Volume 2 Existence — reading the published construction

Part 7 Constructive field theory

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What constructive field theory is — splitting the "making" of quantum field theory into six stages

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The idea that to "make" a quantum field theory is to exhibit a non-Gaussian probability measure satisfying the axioms as the limit of a sequence of approximations. The six stages along that road and the difficulty of each. Which stages each chapter of Part 7 deals with.

Prerequisite chapters — 4-10 The Osterwalder–Schrader axioms and reconstruction. See also 4-02, 4-11 and 5-01. Within Volume 2, read this after 6-04 What is not shown.

Said first — The six steps are an arrangement this book makes following the account of the problem statement (Jaffe–Witten), not standard terminology in the field. This chapter explains a known framework and shows nothing about four-dimensional Yang–Mills. As the preface to Volume 2, Part 7 sets out the tools of constructive field theory and what the literature before 2026 reached, before the published construction (Part 6) is read in detail. How far the published construction got through the steps is treated in 6-01 and 6-04.

The order of this chapter

1. What is it that one makes — a non-Gaussian measure
2. Where the formal expression is broken — three places (a table of calculations)
3. The six stages — a map and how far each theory has got (moving figure)
4. The regularisation and finite-volume steps — what is kept, what is broken
5. The ultraviolet-limit step — renormalisation and uniform estimates
6. The infinite-volume-limit step — clustering
7. The axiom step — reflection positivity
8. The uniform-gap step — "uniformly" is the heart of it
9. Detour: taking a one-dimensional chain through all six stages — moving figure
10. How to read Part 7 — stages and chapters

01

What is it that one makes

Physics textbooks compute assuming the theory exists. The mathematical question comes before that: whether there is an object satisfying the axioms (4-09, 4-10). The problem statement writes that not a single non-trivial relativistic field theory satisfying the axioms is known in four dimensions (JW § 6, 2000).

The problem statement divides the ways of showing existence into two (JW § 6.1). (i) Find an exact solution in closed form. (ii) Solve a sequence of approximate problems and show that the solutions converge to a limit. **Constructive field theory** (constructive quantum field theory) refers to (ii), and at its centre are inequalities independent of the approximation: uniform a priori estimates. Known

In the Euclidean form (4-02), the object to be made is a probability measure μ on the space of distributions $\mathcal{S}'(\mathbb{R}^d)$. If its moments

$$S_n(f_1, \dots, f_n) = \int \phi(f_1) \cdots \phi(f_n)\, \mathrm{d}\mu(\phi) \quad (1)$$

satisfy the OS axioms, the reconstruction theorem gives a Hilbert space and a Hamiltonian ($\phi(f)$ is the field smeared with a test function f). In physics the same measure is written

$$\mathrm{d}\mu(\phi) = \frac{1}{Z}\, e^{-S(\phi)}\, \mathcal{D}\phi \quad (2)$$

This is the target form, but none of the symbols on the right is a mathematical object as it stands.

Example: the free field of mass m is the Gaussian measure with mean 0 and covariance $C = (-\Delta + m^2)^{-1}$ (Δ is the Laplacian, different from the Δ of the mass gap). It satisfies the OS axioms, but has no interaction and is trivial (JW § 6.2). The work is to make a **non-Gaussian** measure. According to the problem statement, the first constructions of the form satisfying the axioms (as non-Gaussian measures) came in the 1970s, for fields in two dimensions with small polynomial interactions (JW § 6.2).

A figure lining up, in eleven rows, what switches with the dimension is 1-12 §10 Figure 4 (opens at d = 4; the bottom row is the YM row of Table 2).

02

Where the formal expression is broken

Expression (2) is broken in three places. (i) $\mathcal{D}\phi$: on an infinite-dimensional space, the only measure that is unchanged by translation and gives every ball a finite weight is 0 (1-09 § 02). (ii) Values at points: in two or more dimensions samples of the free field are distributions, and $\phi(x)^4$ has no meaning (1-11). (iii) Volume: on the whole of $\mathbb{R}^d$ the action is infinite. The action of a scalar field is

$$S(\phi) = \int_{\mathbb{R}^d} \left(\tfrac{1}{2} |\nabla \phi|^2 + \tfrac{1}{2} m^2 \phi^2 + \lambda\, \phi^4 \right) \mathrm{d}x \qquad (3)$$

and for Yang–Mills the field is a connection A and the action is the integral of the square of the curvature F (3-03).

(ii) can be seen in numbers. The variance at a point of the free field on the lattice $a\mathbb{Z}^d$ is

$$G_a(0) = \langle \phi(x)^2 \rangle_a = \int_{[-\pi/a,\, \pi/a]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{1}{m^2 + \frac{4}{a^2} \sum_j \sin^2 \frac{k_j a}{2}} \qquad (4)$$

where the denominator is the lattice version of $m^2 + |k|^2$, and the momenta are cut off at $|k_j| \le \pi/a$.

Computed Table 1 $G_a(0)$ ($m = 1$). Numerical integration of the heat-kernel representation $G_a(0) = a^{2-d} \int_0^\infty e^{-m^2 a^2 s} \, [e^{-2s} I_0(2s)]^d \, \mathrm{d}s$ (I_0 is a modified Bessel function). For $d = 1$ it agrees to 5 digits with the closed form $1/(m\sqrt{m^2 a^2 + 4})$.

a	$d = 1$	$d = 2$	$d = 3$	$d = 4$
1	0.44721	0.25405	0.1705	0.127
1/4	0.49614	0.49320	0.9289	2.431
1/16	0.49976	0.71675	3.9634	39.598
1/64	0.49998	0.93768	16.0950	634.524
$a \to 0$	$\to 0.5$	$\frac{1}{2\pi} \log \frac{1}{a} + 0.2758$	$0.2527/a$	$0.1549/a^2$

In two or more dimensions the variance goes to infinity, and values at points are not random variables. ϕ^4 has to be redefined with a subtraction depending on a, and this is the entrance to the renormalisation of the ultraviolet-limit step.

03

The six stages

Replace the broken places with finite things (regularisation and finite volume), and remove them at the end (the two limits). Once the limits are taken, check the axioms and the spectral gap. The problem statement too writes, for the lattice, that one should

establish the limits of expectation values as the lattice spacing goes to 0 and the volume to infinity (JW § 6.5).

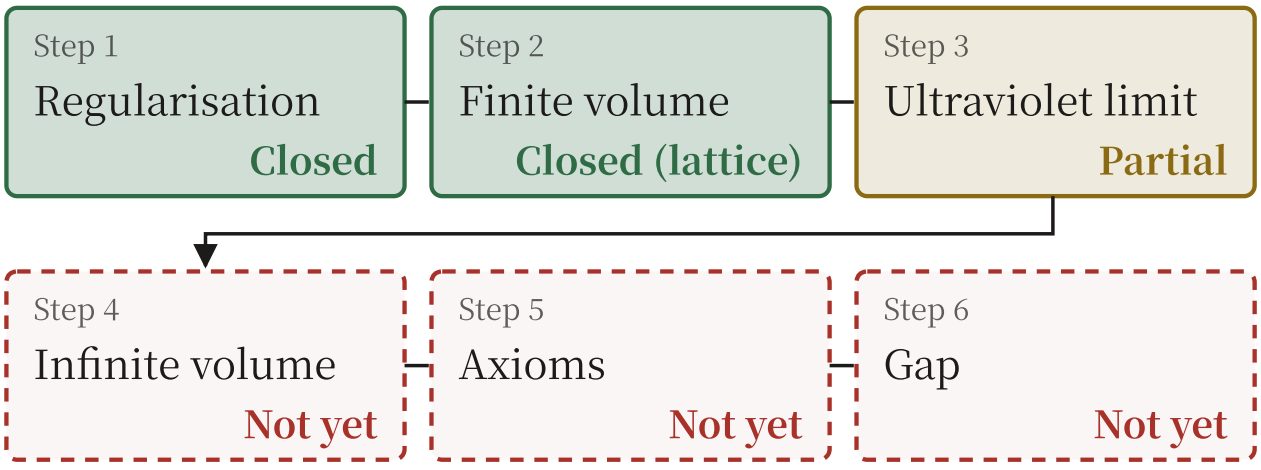
Stage	What it does	What it needs
1 Regularisation	Replace the field by finitely many degrees of freedom	Lattice spacing a
2 Finite volume	Make the measure on a box of side L	Measure $\mu_{a,L}$
3 Ultraviolet limit	$a \to 0$	Renormalisation, uniform estimates
4 Infinite volume	$L \to \infty$	Clustering
5 Axioms	The limit satisfies the OS axioms	Reflection positivity
6 Gap	Gap of the Hamiltonian	Uniform exponential decay

The four steps from the regularisation step to the infinite-volume step make the double limit

$$\mu = \lim_{L\to\infty} \lim_{a\to 0} \mu_{a,L} \quad (5)$$

and the axiom step and the uniform-gap step are properties of μ. There is also the road, as in strong-coupling results on the lattice, of holding a fixed and taking $L \to \infty$ first, in which case the ultraviolet-limit step is left for later. However, the strong-coupling region (where the lattice coupling g is large and $\beta = 2N/g^2$ small) is far from the continuum limit ($\beta \to \infty$), so this road does not as it stands connect to the ultraviolet-limit step. Known

Figure 1 Choosing a theory (with the buttons below or a row of Table 2) changes how far each step has got



YM four dimensions: Ultraviolet stability on the lattice in finite volume (Bałaban 1987–89; with the assumption that the effective coupling enters a sufficiently small range; 7-07). MRS 1993 is an outline of a construction with an infrared cutoff. The continuum finite-volume measure has not been constructed

Known Table 2 How far each theory had got before 2026 (Closed = there is a theorem / Partial / Not yet / — = the question is hard to pose / Not taken = that road does not pass through the step). From the problem statement and primary sources. Years are the publication years of representative papers (the timeline in 16-01 counts by first public appearance, so Sengupta is 1992–93 and Lévy 2001). Selecting a row changes Figure 1.

Theory	1	2	3	4	5
Free field	Closed	Closed	Closed	Closed	Closed
ϕ^4_2 weak coupling	Closed	Closed	Closed	Closed	Closed
ϕ^4_3 weak coupling	Closed	Closed	Closed	Closed	Closed
YM two dimensions	Closed	Closed	Closed	Closed	Partial

Theory	1	2	3	4	5
YM three dimensions	Closed	Closed (lattice)	Partial	Not yet	Not yet
YM four dimensions	Closed	Closed (lattice)	Partial	Not yet	Not yet

Theory	1	2	3	4	5
Lattice YM four dimensions, strong coupling	Closed	Closed	Not taken	Closed	Partial

The current marks, including the published construction (2026), are in 1-12 Table 4 and 7-07 Table 4. In four dimensions, of the six steps of construction, the regularisation, finite-volume (lattice), ultraviolet-limit and axiom steps are closed, the infinite-volume step is partial (up to subsequences of volumes), and the uniform-gap step is not yet done (as marked in 1-12).

The "—" in Table 2 means that the question is hard to pose in its plain form. Two-dimensional Yang–Mills has no propagating degrees of freedom (waves that travel far, the counterpart of light in electromagnetism), so the question of a spectral gap cannot be posed in the same form as in four dimensions (7-02). Known "Not taken" means that the road does not pass through the step. At strong coupling on the lattice a stays fixed, so the road does not pass through the ultraviolet-limit step. For three- and four-dimensional Yang–Mills, the finite-volume step closes by formula (6) on the lattice (§ 04). Building the finite-volume measure directly in the continuum has not yet been done, and that part is recorded as "Partial" for the ultraviolet-limit step and in the summary column.

The four-dimensional YM row of Table 2 is what the literature before 2026 reached. On 2026-11-10, a Lean formalisation that a theory satisfying the Wightman axioms can be built as a scaling limit of four-dimensional Wilson lattice gauge theory was published, together with the committee's assessment report. The committee's assessment is that Lean proves the following: the ultraviolet limit in the form of correlation functions of local observables, the infinite-volume limit along subsequences of volumes, and from there the verification of the axioms and reconstruction (6-01; for reconstruction, 12-08). Convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, and a uniform mass gap (exponential decay) are not shown (6-04).

> 04

The regularisation and finite-volume measure steps

Regularisation can be done with a **lattice** (putting the field on lattice points or edges) or a **momentum cutoff** (restricting Fourier components to $|k| \le \kappa$), and both break something.

Property	Lattice (Wilson action)	Momentum cutoff
Gauge invariance	Kept exactly	Broken. Counterterms are needed to restore it (MRS 1993 § I)
Reflection positivity	Kept (Osterwalder–Seiler 1978)	Easily lost. Ways to recover reflection positivity lost through regularisation are generally scarce (JW § 6.5). MRS 1993 § I says that gauge choices lacking positivity (Feynman type) cannot be used directly from the start, and that OS positivity could probably be shown by cutting off first with a lattice and using a momentum cutoff afterwards
Rotational symmetry	Broken	The cutoff itself can be made rotationally symmetric. However, gauge fixing (choosing one representative from fields related by gauge transformations) and a finite box can break rotations. MRS 1993 uses a gauge that treats a particular direction specially (axial gauge) and an infrared cutoff

On the lattice the finite-volume measure has meaning at once. If the gauge group G is compact,

$$\mathrm{d}\mu_{a,L}(U) = \frac{1}{Z_{a,L}}\, e^{-S_W(U)} \prod_{\ell} \mathrm{d}U_\ell \qquad (6)$$

where ℓ are the edges of the lattice, $U_\ell \in G$, $\mathrm{d}U_\ell$ is the Haar measure (the invariant measure of total weight 1), and S_W is the Wilson action

$$S_W(U) = \beta \sum_P \left(1 - \tfrac{1}{N} \operatorname{Re}\operatorname{tr} U_P\right), \qquad \beta = \frac{2N}{g^2} \quad (G = SU(N))$$

The sum is over the smallest squares of the lattice (plaquettes) P, and U_P is the product of the U_ℓ on the four edges of P taken in order around it (5-01). The integrand is bounded and continuous and the domain compact, so $Z_{a,L}$ is finite and positive (JW § 6.5). Known

Example: a four-dimensional periodic lattice with 16 points on a side has $4 \times 16^4 = 262144$ edges, and `SU(3)` is 8-dimensional over the reals, so formula (6) is an integral in 2,097,152 dimensions. Computed

Making it directly in the continuum is hard, and even three-dimensional Yang–Mills has not yet been constructed on the torus T^3 (Chevyrev 2022). On the lattice, the finite-

volume step is closed by formula (6) ("Closed (lattice)" in Table 2), and instead the recovery of rotational symmetry is passed on to the axiom step.

05

The ultraviolet-limit step

As $a \to 0$, unless the coefficients of the action (the bare couplings) are moved according to a, the limit diverges or collapses to something trivial. This moving is renormalisation (4-04). According to the problem statement, for ϕ^4, if the cutoff-dependent constants are chosen in the form indicated by perturbation theory, expectation values converge even though the constants diverge. In two dimensions a few constants suffice; in three dimensions one needs all the coefficients plus wave-function renormalisation, that is, renormalising the state vectors (vectors of the Fock space of 2-06) by a constant that diverges with the cutoff (JW § 6.2). Known

What is needed can be guessed by counting dimensions. Known Since the action has no units, in units of mass

$$[\phi] = \frac{d-2}{2}, \qquad [\lambda] = 4 - d, \qquad [g^2] = 4 - d \tag{7}$$

(g is the Yang–Mills coupling). For $d < 4$ the coupling has positive mass dimension (**super-renormalisable**), and there are finitely many types of superficially divergent diagrams (pictures of the terms of perturbation theory; 4-03). "Superficially" means that divergence is judged only by counting the powers of momentum in the integral attached to the diagram (4-04; as a textbook, Rivasseau 1991). $d = 4$ is the boundary where the coupling has no units. Table 1 can be read by the same counting. Since $[\phi^2] = d - 2$, $G_a(0)$ grows as a^{2-d}, and for $d = 2$ logarithmically (1-12).

Example: in two dimensions it suffices to redefine the product as $:\phi^4:_a = \phi^4 - 6G_a(0)\phi^2 + 3G_a(0)^2$ (the Wick-ordered product), together with a constant for the vacuum energy (the α, γ of JW § 6.2).

What one wants to show is that expectation values are bounded by constants independent of a,

$$\sup_{a>0} \left|\langle \Phi \rangle_{a,L}\right| \le C_\Phi \tag{8}$$

With a uniform estimate (8), one can take a subsequence of the measures that converges in a weak sense (the measure version of taking a convergent subsequence from a bounded sequence of numbers). The limit, however, need not be unique, and the axioms and the gap are separate questions. The problem statement does not accept

weak existence by compactness alone as a solution unless the properties of the limit are shown separately (JW footnote 2). Four dimensions is treated in 7-07.

06

The infinite-volume-limit step

$L \to \infty$ needs estimates independent of L, and the key is in particular that quantities far apart become nearly independent (**clustering**). The problem statement lists as ways to set up estimates (a) correlation inequalities, (b) symmetries, and (c) convergent expansions (cluster expansions), and says that (a) tends to be limited to scalar fields and abelian gauge theories, and (c) became the central tool (JW § 6.1). The form one wants to show is

$$\left|\langle A\, B_x\rangle_L - \langle A\rangle_L \langle B_x\rangle_L\right| \le C\, e^{-m|x|} \quad (C,\ m \text{ independent of } L\ ) \tag{9}$$

(B_x is B shifted by x). Given a decay rate m, the influence of the edge of the box reaches only about e^{-mL}.

Example: for the one-dimensional chain, the difference between the variance at a point of a ring of length L and the value for the infinite chain is, for large L, almost e^{-mL}/m. For $m = 1$ it is 1.1×10^{-7} at $L = 16$; for $m = 0.1$ it is still 2.5. The rate of approach is set by m. Computed

In four-dimensional Yang–Mills the coupling is thought to become strong at long distances. Physics This view comes from the renormalisation group of perturbation theory (4-04). MRS 1993 § I writes, as the reason for not removing the infrared cutoff, 'this would lead to large values of the coupling constant', and further 'there is no easy solvable model of these phenomenons around which to expand'.

As a metaphor, the scaffolding for the expansion is gone. Precisely, there is neither a solvable model to serve as the starting point of an expansion nor a small parameter to expand in. The problem statement too says that present methods do not suggest how to obtain $T^4 \to \mathbb{R}^4$ (JW § 6).

07

The axiom step

Axioms that hold with the regularisation tend to pass to the limit, and what was broken has to be recovered. The key is **reflection positivity**: for time reversal $\Theta : t \mapsto -t$, for F depending only on the field at $t > 0$,

$$\left\langle\, \Theta F \; F \,\right\rangle \;\ge\; 0 \tag{10}$$

holds. This is the condition for extracting a Hilbert space and a Hamiltonian (5-03).

Formula (10) is an inequality, so it passes to the limit. Known There are three conditions. The F for which the limit is taken can be approximated by functions depending only on the lattice field on the positive-time side. Their expectation values converge. The reflection plane is a lattice plane (a plane through lattice points, or a plane between neighbouring lattice points). Then it is a limit of non-negative values in each $\mu_{a,L}$, so it is non-negative in the limit too. For lattice gauge theory, reflection positivity with respect to lattice planes has been shown (Osterwalder–Seiler 1978; MRS 1993 § I also cites it).

Conversely, ways to recover lost reflection positivity are scarce, and the problem statement writes that setting up the Hilbert space is itself part of the problem (JW § 6.5). Rotational symmetry broken by the lattice has to be shown to recover separately. Known How the published construction satisfies these is read in Part 12 (12-01 Lattice reflection positivity, 12-06 Euclidean invariance, 12-08 Reconstruction and the Wightman axioms).

Example: for a Gaussian measure, reflection positivity is the same as positivity of ΘC restricted to the positive-time side (JW § 6.2). In one dimension $C(t) = e^{-m|t|}/(2m)$, so for $t_i, t_j > 0$

$$\sum_{i,j} f_i \, f_j \, C(t_i + t_j) = \frac{1}{2m} \Big(\sum_i f_i \, e^{-m t_i} \Big)^2 \;\geq\; 0 \tag{11}$$

becomes a square. On a lattice chain with 40 points, trying 20000 random f gives a minimum of 1.4×10^{-8}, never negative. Being a square, this is to be expected; it is a check of the formula. As a control, with the sign-reversed covariance $-C(t_i + t_j)$ the smallest eigenvalue of the matrix is -2.49, and reflection positivity fails. Computed

Stochastic quantisation (7-09) makes the measure as the invariant measure of an equation, so reflection positivity is not part of the output, and has to be shown as a separate task.

08

The uniform-gap step

Next one asks about the gap in the spectrum of the Hamiltonian (4-11). On the Euclidean side it is the rate of decay of correlations. On the lattice the decay rate in the time direction is the gap in lattice units m_{lat}, and the gap in physical units is

$$\Delta = \lim_{a \to 0} \frac{m_{\mathrm{lat}}(a)}{a} \tag{12}$$

One wants to show that this is positive and finite, and moreover uniform in the volume.

On a lattice in a finite box the gap is there automatically. Known The **transfer matrix** T is the integral operator that carries weight from the slice at time t (the set u of values of the lattice field at that time) to the slice v at time $t + a$

$$(Tf)(u) = \int K(u, v)\, f(v)\, \mathrm{d}v$$

The kernel $K(u, v)$ is made as $e^{-S_{t,t+a}(u,v)}$ from the part $S_{t,t+a}(u, v)$ of the action connecting the two times. Reflection positivity makes T a positive operator, and the Hamiltonian is fixed by $T = e^{-aH}$. If the range of field values is bounded and closed (lattice gauge theory with a compact gauge group is such a case) and the kernel is continuous and positive, the largest eigenvalue λ_0 is not degenerate. This is the infinite-dimensional version (a generalisation of Jentzsch's theorem) of the Perron–Frobenius theorem for matrices with all entries positive Known (the generalisation is Schaefer 1974; sources in 5-03). So the ratio to the next eigenvalue λ_1 is $\lambda_1/\lambda_0 < 1$, and there is a gap $-\log(\lambda_1/\lambda_0)/a > 0$ (5-03; the positive self-adjoint transfer matrix of lattice gauge theory is Osterwalder–Seiler 1978). What is asked is uniformity under $a \to 0$ and $L \to \infty$, and the problem statement writes that a gap uniform in the volume needs new ideas (JW § 6.5).

Example: for the chain with $m = 1$, m_{lat}/a approaches 1 as 0.9624 ($a = 1$), 0.9974 ($a = 0.25$), 0.99990 ($a = 0.05$), while m_{lat} in lattice units itself goes towards 0, about 0.05 at $a = 0.05$. In the continuum limit **the gap in lattice units closing is the correct picture**, and what should be looked at is the ratio (12) (5-09). The ratio of eigenvalues of the transfer matrix, $-\log(\lambda_1/\lambda_0)/a$, agrees with the decay rate of correlations, the difference being at most 5×10^{-14}. Computed

09

Detour: taking a one-dimensional chain through all six stages

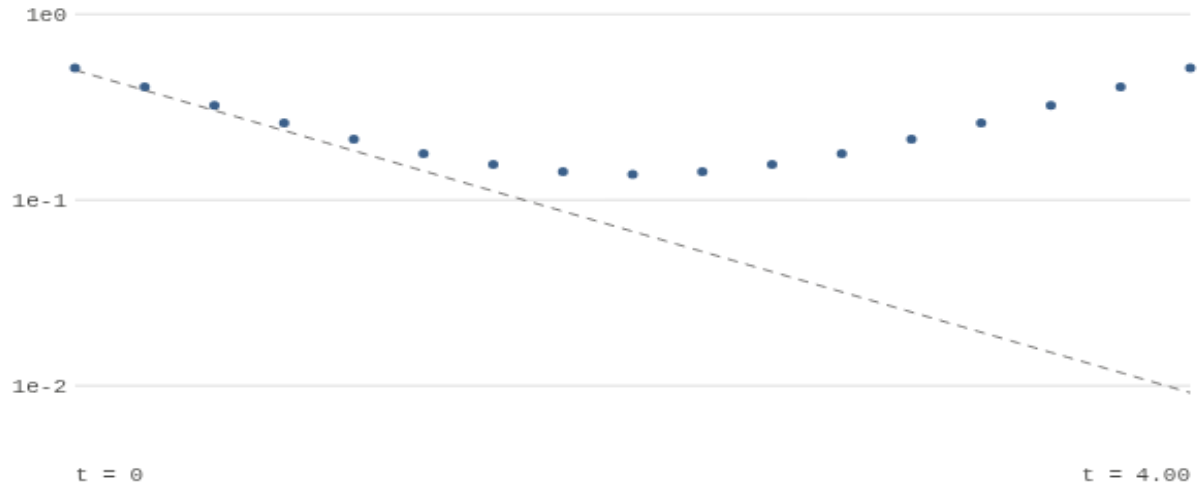
A section you may skip.

Euclidean field theory with $d = 1$ is quantum mechanics: the field $\phi(t)$ is the position of a particle, and the action $\int (\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2)\,\mathrm{d}t$ is the imaginary-time action of the harmonic oscillator (2-06, 2-07). In the regularisation step time is cut into steps of a, in the finite-volume step it is made a ring of length L, and in the ultraviolet-limit and infinite-volume steps $a \to 0$ and $L \to \infty$. The axiom step is formula (11), and the gap of the uniform-gap step is the level spacing m. The correlation on a ring of $N = L/a$ points can be written in closed form.

$$\cosh m_{\mathrm{lat}} = 1 + \tfrac{1}{2}m^2a^2, \qquad \langle \phi_0\, \phi_n \rangle_L = \frac{a\, \cosh\bigl(m_{\mathrm{lat}}(N/2 - n)\bigr)}{2 \sinh m_{\mathrm{lat}}\, \sinh(m_{\mathrm{lat}} N/2)} \quad (13)$$

It agrees, at the level of 10^{-14}, with the values obtained by inverting the matrix of the action directly. Computed

Figure 2 Moving a, L and m changes the correlation on the ring (vertical axis logarithmic) and the lattice gap m_{lat}/a



a = 0.25, L = 4.00 (16 points), m = 1.00, m_lat/a = 0.9974, <φ²>_L = 0.5148 (continuum, infinite 0.5000)

Computed The numbers in this figure were computed on this machine (formula (13)). The points are lattice values, the dashed line the infinite, continuum $e^{-mt}/(2m)$.

Stretching L makes the contribution that goes around the ring (the rise on the right) disappear, and the points fall onto the dashed line. Making m smaller makes the needed L longer in proportion to $1/m$. This is the link between the infinite-volume step and the uniform-gap step, that the gap sets the rate of the infinite-volume limit, and the problem statement too writes that a uniform gap may play a basic role in proving the infinite-volume limit (JW § 5). We look at the two limits in separate tables (both with $m = 1$).

Computed Table 3 Making the lattice spacing a finer (the ultraviolet-limit and uniform-gap steps; infinite chain).

a	m_{lat}/a	$\langle\phi^2\rangle$ **(infinite chain)**
1	0.962424	0.447214
0.25	0.997414	0.496139
0.05	0.999896	0.499844
0.01	0.999996	0.499994
$a \to 0$	$\to 1$	$\to 1/(2m) = 0.5$

Computed Table 4 Stretching the length L of the ring (the infinite-volume step; $a = 0.05$). The value for $L = \infty$ is 0.499844 of the infinite chain (the $a = 0.05$ row of Table 3).

L	$\langle\phi^2\rangle_L$	**Difference from the infinite chain**
1	1.081735	5.8×10^{-1}
2	0.656350	1.6×10^{-1}
4	0.518503	1.9×10^{-2}

L	$\langle\phi^2\rangle_L$	Difference from the infinite chain
8	0.500180	3.4×10^{-4}
16	0.499844	1.1×10^{-7}

Everything goes through in one dimension because values at points are random variables and the ultraviolet-limit step has no divergence (Table 1). Adding an anharmonic term $\lambda\phi^4$ still makes sense as quantum mechanics. Known

10

How to read Part 7

Each row of Table 2 shows the steps being closed in order from the regularisation step. The chapters of Part 7 follow, step by step, which tools gave each "Closed", and where the "Partial" and "Not yet" stop. Reading two-dimensional Yang–Mills and scalar fields as examples where the steps went through, the cluster expansion and the renormalisation group as tools for the steps where they work, and the chapters on three and four dimensions as records of the steps that are stuck, shows where each chapter stands. One chapter may touch several steps, so the correspondence below gives the main steps treated.

