field set is closed and contains the two-step events	28-04
recursion of the activity	Table row	The recursion of the activity z_k, $z_{k+1} \leq \rho z_k + A w_k + B z_k^2$ (together with the existence of the limiting expectation and the large-field weight)	28-04
rarity of supply	Table row	The supply w_k is bounded as $w_k \leq \Phi(u_k)$ from exponential moments of the plaquette variables uniform in the step	28-04

Name	Kind	What it refers to	Introducing chapter
single-scale rarity on the fine lattice	Table row	For $\beta \geq 81$, the probability that all plaquettes of a set X have $\mathrm{Re}\, U_p < 0$ is $\leq q_1^{\lfloor \lvert X \rvert / 6 \rfloor}$ ($q_1 < 1$ independent of the volume)	28-04
moving invariant interval and margin	Table row	The effective coupling stays in a moving invariant interval, and the remainders can be paid for by the margin (inequalities of real functions such as ρW(s) + AΦ(s) + BW(s)^2 ≤ W(s'))	28-04
rarity of two-step events: plaquette part	Table row	For the pushed-forward measure, the probability that the plaquette variable β_m(1 − Re U_p) exceeds the threshold y_h(m) is at most P_1(m) (for large n)	28-04
rarity of two-step events: background-invariant part	Table row	For the pushed-forward measure, the probability that the invariant f of the background field exceeds the threshold f_h(m) is at most S(m) (for large n)	28-04
equicontinuity of observables	Table row	Differences of observable values are bounded uniformly as |s_n(d) − s_n(d')| ≤ τ_n + Ω(dist(d,d')) (τ_n → 0, Ω(r) → 0)	28-04
bracketing of the propagator	Table row	For all steps k and momenta, $c_0\, P/\omega \preceq G_k \preceq P/\omega$, $c_0 = 3/100$	28-04
positivity with background	Table row	The smallest eigenvalue of the constrained Hessian around the background field is at least $\lambda_0(1 - f_1/f_\star)$	28-04
contraction of irrelevant terms	Table row	$\mathcal{M}^{17}(\omega G_k) \preceq \tau\, \omega G_{k+17}$, $\tau < 1/170$, uniformly in k	28-04
coefficient of the one-step decrease of the coupling	Table row	The free-field coefficients b_k converge at rate $(3/5)^k$, and the sum of the transients is finite	28-04
value of the one-loop coefficient	Table row	The limit of the coefficient of the one-step decrease of the coupling is the one-loop value b_∞ = 2b_0 ln 2 (a value from the literature; not rederived for this map)	28-04
contraction of the background map	Table row	The sum of the displacements per graduation, $\sum_j \Delta_j < \infty$ (the free-field part is of the type $\|\psi_l\|^2 \leq 0.608^l$)	28-04
geometry of many-body sums (correlation norm)	Table row	The correlation norm of the plaquette fields, $\Lambda \leq 640/9$	28-04
constrained inverse identity	Table row	The identity for the constrained Green function, C = G − GT*(TGT*)$^{-1}$TG, and its upper and lower estimates	28-04
Constrained Decay Theorem	Table row	The one-step constrained Green function at step k decays with respect to exponential weights, with constants uniform in the step	28-04

Name	Kind	What it refers to	Introducing chapter
zero-order transport of the background	Table row	For the step-by-step plaquette ratio θ_n of the background, sup 4ⁿθ_n < ∞ and Σ4ⁿθ_n^{3/2} < ∞ (the present form is known to be false)	28-04
one-step background-field lemma	Table row	For small fields, the minimiser of the constrained action is unique, analytic, exponentially local, and commutes with coarse gauge transformations	28-04
multistep background field	Table row	Unique and regular in nested small-field regions. The inverse of the linearisation is at most 1 in operator norm, independently of the number of steps	28-04
row sums of the lift uniform in the step	Table row	$\sup \lvert Eg\rvert \le R \sup \lvert g\rvert$, with R uniform in k	28-04
single-scale expansion tools	Table row	The cluster expansion on product white-noise measures, and its convergence	28-04
forest interpolation formula	Table row	The forest interpolation formula used for the cluster expansion (from the literature; its body is not in Lean)	28-04
small-field one-step renormalisation theorem	Table row	Coarse action = $\beta_{k+1} S_W$ (background) + local expansion + many-body terms. The many-body terms are exponentially small in the distance	28-04
gauge fixing step by step	Table row	Landau gauge at each step. The determinant is block lower-triangular. An estimate of the Haar Jacobian	28-04
large-field estimate	Table row	For each step t and each hole γ, $\lvert z_t(\gamma)\rvert \le \exp(-\gamma E \beta S^\flat(\gamma))$, $S^\flat(\gamma) \ge s_* \lvert\gamma\rvert$, with $s_* > 0$ uniform in the step and the volume	28-04
large-field gas	Table row	$\sum_{X \ni x} \lvert z(X)\rvert\, e^{2\lvert X\rvert} \le 1$, and the partition function of the gas equals the corresponding ratio	28-04
class and localisation	Table row	The only local gauge-invariant terms of dimension at most 4 are constants and $\sum \operatorname{tr} F^2$	28-04
induction uniform in the step	Table row	$r_{k+1} \le \tau_k r_k + a_k w_k + s_k$, $w_{k+1} \le b_k r_k + \sigma_k w_k + s'_k$; if the products of the coefficients are less than $1/4$, the iterates stay in the ball	28-04
contraction of the one-step map and matching at the endpoint	Table row	With estimates of the remainder of the one-step map and Lipschitz estimates of the coordinates, the map contracts and the coupling at the endpoint can be matched	28-04
approximation of flowed observables by block variables	Table row	Observables at flow time t are approximated by Lipschitz functions of the block variables at step m, with an error $\varepsilon_m(t) \to 0$ uniform in the lattice	28-04

Name	Kind	What it refers to	Introducing chapter
ultraviolet estimate uniform in the box size	Table row	The premise of the box continuum limit and the premise of ultraviolet stability hold with constants independent of the box size L	28-04
one-step ultraviolet estimate for the bridge	Table row	The one-step image from a rotated fine lattice to the standard coarse lattice lies in the ball of the chain	28-04
independence of the infinite-volume limit from the box shape	Table row	The limits along sequences of tori of different shapes agree	28-04
the limit is not a free field	Table row	The limiting theory is not a free field (which quantity to state it in is undecided)	28-04
non-negativity of θ	Table row	If M is even, $0 \le \theta_{N,M}(\beta)$ for all real β	28-04
single-scale criterion	Table row	At a physical size L_*, $\theta(L_*; T^2_{L_*}) < (s^2)^{-1/(s^2-1)}$ (0.63 for $s=2$, 0.7598 for $s=3$)	28-04
question of the cost of a twist-inserting map	Table row	For a gauge-covariant smooth map T, $\mathbb{E}_+[S\circ T - S - \ln J_T] \le 2\,\mathrm{artanh}(0.7598)$ uniformly in the lattice spacing	28-04
monotonicity of the vortex free energy and renormalisation comparison (short: monotonicity and renormalisation comparison)	Table row	The free energy of twist insertion is monotone in β. $s(N,\beta') \le s(2N,\beta)$	28-04
gap of a small twisted box and uniform acceptance of insertion (short: small-box gap and acceptance)	Table row	A gap $\ge c/L$ uniform in the lattice spacing. The acceptance rate of driven vortex insertion is uniform in the lattice spacing	28-04
from the area law to the gap	Table row	From the band (area exponent) to the uniform decay of block observables	28-04
from the area exponent of vortices to the area law of Wilson loops (short: from the vortex exponent to the area law)	Table row	The step that derives the area law of Wilson loops from the vortex free energy being small with an area exponent (from the literature; the typical-frame version has not been rederived)	28-04
tying the thresholds together	Table row	Tying the scale factor and threshold of the plaquette variables of two-step events to the effective coupling and threshold on the coupling side (for example β_m = 4u_m, y_h(m) = y_s(u_m))	28-04
choice of the small-field set and extension of the background invariant (short: choice of the small-field set)	Table row	The two-step events are contained in the small-field set (this holds if the background invariant is extended by a value larger than the threshold, and is false if it is extended by 0)	28-04

Name	Kind	What it refers to	Introducing chapter
constants of the quadratic type of the background	Table row	The constants of the three quadratic types in the background estimates, which make up the Lipschitz constant of the background map	28-04
geometry of many-body sums (reach constant)	Table row	The row sum R of the correlations of the plaquette fields is finite (the reach constant)	28-04
seam to the vessel of the mass gap	Table row	The implication of the vessel (28-03) assumes weak convergence of the full sequence of measures, and that the subspace of observables is closed under time translations. When used along subsequences, it takes the form "at every limit point"	28-04

Appendices

Name	Kind	What it refers to	Introducing chapter
upper side of the chains · lower side of the chains	Assumption	Collecting the upper halves of the three chains gives the upper side, and the lower halves the lower side	A-06
scale decomposition of the variance	Assumption	The variance can be split scale by scale	A-06
uniform Poincaré at each scale	Assumption	The conditional Poincaré constant of each step is uniform	A-06
upper bound on cross-scale conductance	Assumption	The conductance across steps is bounded	A-06
property that the state space is a single line (short: single-line property)	Assumption	The state space is a single line of scales	A-06
build-check step (a checking step of the Lean reference manual) (short: build-check step)	Name from the literature	The first of the four checking steps of the Lean reference manual: see that the editor's check mark, or lake build, finishes with neither warnings nor errors	A-04
axiom-check step (a checking step of the Lean reference manual) (short: axiom-check step)	Name from the literature	The second of the four checking steps: write #print axioms after the theorem and confirm that the axioms it depends on are only the three standard ones	A-04
kernel-recheck step (a checking step of the Lean reference manual) (short: kernel-recheck step)	Name from the literature	The third of the four checking steps: with lean4checker, pass the proofs saved by the build through the kernel again	A-04
isolated-comparison step (a checking step of the Lean reference manual) (short: isolated-comparison step)	Name from the literature	The fourth of the four checking steps: write only the statements in a trusted place, build the proofs in isolation, pass them again through an independent external checker, and confirm that the statements agree	A-04

