Volume 1 Preparation

Part 0 Entrance — how to read this book

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

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How to read this book — read the strength of a claim from its label, choose an order from the map of prerequisites

> This chapter is a guide (how to read this book).

What this chapter shows — How the book is put together (three volumes, twenty-nine parts and the appendices, 264 chapters) and the form shared by its chapters.
What the five labels (Lean, Paper, Computed, Known, Physics) guarantee and do not guarantee. How they relate to the labels of Volumes 2 and 3.
Which reader starts where. The map of prerequisites, why detours may be skipped, and the reading roads.
What this book claims and what it does not.

Prerequisite chapters — None. The overall picture is in the next chapter, 0-02, and a reading of the problem statement is in 0-03.

Said first — This book does not claim to solve the problem of Yang–Mills existence and the mass gap. Existence was shown by the construction published on 2026-11-10 (the committee's assessment is that it is proved in Lean; Volume 2 reads it; 6-00). The mass gap is open (6-04, 16-02). This book's own results are statements at a fixed lattice spacing, and conditional statements that place assumptions (16-04, § 08). The numbers in § 05– § 07 were computed from the "Prerequisites" column of the contents of 2026-10-15 (166 chapters), and do not include the 98 chapters added after that (most of Volume 2, and Parts 19–22 and 28 of Volume 3, among others).

Order of this chapter

1. What the book is — the one sentence of the problem and the roles of the parts
2. Readers and prerequisite knowledge — from second-year knowledge
3. The key to the labels — what they guarantee and do not guarantee (Table 1)
4. The form of a chapter — the three opening lines, numbering, notation, moving figures
5. The map of prerequisites — how many chapters a given chapter needs (Figure 1, Table 2)

6. How to skip detours — skipping them leaves no prerequisite missing (Detour: how many orderings do not break the prerequisites)
7. Suggested reading orders — entrances for different readers, and roads built from the map of prerequisites (Figure 2, Table 3)
8. What this book claims, and what it does not — the published construction, the open questions, this book's results

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01

What the book is — the one sentence of the problem and the roles of the parts

This book is for reading the Clay Mathematics Institute's prize problem "Yang–Mills existence and mass gap", from second-year university knowledge up to where research stands. The prize problems were announced on 24 May 2000 at the Collège de France in Paris Known , and the core of the official problem statement (the description by Arthur Jaffe and Edward Witten) is the following sentence Known .

> Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.

The mass gap means that the spectrum $\sigma(H)$ of the theory's Hamiltonian H (the energy operator) is empty just above the vacuum energy 0.

$$\sigma(H) \subset \{0\} \cup [\Delta, \infty), \qquad \Delta > 0 \quad (1)$$

Example. The way to (1) is how the volumes are divided. **Volume 1 "Preparation"** (Parts 0–5) gathers the tools. Parts 1 and 2 are the mathematical and physical tools, Part 3 is the classical theory, Part 4 is quantisation and the exact definition of (1) (4-11), and Part 5 is lattice gauge theory. **Volume 2 "Existence — reading the published construction"** (Parts 6–15) reads the construction of four-dimensional Yang–Mills existence published in 2026-11 and the committee's assessment. It starts from what was proved and what was not shown (Part 6, 6-00 onwards), and goes on to the background in constructive field theory (Part 7), objects and the order of limits (Part 8), one step of renormalisation (Part 9), matching the coupling and asymptotic freedom (Part 10), the multi-step estimates and large fields (Part 11), infinite volume and reconstruction (Part 12), the evidence that it is Yang–Mills (Part 13), and the comparison with this book's study (Part 14). Part 15, at the end of the volume, is the record of this book's study attempting ultraviolet control (15-00 onwards). **Volume 3 "Toward the mass gap"** (Parts 16–28 and the appendices) deals with the side that is

open: where things stand and the questions that remain (Part 16, 16-01 onwards), the view from the physical side (Part 17), the dictionary between physical and mathematical vocabulary (Part 18, 18-01 onwards), objects on the lattice and firm theorems (Parts 19–22, 19-00 onwards), the first road, with the diffusive-relaxation-time assumption as its axis (Parts 23 and 24, 23-00 onwards), the second road and the crossover band (Parts 25 and 26), conditional statements (Part 27), the tree of premises (Part 28, 28-00 onwards), and the appendices, such as the table of notation, the glossary and the guide to the literature (A-01 onwards). A word-by-word reading is in 0-03, and why four dimensions is special is in the existing articles What happens only in four dimensions and One sign is different, and in 0-04.

02

Readers and prerequisite knowledge — from second-year knowledge

The intended reader is a second-year student of mathematics or physics. What may be taken as known is multivariable calculus, the basics of linear algebra, complex numbers, ordinary differential equations, introductory mechanics and electromagnetism, and elementary probability. Other tools are prepared in Parts 1 and 2, and a tool not in the prerequisite chapters is defined, minimally, where it is used. If you can read the following three formulas, you can start from Part 1.

$$\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}, \qquad e^{i\theta} = \cos\theta + i\sin\theta, \qquad \det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 - 1 \tag{2}$$

Example. The Gaussian integral becomes the Gaussian measure that represents the free field (1-08), and $e^{i\theta}$, as an element of the group $U(1)$, becomes a gauge transformation of electromagnetism (2-03). The formula on the right finds the eigenvalues of the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, and the eigenvalues are ± 1. If this matrix is read as a Hamiltonian, the difference 2 between the two eigenvalues is the energy gap of a two-level system, the simplest example of a gap (2-05). We recommend that readers from mathematics read Part 2 carefully, and that readers from physics read the second half of Part 1 (1-07–1-11) carefully.

03

The key to the labels — what they guarantee and do not guarantee

This book shows the strength of each claim, sentence by sentence, with a **label**. It does not mix "proved", "conjectured" and "supported numerically" in one sentence, and puts a label on any sentence that might mix them.

Table 1 The five labels.

Label	Meaning	What it does not guarantee	Layer where it mostly appears
Lean	A statement machine-checked in Lean 4, given with its theorem name. The number of `sorry`, the marker for an omitted proof, is 0. `native_decide` (a shortcut that hands a computation to compiled code outside the checker) is not used. The axioms are only Lean's three standard ones (propositional extensionality `propext`, the axiom of choice `Classical.choice`, soundness of quotients `Quot.sound`). For details see A-04 and the Lean guide	That the premises are true	This book's study (5-11, 17-12, Part 15, Volume 3)
Paper	A proof exists on paper, but machine checking has not been done	That there are no errors	This book's study
Computed	The range checked by computation on this machine	That it holds outside the range, or that it is a theorem	All layers (including checks of known formulas)
Known	Known theorems, their restatements, checks against outside literature	That it is a result of this site	Foundations, existing knowledge
Physics	What is accepted in physics but is not a mathematical theorem (numerics, semiclassics, conjectures)	That there is a proof	Established knowledge (Part 17), the map

In Parts 16–28 of Volume 3 and in Part 15, these five labels are refined into five classes suited to the material (Proved here, Cited, Computer-assisted, Reading, Assumption) (23-00 § 01). Roughly, Paper → Proved here or Assumption, Computed → Computer-assisted (interval arithmetic) or Reading (floating point, Monte Carlo), Known → Cited, Physics → Reading or Assumption; Lean is a column outside the labels, in which the theorem name and the assumptions are written on the same row. The "View" label of Part 24 is at the same level as "Reading" (a physical reading of the situation, corresponding to "Physics" among the five labels). The home and the label of each tool are in Table 1 of 16-10.

The labels of Volume 2. Parts 6–14 of Volume 2 use three labels for reading the published construction: Source (the source of the published construction was read), Inferred (this book's inference) and Assessment (according to the committee's assessment). Their meaning, and what each does not guarantee, are in "Labels" in 6-00.

Besides these, the text of Volume 2 also uses the same five classes as Volume 3 (Proved here, Cited, Computer-assisted, Reading, Assumption) and the Lean label, with the same meaning. Volume 2 cites every result of the published construction at the level "the committee's assessment is that it is proved in Lean". This book has neither built nor run that code (6-00 § 02). The Lean label in Table 1 above is attached only to statements that this book's study has machine-checked itself.

The label most easily misunderstood is Lean. What machine checking guarantees is that the conclusion follows logically from the premises, that is, the arrow.

$$P_1 \wedge P_2 \wedge \cdots \wedge P_k \implies Q \tag{3}$$

The truth of the premises $P_1, \ldots, P_k$ is outside the label. If the premises contain measured values or a provisionally chosen form of a function, the conclusion is conditional too, and this book states those premises alongside (23-13). Conversely, even a value re-derived on this machine gets the Known label if the value itself is known (the one-loop coefficient; 4-06).

Example. In this site's paper (5-11), the labels divide stage by stage within a single paper. The chain from the lattice action to a certain lower bound (an inequality containing the constant 27) is Lean , and the theorem names are given in 5-11. The stage that goes from there to a mass gap on the lattice is Paper , and only a sketch at that. That the constant 27 cannot be made smaller than 21 has been checked by exact arithmetic on specific configurations Computed , and is not a Lean theorem. Terms such as what the lower bound means are defined in 5-11. Even within one paper, the label differs from sentence to sentence.

04

The form of a chapter — the three opening lines, numbering, notation, moving figures

Every chapter goes in this order: the three opening lines (what it shows, prerequisites, what is said first), the order of the chapter, numbered sections, and the two closing tables (what it can say, sources and reproduction). Within a section the order is words, formulas, examples. Formulas, tables and figures are numbered afresh in each chapter, and symbols are defined at first appearance and kept consistent with the table of notation (A-01). For example, the action is written this way in every chapter.

$$S[A] = \frac{1}{2g^2} \int_{\mathbb{R}^4} \operatorname{tr}\left(F_{\mu\nu} F_{\mu\nu}\right) d^4x \tag{4}$$

A is the gauge field, $F_{\mu\nu}$ its strength, g the coupling constant, and an index appearing twice is summed from 1 to 4 (3-03). (4) is the physicists' way of writing (the convention

of writing A and F as Hermitian matrices). It is related to the curvature F_{geom} of the geometers' way of writing by $F_{\text{geom}} = -iF$, and written that way the sign of $\operatorname{tr} F^2$ changes (the two ways of writing are in A-01). From Part 3 on we take $\hbar = c = 1$, and say so at the top of the chapter.

Example. Two meanings of the same letter are also stated at the top. β is the inverse coupling $\beta = 2N/g^2$ on the lattice (5-01), and the inverse temperature $\beta = 1/k_B T$ in statistical mechanics (2-08). Moving figures are still at first and move with "Play"; without JavaScript they can be read from the table below the figure. Terms are in A-02.

05

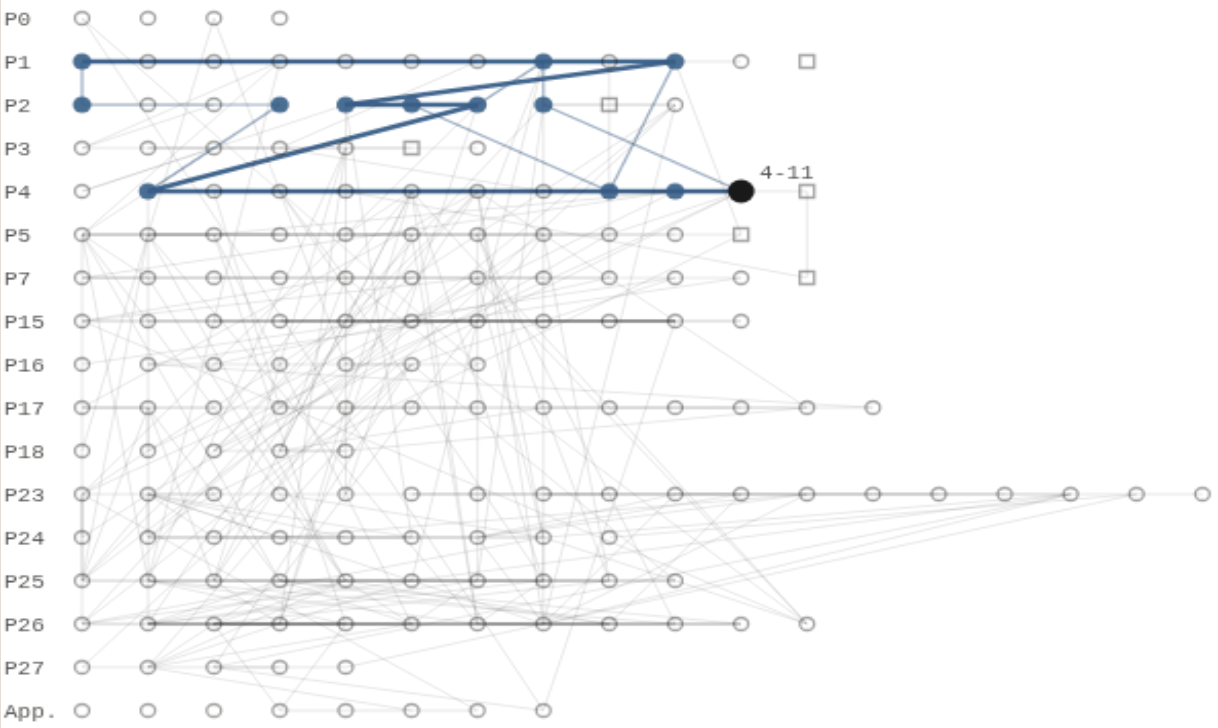
The map of prerequisites — how many chapters a given chapter needs

The "Prerequisites" column of each chapter in the contents lists the chapters to read first. Prerequisites have prerequisites of their own, so the whole set of chapters a chapter needs is the set collected by following the arrows backwards. We call this the **transitive prerequisites**. If the direct prerequisites of chapter c are $\mathrm{Pre}(c)$, the transitive prerequisites $\mathrm{Anc}(c)$ and the number of chapters in the longest chain $\ell(c)$ are determined by the following recursion (for a chapter with no prerequisites, $\varnothing$ and 1).

$$\mathrm{Anc}(c) = \bigcup_{p \in \mathrm{Pre}(c)} \big(\{p\} \cup \mathrm{Anc}(p) \big), \qquad \ell(c) = 1 + \max_{p \in \mathrm{Pre}(c)} \ell(p) \tag{5}$$

If the arrows form no cycle, this recursion terminates. Computed This map is a picture of the 166 chapters of the contents of 2026-10-15 (Volume 1, Part 7, 15-00 to 15-10, 7 chapters of Part 16, Parts 17 and 18, Parts 23–27, and the appendices). Among those 166 chapters there are 336 prerequisite arrows, and no cycle. There are 14 chapters with no prerequisites. The order of the contents of 2026-10-15 (the parts, in today's numbers, in the order 0–5, 7, 17, 18, 16, 23–27, 15, appendices) was arranged to respect the direction of the arrows: apart from 7 arrows in Parts 25–27 (26-01 → 25-06, 25-07; 26-02 → 25-08; 27-02 → 27-01; and the 3 that take the appendices A-04 and A-06 as prerequisites), for every chapter its prerequisite chapters came earlier in the contents. In today's order of the contents, renumbered so that the parts are in reading order, a further 2 arrows (17-13 → 16-05 and 26-11 → 15-00) have the prerequisite coming later Computed . So if you read straight through in today's order of the contents, no prerequisite is missing, provided you turn ahead once to the far end of those 9 arrows. This map does not include the 98 chapters added after that (Parts 6 and 8–14 of Volume 2, 15-11 to 15-14, 16-02 to 16-04 and 16-11, Parts 19–22, 26-12, Part 28, and the correspondence table of notation 24-09). Their prerequisites are in "Prerequisite chapters" at the top of each chapter.

Figure 1 The map of prerequisites of the 166 chapters of the contents of 2026-10-15 (the range is as in the text). Choosing a chapter with the slider colours its transitive prerequisites and draws the longest chain in a thick line. Rows are parts (the sixteen parts in this map, in the order of today's numbers; the last row is the appendices), columns are the order of the contents within a part, and squares are detour chapters. "Play" steps through the chapters in turn



`4-11 The definition of the mass gap | direct prerequisites 4-10, 2-08 | transitive prerequisites 12 chapters | longest chain 7 chapters: 1-01 → 1-10 → 2-05 → 2-07 → 4-02 → 4-10 → 4-11 | chapters that have this chapter as a prerequisite 67`

Computed The numbers in this figure were computed on this machine. The figure follows (5) on the spot from the table of the "Prerequisites" column, and agrees with Table 2 (contents of 2026-10-15).

Example. Computed Table 2 is (5) computed for some chapters. The longest chain is 29 chapters, and leads to "A map to the mass gap" in the appendices (A-08) (the 28-chapter chain up to "What cannot be said, the index of questions" in Part 15, 15-09, with A-08 added). It passes through Parts 1, 2, 4 and 5 into Part 23, goes from the integrated table (23-12) and what remains (23-15) via 24-07 to the summary of ultraviolet control (26-11), and runs through Part 15 in order from 15-00 to 15-06. Up to Part 27 the longest is 26 chapters, leading to 26-10 and 27-01; up to Part 24 it is 19 chapters, leading to 24-07. The longest within Part 23 is 18 chapters, leading to 23-15 and 23-14; in Part 16 it is 14 chapters, leading to the list of open tasks (16-07).

Computed **Table 2** Prerequisites of representative chapters.