- From regularisation to infinite volume: 7-02 two-dimensional Yang–Mills
- All six steps: 7-03 ϕ^4_2, ϕ^4_3 (examples where all six steps went through)
- The infinite-volume step and the uniform-gap step: 7-04 the cluster expansion
- The ultraviolet-limit step: 7-05 the renormalisation group, 7-06 three dimensions, 7-07 four dimensions, 7-08 triviality, 7-12 Detour: 1/0
- The finite-volume and ultraviolet-limit steps by another road: 7-09 stochastic quantisation
- All: 7-10 where four dimensions stops, 7-11 obstacles other than divergence, 16-01 the timeline

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What this chapter can say

	Content
Said	Constructive field theory is a method of exhibiting non-Gaussian measures satisfying the axioms as limits of sequences of approximations, with uniform estimates at its centre (JW § 6.1) Known
Said	The variance at a point of the lattice free field is finite for $d = 1$, and grows as $\log(1/a)$, $1/a$, $1/a^2$ for $d = 2, 3, 4$ ($a \geq 1/64$) Computed

	Content
Said	For the one-dimensional chain the six stages go through in closed form, and the decay rate of correlations agrees with the eigenvalue ratio of the transfer matrix Computed
Not said	The division into six stages is this book's arrangement, not a standard division in the field
Not said	The content of the "Closed" entries in Table 2 (the proofs of the original papers were not followed. ϕ^4_2 and ϕ^4_3 are from the problem statement. The gap for ϕ^4_3 is from the title of the paper)
Not said	A way to advance uniqueness of the infinite-volume limit and the uniform-gap step for four-dimensional Yang–Mills (the problem statement [2000] says there are no ideas pointing the direction; the construction published on 2026-11-10 does not show these two either; 6-04)

Sources and reproduction

Item	Kind	Source / tools
Accounts and quotations of JW (§5, §6, §6.1, §6.2, §6.5, footnote 2)	Primary source (checked in the text)	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000)
The OS axioms, textbook	Bibliographic data only	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83–112; 42 (1975) 281–305. J. Glimm, A. Jaffe, *Quantum Physics*, 2nd ed., Springer (1987)
ϕ^4_2, ϕ^4_3	Bibliographic data only (the reference list of the problem statement)	Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585–632; Glimm–Jaffe, Fortschr. Phys. 21 (1973) 327–376; J. Feldman, K. Osterwalder, "The Wightman axioms and the mass gap for weakly coupled ϕ^4_3 quantum field theories", Ann. Phys. 97 (1976) 80–135
Textbook on perturbation theory and renormalisation	Bibliographic data only (the reference list of the problem statement)	V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton Univ. Press (1991)
Reflection positivity on the lattice, the positive self-adjoint transfer matrix, infinite volume at strong coupling	Bibliographic data and abstract only	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471

Item	Kind	Source / tools
The Perron–Frobenius and Jentzsch theorems	Known theorems (Perron 1907 and Jentzsch 1912: bibliographic data only. The generalisation to kernels on compact spaces is Schaefer 1974 [bibliographic data only, claim checked in secondary sources] and Krein–Rutman 1948 [bibliographic data only])	See 5-03
Exponential decay at strong coupling	Checked in the abstract	H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737
Continuum measure in two dimensions (the plane)	Checked in the abstract and introduction	B. K. Driver, Comm. Math. Phys. 123 (1989) 575–616
Continuum measure in two dimensions (compact surfaces)	Bibliographic data only (Lévy 2003 checked as far as the abstract)	A. Sengupta, Mem. Amer. Math. Soc. 126 no. 600 (1997); T. Lévy, Mem. Amer. Math. Soc. 166 no. 790 (2003), arXiv:math/0101239
Ultraviolet stability in three and four dimensions	1985 and 1989 (CMP 122, 355–392): the opening and the main theorem (the Conditional Ultraviolet Stability Theorem; Theorem 1) checked. 1987 and the rest of the series: bibliographic data only	T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275; 109 (1987) 249–301; 122 (1989) 355–392
Construction with an infrared cutoff	Abstract and § I checked	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383
The three-dimensional measure has not been constructed	Text § 1.1 checked	I. Chevyrev, arXiv:2202.13359 (2022)
Table 1, dimension counts, the one-dimensional chain	Computed	Python on this machine (numpy, scipy): `point_variance.py`, `counts.py`, `chain.py` (the transfer matrix diagonalised with positions on 1601 points)
Figures 1 and 2	Figures	SVG, canvas and plain JavaScript. Figure 1 reads Table 2 (if JavaScript is disabled, a static figure for YM in four dimensions). Figure 2 computes formula (13)

Next chapter to read: 7-02 Solvability in two dimensions and the Yang–Mills measure (an example in which the four steps from regularisation to infinite volume close).

Solvability in two dimensions and the Yang–Mills measure — on the plane the plaquettes become independent, and the measure can be made

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — On a two-dimensional lattice, fixing the gauge on a tree makes the plaquettes independent, and the area law holds exactly at every coupling. In the continuum limit the heat kernel of the group appears, and continuum measures have been constructed on the plane and on closed surfaces. On the torus, stochastic quantisation also arrived at the same measure.

Prerequisite chapters — 5-01 Wilson's lattice. Characters are supplied on the spot, within the range of 1-04. Expectation, independence and densities are in 1-08, white noise and Brownian motion in 1-09, the six stages in 7-01.

Said first — Pure Yang–Mills in two dimensions is not a scaled-down version of four dimensions. It has no propagating degrees of freedom, and the measure sees only area. The core of the solvability cannot be carried over to four dimensions (§ 09). This chapter presents known results.

Order of this chapter

1. Why start with two dimensions — two dimensions is the only case where a continuum measure has been constructed for a non-abelian group
2. Fixing a tree makes the plaquettes independent — the number of edges matches the number of faces
3. The area law is exact at every β — peeling off faces one at a time with a lemma on characters
4. The continuum limit: the heat kernel of the group — area becomes the time of the heat equation
    - Detour: a theory that sees only area — area-preserving deformations leave expectation values unchanged

5. Driver and Gross–King–Sengupta — the field strength becomes white noise, and parallel transport becomes a stochastic differential equation
6. On closed surfaces — the faces are no longer independent, but the measure has been constructed
   - Detour: letting the area go to 0 — the volume of the space of flat connections comes out
7. Seen with the transfer matrix — the lowest excitation is proportional to the length of the circle, and no particles propagate
8. Reconstruction by stochastic quantisation — on the torus, the same measure obtained as an invariant measure
9. What can be carried out, and what cannot — the methods extend, but the independence of faces does not

01

Why start with two dimensions

Two dimensions is, so far, the only case in which a continuum Yang–Mills measure has been constructed for a non-abelian gauge group. Chatterjee (2018) writes that no continuum limit in which Wilson loops behave non-trivially has been constructed in any dimension higher than two. Known The four-dimensional construction published on 2026-11-10 also builds the continuum theory in the form of correlation functions of local observables, and does not build a probability measure or Wilson-loop observables (6-04). In two dimensions the field strength has one component.

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2], \qquad S(A) = \frac{1}{2g^2} \int_{\mathbb{R}^2} |F_{12}|^2 \, d^2x \quad (1.1)$$

A is a Lie-algebra-valued connection (3-02), and $|\cdot|$ a conjugation-invariant length. F has the dimension of an inverse length squared, so for the action to be dimensionless g^2 must have the dimension of an **inverse area**, and g itself provides the scale. In four dimensions g is dimensionless, and the scale is thought to be generated by quantum effects Physics (4-07). The protagonist of this chapter is a property found only in two dimensions: **fixing the gauge suitably makes the degrees of freedom of the faces independent**. In terms of the six steps (7-01), the steps from the regularisation step to the infinite-volume step close, the axiom step is partial (Euclidean invariance; § 05), and the gap of the uniform-gap step is not a question with the same meaning as in four dimensions (§ 07).

02

Fixing a tree makes the plaquettes independent

The set-up is as in 5-01: $U_\ell \in G$ on an edge ℓ, U_p for the product around a face p, and the weight $\prod_p e^{\beta \operatorname{Re}\operatorname{tr} U_p}$ (dU is the Haar measure). The β of this chapter is half of the β_W customary for $SU(2)$ (Wilson action $\beta_W \sum_p \left(1 - \tfrac{1}{2}\operatorname{Re}\operatorname{tr} U_p\right)$; in four dimensions $\beta_W = 4/g^2$): $\beta = \beta_W/2$. β = 1 in § 03 corresponds to $\beta_W = 2$. A gauge transformation $U_{xy} \mapsto g_x U_{xy} g_y^{-1}$ does not change $\operatorname{tr} U_p$. The variables on a tree (a set of edges with no cycles) can be set to 1 by choosing g_x in order from the root, without changing expectation values of gauge-invariant quantities. Known Counting on an $L \times L$ square lattice (free boundary),

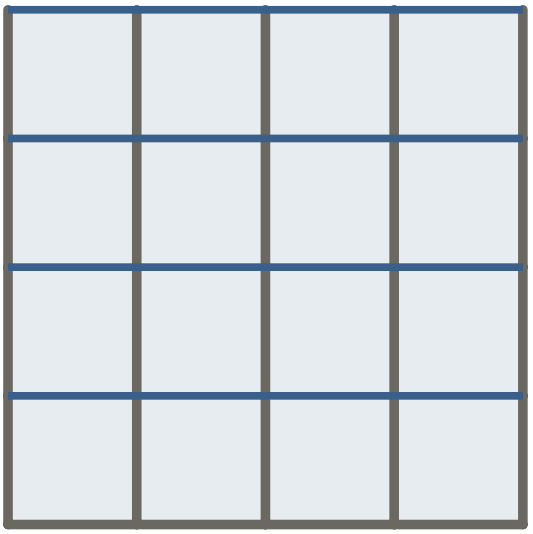
$$V = (L+1)^2, \quad E = 2L(L+1), \quad P = L^2, \qquad E - (V-1) = P. \tag{2.1}$$

the tree has $V - 1$ edges, so the remaining edges are as many as the faces (Euler's formula for the disc). Taking the tree to be the vertical edges and the bottom edges (as in Driver 1989), what remains are the horizontal edges $h_{i,j}$ ($h_{i,0} = 1$), and

$$U_{p(i,j)} = h_{i,j}\, h_{i,j+1}^{-1} \iff h_{i,j+1} = U_{p(i,j)}^{-1} h_{i,j}. \tag{2.2}$$

can be solved from the bottom up in one-to-one correspondence, and since the Haar measure is invariant under multiplication the measure carries over. **The face variables are mutually independent, each with density $\rho_\beta(U) = e^{\beta \operatorname{Re}\operatorname{tr} U}/z(\beta)$.** Known Extending to the whole plane, the infinite-volume limit exists independently of the boundary condition and, on a finite region, agrees with the measure with free boundary (Theorem 7.2 of Driver, shown by integrating out the outer edges using the invariance of the Haar measure; Driver writes that this theorem is essentially contained in Dosch–Müller 1979). Known

Figure 1 The tree gauge — The knob L changes the size of the lattice. "Play" fixes the tree (grey) to 1 and fills the faces from the bottom. Each face filled fixes one edge above it.



L = 4: 25 vertices, 40 edges, 16 faces. Fixing the 24 edges of the tree to 1 leaves 16 edges, matching the number of faces.

Computed The numbers in this figure (numbers of vertices, edges and faces) were computed on this machine. The numbers are formula (2.1).

03

The area law holds exactly at every β

The tool is the **character** $\chi_\lambda(U) = \operatorname{tr} \lambda(U)$ (λ an irreducible unitary representation, of dimension d_λ). It is conjugation-invariant, and $\int_G \chi_\lambda \chi_\mu \, dU = \delta_{\lambda\mu}$. For $SU(2)$ the representation of spin j has dimension $2j+1$, and for an element with eigenvalues $e^{\pm i\theta}$, $\chi_j = \sin((2j+1)\theta)/\sin\theta$ (1-04). The key is a lemma: if the density of X is conjugation-invariant and X is independent of A, then

$$\mathbb{E}[\chi_\lambda(XA)] = w_\lambda \, \chi_\lambda(A), \qquad w_\lambda = \mathbb{E}[\chi_\lambda(X)]/d_\lambda . \tag{3.1}$$

This is because $\mathbb{E}[\lambda(X)]$ commutes with every $\lambda(V)$, so by Schur's lemma it is $w_\lambda I$. When a simple closed curve C encloses n faces, the variables of the enclosed faces appear once each in the tree-gauge U_C, accompanied by conjugations by other face variables. The densities are conjugation-invariant, so (3.1) applies as it is, and moving faces one at a time to the front by cyclicity of the trace and peeling them off gives

$$\frac{\mathbb{E}[\chi_\lambda(U_C)]}{d_\lambda} = w_\lambda(\beta)^n, \qquad \langle W_C \rangle = e^{-\sigma_{\text{lat}} n}, \quad \sigma_{\text{lat}} = -\ln w_{\text{fund}} . \tag{3.2}$$

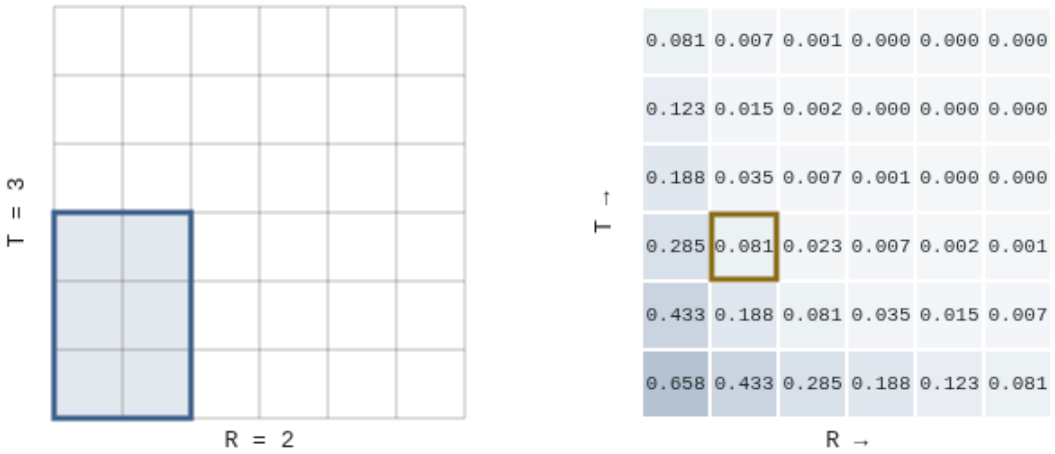
Here $W_C = \chi_{\text{fund}}(U_C)/d_{\text{fund}}$ (the value in the fundamental representation; $\frac{1}{2}\operatorname{tr} U_C$ for $SU(2)$). Migdal (1975) showed that in two dimensions the transformation rule (recursion) for the weights under coarsening the lattice becomes exact, wrote the formula in which the expectation of a character is exponential in the enclosed area A,

$\langle \chi_q \rangle = d_q e^{-g_q^2 A}$ (q a representation, g_q^2 a constant for each representation), and stated that it is "nothing but Wilson's area law". Known The value is fixed by the area alone, and holds **at every β**. This contrasts with four dimensions, where the area law has been shown only on the strong-coupling side (5-02). The coefficient can be written by the following formula in modified Bessel functions, and the difference from numerical integration was at most 2×10^{-16}. Computed

$$SU(2):\ w_j = \frac{I_{2j+1}(2\beta)}{I_1(2\beta)}, \qquad U(1):\ w_n = \frac{I_n(\beta)}{I_0(\beta)} \quad (3.3)$$

I_n is the modified Bessel function, and for integer n, $I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta\, d\theta$. When x is large the ratio $I_n(x)/I_1(x)$ approaches 1, and § 04 uses how it approaches.

Figure 2 The area law (SU(2)) — The knob β changes the value w of a single face, and R and T the shape of the loop. On the right is w^{RT} for the 6×6 rectangles; the same area gives the same value.



β = 2.0 : w = 0.6580、σ = 0.4185。2×3 (area 6, perimeter 10) : ⟨W⟩ = 0.0812。

Computed The numbers in this figure were computed on this machine. The w in the figure is the Bessel series (3.3) summed on the spot, agreeing with values computed with scipy to the fifth decimal place. If JavaScript is disabled, see Table 1.

Table 1 $SU(2)$. "Random" is heat-bath Monte Carlo on a 6×6 lattice (free boundary) **without gauge fixing**. Brackets give the error in the last digit. Computed

β	w	σ_{lat}	2×2 formula	2×2 random	3×3 formula	3×3 random
1	0.4331	0.8367	0.0352	0.0366(11)	0.0005	−0.0002(15)
2	0.6580	0.4185	0.1875	0.1893(12)	0.0231	0.0223(16)
4	0.8192	0.1994	0.4505	0.4514(12)	0.1662	0.1649(29)

The calculation without gauge fixing agrees with the tree-gauge formula within errors.

04

The continuum limit: the heat kernel of the group appears

At lattice spacing a, a loop of area A encloses $n = A/a^2$ faces. Taking the length in (1.1) as $|X|^2 = -2\,\mathrm{tr}\,X^2$, the expansion $U_p \approx e^{a^2 F_{12}}$ gives, for $SU(2)$, $\beta = 2/(g^2 a^2)$. From $I_n(x)/I_1(x) = 1 - \frac{n^2-1}{2x} + O(x^{-2})$, $w_j = 1 - j(j+1)/\beta + O(\beta^{-2})$, so

$$w_j(\beta)^{A/a^2} \longrightarrow e^{-g^2 A\, j(j+1)/2} \qquad (a \to 0). \tag{4.1}$$

The right-hand side is a coefficient of the **heat kernel** Q_t of the group. Δ is the Laplacian on the group (different from the Δ of the mass gap), the operator that has the characters as eigenfunctions, $\Delta\chi_\lambda = -c_\lambda \chi_\lambda$. $c_\lambda \geq 0$ is the Casimir value determined by each representation, $c_j = j(j+1)$ for $SU(2)$. The heat kernel is the solution of the heat equation $\partial_t Q_t = \frac{1}{2}\Delta Q_t$ concentrated at the identity at time 0, and expanded in characters

$$Q_t(U) = \sum_\lambda d_\lambda\, e^{-c_\lambda t/2} \chi_\lambda(U), \qquad Q_s * Q_t = Q_{s+t} \tag{4.2}$$

The right-hand side of (4.1) is the coefficient $e^{-c_j t/2}$ at $t = g^2 A$. That is, **the holonomy of a simple closed curve of area A follows the heat kernel $Q_{g^2 A}$.** Known The semigroup property is "adding areas adds times". In Table 2 each of $j = \frac{1}{2}, 1, \frac{3}{2}$ approaches the heat kernel, and the error becomes 1/4 each time a is halved. Computed

Table 2 $SU(2)$, $g = 1$, area 1, $\beta = 2/a^2$. Values are w_j^n. Computed

a	n	$j = \frac{1}{2}$	$j = 1$	$j = \frac{3}{2}$
1/2	4	0.67919	0.35713	0.14579
1/4	16	0.68527	0.36504	0.15119
1/32	1024	0.68726	0.36783	0.15332
Heat kernel	∞	0.68729	0.36788	0.15335

Taking the face weights to be heat kernels from the start gives, by the semigroup property, a measure independent of how the faces are divided (Migdal's observation). Driver showed that lattices with heat-kernel weights (the Villain action) and with the Wilson action converge to the same continuum measure, using for the latter a group-valued central limit theorem (Borgs–Seiler). Known

Detour: a theory that sees only area

A subsection you may skip.

According to Corollary 6.7 of Driver, expectation values of gauge-invariant quantities do not change under **area-preserving diffeomorphisms** (as long as the arrangement of curves is carried to one of admissible form). As a metaphor, it is a theory on a rubber sheet: stretching the sheet while preserving area changes nothing. Precisely, expectation values can be written as finite-dimensional integrals of products of heat kernels, into which only the areas of the faces and the way the curves cross enter (Theorem 6.4).

05

The continuum measure: Driver and Gross–King–Sengupta

The continuum version of the tree gauge is the **complete axial gauge** ($A_2 = 0$, and $A_1 = 0$ on the x axis). Then $[A_1, A_2] = 0$, so the field strength becomes linear, and the weight e^{-S} becomes Gaussian in F.

$$F_{12} = -\partial_2 A_1, \qquad \mathbb{E}\big[F^a(R)F^b(R')\big] = g^2\delta^{ab}|R \cap R'| \quad (5.1)$$

F is **Lie-algebra-valued white noise** ($F^a(R)$ is the integral over a region R, $|R|$ its area; 1-09), and its values on disjoint regions are independent. It is the continuum version of "independent for each face" on the lattice. Known F is not defined pointwise, so parallel transport becomes a **stochastic differential equation**: an ordinary differential equation whose driving term contains white noise (increments of Brownian motion), solved after fixing one way of taking stochastic integrals. The driving term of parallel transport along a horizontal curve is the integral of F between the curve and the x axis, and each component is a Brownian motion whose time is that area (1-09) (Proposition 3.2 of Driver). This method goes back to Gross–King–Sengupta (1989).

Driver's results are (i) that expectation values of gauge-invariant functions of holonomies can be written as finite-dimensional integrals with products of heat kernels as density (times are the face areas, arguments the way the curves cross; Theorem 6.4), (ii) invariance under area-preserving maps (Corollary 6.7), and (iii) convergence of lattices. Known From Corollary 6.7 follows Euclidean invariance of gauge-invariant expectation values (part of the axioms). Driver does not claim verification of the whole of the Osterwalder–Schrader axioms (4-10), including reflection positivity. The observables are quantities along curves, and Driver writes that Wilson-loop variables are "quite singular for $d = 3$, and especially for $d = 4$".

06

On closed surfaces: Sengupta and Lévy

On a closed surface of genus γ Euler's formula becomes $V - E + P = 2 - 2\gamma$, and after fixing a tree $P - 1 + 2\gamma$ edges remain. The face variables are subject to one overall

constraint, and 2γ variables going around the holes are added, so the faces are not independent. Even so, the measure has been constructed. Sengupta (1992–97) constructed it on compact surfaces (original papers: bibliographic data only), and Lévy (2003) treated the discrete case and the continuum limit in a unified way and showed a **Markov property** with respect to cutting and gluing surfaces (when cut along a curve, fixing the values on the cut makes the two sides not influence each other). Ways of gluing gauge fields on a surface come in topological kinds (1-07); the non-trivial kinds (non-trivial principal bundles) are Lévy (2006), and the general theory of holonomy fields is Lévy (2010). Known The discrete version puts a heat kernel on each face F.

$$\mathbb{E}[f] = \frac{1}{Z}\int f(h)\prod_F Q_{g^2|F|}(h(\partial F))\prod_e dh_e, \qquad Z_\gamma(T) = \sum_\lambda d_\lambda^{2-2\gamma} e^{-c\lambda g^2 T/2} \tag{6.1}$$

$h(\partial F)$ is the product along the boundary of the face, T the total area, and Z is up to a constant factor. Known The Z_1 of the torus has the same form as the $\operatorname{Tr} e^{-\tau H}$ of the Hamiltonian of § 07.

Detour: letting the area go to 0

A subsection you may skip.

If $\gamma \geq 2$, then $Z_\gamma \to \sum_\lambda d_\lambda^{2-2\gamma}$ as $T \to 0$. The measure concentrates on connections of zero curvature (flat connections), and from this limit comes the volume of the **moduli space of flat connections** (the space of flat connections identified under gauge transformations). This is Witten's volume formula, and the exact form of the limit was given by Sengupta (2003) (Theorem 4.1 of the survey). Theorem 4.1 assumes that G is simply connected and $\gamma \geq 2$, and $SU(2)$ with $\gamma \geq 2$ satisfies this. Known For $SU(2)$ the dimensions are $1, 2, 3, \ldots$, so the limit is $\zeta(2\gamma - 2)$. Computed

γ	$T = 1$	$T = 0.1$	$T = 0.01$	$T \to 0$
1	2.2738	7.5201	24.597	Divergent
2	1.2247	1.4713	1.5849	ζ(2) = 1.6449
3	1.0482	1.0767	1.0817	ζ(4) = 1.0823

07

Seen with the transfer matrix: the lowest excitation is proportional to the length of the circle

The mass gap is the gap in the spectrum of the Hamiltonian (4-11). On the lattice we use the **transfer matrix** T: the operator whose kernel is the weight for advancing time by one step, defining the Hamiltonian H by $\mathsf{T} = e^{-aH}$ (5-03). Take space to be a circle

of L edges, and do not make time periodic (an open cylinder). Then edges in the time direction form no cycles and can be fixed to 1, and gauge-invariant states are functions of the conjugacy class of the holonomy once around, $V_1 \cdots V_L$, that is, superpositions of characters. One step of time evolution multiplies each edge by one independent face variable, so using (3.1) L times gives

$$\mathsf{T}\chi\lambda = w\lambda(\beta)^{L}\chi\lambda, \qquad E\lambda = -L\ln w\lambda(\beta) \quad (\text{lattice units},\ E_0 = 0). \tag{7.1}$$

The lowest excitation is $E = \sigma_{\text{lat}} L$ of the fundamental representation, the energy of a "ring of lines of force" going once around the circle. The tension is the same number as in the area law of § 03. For $SU(2)$ at β = 1, $E_{1/2}$ is 3.3469 at $L = 4$ and 6.6938 at $L = 8$. The ratio $E_1/E_{1/2}$ at β = 1, 32, 512 is 2.4044, 2.6664, 2.6667, approaching the Casimir ratio $8/3$. Lemma (3.1) was also checked with random numbers (2×10^6 samples). Computed

The continuum limit can be taken in the same way as in § 04. Holding the physical length $L_{\text{phys}} = La$ fixed and putting $\beta = 2/(g^2 a^2)$ and $-\ln w\lambda = c\lambda/\beta + O(\beta^{-2})$ into (7.1) gives

$$\frac{E\lambda}{a} = -\frac{L_{\text{phys}}}{a^2}\ln w\lambda(\beta) = \tfrac{1}{2}g^2 L_{\text{phys}}\, c\lambda + O(a^2). \tag{7.2}$$

Combined with $\Delta\chi\lambda = -c\lambda\chi\lambda$ (§ 04), the continuum Hamiltonian is a constant multiple of the Laplacian on the group, $-\tfrac{1}{2}g^2 L_{\text{phys}}\Delta$ (acting on class functions). Paper The derivation is this chapter's own; for $SU(2)$ with $g = 1$ and $L_{\text{phys}} = 1$, the value for $j = \tfrac{1}{2}$ at $a = 1/64$ was 0.375011 (limit 0.375), and the difference became 1/4 each time a was halved. Computed

Making the circle longer ($L \to \infty$) sends every eigenvalue other than the vacuum off to infinity, and there are no excitations (particles) propagating with finite energy. We also define the correlation length here. The correlation length is the length ξ_{lat} (in lattice units) when the connected correlation $\langle P_x P_y\rangle - \langle P_x\rangle\langle P_y\rangle$ of the quantity $P_x = \operatorname{Re}\operatorname{tr} U_{p_x}$ of a plaquette p_x decreases like $e^{-|x-y|/\xi_{\text{lat}}}$. On the plane different plaquettes are independent, so the connected correlation is exactly 0, and $\xi_{\text{lat}} = 0$ at every β (what to hold fixed when taking the limit is in 5-09). The four-dimensional question (whether the correlation length diverges with a finite mass remaining in physical units) does not arise in two dimensions.