Name	Kind	What it refers to	Introducing chapter
substitution move, multiplication move, simplification move	Table row	The names of the moves when the three lines on paper (substitute, multiply both sides, simplify) are matched to the four Lean moves	A-04
survey step	Table row	The first of the seven steps a sentence passes through before it appears in the text: collecting literature on each topic, known results and candidate numbers	A-05
cross-checking step	Table row	The second of the seven steps: cross-checking against primary sources, catching errors in bibliographic data, over-paraphrasing and misattribution	A-05
refutation step	Table row	The third of the seven steps: another reader looks for refutations, catching wording stronger than the source, promotion of the degree of certainty and omitted conditions	A-05
reproduction step	Table row	The fourth of the seven steps: reproducing computations, catching miscopied numbers, mixed-up conventions and counterexamples within a finite range	A-05
label step	Table row	The fifth of the seven steps: attaching labels, catching mixtures of "proved", "conjecture" and "numerically supported"	A-05
consistency step	Table row	The sixth of the seven steps: aligning symbols, names and conventions across chapters, catching discrepancies and duplication	A-05
assembly step	Table row	The seventh of the seven steps: assembling and checking mechanically, catching formulas that cannot be rendered, broken links and forbidden words	A-05
gap of definition	Table row	One of the gaps where a Lean proof passes but does not say what was meant: a definition weaker or stronger than intended (for example, defining the gap with ≥ 0 lets 0 pass)	A-04
gap of naming	Table row	One of the gaps where a proof passes but does not say what was meant: the name of the theorem is bigger than the statement	A-04
gap of vacuous truth	Table row	One of the gaps where a proof passes but does not say what was meant: the premises cannot hold simultaneously, so anything can be concluded	A-04
gap of assumptions	Table row	One of the gaps where a proof passes but does not say what was meant: what should be proved is placed among the assumptions, and when it fails is not written	A-04

Name	Kind	What it refers to	Introducing chapter
bracketing of the tension rate (Corollary C of Paper 3) (short: bracketing of the tension rate)	Name from the paper	A corollary of the paper on lattice fractional instantons: bracketing the limit superior and limit inferior of the tension rate	A-07
paper on vortex-insertion identities	Name	The paper, which came out of this book's study, collecting the vortex-insertion identities (Table 4 of A-07)	A-07
paper on the reduction of the band	Name	The paper, which came out of this book's study, collecting the reduction of the band and the numerics for SU(2) and U(1) (Table 5 of A-07)	A-07
paper on lattice fractional instantons	Name	The paper, which came out of this book's study, collecting confinement by lattice fractional instantons (Table 6 of A-07)	A-07

Table 3 Index of names (902 entries). Look things up by name, not by table serial numbers or ranges of labels.

04

Eight readings of the diffusive-relaxation-time assumption (master copy) — why the normalisation is named

The statement "C_P agrees with the square of a length" involves three choices: whether to say it by powers or by a band; whether the length of comparison is the correlation length ξ or the bookkeeping length ξ_{RG}; and whether C_P is measured with the angle $\mathcal{E}_\vartheta$ or the field $\mathcal{E}_A = \mathcal{E}_\vartheta/\beta$. With $2 \times 2 \times 2$ there are eight readings. The following table is the master copy.

Table 4 The eight readings (this table is the master copy; Table 5 of 23-01 is a summary. The two right-hand columns are accounts that presuppose the relaxation-time assumption. The exponent is the power of β multiplying C_P/ξ^2 or $a^2 C_P$)

Level, length, normalisation	Statement	Name in this book	Free approximation	One-loop logarithms Physics
Power, ξ, $\mathcal{E}_\vartheta$	$\ln C_P / \ln \xi \to 2$	Relaxation-time assumption ($\eta_{\mathrm{dyn}} = 0$)	Holds	Does not move
Power, ξ, $\mathcal{E}_A$				
Power, bookkeeping length, $\mathcal{E}_\vartheta$	$\ln C_P / \ln \xi_{\mathrm{RG}} \to 2$	Relaxation-time assumption with the one-scale chain and the bookkeeping-length chain (power)	Holds (the two-loop difference does not show in the power)	Does not move
Power, bookkeeping length, $\mathcal{E}_A$				
Band, ξ, $\mathcal{E}_\vartheta$	$C_P^{(\vartheta)} \asymp \xi^2$	Band form of the relaxation-time assumption [$\mathcal{E}_\vartheta$]	The lower-bound assumption [$\mathcal{E}_\vartheta$] breaks (ratio $g^2/4$, exponent -1)	$-9/22$: the lower-bound assumption [$\mathcal{E}_\vartheta$] breaks

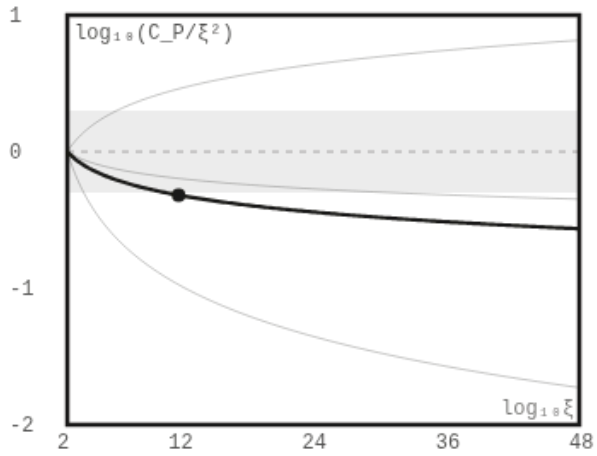
Level, length, normalisation	Statement	Name in this book	Free approximation	One-loop logarithms Physics
Band, ξ, $\mathcal{E}_A$	$C_P^{(A)} \asymp \xi^2$	Band form of the relaxation-time assumption [$\mathcal{E}_A$]	Holds (equality)	$+13/22$: the upper-bound assumption [$\mathcal{E}_A$] breaks
Band, bookkeeping length, $\mathcal{E}_\vartheta$	$a^2 C_P^{(\vartheta)} \asymp 1$	Band form of the relaxation-time assumption [$\mathcal{E}_\vartheta$] with the one-scale chain and the bookkeeping-length chain	Decided by the order of the bookkeeping (23-01 § 07)	$-303/242$
Band, bookkeeping length, $\mathcal{E}_A$	$a^2 C_P^{(A)} \asymp 1$	Band form of the relaxation-time assumption [$\mathcal{E}_A$] with the one-scale chain and the bookkeeping-length chain		$-61/242$

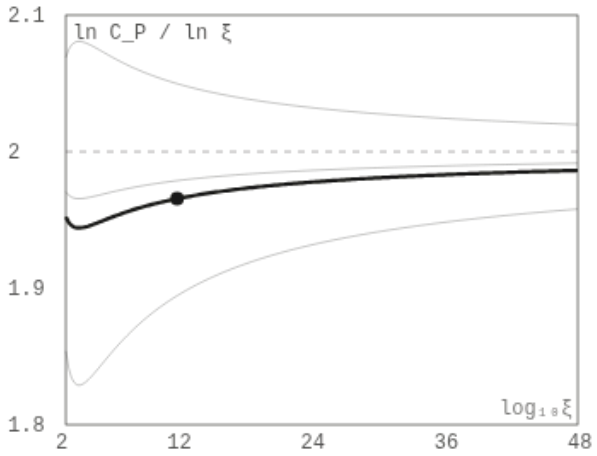
The derivation is in 23-01 The line of construction and the diffusive-relaxation-time assumption and 4-06 § 05; here only the sum is shown (this passage may be skipped). The fractions in the right-hand column are sums of three logarithms: the -1 of the normalisation ($C_P^{(\vartheta)} = C_P^{(A)}/\beta$), the $+13/22$ of the renormalisation of stochastic time (23-01 The line of construction and the diffusive-relaxation-time assumption), and, entering only the rows with the bookkeeping length, the difference between one-loop bookkeeping and two-loop running, $-2 \cdot \frac{51}{121} = -\frac{102}{121}$ ($51/121 = b_1/(2b_0^2)$, independent of N for pure SU(N); 4-06 § 05). All are accounts under the renormalisation group of one-loop perturbation theory, and carry the Physics label.

$$-1 + \frac{13}{22} = -\frac{9}{22}, \qquad \frac{13}{22} - \frac{102}{121} = -\frac{61}{242}, \qquad -\frac{9}{22} - \frac{102}{121} = -\frac{303}{242} \quad (3)$$

We confirmed this sum on this machine in exact rationals Computed . The following figure draws, under this account, how the ratio of the band readings C_P/ξ^2 and the exponent of the power readings $\ln C_P / \ln \xi$ move. β and ξ are tied by the one-loop formula $\xi \approx e^{3\pi^2\beta/11}$ (up to a constant factor, 4-07).

Figure 1 Choosing a reading draws, on the left, the ratio of the band reading C_P/ξ^2 (1 at $\xi = 10^2$; logarithmic vertical axis) and, on the right, the exponent of the power reading $\ln C_P/\ln\xi$. The three buttons choose level, length and normalisation, and the knob chooses $\log_{10}\xi$. The panel with the thicker frame decides the truth of the chosen level. The pale shading on the left is a guide for ratios 1/2–2 (one example of the constants of a band)





Band, ξ, angle graduation → band form of the relaxation-time assumption [angle graduation]: s = −9/22. ξ = 10^12: β = 10.27, ratio 0.4805, power exponent 1.9655. Band: the ratio keeps moving away from 1 (it does not close with a uniform constant)

Computed The numbers in this figure were computed on this machine (the exponent of (3) and the one-loop formula are evaluated on the spot; Table 5 is the Python values).