Chapter	Direct prerequisites	Transitive prerequisites	Longest chain
4-11 The definition of the gap	4-10, 2-08	12	7
5-09 What to hold fixed	5-05, 4-11	26	10

Chapter	Direct prerequisites	Transitive prerequisites	Longest chain
7-10 Where four dimensions stops	7-07	16	10
17-10 Back to four dimensions	17-09	13	11
16-07 The list of open tasks	16-05	25	14
23-12 The integrated table and the dependency graph	23-06, 23-07, 23-09, 23-11	38	17
23-15 What remains and its cost	23-12	39	18
24-07 Because it is four dimensions	24-01, 24-02, 23-03, 23-15, 4-07, 7-07	49	19
26-10 Candidate new viewpoints	26-06, 26-08, 26-09, 25-03, 26-01	86	26
27-01 The conditional statement	16-09, 27-00, 25-04, 25-05, 26-02, 26-05, A-06, 27-02	88	26
15-09 What cannot be said, the index of questions, the dependency diagram	15-00, 15-01, 15-02, 15-03, 15-04, 15-05, 15-06, 15-07, 15-08	60	28
15-10 Systematic attempts	15-05, 15-08	59	27

Counting in reverse shows the foundational chapters. Computed 1-01 (linear algebra) is a transitive prerequisite of 145 chapters, 2-01 (the action) of 132, and 1-10 (Hilbert spaces) of 119. On the other hand, the Prerequisites column lists only chapters needed as tools. Part 3 (classical Yang–Mills) is not among the 16 transitive prerequisites of 7-10. The Prerequisites column is a lower bound on the reading order, not every chapter worth reading.

06

How to skip detours — skipping them leaves no prerequisite missing

There are two kinds of detour. The 6 chapters whose titles carry "Detour" in the contents (1-12, 2-09, 3-06, 4-12, 5-11, 7-12), and subsections inside a chapter whose headings begin with "Detour:". Both are written on the understanding that you can skip them and still go on. A detour subsection inside a chapter appears with its "Detour:" title in the "Order of this chapter" at the top of that chapter, so you can recognise it there and skip it. If the set of detour chapters is Y, then Computed the main-line chapters whose transitive prerequisites contain a detour are only 2-10, which has

the two-dimensional Ising model (2-09) as a prerequisite, the chapters of Parts 23–27 that have 2-10 as a prerequisite, and 7-12.

$$\mathrm{Anc}(c)\cap Y \;\subseteq\; \{\text{2-09},\ \text{4-12},\ \text{5-11}\},\qquad \text{2-09}\in\mathrm{Anc}(c) \iff \text{2-10}\in\mathrm{Anc}(c)\cup\{c\},\quad \text{4-12}\in\mathrm{An}$$

Example. 1-12, 3-06 and 7-12 are prerequisites of no chapter. 4-12 is a direct prerequisite only of 7-12, and 5-11 only of 23-02, so you read 5-11 first only when going on to 23-02. 2-09 is a prerequisite of 2-10 (the relaxation time and the dynamical exponent), and many chapters of Parts 23 to 27 have 2-10 as a prerequisite, so when going on to Part 23 read 2-09 first as well.

Detour: how many orderings do not break the prerequisites

A subsection that may be skipped.

An ordering that respects the prerequisite arrows is called a **linear extension** of the partial order. The order of the contents is one of them. The number of ways to merge two chains of lengths a and b while keeping their orders is $\binom{a+b}{a}$, given by which of the $a+b$ seats go to the first chain Known . Looking at Part 1 with only the arrows inside the part, branches come out of 1-01 (the chain 1-02 → 1-04 and 1-05 → 1-06 join at 1-07, and separately 1-10 → 1-11), and apart from this there are 1-08 → 1-09 and 1-12. Merging the three lumps (9, 2 and 1 chapters), merging the two chains before 1-07, and merging 1-10 → 1-11 into those 6 chapters gives

$$e=\frac{12!}{9!\,2!\,1!}\cdot\binom{5}{2}\cdot\binom{8}{2}=660\cdot 10\cdot 28=184800 \quad (7)$$

ways. **Example.** Computed Counting by dynamic programming over subsets, Part 1 agrees with (7), about 3.9×10^{-4} of 12!. Part 2 has 7560 ways and Part 3 has 20. In Parts 2 and 3 every chapter has at most one prerequisite within the part, so each is a **forest**, and the forest formula $e=n!/\prod_c h(c)$ ($h(c)$ is the number of c together with the chapters in the part that have c as a transitive prerequisite) Known can be used. The values of this formula also agreed with the dynamic programming. Part 1 is not a forest, because 1-07 has two prerequisites, and so it is counted with (7). The $\binom{a+b}{a}$ for two chains is the case of the forest formula where h becomes $1,\dots,a$ and $1,\dots,b$. For the 21 chapters of Parts 1 and 2 there are 44437771778400 ways, about 8.7×10^{-7} of 21!.

07

Suggested reading orders — entrances for different readers, and roads built from the map of prerequisites

Entrances for different readers. This book is arranged to be read through the three volumes in order, but the entrance can be changed according to purpose.

- **Readers who only want to know the problem and where things stand** — 0-02 → 0-03 → 6-00 → 6-01 → 6-04 → 16-02 → 16-04. This goes as far as what has been shown, what is open, and where this book stands.
- **Readers who want to read the published construction as mathematics (mathematics students)** — First read 1-08 to 1-11, 4-09 to 4-11, 5-01, 5-03 and 5-05 of Volume 1, and 7-01, 7-05 and 7-07 on constructive field theory; after that, read Volume 2 in order from 6-00.
- **Readers who want to enter from the physical side (physics students)** — From Parts 3 and 4 of Volume 1 and 5-01 and 5-08, to the view from the physical side (17-01 onwards) and the dictionary (18-01 onwards). After that, 6-01 and 13-01 onwards (the evidence that it is Yang–Mills).
- **Readers who want to read this book's results toward the mass gap** — After 16-01 to 16-04, the road "Reading from the firm places" below, and then the conditional statements (27-00 onwards).
- **Readers who want to follow the roads of this book's study** — The first road (23-00 onwards), the second road (25-00 onwards), the crossover band (26-00 onwards). The attempt at ultraviolet control is Part 15 (15-00 onwards), and the comparison with the published construction is Part 14 (14-01 onwards).

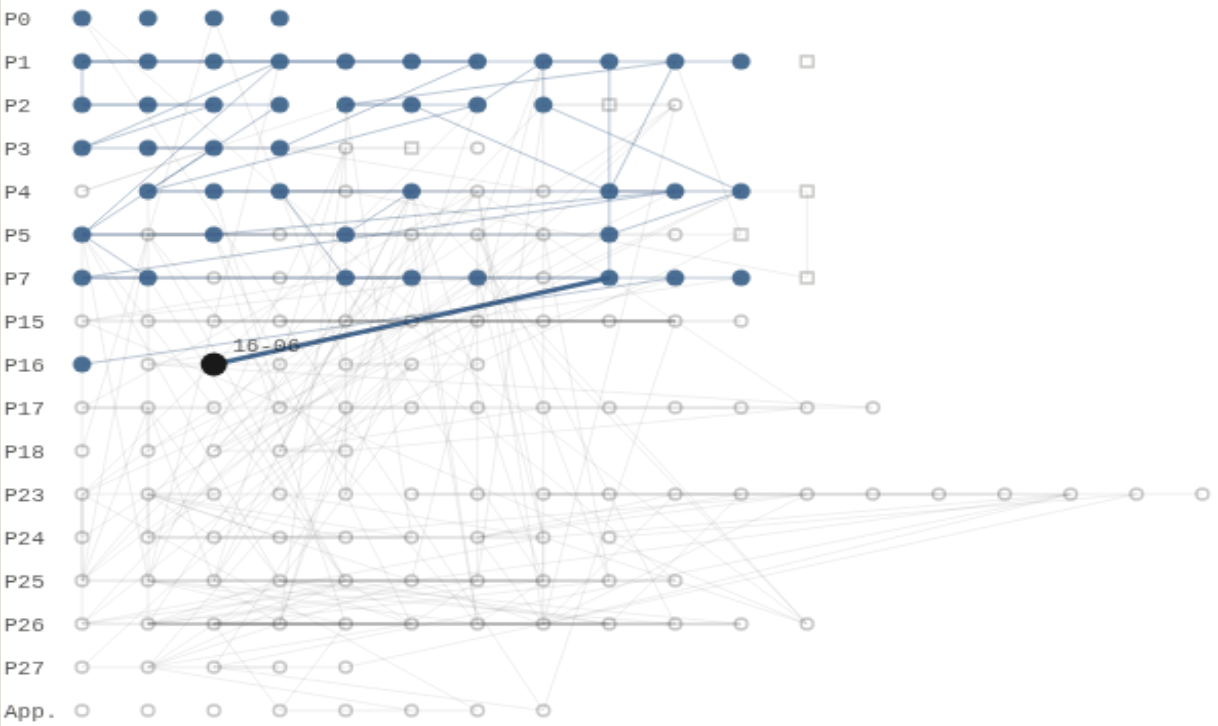
The roads below were built from the map of prerequisites of § 05 (the 166 chapters of 2026-10-15).

To read straight through, the order of the contents is enough. If you are in a hurry, add the transitive prerequisites to the set G of chapters you aim for, and arrange them in the order of the contents. If this sequence contains none of the 9 arrows of § 05 (those whose prerequisite comes later in the contents), no prerequisite is missing.

$$R(G) = G \cup \bigcup_{g \in G} \mathrm{Anc}(g) \qquad \text{(arranged in the order of the contents)} \tag{8}$$

There are five suggestions (Figure 2 shows the first three). The three roads into Parts 25–27 (following Part 25, following the band, from the first road to the band) and the three threads of Part 15 (ultraviolet control) are in "Reading orders" in the contents, and the latter are also in the last three rows of Table 3. **Entrance only** is 7 chapters chosen from the chapters whose prerequisites are empty or only Part 0 (16 chapters including the appendices Computed). **From the mathematical side** goes through the axioms, the lattice and constructive field theory to where four dimensions stops, and **from the physical side** goes through asymptotic freedom, confinement and holography to where physics parts from mathematics. Every road is the sequence built by (8) from the G of Table 3.

Figure 2 Choose a road with the buttons, and advance one chapter at a time with the slider or "Play". Chapters read are coloured, and the prerequisite arrows of the chapter being read are thickened. If a chapter comes before its prerequisite, it is counted as a "missing prerequisite"