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08
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Reconstruction by stochastic quantisation

Stochastic quantisation is the method of not writing the measure directly, but solving a random time evolution $\partial_t A = -\frac{1}{2}\nabla S(A) + \xi$ (ξ is spacetime white noise) that has it as invariant measure. This equation is called the **Langevin equation**. In finite dimensions the invariant measure of this equation is proportional to e^{-S}. Even in two dimensions A is a distribution, so renormalisation and consistency with gauge are needed. The tools for this (regularity structures, the DeTurck method) are treated in 7-09; here we look only at the results. Known

Chevyrev (2019) realised the measure on T^2 as a distributional 1-form, and Chandra–Chevyrev–Hairer–Shen (2022) built a space of distributional connections and gauge orbits (a Polish space, a good metric space on which probability theory can be used as usual) and defined a **Markov process** (a random process whose future distribution is determined by the present state alone) consistent with gauge in the sense of laws. Chevyrev–Shen (arXiv 2023; Comm. Pure Appl. Math. in 2026) showed that the Yang–Mills measure of the trivial principal bundle on T^2 is the **unique invariant probability measure** of this process (Theorem 2.16). As corollaries come long-time existence of the process, that the limits of the Wilson, Villain and Manton lattices are the same measure (the paper writes that for the Villain action this was already known from Lévy 2006), and a decomposition into a Gaussian free field (1-09) and an almost Lipschitz remainder. Known

This road too is not separate from solvability. Chevyrev–Shen use gauge fixing on the lattice and obtain the moment estimates from two-dimensional solvability. The paper calls extending these estimates to higher dimensions or to the two-dimensional Yang–Mills–Higgs model open, and writes that the construction of non-abelian Higgs models is open even in two dimensions. Known In July 2026 two preprints appeared (not refereed; from the abstracts). Chevyrev–Klose–Mohamed give a gauge-fixed representation of optimal regularity for the existing measure, and Chhaibi–Dang–Guedes Bonthonneau–Rivière–Tô say that they used Morse theory (a method of decomposing a surface by the critical points of a function on it) to make the measure on surfaces without passing through lattices, reproducing the formulas of Migdal, Witten and Lévy. Known

09

What can be carried out of two dimensions, and what cannot

The core of the solvability was the counting of § 02. On a d-dimensional lattice, per lattice point, the edges left after removing gauge number $d-1$ and the faces $\binom{d}{2}$: for $d = 2, 3, 4$, 1 and 1, 2 and 3, 3 and 6. In three or more dimensions the face variables are

tied by constraints on each cube (the lattice Bianchi identity) and cannot be taken independently. Known

Property	Two dimensions	Four dimensions
Area law	Every β	Only on the strong-coupling side (5-02, 5-06)
Coupling constant	Dimension of an inverse area	Dimensionless. How β runs is, with some doubts, mathematically open (Chatterjee 2018). The committee's assessment is that the construction published on 2026-11-10 gives the scaling behaviour of the coupling needed for asymptotic freedom (6-01)
Continuum measure	Constructed on the plane and on closed surfaces	Not constructed (the published construction does not build a measure either; 6-04). Not constructed on T^3 even in three dimensions (7-06, 7-07)
Correlation length	0	The heart of the mass gap

What can be carried out is on the side of methods: taking holonomies along curves as observables, expansion in characters (the tool of the strong-coupling expansion), the pattern of proofs of universality of lattice actions, and the framework of stochastic quantisation. What cannot be carried out are the independence of faces, the dependence on area alone, and the absence of propagating degrees of freedom. What the problem statement of Jaffe–Witten cites as "the only complete example" of an interacting gauge theory satisfying the axioms is likewise the two-dimensional abelian Higgs model (Brydges–Fröhlich–Seiler). Known

What this chapter can say

	Content	Label
Said	On a planar lattice the face variables are independent, and the Wilson loop is $w(\beta)^{\text{area}}$ (at every β). Agrees with random numbers without gauge fixing	Known Computed
Said	In the continuum limit holonomies follow the heat kernel $Q_g{}^2{}_A$, with errors proportional to a^2. The lowest excitation on the circle is proportional to the length of the circle, and the continuum Hamiltonian is $-\frac{1}{2}g^2 L_{\text{phys}}\Delta$ (formula (7.2) is this chapter's derivation)	Known Paper Computed
Said	Construction of the continuum measure on the plane and convergence of lattices (Driver, Gross–King–Sengupta). Construction on closed surfaces (Sengupta, Lévy). On T^2, the unique invariant measure of the process of the Langevin equation (Chevyrev–Shen)	Known
Said	Euclidean invariance of gauge-invariant expectation values (from Corollary 6.7 of Driver; part of the axiom step)	Known

	Content	Label
Not said	That this method extends to three or more dimensions. Verification of the whole of the OS axioms including reflection positivity (not claimed by Driver, Sengupta's survey or Chevyrev–Shen). Non-abelian Higgs models (not constructed even in two dimensions)	
Not said	The content of the 2026 preprints (abstracts only). The texts of works cited from bibliographic data only	

Sources and reproduction

Item	Kind	Source / tools
The recursion is exact in two dimensions, $\langle \chi_q \rangle = d_q e^{-g_q^2 A}$ (area A), "nothing but Wilson's area law"	Text § 2	A. A. Migdal, Zh. Eksp. Teor. Fiz. 69 (1975) 810–822 [English translation Sov. Phys. JETP 42 (1976) 413–418]
White noise, Proposition 3.2, Theorem 6.4, Corollary 6.7 (and Euclidean invariance), Theorem 7.2 and the mention of Dosch–Müller, 8.5, 8.8, "quite singular"	Text	B. K. Driver, Comm. Math. Phys. 123 (1989) 575–616
The law of holonomies, the discrete measure, invariance under subdivision, the partition function, Theorem 4.1 (G simply connected, $\gamma \geq 2$) and the attribution of the volume formula	Text of the survey	A. N. Sengupta, arXiv:0707.4136 (2007)
Unification and the Markov property / non-trivial bundles / holonomy fields	Abstracts	T. Lévy, Mem. AMS 166 no. 790 (2003); Probab. Theory Relat. Fields 136 (2006) 171–202; Astérisque 329 (2010)
The stochastic differential equation for parallel transport	Bibliographic data only	L. Gross, C. King, A. Sengupta, Ann. Phys. 194 (1989) 65–112
The infinite-volume limit (in which Theorem 7.2 is essentially contained)	Bibliographic data only (from reference [23] of Driver)	H. G. Dosch, V. F. Müller, Fortschr. Phys. 27 (1979) 547–559
The group-valued central limit theorem	Bibliographic data only (from Theorem 8.8 of Driver)	C. Borgs, E. Seiler, Comm. Math. Phys. 91 (1983) 329–380

Item	Kind	Source / tools
The measure on compact surfaces	Bibliographic data only, not checked	A. N. Sengupta, J. Funct. Anal. 108 (1992) (title and pages not checked); Quantum gauge theory on compact surfaces, Ann. Phys. 221 (1993) 17–52; Gauge theory on compact surfaces, Mem. AMS 126 no. 600 (1997) (titles and pages of 1993 and 1997 from the reference list of the survey)
The volume formula and its exact limit	Bibliographic data only	E. Witten, J. Geom. Phys. 9 (1992) 303–368 (formula (4.72); attribution from the survey); A. N. Sengupta, J. Geom. Phys. 47 (2003) 398–426 (the source of Theorem 4.1 of the survey). Related: E. Witten, Comm. Math. Phys. 141 (1991) 153–209
Abelian Higgs model	Bibliographic data only	D. Brydges, J. Fröhlich, E. Seiler, Ann. Phys. 121 (1979) 227–284; Comm. Math. Phys. 71 (1980) 159–205; Comm. Math. Phys. 79 (1981) 353–399
Stochastic quantisation	Abstracts	Chevyrev, Comm. Math. Phys. 372 (2019); Chandra–Chevyrev–Hairer–Shen, Publ. Math. IHÉS 136 (2022)
Theorem 2.16, the corollaries, the open problems of § 1.2, the statement that the Villain action was known (Lévy 2006)	Text § 1, § 2	Chevyrev–Shen, arXiv:2302.12160v3 (2023); Comm. Pure Appl. Math. (2026) (volume and pages not checked)
The 2026 preprints	Abstracts (not refereed)	Chevyrev–Klose–Mohamed, arXiv:2607.22236; Chhaibi et al., arXiv:2607.24640
No non-trivial continuum limit in higher dimensions, the running of β	Text of the survey	S. Chatterjee, arXiv:1803.01950 (2018)
"The only complete example"	Problem statement § 6.4	Jaffe–Witten, Quantum Yang–Mills theory (Clay)
Checks of the tables, figures, lemma and formula (7.2)	Computed	python3 (numpy, scipy). The random numbers are a heat-bath method with SU(2) represented by quaternions (6×6, free boundary, no gauge fixing, 7000 sweeps, 35 blocks, fixed seed). The figures are plain JavaScript

Next chapter to read: 7-03 Constructing scalar fields.

Constructing scalar fields — until φ^4 satisfies the axioms in two and three dimensions

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The outline of the construction of ϕ^4_2 and ϕ^4_3, the first interacting examples to pass the six stages. That in two dimensions Wick products are enough for the ultraviolet, and the wall was "not bounded below". The road of the field equations and the road of statistical mechanics. How to count the one further renormalisation needed in three dimensions.

Prerequisite chapters — 7-01 What constructive field theory is. We also refer to 4-10, 1-11 and 4-04.

Said first — This introduces known constructions from 1965–1976 and later alternative proofs, and follows the proofs only in outline. The numbers in the tables are computations on finite lattices and with samples, not proofs that the limits exist. Nothing here shows anything about four-dimensional Yang–Mills.

The order of this chapter

1. Why start from scalar fields — the target statement
2. Wick products: which powers become random variables — a table of computations
3. The wall in two dimensions: not bounded below — a moving figure
4. Two roads: field equations and statistical mechanics
5. Weak coupling: the cluster expansion, the mass gap, scattering
6. Three dimensions: one more renormalisation — a moving figure
7. Once more by other roads
8. Detour: strongly coupled ϕ^4_2 and the phase transition
9. What can be carried to four dimensions, and what cannot

01

Why start from scalar fields

On the map of 7-01 (the six stages), the examples of interacting theories in which all six stages closed were ϕ^4 in two and three dimensions. The field carries only one real number at each point, with no gauge symmetry and no constraints, and the difficulty is narrowed to products at a point (the ultraviolet) and volume (the infrared). The Euclidean action is the action of the free field of mass m_0 plus the polynomial of formula (3) of the problem statement.

$$S(\phi) = \int \left(\tfrac{1}{2} |\nabla \phi|^2 + \tfrac{1}{2} m_0^2 \phi^2 + \lambda \phi^4 + \tfrac{1}{2} \sigma \phi^2 + c \right) \mathrm{d}x \quad (1)$$

$\lambda > 0$ is the coupling constant, and $\sigma = \alpha\lambda + \beta\lambda^2$ and $c = \gamma\lambda + \delta\lambda^2 + \varepsilon\lambda^3$ are counterterms made to depend on the cutoff (JW § 6.2). The d in ϕ^4_d is the dimension of space-time, and in two dimensions $P(\phi)_2$, extended to a polynomial P bounded below, was handled by the same methods. The target is one sentence of the problem statement: 'For $0 < \lambda \ll 1$, one can choose $\sigma(\lambda)$, $c(\lambda)$ so the field theory described by (3) exists, is unique, and has a mass equal to m such that $|m - m_0|$ is small' (JW § 6.2). In four dimensions the lattice limit collapses to a Gaussian (7-08). Known

02

Wick products: which powers become random variables

In Table 1 of 7-01 (where it breaks) we saw that in two or more dimensions the variance at one lattice point, $c_a = G_a(0)$, becomes infinite as $a \to 0$. A Wick product is a product with lower powers subtracted so that its expectation is 0, and it is fixed step by step by the following conditions (Summers 2016 § 3.1).

$$:\!\phi^0\!: = 1, \qquad \frac{\partial}{\partial \phi} :\!\phi^n\!: = n :\!\phi^{n-1}\!:, \qquad \int :\!\phi^n\!: \, \mathrm{d}\mu_C = 0 \quad (2)$$

For a single variable with variance c, $:\!x^2\!: = x^2 - c$ and $:\!x^4\!: = x^4 - 6cx^2 + 3c^2$: Hermite polynomials with the scale adjusted. The point is that for a Gaussian field with mean 0 and covariance C

$$\big\langle :\!\phi(x)^n\!: \, :\!\phi(y)^k\!: \big\rangle = \delta_{nk} \, n! \, C(x-y)^n \quad (3)$$

holds. The variance of the smeared $:\!\phi^n\!:(f)$ is $n! \iint f(x) f(y) C(x-y)^n$, and at short distance C behaves like $\log(1/|x|)$ if $d = 2$ and like $|x|^{-(d-2)}$ if $d \geq 3$, so

$$\int_{|x|<1} C(x)^n \, \mathrm{d}^d x < \infty \iff n(d-2) < d \quad (4)$$

In two dimensions every n is finite; in three dimensions only $n \le 2$ is, $n = 3$ is the logarithmic borderline, and $n = 4$ diverges like $1/a$. Known

Computed Table 1 The lattice $\int C^n \approx a^d \sum_x G_a(x)^n$ (m = 1. The side of the periodic box is 8 in two dimensions and 4 in three). The bottom row for two dimensions is the continuum, infinite-volume $\int_0^\infty 2\pi r\,(K_0(r)/2\pi)^n\,dr$.

a	d=2, n=2	n=3	n=4	d=3, n=2	n=3	n=4
1/8	0.08086	0.01571	0.00487	0.04771	0.02128	0.03050
1/16	0.08026	0.01518	0.00453	0.04712	0.02589	0.06776
1/32	0.08007	0.01498	0.00436	0.04678	0.03038	0.14358
1/64	0.08001	0.01491	0.00429	0.04661	0.03482	0.29652
1/256	0.07999	0.01487	0.00425	–	–	–
$\to 0$	0.07958	0.01484	0.00424	converges	log	$\propto$ 1/a

Every two-dimensional column approaches the continuum value. In three dimensions $\int C^2$ converges, the increment of $\int C^3$ each time a is halved approaches about 0.0044 (the continuum prediction is $\log 2/(16\pi^2) = 0.00439$), and $\int C^4$ roughly doubles each time. The two-dimensional $:\phi^4:(f)$ is a random variable, but the three-dimensional one is not. These two columns reappear in § 06 as diagrams with two vertices.

03

The wall in two dimensions: not bounded below

Even if $V = \int_\Lambda :\phi^4: \mathrm{d}x$ (Λ is a finite box) is a random variable, to make the measure $e^{-\lambda V}\,\mathrm{d}\mu C/Z$ we need $e^{-\lambda V}$ to be integrable. The classical ϕ^4 is non-negative, but the Wick product is not. Since $:x^4: = (x^2 - 3c)^2 - 6c^2$,

$$:x^4: \;\ge\; -6c^2, \qquad V_a \;\ge\; -6\,c_a^2\,|\Lambda| \quad (5)$$

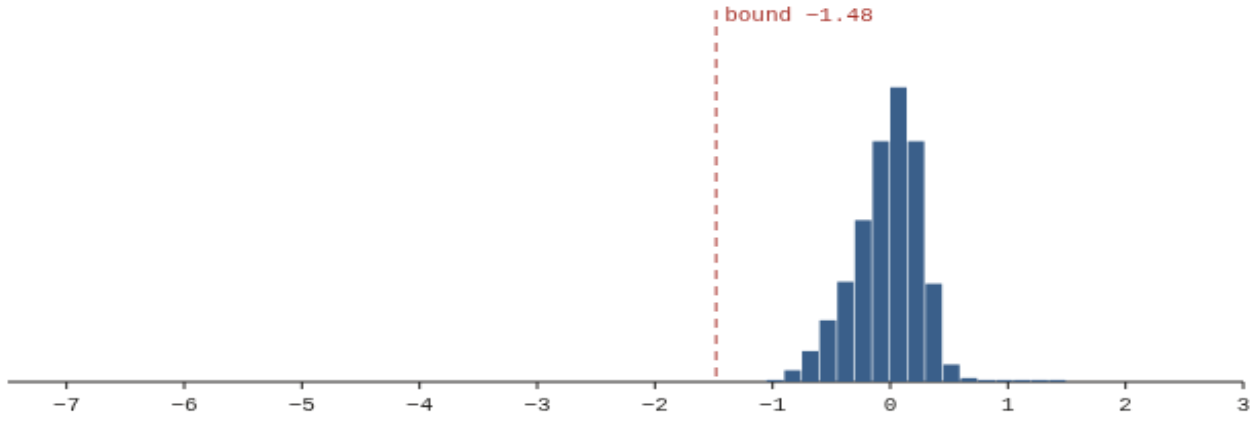
and in two dimensions $c_a = \frac{1}{2\pi}\log\frac{1}{a} + O(1)$, so the lower bound runs away to $-\infty$ at the rate $(\log\frac{1}{a})^2$. Jaffe (2000) says that early progress in two dimensions owed much to Nelson's 1965 paper, which showed the stability (being bounded below) of the Hamiltonian of ϕ^4 with space taken to be a circle. Known

The argument that it is integrable even though the lower bound runs away has three steps. (i) The cut-off V_a is bounded below by formula (5). (ii) The variance of $V - V_a$ becomes small like a power of a, as in the convergence of Table 1. (iii) For a polynomial of degree 4 in a Gaussian field with standard deviation s, the probability of exceeding t is of order $\exp(-\text{const}\cdot(t/s)^{1/2})$. $V < -M$ needs $V - V_a$ to be large and negative, and choosing a to match M makes that probability decrease faster than e^{-pM}. The conclusion is (Summers 2016 § 3.1) Known

$$e^{-\lambda V} \in L^p(\mathrm{d}\mu_C) \qquad (\text{for all } p < \infty) \quad (6)$$

Here $X \in L^p(\mathrm{d}\mu_C)$ means $E[|X|^p] < \infty$, that is, the expectation of the p-th power is finite (E is the expectation in the free-field measure μ_C). For $p = 1$ the normalising constant $Z = E[e^{-\lambda V}]$ is finite, and the measure $e^{-\lambda V}\,\mathrm{d}\mu_C / Z$ can be made.

Figure 1 As a is refined, the lower bound $-6c_a^2$ (dashed) runs away to the left, but in two dimensions the distribution of V_a does not move. Switch to three dimensions and it spreads




```
d = 2, a = 1/4, lower bound −1.48, standard deviation 0.406, 0.1% point −0.969, log E[e^(−V)] =
0.0575 (20000 samples)
```

Computed The numbers in this figure were computed on this machine. $V_a = a^d \sum_x :\phi_x^4:$ was evaluated on 20000 samples of the free lattice field (6000 for a = 1/64 in three dimensions) in a periodic box of side 1 with m = 2. The horizontal axis is V_a, the vertical axis the density.

Computed Table 2 The distributions of Figure 1 (excerpt). The standard deviation is exact from formula (3); the 0.1% point and $\log E[e^{-V}]$ (λ = 1) are from the samples (the error of the latter is ±0.002 in two dimensions and ±0.005–0.17 in three). In brackets, the Gaussian approximation $\mathrm{Var}(V)/2$.

d, a	Lower bound	Standard deviation	0.1% point	$\log E[e^{-V}]$
2, 1/4	−1.48	0.4060	−0.969	0.0575
2, 1/16	−3.15	0.3860	−0.991	0.0479
2, 1/64	−5.37	0.3788	−0.946	0.0505
2, 1/128	−6.70	0.3779	−0.923	0.0500
3, 1/4	−5.91	0.6806	−1.944	0.176 (0.232)
3, 1/16	−97.20	1.3599	−4.152	0.865 (0.925)
3, 1/32	−390.73	1.9229	−5.866	1.746 (1.849)
3, 1/64	−1566.46	2.7194	−8.151	3.54 (3.70)

In two dimensions, even as the lower bound drops from −1.5 to −6.7, the 0.1% point stays at −0.9 to −1.0 and $\log E[e^{-V}]$ settles at about 0.05. The lower bound is far away, but the probability of getting there is extremely small: this is the content of formula (6). In three dimensions, each halving multiplies the standard deviation by $\sqrt{2}$ and

roughly doubles $\log E[e^{-V}]$, approaching the Gaussian approximation. A constant growing like $1/a$ has to be subtracted at order λ^2, and this corresponds to the δ of the problem statement (§ 06). Computed

04

Two roads: field equations and statistical mechanics

The first construction (the Glimm–Jaffe series of four papers, 1968–1972) took the form of solving the field equations as operators in Minkowski space-time. The equation in two dimensions is

$$\partial_t^2\phi - \partial_x^2\phi + m_0^2\phi + 4\lambda\,{:}\phi^3{:} = 0, \qquad {:}\phi^3{:} = \lim_{\kappa\to\infty}\left(\phi_\kappa^3 - 3c_\kappa\phi_\kappa\right) \tag{7}$$

(ϕ_κ is the field with Fourier components cut at $|k| \le \kappa$, $c_\kappa \sim \log\kappa$. Jaffe 2000 § 3). One shows that $H(g) = H_0 + \lambda\int {:}\phi(0,x)^4{:}\,g\,\mathrm{d}x$, cut off in space by a function g, is essentially self-adjoint on Fock space (a unique self-adjoint operator is fixed by the cut-off expression), bounded below, and has a simple lowest eigenvalue, and then uses the fact that signals do not exceed the speed of light to obtain a Poincaré-invariant vacuum as $g \to 1$ (finite-volume step → infinite-volume step. Summers 2016 § 2). The representation is not equivalent to the Fock representation (Haag's theorem of 4-09), but it is equivalent when restricted to a bounded region. Known

In the early 1970s, Nelson's Markov fields (only named here) and the Osterwalder–Schrader axioms (4-10) opened the road of making the Euclidean measure

$$\mathrm{d}\mu_\Lambda = \frac{1}{Z}\,e^{-\lambda U_\Lambda}\,\mathrm{d}\mu_C, \qquad U_\Lambda = \int_\Lambda {:}P(\phi){:}\,\mathrm{d}^2x \tag{8}$$

(given meaning by formula (6)). On a two-dimensional lattice, the difference term, for pairs of neighbouring points $\langle xy\rangle$, becomes

$$\tfrac12\sum_{\langle xy\rangle}(\phi_x-\phi_y)^2 = \sum_{\langle xy\rangle}\left(\tfrac12\phi_x^2+\tfrac12\phi_y^2\right) - \sum_{\langle xy\rangle}\phi_x\phi_y \tag{9}$$

and the last term, inside e^{-S}, appears as $e^{+\phi_x\phi_y}$ and makes neighbours prefer the same sign. It is a ferromagnet whose spins range over all real numbers (the title of Guerra–Rosen–Simon 1975 says 'as classical statistical mechanics'). There correlation inequalities (of GKS type: inequalities saying that correlations are non-negative in systems where neighbours prefer the same sign) apply, and if P has the form of an even polynomial minus $h\phi$ ($h \ge 0$), the Schwinger functions are non-negative and, with suitably chosen boundary conditions, increase with the box. Together with a uniform upper bound, the infinite-volume limit is obtained by monotone convergence (the

infinite-volume step), with no restriction on the size of the coupling (Summers 2016 § 3.1), though the problem statement says that correlation inequalities tend to be limited to scalar bosons and abelian gauge theories (JW § 6.1). Reflection positivity is inherited by the limit (the axiom step, 7-01). Known

05

Weak coupling: the cluster expansion, the mass gap, scattering

Glimm–Jaffe–Spencer (1974) carried it through to the uniform-gap step. When λ/m_0^2 is small enough, they showed the infinite-volume limit and the OS axioms by the cluster expansion, and obtained exponential decay of correlations (formula (9) of 7-01). For the mass operator $M = \sqrt{H^2 - P^2}$ (P is the momentum)

$$\operatorname{spec} M \subset \{0\} \cup \{m\} \cup [m_1, \infty), \qquad m \to m_0, \quad m_1 \to 2m_0 \quad (\lambda \to 0) \tag{10}$$

has been shown (Summers 2016 § 3.1). 0 is the vacuum, the isolated m is the one-particle state, and above m_1 are two or more particles. With an isolated one particle, Haag–Ruelle scattering theory (the general theory that assembles scattering states in the far past and far future from states of isolated particles; only named here) can be used, and Osterwalder–Sénéor (1976) showed that the scattering matrix is non-trivial and that the perturbation series is its asymptotic expansion (Summers 2016 § 3.1). On the other hand, the cluster expansion used in the proofs of existence and of the gap is not a power series in λ but a convergent expansion whose n-th term has the form $\lambda^n G_n(\lambda)$ (G_n bounded as $\lambda \to 0$) (Jaffe 2000 § 4). The perturbation series diverges (4-12) and is only an asymptotic expansion of the exact answer. The mechanism is treated in 7-04. Known

Example: if $\lambda = 0$, the two-point function is $C(x)$, which decays at large distance like $e^{-m_0|x|}$, and the gap is m_0. The weak-coupling result says that this gap is not closed by a small interaction. The source of the gap is the m_0 written into the action (§ 09).

06

Three dimensions: one more renormalisation

The counting of diagrams in perturbation theory gives an idea of what increases (4-03, 4-04). A connected diagram with n vertices and E external lines has $I = 2n - E/2$ internal lines and $L = n - E/2 + 1$ loops. A loop integral carries momentum to the power d and a propagator to the power -2, so the superficial degree of divergence is

$$\omega = dL - 2I = d + (d-4)\,n - \frac{d-2}{2}\,E \tag{11}$$

and the diagrams with $\omega \geq 0$ are the ones that can diverge in the ultraviolet ($\omega = 0$ is logarithmic). A diagram with one vertex consists only of lines joined to itself, and vanishes with Wick products.

Figure 2 Moving the dimension d changes the types of diagram that can diverge (coloured frames; dark for ω > 0, light for ω = 0). Dashed: types with one vertex, which vanish with Wick products. "tree": diagrams with no loop. "—": types with no connected diagram

E\n	n = 1	n = 2	n = 3	n = 4	n = 5	n = 6
E = 0	**ω = 2**	**ω = 1**	**ω = 0**	ω = −1	ω = −2	ω = −3
E = 2	**ω = 1**	**ω = 0**	ω = −1	ω = −2	ω = −3	ω = −4
E = 4	tree	ω = −1	ω = −2	ω = −3	ω = −4	ω = −5
E = 6	—	tree	ω = −3	ω = −4	ω = −5	ω = −6
E = 8	—	—	tree	ω = −5	ω = −6	ω = −7

`d = 3.00, types still divergent after Wick (n ≥ 2): 3`

Computed The values in this figure are formula (11) computed on the spot. Table 3 agrees with the computation on this machine.

Computed Table 3 The types (n, E) with ω ≥ 0, and ω.

d	One vertex (vanishes with Wick products)	Types remaining after Wick
2	(1, 0), (1, 2): 0	none
3	(1, 0): 2, (1, 2): 1	(2, 0): 1, (2, 2): 0, (3, 0): 0
4	(1, 0): 4, (1, 2): 2	E = 0, 2, 4 for every n (ω = 4, 2, 0)

Two of the types remaining in three dimensions are exactly the columns of Table 1. (2, 0) is the vacuum bubble joining two vertices by four lines, $\int C^4$ $(1/a)$, and (2, 2) is the self-energy joining them by three lines, $\int C^3$ (logarithmic). Moving d in Figure 2 from 2.66 to 2.67 brings (2, 0) into the types that can diverge. The borderline is $d = 8/3$, which agrees with the borderline $n = 4$ of formula (4). Computed

The problem statement says that in two dimensions α, γ, and in three dimensions all five, are made to depend on the cutoff (JW § 6.2). Known The problem statement does not say which constant corresponds to which diagram, but reading the order in λ as the number of vertices, α, γ correspond to $n = 1$, β, δ to $n = 2$ and ε to $n = 3$, which matches the number of types in Table 3. Computed Jaffe (2000 § 6) also says that in three dimensions, besides Wick products, one needs a logarithmically divergent mass

counterterm and two vacuum-energy counterterms, divergent linearly and logarithmically. λ itself is not renormalised. Known

Moreover, in three dimensions the Wick-ordered Hamiltonian is not a densely defined operator (an operator defined on a subspace containing vectors arbitrarily close to any vector), the state vectors have to be rescaled by a constant going to 0 with the cutoff (what the problem statement calls wave-function renormalisation), and the representation is not equivalent to the Fock representation (Jaffe 2000 § 6). Jaffe calls this entirely different from two dimensions (§ 04), where it is equivalent to the Fock representation on bounded regions. The tool of control is the localisation of Glimm–Jaffe (1973) into phase-space cells (boxes divided by scale in both position and momentum), and the induction from estimates at one scale to estimates at twice the scale leads to the idea of the renormalisation group (ibid.; 7-05). On that basis Feldman–Osterwalder (1976) and Magnen–Sénéor (1976) independently showed the OS axioms in the limit with the cutoff and volume removed, when λ/m_0^2 is small (Summers 2016 § 3.2; the former puts the mass gap in its title). The restriction on λ/m_0^2 was later removed by other methods by Seiler–Simon (1976) and Park (1977), and Burnap (1977) showed the existence of one-particle states so that Haag–Ruelle scattering theory can be used (Summers 2016 § 3.2; original papers from bibliographic data only). Known

07

Once more by other roads

ϕ^4_2 and ϕ^4_3 then became examples for checking whether new methods can reproduce known results. Each method includes different stages in its output. Brydges–Fröhlich–Sokal (1983) used the lattice, the Schwinger–Dyson equations and correlation inequalities to show existence and non-triviality at weak coupling as subsequential limits of lattice theories. Stochastic quantisation (7-09) makes the measure e^{-S} as the invariant measure (a measure whose distribution does not change under time evolution by this equation) of

$$\partial_t \phi = -\frac{\delta S}{\delta \phi} + \sqrt{2}\,\xi \qquad (\xi \text{ is space-time white noise}) \tag{12}$$

Hairer (2014), with regularity structures, constructed solutions of the stochastic quantisation equation (12) corresponding to Φ^4_3 in finite volume (as local solutions up to a time at which they might blow up). The abstract describes this as solving the problem of constructing a natural Markov process symmetric with respect to that measure, but the text does not show that the solutions are global or that the measure is invariant; it says these are expected. Gubinelli–Hofmanová (2021) showed on $\mathbb{R}^3$ that every limit point is non-Gaussian, reflection positive and translation invariant, and the

abstract says they verified the OS axioms except for rotation invariance and clustering. In finite volume (the three-dimensional torus) there is also a method writing the measure variationally by stochastic optimal control (Barashkov–Gubinelli 2020). Note that the problem statement, for the existence part of the Yang–Mills problem, does not accept weak existence by compactness alone as a solution, unless the properties of the limit (mass gap, axioms) are shown separately (JW footnote 2). Known

08

Detour: strongly coupled φ⁴₂ and the phase transition

A section you may skip.

Two-dimensional ϕ^4 exists for every positive λ, but for $\lambda \gg 1$ the solution splits in two (JW § 6.3). Rearranged, it takes a form with two wells

$$\lambda'\Big(\phi^2 - \frac{1}{\lambda'}\Big)^2 + \tfrac{1}{2}\sigma'\phi^2 + c' \qquad (\lambda' \ll 1) \quad (13)$$

and by setting up a cluster expansion in each well and showing that the probability of going between wells is small, one obtains two phases in which the symmetry $\phi \to -\phi$ is broken (Glimm–Jaffe–Spencer 1976). As the critical value λ_c is approached from below, the gap decreases monotonically and continuously to 0, and near it 'practically nothing has been proved' (JW § 6.3). The gap is a quantity that can close when the coupling is changed, and in the language of 4-11 the critical point is where the correlation length goes to infinity. Pushing the view of formula (9) further, ϕ^4_2 cut off by a lattice and finite volume can be approximated by collections of Ising spins (Simon–Griffiths 1973). This work also extends the Lee–Yang theorem (the theorem that derives, from the location of the zeros of the partition function, that there is no phase transition in an external field) to ϕ^4_2, and from it follows that the vacuum of the model with an external field $h \neq 0$ added is unique (Summers 2016 § 3.1). The gap of the Ising model is in 2-09. Known

09

What can be carried to four dimensions, and what cannot

Known Table 4 Four theories (the mass dimension of the coupling is formula (7) of 7-01, the types of divergence are Table 3).

	φ⁴₂	φ⁴₃	φ⁴₄	Yang–Mills (four dimensions)
Mass dimension of the coupling	2	1	0	0

	φ^4_2	φ^4_3	φ^4_4	Yang–Mills (four dimensions)
Types of diagram that can diverge (ϕ^4 after Wick)	none	3	The three kinds E = 0, 2, 4 at every n (finitely many kinds of counterterm)	Appear at every order (finitely many kinds of counterterm. 4-04)
Coupling at short distance	Acts weakly (super-renormalisable)		Not asymptotically free	Becomes weak (4-06)
Source of the gap	The mass m_0 in the action		—	No mass in the action
Construction	Axioms, gap, scattering	Axioms, gap, one-particle states (Summers § 3.2)	The lattice limit is Gaussian (7-08)	Not yet before 2026 (7-07). The construction published in 2026-11 goes as far as constructing a theory that satisfies the Wightman axioms (6-01). The mass gap is not shown

Four things can be carried over. (1) Reading the form of the counterterms from perturbation theory and making them depend on the cutoff. (2) Piling up estimates scale by scale (phase-space cells became the sequence of renormalisation group steps. 7-05). (3) Properties given by inequalities, such as reflection positivity, are inherited by the limit. (4) If exponential decay of correlations is shown by a weak-coupling expansion, it becomes the gap.