When JavaScript is disabled, it can be read in Table 5.

Table 5 Ratio and exponent under the one-loop account (SU(2); $\beta = \frac{11}{3\pi^2}\ln\xi$; the ratio is $(\beta/\beta(10^2))^s$ and the expone

Computed Physics

Length, normalisation	s	Ratio $\xi = 10^3$	Ratio 10^{12}	Ratio 10^{48}	Exponent 10^3	Exponent 10^{12}
ξ, $\mathcal{E}_\vartheta$	−9/22	0.8472	0.4805	0.2725	1.9442	1.9655
ξ, $\mathcal{E}_A$	+13/22	1.2707	2.8828	6.5400	2.0806	2.0498
Bookkeeping length, $\mathcal{E}_\vartheta$	−303/242	0.6019	0.1061	0.0187	1.8292	1.8945
Bookkeeping length, $\mathcal{E}_A$	−61/242	0.9028	0.6366	0.4488	1.9656	1.9788
(For reference) free approximation, $\mathcal{E}_\vartheta$	−1	0.6667	0.1667	0.0417	1.8636	1.9157

There are two ways to read it. In the ratio columns all four rows keep moving away from 1, and no band reading closes with a uniform constant. In the exponent columns all four rows approach 2, and the power reading does not move. But the approach is slow: even at $\xi = 10^{48}$ the exponent is still 0.008–0.042 away from 2. The last row is the same formula as Table 9 of A-01; the exponents agree, and the ratios agree once the values of A-01 are divided by the value at $\xi = 10^2$. This table does not measure CP of the Yang–Mills measure; it is an example laying out the consequences of the account. The difference between the two kinds of columns in this table is the reason this book placed the relaxation-time assumption at the level of powers.

05

Bookkeeping and exponent quantities — κ, δ, θ, the bookkeeping length, p, η_{dyn}

This book's study in Part 23 kept the books on a step that coarsens the lattice by a factor b with the following map (23-02 Roads tried on the four-dimensional lattice, and walls).

$$\beta \longmapsto \beta - \kappa\big(\ln b - \delta(b)\big), \qquad a^2 C_P \propto a^p, \qquad p = \eta_{\mathrm{dyn}} + (2 - \eta_{\mathrm{dyn}})\,\theta \ \text{(8)}$$

The two numbers in the left formula and the two exponents in the right formula are the quantities of this section. The quantities are not assumptions, but sentences such as "$\sum \delta < \infty$" and "$\eta_{\mathrm{dyn}} = 0$" are assumptions, and the phrasing of Part 23, "the physical-unit gap statement $\Longleftrightarrow \sum \delta < \infty$ and $\eta = 0$", is the algebra of $p = 0$ presupposing the power form and the model of the one-loop kernel.

Table 6 Quantities of bookkeeping and exponents

Symbol	Kind	One-line meaning	Master	Status
κ	Quantity	The slope per step. $\kappa = 4Nb_0 = 11/(3\pi^2) = 0.3715110$ $(N = 2)$	23-02 Roads tried on the four-dimensional lattice, and walls, 4-06 § 02	Known A rewriting of the one-loop coefficient. Agreement to 5 digits in the window of the model is Computed (agreement within the model)
$\delta(b)$	Quantity	The shortfall per step: the amount by which the window falls short of the continuum $\ln b$. Vanishes as c_2/b^2	23-02 Roads tried on the four-dimensional lattice, and walls	A number measured in the model of the one-loop kernel (not a non-perturbative quantity) Computed
$\sum_k \delta(b_k) < \infty$	Assumption	The shortfalls are summable	Same as above, 18-05 § 06	Assumption. For fixed b, $\delta(b) = 0$ is necessary and sufficient Lean fixed_step_bounded_iff
θ	Quantity	The exponent of accumulation of the shortfall. $a\xi \propto a^{\theta}$. For fixed b, $\delta(b)/\ln b$	23-01 The line of construction and the diffusive-relaxation-time assumption	Definition. If $\sum \delta < \infty$ then $\theta = 0$
ξ_{RG}, a	Quantity	The bookkeeping length $b^{k(\beta)}$, and $a := 1/\xi_{\mathrm{RG}}$. At one loop $\xi_{\mathrm{RG}} \approx e^{3\pi^2\beta/11}$	23-01 The line of construction and the diffusive-relaxation-time assumption, 23-02	The definition of Part 23. The one-loop part is Known (the same formula as $1/(a\Lambda)$)
p	Quantity	The power in $a^2 C_P \propto a^p$. The right-hand side of (8)	23-01 The line of construction and the diffusive-relaxation-time assumption	$p = 0 \iff \eta_{\mathrm{dyn}} = 0 \wedge \theta = 0$ (under $0 \le \eta \le 2$, $\theta \ge 0$) Lean drainExp_eq_zero_iff
z, η_{dyn}	Quantity	The dynamical exponent $C_P \asymp \xi^z$ and $\eta_{\mathrm{dyn}} = 2 - z$. The relaxation-time assumption is $\eta_{\mathrm{dyn}} = 0$	2-10 § 06, 18-05 § 02	Definition. The premise $\eta_{\mathrm{dyn}} \ge 0$ includes $z \le 2$

Symbol	Kind	One-line meaning	Master	Status
η_s	Quantity	The power of static correlations. Different from η_{dyn}. They are equal only within $z = 2 - \eta_s$	Same as above, A-01 § 08	For the two-dimensional Ising model, $\eta_s = 1/4$ and $\eta_{\mathrm{dyn}} = -0.1665$ Physics

The symbol η needs care. In the study of Part 23, η originally referred to two quantities. This book fixes the η of the equivalence between the power form and the physical-unit gap statement as η_{dyn}, and uses η_s only for the lower bound by test functions. An η without qualification is η_{dyn}.

Example. Fixing $b = 2$ gives $\delta(2) = 0.1335$ (weight w) and $\theta = \delta(2)/\ln 2 \approx 0.19$ (23-02 Roads tried on the four-dimensional lattice, and walls). Even with $\eta_{\mathrm{dyn}} = 0$, $p = 2\theta \approx 0.39 > 0$, and one falls on the side $a^2 CP \to 0$. If $b_k = k + 2$ is made larger at each step, then $\sum c_2/(k+2)^2 < \infty$ and $\theta = 0$ can be restored.

06

The assumptions of the second road and the band — names, content, status, what is needed to remove them, U(1)/$\mathbb{Z}_2$

We collect in one table the assumptions set by Parts 25–27 (the road from the direction of space, the crossover band, closing). In this section alone, the status column is written with the five grade labels of Part 23 (23-00 § 01: Proved here, Cited, Computer-assisted, Reading, Assumption). Rows at the stage of a line of proof are graded Assumption, and the existence of the line is written in the same column. The stages are the seven stages a proof must contain of 16-09, and the ledger rows are the rows of 27-02 Table 2; both are referred to by name. The last column is the behaviour of the same assumption when placed in $U(1)$ and in four-dimensional $\mathbb{Z}_2$ gauge theory; it is there to show which rows non-abelianness enters.

Table 7 The assumptions of the second road and the band (the conditional statement is in 27-01; the current state of each row and the rule by which it disappears are in 27-02)

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Ultraviolet control (the ultraviolet stage among the stages a proof must contain; the ledger row has the same name)	Multiscale analysis uniform in a, from the lattice spacing a to the scale L: the small-field analyticity condition (locality and analyticity of the effective action), the large-field rarity condition, and the condition that the coupling stays small (the effective coupling stays in $(0, \gamma]$)	Assumption (Bałaban's inequality (0.31) contains a step that the original source says is unpublished)	Completing the write-up of Bałaban's steps, or checking the original source	An assumption relying on asymptotic freedom; it does not apply to U(1) or $\mathbb{Z}_2$
Uniform acceptance of insertion (small-T² confinement stage)	Under small-field frames, the average of $\tanh(W/2)$ for vortex insertions driven by dynamics is at most $1-\delta$ uniformly in a	Assumption (the numbers showing that the work drops as the procedure is lengthened are Reading)	Estimating the distribution of the work at one-loop (determinant) level	—
Base point at small T^2 (small-T² confinement stage)	At a small physical L_0, $\theta(L_0; T_{L_0}^2) \le 1 - \varepsilon_0$, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$	Assumption (line: under ultraviolet control and uniform acceptance of insertion. The form of the tension is Cited formula (2.37) of Tanizaki–Ünsal)	A proof under ultraviolet control and uniform acceptance of insertion	For U(1), the dependence of ε_0 on L_0 goes the other way
Frame decorrelation (of the same family as "uniform block decay" on a small T^2; small-T² confinement stage and band stage)	Correlations between container frames decay, and the Wilson-loop estimate $\mathbb{E}\prod\theta$ can be split into the product of the individual θ	Assumption . If T^2 is fixed, under reflection positivity of z the chessboard replaces it by a single density κ_{cb} Proved here	An estimate of κ_{cb}. Making it uniform in L requires the uniform slab gap	—