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From the mathematical side 48/48: 16-06 Candidates for new concepts | prerequisite 7-09 (chapter 44) | missing prerequisites on this road 0
```

Computed The numbers in this figure were computed on this machine. The figure builds (8) on the spot from the G of Table 3, and agrees with the sequences from python3.

Example. Computed None of the three sequences has a missing prerequisite. From the mathematical side is 48 chapters, from the physical side 39, and 24 are shared. The two roads meet at 16-05 (where the two ways of digging part), and whichever one you come from, you cannot go on without adding chapters of the other road. 16-05 has 24 transitive prerequisites, so it is not on the entrance-only road.

Computed **Table 3** Reading orders. On every road the 4 chapters of Part 0 are added to G. Figure 2 shows the top three. Rows four to eight are the four roads chosen by the role of the chapters (layers) and the road from the tools, and are not in Figure 2. The last five rows are roads into Part 23, Part 24 and Part 15, and are not in Figure 2. The number of chapters includes the transitive prerequisites of the target chapters.

Road	Target chapters G	Chapters	Chapters to add when going on to 16-05
Entrance only	1-12, 16-08, 23-16	7	— (next, to one of the two roads)
From the mathematical side	1-11, 3-04, 4-11, 5-03, 5-09, 7-02, 7-06, 7-09, 7-11, 16-01, 16-06	48	17-04, 17-05, 17-08, 17-09, 17-10, 17-13
From the physical side	3-05, 3-07, 4-07, 5-08, 17-02, 17-03, 17-10, 17-11, 17-13	39	2-06, 4-09, 4-10, 7-01, 7-05, 7-07, 7-10, 7-11

Road	Target chapters G	Chapters	Chapters to add when going on to 16-05
Layer: Foundations	0-02, 0-03, 1-01, 1-02, 1-04, 1-08, 1-10, 2-05, 2-07, 2-08, 3-01, 3-02, 4-02, 4-10, 4-11	24	—
Layer: Existing knowledge	4-06, 4-07, 5-03, 5-04, 5-06, 7-05, 7-07, 7-10, 17-02, 16-01, 24-07, 15-01, 15-02 (sections: 23-08 § 01, 23-08 § 03, 24-05 § 04, 24-06 § 02, 26-04 § 01, 26-08 § 03; not counted)	65	—
Layer: This book's study	5-11, 17-12, 16-09, 16-10, 23-00, 23-01, 23-05, 23-15, 24-04, 25-00, 25-04, 25-05, 26-00, 26-02, 27-01, 15-00, 15-03	101	—
Layer: Map	0-02, 16-05, 16-07, 16-09, 23-15, 25-09, 26-08, 27-02, 27-03, 27-04, 15-08, 15-09	104	—
Starting from the tools (16-10 § 02)	1-10 → 5-03 → 5-11 → 7-04 → 7-05 → 18-04 → 23-08 § 01–§ 03 → A-04 → 24-06 → 15-04 → 15-05	72	—
The road that follows the study in Part 23	23-00 → 23-01 → 23-03 → 23-04 → 23-05 → 23-12 → 23-15 → 23-16 (and their transitive prerequisites)	45	—
The road that follows Part 24	24-00 → 24-01 → 24-03 → 24-04 → 24-05 → 24-08 (and their transitive prerequisites)	49	—
Ultraviolet control (shortest)	26-11 → 15-00 → 15-03 → 15-09	65	— (the transitive prerequisites of 15-09 include every chapter of Part 15)
Ultraviolet control (the computational thread)	15-04 → 15-05 → 15-06	62	—
Ultraviolet control (the thread of the wall)	15-02 → 15-07 → 15-08 → 15-08 § 06 (how the small-field contribution to the coupling is split)	61	—

Adding the transitive prerequisites of 16-07 and 23-15 (what remains and its cost) to the union of the two roads gives 78 chapters, and in the order of the contents of 2026-10-15 nothing is missing. In today's order of the contents, read 17-13, a prerequisite of 16-05, first. In the four rows of Table 3 that enter Part 15 (the road that enters from the tools, and the three ultraviolet rows), in today's order read 26-11 before 15-00 as well

Computed . Chapters you learned in your degree course may be skimmed on any road. **The road that follows the study in Part 23** takes the relaxation-time assumption as its axis, splits the claim of the first step into conditions, and follows in order up to what remains and what is not claimed (as in § 08, it is not a record of having shown anything).

Reading from the firm places — 19-00 → 20-00 → 21-00 → 22-00 → 28-00 (the entrances of Parts 19–22 and Part 28). In order: the objects defined on the lattice, the machine-checked theorems, and the list of premises. The results at one scale (one step of the free field, and linearisation) are in the second half of Part 15 (15-11 onwards). This road is not counted in Table 3 or Figure 2.

08

What this book claims, and what it does not — the published construction, the open questions, this book's results

This book does not claim to solve the problem of Yang–Mills existence and the mass gap. The present situation is written in three parts.

- **Existence.** On 2026-11-10, a Lean proof showing, for every simple compact group, that a relativistic quantum field theory satisfying the Wightman axioms can be made as the scaling limit of four-dimensional Wilson lattice gauge theory was published, together with the assessment of a committee that checked it. The committee assesses this construction as proved. This book has neither built nor run that code, and Volume 2 reads the result at the level "the committee's assessment is that it is proved in Lean" (6-00, 6-01).
- **What is not shown.** According to the same assessment, the following are not shown: the convergence and uniqueness of the infinite-volume limit (the limit is taken along a subsequence), the probability measure of the continuum theory, a uniform mass gap (exponential decay of correlations and a gap in the spectrum), Wilson loops and the area law, and one-particle states. The mass gap is open (6-04, 16-02).
- **This book's results.** The firm theorems of Volume 3 (Parts 20–22) are statements at a fixed lattice spacing: inside the strong-coupling window, the infinite-volume limit, a gap in lattice units and the area law; the vortex floor, which holds at every coupling; and the rarity of large plaquettes on the weak-coupling side. All of them are formalisations of results of known types. The strong-coupling results lie at the opposite end (strong coupling, lattice spacing fixed) from the remaining question (the gap in the continuum limit), and the rarity of large plaquettes is also a statement at one scale before renormalisation; none of them answers that question

(16-04). The roads of Parts 23–27 are conditional statements that place assumptions, and the premises placed are listed in the tree of premises (Part 28).

One point that is easy to mistake. In lattice gauge theory (Part 5), much has been shown with the lattice spacing a fixed, on the side where the inverse coupling β is small (strong coupling). What the problem needs is the opposite side.

$$\underbrace{\beta \text{ small (strong coupling)}}_{\text{the side shown on the lattice}} \quad \longleftrightarrow \quad \underbrace{\beta \to \infty, \;\; a \to 0}_{\text{the side the problem needs}} \tag{9}$$

Here is one sentence of this book. **A proof of the mass gap is a proof of a lower bound.** Trial states and finite measurements give only upper bounds on the gap; a lower bound needs an argument about all states (the box "The asymmetry of upper and lower bounds" in 1-10).

To show the mass gap on the road that enters from the lattice, what has to be shown is that $\Delta = m_{\text{lat}}/a$, the mass in lattice units m_{lat} divided by a, stays positive as $a \to 0$. That the limiting theory satisfies the axioms and is non-trivial was shown by the construction published in 2026-11 (at the level of the committee's assessment; 6-01). That construction, however, says nothing about the gap (5-09, 23-01, 0-03).

Example. On the strong-coupling lattice, the area law for Wilson loops has been shown (Osterwalder–Seiler 1978) Known . A Wilson loop is an observable made by multiplying the parallel transports of the gauge field along a closed path, and the area law means that its expectation decreases exponentially in the area enclosed by the path. It is taken as a sign of quark confinement Physics (5-02). But the area law holds at strong coupling in every lattice gauge theory, including those with the abelian group $U(1)$, so on its own it is not evidence of confinement in the non-abelian theory (5-06) Known . What has been shown on the strong-coupling side about the infinite-volume limit is treated in 5-04. The paper in the example of section 03 is also a result on the strong-coupling side, and claims nothing about the continuum limit.

The part of study with the relaxation-time assumption as its axis (23-00 onwards) sets out the range checked by computation, computer assistance and Lean, together with the assumption rows, and also contains ideas not yet verified (23-17), distinguished by labels. The second half of the first road (Part 24, 24-00 onwards) gives views other than the relaxation-time assumption and new views that take the relaxation-time assumption as a premise. What is not claimed is in 23-16; how to read Lean is in the Lean guide and the Lean record; the papers are in the list of papers.

What this chapter can say

What is said	What is not said (where it is treated)
Computed What was counted from the "Prerequisites" column of the contents of 2026-10-15 (166 chapters): 336 arrows, no cycles, the order of the contents of 2026-10-15 respects the arrows (except 7 in Parts 25–27; 9 in today's order), Table 2, (6), the number of linear extensions, no missing prerequisites in the sequences of Table 3	The Prerequisites column records dependencies at the design stage and does not necessarily capture the full content of the chapters (the numbers change if the contents are revised). That the five roads are the best (they are suggestions, not an optimisation). That detours inside chapters can be skipped is a commitment, and has not been computed
Known The one sentence of the problem, and the date and place of the announcement. The area law on the strong-coupling lattice, and that it is a result on the opposite side from the continuum limit	The proof of the mass gap (unsolved; 6-04, 16-02). The construction of the continuum limit was shown by the construction published in 2026-11 (at the level of the committee's assessment; 6-00). The infinite-volume limit on the strong-coupling side (5-04)
The meaning of the labels, and that the Lean label does not guarantee the truth of the premises (a commitment of this book)	The premises and theorem names of individual Lean statements (23-13, A-04; for the paper, 5-11)

Sources and reproduction

Item	Kind	Source / tool
The one sentence of the problem	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement PDF of the Clay Mathematics Institute). The sentence in § 4 was checked against the original. The translation is in 0-03
Date and place of the announcement	Known	J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, CMI / AMS, preface (p. viii): 'The prize was announced at a meeting on May 24, 2000, at the Collège de France.'. Checked via the source table of 0-03 (the PDF of the book published by the Institute)
How the labels divide in the paper	Lean Paper Computed (by stage)	This site's paper (5-11). The labels by stage were checked against the manuscript of the paper in the review of this book's material
The area law on the strong-coupling lattice	Known	K. Osterwalder, E. Seiler, “Gauge field theories on a lattice”, Ann. Phys. 110 (1978) 440–471. Bibliographic data only (attributing the area law to this paper rests on Chatterjee § 4 in the next row)

Item	Kind	Source / tool
That the area law holds at strong coupling in every lattice gauge theory, including those with $U(1)$	Known	S. Chatterjee, “Yang–Mills for probabilists”, arXiv:1803.01950 (2018) § 4. The survey of this book's material checked the text of § 4: 'the area law holds for any lattice gauge theory at sufficiently small β. A rigorous proof was given by Osterwalder and Seiler ... this also implied that the area law at small β cannot be evidence for quark confinement'
The forest formula, $\binom{a+b}{a}$	Known	D. E. Knuth, *The Art of Computer Programming*, Vol. 3, Addison–Wesley, 1973, p. 70. Bibliographic data only (attributing the formula for linear extensions of forests to this page rests on the text of S. C. Billey, J. P. Swanson, arXiv:2010.12701 § 1.3 and § 2.3). $\binom{a+b}{a}$ is the case of two chains
Table 2, Table 3, (6), (7), the numbers in the text, the figures	Computed	Computed with python3 from the "Prerequisites" column of the contents of 2026-10-15 (separators ",", "、", "・"; the table of notation for Part 24, 24-09, is not counted as a chapter). Linear extensions by dynamic programming over subsets, cross-checked with the forest formula and (7). The figures are traced on the spot by JavaScript in the browser from the same table

Next chapter: 0-02 One picture.

One picture — the tools go furthest at short distances, and let go at long distances

> In this chapter, § 01– § 05 are established knowledge, § 06 and § 07 are a map of open questions and § 08 is guide material.

What this chapter shows — The two jobs the problem asks for, and why the gap shows up neither in classical theory nor in perturbation theory.
In one picture that lines up the scales of length, how far the known tools go, and where mathematics and physics let go of each other.
Which part of this book treats which part of the picture.

Prerequisite chapters — None. Formulas are defined where they appear. A word-by-word reading of the problem statement is in 0-03, and the two four-dimensional worlds in 0-04.

Said first — This is a guide-map chapter. This book does not claim to solve the problem of Yang–Mills existence and the mass gap. Existence was shown by the construction published in 2026-11 (the committee's assessment is that it is proved in Lean; 6-00), and the mass gap is open (6-04). Figure 1 is a schematic based on a one-loop approximate formula (at the level of physics), and the boundaries and widths of the bands are choices made for explanation. Figure 2 is an example of a field without interaction.

Order of this chapter

1. The question
2. A mass invisible in classical theory (Table 1)
3. One picture: the ladder of scales — weak at short distances, strong at long distances (Figure 1, Table 2)
4. A finite box and a uniform gap (Figure 2, Table 3)
5. How far the world has come (Table 4)
6. What there is besides divergence (Table 5, Detour: if infinity were given a symbol)
7. Two ways of digging and seven seams (Table 6)
8. The map of this book

01

The question — what to construct, and what to show

On 24 May 2000, in Paris, the Clay Mathematics Institute announced seven prize problems. The official problem statement for this problem was written by Jaffe and Witten. Yang–Mills theory is a gauge theory that uses a non-abelian group, and is the framework of the description of the strong and weak forces (3-02). The core sentence is this. Known

> Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$ (§ 4).

"Non-trivial" means that the theory does not reduce to a free field (a field without interaction; its probability weight is **Gaussian**, the normal distribution extended to infinitely many variables) (0-03).

There are two jobs. **Constructing** — "existence" includes establishing properties at least as strong as the axioms of quantum field theory (the Wightman axioms, the Osterwalder–Schrader axioms). The axioms are a list, in the language of mathematics, of the conditions a quantum field theory should satisfy. Wightman's are written on the side of spacetime (Minkowski space), and Osterwalder–Schrader's on the side where time is made imaginary (Euclidean space) (4-09, 4-10). **Showing** — for the energy operator H (the Hamiltonian),

$$\sigma(H) \subset \{0\} \cup [\Delta, \infty), \qquad \Delta > 0 \quad (1)$$

$\sigma(H)$ is the spectrum, which you may think of as the set of values the energy can take (1-10). 0 is the energy of the vacuum, and formula (1) says that every excitation has energy at least Δ.

Example. The energy of the harmonic oscillator, measured from the vacuum, is $0, \hbar\omega, 2\hbar\omega, \ldots$, with a gap $\hbar\omega$; the $p^2/2m$ of a free particle on a line comes arbitrarily close to 0, so there is no gap. In relativity the energy of a particle at rest is its mass ($E = mc^2$), so the gap corresponds to the mass of the lightest excitation. Known To be exact, the gap in the problem statement is defined by the spectrum, not by particles. The object is pure Yang–Mills theory without quarks, and confinement (that things carrying colour charge cannot be taken out on their own) is an extension of the problem (0-03).

02

A mass invisible in classical theory — a scale that shows up in no power

According to the problem statement, the classical Yang–Mills equations describe massless waves travelling at the speed of light (§ 1), and in four dimensions the mass gap is not visible classically (§ 6.6). Known The only constant in the action is the dimensionless coupling g, and a mass cannot be made from the classical theory. Even so, physics holds that in the quantum theory one scale is born (dimensional transmutation). In the one-loop approximation

$$\Lambda = \mu \, \exp\!\Big(-\frac{1}{2 b_0 \, g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{3} \cdot \frac{1}{16\pi^2} \quad (2)$$

(μ is a reference scale, $g^2(\mu)$ the coupling there, and N is N for $SU(N)$). The gap is expected to be $\Delta = c\Lambda$ (c a number), but this is not a theorem. Physics

The function $e^{-c/x}$ ($c > 0$) vanishes faster than any power as $x \to 0$.

$$\lim_{x \to 0+} \frac{e^{-c/x}}{x^n} = 0 \qquad (n = 0, 1, 2, \dots) \quad (3)$$

So all its Taylor coefficients at $x = 0$ are 0, and perturbation theory, which expands in power series in g^2, sees Λ as 0 at every order. Known What is missing is not an infinite value but a quantity that does not appear in the series (Table 1).

Computed Table 1 Pure $SU(3)$, formula (2). $1/(2b_0) = 24\pi^2/33 \approx 7.178$

g^2	Λ/μ	$\div (g^2)^{10}$	$\div (g^2)^{20}$
0.5	5.83e−7	5.97e−4	0.611
0.3	4.06e−11	6.88e−6	1.17
0.1	6.71e−32	6.71e−22	6.71e−12

The ratio divided by the 20th power exceeds 1 at $g^2 = 0.3$, but tends to 0 as $g^2 \to 0$. Continued in 4-07, 7-11 and 4-12.

03

One picture: the ladder of scales

Put the length ℓ (logarithmic) on the horizontal axis, and the strength of the coupling at that length on the vertical axis. In four-dimensional Yang–Mills the coupling becomes weaker at shorter distances (asymptotic freedom; Gross–Wilczek, Politzer, 1973). Known At one loop, for $\alpha = g^2/4\pi$,

$$\alpha(\ell) = \frac{1}{2 b_0' \ln\!\big(1/(\ell\Lambda)\big)}, \qquad b_0' = 4\pi b_0 = \frac{11N}{12\pi}, \qquad \ell\Lambda < 1 \quad (4)$$

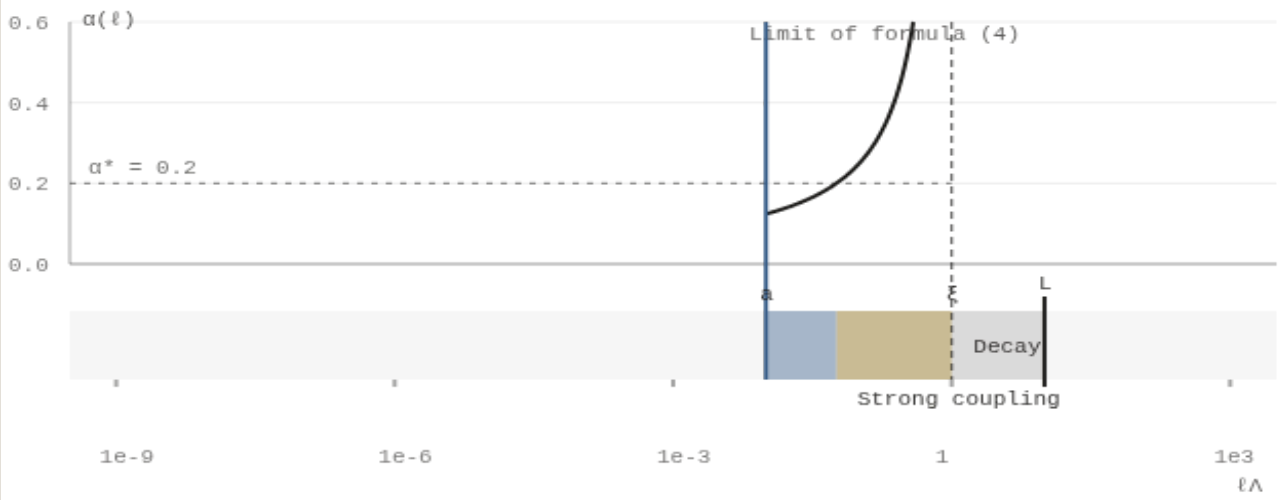
(written with g^2, this is formula (2)). Physics As $\ell \to 0$, α vanishes only at a logarithmic rate. In two and three dimensions g^2 has the dimension of mass to the power $4 - d$, and the dimensionless strength $g^2 \ell^{4-d}$ vanishes at a power rate, so four dimensions is the boundary (0-04).

A figure that sets the same $4 - d$ beside other quantities of the book and moves them with a single slider is 1-12 §10 Figure 4 (opens at d = 4).

The marker on the long side is the **correlation length** ξ. It is the length when the correlation between two distant points decreases at the rate $e^{-r/\xi}$ in the distance r, and it is related to the gap by $\xi = 1/\Delta$ (in units of $\hbar = c = 1$; 4-11).

To construct the theory, cut the short side off with a lattice spacing a and the long side with a box of side L, and take $a \to 0$ (the ultraviolet limit) and $L \to \infty$ (the infinite-volume limit). Doubling the length is counted as one step (one step of the renormalisation group; 7-05).

Figure 1 Making a smaller and L larger with the sliders lengthens the weak-coupling band on the left and the decay band on the right. The strong-coupling band in the middle does not depend on a. "Play" takes the two limits at the same time



log₁₀(aΛ) = −2.0, α(a) = 0.124, steps: weak 2.5 / strong 4.1 / ξ–L 3.3

Computed The numbers in this figure were computed on this machine (formula (4), $N = 3$). The guides $\alpha_* = 0.2$ and $\xi\Lambda = 1$ are choices made for explanation. If JavaScript is disabled, see Table 2.

Computed Table 2 Number of steps for formula (4), $N = 3$, $\alpha_* = 0.2$

$\log_{10}(a\Lambda)$	$\alpha(a)$	Weak coupling	Strong coupling
−1	0.248	0	3.3
−2	0.124	2.5	4.1
−3	0.083	5.8	4.1
−6	0.041	15.8	4.1

$\log_{10}(a\Lambda)$	$\alpha(a)$	Weak coupling	Strong coupling
−9	0.028	25.8	4.1

Each time a is divided by 10 the number of steps grows by about 3.3, and all the added steps are weak-coupling steps. The strong-coupling steps do not depend on a and are about 4.1 (with the guides set to 0.1 and 0.3 they are 8.2 and 2.7, and again do not depend on a). Making L larger adds $\log_2(L\Lambda)$ steps on the side longer than ξ. Computed

The width of the bands changes with how ξ and α_* are placed. In physics numerics, the mass m of the lightest glueball of pure $SU(3)$ (a bound state made only of gluons, that is, particles of the Yang–Mills field) is estimated as $r_0 m \approx 4.21$ (Morningstar–Peardon 1999), and Λ of the same theory, defined as $\overline{\text{MS}}$, as $r_0\Lambda \approx 0.602$ (Capitani–Lüscher–Sommer–Wittig 1999). r_0 is a ruler for length fixed from the force between static quarks. Setting $\xi = 1/m$ gives $\xi\Lambda \approx 0.14$, and ξ is expected to fall on the side shorter than $1/\Lambda$. Physics Its definition differs from the Λ of formula (4), so this is a guide, but with this value the strong-coupling band is about 1.3 steps at $\alpha_* = 0.2$ and almost 0 steps at $\alpha_* = 0.3$. Computed The numbers for the width of the bands are not physical quantities. What is robust in this picture is rather that the width does not depend on a.

The weak-coupling band is the side where there is a foothold for expansion, and it is also where mathematics has gone furthest (Bałaban; not complete; § 06, 7-07). The strong-coupling band has no small parameter and does not vanish even as $a \to 0$. In the band longer than ξ, the key is to show exponential decay of correlations in physical units, uniformly in the box (§ 04). The body of the problem is to join the three bands by estimates that do not break down in either limit. To be exact, formula (4) is a formula of perturbation theory, and this picture is a picture at the level of physics. Physics

04

A finite box and a uniform gap

What the problem statement asks for is the gap of the theory on $\mathbb{R}^4$. As a road towards it, the problem's authors consider finite-volume approximations, and name, as places where present methods do not show the way, "a mass gap uniform in the volume" and "the infinite-volume limit $T^4 \to \mathbb{R}^4$" (§ 6; T^4 is a four-dimensional box with opposite faces glued). They write that the former needs new ideas, and that if it were obtained the study of $T^4 \to \mathbb{R}^4$ would become possible (end of § 5, § 6.5). Known

With the lattice spacing a and the side of the box L fixed, that the lattice gap is positive is the easy side (the detour in 17-13). What the problem's authors see as the key is that

the gap of the continuum theory (after $a \to 0$, or uniformly in a as well) is bounded from below independently of the size of the box,

$$\inf_{L} \ \Delta_{\mathrm{phys}}(L) > 0 \quad (5)$$

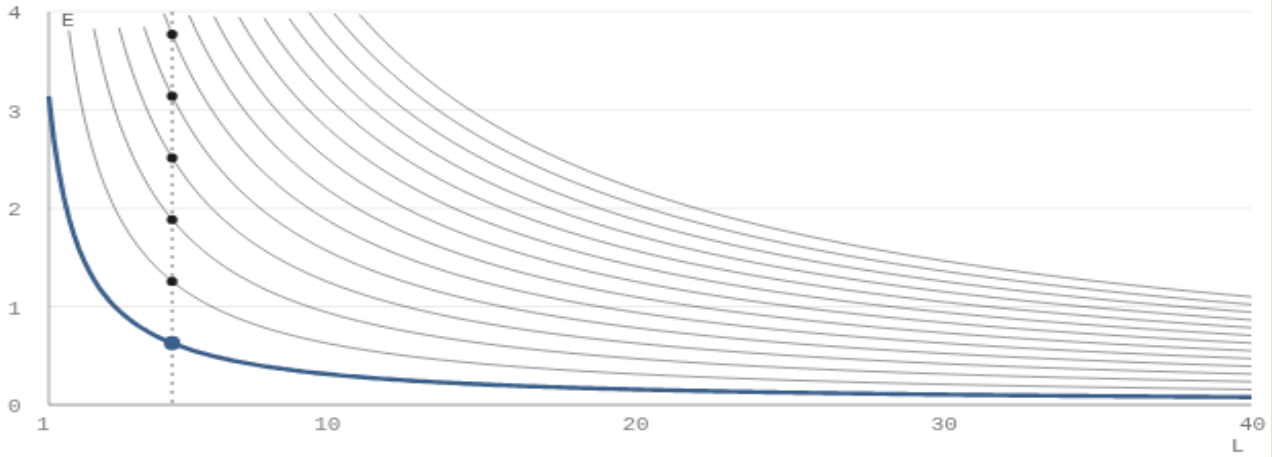
(§ 5, § 6.5). $\Delta_{\mathrm{phys}}(L)$ is the gap in a box of side L measured on a physical scale such as Λ, not measured in lattice units $1/a$. This distinction matters. If a is fixed and one measures in lattice units, exponential decay in infinite volume has already been shown on the strong-coupling side (Table 4). What is needed is that the gap measured in physical units is uniform in both $a \to 0$ and $L \to \infty$ (5-09).

Let us see the difference with a field without interaction (a free field). The one-particle levels in a box of length L with the field 0 at both ends are

$$\omega_n(L) = \sqrt{(n\pi/L)^2 + m^2}, \qquad n = 1, 2, 3, \ldots \quad (6)$$

and the gap is ω_1 ($\hbar = c = 1$). If $m = 0$, the gap π/L exists in every box but collapses as $L \to \infty$; if $m > 0$, it is always larger than m and satisfies a uniform lower bound of the form of formula (5). Known

Figure 2 Moving the box length L with the slider moves the levels (dots) along the curves ω_n(L). The buttons switch the mass. At mass 0 the gap ω_1 collapses; at mass 1 it does not come below the dashed line E = 1