At the centre of what cannot be carried over is the source of the gap. For ϕ^4_2 and ϕ^4_3, $m_0 > 0$ was written into the action and it was shown that $|m - m_0|$ is small. The Yang–Mills action has no mass term, and the expected scale of the gap depends on the coupling in the form $e^{-\text{const}/g^2}$, which cannot be expanded in powers of g (4-07). The mass that would be the foothold of a weak-coupling expansion is absent from the start. In addition, in four dimensions divergent diagrams appear at every order, and the field must preserve gauge invariance as a connection (5-01). Known Physics

On the other hand, the problem statement cites the fact that four-dimensional ϕ^4 is not asymptotically free as evidence of triviality, and names Yang–Mills, whose coupling becomes weak at short distance, as the natural candidate for constructive field theory in four dimensions (JW § 6.2). Where it stops is seen in 7-10.

What this chapter can say

	Content
Said	The lattice $\int C^n$ converges for n = 2, 3, 4 in two dimensions; in three dimensions n = 2 converges, n = 3 is logarithmic and n = 4 grows like 1/a (a $\geq$ 1/256, 1/64) Computed
Said	In two dimensions, even as the lower bound $-6c_a^2$ drops, the left tail of V_a and $\log E[e^{-V_a}]$ do not move; in three dimensions $\log E[e^{-V_a}]$ grows like 1/a (a $\geq$ 1/128, 1/64) Computed
Said	The types of diagram that can still diverge after Wick are none for d = 2 and three for d = 3, and for d = 4 the three kinds E = 0, 2, 4 appear at every n. The three in three dimensions match the descriptions of the counterterms in the problem statement and in Jaffe (2000), reading the order of the constants as the number of vertices Computed Known
Said	Weakly coupled ϕ^4_2 and ϕ^4_3 satisfy the axioms and have a mass gap, and strongly coupled ϕ^4_2 has two phases (according to the problem statement and the survey) Known
Not said	The proofs of formula (6), the cluster expansion and phase-space cells (outline only; original papers from bibliographic data only)
Not said	The existence of limits from the numbers in Tables 1 and 2 (computations with finite a and finite samples)
Not said	Near the critical point; rotation invariance and clustering on the road of stochastic quantisation (later literature not followed)
Not said	Global solutions and invariance on the road of stochastic quantisation (literature after Hairer 2014 not followed)

Sources and reproduction

Item	Kind	Source / tools
Problem statement § 6.1–6.3, footnote 2	Primary source (text checked)	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000)
History, formula (7), counterterms and representation in three dimensions, phase-space cells	Survey (text checked)	A. Jaffe, Constructive quantum field theory, in *Mathematical Physics 2000*, World Scientific (2000)
Formulas (2), (6), (8), (10), the order of steps, correlation inequalities, scattering and the extension of the Lee–Yang theorem in § 3.1, removal of the restriction and one-particle states in § 3.2	Survey (text checked)	S. J. Summers, A Perspective on Constructive Quantum Field Theory, arXiv:1203.3991 (v2, 2016)

Item	Kind	Source / tools
Original papers in two dimensions	Bibliographic data only (reference lists of the problem statement and the survey)	Nelson, proceedings of the 1965 conference *Mathematical Theory of Elementary Particles* (ed. R. Goodman, I. Segal, MIT Press 1966). Glimm–Jaffe, Phys. Rev. 176 (1968) 1945–1951; Ann. Math. 91 (1970) 362–401; Acta Math. 125 (1970) 203–267; J. Math. Phys. 13 (1972) 1568–1584. Nelson, Proc. Sympos. Pure Math. XXIII, AMS (1973) 413–420. Osterwalder–Schrader, Comm. Math. Phys. 31 (1973) 83–112; 42 (1975) 281–305. Guerra–Rosen–Simon, Ann. Math. 101 (1975) 111–259. Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585–632. Osterwalder–Sénéor, Helv. Phys. Acta 49 (1976) 525–535
Original papers in three dimensions	Bibliographic data only	Glimm–Jaffe, Fortschr. Phys. 21 (1973) 327–376. Feldman–Osterwalder, Ann. Phys. 97 (1976) 80–135. Magnen–Sénéor, Ann. Inst. H. Poincaré A 24 (1976) 95–159. Seiler–Simon, Ann. Phys. 97 (1976) 470–518. Park, J. Math. Phys. 18 (1977) 354–366. Burnap, Ann. Phys. 104 (1977) 184–196
Other constructions	Brydges–Fröhlich–Sokal from bibliographic data and a summary of the abstract only. Hairer checked in the abstract and the text of arXiv version v4 (Remark 1.18), the others in the abstracts	Brydges–Fröhlich–Sokal, Comm. Math. Phys. 91 (1983) 141–186. Hairer, Invent. Math. 198 (2014) 269–504, doi:10.1007/s00222-014-0505-4, arXiv:1303.5113. Barashkov–Gubinelli, Duke Math. J. 169 (2020) 3339–3415, arXiv:1805.10814. Gubinelli–Hofmanová, Comm. Math. Phys. 384 (2021) 1–75, arXiv:1810.01700
Phase transition	Bibliographic data only	Glimm–Jaffe–Spencer, Ann. Phys. 101 (1976) 610–630, 631–669. Simon–Griffiths, Comm. Math. Phys. 33 (1973) 145–164. Decrease towards the critical point (references of JW § 6.3): Glimm–Jaffe, Phys. Rev. D 10 (1974) 536–539. Guerra–Rosen–Simon, Comm. Math. Phys. 41 (1975) 19–32. McBryan–Rosen, Comm. Math. Phys. 51 (1976) 97–105
ϕ^4 in four dimensions	Bibliographic data only (treated in 7-08)	Aizenman–Duminil-Copin, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973
Tables 1–3, Figure 1	Computed	Python (numpy, scipy) on this machine: `wick_powers.py`, `wick_mc.py` (free fields on periodic lattices generated by FFT, fixed random seed. Run separately for each (d, a), each within 5 minutes; the heaviest, a = 1/64 in three dimensions, takes about 3 minutes 40 seconds), `fig_data.py`, `power_count.py`

Item	Kind	Source / tools
Figures 1 and 2	Figures	canvas, SVG and plain JavaScript. Figure 1 is the frequencies of the samples of Table 2 (densities in steps of 0.01), Figure 2 computes formula (11)

Next chapter to read: 7-04 The cluster expansion (the mechanism of the tool used in § 05).

The cluster expansion — a tool that splits into weakly linked small lumps and rewrites the logarithm as a convergent sum

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The formula that writes the logarithm of the partition function as a sum over "connected lumps" (clusters), and its convergence condition. That convergence gives the infinite-volume limit, analyticity and exponential decay of correlations all at once. That continuum fields, too, can be decoupled cube by cube.

Prerequisite chapters — 7-03. We also use the infinite-volume step and the uniform-gap step of 7-01, 4-11 and 1-08 (Gaussian measures). The language of graphs is defined on the spot.

Said first — This is a tool for when there is a small parameter (weak coupling, high temperature, strong coupling on the lattice), and it does not reach the infinite-volume limit or the gap of four-dimensional Yang–Mills (the infinite-volume limit of the construction published on 2026-11-10 is taken along subsequences of volumes by compactness, not by an expansion; 12-05). The computations concern a one-dimensional model and small graphs.

The order of this chapter

1. What the tool is for — if independent, the logarithm is a sum. Weak links are added in as corrections
2. The smallest example: the one-dimensional hard-core lattice gas — coefficients, radius of convergence 1/4, the singularity on the side of negative activity (a table of computations)
3. Polymer models — the common form of a gas of incompatible lumps
4. Taking the logarithm: Ursell functions and connected graphs — only connected clusters remain
5. Why it can converge: cancellation and trees — the signs cancel, and only the number of trees remains (a moving figure)

6. Criteria for convergence — inequalities such as Kotecký–Preiss, and the range they guarantee (a moving figure)
7. Detour: the same inequality as the local lemma of probability — the Lovász local lemma and the zeros of the independent set polynomial
8. What the expansion gives: infinite volume, analyticity, exponential decay — the decay rate of correlations is the "mass gap" of this model
9. To continuum fields: cut into cubes and decouple by interpolation — rewritten as a gas of lumps of cubes
10. Uses and limits for gauge fields — works on the strong-coupling side, and does not reach the infrared in four dimensions

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01

What the tool is for

What the infinite-volume step and the uniform-gap step need are estimates independent of the size of the box, and quantities far apart becoming nearly independent (formula (9) of 7-01). The problem statement says that because perturbation series usually diverge, expansion methods collectively called "cluster expansions" were developed, and writes 'Refined estimates yield the rate of exponential decay of Green's functions, magnitude of masses, …' (JW § 6.1). Known

The core is a single point. If the blocks are independent of each other, the partition function is a product and the logarithm is a sum.

$$Z_\Lambda = \prod_j Z_j \quad \Longrightarrow \quad \log Z_\Lambda = \sum_j \log Z_j \tag{1}$$

If it is a sum, quantities per unit volume do not depend on the box, and distant blocks have no effect. Real blocks are weakly linked, and the cluster expansion adds those links in as corrections for each "set of connected blocks" and shows that the sum converges. As a metaphor, it is a thin gas that hardly ever collides. More precisely, two things are needed: (i) the correction linking n of them is of the order of a small number to the power n, and (ii) the number of ways of linking, divided by $n!$, is of exponential order (§ 05).

Example: if each block is independently "empty" or "one particle (weight z)", then $Z = (1+z)^N$. $\log Z/N = z - \frac{z^2}{2} + \cdots$ converges for $|z| < 1$ and stops at $z = -1$, where $Z = 0$. The reach of the series is decided by where the partition function vanishes for complex z.

02

The smallest example: the one-dimensional hard-core lattice gas

This example is used throughout the chapter. Place particles at the points $x = 1, \dots, N$ ($n_x \in \{0,1\}$), forbid neighbours from being occupied together (**hard-core repulsion**), and give each particle the weight z (the **activity**).

$$\Xi_N(z) = \sum_{\substack{n \in \{0,1\}^N \\ n_x n_{x+1} = 0}} z^{\,n_1 + \cdots + n_N} \tag{2}$$

If points N and 1 are also regarded as neighbours to make a ring, one can write $\Xi_N = \operatorname{tr} T^N$ with the transfer matrix $T = \begin{pmatrix} 1 & \sqrt{z} \\ \sqrt{z} & 0 \end{pmatrix}$ (transfer matrices are in 2-08 § 03; the example of the Ising model is in 2-09, and lattice gauge theory in 5-03). The entries are $T_{nn'} = z^{(n+n')/2}$ ($n, n' \in \{0,1\}$; only $n = n' = 1$ gives 0). So in $\operatorname{tr} T^N = \sum_{n_1,\dots,n_N} \prod_{x=1}^{N} T_{n_x n_{x+1}}$ ($n_{N+1} = n_1$), the weight of each n_x is picked up twice, half each time, giving z^{n_x}, and simultaneous occupation of neighbours vanishes with 0. This is exactly formula (2) on the ring. Of the eigenvalues $\lambda_\pm = \tfrac{1}{2}(1 \pm \sqrt{1+4z})$, the larger survives. We call the following $p(z)$ the **pressure** (the physical pressure times the inverse temperature, the log partition function per unit volume).

$$p(z) = \lim_{N \to \infty} \frac{1}{N} \log \Xi_N = \log \frac{1 + \sqrt{1+4z}}{2} = \sum_{n \geq 1} b_n z^n, \qquad b_n = \frac{(-1)^{n-1}}{n} \binom{2n-1}{n} \tag{3}$$

Computed Table 1 The coefficients b_n (agree with the closed form up to $n \leq 24$ by series operations with rational numbers).

n	2	3	4	6	20	60
b_n	−3/2	10/3	−35/4	−77	-3.4×10^{9}	-8.1×10^{32}
$\lvert b_n / b_{n-1} \rvert$	1.500	2.222	2.625	3.056	3.705	3.901

The ratio approaches 4, and the radius of convergence is $1/4$. The radius of convergence of a power series is the distance to the nearest point to the origin at which the function stops being analytic in the complex plane (a theorem of complex analysis) Known . Here that point is $z = -1/4$, where the inside of the square root vanishes and the eigenvalues coincide. There is no phase transition for $z > 0$, and yet the series cannot go beyond $1/4$. **The reach of the expansion is decided by a singularity on the side of non-physical negative activity.** The zero of Ξ for a chain of length n nearest to the origin is $-1/(4\cos^2 \frac{\pi}{n+2})$ (Scott–Sokal 2005, Ex. 3.2), which for $n = 5, 10, 20$ is $-0.3080, -0.2679, -0.2552$. Computed

03

Polymer models

We extract the structure. Prepare a collection $\mathcal{P}$ of finite lumps (**polymers**) γ, a relation of being **incompatible**, $\gamma \not\sim \gamma'$ (overlapping, touching and so on; each is incompatible with itself), and complex activities z_γ, and count only the sets in which every two are compatible.

$$\Xi_\Lambda = 1 + \sum_{n\ge 1} \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\in\Lambda^n} z_{\gamma_1}\cdots z_{\gamma_n} \prod_{j<k} \mathbf{1}\{\gamma_j \sim \gamma_k\} \tag{4}$$

Λ is a finite collection, and $\mathbf{1}\{\cdot\}$ is 1 if the condition is true and 0 if false. This formalism is said to have been introduced by Gruber–Kunz (1971) (Fernández–Procacci 2007 § 1). Known

Example: in § 02, a polymer = one occupied point, incompatible = the same point or a neighbour. In the high-temperature Ising model without external field (2-08), the polymers are connected sets of edges with even degree at every point, they are incompatible if they share a vertex, and the activity is $(\tanh\beta)^{\text{number of edges}}$. Strong coupling in lattice gauge theory (5-04) has connected sets of plaquettes, and continuum fields at weak coupling (§ 09) have connected sets of cubes. The reason the lumps are small becomes the small parameter of the expansion.

04

Taking the logarithm: Ursell functions and connected graphs

In the device that goes by the name of Mayer (said to go back to Mayer 1937), one writes $\mathbf{1}\{\gamma_j \sim \gamma_k\} = 1 + \zeta_{jk}$ (-1 if incompatible, 0 if compatible) and expands the product. Each term corresponds to a **graph** (points, and edges joining pairs of points) on the points $1,\dots,n$, and in the logarithm only connected graphs remain. The reason is that the weight of a graph splits into the product of the weights of its connected components. Organising by the ways of dividing the points into connected components, between the sums over all n with $1/n!$ attached, $\text{sum over all graphs} = \exp\big(\text{sum over connected graphs}\big)$ (the **exponential formula**) holds, and taking the logarithm leaves only the connected ones. Known

$$\log\Xi_\Lambda = \sum_{n\ge1}\frac{1}{n!}\sum_{(\gamma_1,\dots,\gamma_n)} \varphi^{\mathrm{T}}(\gamma_1,\dots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n}, \qquad \varphi^{\mathrm{T}} = \sum_{\substack{G\subset G_\gamma \\ \text{connected, spanning}}} (-1)^{|E(G)|}$$

G_γ is the graph joining j,k when $\gamma_j \not\sim \gamma_k$, the sum is over connected subgraphs using all the points, and $|E(G)|$ is the number of edges (1 for $n=1$). φ^{T} is the **Ursell**

function (unrelated to the field ϕ), and it is 0 if G_γ is not connected. Only sets connected by incompatibility, that is **clusters**, contribute (Fernández–Procacci (2.3) (2.4). Still a formal series).

Example: in § 02, averaging over translations, $b_n = \frac{1}{n!} \sum_{x_2,\dots,x_n} \varphi^{\mathrm{T}}(0, x_2, \dots, x_n)$. For $n = 2$, each of $x_2 = -1, 0, 1$ gives -1 with one edge, so $b_2 = -3/2$. The $(0, 0)$ forbidden in configurations appears in the logarithm. Summing by brute force up to $n \leq 6$ agreed with Table 1 in rational numbers. Computed

05

Why it can converge: cancellation and trees

The sum in formula (5) is huge. When every two are incompatible, G_γ is the **complete graph** K_n (joining all pairs), there are nearly as many connected spanning subgraphs as graphs in all, and if they remained as ± 1 the sum would not converge even after dividing by $n!$. But the sum cancels, and only $(-1)^{n-1}(n-1)!$ remains.

Computed Table 2 K_n (brute force for $n \leq 6$, a recurrence for $n = 10$).

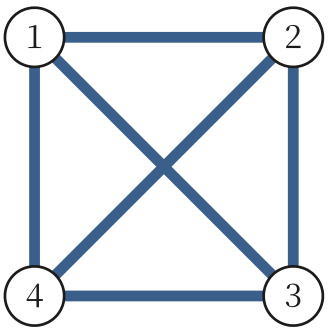
n	Connected spanning subgraphs	Spanning trees n^{n-2}	Alternating sum
4	38	16	−6
10	34 496 488 594 816	100 000 000	−362 880

The mechanism is due to Penrose (1967). Dividing the connected spanning subgraphs into intervals $[T, R(T)]$, one for each **spanning tree** (a subgraph without cycles connecting all the points) T, the sum of signs within an interval vanishes by the factor $\prod_{e \in R(T) \setminus T}(1 - 1)$, and only the trees with $R(T) = T$ (Penrose trees) remain (Fernández–Procacci, Prop. 5). Known

$$\varphi^{\mathrm{T}}(\gamma_1, \dots, \gamma_n) = (-1)^{n-1}\, \#\{\text{Penrose trees}\} \quad \Longrightarrow \quad |\varphi^{\mathrm{T}}| \leq \#\{G_\gamma \text{ 's spanning trees}\} \tag{6}$$

The number of Penrose trees agreed with the absolute value of the alternating sum by brute force for $n \leq 6$ Computed . Even for K_n there are n^{n-2} spanning trees, and $n^{n-2}/n!$ is of order e^n. **Since only an exponential remains after dividing by** $n!$, (5) can converge if the activities are small. Identities that write the sum in terms of trees or forests from the start were refined from Brydges–Federbush (1978) to the forest formula of Brydges–Kennedy (1987) (Abdesselam–Rivasseau 1994).

Figure 1 Choose a graph and press "Play": the connected spanning subgraphs are counted in order of increasing number of edges, and the running total of the signs $(-1)^{|E|}$ changes (bar: green +1, red −1)



K4: 38 subgraphs, no. 38

6 edges → sign +1

running total -6



K4: 38 subgraphs, no. 38, 6 edges → sign +1, running total -6

Computed The numbers in this figure were computed on this machine. The running total for K_4 is −16 at 3 edges (16 spanning trees), −1 at 4 (15 of them), −7 at 5 (6 of them) and −6 at 6. C_4 gives −3, P_4 −1 and K_3 +2.

06

Criteria for convergence

Summing up the tree estimates gives a condition. The widely used condition of Kotecký–Preiss (1986) assigns to each lump a "budget" $a(\gamma) \geq 0$ and requires

$$\sum_{\gamma' \not\sim \gamma} |z_{\gamma'}|\, e^{a(\gamma')} \;\leq\; a(\gamma) \qquad \text{(for all } \gamma\text{)} \tag{7}$$

(the left-hand side includes γ itself). If it holds, the series obtained by differentiating $\log \Xi_\Lambda$ with respect to z_{γ_0} (the sum of clusters containing the lump γ_0; the sum per polymer) is at most $e^{a(\gamma_0)}$ independently of Λ, even with each term replaced by its absolute value (Fernández–Procacci (2.6)(2.7)(2.14)(2.15)). $\log \Xi_\Lambda$ itself grows with the volume, but because this per-polymer upper bound does not depend on Λ, (5) converges absolutely with estimates uniform in Λ (ibid., the sentence right after (2.7)). The change in the logarithm when the lump γ_0 is removed is also at most $|z_{\gamma_0}|e^{a(\gamma_0)}$ (ibid. (2.11)). Read it as "the total weighted activity of the incompatible partners fits within one's own budget". The conditions of the same type of Dobrushin (1996) and Fernández–Procacci (2007) widen the range of application in that order. Known

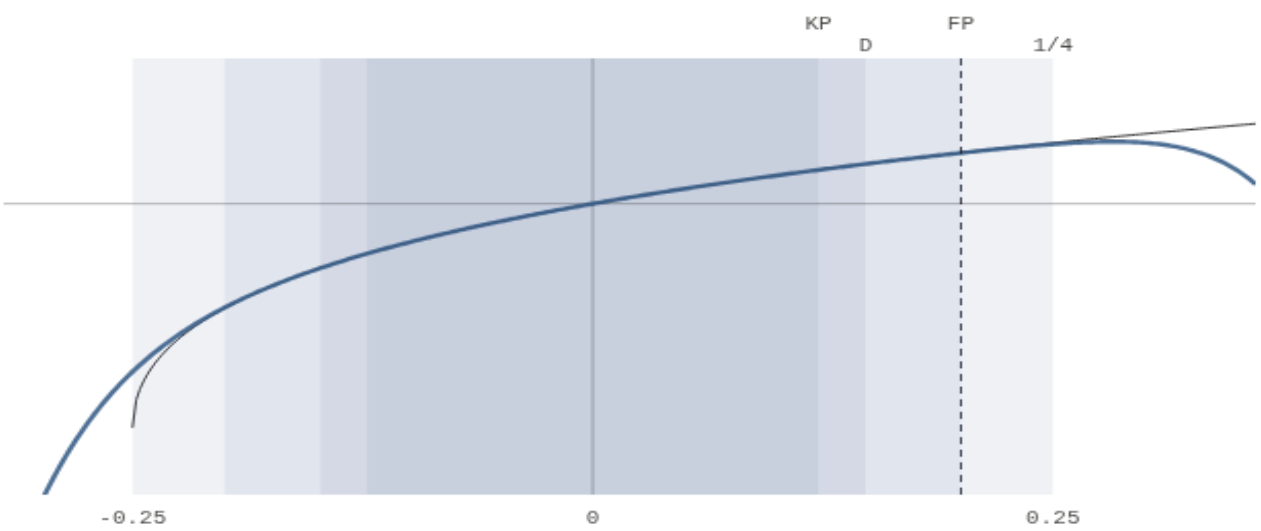
Computed Table 3 The range guaranteed for the gas of §02 (maximum degree $\Delta = 2$). Numerical maximisation agrees with the formulas of Fernández–Procacci Table 1.

Condition	Form for this gas	Guaranteed $\lvert z\rvert$
Kotecký–Preiss	$3\lvert z\rvert e^{a} \leq a$	$1/(3e) = 0.1226$

Condition	Form for this gas	Guaranteed $\lvert z\rvert$
Dobrushin	$\lvert z\rvert \le \mu/(1+\mu)^3$	$4/27 = 0.1481$
Fernández–Procacci	$\lvert z\rvert \le \mu/(1+3\mu+\mu^2)$	$1/5 = 0.2000$
Exact radius	Formula (3)	$1/4 = 0.2500$

$1+3\mu+\mu^2$ counts the mutually compatible subsets of three incompatible points (the empty set, the three single points, $\{x-1, x+1\}$), and $(1+\mu)^3$ is coarser by also counting the sets that are not. The $1/4$ in the last row is the exact value $(\Delta-1)^{\Delta-1}/\Delta^{\Delta}$ on the infinite tree with every degree Δ (Shearer / Scott–Sokal; the starred entry in Fernández–Procacci Table 1) with $\Delta = 2$ put in. This tree with $\Delta = 2$ is the line of integers itself, so it agrees with the $1/4$ of formula (3). All three conditions above are **sufficient conditions**.

Figure 2 As the number of terms N increases, the partial sum of series (3) (thick line) lies on the exact pressure (thin line) for $\lvert z\rvert < 1/4$, and runs wild outside (bands are the range of each condition)




```
N = 10, z = 0.200: partial sum 0.15732, difference 3.86e-4
```

Computed The numbers in this figure were computed on this machine. $\lvert S_N(z) - p(z)\rvert$ for $N = 5, 20, 40$ is 3e−3, 1.5e−5, 6.4e−8 at $z = 0.2$ and 0.029, 0.063, 0.88 at $z = 0.3$.

07

Detour: the same inequality as the local lemma of probability

A section you may skip.

The **Lovász local lemma** of combinatorics (Erdős–Lovász 1975) is a condition for the probability that none of the "bad events" A_x occurs to be positive. The way the events depend on each other is represented by a graph. Write $\Gamma(x)$ for the set of points adjacent to the point x.

Dependency graph: when each A_x is independent of the events other than x and its neighbours $\Gamma(x)$ (taken all together), this graph is called a dependency graph. The theorem has

the following form (Scott–Sokal 2005, Thm 1.1; A_x^c is the event that A_x does not occur).

$$0 \le r_x < 1, \quad \mathbb{P}(A_x) \le r_x \prod_{y \in \Gamma(x)} (1 - r_y) \text{ (for all } x) \implies \mathbb{P}\Big(\bigcap_x A_x^c\Big) \ge \prod_x (1 - r_x) > 0 \tag{8}$$

Scott–Sokal showed the following equivalence. That the conclusion of the local lemma (the probability that none occurs is positive) holds for **every** family of events with dependency graph G and $\mathbb{P}(A_x) \le p_x$ is equivalent to the partition function of the hard-core gas on G not vanishing on the polydisc of radii p_x (the tuples of complex numbers (z_x) with $|z_x| \le p_x$ for every x). This partition function is the sum $\sum_S \prod_{x \in S} z_x$ over sets S of mutually non-adjacent points (the **independent set polynomial**; the multivariate version with an activity for each point), and if G is a path it becomes the gas of § 02. They also show that the usual proof of the local lemma corresponds to the induction on "having no zeros" found implicitly by Shearer (1985) and explicitly by Dobrushin (abstract).

Setting $r = \mu/(1+\mu)$, the hypothesis of (8) is Dobrushin's form in Table 3, and for a sequence of events depending only on their left and right neighbours $p \le 4/27$ is enough. The best is $1/4$, bounded from above by Example 8.1 of Scott–Sokal § 8.4 (the zeros of finite paths approach $-1/4$) and from below by the exact radius of Table 3 (Shearer's value $(\Delta-1)^{\Delta-1}/\Delta^{\Delta}$ with $\Delta = 2$). The symmetric version often seen in textbooks also follows from here. When each event depends on at most d other events and every probability is at most p, setting $r_x = 1/(d+1)$ in (8) gives $r(1-r)^d = \frac{1}{d+1}\big(\frac{d}{d+1}\big)^d \ge \frac{1}{e(d+1)}$, so $ep(d+1) \le 1$ is enough. For $d = 2$ this is $1/(3e)$, the Kotecký–Preiss value of Table 3. In this way all four values of Table 3 also appear on the side of the local lemma. Known

08

What the expansion gives: infinite volume, analyticity, exponential decay

If condition (7) holds independently of the volume, three things follow. (i) The limit of the pressure exists and is analytic in the disc. (ii) Only clusters that touch the edge or go round feel the size of the box. (iii) In truncated correlations only clusters touching both x and y, that is, connected from x to y by a chain of incompatibility, remain, and each link contributes a factor of order $|z|$ (Ueltschi 2004, Thm 3). Known

$$\big|\langle A_x B_y\rangle - \langle A_x\rangle\langle B_y\rangle\big| \;\le\; C\, e^{-m|x-y|} \tag{9}$$

Example: counting the gas of § 02 on a ring of 14 points, the coefficients of $\log \Xi_{14}/14$ agree with b_n for $n \le 13$, and first differ at $n = 14$ (going round needs all 14 points). The correction is $\frac{1}{N}\log\big(1 + (\lambda_-/\lambda_+)^N\big)$, and for small z, $\lambda_-/\lambda_+ \approx -z$, so it is of order $(-z)^N/N$ (the difference in the coefficient at $n = 14$ is exactly $1/14$). The truncated correlation of the density is proportional to $(\lambda_-/\lambda_+)^{|x-y|}$, and the decay rate is $m(z) = \log(\lambda_+/|\lambda_-|)$. Computed

Computed Table 4 The decay rate $m(z)$ and the estimate from chains $\log(1/|z|)$.

z	−0.249	−0.1	0.01	0.25	1
$m(z)$	0.1267	2.0634	4.6249	1.7627	0.9624
$\log(1/\lvert z\rvert)$	1.3903	2.3026	4.6052	1.3863	0

For small z, $m \approx \log(1/z)$ and the picture of chains is right; as $z \to -1/4$, $m \to 0$ (decay stops at the singularity that fixed the radius). $z = 1$ is outside the radius and still has $m > 0$: convergence is a sufficient condition.

$m(z)$ is the ratio of eigenvalues of the transfer matrix, the "mass gap" of this model (the uniform-gap step of 7-01, 4-11). In field theory too, if truncated correlations in the time direction decay like e^{-mt} uniformly in the volume, and the vectors ψ (with the vacuum component subtracted) made from the observables for which this decay can be shown are dense in the subspace orthogonal to the vacuum, then the Hamiltonian H of the OS reconstruction (4-10) has no spectrum in $(0, m)$. Known

The tool is the **spectral measure**. For each ψ there is a measure μ_ψ on $[0,\infty)$ with $\langle \psi, e^{-tH}\psi\rangle = \int e^{-t\lambda}\,\mathrm{d}\mu_\psi(\lambda)$ (the spectral theorem of 1-10; if there is a basis of eigenvectors e_k, μ_ψ puts the weight $|\langle e_k,\psi\rangle|^2$ on each eigenvalue λ_k). If $\int e^{-t\lambda}\mathrm{d}\mu_\psi \le Ce^{-mt}$ holds for all t, then looking at $t\to\infty$, μ_ψ cannot put weight on $\lambda < m$. Since this holds for a dense set of ψ, no spectrum remains in $(0,m)$. The converse is formula (2) of § 5 of the problem statement.