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Reflection positivity of z (small-T^2 confinement stage and band stage)	The ensemble of z summed over twists is positive under reflections through lattice sites in directions 3 and 4	Proved here (no longer an assumption. The algebra of the kernel is in Lean `ReflPos.refl_pos`, `refl_cs`. The chapter giving the paper proof is 26-04)	—	Holds in the same way for $\mathbb{Z}_2$ and U(1) (a structural fact)
Contraction through good blocks (small-T^2 confinement stage and band stage)	Passing through a good block, the influence of the frame contracts to e^{-mw} in the gauge-invariant distance	Assumption	Bałaban-type analysis	—
Lateral-widening inequality, strong form (band stage; a form within the ledger row "positivity of the 't Hooft tension")	With the container fixed, widening the transverse T^2 by a factor s multiplies the sector bias θ by at most Cs^2	Assumption. The numbers are consistent Reading. It does not follow from the structural inputs alone (a negative lemma, 26-04)	Input from non-abelian dynamics (self-dual clusters, asymptotic freedom)	**Breaks** (for U(1) the flux spreads uniformly; for $\mathbb{Z}_2$, surface tension)
Lateral-widening inequality, weak form (band stage; a form within the ledger row "positivity of the 't Hooft tension")	A form of the strong form with a finite total loss (the slab tension σ_{eff} is "almost monotone")	Assumption (almost the same content as the band)	As above	Breaks
Uniform slab tension (band stage; the weak form of the ledger row "positivity of the 't Hooft tension")	The string tension of the slab $\sigma_{\text{eff}}(L)$ is positive uniformly in L	Assumption. The Creutz ratio grows and saturates for N $T = 4 \to 10$ Reading (conditions in 25-08)	The band itself	**Breaks** (for U(1), $\sigma \sim e^2/L^2$ goes to 0)

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Uniform slab gap (band stage; the ledger row "uniform decay of block observables")	The mass gap of the slab is uniform in L: correlations of block observables decay uniformly in L (the ledger row of 27-02)	Assumption	A comparison of $\kappa_{\mathrm{cb}}(L)$ across scales	—
Limit to $\mathbb{R}^4$ (R^4 stage; one of the gaps remaining outside the ledger)	The infinite-volume limit, the OS axioms, recovery of rotation invariance, non-triviality	Assumption (mostly standard lines; the infrared side of rotation and uniqueness of the limit are open)	—	—
Tying the two roads together (tying the gap and confinement; spans the small-T^2 gap stage, the small-T^2 confinement stage and the R^4 stage; one of the gaps remaining outside the ledger)	Obtaining both the area law and the gap of the transfer matrix for the same theory on $\mathbb{R}^4$	Assumption (the statement is not yet fixed; the area law alone does not give a gap)	The uniform slab gap and the limit to $\mathbb{R}^4$	U(1): no confinement and no gap
All compact simple groups (general-group stage; one of the gaps remaining outside the ledger)	$SU(N)$ and groups without centre	Assumption	—	—

Name	Content (one line)	Status	What is needed to remove it	Behaviour for U(1), $\mathbb{Z}_2$
Diffusive-relaxation-time assumption (the upper side is the upper-bound assumption, the lower side the lower-bound assumption; the ledger row "uniformity of the relaxation-time upper bound and the one-sided bridge")	The time for fluctuations to even out grows as the square of the correlation length: $C_P \asymp \xi^2$ (dynamical exponent $z = 2$). The upper-bound assumption, together with the one-sided bridge, gives a lower bound on the gap in physical units	Assumption (23-01). The remaining items of the first step are in 23-15 Table 1 and the subsection on the road of blocks and mixing	The remaining items of Part 23 (23-15), and a form uniform in β (crossing the band in the language of time)	See 23-01, 24-01
Sector bias $\theta(R; T_L^2)$ (quantity)	The absolute value of the frame-conditioned expectation of the sector sign of the container $D \times T_L^2$ (side of D equal to R), $\lvert\mathbb{E}[\sigma_D \mid \Phi]\rvert$	Definition (25-03)	—	Constant for U(1); expected to approach 1 for four-dimensional $\mathbb{Z}_2$ Reading (physics)
Chessboard density κ_{cb} (quantity)	The per-block value (in the volume limit) of $\mathbb{E}\prod_t \theta_t$ when the plane is tiled with blocks of equal size and a container is placed in every block	Definition (26-05)	—	—

A note on symbols. The θ and κ_{cb} of Table 7 are different quantities from the θ (exponent of accumulation of the shortfall) and κ (one-step slope) of Table 6. The sector bias is written with arguments as $\theta(R; T_L^2)$, and the chessboard density with a subscript as κ_{cb}. The point to read in this table is that the rows marked "Breaks" in the non-abelian column gather in the two rows of the lateral-widening inequality and the row of the uniform slab tension. These rows (in the ledger, the row of positivity of the 't Hooft tension and the row of the renormalisation comparison among the inputs of the reduction) are where non-abelian dynamics enters; the other rows are structural facts or rows depending on the ultraviolet wall.

What this chapter can say

What is said	What is not said
The names and introducing chapters of Table 3 are taken from the table of names for the whole book (one name for one object)	The one line of "what it refers to" is a guide. The exact statements, assumptions and statuses are in the introducing chapters
Tables 1 and 2 are the conventions for names and symbols of the chapters of Volume 2	The full correspondence with the identifiers in the source of the published construction (where to find it is in the sources of each chapter)
Proved here For SU(2), $s = \kappa = 11/(3\pi^2)$	The values of the κ of Volume 3 for other groups (the theorems of Volume 3 are for SU(2) only)

Sources and reproduction

Item	Kind	Source / tools
Names and introducing chapters	This book	The text of each chapter. Names are made uniform across the whole book
Names and symbols of Volume 2	Source	The chapters of Parts 6–14, which read yang-mills-group/OpenAI-YM-20261028 (submitted 2026-10-28, version 3456e5e)
Values of s and κ	Proved here	10-01 ($C_g = 8$), 4-06 § 02 ($\kappa = 4Nb_0$)
Tables 4 and 5 (the eight readings and the one-loop account)	Computed, Physics	23-01, 4-06 § 05. The sums of fractions were checked on this machine as exact rationals

Next chapter to read: A-07 Papers that came out of this book.

Papers that came out of this book

> This chapter is this book's own investigation. § 01 and § 07 are guide material.

What this chapter shows — A list of the five papers that came out of this book's study. For the two published on this site's papers page (the one-sided bridge, one-link integration) and the three at the draft stage (the paper on vortex-insertion identities, the paper on the reduction of the band, the paper on lattice fractional instantons), we list the title, the abstract and the main theorems, attach to each main theorem the book's label, its conditions and the reach of machine checking, and show which chapters of the book treat it.

Prerequisite chapters — None. How to read the labels is in 23-00 § 01, and how to read the Lean column is in A-04.

Said first — None of the papers claims a solution of the problem of Yang–Mills existence and the mass gap. The three drafts have not been through checking by outside mathematicians, and the book's labels are not copies of the drafts' statements (only what has been placed in a chapter of the book after its proof was checked is Proved here). The titles of the papers are quoted as in the original (English), and the abstracts and statements are shortened (the names and numbers of theorems are those of the originals).

Order of this chapter

1. The list of five — title, stage, pages, machine checking, chapters of the book (Table 1)
2. The one-sided bridge — the inverse square of the gap bounds the Poincaré constant from below (Table 2)
3. One-link integration — integrating one link before Bakry–Émery (Table 3)
4. Paper on vortex-insertion identities — the vortex-insertion identities (Table 4)
5. Paper on the reduction of the band — the reduction of the band, and numerics for $SU(2)$ and $U(1)$ (Table 5)
6. Paper on lattice fractional instantons — lattice confinement by fractional instantons (Table 6)
7. The papers and the Lean bundles — names of bundles, numbers of theorems, what they guarantee (Table 7)

01

The list of five

The "stage" in Table 1 is either published (the PDF and LaTeX source are on this site's papers page) or draft (material for the study in this book, not published). Pages are pages of the PDF. The authorship follows the form of publication.

Table 1 Papers that came out of this book (as of 2026-10-15)

Paper	Title (original)	Stage	Pages	Machine checking	Chapters of the book
The one-sided bridge	The inverse squared spectral gap bounds the Langevin Poincaré constant of Euclidean path measures from below	Published (2026-10-09, list of papers, PDF)	12	The algebraic skeleton (bundle EuclideanPath.GapBridge, 17 declarations)	23-01 § 06, 24-04 § 02, 24-05
One-link integration	One-link integration before Bakry–Émery for SU(2) lattice Yang–Mills — An explicit strong-coupling threshold β < √(2/27) in dimension four, and why complete families of links exist only for d ≤ 4	Published (2026-09-21, list of papers, PDF)	24	The chain from the Wilson action to the lower bound on the Hessian, and the combinatorics of complete families	5-11, 23-02 § 02

Paper	Title (original)	Stage	Pages	Machine checking	Chapters of the book
Paper on vortex-insertion identities	Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds	Draft (2026-10-13)	3	Algebra on a finite configuration space	25-05, 25-06
Paper on the reduction of the band	Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics	Draft (2026-10-13)	9	The flux estimate, the electric-flux form, the doubling iteration, the algebra of reflection positivity	26-01, 26-02, 26-04, 26-05, 25-07, 26-09
Paper on lattice fractional instantons	Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus	Draft (2026-10-14)	7	The factorisation of the column representation and the binomial estimate	25-02 § 06, 26-06 § 04, 27-02 upper end of the renormalisation comparison

The "window" in the original title of the paper on the reduction of the band is, in this book's vocabulary, "the crossover band". It is a different thing from the windows of Parts 23 and 24 (an interval of sites, a range of $\hbar$, an interval of times), and in the text we write "band" (26-00 § 02). The "rigorous" in the original title of the paper on lattice fractional instantons is the original's word, and is independent of the book's label.