m = 0, L = 5.0, gap ω_1 = 0.6283, levels with E < 1: 1

Computed The numbers in this figure were computed on this machine (formula (6)). If JavaScript is disabled, see Table 3.

Computed Table 3 The gap $\omega_1(L)$ of formula (6)

L	$m = 0$: π/L	$m = 1$	Number of levels with $E < 1$ at $m = 0$
1	3.1416	3.2969	0
10	0.3142	1.0482	3
100	0.0314	1.0005	31
1000	0.0031	1.0000	318

At $m = 1$ the levels with $L^2(\omega_1 - 1) \to \pi^2/2$, and at $m = 0$ those with $E < 1$, increase at the rate L/π. Computed To be exact, this m was put in by hand, and the Yang–Mills gap must be born from the interaction. The other face of the gap is the decay of correlations: formula (2) in § 5 of the problem statement says that for local quantities with vacuum expectation 0, for any positive C smaller than Δ, correlations at sufficiently large distances are bounded by e^{-Cr} (0-04, 2-09).

05

How far the world has come

The job of constructing can be divided into six stages (7-01). ① regularisation (cutting off short distances, with a lattice or otherwise), ② the finite-volume measure $\mu_{a,L}$ (the probability weight on field configurations; 1-08), ③ $a \to 0$, ④ $L \to \infty$, ⑤ the axioms, ⑥ the gap. ①–④ construct

$$\mu = \lim_{L\to\infty} \lim_{a\to 0} \mu_{a,L} \quad (7)$$

and ⑤⑥ are properties of μ.

Known Table 4 What has been reached, theory by theory

Theory	What has been reached	Literature
YM, two dimensions	The continuum measure exists (plane, closed surfaces, torus), by an exactly solvable structure. The axioms are outside the scope	Driver 1989, Gross–King–Sengupta 1989, Lévy, Chevyrev–Shen
YM, three dimensions	Up to ultraviolet estimates on the lattice uniform in the spacing. The continuum measure is not constructed even on T^3	Bałaban 1985, Chevyrev 2022
YM, four dimensions	Ultraviolet stability in finite volume (Bałaban; with assumptions, at the level of convergent subsequences), and a sketch of a continuum construction that keeps an infrared cutoff (MRS). The construction published in 2026-11 reaches ③ and ⑤, and ④ at the level of subsequences (the committee's assessment is that it is proved in Lean). The uniqueness in ④, and ⑥, are not done	Bałaban 1984–89, Magnen–Rivasseau–Sénéor (MRS) 1993, the published construction and the committee's assessment 2026 (6-00)
Lattice YM, strong coupling	With a fixed (in lattice units), infinite volume and exponential decay. The side opposite to the continuum limit	Osterwalder–Seiler 1978, Shen–Zhu–Zhu 2023
φ^4	In two and three dimensions non-trivial theories satisfying the axioms exist. In four dimensions, the scaling limit near the critical point of the single-component φ^4 with a lattice ultraviolet cutoff is Gaussian (a free field)	Problem statement § 6.2, Aizenman–Duminil-Copin 2021

Terms in Table 4: "ultraviolet stability" means that quantities do not run wild as the lattice spacing is made small, and convergent subsequences can be taken (uniqueness of the limit is not included). "Gaussian" is the type of the free field in §01. "Axioms" are the Wightman and Osterwalder–Schrader axioms of §01.

How to read Table 4. In the row for four-dimensional Yang–Mills, the construction published in 2026-11 passes ③ and ⑤, and ④ has come as far as the level of subsequences of volumes (6-01). The gap of ⑥ is untouched (6-04). The strong-coupling row shows something like ④ and ⑥, but it is about a lattice with a fixed and does not pass through ③, so it is a different proposition from formula (5) of § 04. The φ^4 row shows that there are theories whose limit becomes a free field in four dimensions. For four-dimensional Yang–Mills, the published construction made a non-trivial limit (at the level of the committee's assessment; 6-01).

The problem statement of 2000 says that not a single non-trivial relativistic field theory satisfying the axioms is known in four dimensions. Douglas's report of 2004 places what has been reached at Bałaban and MRS, and Chatterjee's 2018 survey says that a construction of a continuum limit above two dimensions in which Wilson loops (the quantity made by multiplying the rotations by the gauge field along a closed path and taking the trace; 5-01) behave non-trivially does not yet exist. Known The construction published in 2026-11 made a non-trivial theory satisfying the Wightman axioms in four dimensions (the committee's assessment is that it is proved in Lean; 6-00). Wilson loops, however, are outside that construction, and a continuum limit in Chatterjee's sense has still not been made (6-04). "Two and three dimensions can be done" is inaccurate for non-abelian Yang–Mills: in three dimensions the continuum measure has not been constructed even in finite volume (T^3) (7-02, 7-06, 16-01).

06

What there is besides divergence

"It is hard because mathematics does not allow the handling of infinities" is one side of it. Handling divergences is a matter of the left band in Figure 1, and it is the most advanced (Bałaban's programme is unfinished: its main theorem assumes that the effective coupling is small, and the theorem removing that assumption is unpublished; 7-07. The construction published in 2026-11 controlled the ultraviolet by a differently assembled renormalisation; Parts 9–11). Where the problem's authors asked for new ideas is the place in § 04 that is not about divergence. Known

Known Table 5 Obstacles that cannot be reduced to divergence (7-11)

1	No small parameter in the infrared	There is no easy solvable model to put at the centre of an expansion (MRS). The lattice strong coupling that can be handled rigorously is on the side opposite to the continuum limit
2	Invisible to perturbation theory	The scale of the gap is of the type in Table 1

3	Non-triviality	φ^4_4 (the four-dimensional φ^4; not asymptotically free) has a Gaussian limit near the critical point with a lattice cutoff (7-08). For four-dimensional Yang–Mills, the construction published in 2026-11 made a non-trivial limit (at the level of the committee's assessment; 6-01)
4	The state space is curved	The orbit space is not linear, and one cannot choose one field from each orbit continuously (a global gauge fixing) (on closed spaces such as S^4; Singer 1978; the setting is in 3-04)
5	Few regularisations preserve reflection positivity	The lattice preserves it but breaks rotational symmetry. Many continuum regularisations break it (§ 6.5)
6	The structure of infinitely many degrees of freedom	Under standard conditions, the interaction picture does not exist (Haag's theorem)

Terms in Table 5: the "orbit space" is the space made by regarding fields related by gauge transformations as the same. "Reflection positivity" is the positivity condition for recovering the Hilbert space of quantum mechanics from a measure on Euclidean space (4-10). The "interaction picture" is the way of writing time evolution by taking the state space of the free field as it is and adding the interaction to it (4-09).

How to read Table 5. 1 and 2 concern the middle and right bands of Figure 1, that is, the side where the coupling is not small. 3 is the job of showing that the limit constructed has not collapsed to a free field. 4–6 are structures carried by the framework itself of gauge theory and infinitely many degrees of freedom. Each is a different problem from "how to handle infinities", and handling divergences alone does not settle them. Known

"Because it is non-abelian" also, to be exact: for continuous non-abelian groups in three or more dimensions, exponential decay in infinite volume has been shown only on the strong-coupling side, and weak-coupling results are limited to finite groups (Adhikari–Cao; 5-06, 5-10).

Detour: if infinity were given a symbol

A subsection that may be skipped.

"Would anything change if $1/0$ were made a symbol?" As far as known cases go, it is thought likely that nothing would change. Handling divergences inside perturbation theory is already rigorous (BPHZ, Epstein–Glaser, Connes–Kreimer; renormalisability by 't Hooft–Veltman 1972 and others), and the $0/0$ that appears has been handled not by assigning a value but by rates of approach or by changing the geometry. The scale of the gap is of the type e^{-1/g^2}. This is a sorting of cases, not an impossibility theorem (7-12).

07

Two ways of digging and seven seams

The mathematical side starts from "the theory does not yet exist" and constructs in the order existence → axioms → gap (Parts 5 and 7). The physical side starts from "the theory exists" and measures glueball (§ 03) masses and the string tension (Part 17). As a metaphor, it is a tunnel dug from both ends. With the construction published in 2026-11, the mathematical side has dug as far as existence and the axioms (6-00), but at the gap it has not yet broken through. To be exact, the numbers of physics are checks of consistency on the assumption of existence (17-01, 16-05).

Table 6 The seven seams

1	The premise of existence	That numbers appear to converge is not a proof of existence	17-13
2	One scale	Physics holds that the gap and the string tension are both constant multiples of Λ. Not a theorem	17-12
3	Different shores	Shores with a foothold: perturbation theory (formal series), lattice strong coupling (theorems), holography ($N \to \infty$, $\lambda \to \infty$, at the level of physics). The theory of the problem statement is on none of the shores	17-13
4	Finite and infinite	Numbers are extrapolations. What is needed is uniform estimates	5-09
5	Spacetime and space	Construction is on the Euclidean side. The way back to spacetime is reflection positivity	0-04
6	The word "particle"	Physics counts particles; the problem statement asks for a gap in the spectrum	4-11
7	Extra dimensions	The gap turns into geometry, but the order of limits for returning to four dimensions is not worked out	17-10

Terms in Table 6: "shore" is a metaphor; exactly, it is a range of parameters where theorems or controlled approximations are available. $\lambda = g^2 N$ is the 't Hooft coupling. "Turns into geometry" is also a metaphor; exactly, in holography the gap is computed as an eigenvalue of a wave equation on a geometry that includes extra dimensions (the leading term in $1/N$; at the level of physics; 17-09).

How to read Table 6. All seven are places where a step taken for granted in physics becomes, in mathematics, a proposition that needs a proof. 1 and 4 concern the relation between numerical computation and limits, 2 and 6 what is called a "mass", 3 and 7 the distance between computable models and the theory of the problem statement, and 5 the translation between spacetime and Euclidean space. The rightmost column gives the chapter that treats each in detail.

This book's study has set to work around 2: it cut out "one scale" as the diffusive-relaxation-time assumption, but has not proved it (23-01; the correspondence with the equivalence principle is an analogy; 17-12). The remaining six are only named.

08

The map of this book

This book is in three volumes. Volume 1 "Preparation" (Parts 0–5) gathers the tools for reading the pictures of this chapter. The remaining two chapters of Part 0 read the words of the problem statement (0-03) and the two four-dimensionalities (0-04). Parts 1 and 2 are the mathematical and physical preparation, and provide the tools (spectrum, measure, path integral, statistical mechanics) for reading formulas (1) and (7) and § 04. Part 3, classical Yang–Mills theory, deals with § 02; Part 4, quantisation, with formulas (2) and (4) and the axioms; and Part 5, lattice gauge theory, with a (the lattice spacing) in Figure 1 and the strong-coupling side.

Volume 2 "Existence — reading the published construction" (Parts 6–15) reads the construction of 2026-11 that rewrote the four-dimensional row of Table 4. In order: what was proved and what was not shown (Part 6), the literature of Tables 4 and 5 against the background of constructive field theory (Part 7), the objects from the lattice to the continuum and the order of limits (Part 8), one step of renormalisation (Part 9), matching the coupling and asymptotic freedom, which is the left band of Figure 1 (Part 10), the multi-step estimates and large fields (Part 11), infinite volume and reconstruction (Part 12), the evidence that it is Yang–Mills (Part 13), and the comparison with this book's study (Part 14). Part 15, at the end of the volume, is the record of this book's study attempting ultraviolet control (15-00 onwards).

Volume 3 "Toward the mass gap" (Parts 16–28 and the appendices) deals with the side that is still open: the middle and right bands of Figure 1, and formula (5). In order: where things stand and the questions that remain (Part 16; the continuation of § 05 and Table 6), the view from the physical side (Part 17), the dictionary between physical and mathematical vocabulary (Part 18), objects on the lattice and firm theorems (Parts 19–22), the two roads of this book's study and the crossover band (Parts 23–26; including the area around 2 of Table 6), and conditional statements and the tree of premises (Parts 27 and 28). The appendices are the table of notation, the glossary, the guide to the literature, the guide to Lean, and so on. Chapters and sections marked "Detour" may be skipped. Suggested reading orders are in 0-01.

What this chapter can say

What is said	What is not said (where it is treated)
Known The relevant passages of the problem statement (checked against the PDF)	The word-by-word reading (0-03)

What is said	What is not said (where it is treated)
Known Formula (3). Computed Table 1	$\Delta = c\Lambda$ is the expectation of physics (4-07)
Computed Figure 1, Table 2 (the number of steps within formula (4))	Formulas (2) and (4) are not theorems. The boundaries and widths of the bands are choices of drawing. $\xi\Lambda \approx 0.14$ is a guide via Λ, which is defined differently (Physics)
Computed Figure 2, Table 3 (free field)	Shows nothing about the Yang–Mills gap
Known Tables 4 and 5	Not a complete survey of the literature (16-01)
—	This book shows neither the continuum limit nor the gap (the continuum limit was shown by the published construction, at the level of the committee's assessment; 6-00). The diffusive-relaxation-time assumption remains an assumption

Sources and reproduction

Item	Kind	Source / tool
The one sentence, § 1– § 6.6	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement PDF of the Clay Mathematics Institute; the relevant passages were read). As a book, included in J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, Clay Mathematics Institute / American Mathematical Society (2006) pp. 129–152 (contents checked in the PDF published by CMI, year of publication on the AMS book page)
Date and place of the announcement	Known	The Millennium Prize Problems page of the Clay Mathematics Institute website (read; 24 May 2000, Collège de France, Paris). The preface of the same book gives the same date and place (checked in the PDF published by CMI)
Asymptotic freedom	Known	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343 / H. D. Politzer, ibid. 1346. Abstracts checked (the former: a wide class of non-abelian gauge theories has the asymptotic form of a free field up to logarithmic corrections; the latter: perturbation theory becomes arbitrarily good in the deep Euclidean region)
The guide for $\xi\Lambda$ in § 03	Physics	C. Morningstar, M. Peardon, Phys. Rev. D 60 (1999) 034509 Table VII (bibliographic data only; the survey of the material checked the text) / S. Capitani, M. Lüscher, R. Sommer, H. Wittig, Nucl. Phys. B 544 (1999) 669 eq. (6.2) (the text was read). Ratios and numbers of steps by python3
b_0	Physics	Particle Data Group, Phys. Rev. D 110 (2024) 030001, QCD chapter § 9.1.1. Bibliographic data only (checked by the survey of the material)

Item	Kind	Source / tool
Table 4	Known	B. K. Driver, CMP 123 (1989) 575 / L. Gross, C. King, A. Sengupta, Ann. Phys. 194 (1989) 65 / T. Lévy, Mem. AMS 166 no. 790 (2003) / I. Chevyrev, H. Shen, arXiv:2302.12160 / T. Bałaban, CMP 102 (1985) 255, CMP 122 (1989) 355 / J. Magnen, V. Rivasseau, R. Sénéor, CMP 155 (1993) 325 / I. Chevyrev, arXiv:2202.13359 / K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 / H. Shen, R. Zhu, X. Zhu, CMP 400 (2023) 805 / M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163 / M. R. Douglas, report for the Clay Mathematics Institute (2004) / S. Chatterjee, arXiv:1803.01950. Bibliographic data only (the survey of the material checked them against the originals; CMP = Comm. Math. Phys.) / The four-dimensional row of 2026-11: yang-mills-group/OpenAI-YM-20261028 (the Lean source) and the committee's assessment, docs/assessment-1.pdf in yang-mills-group/YM (2026-11-10). How to read them, and at what level, is in 6-00
Table 5, § 06	Known	I. M. Singer, CMP 60 (1978) 7 / R. Haag, Mat.-Fys. Medd. Danske Vid. Selsk. 29 (1955) no. 12 (via secondary sources) / A. Adhikari, S. Cao, Ann. Probab. 53 (2025) 140 / G. 't Hooft, M. Veltman, Nucl. Phys. B 44 (1972) 189. Bibliographic data only
Tables 1–3, Figures 1 and 2	Computed	python3 (formulas (2), (4), (6)). The figures are drawn by the browser from the same formulas
Restatements of terms (axioms, reflection positivity, orbit space, interaction picture)	Known	Detailed definitions and sources in 4-09, 4-10 and 3-04 of this book

Next chapter: 0-03 Reading the problem statement word by word.

Reading the problem statement word by word — each word closes a loophole to an easy answer

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — Where the core sentence of the problem statement, the paragraphs that fix its words, and the two footnotes are, and what they say.
What each of the words "any compact simple group", "$\mathbb{R}^4$", "exists", "non-trivial", "mass gap" and "short distances" demands.
A list of the "easy answers" each word directly shuts out (Table 5).

Prerequisite chapters — None. Each word is made into a tool in later chapters (the destination is given at the end of each section). The overall picture is in the previous chapter, 0-02, and the difference between the two four-dimensional worlds is in the next, 0-04.

Said first — This chapter only reads the problem statement, and shows nothing about the problem. This book does not claim to solve this problem. The existence side was shown by the construction published in 2026-11 (the committee's assessment is that it is proved in Lean; 6-01 sets the words of this chapter against the construction), and the mass gap is open. The English quotations are the original text, checked against the official problem statement PDF of the Clay Mathematics Institute, and the restatements are this chapter's. Table 5 is this chapter's reading, not something the problem's authors stated in that form.

Order of this chapter

1. The core sentence — where it is written, and how it is built
2. Any compact simple group — all of them, down to the exceptional groups (Table 1; Detour: for non-compact groups)
3. On $\mathbb{R}^4$ — neither a box nor a lattice, but an infinitely wide continuous four dimensions
4. Exists — constructing something that satisfies the axioms (Table 2)
5. Non-trivial — not collapsed to a Gaussian

6. Mass gap — the gap in the spectrum and $m < \infty$ (Figure 1, Table 3)
7. Agreement at short distances — asymptotic freedom names "Yang–Mills"
8. The two footnotes — quantum operators, and existence by compactness alone (Figure 2, Table 4)
9. What is not written — extensions of the problem, QCD, formulations on the lattice
10. What each word shuts out — a word-by-word list (Table 5)

01

The core sentence — where it is written, and how it is built

On 24 May 2000, at a meeting at the Collège de France in Paris, the Clay Mathematics Institute announced seven prize problems (the Institute's website, and the preface of the book that contains the problem statements). The official problem statement for one of them is Arthur Jaffe and Edward Witten's description "Quantum Yang–Mills Theory" (14 pages) Known . Below, § is a section of the problem statement, and "section 06" is a section of this chapter. The problem is the one sentence in § 4.

> *Yang–Mills Existence and Mass Gap.* Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in [45, 35].

Restated: **For any compact simple gauge group G, prove that a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$.** Existence includes establishing properties at least as strong as the axiomatic properties cited in reference [45] (the book of Streater–Wightman) and [35] (the two papers of Osterwalder–Schrader).

Writing the skeleton as a logical formula makes the order of the quantifiers visible ($\sigma(H)$ is the spectrum of the Hamiltonian H; section 06).

$$\forall G \; \exists \mathcal{T}_G : \quad \mathcal{T}_G \text{ satisfies the axioms, is non-trivial, agrees at short distances, and has } \sigma(H) \cap$$

One theory $\mathcal{T}_G$ is needed for each group, and the gap Δ_G may differ from group to group. § 4 also imposes one more thing: that the supremum of the gap m_G is finite ($m_G < \infty$; section 06). The meaning of the words is fixed by the surrounding paragraphs. Quantum fields and the axioms are in § 3, local fields and the short-distance condition

in the first paragraph of § 4, the mass gap in the second paragraph of § 4, clustering and the extensions of the problem in § 5, and finite-volume estimates and the footnote on weak existence (footnote 2 of Jaffe–Witten) in § 6 and § 6.5.

Example. For $G = SU(3)$, what is asked for is a theory of only the 8 kinds of gauge field (gluons), with no quarks (8 is the dimension of $SU(3)$; Table 1). QCD with quarks added is not the object (section 09).

02

Any compact simple group — all of them, down to the exceptional groups

The **gauge group** is the group of symmetries by which the values of the field may be rotated independently at each point (3-01). **Compact** means bounded and closed as a group of matrices. **Simple** is meant in the sense of Lie groups: the Lie algebra (1-03) is not commutative and has no non-trivial ideal (a subspace that bracketing with any element does not take outside). $SU(2)$ has the centre $\{\pm 1\}$, so as an abstract group it is not simple. In the problem statement, § 1 names $G = SU(3)$ as the gauge group of the strong force (it has the centre $\mathbb{Z}_3$, so it is not simple as an abstract group), and § 6.6 also treats $SU(N)$, so we read it in the sense of Lie groups.

A gauge field has as many components as the dimension of the group. Choosing a basis T^a of the Lie algebra, the bracket (commutator) can be expanded in the basis as $[T^a, T^b] = i f^{abc} T^c$. These coefficients f^{abc} are the **structure constants**, the multiplication table of the Lie algebra (1-03). With them,

$$A_\mu = \sum_{a=1}^{\dim G} A_\mu^a \, T^a, \qquad \sum_{c,d} f^{acd} f^{bcd} = C_A \, \delta^{ab} \tag{2}$$

C_A is 0 for a commutative group and positive for a compact simple group Known . Its value changes with how the size of the basis is taken, so we fix a normalisation. The basis T^a is taken orthonormal for an inner product invariant under the group action, and that inner product is chosen so that the squared length of the longest root (1-04) is 1. For $SU(N)$, the standard normalisation of physics, $\operatorname{tr} T^a T^b = \tfrac{1}{2}\delta^{ab}$ in the fundamental representation, corresponds to this. In this normalisation C_A equals the **dual Coxeter number** $h^\vee$, a positive integer determined by each group, and for $SU(N)$ it is N Known . In the normalisation common in mathematics, "squared length 2", it becomes $C_A = 2h^\vee$ (checked from the structure constants for $SU(2), SU(3), SU(4)$, $SO(5), SO(7), SO(8)$ Computed). The compact simple Lie algebras are exhausted by the series A_n, B_n, C_n, D_n and the exceptional types G_2, F_4, E_6, E_7, E_8 (the Killing–Cartan classification; 1-04) Known , and "any" means all of these.

Computed **Table 1** The dimension and the positive integer $h^\vee$ determined by each group (list of values in 1-04). Every row agrees with the standard values. The procedure of computation is in the source table. b_0 is the one-loop coefficient of section 07.

Group	Type	Dimension	$C_A = h^\vee$	$b_0 = 11C_A/12\pi$
$SU(2)$	A_1	3	2	0.5836
$SU(3)$	A_2	8	3	0.8754
$Sp(3)$	C_3	21	4	1.1671
$SO(8)$	D_4	28	6	1.7507