09

To continuum fields: cut into cubes and decouple by interpolation

For continuum fields, the covariance $C(x,y)$ of the free field links even distant points. Cut the box into unit cubes b_i (indicator functions χ_i), and multiply the parts straddling cubes by $s_{ij} \in [0,1]$.

$$C(\mathbf{s})(x,y) = \sum_i \chi_i(x) C(x,y) \chi_i(y) + \sum_{i \neq j} s_{ij}\, \chi_i(x) C(x,y) \chi_j(y), \qquad s_{ij} = s_{ji} \tag{10}$$

If all the s are 1 this is the original covariance, and if all are 0 the cubes are independent (Abdesselam–Rivasseau 1994 (V.A.5)). The two are joined by Taylor's formula (ibid. (III.1)). The index ℓ in formula (11) runs over pairs of cubes $\{i,j\}$, and $s_\ell = s_{ij}$. $f(\mathbf{s})$ is the partition function with covariance $C(\mathbf{s})$, I is a set of pairs, and $\psi_I(h)$ is the point with $s_\ell = h_\ell$ for $\ell \in I$ and $s_\ell = 0$ for $\ell \notin I$.

$$f(1,\dots,1)=\sum_{I\subset\{\text{pairs}\}}\prod_{\ell\in I}\int_0^1 \mathrm{d}h_\ell\,\Big(\prod_{\ell\in I}\frac{\partial}{\partial s_\ell}\Big)f\big(\psi_I(h)\big)\qquad (11)$$

For a single pair, $f(1)=f(0)+\int_0^1 f'(h)\,\mathrm{d}h$, and (11) repeats this pair by pair. $\partial/\partial s_\ell$ ($\ell=\{i,j\}$) brings down one line joining the cubes b_i, b_j, of size about $|C|$, and with mass m_0 about $e^{-m_0|x-y|}$. The differentiated pairs form lumps, the integral splits lump by lump, and it becomes a gas whose polymers are connected sets of cubes Y (ibid. (V.A.7)(V.A.8)).

However, the interpolated covariance must stay positive definite (the condition for it to make sense as a Gaussian measure), so in practice one uses the **forest formula** (the Taylor forest formula of Brydges–Kennedy, ibid. Thm III.1). This is the version in which the parameter joining two cubes is set to the minimum of h_ℓ along the path joining them in the tree of the forest, and this keeps positive definiteness (ibid. (V.A.9), Thm IV.5). In this form, for any constant K, with an ultraviolet cutoff in place and the coupling g taken small enough (depending on K), $\sum_{Y\ni 0}|A(Y)|K^{|Y|}\le 1$ (ibid. (V.A.10); $|Y|$ is the number of cubes in Y). It was Glimm–Jaffe–Spencer who brought this type of expansion into constructive field theory (ibid. § I), and the results are in 7-03 § 05 and § 08. Known

Example: the tool is **Gaussian integration by parts** (Gaussian distributions are in 1-08 § 06). For a standard normal ϕ, the derivative of the density $e^{-\phi^2/2}$ is $-\phi\,e^{-\phi^2/2}$, so integration by parts gives $\mathbb{E}[\phi F(\phi)]=\mathbb{E}[F'(\phi)]$. Extending this to Gaussian variables with mean 0 and covariance matrix $C(s)$ gives $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_s[H]=\frac12\sum_{a,b}\frac{\mathrm{d}C_{ab}}{\mathrm{d}s}\,\mathbb{E}_s[\partial_a\partial_b H]$ Known . For Gaussian variables (ϕ_1,ϕ_2) with variance 1 and covariance sc, $\mathrm{d}C_{12}/\mathrm{d}s=\mathrm{d}C_{21}/\mathrm{d}s=c$, so $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_s[F(\phi_1)G(\phi_2)]=c\,\mathbb{E}_s[F'(\phi_1)G'(\phi_2)]$. At $s=0$ the two are independent, so integrating in s from 0 to 1 gives

$$\operatorname{Cov}\big(F(\phi_1),G(\phi_2)\big)=c\int_0^1\mathbb{E}_s\big[F'(\phi_1)\,G'(\phi_2)\big]\,\mathrm{d}s,\qquad \big|\operatorname{Cov}\big|\le|c|\,\sup|F'|\,\sup|G'|\qquad (12)$$

and the correlation of lumps is bounded by the covariance $|c|$ that links them. For $F=G=\cos$ the value is $e^{-1}(\cosh c-1)$ (0.15932358 at $c=0.9$), and numerical integration of the left- and right-hand sides agreed to 8 digits. Computed

10

Uses and limits for gauge fields

Strong coupling on the lattice. For small β the expansion in lumps of plaquettes works (5-02, 5-04), and the existence and analyticity of the infinite-volume limit and estimates of confinement have been shown (Osterwalder–Seiler 1978). But Chatterjee

(2018) cautions that this area law holds for every group including $U(1)$, and is not evidence of non-abelian confinement. Known

Phase-space expansion. When there is no small parameter at the unit scale, the field is split by scale, expanded in the "phase-space cells" of each scale, and the results piled up (Battle–Federbush 1982; the 'phase-space localization' of JW § 6.1). In lattice gauge theory, Bałaban's series of papers is this (7-05). Known

The infrared limit. In four dimensions the coupling becomes strong at long distances. Magnen–Rivasseau–Sénéor 1993 § I writes 'there is no easy solvable model of these phenomenons around which to expand', and the problem statement also says that a gap uniform in the volume needs new ideas (JW § 6.5). The small β where the expansion converges is the shore opposite to the continuum limit $\beta \to \infty$. More precisely, the expansion converges on a finite interval on the side of small β, and the continuum limit is taken at $\beta \to \infty$ (5-09).

Seam The reach, and where we want to go

In § 02 the singularity that stopped the expansion was on the side of negative activity. On the four-dimensional $U(1)$ lattice there really is a phase at large β in which the area law fails (Guth 1980; a complete proof and the massless phase are Fröhlich–Spencer 1982). For the non-abelian case (the Wilson action of $SU(2)$ and $SU(3)$), it is believed that there is no phase transition that loses confinement Physics , but it is not a theorem. Where the convergence of the expansion stops says nothing about whether there is a phase transition beyond it. In the organisation of this book, what is missing is one step between the weak-coupling ultraviolet (7-05 to 7-07) and the strong-coupling expansion, where the coupling becomes $O(1)$ (7-10).

What this chapter can say

	Content
Said	The formula of the expansion, Penrose's identity, the three convergence conditions and their consequences (analyticity, exponential decay) Known
Said	The coefficients, radius $1/4$, thresholds and decay rate of the one-dimensional hard-core lattice gas. Agreement of the alternating sum of K_n with the Penrose trees ($n \le 6$). Formula (12) Computed

	Content
Not said	The content of the convergence proofs for continuum fields and for strong coupling on the lattice (original papers not followed; 5-04). An expansion usable in the infrared in four dimensions (none is known)

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement (§ 5, § 6.1, § 6.5)	Text checked	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000)
Formulas (4)–(7), the conditions of Dobrushin and FP, the formulas of Table 3	Text checked	R. Fernández, A. Procacci, CMP 274 (2007) 123–140, arXiv:math-ph/0605041
Formulas (10)(11), the forest formula / decay of correlations	Text checked	A. Abdesselam, V. Rivasseau, arXiv:hep-th/9409094 (1994). D. Ueltschi, Moscow Math. J. 4 (2004) 511–522
The local lemma, the equivalence, the zeros of chains, $\lambda_c \leq 1/4$, Shearer's value	Abstract, § 1, § 4.1, § 5.2, § 8.4 checked	A. D. Scott, A. D. Sokal, J. Stat. Phys. 118 (2005) 1151–1261, arXiv:cond-mat/0309352
Original sources, lineage	Bibliographic data only	Kotecký–Preiss, CMP 103 (1986) 491. Dobrushin, AMS Transl. (2) 177 (1996) 59. Gruber–Kunz, CMP 22 (1971) 133. Penrose, in *Statistical Mechanics: Foundations and Applications* (T. A. Bak ed.), Benjamin (1967) 101–109. Mayer, J. Chem. Phys. 5 (1937) 67. Brydges–Federbush, J. Math. Phys. 19 (1978) 2064. Brydges–Kennedy, J. Stat. Phys. 48 (1987) 19. Battle–Federbush, Ann. Phys. 142 (1982) 95. Glimm–Jaffe–Spencer, in *Constructive Quantum Field Theory* (G. Velo, A. Wightman eds.), Lecture Notes in Physics 25, Springer (1973) (Part II: The cluster expansion), and Ann. Math. 100 (1974) 585. Erdős–Lovász, in *Infinite and Finite Sets* II, Colloq. Math. Soc. J. Bolyai 10, North-Holland (1975) 609–627. Shearer, Combinatorica 5 (1985) 241
Strong coupling on the lattice, the phases of $U(1)$, the quotation on the infrared	From the abstracts and the checks of this book's materials	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (text not obtained). Chatterjee, arXiv:1803.01950 § 4. Guth, PRD 21 (1980) 2291 (bibliographic data from Chatterjee's reference list). Fröhlich–Spencer, CMP 83 (1982) 411. Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325

Item	Kind	Source / tools
The exponential formula, spectral measures, Gaussian integration by parts, radius of convergence and singularities	Known	Standard facts (1-08, 1-10)
Tables, zeros, rings, formula (12), figures	Computed	Python on this machine: `hardcore1d.py`, `graphs.py`, `decouple.py`. The figures are plain JavaScript

Next chapter to read: 7-05 The sequence of renormalisation group steps (piling up the expansion into a sequence of scales).

The sequence of renormalisation group steps — integrating out one scale at a time, splitting into small and large fields as we go

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The idea of controlling the ultraviolet limit by repeating a transformation that coarsens the lattice one step at a time (blocking). Splitting the field at each step into "small fields" and "large fields". That gauge fields need averages of group elements and local gauges.

Prerequisite chapters — 7-01 (the ultraviolet-limit step) and 4-04 (the running of the coupling). We also use the polymers of 7-04, the b_0 of 4-06 § 02, and two-dimensional $U(1)$ from 7-02 (independence of the plaquette angles, the heat kernel). Plaquettes and the Wilson action are the language of 5-01, but what is needed is defined on the spot.

Said first — This explains the skeleton of the method and does not follow Bałaban's proofs. In four dimensions even the main theorem of this method carries a hypothesis, and it does not reach uniqueness of the continuum limit, infinite volume or the mass gap (§ 07, § 09). How the construction published on 2026-11-10 handles one step of renormalisation and large fields is read in Parts 9 and 11. The details for gauge fields are within the range of the introductions and the statements of theorems in the primary sources, and an exposition for scalar fields (Dimock). The computations concern two-dimensional $U(1)$, free fields and the one-loop running, not numbers for non-abelian theories in four dimensions.

The order of this chapter

1. Why a sequence of steps — there is no small number common to all scales, so the scales are integrated one step at a time (formula (1))
2. One step of the transformation: blocking — fixing the averages loosely and integrating the fine variables changes the form of the action in one step (Table 1)

3. For free fields one step can be written exactly — minimal configurations, and the locality of the Green's function produced by the constraint on averages alone (Table 2)
4. Small fields and large fields (a moving figure) — the region is split at each step, and large fields carry a small factor (Figure 1, Table 3)
5. What gauge fields add — cannot be rescaled, cannot be added and divided, the gauge is chosen locally (formula (8))
6. The running of the coupling and the number of steps (a moving figure) — in four dimensions the sum of corrections keeps growing with the number of steps (Figure 2, Table 4)
7. The form of the induction — keep estimates of the same form at each step, with constants independent of the step and the lattice spacing (formula (10))
8. Detour: the name "phase-space cells" — counting the degrees of freedom integrated at each step (formula (11))
9. Another road, and where it stops — construction with a continuum cutoff, and the scale where the coupling becomes $O(1)$

01

Why a sequence of steps

What we wanted to show in the ultraviolet-limit step was that expectation values are bounded independently of the lattice spacing a (formula (8) of 7-01). In four-dimensional Yang–Mills, by asymptotic freedom (4-06), the coupling g is small at fine scales and becomes $O(1)$ at coarse scales. Every scale has a small number, but there is no small number common to all scales. So the scales are integrated one step at a time.

$$a \;\to\; La \;\to\; L^2 a \;\to\; \cdots \;\to\; L^K a = a_0, \qquad K = \log_L \frac{a_0}{a} \quad (1)$$

The transformation with factor L is repeated K times, and at each step only the coupling of that scale is used as the small number (L is unrelated to the size of the box in 7-01, and a_0 is fixed). Writing ρ_k for the weight (effective density) after step k, $Z = \int \rho_0 = \int \rho_1 = \cdots = \int \rho_K$. The problem statement explains this transformation as "integrating out the large-momentum degrees of freedom and restoring the scale" (JW § 6.5), and the introduction of Bałaban 1985 says it follows the form proposed by K. Wilson. Known

Example: with $L = 2$, $a_0 = 1$ and $a = 2^{-20}$, $K = 20$. Each halving of a adds one step, so what must be shown are estimates **independent of** K.

02

One step of the transformation: blocking

In one step, one fixes the variables of the coarse lattice and integrates the fine variables so as to keep those values. Written for a scalar field, the average over a block $B(y)$ of side L points is (2), and Dimock, who expounds Bałaban's method, fixes the average loosely with a Gaussian weight ($\alpha > 0$ is arbitrary, N a constant that preserves the integral).

$$(Q\Phi)(y) = \frac{1}{L^d} \sum_{x \in B(y)} \Phi(x) \quad (2)$$

$$\tilde{\rho}_{k+1}(\Phi') = \frac{1}{N} \int \exp\Big(-\frac{\alpha}{2L^2} \big\| \Phi' - Q\Phi \big\|^2 \Big) \, \rho_k(\Phi) \, \mathrm{d}\Phi \quad (3)$$

Then the lattice is shrunk to unit spacing and the field rescaled, giving ρ_{k+1} (Dimock I (26)(27)). Known

Example: look at two-dimensional $U(1)$. An element of $U(1)$ can be written as $e^{i\theta}$ with the angle θ. A unit square p of the lattice is called a **plaquette**; the product of the variables on its four sides taken in order along the orientation is written $U(\partial p) = e^{i\theta_p}$, and θ_p is called the angle of the plaquette (5-01 § 03). Putting the weight $e^{\beta \cos \theta_p}$ on each plaquette is the **Wilson-type** action (β is the inverse lattice coupling; the larger it is, the weaker the coupling. A different quantity from the β function of § 06). In two dimensions with free boundary the plaquette angles are independent (7-02 § 02), and the angle of a 2×2 block is the sum of the four angles. For a sum of independent angles, the Fourier coefficients of the distribution are the products of the individual coefficients. Using the coefficient of one angle

$$c_n(\beta) = \frac{\int_{-\pi}^{\pi} e^{in\theta} \, e^{\beta \cos\theta} \, \mathrm{d}\theta}{\int_{-\pi}^{\pi} e^{\beta \cos\theta} \, \mathrm{d}\theta}$$

the coefficient of the angle of the block becomes $c_n(\beta)^4$ in one step. Setting $I_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} e^{\beta \cos\theta} \, \mathrm{d}\theta$, $c_n = I_n(\beta)/I_0(\beta)$, and I_n is a special function called the modified Bessel function. If the action were still of Wilson type after one step, the β' fixed by c_1 should also match c_2.

Computed Table 1 One step ($L = 2$) of two-dimensional $U(1)$ with Wilson type. β' is the value fixed by $c_1(\beta') = c_1(\beta)^4$, and the mismatch is $c_2(\beta')/c_2(\beta)^4 - 1$.

β	2	10	50	100	400
β'	0.488	2.996	12.892	25.383	100.377

β	2	10	50	100	400
Mismatch	2.43	6.6e−2	2.8e−4	3.2e−5	4.8e−7

It does not match. The form of the action changes in one step, and new terms are born. Computed If the weight is chosen from the start so that its coefficients are $c_n = e^{-n^2/2\beta}$, then $c_n^4 = e^{-n^2/(2\cdot\beta/4)}$, and the form is kept exactly. This weight is the **heat kernel** on the circle (also called Villain type): the distribution on the circle obtained by adding up copies of the density of the normal distribution with variance $1/\beta$ shifted by 2π, that is, the normal distribution **wrapped** round the circle (7-02 § 04). In general one follows the main local terms and gathers the rest into corrections $E_k(X)$ for each lump X (Dimock I (105)(108); the form of the polymers of 7-04). Known

03

For free fields one step can be written exactly

If the weight is Gaussian (a free field), (3) can be written in closed form for any number of steps. The density after k steps is determined by the configuration ϕ_k that minimises the action of the fine field given the coarse values Φ (the **minimal configuration**, or background field) (Dimock I (45)–(52)).

$$\rho_k(\Phi) = Z_k\, e^{-S_k(\Phi,\,\phi_k)}, \qquad \phi_k = \alpha_k G_k Q_k^{\mathsf T}\Phi, \qquad G_k = \big(-\Delta + \bar\mu_k + \alpha_k Q_k^{\mathsf T} Q_k\big)^{-1} \tag{4}$$

Δ is the lattice Laplacian (not the Δ of the mass gap), Q_k the average over k steps, $Q_k^{\mathsf T}Q_k$ the projection onto functions constant on blocks, $\bar\mu_k$ the mass squared, and $\alpha_k = \alpha(1-L^{-2})/(1-L^{-2k})$ (Dimock I (32)) a constant gathering the weights of k steps. Fluctuations about the minimal configuration are Gaussian variables, and integrating them leaves a constant Z_k. The point is the locality of G_k. Even if $\bar\mu_k$ is close to 0, the term acting on the averages plays the role of a mass, and when x is in a cube Δ_y and the support of f in a cube $\Delta_{y'}$ (Δ_y is a large cube of side $M = L^m$, and $d(y,y')$ the distance measured in units of these cubes)

$$\big|(G_k f)(x)\big| \le C\, e^{-\gamma\, d(y,y')}\, \|f\|_\infty \tag{5}$$

holds (Dimock I (85), where $\gamma = \tfrac12\gamma_0$. The literature is Bałaban, CMP 89 (1983) and others). Known The massless $(-\Delta)^{-1}$ changes in two dimensions only with the logarithm of the distance.

Example: on a two-dimensional periodic lattice (32 blocks on a side), we lined up $(-\Delta + \ell^{-2}P)^{-1}$ (P is the projection onto the averages over blocks of side ℓ) by the distance r in block units.

Computed Table 2 $G(r)/G(0)$. The bottom row is γ from fitting $e^{-\gamma r}$ to the values for $r = 4$ to 12 (the decay rate per block). The rightmost column puts the mass ℓ^{-2} everywhere.

r	$\ell = 2$	$\ell = 4$	$\ell = 8$	**Mass ($\ell = 8$)**
1	0.181	0.135	0.111	0.111
4	4.29e−3	3.12e−3	2.53e−3	2.94e−3
8	5.10e−5	3.47e−5	2.76e−5	3.87e−5
Decay rate	1.089	1.106	1.111	1.064

The decay rate is about 1.1 per block and hardly depends on ℓ. Although the constraint acts only on the block averages, it is about the same as putting a mass everywhere (1.055–1.064 for $\ell = 2, 4, 8$ by the same fit), and if anything slightly faster. This locality, independent of the step, is why the cluster expansion (7-04) can be used at each step. Computed

04

Small fields and large fields

Analysis in perturbative form can only be used when the field is small. The distinctive feature of Bałaban's method is that at each step the lattice is split into the **small-field region** Ω and its outside, and the sum is taken over all ways of splitting (Dimock II § 1).

$$1 = \sum_{\Omega} \zeta(\Omega^c, \Phi)\, \chi(\Omega, \Phi) \quad (6)$$

χ is the indicator function that the small-field condition holds on Ω, and ζ that the condition fails somewhere in each cube of Ω^c. In the words of the introduction of Dimock I (§ 1.1), 'The large field region is treated crudely but makes a small contribution due to the fundamental stability of the interaction'. The next step splits within the small-field region of the previous step, so the regions shrink, $\Omega_1 \supset \Omega_2 \supset \cdots$. Known

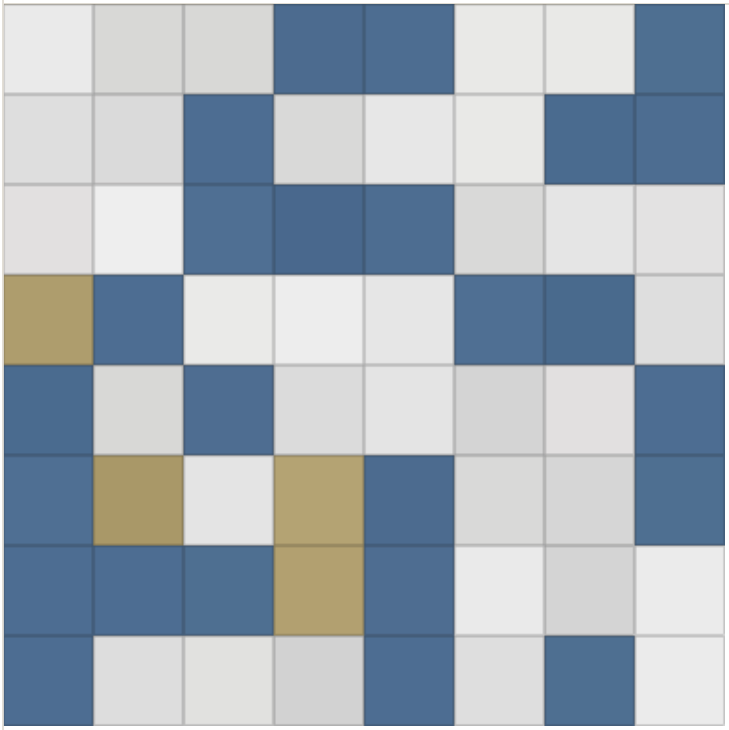
For gauge fields the condition is put on plaquettes. At the first step (bare coupling g_0), Bałaban 1985 imposes on each plaquette p

$$|U(\partial p) - 1| < \varepsilon_1, \qquad \varepsilon_1 = g_0\, p(g_0), \qquad p(g) = c\,(1 + \log g^{-1})^{p_0}, \quad p_0 > 2 \quad (7)$$

(c is a sufficiently large constant), and takes as the small-field region the union of blocks sufficiently far from the plaquettes where it fails (ibid. (4)(7) on pp. 256–257; notation adapted to this book). Known

We estimate the sizes as a rough guide. The weight is roughly $e^{-|U(\partial p)-1|^{2}/g^{2}}$, so the typical size is about g, and the threshold is $p(g)$ times that. Large fields then carry, as a rough guide, a factor of order $e^{-O(1)\,p(g)^{2}}$. This is smaller than any power of g. The exposition for scalar fields says that a factor $e^{-O(1)\,p_0^{2}}$ appears for each large-field cube (Dimock II § 1; the $p_0 = (-\log \lambda_0)^{p}$ there is the size of the threshold, a different symbol from the exponent p_0 in (7)). Known **Example:** with $c = 1$, $p_0 = 3$ and $g = 0.1$, $p \approx 36.0$. Taking $O(1)$ to be 1, $e^{-p^{2}} \approx e^{-1298}$, which is in a different league from $g^{10} = e^{-23}$. Computed

Figure 1 As the step k advances, how the size of the block angles (shade of the background), large fields (■ dark fill) and the parts that left the small-field region at earlier steps (■ light fill) grow



Step 2 (4×4), β_k = 4.000, ε = 0.50, large fields 25/64, outside the region 29/64

Computed The numbers in this figure were computed on this machine (a 32×32 sample of standard normals embedded at about 0.1 resolution; the angles $\theta_p = z_p/\sqrt{\beta}$ and the block sums are computed on the spot).

Computed Table 3 The sample of Figure 1 ($\beta = 64$, $\varepsilon = 0.5$; step 0 is 0/1024). "Exact" is the probability under the wrapped normal distribution with variance $4^{k}/\beta$.

Step k	β_k	Large fields (sample)	Same (exact)	Outside the region
1	16	13/256	0.043	13/256
2	4	25/64	0.312	29/64
3	1	13/16	0.613	15/16

The model of the figure is the heat-kernel type of § 02, and the effective coupling at step k is exactly $\beta_k = \beta/4^{k}$. In two dimensions the coupling strengthens at each step, and with the threshold fixed, large fields fill everything within a few steps. The place to

stop before the coupling becomes $O(1)$ is the a_0 of (1). In four dimensions the strengthening is much slower (§ 06).

05

What gauge fields add

Gauge fields add three things. Known **(i) Group elements cannot be rescaled.** Bałaban 1985 writes 'we use improvement of regularity properties of typical gauge fields instead of non-existing scalings'. **(ii) Group elements cannot be added and divided.** The variables on coarse edges are made from those on fine edges, but the average of group elements leaves the group. How to make the average is itself the subject of a whole paper (Bałaban, CMP 98 (1985) 17–51, bibliographic data only). For charged fields, one averages gauge-covariantly by multiplying by phases along paths (formula (20) of Dimock's paper on QED). **(iii) The gauge is chosen locally, scale by scale.** To write $U = e^{igA}$ and expand, the gauge must be fixed, but no globally continuous choice exists (Singer 1978, 3-04, 4-08). On the lattice the group is compact, so the definition of Z needs no gauge fixing, and only local choices are needed. According to the problem statement, the gauge is chosen for each place in space-time and each scale ('phase cells'), and is rechosen with the steps from an ultraviolet gauge to an infrared gauge (JW § 6.5; describing the work in three dimensions).

The reason it is local is elementary. Define the **operator norm** of a matrix A as the factor by which it stretches vectors the most, $\|A\| = \max_{|v|=1} |Av|$. A unitary matrix does not change lengths, so $\|U\| = 1$. For unitary matrices U, V, W

$$\|UV - 1\| \le \|U - 1\| + \|V - 1\|, \qquad \|WUW^{-1} - 1\| = \|U - 1\| \quad (8)$$

(the former from $UV - 1 = U(V - 1) + (U - 1)$ and $\|U(V-1)\| \le \|U\|\,\|V-1\| = \|V - 1\|$, the latter because neither W nor W^{-1} changes lengths). Paper Under a gauge transformation the plaquette variable changes to the form $WU(\partial p)W^{-1}$, so by the latter, condition (7) measured in the operator norm does not change under gauge transformations. On a two-dimensional block of side ℓ, setting the vertical edges and the bottom horizontal edges to 1 (complete axial gauge), a horizontal edge at height y is the product of the inverses of the variables of the y plaquettes directly below it, so repeated use of (8) gives $\|U_b - 1\| < \ell\,\varepsilon_1$. Paper **Example:** for $\varepsilon_1 = 0.05$, $\ell = 8$ gives within 0.4. $\ell = 64$ gives 3.2, which says nothing since always $\|U - 1\| \le 2$. The gauge is effective over a range of about a block.

06

The running of the coupling and the number of steps

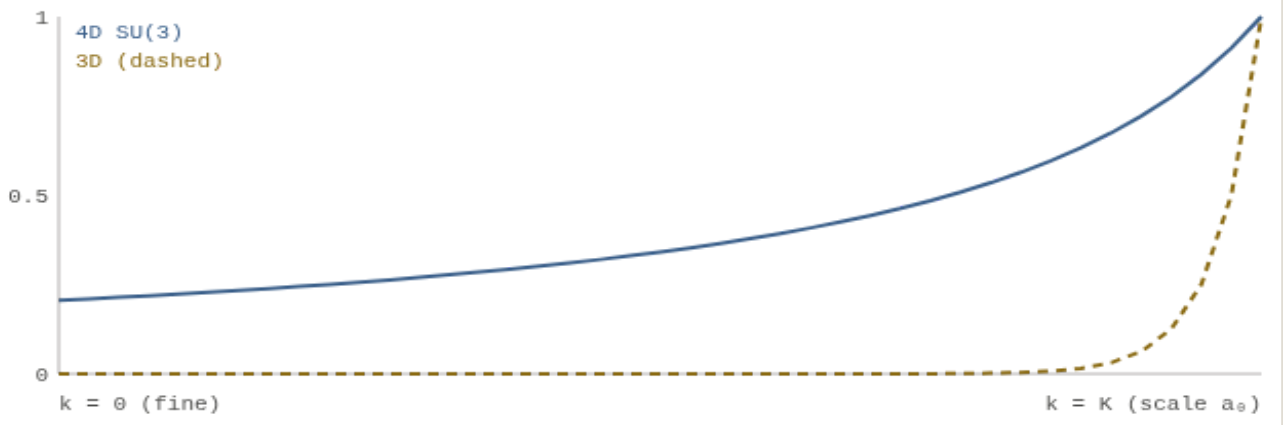
The main thing that changes from step to step is the effective coupling g_k. In four dimensions, cutting into steps the one-loop running ($1/g^2(\mu) = 1/g^2(\mu_0) + 2b_0 \ln(\mu/\mu_0$), obtained by integrating the β function $\mu\, \mathrm{d}g/\mathrm{d}\mu = -b_0 g^3 + O(g^5)$, $b_0 = 11N/48\pi^2$ of 4-06 § 01; the same formula from the physics side is formula (2) of 17-01) gives

$$\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - 2b_0 \ln L \quad (9)$$

and it strengthens by a fixed amount at each coarsening. In three dimensions g^2 has the dimension of mass, and the dimensionless coupling at step k, $g_k^2 = g^2 L^k a$, is geometrically small on the fine side. Known

From here on this is a rough-guide argument using only formula (9); it is not Bałaban's estimate itself, and not a claim at the level of a theorem. The difference shows up in the sum. If each step produces a correction of order g_k^2, then in three dimensions $\sum_k g_k^2$ is finite, but in four dimensions $g_k^2 \approx 1/(2b_0 \ln L \cdot (K-k))$ and the sum keeps growing like the logarithm of the number of steps, so it cannot be piled up as an error. What this rough guide shows is that corrections of that size must be renormalised into the running of the coupling itself and followed. If the remainder is of order g_k^3, the sum stays bounded. The title of the paper extending to four dimensions (Bałaban, CMP 109 (1987); from the reference list of JW) contains 'a coupling constant renormalization in 4D'. Known

Figure 2 Moving the number of steps K (a = 2^−K): how the sequence of effective couplings g_k² (g² = 1 at the right end) and its sums change. In three dimensions g_k² = 2^(k−K)



K = 40, four dimensions: g_0^2 = 0.206, Σg² = 16.99, Σg³ = 11.88, three dimensions: Σg² = 2.000

Computed The numbers in this figure were computed on this machine (partial sums of formula (9); they agree with Table 4).

Computed Table 4 $SU(3)$, $L = 2$, $g_K^2 = 1$ at the right end. g_0^2 is the value at the finest step.