02

The one-sided bridge — the inverse square of the gap bounds the Poincaré constant from below

Abstract. For $H = -(\hbar^2/2)\Delta_M + V$ on a finite-dimensional configuration space (a compact connected Riemannian manifold, or $\mathbb{R}^n$ with a constant metric and a smooth confining potential), assume a discrete spectrum, a simple positive ground state ψ_0 and a gap $\Delta = E_1 - E_0 > 0$. For the Euclidean path measure stationary over all time, the Poincaré constant of the Dirichlet form of stochastic quantisation (Langevin) on smooth time-averaged functionals satisfies $C_P \geq \kappa(\hbar/\Delta)^2$. The test function is the time integral of $F = \psi_1/\psi_0$, and among functionals local in time it is the best. The converse is false in general: for a free particle on the circle, $C_P = \infty$ while $\Delta > 0$. It touches neither four dimensions nor gauge fields.

Table 2 The main theorems of the one-sided bridge (the book's names are those of 23-01 and 24-04)

Number in the paper	Name in the book	Statement (with its conditions in the same row)	Label	Lean
Theorem 3.1	One-Sided Bridge Proposition (Theorem 3.1 of the paper)	Under the setting above and three assumptions (the ground-state assumption, the stationary path-measure assumption and the metric comparison assumption; (A1)–(A3) in the paper: discrete spectrum and simplicity of the ground state, a stationary path measure, the comparison constant κ of the Langevin metric), $C_P \geq \kappa\hbar^2/\Delta^2$. With time taken as continuous	Proved here (23-01 § 06)	Only the algebraic step: `poincare_lower_of_gap` (assumptions: $0 < p, m, \kappa, E$, $2p/m \leq V$, $E \leq 2mp/\kappa$, $V/E \leq C_P$), `window_ratio_closed` (assumptions: $0 < m$, $0 < T$). The path measure, the covariance formula and the ground-state identity are outside Lean
Corollary 3.6	The corollary of the bridge	Under the same assumptions, if $C_P \leq KL^2$ then $(\Delta L/\hbar)^2 \geq \kappa/K$. The paper gives no name to this assumption $C_P(\mathcal{D}) \leq KL^2$ (the assumption of Corollary 3.6 of the paper; of the same type as the upper side $\xi^2\uparrow$ of the book's diffusive-relaxation-time assumption)	Proved here (algebra on top of the assumptions)	`gap_lower_of_poincare_upper` (assumptions: $0 < m$, $\kappa/m^2 \leq C_P$, $C_P \leq KL^2$)
Proposition 4.1	(2) of 24-04	With the canonical metric ($\kappa = 1$), the supremum of the Rayleigh quotient of functionals local in time is exactly $(\hbar/\Delta)^2$, and it is not attained	Proved here (24-04 § 02)	`local_rayleigh_le` (assumptions: $0 < m$, $m \leq a_k$, $0 \leq c_k$), `fourier_weight_le` (assumption: $0 < m \leq a$). Only finite sums and pointwise inequalities

Number in the paper	Name in the book	Statement (with its conditions in the same row)	Label	Lean
Proposition 5.1	Free-Particle-on-a-Circle Counterexample	For a free particle on the circle, $\Delta > 0$ and $C_P = \infty$ (because of the winding number). The converse inequality is false in general	Proved here (23-01 § 06 (5))	`rotor_ratio_unbounded` (assumption: $0 < r$), `rotor_gap_pos` (assumption: $0 < \hbar$)

Scope. Wilson's lattice with discretised time, gauge fields, infinite volume and the continuum limit are outside the statements of the paper. To use it for four-dimensional lattice gauge theory, the time-step comparison assumption ((D) in the paper "one-sided bridge") on the lattice with discretised time remains (24-04 § 08).

03

One-link integration — integrating one link before Bakry–Émery

Abstract. For the $SU(2)$ Wilson action on $(\mathbb{Z}/L)^4$, if the links of a "complete family", containing exactly one link of every plaquette, are integrated first, the Haar integral splits star by star, and the marginal density of the remaining links takes the form $e^{-\Phi}$. The second derivative of Φ along geodesics is at least $-27\beta^2 \sum_e \|X_e\|^2$ for every β and every configuration. Together with the Ricci curvature 2 of the unit S^3 this gives Bakry–Émery curvature $2 - 27\beta^2$ and the threshold $\beta < \sqrt{2/27} = 0.2722$. A complete family exists in $\mathbb{Z}^d$ only when $d \le 4$. The mass-gap step is not a Lean theorem; it is on paper, and moreover a sketch.

Table 3 The main theorems of one-link integration (detailed labels in 5-11)

Number in the paper	Statement (with its conditions in the same row)	Label	Lean
Theorem 1	For L even with $L \ge 3$, and any Haar probability measure on the unit quaternions: (i) integrating the links of a complete family splits star by star and gives $e^{-\Phi}$; (ii) the second derivative of Φ is at least $-27\beta^2 \sum_e \|X_e\|^2$ for every β and every configuration	Lean	`WilsonOneLink.integral_star_links`, `StaplePerturbation.hess_gauge` (assumptions as in the type signatures of § 11 of the paper)
Theorem 2, 2′	A complete family exists in $\mathbb{Z}^d$ only when $d \le 4$. On the torus of period L this is equivalent to "$d \le 4$ and L even". In the version for k-cells, $d \le 3k + 1$, and the density is exactly $1/(2(k+1))$	Lean	`no_perfectZ_of_five` (assumption: $5 \le d$), `perfectK_one_iff` (assumption: $2 \le d$) and others
Corollary 3 (a)	For $\beta < \sqrt{2/27}$, the marginal measure satisfies the Bakry–Émery condition with curvature at least $2 - 27\beta^2$, and has a logarithmic Sobolev inequality uniform in the volume and the boundary conditions	Cited (the Bakry–Émery theorem, the Ricci curvature of S^3) + Theorem 1 of this paper (the row above)	—
Corollary 3 (b)	Uniqueness of the infinite-volume Gibbs measure and exponential decay of correlations (a mass gap in lattice units)	Assumption (a sketch following the earlier logarithmic-Sobolev arguments at strong coupling; not written out line by line)	—

Scope. All that can be said is that "the constant improves within the Bakry–Émery method"; it does not say that the range of strong coupling has been widened. It says nothing about the continuum limit (5-11).

04

Paper on vortex-insertion identities — the vortex-insertion identities

Abstract (draft). The $SU(2)$ Wilson lattice gauge theory is written in the twist-summed ensemble of Mack–Petkova, and a family of identities without approximation is shown for local centre-vortex insertions. For an involutive insertion τ aligned with the sector (change of action ΔS) and an observable H odd under τ: $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ and $\mathbb{E}[H\sqrt{\rho}] = 0$ ($\rho = e^{-\Delta S}$). Combining insertions in separated cells and averaging over the position of the

insertion gives $|\langle W(C)\rangle| \le \mathbb{E}\prod_c |\bar{t}_c|$ (each factor in $[-1,1]$). For insertions driven by reversible heat-bath dynamics, a Crooks-type identity $\mathbb{E}[H] = \mathbb{E}[H\tanh(W/2)]$ holds for the work W. The slab-confinement theorem for every β is recovered as the special case of a thin insertion. The draft places these as rewritings within the framework of Tomboulis–Yaffe, Mack–Petkova and Kovács–Tomboulis, and says nothing about the continuum limit or about the problem itself.

Table 4 The main theorems of the paper on vortex-insertion identities (the book's labels are those of 25-05. The conditions throughout: a finite configuration space, positive weights, a bijection that flips the sector)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
bounded identity	If τ is an involution and $H\circ\tau = -H$, then $\mathbb{E}[H] = \mathbb{E}[H\,t]$, $t = \tanh(\Delta S/2)$	Proved here	BoundedInsertion (`Bounded.single`) (assumptions: τ is an involution, the weights are positive, $H\circ\tau = -H$)	25-05 § 04
Bhattacharyya identity	Under the same conditions, $\mathbb{E}[H\sqrt{\rho}] = 0$	Proved here	SecondMoment (`SecondMoment.sqrt_identity`) (the same)	25-05 § 05
averaged bound	For insertions in separated cells averaged over position, if $\lvert H\rvert \le 1$ and it is odd under every insertion, then $\lvert\mathbb{E}[H]\rvert \le \mathbb{E}\prod_c \lvert\bar{t}_c\rvert$	Proved here	BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (assumptions: the t of other cells is invariant under the insertion, $\lvert H\rvert \le 1$)	25-05 § 04
Paley–Zygmund form	For one cell, $\mathbb{E}[H]^2 \le 1 - \mathbb{E}[Y]^2/\mathbb{E}[Y^2]$, $\mathbb{E}[Y^2] \le 1$	Proved here	SecondMoment (`SecondMoment.pz_bound`) (does not include the part $\mathbb{E}[Y^2] \le 1$)	25-05 § 05
slab confinement (corollary)	With thin insertions, $\lvert\langle W(C)\rangle\rvert \le \tanh(\beta N^2)$ $A(C)$ (slab, N and the lattice spacing fixed, every β)	Proved here	— (the bundle for the slab theorem is in 25-04)	25-05 § 06, 25-04
driven identity	If $H = h(\mathrm{frame})\,\sigma$, then $\mathbb{E}[H] = \mathbb{E}[H\tanh(W/2)]$ (W is the work of an insertion driven by reversible heat bath)	Proved here (the Crooks and Jarzynski identities are Cited)	—	25-06
Numerics	On $2\times2\times8\times8$, $\beta = 2.3$, $n12 = 1$, with a disc of radius 2, driving with 16 heat-bath steps takes the mean work $10.7 \to 2.9$ and the mean of $\tanh(W/2)$ $0.99 \to 0.69$	Reading (the draft does not state the number of seeds or the errors)	—	25-08

Novelty. The draft lists things such as "local identities in the twist-summed ensemble" as its own contributions, but the book has not checked them against the earlier literature and does not claim novelty (25-05 § 08 (d)).