G_2	G_2	14	4	1.1671
F_4	F_4	52	9	2.6261
E_7	E_7	133	18	5.2521
E_8	E_8	248	30	8.7535
$U(1)$ (not simple)	Commutative	1	0	0

The general form of the one-loop coefficient is $b_0 = (11C_A - 4n_f T_R)/(12\pi)$ (n_f is the number of kinds of quark, and T_R a positive number determined by the representation of the quarks; PDG 2024) Known . In a theory without quarks, $n_f = 0$ gives $b_0 = 11C_A/12\pi$, and from $C_A > 0$ it is $b_0 > 0$ for every simple group. At the level of one-loop perturbation theory, the coupling becomes weaker at shorter distances (section 07). The numerical value of b_0 depends on the normalisation of the coupling g (the normalisation of C_A above), so comparing its size between groups has no meaning. The point is that it is positive. For $U(1)$ it is $b_0 = 0$, and that theory is the free field of the massless photon. There is no gap. For a product of two simple groups the action splits into a sum, giving two theories that do not act on each other.

Detour: for non-compact groups

A subsection that may be skipped.

For the action $\frac{1}{4g^2}\int \mathrm{Tr}\, F \wedge *F$ (formula (1) of the problem statement writes it as the Lagrangian L; its content is the integrated action), the problem statement says only that Tr is "an invariant quadratic form on the Lie algebra". Requiring it to be definite is this chapter's argument, and the reason is that it is needed for the classical energy to be positive. The elements of $\mathfrak{su}(2)$ are anti-Hermitian, so $\mathrm{tr}\, X^2 \le 0$. In the Lie algebra of $SL(2,\mathbb{R})$,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : \ \mathrm{tr}\, X^2 = 2, \qquad Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \ \mathrm{tr}\, Y^2 = -2$$

both signs appear, and the invariant quadratic form (limited to constant multiples of $\operatorname{tr} XY$) is indefinite. Real Lie algebras with a positive-definite invariant inner product are limited to the Lie algebras of compact groups Known .

03

On $\mathbb{R}^4$ — neither a box nor a lattice, but an infinitely wide continuous four dimensions

$\mathbb{R}^4$ is the set of all quadruples of real numbers; it is not a finite box, not a periodic box (the four-dimensional torus T^4), and not a lattice. On the approximations, symmetry is lost. The lattice breaks continuous translations and rotations, and T^4 keeps translations but breaks rotations.

The sentence in § 4 does not state the signature of $\mathbb{R}^4$. § 3 introduces the Wightman axioms as a theory on "$\mathbb{R}^4$ with Minkowski signature", and § 6, in a parenthesis describing the state of constructive field theory, writes "in the Euclidean formulation, replace Euclidean space-time $\mathbb{R}^4$ by $\mathbb{R}^2$ or $\mathbb{R}^3$", from which one can read that the stage of the Euclidean version is $\mathbb{R}^4$ Known . There are two ways of measuring length.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 \ \text{(Euclidean)}, \qquad -t^2 + x^2 + y^2 + z^2 \ \text{(Minkowski)} \ (3)$$

The two are connected by the theory of Osterwalder and Schrader (§ 3; formally, the substitution $t = -i\tau$). How this works is treated in 0-04 and 2-04 and in the existing article One sign is different.

Example. § 6 says of T^4 that "the construction is not complete, but there is good indication that known methods can be extended", and that methods for obtaining the $T^4 \to \mathbb{R}^4$ limit are not suggested by present techniques Known . This gap goes straight into the problem. Why four is special is in 1-12 and in the existing article What happens only in four dimensions.

Symmetry is not the only thing lost in approximation. On reflection positivity § 6.5 writes 'few methods exist to recover reflection positivity in case it is lost through regularization—such as with dimensional regularization, ...', giving dimensional regularisation and Pauli–Villars as examples Known . The lattice keeps reflection positivity and breaks rotations, and continuum regularisations do the opposite (4-04, 4-10, 7-11).

04

Exists — constructing something that satisfies the axioms

"Exists" does not mean that some limit exists. § 4 says to **define** a quantum field theory in the sense of § 3, with local fields corresponding to **gauge-invariant local polynomials** in the curvature F and its covariant derivatives Known . The curvature is written $F = dA + A \wedge A$ in § 1, and the local polynomial § 4 gives as an example is $\mathrm{Tr}\, F_{ij}F_{kl}(x)$.

$$F = dA + A \wedge A \ \ (\S 1), \qquad \mathrm{Tr}\, F_{ij}F_{kl}(x) \ \ (\S 4) \ (4)$$

The gauge field A itself is not asked for. A changes under gauge transformations and is not an observable. A "quantum field" is defined in § 3 as "an operator-valued distribution on space-time". The value at a point $\mathcal{O}(x)$ has no meaning; $\mathcal{O}(f) = \int \mathcal{O}(x) f(x)\, d^4x$, smeared with a test function f, is an operator (1-11).

Table 2 The properties included in "existence" (a summary of § 3 of the problem statement; the exact form is in references [45] and [35]; in this book, 4-09 and 4-10).

Property	Content
The space of states	A Hilbert space $\mathcal{H}$ carries a representation of the Poincaré group (the group combining translations and Lorentz transformations of spacetime; 2-04). H and P are the generators of time and space translations
Energy	$0 \le H$
vacuum	A vector Ω invariant under the Poincaré group, unique up to phase
Fields	Gauge-invariant functions of the field act as operators on $\mathcal{H}$ and transform covariantly
Locality	Fields in regions that cannot be joined by a light signal are independent (the operators commute)
Euclidean version	Conversion back and forth with a Euclidean-invariant theory. The key is reflection positivity

Example. The free Klein–Gordon field was the field that served as a model for the first axiom system (§ 6.2), but it is excluded by "non-trivial", next. § 6.5 states plainly that "establishing the Hilbert space of quantum mechanics is part of the solution of this Millennium problem" Known .

05

Non-trivial — not collapsed to a Gaussian

The problem statement does not define "non-trivial". It is usually read as "not a free field (a field without interaction)". § 6.2 says that the **Gaussian measure** with mean 0 and covariance $(-\nabla^2 + m_0^2)^{-1}$ (∇^2 the Laplacian) gives the free field of mass m_0, and that non-linear fields need non-Gaussian measures Known . Whether something is

Gaussian can be told from higher correlations. For a Gaussian random variable X with mean 0,

$$\frac{\mathbb{E}[X^4]}{(\mathbb{E}[X^2])^2} = 3 \quad (5)$$

and in several variables too, four-point expectations can be written as sums of products of two-point expectations (Isserlis' formula; 1-08) Known .

Example. Computed For the density $\propto e^{-x^4}$ the ratio (5) is 2.18844 (agreeing with $\Gamma(\frac{1}{4})\Gamma(\frac{5}{4})/\Gamma(\frac{3}{4})^2$). For the density $\propto e^{-x^2/2-\lambda x^4}$, with $\lambda = 1,\ 0.1,\ 0.01$ it is 2.319, 2.537, 2.843, returning to 3 as $\lambda \to 0$. **Triviality** of a field theory is the phenomenon in which each approximation is non-Gaussian, yet in the limit ratios of this kind return to the Gaussian value.

To be exact, the one-variable ratio is a guide. For four-dimensional $\lambda\varphi^4$, it is a theorem that if the ultraviolet is cut off by a lattice and the scaling limit near the critical point is taken with lattice spacing 0 and infinite volume, the limit is Gaussian (Aizenman–Duminil-Copin 2021; 1-08) Known . Yang–Mills differs in that its coupling becomes weak at short distances, and a non-trivial limit has been expected Physics . In four dimensions, the construction published in 2026-11 made a non-trivial limit (the committee's assessment is that it is proved in Lean; 6-01). A free field with mass has a gap (Table 3), so the gap alone cannot shut out free fields (7-08). § 6 says that at present not a single non-trivial relativistic field theory satisfying the Wightman (or other reasonable) axioms is known in four dimensions (as written in 2000) Known . The committee assesses that the construction published in 2026-11 made such a theory for four-dimensional Yang–Mills (6-00).

06

Mass gap — the gap in the spectrum and $m < \infty$

The definition in § 4 is written in terms of the spectrum $\sigma(H)$ of the Hamiltonian H (what corresponds to the set of eigenvalues for a matrix; 1-10). The vacuum is an eigenvector with energy 0, and the energy is never negative. On top of that, having an empty interval just above 0 is called a gap.

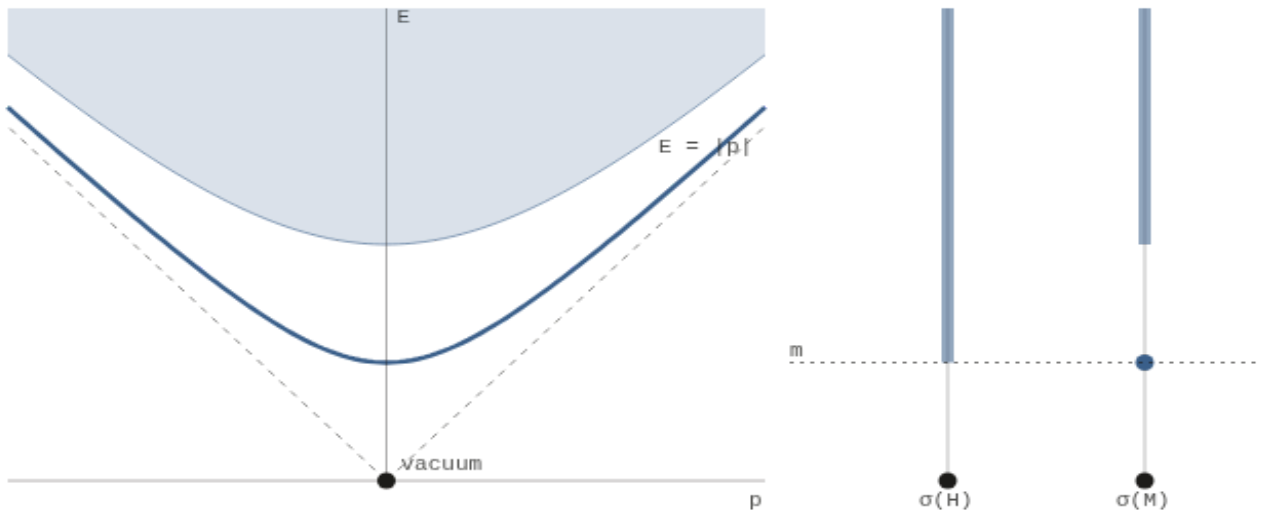
$$H\Omega = 0, \qquad \sigma(H) \subset [0, \infty), \qquad \sigma(H) \cap (0, \Delta) = \varnothing, \qquad m = \sup \Delta < \infty \quad (6)$$

$\sup \Delta$ is the supremum of all Δ satisfying the condition, and we call it the **mass** m Known . The problem statement does not give the reason for imposing $m < \infty$. Reading it from the formula shows what it excludes. If $\sigma(H) = \{0\}$, every Δ satisfies the condition, and $m = \infty$. **Self-adjoint** (1-10) is the property of Hermitian matrices

(matrices whose eigenvalues are all real) extended to infinite-dimensional operators. For self-adjoint H, $\sigma(H) = \{0\}$ is the same as $H = 0$, so $m < \infty$ means "there is some state of positive energy above the vacuum".

The reason for calling it "mass" lies in the relation between energy and momentum ($\hbar = c = 1$; 2-04). A particle of mass μ has $E = \sqrt{|p|^2 + \mu^2} \geq \mu$. For the **mass operator** $M = \sqrt{H^2 - P^2}$ of § 6.2, $H \geq M$, and at momentum 0, $H = M$. A state with $M > 0$ can have its momentum brought to 0 by a Lorentz transformation (using the Lorentz covariance of the axioms), so the gap of H and the gap of M have the same width Known . If there is a state with $M = 0$ besides the vacuum, both gaps are 0, and the conclusion is the same. This equivalence is not written in the problem statement itself.

Figure 1 Moving m with the slider raises and lowers the hyperbola E = √(p² + m²) in the energy–momentum plane on the left, and opens and closes the gaps of σ(H) and σ(M) on the right. Switching the isolated one-particle state on and off with the button does not change σ(H)



m = 1.00, σ(H) = {0} ∪ [1.00, ∞), σ(M) = {0} ∪ {1.00} ∪ [2.00, ∞)

Computed The figure is drawn on the spot from the formula $E = \sqrt{p^2 + m^2}$ (momentum in one direction only; the shading is continuous spectrum).

Table 3 The spectra of four examples.

Example	$\sigma(M)$	$\sigma(H)$	Verdict
One kind of particle of mass m (no bound states)	$\{0\} \cup \{m\} \cup [2m, \infty)$	$\{0\} \cup [m, \infty)$	Gap
Only continuous spectrum above the gap	$\{0\} \cup [m, \infty)$	$\{0\} \cup [m, \infty)$	Gap
The massless free field (photon)	$[0, \infty)$	$[0, \infty)$	No gap
The vacuum only	$\{0\}$	$\{0\}$	Excluded by $m = \infty$

Rows 1 and 2 have the same $\sigma(H)$. What the problem statement asks for is only the gap of $\sigma(H)$; an isolated one-particle state is an extension of the problem (section 09). § 5 gives **clustering** as a consequence of the gap. For local fields $\mathcal{O}$ and $0 < C < \Delta$ with $\langle \Omega, \mathcal{O}\Omega \rangle = 0$, if the two points are far enough apart,

$$|\langle \Omega, \mathcal{O}(x)\, \mathcal{O}(y)\, \Omega \rangle| \leq e^{-C|x-y|} \quad (7)$$

Known . The gap determines how fast fluctuations at distant places approach independence. The body of the definition is in 4-11, and how to read it for finite systems is in 2-05.

07

Agreement at short distances — asymptotic freedom names "Yang–Mills"

Quantum Yang–Mills theory has no definition before it is constructed. What ties the constructed object to "the quantum Yang–Mills theory of G" is the condition in the first paragraph of § 4. The correlation functions of the quantum fields must agree **at short distances** with the predictions of asymptotic freedom and perturbative renormalisation theory found in textbooks (original: 'should agree at short distances with the predictions of asymptotic freedom and perturbative renormalization theory, as described in textbooks'). The predictions include the existence of a stress tensor and an operator product expansion, whose local singularities have the form predicted by asymptotic freedom Known .

Asymptotic freedom is the property that the strength of the coupling changes with scale and becomes weak at short distances (large momentum μ) (Gross–Wilczek and Politzer, 1973; references [21] and [38] of the problem statement). In a theory without quarks, $\alpha = g^2/4\pi$ runs in the following way (formula (9.3) of PDG 2024 with $n_f = 0$; on the right, the one-loop solution) Known .

$$\mu^2 \frac{d\alpha}{d\mu^2} = -b_0 \alpha^2 - b_1 \alpha^3 - \cdots, \qquad b_0 = \frac{11 C_A}{12\pi}, \qquad \alpha(\mu^2) \approx \frac{1}{b_0 \ln(\mu^2/\Lambda^2)} \quad (8)$$

Λ is a constant of integration, the scale at which the coupling defined in perturbation theory diverges.

Example. Computed For $SU(3)$, $b_0 = 0.8754$, and when $\mu/\Lambda = 10,\ 100,\ 1000$, $\alpha = 0.248,\ 0.124,\ 0.083$. Each time the momentum grows tenfold, $1/\alpha$ grows by the same amount $2b_0 \ln 10 \approx 4.03$. The weakening is logarithmic, and slow.

Solving (8) gives $\Lambda = \mu\, e^{-1/(2b_0\alpha)}$, and $e^{-1/x}$ approaches 0 faster than any x^k as $x \to 0^+$, so all coefficients of the power series in α are 0 Known . The classical action has no constant with the dimension of mass, and physics holds that the mass of the quantum

theory takes the form $\Delta = c\Lambda$ Physics . This is consistent with "the mass gap is not visible classically" in § 6.6. The problem statement does not specify the value of the coupling constant g (4-06, 4-07).

08

The two footnotes — quantum operators, and existence by compactness alone

The footnote on local fields (footnote 1 of Jaffe–Witten; § 4) says that, because of the subtleties of renormalisation, there is no natural one-to-one correspondence between classical "differential polynomials" and quantum operators, and that the spaces of those of dimension at most d are expected to correspond Known . The dimension is the mass dimension: A has 1, F has 2, and $\operatorname{Tr} F^2$ has 4.

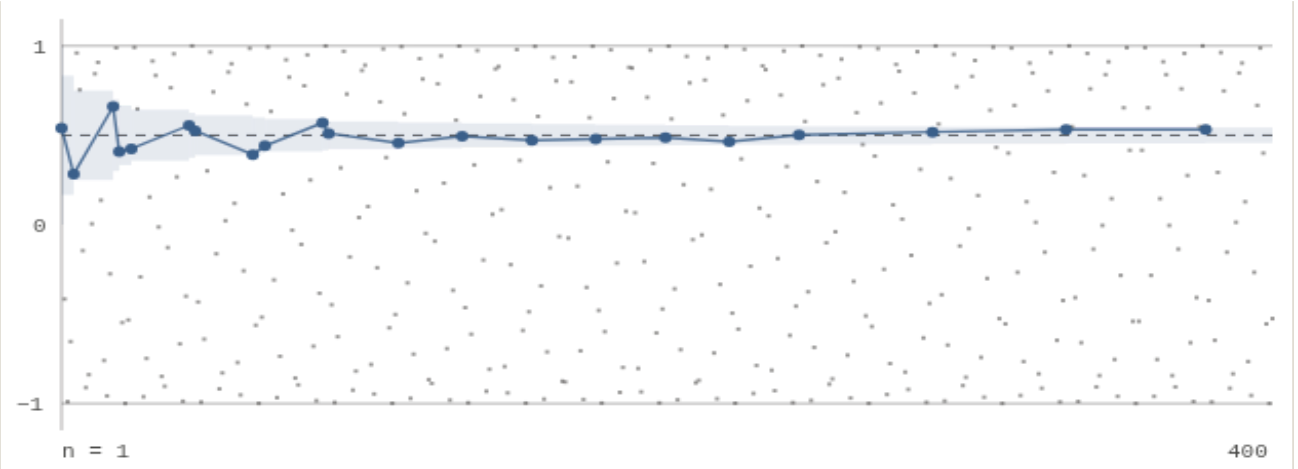
The footnote on weak existence (end of § 6.5) says that **weak existence (compactness)** is specifically excluded as a solution of the existence part, unless the properties of the limit (the mass gap and the axioms) are established by other methods Known . The text to which the footnote is attached says, in effect, "showing a mass gap uniform in the volume needs new ideas; with that, one could study the $T^4 \to \mathbb{R}^4$ limit". Uniform means that the right-hand side holds for the gap Δ_L for each box size L.

$$\Delta_L > 0 \ \text{(each } L\text{)} \qquad \text{rather than} \qquad \inf_L \Delta_L > 0 \ \text{(9)}$$

The left-hand side alone holds even for the massless free field. With the field set to 0 on the walls of a box of side L, the energy of the lowest vibration is $\sqrt{3}\,\pi/L$, and the gap closes as $L \to \infty$ (2-05).

A compactness argument is an argument of the type "a bounded sequence has a convergent subsequence" (Bolzano–Weierstrass). For a sequence of probability measures too, given a uniform estimate (tightness), a weakly convergent subsequence can be taken (Prokhorov's theorem) Known . **Tight** means that every measure in the sequence puts almost all of its probability (at least $1 - \varepsilon$) on a common compact set (for $\mathbb{R}^n$, a bounded closed set), and **weak convergence** means that the expectations of all bounded continuous functions converge. What is weak is that the limit can change with the choice of subsequence. Whether the limit is unique, and which of the gap or the axioms it satisfies, cannot be known from compactness alone (some properties, like reflection positivity, are preserved under weak convergence; 5-03). Figure 2 uses the sequence $a_n = \cos n$ to show how the limit is not unique.

Figure 2 Moving the target t with the slider extracts, from the same sequence a_n = cos n (dots), a subsequence converging to t (coloured dots and line). The band is the error allowed for the next term. "Play" moves t back and forth



Computed The numbers in this figure were computed on this machine. Rule: n_1 is the smallest n with $|\cos n - t| < 1/2$, and n_{k+1} is the smallest n with $n > n_k$ and $|\cos n - t| < 1/(k+2)$. The figure computes the same rule on the spot, and agrees with the python3 results (Table 4).

Computed **Table 4** Subsequences extracted with $n \le 400$.

t	First 8 terms n_k	Terms	$\cos n_k$ of the last term
1.0	1, 6, 7, 12, 13, 19, 25, 31	40	0.9873
0.5	1, 5, 18, 20, 24, 43, 45, 64	21	0.5328
−1.0	3, 9, 10, 16, 22, 28, 35, 41	41	−0.9998

The remainders of n divided by 2π are uniformly distributed (Weyl's equidistribution theorem), so a subsequence converging to any $t \in [-1, 1]$ can be taken Known . The difference between the empirical distribution of $\cos n$ for $n \le 100000$ and the arcsine distribution was below 0.0001 in all 10 intervals Computed . Boundedness alone does not make the limit unique.

To be exact, in the lattice method there is a measure μ_a for each lattice spacing a, and one wants to take $a \to 0$. Even if a subsequential limit is obtained from uniform estimates, whether it is unique, and whether it satisfies the axioms and the gap, are separate questions, and the footnote on weak existence says that these must be shown. It has been stated that what can be expected from the known results on the ultraviolet side in four dimensions is also a limit by a compactness argument using subsequences, and it is also written that the limit need not be unique (the introduction of Magnen–Rivasseau–Sénéor 1993; 7-07) Known .

In the construction published in 2026-11 too, the infinite-volume limit is taken along a subsequence by compactness, and uniqueness is not shown. The committee's assessment notes that, in this footnote, the problem statement regards subsequential limits alone as not enough, and then states that the reconstruction and the theorems on local fields give information beyond compactness (6-01; at the level "the committee's assessment is that it is proved in Lean"). The mass gap is not shown (6-04).

09

What is not written — extensions of the problem, QCD, formulations on the lattice

§ 5 lists the following as **natural extensions**, separate from the body of the problem Known . The existence of isolated one-particle states (an "upper gap" in addition to the mass gap). A proof of confinement. The existence of other four-dimensional gauge theories with added fields that preserve asymptotic freedom, and their dynamics (gap, confinement, chiral symmetry breaking). Extending the existence theorem to arbitrary four-dimensional manifolds. Written as a formula, the upper gap means that the spectrum of M has the following form (this chapter's way of writing; it corresponds to "m is an isolated eigenvalue of M" in § 6.2).

$$\sigma(M) = \{0\} \cup \{m\} \cup [m', \infty), \qquad m < m' \tag{10}$$

In a theory with only one kind of particle of mass m and no bound states, the two-particle continuous spectrum begins at $m' = 2m$ (row 1 of Table 3).

§ 1 names three properties that QCD should have at the quantum level: the mass gap, quark confinement and chiral symmetry breaking. Only the first entered the body of the problem, and the object, too, is pure Yang–Mills without quarks. The problem statement does not ask for the **value** of the gap either. Glueball masses and their ratio to the string tension are objects of physical measurement (5-08).

Example. Chatterjee's (2018) lattice correlation-length problem (Chatterjee's Problem 5.1) asks that on the lattice (β is the inverse of the lattice coupling; the larger it is, the weaker the coupling; 5-01) the correlations of plaquette variables decay exponentially (the correlation length $\xi(\beta)$ is finite), and that $\xi(\beta) \to \infty$ as $\beta \to \infty$. It is a formulation on the lattice, and a different thing from Clay's requirements, which include the continuum limit, $\mathbb{R}^4$, the axioms and non-triviality. Even if it were shown, that alone would not solve the Clay problem (5-09, 16-08).

10

What each word shuts out — a word-by-word list

We take the conditions of (1) word by word, together with the $m_G < \infty$ mentioned in section 01, and list what each word directly shuts out.

$$\forall G \; \exists \mathcal{T}_G \; \big[\, \text{axioms} \wedge \text{non-trivial} \wedge \text{short distances} \wedge \Delta_G > 0 \wedge m_G < \infty \, \big] \tag{11}$$

Table 5 A word-by-word list (this chapter's reading).

Word	What the word directly shuts out	Section
Any	A construction for only one convenient group (for example $SU(2)$)	02
Compact	Groups whose invariant quadratic form is indefinite, so that the classical energy is not positive	02
Simple	$G = U(1)$. The free electromagnetic field has no gap, so including it would make the claim false	02
$\mathbb{R}^4$	Theories on T^4 or on a lattice. On a finite lattice the integrals are finite-dimensional (§ 6.5) and always exist	03
Existence (axioms)	Something for which merely some limit exists, with neither a Hilbert space nor a Hamiltonian	04
Non-trivial	Limits collapsed to a Gaussian. The free field with mass satisfies the axioms and also has a gap	05
$\Delta > 0$	The massless free field. It satisfies the axioms but has no gap	06
$m < \infty$	A theory with only the vacuum ($\sigma(H) = \{0\}$)	06
Short distances	Another theory, unrelated to Yang–Mills, that has a gap	07
The footnote on weak existence	Compactness limits taken along subsequences (with neither uniqueness nor the gap and the axioms shown)	08

The words of Table 5 fall into two kinds according to what happens when they are removed. One kind makes the claim **false** when removed. "Simple" is one: as in the example below, a theory without a gap gets mixed in. The other kind makes the claim **easy** when removed. Removing "$\mathbb{R}^4$" leaves only finite-dimensional integrals on a finite lattice; removing "existence (axioms)" lets through something for which merely a limit exists; removing "the footnote on weak existence" makes a subsequential limit enough. Removing "$m < \infty$" lets a theory with only the vacuum satisfy the gap condition in form. In this chapter's reading, most of the words of the problem statement are of the latter kind, and close the loopholes to easy answers one by one.

The shutting-out overlaps. The free field with mass is caught by both "non-trivial" and "short distances". Even when one word is removed, another word may shut out the same thing. Table 5 is a guide to the role of each word.

Example. Removing "simple" means showing a gap for $G = U(1)$ as well, but that theory is the free electromagnetic field, which, as row 3 of Table 3 shows, has no gap. The claim with the word removed is false by this one example.

What this chapter can say

What is said	What is not said (where it is treated)
Known Where the sentence, the definitions, the footnotes and the extensions of the problem are in the problem statement, and what they say (checked against the Clay PDF)	The meaning of words the problem statement does not define ("non-trivial" and others) is this chapter's reading
Computed The dimensions and $h^\vee$ of Table 1 obtained from root systems, agreeing with the standard values. That $C_A = h^\vee$ in the normalisation of formula (2), checked for six groups	That the classification is exhausted by these, and a proof of $C_A = h^\vee$ for general groups (not proved in this book; list of values in 1-04)
Known That $m < \infty$ and $\sigma(H) \neq \{0\}$ are the same. The examples of Table 3	That the H of four-dimensional quantum Yang–Mills exists at all (the construction published in 2026-11 made it by reconstruction, at the level of the committee's assessment; 6-00; its gap is not shown)
Computed Table 4, the ratio (5), the numbers of (8), the empirical distribution of $\cos n$	A proof of Prokhorov's theorem (not treated in this book)
Physics $\Delta = c\Lambda$, the expectation of non-triviality	Neither has a proof (4-07, 17-12)