K	g_0^2	$\sum g_k^2$ (4D)	$\sum g_k^3$ (4D)	$\sum g_k^2$ (3D)
10	0.509	7.76	6.63	1.999
40	0.206	16.99	11.88	2.000
160	0.061	29.54	16.12	2.000
320	0.031	36.38	17.56	2.000

Example: to go from $g^2 = 1$ at the right end to $g^2 = 0.1$ at the finest step takes about 93 steps for $SU(3)$ ($a \approx 9 \times 10^{-29}$) and about 140 steps for $SU(2)$. $\sum g_k^2$ grows by about 7.2 each time K doubles, while $\sum g_k^3$ settles at about 21.2 and the three-dimensional $\sum g_k^2$ at 2. Computed

07

The form of the induction: what is assumed and what is shown at each step

The proof is an induction on the step k. The hypothesis is that "ρ_k satisfies a fixed form and estimates". The form consists of the main action on the small-field region, exponentially small corrections $E_k(X)$ localised on lumps X, and the sum over large-field regions (Dimock II § 1). One shows that the same form and estimates hold after one step (3), and the key is that the constants do not depend on k or a. The estimate of the main theorem of the three-dimensional paper has the following form (Bałaban 1985 (5); notation adapted to this book).

$$\chi(U)\,\exp\Big(-\frac{1}{g_k^2}A\big(U_k(U)\big)-C\,|T_k|\Big)\;\le\;\rho_k(U)\;\le\;\exp\big(C\,|T_k|\big) \tag{10}$$

A is the Wilson action, $U_k(U)$ the minimal configuration, χ the indicator function of the small-field condition, $|T_k|$ the number of points of the lattice at step k, and C depends neither on a nor on k. Integrating gives upper and lower bounds on the partition function uniform in a (ibid. (6)). In four dimensions, the 1989 paper states ultraviolet stability in four dimensions as its main theorem (the Conditional Ultraviolet Stability Theorem; Theorem 1 of Bałaban 1989) under the hypothesis 'If the sequence of the effective coupling constants is contained in an interval]0, γ] with a sufficiently small positive γ'. Known

Example: for three-dimensional ϕ^4, Dimock's ultraviolet stability theorem (Dimock's Theorem 1) is of the same type: with the counterterms chosen, $e^{-c\,\mathrm{Vol}} \le Z/Z_{\mathrm{free}} \le e^{c\,\mathrm{Vol}}$ holds independently of the ultraviolet cutoff and the volume (7-03 § 06).

Seam The type of the estimate, and what it lets us say

What follows from estimates of type (10) goes only as far as uniform boundedness of the partition function and the effective action. The 1989 paper says in its

introduction that the proof of the basic inequality that removes the hypothesis (Theorem 2 of Bałaban 1989, by a second-order perturbative calculation) is unpublished, and leaves the analysis of expectation values such as Wilson loops as future work. MRS 1993 comments that from this the ultraviolet limit of smoothed Wilson loops should follow at the level of subsequences, but not necessarily uniquely. The problem statement does not accept weak existence by compactness alone as a solution, unless the properties of the limit are shown separately (JW footnote 2). Three dimensions in 7-06, four dimensions in 7-07.

08

Detour: the name "phase-space cells"

A section you may skip.

This method is also called "phase cell renormalization" (Chatterjee 2018 § 6). Known We explain what the name means in the words of this chapter. The fluctuations at step k are localised in space to blocks of side $L^k a$, and in momentum to about $1/(L^k a)$. So one counts roughly one degree of freedom for each box in which the product of the widths in position and momentum is $O(1)$: a phase-space cell. The content is the exponential localisation of (5). For three-dimensional scalar fields, the same idea appeared as the localisation into phase-space cells of Glimm–Jaffe (1973) (7-03 § 06).

$$n_k = L^{d(K-k)}\bigl(1 - L^{-d}\bigr), \qquad \sum_{k=0}^{K-1} n_k = L^{dK} - 1 \tag{11}$$

n_k is the number of variables integrated at step k per unit volume. **Example:** in four dimensions with $L = 2$ and $K = 20$ there are $2^{80} \approx 1.2 \times 10^{24}$ points, and the first step integrates 15/16 of them. Computed The same words also appear in 7-04 § 10.

09

Another road, and where this method stops

Magnen–Rivasseau–Sénéor (MRS 1993) treated four-dimensional $SU(2)$ with a continuum momentum cutoff (with an infrared cutoff, in the trivial topological sector). They use axial gauge for large fields and a background-field-dependent gauge for small fields, and restore with counterterms the gauge invariance broken by the cutoff. The introduction states explicitly that the paper does not give a detailed proof ('we do not provide a detailed proof'). We saw the introduction's comment on Bałaban's results in the Seam of § 07. Known The advantage of starting from the lattice is that gauge invariance and reflection positivity are kept (JW § 6.5, 5-03). In exchange, the

task remains of restoring the broken rotation symmetry in the limit (the axiom step of 7-01).

Where it stops is the right end of (1): at scales longer than a_0 the expansion has no foothold. MRS give as their reason for not removing the infrared cutoff 'this would lead to large values of the coupling constant, and presumably to so-called non-perturbative effects corresponding to confinement', and the problem statement says that a gap uniform in the volume needs new ideas (JW § 6.5). **Example:** increasing K in Figure 2 does not move $g_K^2 = 1$ at the right end. A way of adding steps beyond the right end is not among the tools of this chapter. Continued in 7-10 and 7-11.

What this chapter can say

	Content
Said	The forms of formulas (3)–(7) and (10) (from Dimock's exposition and the statements of the theorems of Bałaban 1985). The hypothesis of the 1989 main theorem (from the introduction) Known
Said	The values in Tables 1–4 and the figures (two-dimensional models and the arithmetic of formula (9)) Computed
Said	Inequality (8) and the estimate of edges in the two-dimensional complete axial gauge (proved on the spot) Paper
Not said	The content of the averages, the choice of gauge and the handling of large fields for gauge fields (the texts of many papers in the series not obtained)
Not said	The estimate of the large-field factor $e^{-O(1)p(g)^2}$ in § 04 and the argument on the sum of corrections in § 06 (both rough guides, not at the level of theorems)
Not said	Completion of the proof in four dimensions by Bałaban's method (the main theorem carries a hypothesis; 7-07). That Table 4 is the actual sequence of effective couplings. Infinite volume and the gap

Sources and reproduction

Item	Kind	Source / tools
The transformation, gauges, reflection positivity, footnote 2	Text checked	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) § 6.5
The main theorem in three dimensions, (7)(10), Wilson's renormalisation group	Text pp. 255–257 checked	T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275

Item	Kind	Source / tools
The hypothesis of the main theorem in four dimensions, the unpublished basic inequality (Theorem 2)	Abstract and introduction checked	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392
Other papers of Bałaban's series (averages, gauge fixing, four dimensions I, II and others)	Bibliographic data only	CMP 89 (1983); 95, 96 (1984); 98, 99 (two papers), 102 (277–309) (1985); 109 (1987) 249–301; 116, 119 (1988); 122 (1989) 175–202 (cross-checked with the reference lists of Dimock I and Chatterjee)
(2)–(6), the form of the induction, the theorem for ϕ^4_3, averages for charged fields	I § 1–3, II § 1, QED § 2.3 checked; III from the abstract	J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705; Quantum electrodynamics on the 3-torus I, arXiv:math-ph/0210020
The continuum cutoff, the comment on Bałaban, the reservation about the infrared, the explicit statement that there is no detailed proof	pp. 325–327 checked	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383
The name phase cell renormalization	§ 6 checked	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950
The impossibility of global gauge fixing	Bibliographic data only	I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12
Localisation into phase-space cells for three-dimensional scalar fields	Bibliographic data only (attribution from 7-03 § 06)	J. Glimm, A. Jaffe, Fortschr. Phys. 21 (1973) 327–376
Tables 1–4, the sample of the figure, the numbers in § 04 and § 08	Computed	Python (numpy, scipy) on this machine: `wilson_step.py`, `greens.py`, `villain2d.py`, `running.py`, `numbers.py` (the numbers in § 04 and § 08). The figures are canvas and plain JavaScript

Next chapter to read: 7-06 Ultraviolet stability of three-dimensional Yang–Mills (the place where this method was first used: what was shown, and what is not yet constructed).

Ultraviolet stability of three-dimensional Yang–Mills — the lattice estimates became uniform, but there is no measure yet

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That in three dimensions the coupling constant has the dimension of an inverse length, and the diagrams that can diverge are finitely many. What the theorem of Bałaban (1985) estimated uniformly on the lattice. Why there is still no continuum measure and no observables.

Prerequisite chapters — 7-05 The sequence of renormalisation group steps. We also refer to 7-01, 5-01 and 4-04.

Said first — This introduces known results and follows the proofs only in outline. The numbers in the tables and figures are computations for free (Gaussian) fields and for the counting of steps, and show nothing about the measure of three-dimensional Yang–Mills. The non-abelian three-dimensional continuum measure has not been constructed, even in finite volume.

The order of this chapter

1. Three dimensions in between — two dimensions is constructed, for four a construction in the form of correlation functions was published in 2026-11, three has reached uniform ultraviolet estimates
2. The coupling constant has a length — the coupling seen on the lattice is $g^2 a$, weaker the finer it is
3. Counting divergences — the types that can diverge are finitely many, and one constant is the only counterterm needed. A table of computations
4. The theorem of Bałaban 1985 — upper and lower bounds on the effective densities independent of the lattice spacing. A moving figure
5. The outline of the proof — small fields and large fields, local axial gauges, the one place semisimplicity is used

6. What the theorem does not say — the distance from uniform estimates to limits, observables and axioms
7. The wall of observables: Wilson loops in three dimensions — line integrals collapse, but with smoothed fields a value remains. A table of computations, a moving figure
8. Another road: stochastic quantisation and the state space — the road approaching the measure from the equation, and the state space made from smoothed fields
9. Detour: lattice numerics in 2+1 dimensions — numerically there are glueball masses and a string tension (at the level of physics)
10. Where three dimensions stands — what has been reached at each stage, and how it differs from four dimensions

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01

Three dimensions in between

Pure Yang–Mills in two dimensions is solved exactly on the lattice, and the continuum measure has been constructed (7-02). In four dimensions, in the line of work of Bałaban and MRS, the ultraviolet side had come as far as "the level of control" in finite volume, and was unfinished (7-07). The committee's assessment is that the construction published on 2026-11-10 built a theory satisfying the Wightman axioms in the form of correlation functions of local observables (6-01). A continuum probability measure has not been built (6-04). Three dimensions lies in between, and corresponds to 2+1-dimensional space-time.

The survey says that two and three dimensions are thought to be 'substantially simpler' because they are super-renormalisable (only finitely many diagrams can diverge; § 03), and then writes 'a construction of the 3D YM measure and a description of its gauge-invariant observables, even on T3, remains open' (Chevyrev 2022 § 1.1). Known

To say it first in one sentence: what has been shown in three dimensions is **uniform estimates** on the lattice, and what remains is turning the estimates into a **limit** and deciding what to measure in the limit (**observables**). The place is the ultraviolet-limit step of the six steps. Saying "it has been done in two and three dimensions" is accurate for scalar fields (7-03), and for abelian groups too the continuum limit has been shown (Table 3). For the non-abelian case, "three dimensions has reached uniform ultraviolet estimates" is accurate.

02

The coupling constant has a length

In this chapter the coupling is put in front of the action, and A does not contain g. A is a 1-form with values in anti-Hermitian matrices (matrices satisfying $X^\dagger = -X$; the Lie algebra of $SU(N)$ has this form, 1-03), $F_A = dA + A \wedge A$ and $|F_A|^2 = -\sum_{\mu,\nu} \operatorname{tr} F_{\mu\nu}^2 \geq 0$ (3-03). The relation to the Hermitian A_μ of the physics chapters (3-02, written with the coupling absorbed) is $A = -iA_\mu dx^\mu$, $F_A = -iF$.

The action $S(A) = \frac{1}{2g^2} \int |F_A|^2 \, d^d x$ is a dimensionless number. A has the dimension of length to the power -1, the curvature to the power -2, and the volume element to the power d, so the dimension of the coupling constant, and the dimensionless coupling seen on a lattice of spacing a, are fixed.

$$[g^2] = (\text{length})^{d-4}, \qquad g_0^2 = g^2 a^{4-d} = g^2 a \quad (d = 3) \ (1)$$

The formula on the right is the starting point of Bałaban (1985). $g_0^2 = g^2 a$ is a relation fixed by dimensions, and the constant of proportionality depends on the normalisation of the action. In this book's conventions (5-01, appendix A-01), writing the Wilson action of $SU(N)$ as $\beta \sum_p \left(1 - \frac{1}{N} \operatorname{Re} \operatorname{tr} U_p\right)$ gives $\beta = 2N/g_0^2$. The normalisation of tr is not necessarily the same as in the paper's action, so it may differ by a constant factor from the paper's g_0. In either convention, the three-dimensional $\beta = 2N/(g^2 a)$ is proportional to $1/a$. Known

Example: in units with $g^2 = 1$, $a = 2^{-20}$ gives $g_0^2 \approx 10^{-6}$. The finer the lattice, the weaker the coupling becomes by itself, without waiting for the logarithmic slowness of asymptotic freedom (4-06). At the length $1/g^2$ the coupling is of order 1, and since this is the only length scale, if the theory exists and has a mass, it is a constant multiple of g^2 (for four dimensions see 4-07). Physics

For how this book uses this, see 15-01 § 01 (in four dimensions the sum of losses diverges logarithmically).

03

Counting divergences

We count the diagrams that can diverge in perturbation theory (4-04; for scalar fields 7-03). In perturbation theory one substitutes $A \to gA$. The action takes the form $\frac{1}{2} \int |dA|^2$ with a three-point vertex carrying g and a four-point vertex carrying g^2. A one-particle-irreducible diagram (one that does not fall into two pieces when any single line is cut) with E external gluon lines and L loops carries g^{E-2+2L}, so subtracting the dimension of the power of g from the dimension of momentum,

$$\omega(E, L) = d - E + (4 - d)(1 - L) \quad (2)$$

is the superficial degree of divergence. In four dimensions $\omega = 4 - E$, independent of L, and divergences appear at every order. In three dimensions $\omega = 4 - E - L$, which drops by 1 with each additional loop.

Computed Table 1 $\omega(E, L)$ in three dimensions. Bold is $\omega \geq 0$. $E = 1$ vanishes by colour symmetry. External lines are gluons only (types with external ghost lines are omitted).

E\L	1	2	3	4	5
0 (vacuum)	**3**	**2**	**1**	**0**	−1
2	**1**	**0**	−1	−2	−3
3	**0**	−1	−2	−3	−4
4	−1	−2	−3	−4	−5

With external lines restricted to gluons, there are seven types with $\omega \geq 0$ in three dimensions (three in two dimensions, infinitely many in four).

One constant is the only counterterm

Of the seven, the divergences of the three other than the vacuum (one and two loops with $E = 2$ and one loop with $E = 3$) need no counterterm under a regularisation that keeps gauge invariance. The reason lies in the form of the counterterms. A counterterm is a gauge-invariant local term (the integral of a polynomial in the field and its derivatives at one point) whose coefficient can be written as a power of g^2. In three dimensions g^2 has the dimension of length to the power -1, so with the conventions of § 02 the terms are restricted to dimension 3 or less, and the only such term is a constant. The candidate term linear in F, of dimension 2, would have to be a linear form on the Lie algebra invariant under the adjoint action ($X \mapsto hXh^{-1}$), and for a semisimple group there is only 0 (in matrices, $\operatorname{tr} F = 0$). The candidate of dimension 3, the Chern–Simons term, changes sign under reversal of orientation (parity), so it does not arise from an action invariant under reversal. Known

This argument belongs to continuum perturbation theory. On the lattice rotation symmetry is broken (only the rotations and reflections of the cubic lattice remain), and it cannot be applied as it is. What corresponds to it on the lattice is Bałaban's "one place semisimplicity is used", seen in § 05. Bałaban's starting density also contains, besides the Wilson action, only constants including 'vacuum energy renormalization counterterms'. A vacuum diagram with L loops is of order $g_0^{2(L-1)}$, and the last one, $L = 4$, corresponds to g_0^6, which matches the paper's expansion up to order 6 in g_0 (a cross-check made in this chapter).

04

The theorem of Bałaban 1985

The statement of the theorem

A torus $\mathbb{T}$ of fixed side is approximated by lattices of spacing a (Bałaban's paper writes the lattice spacing as ε). The starting density is

$$\rho_0(U) = \exp\big[- A(U)/g_0^2 - E_0\big]$$

(formula (1) of the paper), where A is the Wilson action and E_0 is a constant including the normalisation and the vacuum-energy counterterms (E in the paper; we write E_0 to distinguish it from the number of external lines in § 03). This corresponds to the single constant being enough as counterterm in § 03. The renormalisation transformation T (blocking that coarsens by a factor L; 7-05) produces $\rho_{k+1} = T\rho_k$, and one stops at $L^K a = a_0$ (a_0 a positive constant fixed by g alone). Ultraviolet stability means that the upper and lower bounds on the last density do not depend on a, and formula (3) of the paper is

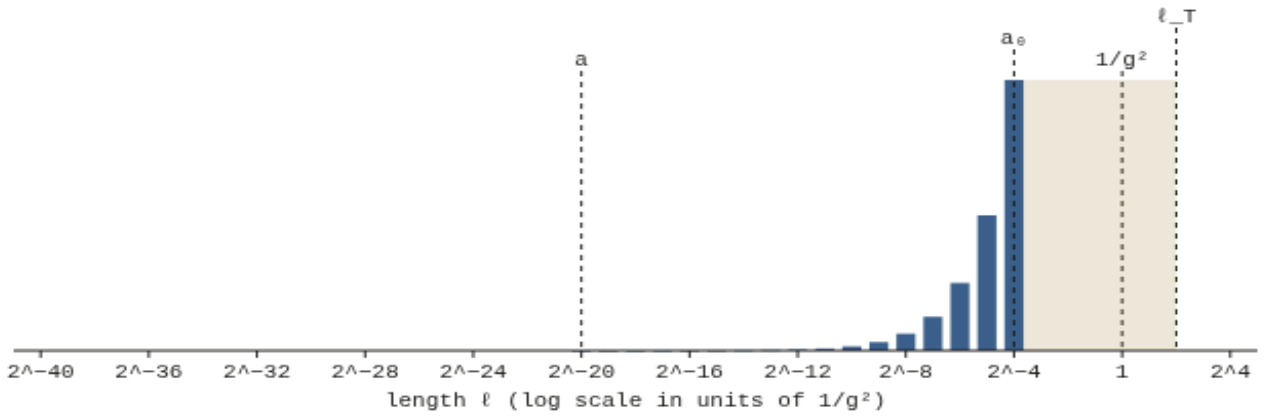
$$\chi(U)\, e^{-O(1)|\mathbb{T}|} \;\le\; \rho_K(U) \;\le\; e^{O(1)|\mathbb{T}|} \quad \text{(3)}$$

Here $U(\partial p)$ is the product of the link variables once round the boundary of the plaquette p, χ is the function (indicator function) that is 1 when $|U(\partial p) - 1|$ is small on every plaquette of the last coarse lattice (spacing a_0) and 0 otherwise, $|\mathbb{T}|$ is the volume, and $O(1)$ is a constant fixed by g and a_0 alone. Known

The paper notes that this $O(1)$ goes to ∞ as $g \to 0$, rewrites it in a form with the terms of the action divergent in g written out, and extends it to all steps k. That inequality (5) is the main theorem (its form is formula (10) of 7-05): 'The lattice approximations of the three-dimensional pure Yang-Mills theory with a semi-simple compact group Lie G are ultraviolet stable in the sense that the sequence of densities ρk ... satisfies the bounds (5).' The corollary is 'uniform in ε bounds for the partition function'. The author cautions that 'the proof is very sketchy'. Known

The sum over steps

Figure 1 Refining the lattice spacing a adds steps (blue bars = the coupling g_k^2 seen at that step) at the left end. The yellow band is the scales left outside the steps (from a_0 to the side of the torus $\ell_T = 4/g^2$)



a = 2^−20/g², number of steps K = 16 g_0^2 = 9.5e-7 Σ g_k^2 = 0.124999 (below 0.125), points of the fine lattice ≈ 10^19.9

Computed The numbers in this figure were computed on this machine (in units of $g^2 = 1$; $L = 2$ and $g^2 a_0 = 1/16$ are values chosen for illustration).

The coupling seen on the lattice at step k is $g_k^2 = g^2 L^k a$ (the paper's $g_k = g(L^k a)^{1/2}$), shrinking by a factor $1/L$ at each step towards the fine side. A new step is only added at the left end with a smaller coupling than before, and the sum does not exceed $\frac{L}{L-1} g^2 a_0$ (for $g^2 a = 2^{-8}, 2^{-20}, 2^{-40}$ the steps are 4, 16, 36 and the sums 0.121094, 0.124999, 0.125000). Computed

In four dimensions the same sum grows like the logarithm of the number of steps (§ 06 of 7-05; about 7.2 more each time the number of steps doubles). Above a_0 there is a finite-dimensional integral on a lattice whose number of points does not depend on a (262144 points in the figure), and since it is over a compact group it is bounded. But if the torus is larger than $1/g^2$, scales where the coupling exceeds 1 enter there. Computed

05

The outline of the proof

The proof stands on a group of earlier papers (propagators, averaging, spaces of regular configurations, variational problems) and follows the author's own method for the three-dimensional Higgs model. Known

Small fields and large fields. For each plaquette, 1 is partitioned according to whether $|U(\partial p) - 1|$ is small or large (1 is written as the sum of two functions, "small" and "large", and each case is estimated). A large plaquette makes $A(U)/g_0^2$ large, so in the end a small factor $\exp(-\frac{1}{4} p^2(g_j))$ appears for each large-field plaquette. $g_j = g(L^j a)^{1/2}$ is the coupling at step j, and $p(g)$ is the multiplier for the large-field threshold, which grows like a power of $\log g^{-1}$ (formula (7) of 7-05). This factor becomes small faster than any power of g_j.

Inside the small fields. In each block one takes an axial gauge (a gauge setting the link variables in a chosen direction to 1 within the block) by the Faddeev–Popov procedure (a method of fixing the gauge to one and multiplying the integral by a compensating factor; 4-08) (locally, since there is no global gauge fixing; 3-04), and splits into the configuration minimising the action (the background field) and fluctuations. The fluctuations become a Gaussian measure with exponentially decaying covariance plus local interactions, and since $g_0 = g a^{1/2}$ is small, an expansion to order 6 in g_0 and vacuum diagrams with up to 6 vertices are enough. The paper writes that this integral could properly be expressed by the cluster expansion (7-04), but it uses the simplifications given by super-renormalisability.

Two features. Lattice fields with values in a group cannot be rescaled, so instead of scale transformations one uses 'improvement of regularity properties of typical gauge fields'. Semisimplicity is used in just one place: from the fact that the only element of the Lie algebra invariant under the adjoint action is 0, one derives 'there are no relevant variables in the effective action' (there are no quantities in the effective action that grow as the steps advance and so need to be tracked). This is the lattice-side way of saying the "the only counterterm is a constant" of § 03 (this chapter's reading). The problem statement introduces this work as choosing the gauge again for each phase-space cell and each scale, from ultraviolet-oriented to infrared-oriented (JW § 6.5). Known

06

What the theorem does not say

Formula (3) says only that the effective densities and the partition function are uniformly bounded, and says nothing about the limit.

$$\sup_a |\log Z_a| < \infty \quad /\!\!\Longrightarrow \quad \lim_{a\to 0}\langle \mathcal{O}\rangle_a \text{ exists} \quad /\!\!\Longrightarrow \quad \text{axioms and gap} \ (4)$$

From estimates to a limit. From uniform estimates, compactness (the property that a convergent subsequence can be taken from a bounded sequence) gives the prospect of subsequential limits, but not uniqueness (there is also a usage that calls the existence of subsequences "ultraviolet stability"; Chatterjee 2018 § 6). Footnote 2 of the problem statement says that for the four-dimensional problem 'weak-existence (compactness)' alone is not accepted as a solution (unless the properties of the limit are shown separately). Known

What to measure. The problem statement writes 'These results need to be extended to the study of expectations of gauge-invariant functions of the fields' (JW § 6.5). In three dimensions this is the problem of defining observables (next section).

Beyond. The constant $O(1)|\mathbb{T}|$ is proportional to the volume and says nothing about infinite volume (the infinite-volume step). Reflection positivity holds on the lattice (Osterwalder–Seiler 1978, 5-03), but recovering rotation symmetry and the gap (the axiom step and the uniform-gap step) are untouched. The paper of Chevyrev–Shen, which treats the dynamics of stochastic quantisation in two dimensions, also writes in § 1.2 of its 2026 version 'the construction of the 3D YM measure in the continuum is open, even in finite volume'. Known

07

The wall of observables: Wilson loops in three dimensions

Line integrals are not defined

In two dimensions the Wilson loop $W(\gamma) = \operatorname{tr} \mathcal{P} \exp \oint_\gamma A$ is the natural observable, and the area law holds exactly (7-02). Three-dimensional Yang–Mills is thought to behave at short distances like the Gaussian free field (GFF; 1-09), whose covariance is of order $1/|x-y|$, so the variance of the line integral over a segment is $\int_0^1 \int_0^1 ds\, dt/|t-s| = \infty$ (Chevyrev 2022 § 3.1; the abstract of Cao–Chatterjee also says 'line integrals of the GFF do not exist in such dimensions'). Known

We check this in the simplest case. Making the U(1) free field (a Gaussian measure) on a lattice, with a square loop of side 1 and coupling $e^2 = 1$, independently of the gauge $\langle W \rangle = \exp(-\frac{1}{2} \operatorname{Var} \oint A)$, and

$$\operatorname{Var} \oint_\gamma A = \oint_\gamma \oint_\gamma G(x-y)\, dx \cdot dy, \qquad G_t(r) = \frac{\operatorname{erf}\left(r/\sqrt{8t}\right)}{4\pi r} \quad (d=3) \tag{5}$$

G is the inverse of the lattice Laplacian, and G_t is the continuum version of the covariance of the field smoothed by the heat flow $A \mapsto e^{t\Delta} A$ (the linearisation of the Yang–Mills flow). $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds$ is the error function, which approaches 1 when x is large. The smoothing width is about $\sqrt{8t}$, the length in the argument of erf; for two points farther apart than that, G_t is almost the same as $1/(4\pi r)$, and for two points closer together it is held to a finite value.

Computed Table 2 $\langle W \rangle$ for the square of side 1. t is the time of the heat flow. The bottom row is formula (5) integrated in the continuum.

a	Two dimensions	Three dimensions	Three dimensions t = 0.01	Four dimensions
1/4	0.6065	0.4837	0.6919	1.96e−1
1/16	0.6065	0.3150	0.7396	5.22e−4
1/128	0.6065	0.1631	0.7421	1.67e−28

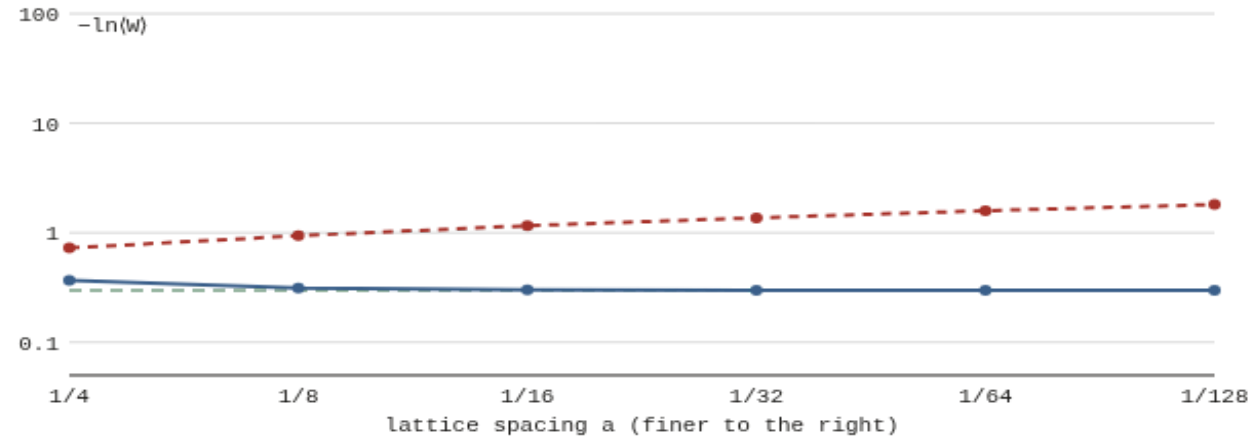
a	Two dimensions	Three dimensions	Three dimensions t = 0.01	Four dimensions
Continuum	$e^{-1/2}$	0	0.7421	0

Reading Table 2

In two dimensions, on every lattice $\mathrm{Var} = 1$ (the area) and $\langle W \rangle = e^{-1/2}$ comes out exactly. In three dimensions Var grows in proportion to $\ln(1/a)$, and $\langle W \rangle \to 0$. The slope of Var against $\ln(1/a)$ goes from 0.613 to 0.635 (approaching $2/\pi \approx 0.637$). The increment each time a is halved goes from 0.425 to 0.440, approaching $(2/\pi)\ln 2 \approx 0.441$. In four dimensions $\mathrm{Var} \approx 1/a$. Computed

With smoothing, the lattice values converge to the continuum values in both three and four dimensions. This is the smallest example of the starting point of Cao–Chatterjee and CCHS, which define observables as "Wilson loops of the smoothed field". Computed

Figure 2 Moving the time of the heat flow t: the smoothed Wilson loop (blue) settles to the continuum value (green dashed) as the lattice is refined. The red dashed line is unsmoothed



d = 3 t = 0.01, a = 1/128: unsmoothed −ln⟨W⟩ = 1.81, smoothed value 0.298 (continuum 0.298)

Computed The numbers in this figure were computed on this machine (the same computation as Table 2).