05

Paper on the reduction of the band — the reduction of the band, and numerics for $SU(2)$ and $U(1)$

Abstract (draft). Heads towards lattice confinement for four-dimensional $SU(2)$ by the road of the slab $T^2_L \times \mathbb{Z}^2$ through a small twisted torus T^2_L. It states four points. (i) The naive condition "$\theta(L,L) \le \theta_0 < 1$ uniformly in L" is also satisfied by four-dimensional $U(1)$, so it does not imply confinement; the correct target is $\ln(1/\theta(L,L)) \ge cL^2$ (a local form of the 't Hooft criterion). (ii) Combined with nesting in the planar directions, the target reduces to a single lateral-widening inequality. This inequality fails for $U(1)$ and $\mathbb{Z}_2$, and for $SU(2)$ in the dilute regime its sign is set by $11/3 - 2 = 5/3 > 0$. (iii) Reflection positivity, the identities of centre and duality, and positivity of the transfer matrix hold just as well for $\mathbb{Z}_2$ and $U(1)$, so structural inputs alone cannot show the lateral-widening inequality. (iv) Monte Carlo on the slab shows the predicted contrast. At one loop, the comparison from maps isometric in the fluctuations can give only the $U(1)$ direction $\beta' \ge \beta + 0.023$, while maps that reshape the fluctuations give $\beta - \beta' = 0.258$. It shows neither confinement in the continuum limit nor the mass gap.

Table 5 The main statements of the paper on the reduction of the band (the book's labels are those of the chapters on the right)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
uniform θ_0 is insufficient	From the naive form of the band $\theta \le \theta_0$ only the lower bound on the tension $\ln(1/\theta_0)/L^2 \to 0$ follows. The Coulomb phase of $U(1)$ satisfies the naive form of the band	Proved here (the logic of the shortfall) / Reading (the count for $U(1)$, a physical estimate; the draft also states it is heuristic)	—	26-01
conditional theorem	Under the three of the base point on a small T^2, frame decorrelation and the lateral-widening inequality, $\theta(R,R) \le e^{-cR^2}$, and an area law with tension uniform in L	Proved here (conditional. The draft's proof is a sketch; the book rewrote the assumptions as numerical inequalities and placed a proof)	Doubling (`Doubling.iterate`, `Doubling.decay`) (assumptions: $c \ge 0$, $t_k \ge 0$, $t_{k+1} \le c^3 t_k^4$. Only the algebra of the iteration)	26-02 § 03
necessary condition	The lateral-widening inequality (strong form) contains that the slab tension $\sigma_{\mathrm{eff}}(L)$ does not decrease in L (stronger than confinement)	Proved here	—	26-02 § 05
the inequality separates the theories	The lateral-widening inequality fails for $U(1)$ and $\mathbb{Z}_2$, and for $SU(2)$ in the dilute regime it grows as $\ln(1/\theta) \propto A_D L^{5/3}$	Reading (a physical estimate; the draft also states it is heuristic)	—	26-03
no-go from structure	The flux estimate, the electric-flux form, positivity of the transfer matrix, the identities of centre and duality and the lattice symmetries hold just as well for $\mathbb{Z}_2$ and $U(1)$, so from this set of inputs alone the lateral-widening inequality does not follow	Proved here (in the form "from this set of inputs alone")	FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`) (for $\theta \le 1$, positivity of the twisted partition functions is assumed)	26-04
site-reflection positivity	The z ensemble with the twist fixed is positive under reflections through lattice points in directions 3 and 4 (M even). Transverse reflections are not positive because of the twist	Proved here (reflection positivity of z. The proof is placed in 26-04 § 03. This row is closed)	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (assumption: $A \ge 0$. The algebra of the core)	26-04, 27-02 reflection positivity of z
chessboard (corollary)	With L fixed, frame decorrelation is replaced by a single density κ_{cb}, and $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$	Cited (the chessboard estimate) + Proved here (the statement with L fixed)	—	26-05
flux bound	For a stationary kernel that flips the sector, $\theta \le 1 - a$. For a one-step kernel $a \lesssim \frac{1}{2}$, so it gives only an estimate of the naive form of the band	Proved here	FluxLemma (`FluxLemma.flip_le_one_sub_abs`)	25-06
The one-loop comparison	The comparison isometric in the fluctuations gives $\beta' \ge \beta + 0.023$ (per doubling), and the comparison that reshapes the fluctuations gives $\beta - \beta' = 0.258$	Proved here (at the one-loop level)	—	25-07, 26-06 § 03

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Numerics	Widening the T^2 raises the free energy of the global flux for $U(1)$ and does not raise it for $SU(2)$. The slab tension increases and saturates. In the lattice-doubling test, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ for $SU(2)$ ($\beta = 2.5$–3.0, 1–3 seeds), and in the opposite direction for $U(1)$	Reading (the conditions on lattice sizes and errors are in 26-00 § 03 (ii) and the rows of 26-09)	—	26-09, 26-06 § 05

A planned addition in the record. Adding to the paper on the reduction of the band the scan with step functions and the $U(1)$ control (the sign flip at $\beta_c \approx 1.01$) is listed as one row in the order of work (27-03 § 06).

06

Paper on lattice fractional instantons — lattice confinement by fractional instantons

Abstract (draft). In four-dimensional $SU(2)$ Wilson lattice gauge theory on $T^2_N \times T^2_M$, with the 't Hooft twist $n_{12} = 1$ put on the small torus T^2_N, it considers the bias $\theta(N, M, \beta)$ of the flux n_{34} (the quantity that bounds large planar Wilson loops through the Tomboulis–Yaffe inequality). At fixed N, an area law with rate $\approx e^{-2\beta N^2}$ follows elementarily from the cost of the surface of a thin $\mathbb{Z}_2$ vortex. The draft shows a lower bound that replaces the rate by the minimal action $S_*(N, \ell)$ of a fractional instanton inside a frame frozen to the vacuum. At fixed M it also shows the bound in the opposite direction, sandwiching the exponential rate between $S_{\min}(N, M)$ and $S_*(N, M/2)$. It is not uniform in N and says nothing about the continuum limit.

Table 6 The main theorems of the paper on lattice fractional instantons (the book's labels are those of 25-02)

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Confinement Lower Bound by Fractional Instantons (Theorem A of the paper, confinement direction)	For all $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, $-\ln\theta \geq c(N,\ell)\,\beta^{-a} e^{-\beta(S_*(N,\ell)+\varepsilon(\beta))} M^2$, $\varepsilon(\beta) = O(\sqrt{\ln\beta/\beta})$. It beats the thin-surface estimate only beyond a large β that is not made explicit	Assumption (proof in the draft. It uses the Chessboard Estimate and the Morse–Bott lemma by citation, and the rigidity constants are only shown to exist. Not raised until a proof is placed in the text)	Only the factorisation of the column representation and the binomial estimate (ColumnChessboard (`ColumnChessboard.signed_sum`, `ColumnChessboard.binomial_bound`) and others)	25-02 § 06, 26-06 § 04
Reverse Estimate at a Fixed Plane (Theorem B of the paper, fixed-M reverse bound)	For all $N, M \geq 2$, $\beta \geq 1$, if $x = 2\exp(-\beta S_{\min}(N,M) + N^2M^2(12\ln\beta + 34)) \leq \frac{1}{2}$ then $-\ln\theta \leq 2x$. A statement with M fixed	Assumption (the same)	—	25-02 § 06
Bracketing of the Tension Rate (Corollary C of the paper)	At fixed N and $M \in 2\ell\mathbb{N}$, the lower and upper limits of $-\ln(-\ln\theta)/\beta$ as $\beta \to \infty$ lie between $S_{\min}(N,M)$ and $S_*(N,\ell)$	Assumption (on top of the Confinement Lower Bound by Fractional Instantons and the Reverse Estimate at a Fixed Plane)	—	27-02 upper end of the renormalisation comparison
trivial bound (proposition)	For every frame $\lvert\theta_t\rvert \leq \tanh(\beta N^2)$, and at $\ell = 2$, $-\ln\theta \geq \frac{M^2}{2} e^{-2\beta N^2}$	Proved here (of the same type as the slab-confinement theorem)	—	25-04

Name in the paper	Statement (with its conditions in the same row)	Label	Lean	Chapter of the book
Upper bounds with certificates	$S_*(2,12) \le 0.6762\pi^2$, $S_*(3,12) \le 0.9193\pi^2$, $S_*(4,8) \le 0.9707\pi^2$ (estimates of the action of concrete configurations; only in the upper-bound direction. The value for $N=4$ is one where cooling has not converged)	Computer-assisted (upper bounds only; not attached to the minimum itself)	—	25-02 § 05

What is open. The converse bound uniform in M (an upper bound by a dilute gas), and the agreement $S_* = S_{\min}$. The draft reduces the former to a single positivity lemma (27-03 question "reverse direction of the fractional-instanton lower bound", question "agreement of the minimal actions"). The direction of increasing N is the same problem as the middle of the band.

07

The papers and the Lean bundles — names of bundles, numbers of theorems, what they guarantee

Table 7 lists the machine-checked part of each paper by bundle (a set of Lean files). Every bundle guarantees only the algebra of finite sums and real numbers from the list of assumptions to the conclusion; the Haar measure, path measures, the topology of sectors, that θ is the bias of a sector, and positivity of the transfer matrix are outside Lean. The location and naming of the bundles follow appendix A-04.