—	Table 5 is a reading, not a statement of the problem's authors. This chapter shows nothing about the problem

Sources and reproduction

Item	Kind	Source / tool
The sentence, § 1–§ 6.6, footnotes	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement of the Clay Mathematics Institute, PDF on the Institute's website; checked in full). In: J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, CMI / AMS, pp. 129–152 (year of publication not checked)
Date and place of the announcement	Known	The Clay Mathematics Institute website (the Millennium Prize Problems page) / the preface of the book above (Introduction, p. viii): 'The prize was announced at a meeting on May 24, 2000, at the Collège de France.' (checked in the PDF of the book published by the Institute)
(8), the general form $(11C_A - 4n_f T_R)/(12\pi)$ of b_0, Λ	Known	S. Navas et al. (Particle Data Group), Phys. Rev. D 110 (2024) 030001, QCD chapter § 9.1.1, formula (9.3). The relevant passage was read (the same passage as in the source table of 4-06)
Asymptotic freedom	Known	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343 / H. D. Politzer, ibid. 30 (1973) 1346. Bibliographic data only

Item	Kind	Source / tool
Axiom systems	Known	R. Streater, A. Wightman, *PCT, Spin and Statistics and All That*, Benjamin, 1964 / K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83–112, 42 (1975) 281–305. Bibliographic data only (content as in § 3 of the problem statement)
Triviality of φ^4_4	Known	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973. Abstract checked
What is expected for the ultraviolet limit	Known	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383. Bibliographic data only (the relevant passage of the introduction is in the source table of 7-07)
Lattice correlation-length problem (Chatterjee's Problem 5.1)	Known	S. Chatterjee, "Yang–Mills for probabilists", arXiv:1803.01950 (2018). Abstract checked. The text of the lattice correlation-length problem is in the source table of 5-09
Definitions of $C_A = h^\vee$ (in the normalisation where the squared length of the longest root is 1) and $h^\vee$	Known	V. G. Kac, *Infinite Dimensional Lie Algebras*, 3rd edition (1990). Bibliographic data only (the same as in the source table of 1-04)
Classification, $C_A > 0$, invariant inner products, Isserlis, Weyl, Prokhorov	Known	Standard facts (no bibliography given)
Tables 1 and 4, numbers in the text, figures	Computed	python3. Table 1 by exact rational computation: generate all positive roots from the Cartan matrix; dimension = rank + $2\times$ (number of positive roots), $h^\vee = (\theta, \theta+2\rho)/(\theta,\theta)$ (θ is the highest root, 2ρ the sum of the positive roots; as a ratio it does not depend on the normalisation of the inner product). The normalisation of formula (2): structure constants built from the generators of the defining representation, checked numerically for $SU(2), SU(3), SU(4), SO(5), SO(7), SO(8)$ to give $C_A = h^\vee$ with the inner product where the squared length of the longest root is 1, and $2h^\vee$ with the one where it is 2. Numerical integration by SciPy. Figures by JavaScript in the browser (Figure 1 from the formula, Figure 2 by the same rule as Table 4)

Next chapter: 0-04 Four dimensions in mathematics and four dimensions in physics. For digging deeper into each word, go to the chapters named at the end of each section.

Four dimensions in mathematics and four dimensions in physics — the same R⁴ has two rulers, and the problem asks for a theory that works with both

In this chapter, § 01– § 08 are established knowledge and § 09 is a map of open questions.

What this chapter shows — That the $\mathbb{R}^4$ of the problem statement can be read in two ways: as Euclidean four dimensions (the mathematical side) and as the four dimensions of spacetime (the physical side). That what joins the two is imaginary time and reflection positivity, and that the mass gap too has two faces, "the gap of the Hamiltonian" and "exponential decay of correlations". That the "four" that matters in the problem is the four at which the dimension of the coupling constant vanishes.

Prerequisite chapters — None. The vocabulary of quantum mechanics is set out minimally in section 04, and reflection positivity in section 06 (for details, 2-05 and 4-10). The examples of sections 06 and 07 only use the modified Bessel function $I_k(J) = \frac{1}{2\pi}\int_0^{2\pi} e^{J\cos\theta}\cos k\theta\, d\theta$ $(k = 0, 1)$ as numbers. Having read the articles What happens only in four dimensions and One sign is different makes the connections easier to see.

Said first — This is a chapter that guides you along the road. The substitution of imaginary time is a heuristic, and the rigorous bridge (the Osterwalder–Schrader reconstruction) is treated in 4-10. The computation of section 07 is an example in the quantum mechanics of a single particle. For four-dimensional Yang–Mills, the Hilbert space at the end of the way back was made by reconstruction in the construction published in 2026-11 (at the level of the committee's assessment; section 09), but a gap on it has not been shown. This chapter shows nothing about the mass gap.

Order of this chapter

1. The R^4 of the problem statement — the signature is not written, and two axiom systems are specified
2. The same points, two rulers — a rotation by an imaginary angle becomes a boost
3. Where the cases of the two articles matter in the problem (Table 1)

4. Rotating time — oscillation turns into decay because the energy is positive (Figure 1, Table 2)
5. A dictionary between the two four dimensions (Table 3, Detour: self-duality lives only in imaginary time)
6. Reflection positivity — recovering a Hilbert space from probability
7. The two faces of the mass gap — counting with a particle on a circle (Figure 2, Table 4)
8. The "four" that matters in the problem — what is counted is the Euclidean d
9. Three barriers on the way back — mismatches peculiar to Yang–Mills

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01

The R⁴ of the problem statement — the signature is not written

The problem statement (the Clay Mathematics Institute's description by Jaffe–Witten) asks to show that for any compact simple group G a non-trivial quantum Yang–Mills theory exists "on $\mathbb{R}^4$" and has a mass gap $\Delta > 0$ (a word-by-word reading is in 0-03). For this $\mathbb{R}^4$, the way of measuring length is not written.

The way of measuring appears in the surrounding sections. § 3 introduces the axioms as 'Gårding and Wightman gave mathematically precise axioms for quantum field theories on R4 with a Minkowski signature', and § 6 adds '(Equivalently in the Euclidean formulation one replaces Euclidean space-time R4 by R2 or R3.)'. Existence includes 'axiomatic properties at least as strong as those cited in [45, 35]', where [45] is Streater–Wightman (the axioms on the side of spacetime) and [35] is Osterwalder–Schrader (the axioms on the Euclidean side) Known . One can say that the same $\mathbb{R}^4$ has two rulers. To be exact, the set of points is the same, and what differs is the quadratic form that measures squared length ($c = 1$).

$$Q_{\mathrm{E}}(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2$$
$$Q_{\mathrm{M}}(t, x) = -t^2 + x_1^2 + x_2^2 + x_3^2 \quad (1)$$

Example. Reading the last coordinate of the point $(1, 0, 0, 1)$ as time, $Q_{\mathrm{E}} = 2$ but $Q_{\mathrm{M}} = 0$, and it is joined to the origin exactly by light. $(0, 0, 0, 1)$ has $Q_{\mathrm{E}} = 1$, $Q_{\mathrm{M}} = -1$ (timelike). There is also the convention of writing the signature as $(+, -, -, -)$; the content is the same.

02

The same points, two rulers — a rotation by an imaginary angle becomes a boost

Fixing a ruler fixes the group of transformations that preserve it. $SO(4)$ is the group of 4×4 real matrices of determinant 1 that leave Q_{E} unchanged, and $SO(3,1)$ the one that leaves Q_{M} unchanged. The former is the rotation group of four dimensions and the latter the Lorentz group; together with translations they are the Euclidean group and the Poincaré group. Both have one direction for each of the $\binom{4}{2} = 6$ coordinate planes (What happens only in four dimensions). The difference is in the 3 planes that contain time, where rotations become boosts in spacetime (2-04).

$$\begin{aligned} \tau' &= \tau\cos\alpha - x\sin\alpha \\ x' &= \tau\sin\alpha + x\cos\alpha \\ &\downarrow\quad \tau = it,\ \ \alpha = i\varphi \\ t' &= t\cosh\varphi - x\sinh\varphi \\ x' &= x\cosh\varphi - t\sinh\varphi \end{aligned} \tag{2}$$

This is because $\cos(i\varphi) = \cosh\varphi$ and $\sin(i\varphi) = i\sinh\varphi$ Known . This φ is called the rapidity. A point with $x' = 0$ in the lower formula of (2) moves along $x = t\tanh\varphi$, so the velocity of the boost is $v = \tanh\varphi$. Example. Putting $\varphi = 0.7$, $(t, x) = (0.4,\ 1.3)$ into the left gives $t' = -0.484091,\ x' = 1.328286$, agreeing with the right Computed . The angle of a rotation goes once round in 2π, but the rapidity has no upper bound (as $\varphi \to \infty$, v only approaches 1). Restated in terms of quadratic forms, $SO(4)$ can move a point only on a sphere where Q_{E} is constant, but $SO(3,1)$ can carry it arbitrarily far on the hyperboloid $Q_{\mathrm{M}} = -1$. That $SO(4)$ is compact and $SO(3,1)$ non-compact is an expression of this difference (One sign is different § 01).

03

Where the cases of the two articles matter in the problem

The article What happens only in four dimensions looked at Euclidean $\mathbb{R}^4$ and its topology, and One sign is different at spacetime. Not every case matters for the problem. What matters most on the Euclidean side is that the 6 planes form 3 pairs of "perpendicular planes", so that the field strength F splits in two (1-12). The field strength is the generalisation to Yang–Mills (3-02) of $F_{\mu\nu}$, which combines the electric and magnetic fields of electromagnetism (2-03), and it has one component for each coordinate plane. In electromagnetism, the 3 components of the electric field correspond to the 3 planes containing time, and the 3 components of the magnetic field to the 3 purely spatial planes. Writing the operation that moves the component of

each plane to the perpendicular plane as the Hodge star $*$, the way the planes are divided becomes exactly the way F is divided.

$$\binom{4}{2} = 6 = 3 + 3$$
$$F = F^{+} + F^{-}, \qquad F^{\pm} = \tfrac{1}{2}\left(F \pm *F\right) \quad (3)$$

As examples, we list how the cases show up in the problem.

Table 1 The cases of the two articles and the problem (E = Euclidean, M = spacetime)

Case		Relation to the problem	This book
The axis disappears from rotations	E	Splits into $F^{\pm}$. The classification of instantons	3-05
24-cell	E	Finite subgroups of SU(2). Lattice models with finite groups	5-10
Exotic $\mathbb{R}^4$	E	Classical equations tell topologies apart (Donaldson). Does not appear in the problem statement	3-06
Rotation stops being rotation; introducing imaginary numbers	E, M	A heuristic joining the two four dimensions	Section 04
The light cone and causality	M	The axiom that fields in regions that cannot be joined by light commute	4-09
Gravity, tails of waves	M	The problem is only about flat spacetime; does not appear	—

How to read the table. The top three rows (E) matter as tools and models in Part 3 (classical Yang–Mills) and Part 5 (the lattice). The fourth row (E, M) is used in sections 04–07 of this chapter to join the two four dimensions. Of the M rows, the light cone enters the axioms of Part 4 (4-09), and the gravity row is outside the problem.

The "four" that matters in the problem is not a phenomenon peculiar to four dimensions like exotic structures, but the four at which the dimension of the coupling constant vanishes (section 08). However, the proof of the existence of exotic $\mathbb{R}^4$ uses, together with Freedman's topological theorem, the same Yang–Mills equations (the space of their anti-self-dual solutions) (1-12, 3-06). Classical theory uses the solutions of the equations as a tool for seeing topology; quantum theory uses the action as a probability weight.

04

Rotating time — oscillation turns into decay because the energy is positive

The first tool for going back and forth between the two four dimensions is to replace time t by an imaginary number (Wick rotation; the name is said to come from Wick

1954; for the form in field actions, 4-02 § 02 is the authoritative place). Putting $t = -i\tau$ into (1) turns Q_{M} into Q_{E}. What changes can be seen in time evolution in quantum mechanics.

A state is a vector of length 1 in a Hilbert space (a complex vector space with an inner product, closed under limits; 1-10), and we assume it can be expanded in the eigenvectors ψ_n (eigenvalues E_n) of the energy operator H (the Hamiltonian). The component with energy E_n turns with phase $e^{-iE_n t}$ ($\hbar = 1$). Taking the lowest state as the vacuum Ω (energy 0) and writing $O\Omega = \sum_n c_n \psi_n$ for the vector obtained by applying an observable O, the correlation across a time t is

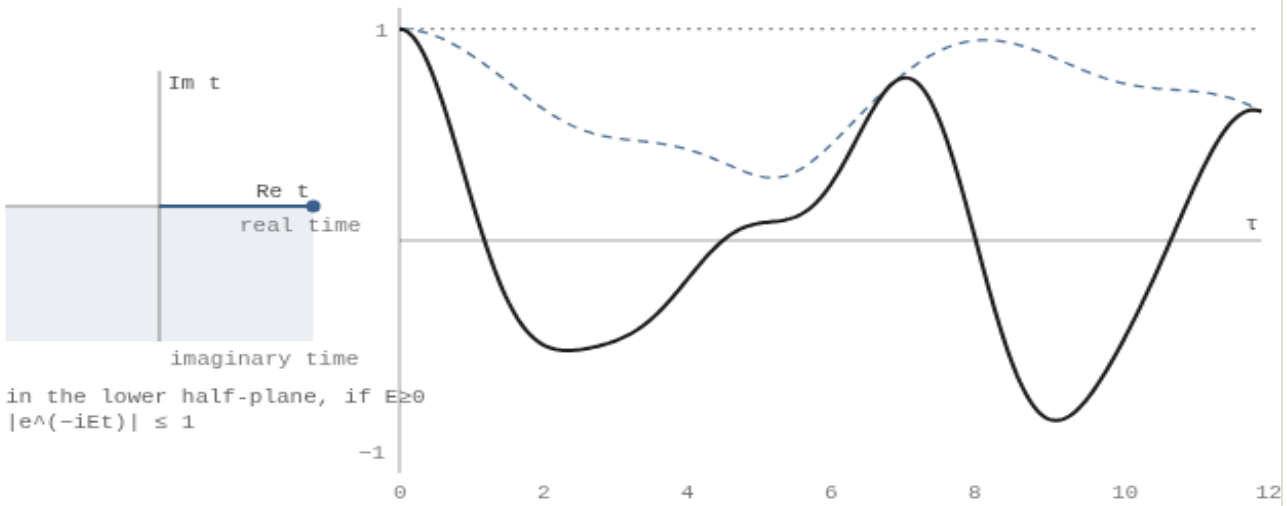
$$C(t) = \langle O\Omega,\ e^{-iHt}\, O\Omega \rangle = \sum_n |c_n|^2 e^{-iE_n t}$$

$$t = \tau e^{-i\vartheta} \implies \left| e^{-iE_n t} \right| = e^{-E_n \tau \sin\vartheta} \qquad (4)$$

ϑ is the angle by which time is rotated in the complex plane: $\vartheta = 0$ is real time and $\vartheta = 90^\circ$ imaginary time. If every E_n is at least 0, then rotating t into the lower half-plane keeps each term no greater than 1, and at $\vartheta = 90^\circ$ oscillation turns into decay $e^{-E_n\tau}$. If there were a negative E_n, that term would grow and the sum would lose its meaning. § 3 of the problem statement writes 'Wightman used positive energy to establish analytic continuation of expectations of Minkowski field theories to Euclidean space.' Known .

Example. We draw the connected part C_{conn}, with the vacuum term removed, for a model with $E_n = 1,\ 1.7,\ 2.6$ and weights $0.6,\ 0.3,\ 0.1$. The gap is $\Delta = 1$.

Figure 1 Moving the angle ϑ from 0° to 90° turns the direction of time (left) towards the imaginary axis, and the oscillation of the real part of the correlation (solid line, right) turns into decay. The dashed line is |C| and the dotted line e^(−Δτ sinϑ). Adding "negative energy" turns it into growth



ϑ = 0°, rate of decay Δ·sinϑ = 0.000, rate of oscillation Δ·cosϑ = 1.000, |C(τ=8)| = 0.94469

Computed The numbers in this figure were computed on this machine (the sum (4) computed in the browser; Table 2 by python3). "Negative energy" is a term with $E = -0.3$ and weight 0.03.

Table 2 $|C_{\text{conn}}(\tau e^{-i\vartheta})|$ (the last column is the case with negative energy added)

ϑ	τ = 2	τ = 8	$e^{-8\sin\vartheta}$	With negative, τ = 8
0°	0.62270	0.94469	1	0.92401
30°	0.23920	0.01104	0.01832	0.08978
60°	0.11873	0.00059	0.00098	0.24004
90°	0.09176	0.00020	0.00034	0.33090

In real time it is still 0.94 at $\tau = 8$, and the correlation does not decrease. The more it is rotated, the faster it decreases; with negative energy added, the more it is rotated, the more it grows. The upper bound $|C_{\text{conn}}| \le e^{-\Delta\tau\sin\vartheta}$ follows from the triangle inequality applied to each term of (4), since the weights sum to 1 and $E_n \ge \Delta$ Known . It was also checked at all points of a fine grid in $0 \le \vartheta \le 90°$ and $0 \le \tau \le 12$ Computed .

05

A dictionary between the two four dimensions

Moving to imaginary time replaces the vocabulary of quantum theory with the vocabulary of probability. § 3 of the problem statement sums up that Symanzik read Euclidean expectations as 'a statistical mechanical ensemble of classical Markov fields [46], with a probability density proportional to exp(−S)', and that Osterwalder and Schrader gave a general theory of the equivalence of the two axiom systems Known (the exact direction and conditions of the theorem are in 4-10).

$$e^{iS[\phi]} \longrightarrow e^{-S_{\mathrm{E}}[\phi]}$$
$$e^{-iHt} \longrightarrow e^{-H\tau} \quad (5)$$

The e^{iS} on the left (S is the action and ϕ the field; 2-01, 2-02) oscillates with size 1 and is not a probability. The $e^{-S_{\mathrm{E}}}$ on the right (S_{E} is the action moved to imaginary time) is positive, and when normalised can be a probability weight (2-07, 4-02). This is why on the mathematical side Yang–Mills becomes "the problem of constructing a probability measure on the space of connections" (a connection is the gauge field A; for details, 1-07). As examples, we list the main rows of the correspondence.

Table 3 A dictionary between the two four dimensions (the rigorous form is in 4-10)

Four dimensions of physics (spacetime)	Four dimensions of mathematics (Euclidean)
The Lorentz group $SO(3,1)$	The rotation group $SO(4)$
Time evolution e^{-iHt}	$e^{-H\tau}$. On the lattice, the transfer matrix (5-03)
The weight of the path integral e^{iS}	The probability weight $e^{-S_{\mathrm{E}}}$ (2-08)

Four dimensions of physics (spacetime)	Four dimensions of mathematics (Euclidean)
Vacuum expectations of products of fields (Wightman functions)	Moments of a probability measure (Schwinger functions)
A Hilbert space and positivity of energy $H \geq 0$	Reflection positivity (section 06)
Fields in regions that cannot be joined by light commute	Schwinger functions do not change under permutation of their arguments (for bosonic fields)
Mass gap: the spectrum of H is absent from $(0, \Delta)$	Connected correlations decrease exponentially (section 07)

Every row finally passes through the single point "with reflection positivity, one can get back to a Hilbert space".

Detour: self-duality lives only in imaginary time

A subsection that may be skipped.

There are also rows the dictionary cannot translate. The sign of applying the Hodge star $*$ (the operation that moves the component of each plane to the perpendicular plane; 1-12) twice changes with the ruler.

$$
\begin{aligned}
** &= +1 \quad \text{(Euclidean)} \\
** &= -1 \quad \text{(spacetime)} \\
* &: (E, B) \mapsto (-B, E) \quad \text{(spacetime)}
\end{aligned}
\tag{6}
$$

In Euclidean space there are real fields satisfying $F = *F$ (self-dual), and instantons are its solutions (3-05). In spacetime the eigenvalues of $*$ are $\pm i$, so for a real field, $*F = \pm F$ implies $F = 0$. Example. Building $*$ as a six-component matrix, the eigenvalues were ± 1 in Euclidean space and $\pm i$ with signature $(-,+,+,+)$, three of each Computed (the sign on the right of (6) is by convention). Reading instantons as tunnelling in imaginary time is a semiclassical interpretation Physics .

06

Reflection positivity — recovering a Hilbert space from probability

On the Euclidean side there is only a probability measure (expectation $\mathbb{E}$), with neither a Hilbert space nor a Hamiltonian. To recover them an inner product is needed, and the material is the mirror of time $\Theta : \tau \mapsto -\tau$. Let X, Y be a quantity determined only by the fields on the side $\tau > 0$, and ΘX the result of reversing the sign of every time in X (a quantity on the side $\tau < 0$), and set

$$
\begin{aligned}
\langle X, Y \rangle &:= \mathbb{E}\big[\, \Theta X \; Y \,\big] \\
\langle X, X \rangle &\geq 0
\end{aligned}
\tag{7}
$$

The second half is reflection positivity (also called OS positivity). If it holds, $\langle \cdot,\cdot \rangle$ becomes an inner product (after identifying vectors of length 0), and the operation of shifting by s in the direction of τ acts as e^{-sH} $(H \geq 0)$ Known . § 6.5 of the problem statement writes 'Establishing a quantum mechanical Hilbert space is part of the solution to this Millennium problem.'

Example. Put a particle rotating on a circle (moment of inertia I, angle θ) on an imaginary-time lattice with step a. In this chapter θ is used for the angle of this particle (it is different from the angle of the θ term in 3-05 and elsewhere). Writing the angle at time ka as θ_k and approximating one step of the action $\int \frac{I}{2}\dot\theta^2 d\tau$ by $\frac{I}{a}\big(1-\cos(\theta_{k+1}-\theta_k)\big)$, the weight is

$$\prod_k e^{J\cos(\theta_{k+1}-\theta_k)}, \qquad J=\frac{I}{a}$$

$$e^{J\cos u} = \sum_{n\in\mathbb{Z}} I_n(J)\, e^{inu} \tag{8}$$

$$I_n(J) = \frac{1}{\pi}\int_0^{\pi} e^{J\cos u}\cos nu\, du$$

J is the strength with which the angles at neighbouring times are pulled into line, and in statistical mechanics this chain of weights is called the one-dimensional XY model (the vocabulary of partition functions and transfer matrices is in 2-08 and 2-09). I_n is a modified Bessel function, and since the coefficients of its power series are all positive, $I_n(J)>0$ Known . The transfer matrix T is the operator whose integral kernel is the weight of one step.

$$(Tf)(\theta) = \int_0^{2\pi} K(\theta,\theta')\, f(\theta')\, d\theta', \qquad K(\theta,\theta') = e^{J\cos(\theta-\theta')}$$

Substituting the expansion (8), T maps $e^{in\theta}$ to itself with eigenvalue $2\pi I_n(J)$ (agreeing to 12 digits with a numerical computation that divides θ into 256 points Computed). If the plane of reflection is placed between lattice points, the left-hand side of (7) becomes a quadratic form sandwiching T, and since the eigenvalues are positive it is at least 0. In this way $H_a = -\frac{1}{a}\log\big(T/2\pi I_0(J)\big) \geq 0$ is determined. Reflection positivity in Wilson's lattice gauge theory was shown by Osterwalder–Seiler 1978 (for the construction of the transfer matrix, also Lüscher 1977; 5-03).

07

The two faces of the mass gap — counting with a particle on a circle

The definition of the mass gap is on the side of spacetime; § 4 of the problem statement writes 'A quantum field theory has a mass gap Δ if H has no spectrum in the

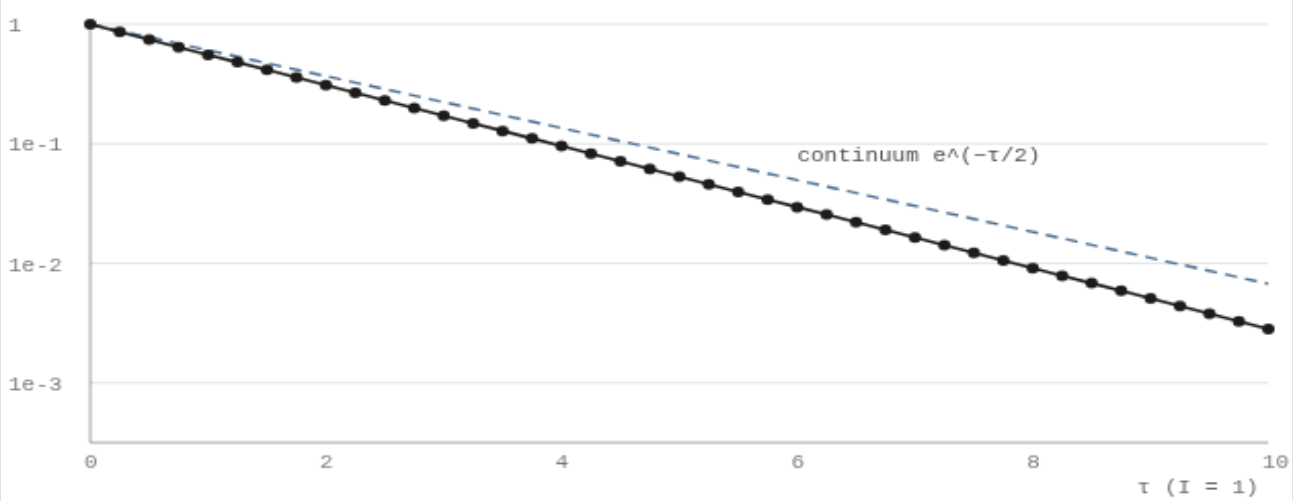
interval $(0,\ \Delta)$ for some $\Delta > 0$.' (H is the Hamiltonian; the spectrum $\operatorname{spec} H$, in finite dimensions, is the set of eigenvalues). On the Euclidean side the same thing appears as decay of correlations. We call this the two faces of the mass gap. To be exact, they are a condition on the spectrum of the Hamiltonian and exponential decay of connected correlations (Schwinger functions with the vacuum contribution subtracted; Table 3). Setting $\vartheta = 90^\circ$ in (4),

$$\begin{aligned} \operatorname{spec} H \subset \{0\} \cup [\Delta, \infty) \implies \\ C_{\mathrm{conn}}(\tau) = \sum_{E_n \geq \Delta} |c_n|^2 e^{-E_n \tau} \\ \leq C_{\mathrm{conn}}(0)\, e^{-\Delta \tau} \end{aligned} \tag{9}$$

If O overlaps with the lowest excitation, $-\frac{d}{d\tau} \log C_{\mathrm{conn}}$ approaches Δ; in the model of Figure 1 it was 1.178, 1.086, 1.021, 1.001 at $\tau = 1, 2, 4, 8$ Computed . Measuring the masses of glueballs (candidates for excitations corresponding to Yang–Mills particles) on the lattice uses this face (5-08). In Euclidean space time and space are carried into each other by rotations, so decay in the direction of τ is also decay in spatial directions, which leads to the clustering of § 5 of the problem statement Known . A local observable (a quantity determined only by the fields in a bounded region) rotated is again a local observable, so decay at the same rate holds in every direction. But this is a matter for after the reconstruction has given a Hilbert space and a Hamiltonian.

Example. Take $O = e^{i\theta}$ for the particle of section 06. The eigenvalues of $H = -\frac{1}{2I}\frac{d^2}{d\theta^2}$ are $\frac{n^2}{2I}$, the gap is $\Delta = \frac{1}{2I}$, and $O\Omega$ has only components with $n = 1$, so $C_{\mathrm{conn}}(\tau) = e^{-\Delta\tau}$. On the lattice, an infinitely long chain gives $\langle e^{i(\theta_k - \theta_0)} \rangle = (I_1/I_0)^k$, and the rate of decay read as $\tau = ka$ is $\Delta(a) = -\frac{1}{a} \log \frac{I_1(J)}{I_0(J)}$. This is the gap on the lattice measured in physical units (the smallest positive eigenvalue of H_a in section 06), and in lattice units it is $a\,\Delta(a)$ (the m_{lat} of Appendix A-01).

Figure 2 Making the step a = I/J finer brings the slope of the lattice correlation (solid line; dots too on coarse lattices) close to the continuum e^(−Δτ) (dashed line). The vertical axis is logarithmic. Clicking a row of Table 4 also changes J