08

Another road: stochastic quantisation and the state space

The road of solving an equation

Instead of making the measure directly, there is the road of solving a stochastic partial differential equation that should have it as its invariant measure (a measure whose distribution does not change as time passes) (7-09). Chandra–Chevyrev–Hairer–Shen (CCHS 2024), for three-dimensional Yang–Mills–Higgs, extended gauge equivalence to a non-linear metric space of distributions, constructed local solutions (solutions existing only for a short time) of the equation renormalised with regularity structures (the theory of solutions of singular stochastic partial differential equations, treated in

7-09), and showed 'there is a unique choice of renormalisation counterterms such that these solutions are gauge covariant in law'. Known

The Higgs field may be absent, and a statement restricted to pure Yang–Mills is in Chevyrev 2022 Remark 1.2. As a result, a Markov process on the orbit space (a stochastic process whose future behaviour is determined only by the present state) is defined, with the possibility of finite-time blow-up (the solution becoming infinite in finite time).

The state space

How the state space is made is the answer to the previous section. Line integrals cannot be used, so the deterministic Yang–Mills flow (with the DeTurck term; a term in the gauge direction added to put it in the form of a heat equation) is taken as "smoothing compatible with gauge", and gauge equivalence is defined for the fields after smoothing (Chevyrev 2022 § 3.1). The space of orbits is a Polish space (a space with a complete metric and a countable dense subset) in two dimensions, but in three dimensions it is only known to be 'completely Hausdorff (and separable)' (any two distinct points can be separated by a continuous function, and there is a countable dense subset) (ibid. Remark 1.5). In the non-abelian case the space of orbits is non-linear, and the difficulty begins with the state space (ibid. § 1.1, 3-04).

What remains is the invariant measure, and the survey writes that showing it exists uniquely would give a construction of the three-dimensional Yang–Mills measure in finite volume (ibid. § 4). Cao–Chatterjee (2024) showed that, assuming the theory behaves like the GFF at short distances, the theory can be realised on the unit torus as 'random distributional gauge orbits', and smoothed Wilson loops can be defined. This too is a result about the state space. Known

09

Detour: lattice numerics in 2+1 dimensions

A section you may skip.

On the physics side, three-dimensional Yang–Mills is thought to have confinement and a mass gap. Teper (arXiv 1998, Phys. Rev. D 1999) computed lattice theories with SU(2) to SU(5) in 2+1 dimensions by the Monte Carlo method (5-07), and extrapolated the ratios of glueball masses to the string tension to the continuum limit. According to the abstract, the ratios hardly depend on N, and the string tension does not go to 0 even for large N. Since $\beta = 2N/(g^2 a)$, the correlation length in lattice units $\xi/a = \beta/\left(2N\,(m/g^2)\right)$ only grows in proportion to β (exponentially in four dimensions; 5-05). Physics

The problem statement lists Feynman's paper in 2+1 dimensions (1981) among the literature on the view that the gap comes from the quartic term $(A \wedge A)^2$ (JW § 6.6). Three-dimensional U(1)

lattice gauge theory has been shown to confine static charges at every coupling with the lattice spacing kept fixed (Göpfert–Mack 1982; for four-dimensional U(1) see 5-06). Known Its continuum limit is the free electromagnetic field (Gross 1983), so this confinement is thought not to survive in the continuum limit. Physics

10

Where three dimensions stands

Known Table 3 What has been reached in three dimensions (stages as in 7-01; categories as in Table 2 of 7-01).

Stage	Non-abelian Yang–Mills	Abelian U(1) (for reference)
1 Regularisation	Closed	Closed
2 Finite-volume measure	Closed (lattice). Making it directly in the continuum is counted under "Partial" in the ultraviolet-limit step	Closed (lattice)
3 Ultraviolet	Partial: uniform lattice estimates (Bałaban 1985). The continuum measure is not constructed. Stochastic quantisation as far as the dynamics (CCHS 2024)	Continuum limit (Gross 1983, in a different kind of convergence; the limit is a free field). With Higgs, King 1986
4 Infinite volume	Not yet	Limit with Higgs (King 1986)
5 Axioms	Not yet	Not checked here
6 Gap	Not yet (present numerically. Physics)	Confinement at every coupling on the lattice (a fixed) (Göpfert–Mack 1982). This property is thought not to survive in the continuum limit (the ultraviolet-limit step) Physics

The U(1) column is for reference. What is written in the row for the uniform-gap step is confinement (the force between static charges not weakening with distance), and confinement and the mass gap are different properties (17-02 § 02).

The difference from four dimensions (7-07) is that what weakens the coupling in the ultraviolet is the dimension itself. What remains in common is turning estimates into limits, deciding the quantities to measure in the limit, and estimates uniform in the volume. In infinite volume, three dimensions too has the coupling of order 1 at the length $1/g^2$, and enters the same kind of difficulty as the infrared in four dimensions (this chapter's reading; 7-10).

What this chapter can say

	Content
Said	The coupling seen on the lattice is $g^2 a$. With external lines restricted to gluons, there are seven types that can diverge Computed . From the dimensions of gauge-invariant local terms, the only counterterm needed is one vacuum-energy constant Known
Said	The theorem of Bałaban (1985): upper and lower bounds on the effective densities uniform in the lattice spacing, and uniform estimates of the partition function (text checked) Known
Said	The Wilson loop of the U(1) free field collapses at the rate $\ln(1/a)$ in three dimensions and $1/a$ in four, and when smoothed by the heat flow converges to the continuum value (in the range $a \geq 1/128$) Computed
Said	The non-abelian three-dimensional continuum measure is not constructed, even in finite volume (Chevyrev 2022, Chevyrev–Shen) Known
Not said	The details of Bałaban's proof (the author himself calls it a sketch). Whether smoothed Wilson loops have a limit in an interacting theory (Table 2 is a free field)
Not said	The mass gap and confinement in three dimensions (the numerics are at the level of physics). Complete coverage of the literature from 2024 on

Sources and reproduction

Item	Kind	Source / tools
§ 02– § 05	Primary source (text checked)	T. Bałaban, Ultraviolet stability of three-dimensional lattice pure gauge field theories, Comm. Math. Phys. 102 (1985) 255–275
Quotations in § 05, § 06, § 09	Primary source (text checked)	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000)
Quotations in § 01, § 07, § 08	Survey (text checked)	I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (2022) § 1.1, Remark 1.2, Remark 1.5, § 3.1, § 4
§ 06, the abelian group in Table 3	Survey (text checked)	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (2018) § 4, § 6
The status of the continuum measure in § 06	Primary source (§ 1.2 text checked)	I. Chevyrev, H. Shen, Invariant measure and universality of the 2D Yang–Mills Langevin dynamic, Comm. Pure Appl. Math. (2026), arXiv:2302.12160 (v3, 2026)
§ 07, § 08, § 09	Checked in the abstracts	A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Stochastic quantisation of Yang–Mills–Higgs in 3D, Invent. Math. 237 (2024) 541–696, arXiv:2201.03487. S. Cao, S. Chatterjee, A state space for 3D Euclidean Yang–Mills theories, Comm. Math. Phys. 405 (2024), arXiv:2111.12813. M. Teper, Phys. Rev. D 59 (1999) 014512, arXiv:hep-lat/9804008

Item	Kind	Source / tools
Abelian groups, Feynman, reflection positivity	Bibliographic data only	L. Gross, CMP 92 (1983) 137–162. C. King, CMP 102 (1986) 649–677; 103 (1986) 323–349. M. Göpfert, G. Mack, CMP 82 (1982) 545–606 (this one checked as far as the abstract: Villain action, positive string tension at every coupling. INSPIRE. Text not read. The year is the Crossref publication year; INSPIRE gives 1981). R. P. Feynman, Nucl. Phys. B 188 (1981) 479–512. K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471
Tables, figures	Computed	Python on this machine: `powercount.py` (Table 1), `ladder.py` (the sum over steps in § 04, Figure 1), `wilson_free.py` (Table 2, Figure 2. Integrates the lattice Green's function with scipy from the Bessel representation of the heat kernel, reproducing Var = 1 in two dimensions to 1e−10. About 10 seconds). The figures are canvas and plain JavaScript

Next chapter to read: 7-07 Four dimensions (the four-dimensional version of the same method).

Four dimensions: Bałaban and Magnen–Rivasseau–Sénéor — the ultraviolet side has reached "the level of control" in finite volume, but in these two lines of work the construction is not complete

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That in four dimensions the coupling weakens only logarithmically, so the coupling itself must be renormalised. What Bałaban's 1989 main theorem assumes and what it concludes. What MRS 1993 did with a continuum cutoff. How far "the level of control" reaches, and what lies beyond it.

Prerequisite chapters — 7-05 The sequence of renormalisation group steps. We also refer to 7-06, the ultraviolet-limit step of 7-01 and 4-06.

Said first — This introduces known results. Bałaban's four-dimensional series is treated from bibliographic data only, apart from the abstract and introduction of the 1989 paper. MRS is read as far as the introduction and the section on the set-up. The numbers in the tables and figures are arithmetic of the running of the coupling. In the two lines of work in this chapter, the construction of four-dimensional Yang–Mills is not complete, even on a torus of finite volume. On 2026-11-10, a Lean formalisation of a construction with a different design was published, together with the committee's assessment report. It is read in Part 6 and Parts 8–14; this chapter records what had been reached before it (only the right-hand column of Table 4 is the published construction).

The order of this chapter

1. What changes in four dimensions — the coupling weakens only logarithmically
2. Bałaban's four-dimensional series — from the foundations to four dimensions, 13 papers over six years
3. Reading the 1989 main theorem — a theorem with a hypothesis
4. The hypothesis in numbers — a moving figure, a table

5. Three readings of "ultraviolet stability" — estimates, subsequences, uniqueness
6. MRS 1993: with a continuum cutoff — axial gauge and the Slavnov identities
7. Setting the two roads side by side — the lattice and the continuum cutoff, the first upper bound on the field
8. What "the level of control" means — a moving figure, a table
9. Detour: how to check a proof of several hundred pages

01

What changes in four dimensions

In three dimensions, refining the lattice weakened the coupling by itself (7-06 § 02). In four dimensions the coupling g has no dimension of length, and only asymptotic freedom weakens the coupling at fine scales (4-06). The coupling $g(\ell)$ at length scale ℓ obeys, for pure gauge $SU(N)$,

$$\ell \frac{dg}{d\ell} = b_0 g^3 + b_1 g^5 + \cdots , \qquad b_0 = \frac{11N}{48\pi^2}, \qquad b_1 = \frac{17N^2}{384\pi^4} \quad (1)$$

(the PDG formula (9.3) converted to the conventions of this book). Known Keeping only one loop,

$$\frac{1}{g^2(\ell)} = \frac{1}{g^2(\ell')} + 2b_0 \ln \frac{\ell'}{\ell} \quad (2)$$

and each halving of the scale increases $1/g^2$ by $2b_0 \ln 2$. **Example:** halving the lattice spacing from $g^2 = 0.2$, for $SU(3)$ $1/g^2$ goes from 5 to 5.0966 and g^2 decreases by 1.9%; for $SU(2)$ it decreases by 1.3%. In three dimensions $g_0^2 = g^2 a$ would halve. Computed

Because of this slowness, the sum of corrections of order g_k^2 at each step keeps growing like the logarithm of the number of steps, and cannot be piled up as an error (7-05 § 06). The corrections must be renormalised into the coupling itself, and the title of the first paper of the four-dimensional series contains 'a coupling constant renormalization in four dimensions'. Known

02

Bałaban's four-dimensional series

Bałaban's work is a series that piles four-dimensional papers (1987–89) on foundations shared with three dimensions (1983–85), and the survey calls it 'a long series of papers

spanning six years' (Chatterjee 2018 § 6). Known

Known Table 1 The order of the series (bibliographic data cross-checked with Chatterjee 2018, Dimock 2011 and Crossref).

Year	Journal CMP	Content (from the titles)	Checked
1983–85	89, 95, 96, 98, 99 (two papers), 102	Foundations: propagators, averaging, gauge fixing, background fields and so on	Bibliographic data only
1985	102, 255–275	Ultraviolet stability in three dimensions (7-06 § 04)	Text pp. 255–257
1987	109, 249–301	The effective action in the small-field approximation, and coupling constant renormalisation in four dimensions	Bibliographic data only
1988	116, 1–22; 119, 243–285	Same, II: cluster expansions; convergent renormalisation expansions	Bibliographic data only
1989	122, 175–202	Renormalisation of large fields I: the basic step of the R operation	Bibliographic data only
1989	122, 355–392	Renormalisation of large fields II: localisation, exponentiation and estimates	Abstract and introduction

The problem statement calls this extension 'a qualitative breakthrough', and summarises it as 'an analysis of asymptotic freedom to control the renormalization group flow as well as obtaining quantitative estimates on effects arising from large values of the gauge field' (JW § 6.5). Going by the titles, the 1987 paper handles small fields and the running of the coupling, and the two 1989 papers handle large fields (the Ω^c of 7-05 § 04). Known

For how this book uses this, see 15-02 § 01 (a map of the five stages of the series).

03 Reading the 1989 main theorem

This is the theorem in the introduction of the last paper (the Conditional Ultraviolet Stability Theorem; Theorem 1 of Bałaban 1989).

'Theorem 1. If the sequence of the effective coupling constants is contained in an interval]0, γ] with a sufficiently small positive γ, then the effective densities ρ_k have the form, and satisfy all the conditions and bounds, described in Sect. 2 [III].'

The sequence of effective couplings is the couplings g_k for each step $k = 0, \dots, K$ (from the fine steps to the unit lattice). "Form, conditions and estimates" refers to the form of the induction in 7-05 § 07. Extracting only the logical form,

$$\underbrace{0 < g_k \le \gamma \ \ (k = 0, \dots, K)}_{\text{hypothesis}} \implies \underbrace{\rho_k \text{ satisfies the inductive representation and estimates}}_{\text{Theorem 1}} \tag{3}$$

The author continues that the hypothesis can be removed because it follows from basic inequalities that come from Theorem 2 of an earlier paper, but 'The proof of

Theorem 2, which is based on second order perturbative calculations, is very awkward and long ... and has not been published yet, so we have Theorem 1 with the assumption' (Theorem 2 is the basic inequality for removing the hypothesis). As of 1989, ultraviolet stability in four dimensions is a **theorem with a hypothesis**. Within what we searched (the bibliographic data of the author's papers from 1989 on), no paper containing the proof of the basic inequality (Theorem 2) was found. Known

From the theorem follow 'the ultraviolet stability bounds of the same type as for superrenormalizable models in [16]', with constants $E_{\pm}$ 'independent of k, T_{η}, U_{k}'. The analysis of expectation values of loop variables and the like is left as future work. Known

This means estimates of the same type as for super-renormalisable models (reference [16] of the original paper). Three-dimensional Yang–Mills is super-renormalisable (formula (7) of 7-01), so the form can be read as being of the same type as formula (10) of 7-05. T_{η} is the paper's lattice and $|T_{\eta}|$ its number of points, corresponding to the $|T_{k}|$ of formula (10) of 7-05. U_{k} is the field configuration at step k. The point is that the constants depend neither on the step, nor on the size of the lattice, nor on the field (this paragraph is this chapter's reading).

Example: the effective couplings g_{k} of the paper are defined step by step within the paper, and are not the running (2) itself. Reading them with the one-loop running (2), g_{k} is larger on the coarser side, and the maximum of the sequence is g_{K} at the unit lattice, so hypothesis (3) shrinks to the single condition $g_{K} \leq \gamma$ (this chapter's reading).

04

The hypothesis in numbers

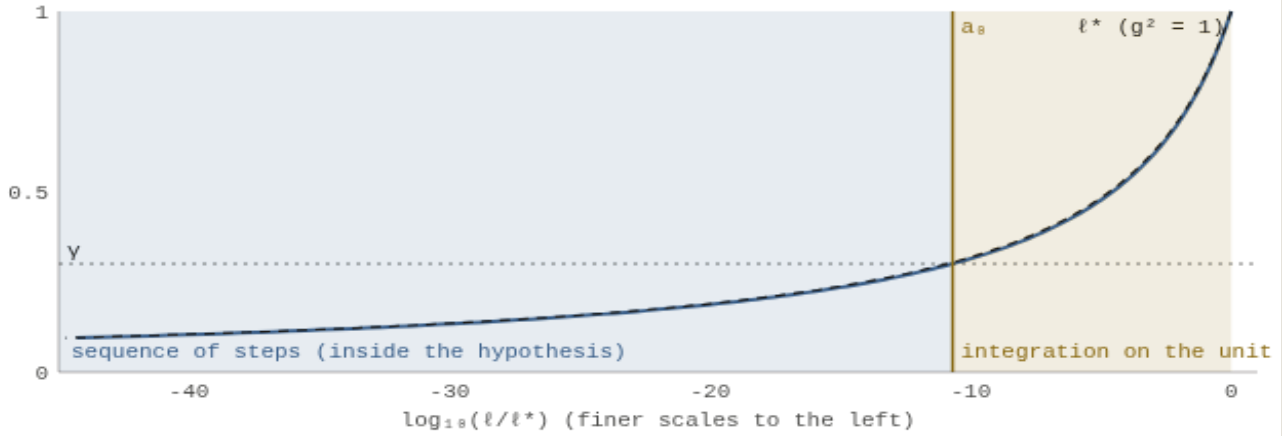
If the scale of the unit lattice where the sequence of steps stops is a_{0} (formula (1) of 7-05), the hypothesis is that the coupling is small at the scale of the unit lattice, written here as $g^{2}(a_{0}) \leq \gamma$. Extending formula (1) to the coarse side, write ℓ_{*} for the length at which $g^{2} = 1$. ℓ_{*} is a **rough-guide length**, extended to where perturbation theory cannot be trusted, and an indication of the scale at which physics thinks confinement and masses appear. Physics Integrating formula (1) with $u = g^{2}$,

$$\ln \frac{\ell_{*}}{a_{0}} = \int_{\gamma}^{1} \frac{du}{2u^{2}\,(b_{0} + b_{1}u)} \;\approx\; \frac{1}{2b_{0}}\left(\frac{1}{\gamma} - 1\right) \quad (4)$$

The right-hand one is the one-loop approximation. The statement of the theorem only says that γ is 'sufficiently small', so the γ below are values for illustration. Here γ is

used as an upper bound on g^2 (it differs from the $g_k \leq \gamma$ of formula (3) only by a relabelling of γ).

Figure 1 Moving γ changes the distance between the scale a_0 where the sequence of steps stops (vertical line) and the rough guide ℓ^* for $g^2 = 1$ (right end)



SU(2), γ = 0.30, $\log_{10}(a_0/\ell^*)$ = −10.70 (two loops), −10.91 (one loop), points of the unit lattice on a torus of side $\ell^* \approx$ 10^42.8

Computed The numbers in this figure were computed on this machine (formula (4); solid line two loops, dashed line one loop; agrees with Table 2).

Computed Table 2 $\log_{10}(\ell_*/a_0)$ (one loop / two loops), and the number of points of the unit lattice on a four-dimensional torus of side ℓ_* (two loops).

γ	SU(2)	SU(3)	Points SU(2)	Points SU(3)
0.5	4.68 / 4.55	3.12 / 3.00	10^18.2	10^12.0
0.3	10.91 / 10.70	7.27 / 7.06	10^42.8	10^28.2
0.2	18.70 / 18.42	12.47 / 12.18	10^73.7	10^48.7
0.1	42.08 / 41.67	28.06 / 27.64	10^166.7	10^110.6

In the range of Table 2 ($\gamma = 0.1$ to 0.5), a_0 lies 3 to 42 orders of magnitude finer than ℓ_*, and the four-dimensional torus of side ℓ_* has 10^{12} to 10^{167} points of the unit lattice.

Computed

How to read it (this chapter's reading). The sequence of steps controls only scales finer than a_0. Beyond that is a finite-dimensional integral on the unit lattice, which is bounded since the group is compact, but the constant has the form $e^{E|T|}$. $|T|$ is the number of points of the unit lattice, corresponding to the last step of the $|T_k|$ of formula (10) of 7-05 (the $|T_\eta|$ of § 03). For a torus of side ℓ_*, $|T|$ is of the size of the number of points above. Ultraviolet stability is not information about the scale ℓ_*.

> 05

Three readings of "ultraviolet stability"

The same words are used with different strengths. Known **As estimates:** 'The ultraviolet stability means that the actions ρ_K have bounds independent of the lattice spacing ε' (Bałaban 1985). **As subsequences:** 'The existence of a convergent subsequence is known as ultraviolet stability' (Chatterjee 2018 § 6). **As "a form":** for MRS and Bałaban, 'a form of ultraviolet stability' was shown (Chevyrev 2022 § 1.1).

The tool that goes from estimates to subsequences is compactness, and its simplest form is the Bolzano–Weierstrass theorem.

$$\sup_K |x_K| < \infty \implies \text{for some } K_1 < K_2 < \cdots,\ x_{K_j} \text{ converges} \quad (5)$$

The limit need not be unique. **Example:** $x_K = \sin K$ has $|x_K| \le 1$ but does not converge: at $K = 33, 77$ it is 0.99991, 0.99952, and at $K = 11, 55$ −0.99999, −0.99976, so one can take a subsequence approaching 1 and one approaching −1. Computed

The introduction of MRS comments on Bałaban's result as follows. 'From this result the existence of an ultraviolet limit for gauge invariant observables such as "smoothed Wilson loops" should follow, at least through a compactness argument using a subsequence of approximations; but the limit is not necessarily unique.' Footnote 2 of the problem statement does not accept weak existence by compactness as a solution, unless the properties of the limit (gap and axioms) are shown separately (JW § 6.5; see also 7-06 § 06). Known

06

MRS 1993: with a continuum cutoff

Magnen–Rivasseau–Sénéor did not use a lattice, and started from a continuum momentum cutoff. The object is the regularised axial-gauge Schwinger functions of pure four-dimensional $SU(2)$, in the trivial topological sector, with a fixed infrared cutoff, and with the ultraviolet cutoff removed. Known We define the words in turn.

- **Axial gauge**: a choice of gauge that sets the component in one coordinate direction to 0 ($A_4 = 0$ and so on; gauge fixing is in 4-08).
- **Trivial topological sector**: treating only configurations with instanton number (3-05) 0.
- **Schwinger functions**: the expectation values of products of fields with one gauge fixed, $\langle A(x_1) \cdots A(x_n) \rangle$.

The centre is the estimate of the size of the field. The counterterms restoring the gauge invariance broken by the cutoff hold the field to about λ^{-1} (λ is the coupling). In axial gauge, the measure after gauge fixing remains a positive weight. This is the **positivity** of axial gauge, and MRS give 'lack of positivity' as the reason Feynman gauge and the like cannot be used as starting points. Together with this positivity, the field is about $\lambda^{-1/2}$ in the sense of probability. So one treats it by splitting into small-field and large-field regions. Written with the curvature of MRS,

$$F = dA - \lambda[A,A], \qquad \frac{\lambda|A|^3}{|A|^2} = \lambda|A|, \qquad \frac{\lambda^2|A|^4}{|A|^2} = \lambda^2|A|^2 \quad (6)$$

Expanding the action $|F|^2$ gives quadratic, cubic and quartic terms. The two on the right are the ratios of the cubic and quartic terms to the quadratic term, rough guides to size without counting derivatives (this chapter's reading). At $|A| \sim \lambda^{-1}$ both ratios are 1, and at $|A| \sim \lambda^{-1/2}$ they are $\lambda^{1/2}$ and λ. At λ^{-1} the non-linear terms are as large as the quadratic term, and MRS too write that information at λ^{-1} alone is 'not enough to start perturbation theory'. **Example:** for $\lambda = 0.01$, at $|A| \sim 100$ the ratios are 1 and 1, and at $|A| \sim 10$ 0.1 and 0.01. Computed

In the small-field region the gauge is replaced by a background-field-dependent one suited to perturbative calculation. The conclusion is 'The ultraviolet limit as $\rho \to \infty$ of the Schwinger functions … exists; furthermore these functions in the ultraviolet limit satisfy the Slavnov identities'. The Slavnov identities are relations that invariance under small gauge transformations leaves among gauge-fixed correlation functions (4-08). Known

The authors themselves also write the reservations. 'we do not provide a detailed proof of this statement', and a detailed proof 'would presumably be several hundred pages long'. It is restricted to axial gauge, does not treat instantons, and does not remove the infrared cutoff. Removing it would make the coupling large, leading 'presumably to so-called non-perturbative effects corresponding to confinement'. Known

07

Setting the two roads side by side

MRS say the role of their cutoff is 'quite similar to the role played by compactness of integration over the gauge group in Balaban'. As formulas,

$$\text{lattice: } U_\ell \in G \ \text{(compact)} \Rightarrow \|U_\ell - 1\| \le 2, \qquad \text{continuum: counterterms} \Rightarrow |A| \lesssim \lambda^{-1} \quad (7)$$

and both are first upper bounds on the field. Known The lattice side is written with group elements and the continuum side with Lie algebra elements, so we match the

sizes. Writing $U_\ell = e^{igB_\ell}$ (B_ℓ a Lie algebra element) for $SU(2)$, the rotation angle can be taken from 0 to π, so $|B_\ell| \le \pi/g$. That is, on the lattice side too the Lie-algebra field is held to about $1/g$, and both sides take the form "field $\lesssim$ 1/coupling" (this chapter's reading). **Example:** for $SU(2)$, $\|U-1\| = 2$ happens at $U=-1$, rotation angle π. Computed Also in common is that upper bounds alone cannot start perturbation theory.

Known Table 3 The two roads.

	Bałaban (1987–89)	**MRS (1993)**
Regularisation	Lattice (Wilson action)	Momentum cutoff (breaks gauge invariance, restored by counterterms)
Group, infrared	Compact Lie group G (the set-up of the three-dimensional paper; details in four dimensions not checked), on a torus	$SU(2)$, fixed infrared cutoff
What was controlled	Upper and lower bounds on the effective densities and the partition function, independent of the lattice spacing	The ultraviolet limit of axial-gauge Schwinger functions and the Slavnov identities
Gauge	Chosen locally for each place and scale (the idea is in 7-05 § 05; the concrete choice in the four-dimensional papers not checked in the text)	Axial for large fields, background-field-dependent for small fields
Reflection positivity	Holds for the Wilson measure at each lattice spacing (JW § 6.5, 5-03). Holding in the limit is not treated	Not yet. The authors say it could probably be shown by taking a lattice as the first cutoff
Proof	The main theorem carries a hypothesis (§ 03)	No detailed proof (the authors)

MRS point out that Bałaban's work does not treat gauge-fixed Schwinger functions, and write 'one can consider in fact that the ultraviolet problem for the Yang-Mills field theory is not yet understood until this point is clarified'. Conversely, MRS leave reflection positivity and the infrared for later. Known

08

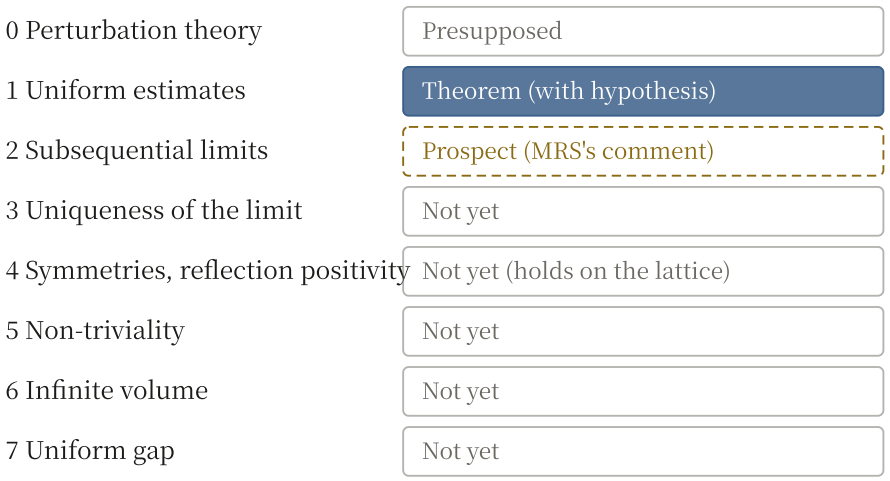
What "the level of control" means

We divide what lies inside and beyond the ultraviolet-limit step (7-01) into eight levels, from the level of perturbation theory to the level of a uniform gap (the rows of Table 4. The level of non-triviality is that the limit is not a free (Gaussian) field, and the problem statement further requires the short-distance behaviour to agree with the predictions of asymptotic freedom. The level of a uniform gap is a mass gap uniform in the volume). The numbers in formula (8) are the row numbers of Table 4. Putting the words of the literature on arrows,

$$1 \xrightarrow{\text{should follow (MRS)}} 2 \xrightarrow{\text{not necessarily unique}} 3 \longrightarrow 4,5 \longrightarrow 6,7 \quad (8)$$

and only the level of uniform estimates has a theorem (in four dimensions, with a hypothesis). Known

Figure 2 Choosing a work changes the status of each level (theorem, with hypothesis, sketch, prospect, not yet)

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Bałaban four dimensions: the level of uniform estimates is a theorem with a hypothesis, the level of subsequential limits "should follow"
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Known Table 4 The levels of control ("prospect" is what the literature says "should follow" or "could probably be shown". What the problem statement asks for is the existence and properties of the theory, and the level of uniform estimates is a step of method). The right-hand column is the construction published on 2026-11-10, where "Theorem" means "the committee's assessment is that it is proved in Lean" (6-01, 6-04).