Table 7 The Lean bundles of the papers (the number of theorems is, for the one-sided bridge, the number of rows of the axiom record; for the bundles of Parts 25 and 26, the number of theorem and lemma declarations in the files)

Paper	Bundle (files)	Theorems	Contents	Axioms
The one-sided bridge	EuclideanPath.GapBridge	17	The closed form of the ratio for an interval of times, the abstract form of the Poincaré lower bound, sums over families local in time, the corollary of the bridge, the rotor	A subset of the three standard axioms (the published axiom record, A-04 Table 4c)
One-link integration	LatticeGaugeOneLink (StarInequality, StapleCurve, OneLinkSeries, CompleteFamilyLattice, StaplePerturbation, WilsonOneLink), PerfectFamily, PlaquetteCounting	— (the axiom record in the list of papers)	The chain from the Wilson action to the lower bound on the Hessian of Φ, the combinatorics of complete families	A subset of the three standard axioms (the published axiom record)

Paper	Bundle (files)	Theorems	Contents	Axioms
Paper on vortex-insertion identities	InsertionIdentity (namespace Insertion, 5), BoundedInsertion (namespace Bounded, 5), SecondMoment (2), MultiCell (3)	15	Identities for the insertion ratio, the tanh form, averaging over position, the second moment, composing cells ("odd under insertion" is an assumption)	A subset of the three standard axioms (passed the re-check on the side of this book that reads the axioms again)
	FlipLemma (2)	2	The estimate of the flip probability and the cutoff formula (the flow lemma of 25-06)	
	Containers (2)	2	The factorisation into a product of containers and the container estimate (25-03)	
Paper on the reduction of the band	FluxLemma (1)	1	The flux estimate in the form $\theta \le 1 - \lvert a \rvert$	
	FluxDuality (3)	3	Decomposition of the electric flux, the estimate of the twisted sum, the range of the bias (positivity is assumed for $\theta \le 1$)	
	Doubling (2)	2	The doubling iteration $t_{k+1} \le c^3 t_k^4 \Rightarrow c t_k \le (c t_0)^{4^k}$, and decay	
	ReflPos (2)	2	The algebra of the core of reflection positivity (Cauchy–Schwarz on weights with $A \ge 0$)	
Paper on lattice fractional instantons	ColumnChessboard (5)	5	Signed and unsigned sums of the column representation, expansion of the product, the binomial estimate, the exponential estimate	

The bundles of Parts 25 and 26 come to 11 files and 32 theorems and lemmas in all. In addition there is a separate bundle (150 theorems) containing the Slab Confinement Theorem and the container lemmas (`SlabConfine` , `SlabAverage` , `FrameLemma` , `CommutingFlips` and others), which 25-03 and 25-04 refer to. It is not a bundle of the papers, and it has passed the same re-check. The names and location of the bundles will be decided when they are added to the table of bundles in appendix A-04.

What this chapter can say

What is said	What is not said
The titles, stages and page counts of the five papers, and the statements and conditions of their main theorems (Tables 1–6)	The correctness of the proofs of the three drafts. Rows whose label in the book is Proved here rest on proofs checked in chapters of the book, and rows labelled Assumption rest on the statements of the drafts
The reach of machine checking for each paper (Table 7)	That the Lean bundles check the statements of the papers themselves (measures, sectors, limits)
That none of the papers claims a solution of the problem of existence and the mass gap	Outside checking of the papers (none of them has been through checking by outside mathematicians)

Sources and reproduction

Item	Kind	Source / tools
Titles, abstracts and statements of the one-sided bridge and one-link integration	This book	The PDFs and LaTeX sources on this site's papers page (list of papers). The correspondence between theorem numbers and Lean declarations is the correspondence table of the one-sided bridge paper (§ 9)
Titles, abstracts and statements of the three drafts	This book	The LaTeX source of each draft (paper on vortex-insertion identities: 2026-10-13, 3 pages; paper on the reduction of the band: 2026-10-13, 9 pages; paper on lattice fractional instantons: 2026-10-14, 7 pages). The book's labels are from the classification tables of the chapters in the right-hand column

Item	Kind	Source / tools
Numbers of theorems in the Lean bundles	This book	For the one-sided bridge, the published axiom record (17 rows). For the bundles of Parts 25 and 26, the theorem and lemma declarations in each file were counted (11 files, 32)
The twist-summed ensemble, the container identity	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the source table of 25-03)
Upper bounds on Wilson loops (from the vortex free energy)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442 (the source table of 26-01)

Terms used in this chapter — published, draft, the main theorem, label, bundle, the twist-summed ensemble, the crossover band, the lateral-widening inequality, fractional instanton, container. Their meanings are in the A-02 Glossary.

Next page to read: A-08 A map to the mass gap. To read alongside: Contents. Which condition rows of the book the statements of the papers bear on is in 27-02 The ledger of conditions.

A map to the mass gap — with the published construction as the foundation

This chapter is a map of what is unsolved. The reading of the published construction is in Parts 6–13, and the theorems of this book's study are in Parts 19–22.

What this chapter shows — With the construction of four-dimensional Yang–Mills theory published on 2026-11-10 taken as the foundation, what must be guaranteed on the way to a proof of the mass gap. In three tables we set out what is cited as the foundation, the seams that remain on the ultraviolet side, what remains on the infrared side (the crossover band), and where the theorems closed by this book's study stand.

Prerequisite chapters — 6-04 What is not shown — a bridge to Volume 3, 16-03 The infrared edge of the published construction, 16-04 Where this book stands. For how to read the labels, see 6-00.

Said first — The standing of the theorems of the published construction is "the committee's assessment is that they are proved in Lean". What this book has read goes as far as the definitions and the form of the statements, and the committee itself writes that it does not yet claim to understand the proof completely Assessment . The "Now" column is written within the range in which this book has read the source and the assessment. This map is a set of signposts, not a scale of progress. This book does not claim to solve the problem of Yang–Mills existence and the mass gap.

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The labels used throughout the book (Cited , Lean , Computed and so on) have the same meanings as before (for details, see "Labels" in 6-00).

Order of this chapter

1. Role of this chapter
2. What must be shown — one line, and the shape of the question
3. The foundation — what is cited from the published construction (Table 1)

4. What remains on the ultraviolet side — the seams (Table 2)
5. The infrared side — the crossover band (Table 3)
6. Guarantees needed beyond the tables
7. Where the theorems closed by this book's study stand
8. What cannot be said

01

Role of this chapter

The existence of four-dimensional Yang–Mills theory: for the construction published on 2026-11-10 (Lean code and the committee's assessment), the committee's assessment is that it is proved in Lean (Part 6). The mass gap is not shown (6-04, 16-02).

This chapter takes that construction as the foundation and sets out again what must be guaranteed on the way to a proof of the mass gap. It reads as follows.

1. In § 02 we write in one line what must be shown.
2. In Table 1 of § 03 we list what is cited from the published construction as the foundation.
3. Table 2 of § 04 is what remains on the ultraviolet side, and Table 3 of § 05 what remains on the infrared side. On the ultraviolet side these are "seams"; on the infrared side, the "core".
4. In § 07 we write where on this map the theorems closed by this book's study stand.

02

What must be shown

One line — The correlations of renormalised lattice observables decay at an exponential rate, in physical units, uniformly in the cutoff and the volume.

The decay is needed for all members of a family that is dense in the Osterwalder–Schrader space (field words of every degree). The prefactor may depend on the observable; the rate must be uniform. Once this holds, the mass gap of the limiting theory and the one-dimensionality of that theory's zero-energy space (uniqueness of the vacuum) follow from known general theory Cited . Uniqueness of the infinite-volume limit does not follow from this. What the published construction has goes only as far as an estimate at a polynomial rate, uniform over the phases Source (12-08).

The shape of the question — The published construction takes the continuum limit while holding the coupling of the coarsest step (the lattice of spacing 1) at a reference value $\bar{\beta}$ (16-03). So, speaking of one construction, the main part of the remaining question is expected to be brought down to a question on the lattice: "does the effective theory at the spacing-1 step have exponential decay at long distances?" Inferred . To restate it that way, however, the top three rows of Table 2 are needed.

- The theory at the spacing-1 step is not the Wilson action but a theory of an effective action with a representation, and the reference coupling is a coefficient of it. The representation is not unique Source (10-01).
- "Long distances" means, by one-loop extrapolation, of the order of $e^{\bar{\beta}/s}$ in units of spacing 1 (s is the one-loop slope) Inferred .
- From just one of the conditions imposed on the reference coupling ($x^{-1/2} e^{(\log\log x)^2} \leq 1$ for all $x \geq \bar{\beta}$), $\bar{\beta} \geq e^{13.7} \approx 9 \times 10^5$ is needed Computed . In the standard conventions for SU(2) this is about $g^2 \leq 4.5 \times 10^{-6}$ (conversion as in 8-01). So the distance above is an astronomical length, at least $e^{2.4 \times 10^6}$.

The remaining question is about as hard as, or harder than, the long-open question of the mass gap of weakly coupled lattice gauge theory Inferred .

03

The foundation — what is cited from the published construction

Item	Form in the published construction	Caution
Continuum limit in a finite periodic box	Wilson's four-dimensional lattice, any compact simple group. In the cutoff, convergence along the whole sequence of spacings Source (8-05, 11-10)	Only cubic boxes with periodic boundary conditions
Infinite volume, phase decomposition, reconstruction	In the volume, along a subsequence. Wightman axioms for each phase Assessment (12-05, 12-07, 12-08)	No convergence or uniqueness of the limit
Running of the coupling on the weak-coupling side	The coupling at each step lies between half and twice "the reference value + the one-loop slope × the logarithm of the scale ratio" (as a coefficient in the representation) Source (10-05, 11-00)	Not a quantity determined by the measure (a coefficient of the effective action). It works only where the coefficient is larger than the entry value of the one-step estimate. The one-step estimate is for odd blocks with a large side
All groups	Existence, for any compact simple group Assessment	—

Table 1 The foundation. The standing of the theorems of the published construction is "the committee's assessment is that they are proved in Lean".