```
J = I/a = 4, a = 0.25, Δ(a) = 0.586941, Δ(a)/Δ = 1.173882 (Δ = 0.5)
```

Computed The numbers in this figure were computed on this machine (I_1/I_0 and $\Delta(a)$ are the values of Table 4 computed with python3 and SciPy; the figure only joins the points $(ka,\ (I_1/I_0)^k)$).

Table 4 The lattice for a particle on a circle ($I = 1$, $\Delta = 0.5$, step $a = 1/J$). The last column is the ratio to the second level $E_2 = 2$

J	Δ(a)/Δ	1 + 1/(2J)	E₂(a)/E₂
1	1.613125	1.500000	1.116436
2	1.439436	1.250000	1.196582
4	1.173882	1.125000	1.130427
8	1.071311	1.062500	1.065295
16	1.033214	1.031250	1.032022
32	1.016093	1.015625	1.015824
64	1.007927	1.007813	1.007863

On coarse lattices ($J \le 2$) $\Delta(a)/\Delta$ is 1.4–1.6, a different number from the gap. Making the lattice finer brings it to 1 at almost the same rate as $1 + \frac{1}{2J}$, and the second level approaches too. The correlation computed from products of transfer matrices on a chain of period 400 also agreed with $(I_1/I_0)^k$ to 10 digits Computed . In this example the existence of a gap is clear from the start. What is needed for four-dimensional Yang–Mills is that $\Delta(a)$ stays positive in the limit where the decay seen in lattice units $a\,\Delta(a)$ goes to 0 (5-09). The Hilbert space of the limit was made by reconstruction in the construction published in 2026-11 (at the level of the committee's assessment; section 09).

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08
```