04

What remains on the ultraviolet side — the seams

Ultraviolet control, as existence, has entered the foundation. What remains are the seams that put the ultraviolet results into a form the infrared argument can use.

What is to be guaranteed	Now	Needed for the mass gap?
Fixing the theory at the spacing-1 step as an object (the limit with the cutoff removed, a representation of the same form, consistency between steps)	That the ingredients and density of the representation converge in the cutoff is in the statements Source . The limiting object and its representation are not in the statements (11-10, 16-03)	Needed
Local estimates uniform in the volume	The one-step ledger is in a form independent of the volume. The estimates at the level of densities contain exponents of the volume, and an estimate uniform in the volume for local observables is not in what we have read Source	Needed
The bridge for observables (continuum correlation = expectation of the image of the observable at the spacing-1 step)	There is an estimate of the difference between the read-out at fine steps and the true correlation. The image and the equality at the spacing-1 step are not there Source	Needed
Translation into quantities determined by the measure (the coupling as a coefficient of the representation ↔ vortex ratios, slab tension, coarse Wilson loops)	None	Needed on this book's route
Twisted boundary conditions and elongated boxes	None (only periodic cubic boxes)	Needed on this book's spatial route (thickening from a small twisted box, Parts 25–27). The conditions of that route depended on ultraviolet control on a small twisted box, so this becomes the first question
How reflection positivity is carried	Present only for the original Wilson measure Source (12-01). Not stated for the measures at renormalised steps	Depends on the route (not needed if the decay is stated for correlations of the original lattice)

What is to be guaranteed	Now	Needed for the mass gap?
Uniqueness of the infinite-volume limit, independence of the sequence of spacings and the reference value	None	Not needed for the mass gap of one theory. If uniform exponential decay holds, that limit has clustering Inferred
Understanding of the foundation, and independent verification	What this book has read goes as far as the definitions and the form of the statements. The committee also writes that it "does not yet understand it completely" Assessment	Reliability of the foundation

Table 2 What remains on the ultraviolet side. The top three are the main line; the fourth and fifth are for this book's route.

05

The infrared side — the crossover band

The core is the crossover band between the strong-coupling end (the side where the mass gap is easier to show) and the weak-coupling end (the side where the continuum limit is taken) (26-00).

What is to be guaranteed	Now
Fixing the two ends of the band	The strong side is this book's Lean window: the infinite-volume limit and the mass gap for $\lvert\beta\rvert < 1/19$, the area law for $\beta < 1.1 \times 10^{-3}$ in a finite box and $\beta < 1.2 \times 10^{-4}$ in infinite volume Lean (20-03, 20-06). The weak side is the lower bound on the coefficient of the effective action required by the one-step estimate of the published construction (not the β of the Wilson action). The entry value is in an existence form, "it holds if large enough", and the source gives neither a formula nor a number Source . From one of the conditions on the reference coupling alone, at least 9×10^5 Computed . The two ends are stated for different quantities and different classes of action, so restating them in the same quantity is also needed
The core of the band — a quantity that is preserved across scales, distinguishes from U(1), and gives decay on the strong side	None. This book's form is the tension-doubling inequality for slab tension (the reduction is in Lean; the lateral-widening inequality itself is open, 26-02). Another form is a mixing condition in a finite box (the time-direction route, 26-07)
Extending the strong-coupling end to classes of effective actions	For the Wilson action it is in Lean (infinite volume, mass gap, area law, vortex floors). Not for classes of effective actions. The window is narrow
Bringing the decay back to the original observables	None

What is to be guaranteed	Now
From tension (area law) to the mass gap	False as a general implication. What should be added has not yet been stated
From decay to the spectrum	General theory is known Cited . For the strong-coupling lattice it is in Lean (20-04). Not within the reconstruction framework of the published construction
Groups without a centre	The vortex tools cannot be used. The mass gap for all groups needs other tools

Table 3 The infrared side.

How to read it.

- The published construction gives no foothold inside the band. It carries every step at weak coupling and stops at spacing 1 Inferred (16-03).
- At the weak end there is a non-perturbative input that "the finer the step, the weaker the coupling" (as a coefficient of the effective action; an input that uses non-commutativity) Source . But it works only at the very weak-coupling end, where the coefficient is 9×10^5 or more.
- The "input that distinguishes from U(1)" missing from the argument on the band is needed in the middle section (around $\beta \approx 1$, where U(1) has its transition), and the input above does not address it Inferred .
- To use the input above in this book's doubling inequality, four things are needed Inferred : (i) the relation between the coefficient of the representation and the ratio of twisted partition functions and slab tension, (ii) a comparison between the effective action and the Wilson action, (iii) transfer to the one-step map with sides 2 and 3, (iv) the construction with twisted boundary conditions.
- To reach the mass gap rather than confinement, the row "from tension to the mass gap" of Table 3 is needed. The tension route alone is not enough.
- The map of ideas this book's study has tried on the band (alive, broken, undecided) is in 16-11, and the seven models on which arguments about the band are tested are in 26-12.

06

Guarantees needed beyond the tables

Besides the rows of Tables 1–3, we list guarantees that will be needed when a proof is written out Inferred .

1. Decay at a uniform rate for field words of every degree (the prefactor may depend on the observable).

2. The order of volumes: uniformity where the side of the box is much larger than the correlation length. On the route with twisted boundary conditions, also the order of limits for elongated boxes.
3. Identification of the state: if the decay is obtained with twisted boundary conditions or in a pure phase, the theory built from that state must satisfy the axioms and be the same as the theory of the published construction. Where decay of a mixed state is needed, decay in each pure phase may not be enough (the low-temperature Ising model is an example).
4. Lorentz covariance: used to turn the Hamiltonian gap into a mass gap. In the published construction it is present for the component theories Assessment .
5. General SU(N): this book's Lean theorems are for SU(2) only.

07

Where the theorems closed by this book's study stand

The Lean theorems of this book's study are all statements at fixed lattice spacing, and are formalisations of results of known types Lean (listed in Table 1 of 16-04). On this map they stand in the following places.

- **The strong-coupling end**: exponential decay of correlations, the infinite-volume limit and clustering, the mass gap of the reconstruction container (in lattice units), the area law for Wilson loops, an upper bound on the vortex free energy (20-02–20-06). This is the strong end of the first row of Table 3. The rate per unit area of the upper bound on the vortex free energy does not depend on the transverse size, but only within the strong-coupling window, and the prefactor (the transverse area) remains.
- **At any coupling**: reflection positivity, the thin-vortex floor and the thick-vortex floor, pairs of Polyakov loops, continuity, limits and matching of vortex ratios (Part 21). None of these is uniform in the transverse size.
- **The weak-coupling side**: rarity of large surfaces at one scale (22-01). It says nothing about scales after renormalisation.
- **Specific to this book's design**: a fixed block map independent of the cutoff (consistency between steps written as a push-forward identity), reflection positivity of the pushed-forward measure, matching by quantities determined by the measure (vortex ratios, averages over coarse surfaces) (14-06). The matching is still at the stage of design choices.

What this book's study has on the band is a conditional implication and a list of premises (27-01, 28-04). It has not shown that the premises are met. The question aimed at by the conditional implication about weak ultraviolet existence in a small box

that this book's study assembled (2026-10-30) has been answered in a stronger form by the theorems of the published construction Inferred . The premises of that implication are statements about this book's design and do not follow from the results of the published construction.

There are three candidates for the next move Inferred . (1) Read how the ratio of twisted partition functions comes out of the representation of the published construction. (2) Read the entry value of the one-step estimate (the weak end of the band). (3) Widen the window at the strong-coupling end and extend it to classes of effective actions.

08

What cannot be said

1. Even if all rows of the tables were filled, we do not say that this would solve the problem. The tables list the guarantees visible within what this book has read, and we do not say they are exhaustive.
2. The rows cannot be counted and read as "so many percent left".
3. The "Now" column covers what was read of the definitions and the form of the statements of the published construction. This book has not checked that the content of the proofs is correct.
4. This book's strong-coupling results (the infinite-volume limit, the mass gap, the area law, the vortex floors) are statements at the end opposite to the remaining question (strong coupling, fixed lattice spacing).

A cross-cutting list of what is not claimed is in 27-04, and what the published construction does not show is in 6-04.

What this chapter can say

Can say	Cannot say
Source The "Now" column of Tables 1–3 was written within the range in which Parts 6–13 read the source and the assessment of the published construction	That the content of the proofs in the source is correct. Statements that may be in parts not read
Computed The lower bound 9×10^5 on the reference coupling is a lower bound derived from one condition only	The true entry value of the one-step estimate (the source gives only existence)
Inferred The expectation that the main part of the remaining question can be brought down to a long-distance question for the effective theory at the spacing-1 step	That it can be restated that way (the top three rows of Table 2 are needed). The comparison of difficulty
Lean The range of this book's theorems in § 07 (fixed lattice spacing)	That they cross the band. This chapter adds no new results

Sources and reproduction

Item	Kind	Source, tool
Published code	Source	yang-mills-group/OpenAI-YM-20261028 (Apache-2.0, submitted 2026-10-28, revision 3456e5e). Locations in the chapters of Parts 8–13
Assessment	Assessment	docs/assessment-1.pdf in yang-mills-group/YM (2026-11-10, revision 55ba9fc)
Committee statement	Cited	The third statement on the Kavli Institute for Theoretical Physics page (2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov
Condition on the reference coupling	Source, Computed	`Bar.threshold` in the source (`YM/Price/Intent.lean`). The smallest $\log \bar{\beta}$ for which $(\log\log x)^2 \le \frac{1}{2}\log x$ holds for all $x \ge \bar{\beta}$ was found in floating point (about 13.71)
Range of this book's theorems	Lean	Table 1 of 16-04, Table 4g of A-04
Problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). How to read it: 0-03

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