The "four" that matters in the problem — what is counted is the Euclidean d

The counting of dimensions is done on the Euclidean side, over d equivalent coordinates. With $c = 1$, time too is measured as a length, so 3+1-dimensional spacetime is $d = 4$. Write the d-dimensional Euclidean action as $S_d[A] = \frac{1}{2g^2}\int_{\mathbb{R}^d} \operatorname{tr} F_{\mu\nu}F_{\mu\nu}\, d^dx$ (summed over μ, ν). This is the form of the action in Appendix A-01; here we take the convention that puts the coupling g in front of the action and does not include it in F. With $\hbar = c = 1$, writing the mass dimension as $[\cdot]$ (for the derivation, 2-04 and 1-12),

$$\begin{gathered}[g^2] = 4 - d \\ S_d[A_\lambda] = \lambda^{4-d}\, S_d[A] \\ \left(A_\lambda(x) = \lambda A(\lambda x)\right)\end{gathered} \tag{10}$$

and only in four dimensions does the classical action have no scale of length. A mass cannot be built from classical constants, so the gap is expected to be a constant multiple $\Delta = C\Lambda$ (C a dimensionless constant) of a scale Λ born from quantum effects Physics . This is what lies behind 'the mass gap is not classically visible' in § 6.6 of the problem statement (4-07). A figure that moves the dimension d continuously and sets it beside other quantities that switch at four is the dimension slider in 1-12 § 10.

Example. In three dimensions $[g^2] = 1$, so g^2 itself becomes a mass scale, and by this counting it falls on the easy side. Chevyrev (arXiv:2202.13359 § 1.1) also writes 'The cases d = 2, 3 are considered substantially simpler than d = 4', but in the same section the construction of the three-dimensional measure 'even on T3, remains open' Known . In two dimensions the continuum measure has been constructed; in four it has not been constructed as a measure (what the construction published in 2026-11 made is at the level of correlation functions; section 09. Where things stand by dimension is in 7-10, misconceptions in 16-08).

09

Three barriers on the way back — mismatches peculiar to Yang–Mills

For the particle on a circle the dictionary could be used as it is. In this chapter we call the way back the road that starts from a theory constructed on the Euclidean side and follows the dictionary in reverse to a theory on the side of spacetime. To be exact, the way back means obtaining, from a Euclidean probability measure by the OS reconstruction, a theory satisfying the Wightman axioms (one with a Hilbert space and $H \geq 0$). For Yang–Mills there are three barriers along it: observables, regularisation, volume. The whole road, set out, is as follows.

$$\underbrace{\text{lattice, regularisation}}_{\text{reflection positivity}} \xrightarrow{a\to 0} \text{continuum measure}$$
$$\xrightarrow{T^4\to\mathbb{R}^4} \text{Euclidean-invariant}$$
$$\xrightarrow{\text{reconstruction}} (\mathcal{H},\, H\geq 0) \xrightarrow{?} \Delta > 0 \qquad (11)$$

The regularisation at the left end puts a minimal length into the theory (the lattice step a or the like) so that infinities do not appear. T^4 is a four-dimensional box with each side joined periodically, to keep the volume finite. For four-dimensional non-abelian Yang–Mills, the construction published in 2026-11 went through everything up to just before the last arrow of (11). The continuum object, however, is not a measure but correlation functions, and the volume limit is along a subsequence (the committee's assessment is that it is proved in Lean; 6-00, 12-06, 12-08). The last arrow, the "?", the mass gap, has not been shown (6-04). § 6 of the problem statement of 2000 writes of the construction on T^4 'while the construction is not complete'. In two dimensions the road goes through from the lattice to the continuum measure on the plane $\mathbb{R}^2$ (7-02). The three barriers are as follows.

Observables. Only gauge-invariant quantities have meaning, that is, quantities whose values do not change under gauge transformations (the operation of choosing a group element at each point and replacing the field; 3-01). Local ones are composite quantities such as $\mathrm{Tr}\, F_{ij}F_{kl}(x)$ (§ 4 of the problem statement). Chatterjee (arXiv:1803.01950 § 1), citing Seiler, writes 'the key observables in these theories do not take values at points, but at curves', and Driver 1989 says that Wilson loops are 'quite singular in d = 3 and especially d = 4' Known . This barrier bears on every stage of (11).

Regularisation. The barrier at the left end of (11). Wilson's lattice preserves reflection positivity but reduces rotations to steps of 90°. § 6.5 of the problem statement writes that there are few methods for recovering reflection positivity lost through regularisation (quoted in 0-03 § 03), and Magnen–Rivasseau–Sénéor 1993, which uses a continuum regularisation, too stops at considering that OS positivity (reflection positivity) can be shown with additional work Known .

Volume. The barrier at the second arrow of (11). Even on T^4, if there is reflection positivity, a finite-volume Hilbert space and transfer matrix are obtained, but a theory with Poincaré symmetry on $\mathbb{R}^4$ needs $T^4 \to \mathbb{R}^4$. § 6 of the problem statement writes 'Nor do present methods suggest how to obtain the existence of the infinite volume limit T4 → R4.' Known . The construction published in 2026-11 took the infinite-volume limit along a subsequence by compactness (12-05). That the limit does not depend on the subsequence has not been shown (6-04).

Example. For non-abelian Yang–Mills in three and four dimensions, a unique non-trivial continuum limit has not yet been shown (in four dimensions, a non-trivial limit was made by the construction published in 2026-11, but its uniqueness has not been

shown; in three dimensions it is not constructed even on T^3; section 08). The final "?" of (11) is the body of the mass gap, but the gap is not a single arrow left over at the end. § 6.5 of the problem statement writes 'New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time', and goes on to expect that if a gap independent of the volume (uniform however large the box is made) were shown, it would become possible to study the $T^4 \to \mathbb{R}^4$ limit Known . So the barriers of the gap and of volume are entangled (7-11, 16-05).

What this chapter can say

Content	Status
That the $\mathbb{R}^4$ of the problem statement has no specified signature, and that the axiom systems of both sides are specified as [45, 35]	Said Known (problem statement § 3, § 4, § 6)
(2), (6), (8); that with positive energy it becomes decay in imaginary time (4); that the gap gives an upper bound on decay (9)	Said Known (Figures 1–2, Tables 2 and 4 are Computed)
The dictionary (Table 3), and the procedure (7) for building a Hilbert space from reflection positivity	Known as a correspondence. The form of the theorem is treated in 4-10; this chapter only states it
That the gap is born from quantum effects in the form $\Delta = C\Lambda$	At the level of physics Physics (4-07)
That each arrow of (11) holds for four-dimensional Yang–Mills	The last arrow (the mass gap) is unsolved. The arrows before it were passed at the level of correlation functions by the construction published in 2026-11 (at the level of the committee's assessment; the volume limit along a subsequence; 6-00). This chapter shows nothing

Sources and reproduction

Item	Kind	Source / tool
The numbers of (2) and (6), Tables 2 and 4, Figures 1–2, the effective mass of section 07 and the check with chains	Computed	python3 (NumPy, SciPy). Figure 1 computes (4) in the browser; Figure 2 draws the values of Table 4

Item	Kind	Source / tool
Quotations from the problem statement	Known	A. Jaffe, E. Witten, “Quantum Yang–Mills Theory”, the official problem statement of the Clay Mathematics Institute (problem announced in 2000). In: J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, CMI / AMS, pp. 129–152 (checked by contents and pages; year of publication not checked). Text checked
The axioms of both sides, and reconstruction	Known	R. F. Streater, A. S. Wightman, *PCT, Spin and Statistics and All That* (1964). K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83–112; 42 (1975) 281–305. Bibliographic data only
Reflection positivity and the transfer matrix on the lattice	Known	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471. Checked from the abstract (physical positivity of the Schwinger functions of lattice gauge theory, a positive self-adjoint transfer matrix). M. Lüscher, “Construction of a selfadjoint, strictly positive transfer matrix for Euclidean lattice gauge theories”, Comm. Math. Phys. 54 (1977) 283–292. Checked from the abstract
Wick rotation	Known	G. C. Wick, Phys. Rev. 96 (1954) 1124–1134. Checked from the abstract (analytic continuation to purely imaginary values of the relative time, rotation of the integration contour). Attributing the name "Wick rotation" is a convention
Observables on curves / where two and three dimensions stand	Known	S. Chatterjee, arXiv:1803.01950 § 1 / I. Chevyrev, arXiv:2202.13359 § 1.1. Text checked
Singularity of Wilson loops	Known	B. K. Driver, Comm. Math. Phys. 123 (1989) 575–616. The relevant sentence of the introduction checked
Construction in four dimensions with an infrared cutoff	Known	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383. The passage on OS positivity in § I checked
Modified Bessel functions	Known	NIST DLMF 10.32.3 (integral representation) checked. Power series in § 10.25

Next chapter: 1-01 Vector spaces and linear maps. This chapter continues in 2-04, 4-02, 4-10 and 4-11.