Level	Bałaban 4D	MRS	Problem statement	Published construction (2026-11-10)
0 Perturbation theory	Presupposed	Presupposed	Presupposed	Presupposed
1 Uniform estimates	Theorem (with hypothesis)	Sketch	(usually needed as a method)	Theorem (multistep estimates; Part 11)
2 Subsequential limits	Prospect	Sketch	Not enough alone (footnote 2)	Theorem
3 Uniqueness	Not yet	Sketch	Required	Converges along the whole sequence of spacings chosen by the family. Universality with respect to other sequences of spacings not yet (6-04)
4 Symmetries, reflection positivity	Not yet (holds for the Wilson measure at each lattice spacing)	Slavnov a sketch, OS positivity a prospect	Required	Theorem (each component of the phases satisfies a variant of the OS axioms; Part 12)

Level	Bałaban 4D	MRS	Problem statement	Published construction (2026-11-10)
5 Non-triviality	Not yet	Not yet	Required	Theorem
6 Infinite volume	Not yet	Not yet (the infrared cutoff is not removed)	Required ($\mathbb{R}^4$)	Theorem along subsequences of volumes. Convergence and uniqueness not yet (6-04)
7 Uniform gap	Not yet	Not yet	Required ($\Delta > 0$)	Not yet (only polynomial estimates; 6-04)

"Has reached the level of control" means the state in which the level of uniform estimates goes through (in four dimensions, with a hypothesis) and parts of the levels of subsequential limits, uniqueness, and symmetry and reflection positivity are written as prospects or sketches. The problem statement writes of T^4 'while the construction is not complete, there is ample indication that known methods could be extended to construct Yang–Mills theory on T4', and says that a detailed construction on a compact space alone would be 'a major breakthrough' (JW § 6). A 2004 report also writes 'The state of the art remains the works of Balaban and of Magnen, Rivasseau and Sénéor' (Douglas 2004). Known The committee's assessment is that the construction published on 2026-11-10, as in the right-hand column of Table 4, reaches the levels from uniform estimates to non-triviality, and infinite volume along subsequences of volumes. It does not reach the level of a uniform gap (6-01).

Example: as a yardstick, for weakly coupled three-dimensional ϕ^4, everything corresponding to the levels from uniform estimates to a uniform gap is a theorem (JW § 6.2, 7-03 § 06). What corresponds to the level of non-triviality is that the limiting measure is non-Gaussian ('a remarkable non-Gaussian measure' of JW § 6.2). That the level of non-triviality is a property that must be shown separately is treated in 7-08, and that the setters ask for new ideas at the level of infinite volume and the level of a uniform gap in 7-10.

09

Detour: how to check a proof of several hundred pages

A section you may skip.

The two works share one difficulty: length. Adding up the pages of the papers in Table 1 gives

$$\underbrace{221}_{\text{7 foundation papers}} + \underbrace{21}_{\text{three dimensions}} + \underbrace{184}_{\text{4D, 5 papers}} = 426 \text{ pages} \quad (9)$$

(sums of page ranges). Computed MRS judge Bałaban's work 'not easily accessible ... the results are scattered over many publications; hence to check the consistency of all the arguments is

very difficult', and conclude that writing their own detailed proof would be 'an extremely valuable task'. Known

One response is rewriting in an easier model: Dimock, saying 'others have been slow to adopt the approach', expounded the method for three-dimensional ϕ^4 (7-05). Another is tools that check long proofs by machine; the idea is in the guide to Lean.

What this chapter can say

	Content
Said	The statement and hypothesis of the Conditional Ultraviolet Stability Theorem (Theorem 1) and the unpublished basic inequality (Theorem 2), the set-up and reservations of MRS 1993, the assessments of the setters and the surveys (checked in primary sources) Known
Said	The arithmetic of formulas (1)(2)(4)(6), Table 2 and Figure 1, the examples and numbers in § 01 and § 04– § 07, formula (9) Computed
Not said	The content of Bałaban's four-dimensional proof and the details of the set-up. The value of γ. That the a_0 of Table 2 coincides with the scale of the paper's unit lattice (this chapter's reading)
Not said	Publication of the proof of the basic inequality (Theorem 2) (not found within what we searched = the bibliographic data of the author's papers from 1989 on). Whether there has been progress on the four-dimensional ultraviolet side in the literature since 2004 (the 2026 survey was read from bibliographic data and abstract only, and not used for assessment; the construction published on 2026-11-10 is read in Volume 2)
Not said	Uniqueness of the limit, axioms, non-triviality, infinite volume and the mass gap in four dimensions. On none of these does this chapter, or this book, show anything of its own. For the construction published on 2026-11-10, the committee's assessment is that the axioms, non-triviality and infinite volume along subsequences of volumes are proved in Lean. Uniqueness of the volume limit and the mass gap are not shown (6-04)

Sources and reproduction

Item	Kind	Source / tools
Theorem 1, Theorem 2	Abstract and introduction pp. 355–356	T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392
The definition of ultraviolet stability (three dimensions)	Text pp. 255–257	T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275
Other papers of the series (Table 1)	Bibliographic data only	CMP 89 (1983); 95, 96 (1984); 98, 99, 102 (1985); 109 (1987); 116, 119 (1988); 122 (1989) 175–202. Titles cross-checked with Chatterjee's reference list and Crossref

Item	Kind	Source / tools
Whether the basic inequality (Theorem 2) was published	Bibliographic data only	Crossref author search (T. Bałaban, 1989 and later). The only paper involving gauge fields is T. Bałaban, M. O'Carroll, R. Schor, J. Math. Phys. 32 (1991) 3199–3208 (block renormalisation of fermions in an external gauge field), and nothing corresponding to the proof of the basic inequality was found
The content of MRS and its comment on Bałaban	p.325–328	J. Magnen, V. Rivasseau, R. Sénéor, Construction of YM_4 with an infrared cutoff, Comm. Math. Phys. 155 (1993) 325–383
The setters' assessment, footnote 2, the non-Gaussianity of ϕ^4_3	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) § 6, § 6.2, § 6.5
'state of the art'	Text	M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem, Clay Mathematics Institute (2004)
The usage of ultraviolet stability, reference list	§ 6	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950
'a form of ultraviolet stability'	§ 1.1	I. Chevyrev, Stochastic quantisation of Yang–Mills, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359
Motivation of the exposition, reference list	§ 1.1	J. Dimock, The renormalization group according to Balaban, I. Small fields, arXiv:1108.1335
b_0, b_1	§ 9.1.1 formula (9.3)	S. Navas et al. (Particle Data Group), Phys. Rev. D 110 (2024) 030001
The 2026 survey	Bibliographic data and abstract only (not used for assessment)	M. R. Douglas, The Yang–Mills Millennium problem, Nature Reviews Physics 8 (2026) 86–97
Table 2, Figure 1, examples, formula (9)	Computed	Python on this machine: `running_gamma.py` (formula (4), checked against numerical integration), `ratios_quartic.py` (the ratios of formula (6) and $\lvert B\rvert \le \pi/g$ for $SU(2)$)

Next chapter to read: 7-08 Triviality of ϕ^4 in four dimensions (why the level of non-triviality is a property that must be shown separately).

Triviality of φ⁴ in four dimensions — the limit made on the lattice collapses to a Gaussian

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — That "trivial" means Gaussian. The counting that makes four dimensions the borderline, and a way of seeing it through the meeting of two random walks. The hypotheses and conclusion of the theorem of Aizenman–Duminil-Copin (2021). Why "non-trivial" becomes a separate requirement for Yang–Mills.

Prerequisite chapters — 7-03 Constructing scalar fields (see also 2-08 and 7-07). Only the one-loop formula of § 08 uses 4-04 Divergences and renormalisation (and Parseval's identity from 1-11).

Said first — The proofs are followed only in outline. The numbers in the tables are computations for free fields and simple random walks, not for the Ising model. § 08 and § 09 are a one-loop picture, not theorems. This chapter shows nothing about whether Yang–Mills is trivial or non-trivial.

The order of this chapter

1. What trivial means: Gaussians and Wick's rule
2. From the lattice to the continuum: approaching the critical point
3. Counting dimensions: the tree-diagram bound and L^{4-d}
4. Random currents: the deviation from Wick is the meeting of clusters
5. The meeting of two random walks: rare, but they meet many times at once
6. The bubble diagram: the logarithm of four dimensions
7. Reading the theorem: its range and a chronology
8. The physics picture: the one-loop running coupling
9. Reading the table and figure: the collapsing coupling and the Landau pole (Detour: the side that becomes free in the infrared)
10. The requirement of non-triviality: what Yang–Mills needs

01

What trivial means: Gaussians and Wick's rule

A Euclidean field theory is a probability measure, and its content appears in the correlation functions $S_n(x_1,\dots,x_n)=\langle\Phi(x_1)\cdots\Phi(x_n)\rangle$ (7-01). For a Gaussian measure, the correlations at an even number of points are determined just by dividing the two-point functions into pairs (1-08).

$$S_{2n}(x_1,\dots,x_{2n})=\sum_{\pi}\prod_{j=1}^{n}S_2\big(x_{\pi(2j-1)},x_{\pi(2j)}\big) \tag{1}$$

The sum is over all the ways π of grouping the $2n$ points in twos. This Wick rule means there is no interaction, and for that reason Gaussian fields have been called **trivial** (ADC § 1.1). The first quantity measuring the deviation is the four-point Ursell function. Known

$$U_4(x,y,z,t)=S_4(x,y,z,t)-\big[S_2(x,y)S_2(z,t)+S_2(x,z)S_2(y,t)+S_2(x,t)S_2(y,z)\big] \tag{2}$$

In models with the Lee–Yang property (the zeros of the partition function in the magnetic field lie on the imaginary axis; the Lee–Yang theorem of 2-08), Gaussianity follows if U_4 vanishes in the limit (Newman 1975, ADC § 6.3). Known **Example:** for a single spin taking ±1 with equal probability, $\langle\sigma^4\rangle-3\langle\sigma^2\rangle^2=-2$, and for a sum of N independent ones the ratio U_4/Var^2 collapses like $-2/N$ (1-08). Spins near the critical point are strongly correlated, so this reason cannot be used, and three-dimensional ϕ^4 gives a non-Gaussian measure (JW § 6.2, 7-03).

02

From the lattice to the continuum: approaching the critical point

The object is the model with a real number ϕ_x at each point of the lattice $\mathbb{Z}^d$, with neighbours linked by a ferromagnetic coupling $J\ge0$ (ADC formula (1.12); the lattice version of formula (1) of 7-03).

$$\langle F\rangle=\frac{1}{Z}\int F(\phi)\,\exp\Big(\sum_{\{x,y\}}J_{xy}\phi_x\phi_y\Big)\prod_x e^{-\lambda\phi_x^4+b\phi_x^2}\,\mathrm{d}\phi_x \tag{3}$$

As $b=2\lambda\to\infty$, the distribution at a point concentrates on ±1 and it becomes the Ising model (ADC formula (1.20)). Taking the scale at which we look to be L in lattice units (the lattice spacing in continuum units is $a=1/L$), we follow the smeared sum divided

by the standard deviation $\sqrt{\Sigma_L}$ of the sum over the box Λ_L ($\Lambda_L = [-L, L]^d$, ADC formulas (1.14)(1.15)).

$$T_{f,L}(\phi) = \frac{1}{\sqrt{\Sigma_L}} \sum_{x \in \mathbb{Z}^d} f(x/L)\, \phi_x, \qquad \Sigma_L = \Big\langle \Big(\sum_{x \in \Lambda_L} \phi_x \Big)^2 \Big\rangle \quad (4)$$

The volume is first taken to infinity and then $L \to \infty$, and (J, λ, b) may be moved along the way. To get smooth correlations at scale L, one approaches the critical point to make the correlation length at least of order L. If the correlation length stays finite in lattice units, the limit is white noise (ADC footnote 1, 2-08). For the Ising model ($b = 2\lambda \to \infty$), J is written βJ, β is called the inverse temperature, and β_c the critical value at which the correlation length diverges (2-08). $S_2 \to 0$ at large distance is the side $\beta \le \beta_c$ (ADC formula (1.10)), and this chapter is restricted to that side. Known

Marginal Triviality Theorem (THEOREM 1.2 of Aizenman–Duminil-Copin 2021)

For $d = 4$, every random field reachable by the construction above with $S_2 \to 0$ at large distance is a generalised Gaussian process. Known

A **generalised Gaussian process** is a distribution-valued random field for which every finite collection of $T f(\Phi)$, one for each test function f (the limit as $L \to \infty$ of the $T_{f,L}$ of formula (4)), follows a multivariate Gaussian distribution (Gaussian measures in 1-09).

03 Counting dimensions: the tree-diagram bound and L^{4-d}

Triviality in five or more dimensions was shown by two inequalities (Aizenman 1981, 1982, Fröhlich 1982): the **tree-diagram bound**, which bounds the four-point deviation by "the diagram in which four legs meet at one point u", and the infrared bound, by which the two-point function does not exceed the free-field form even at the critical point (Ising model, ADC formulas (1.22)(1.23)).

$$|U_4(x,y,z,t)| \le 2 \sum_{u \in \mathbb{Z}^d} S_2(u,x) S_2(u,y) S_2(u,z) S_2(u,t), \qquad S_2(x,y) \le \frac{C}{|x-y|^{d-2}} \ (\beta = \beta_c$$

If the four points are at mutual distances of order L, then $S_4 \sim S_2^2 \sim L^{-2(d-2)}$; on the right-hand side the sum over u gives L^d and the four factors $L^{-4(d-2)}$.

$$\frac{|U_4|}{S_4} \lesssim \frac{L^d\, L^{-4(d-2)}}{L^{-2(d-2)}} = L^{4-d} \quad (6)$$

For $d > 4$ this goes to 0, and apart from details this is the backbone of the theorem in five or more dimensions (ADC § 1.3). For $d = 4$ it is L^0, and the primary source writes 'clearly inconclusive for d = 4'. The problem statement also writes that in five or more dimensions scalar fields with a quartic self-interaction do not exist, and that the proof applies to one-component fields (JW § 6.2). Known The same exponent also appears in the dimension of the coupling $[\lambda] = 4 - d$ in formula (7) of 7-01; for $d > 4$ it points to the coupling ceasing to act at long distances. Physics

04

Random currents: the deviation from Wick is the meeting of clusters

The origin of the tree-diagram bound is the **random current representation**. Taylor-expanding the weight $e^{\beta J \sigma_x \sigma_y}$ edge by edge, viewing the degree $n(x,y)$ as an integer on the edge (a current), and summing over spins, only the configurations survive in which the set ∂n of points with odd total degree (sources) coincides with the set A of points whose correlation is taken (ADC formula (3.5)).

$$\Big\langle \prod_{x \in A} \sigma_x \Big\rangle = \frac{\sum_{\partial n = A} w(n)}{\sum_{\partial n = \varnothing} w(n)}, \qquad w(n) = \prod_{\{x,y\}} \frac{(\beta J_{xy})^{n(x,y)}}{n(x,y)!} \tag{7}$$

Configurations without sources are collections of loops, and the two-point correlation is "the effect of inserting one path joining x and y". An identity that exchanges sources (the switching lemma) gives the following (ADC formula (3.11)).

$$U_4(x,y,z,t) = -2\, S_2(x,y)\, S_2(z,t)\, \mathbf{P}^{xy,zt}\big[\, \mathcal{C}(x) \cap \mathcal{C}(z) \;/= \varnothing \,\big] \tag{8}$$

$\mathbf{P}^{xy,zt}$ is the law of two independent currents n_1, n_2 with sources $\{x,y\}$ and $\{z,t\}$ respectively (the weights of formula (7) turned into probabilities). $\mathcal{C}(x)$ is the set of points connected to x by following edges where the sum $n_1 + n_2$ is positive, and likewise $\mathcal{C}(z)$. $\mathcal{C}(x) \cap \mathcal{C}(z) \;/= \varnothing$ means that x and z are connected in $n_1 + n_2$. The deviation from Wick's rule is exactly the probability of this meeting. Known

The two clusters are not independent. So one adds currents without sources and replaces them by two independent clusters. Doing so can only increase the probability of intersecting (ADC formulas (3.13)(3.14)). Bounding that probability by the expected number of intersection points and using the switching lemma once more gives the first inequality of formula (5) (the tree-diagram bound) (ADC § 3.2). Known

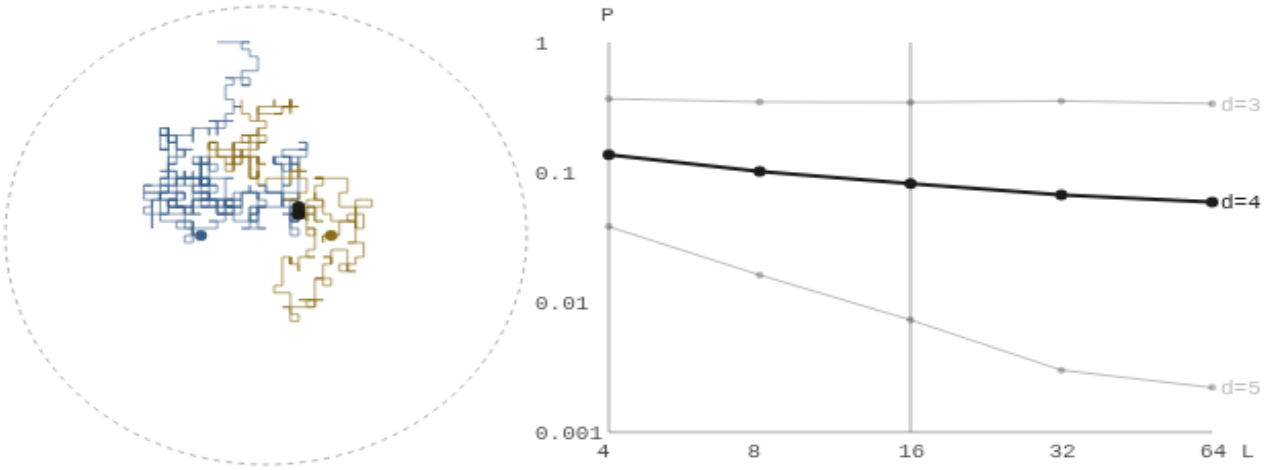
05

The meeting of two random walks: rare, but they meet many times at once

Intuition comes from picturing the two clusters as random walks joining x and y, and z and t (the primary source also uses this analogy as a guide). In four dimensions, consider two trajectories starting from two points a distance L apart and walking until they leave a region of radius about L (about L^2 steps). The probability of intersecting goes to 0 like $1/\log L$, yet the expected number of intersection points does not become smaller than a positive constant independent of L. Intersections are rare, but when they intersect the number of intersection points grows like $\log L$ (ADC § 3.2). Known

The rough guide is that the trajectory of a random walk is two-dimensional (about L^2 points within radius L), and the borderline for two two-dimensional objects to meet is $2 + 2 = 4$. More precisely, the content of this guide is that the expected number of intersection points behaves like the L^{4-d} of formula (6) (the E of Table 2).

Figure 1 Choosing the dimension and L changes the two random walks (projected onto the first two coordinates) and the points that coincide in d dimensions (black dots). On the right (below on narrow screens) is the intersection probability P (the computation of Table 2; logarithmic vertical axis). The number of intersection points is the value for one sample, not an average



d = 4, L = 16, intersections in this sample: 5 (sample value), average: P = 0.082, E = 0.3

Computed The points P and Table 2 were computed on this machine. The pictures of random walks are samples from random numbers with fixed seeds (the first picture was chosen to be an intersecting example). Most of the places that look overlapping in the projection do not coincide in d dimensions.

Computed Table 2 Two simple random walks started from $\pm(L/2)e_1$ and run until they leave the ball of radius $2L$: the mean number of common lattice points E, the intersection probability P, and the mean given intersection E/P (6000 pairs each; the standard error of P is 0.003–0.0044; $L = 8, 32$ are in Figure 1).

L	E (d=3)	E (d=4)	E (d=5)	P (d=4)	E/P (d=4)
4	1.75	0.32	0.064	0.137	2.35
16	5.86	0.30	0.013	0.082	3.67
64	23.03	0.27	0.004	0.059	4.61

E is proportional to L in three dimensions, decreases like $1/L$ in five, and is nearly constant in four, just as the L^{4-d} of formula (6). In four dimensions P decreases slowly and E/P increases: the shape "rare, but when they meet they meet many times at once". With finite L this does not confirm the rate $1/\log L$. Computed

06

The bubble diagram: the logarithm of four dimensions

The key to improving the tree-diagram bound in four dimensions is the bubble diagram B_L cut off at distance L (ADC formula (1.25)): the diagram that joins two two-point functions and goes once round. For the free-field two-point function G (the inverse of the lattice Laplacian) it is proportional to the expected number of times two random walks from the same point visit the same point, and since $G(x) \approx 1/(4\pi^2|x|^2)$ far away in four dimensions, it grows logarithmically. Known

$$B_L = \sum_{x \in \Lambda_L} S_2(0,x)^2, \qquad \sum_{x \in \Lambda_L} G(x)^2 \approx \int_{1<|x|<L} \frac{\mathrm{d}^4 x}{16\pi^4 |x|^4} = \frac{\ln L}{8\pi^2} + \text{const} \tag{9}$$

Computed Table 1 The lattice bubble of the free field $\sum_{x \in [-L,L]^d} G(x)^2$ (infinite lattice, mass 0, $G(x) = \int_0^\infty \prod_i e^{-2t} I_{x_i}(2t)\,\mathrm{d}t$ integrated numerically. I_n is the modified Bessel function, and $e^{-2t} I_n(2t)$ is the probability that a one-dimensional continuous-time random walk jumping left and right each at rate 1 is at n at time t). The increment is the difference when L doubles; formula (9) gives $\ln 2/(8\pi^2) = 0.00878$.

L	d=3	d=4	Increment for d=4	d=5
4	0.4504	0.05123	0.00738	0.01866
8	0.8388	0.05926	0.00803	0.01898
16	1.6161	0.06766	0.00839	0.01916
32	3.1711	0.07624	0.00858	0.01925

The four-dimensional increment approaches 0.00878; in three dimensions the increment doubles each time ($\propto L$), and in five it halves and converges. Computed The improved tree-diagram bound (Theorem 1.3 of Aizenman–Duminil-Copin) gives, for the four-dimensional nearest-neighbour ferromagnetic Ising model with $\beta \le \beta_c$, $L \le \xi(\beta)$, and four points mutually farther apart than L, with some $c, C > 0$, the following (ADC formula (1.24)).

$$|U_4(x,y,z,t)| \le \frac{C}{B_L(\beta)^c} \sum_{u \in \mathbb{Z}^4} S_2(u,x) S_2(u,y) S_2(u,z) S_2(u,t) \tag{10}$$

If the two-point function nearly satisfies the infrared bound (the case expected in four dimensions), B_L is of order $\log L$ and $|U_4|/S_4 = O((\log L)^{-c})$; if it is much smaller, formula (5) alone suffices. Beyond four points, it closes by the characteristic function

of the quantity in formula (4) being close to Gaussian with error $(\log L)^{-c}$ (the proposition that the characteristic function is close to Gaussian; ADC Proposition 1.4). The centre of the proof is a multi-scale analysis showing that, given intersection, the number of intersection points grows at least like $(\log L)^{c}$ (ADC § 4, § 6). Known

07

Reading the theorem: its range and a chronology

The Marginal Triviality Theorem (Theorem 1.2 of ADC) does not depend on the choice of counterterms. The result rests not on computing counterterms but on dimension-dependent relations among the Schwinger functions (the S_n of § 01) that can appear in any limit (ADC § 1.1). It includes "hard" ϕ^4 and the Ising model (the Griffiths–Simon construction, which makes ϕ^4 from averages of blocks of spins). On the other hand, the object is restricted to limits of nearest-neighbour, ferromagnetic, one-component models with a lattice cutoff, and it is not a theorem about every method of construction. Nor does it give the exact size of the logarithmic correction or the covariance of the limit (ADC § 1.2). Known

Known Table 3 The main results on triviality (classification from ADC §1.1–1.2. §1.1 also counts some of the small-λ results below among the partial results for $d = 4$. Journals at the end).

Year	Authors	Content
1981–82	Aizenman; Fröhlich	Gaussian for $d > 4$
1983	Aizenman–Graham; Aragão de Carvalho–Caracciolo–Fröhlich	Partial results for $d = 4$
1985–87	Gawędzki–Kupiainen; Feldman–Magnen–Rivasseau–Sénéor; Hara–Tasaki	Small λ (renormalisation group)
2014	Bauerschmidt–Brydges–Slade	Small λ, n components
2021	Aizenman–Duminil-Copin	$d = 4$, all λ and Ising

Since then, a one-page correction (2024) and an extension to effective dimension four and above (Panis 2026) have appeared (bibliographic data only).

08

The physics picture: the one-loop running coupling

The reason on the physics side is that the quartic coupling in four dimensions is not asymptotically free (JW § 6.2). What follows is a one-loop picture, not a theorem. Physics If you have not read 4-04, you may accept just the right-hand end of formula (11) and read on. Write the interaction as $\frac{u}{4!}\phi^4$ (for formula (3), $u = 24\lambda$). At one loop the four-point coupling shifts by one inserted bubble; there are three ways of inserting

it (the three pairs of formula (2)), each with symmetry factor $\frac{1}{2}$, and the bubble is the same $\frac{1}{8\pi^2}\ln$ as in formula (9) (the sum over positions $\sum G^2$ and the integral over momenta are equal by Parseval's identity; 1-11). With Λ the cutoff and μ the scale we look at,

$$u(\mu) \approx u(\Lambda) - \frac{3}{2}u(\Lambda)^2\frac{\ln(\Lambda/\mu)}{8\pi^2}, \qquad \frac{1}{u(\mu)} = \frac{1}{u(\Lambda)} + \frac{3\ln(\Lambda/\mu)}{16\pi^2} \implies u(\mu) < \frac{16\pi^2}{3\ln(\Lambda/\mu)} \quad (11)$$

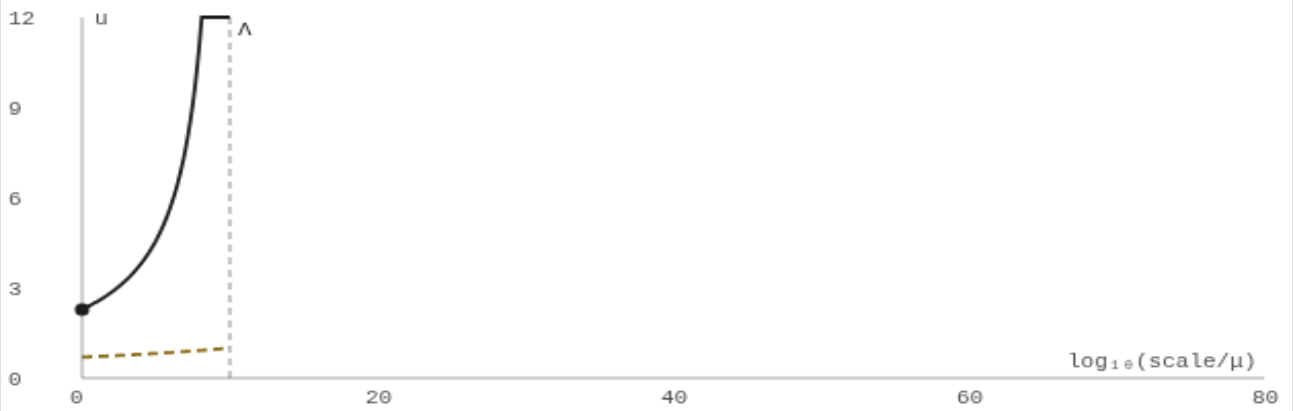
(the middle is the integral of the flow equation $\mu\, du/d\mu = 3u^2/16\pi^2$). However large the bare coupling $u(\Lambda)$ is made, raising the cutoff sends $u(\mu)$ to 0. Conversely, keeping $u(\mu) = 1$ and extending upwards, the denominator becomes 0 at $\ln(\Lambda/\mu) = 16\pi^2/3 \approx 52.6$ ($\Lambda/\mu \approx 10^{22.9}$) (the Landau pole). For Yang–Mills the sign is reversed (formula (2) of 7-07), and fixing $g^2(\mu)$ and raising Λ only sends the bare coupling $g^2(\Lambda)$ to 0. Physics Computed

09

Reading the table and figure: the collapsing coupling and the Landau pole

We look at formula (11) in numbers. The two left columns of Table 4 are ϕ^4 and the right end is Yang–Mills; Figure 2 draws the same quantities as functions of the scale.

Figure 2 Raising the cutoff Λ (vertical line): for ϕ^4 the coupling at the scale μ (left end) collapses, and for Yang–Mills the bare coupling (on the vertical line) goes to 0 (one loop). For ϕ^4 the solid line is bare coupling $u(\Lambda) = \infty$, the dashed line $u(\Lambda) = 1$



φ⁴, $\log_{10}(\Lambda/\mu) = 10$, $u(\mu) < 2.286$ (even with bare coupling ∞), if $u(\Lambda) = 1$: 0.696

Computed The numbers in this figure were computed on this machine (formula (11) and formula (2) of 7-07; they agree with Table 4).

Computed Physics Table 4 $u(\mu)$ of formula (11), and for $SU(3)$ the bare coupling $g^2(\Lambda)$ keeping $g^2(\mu) = 1$ ($b_0 = 11 \cdot 3/48\pi^2$).

$\log_{10}(\Lambda/\mu)$	$u(\Lambda)=\infty$	$u(\Lambda)=1$	$g^2(\Lambda)$
2	11.430	0.920	0.609
10	2.286	0.696	0.238
40	0.572	0.364	0.072
80	0.286	0.222	0.038

The rate of collapse is $1/\ln\Lambda$, the same type of slowness as the $(\log L)^{-c}$ of formula (10). The one-loop formula cannot be trusted where the coupling is large, and is not a proof of triviality. In 2000 the problem statement wrote 'all insights from random walks, perturbation theory, and renormalization analysis point toward triviality' (JW § 6.2). The answer as a theorem is § 02– § 07. Known

Detour: the side that becomes free in the infrared

A subsection you may skip.

The primary source writes that what may be regarded as a 'disappointment' from the point of view of constructive field theory is, from the point of view of statistical mechanics, 'a positive and constructive result' (ADC § 1.2). This is because it answers that the large scales of the four-dimensional critical Ising model are described by Gaussians. Known

Reading formula (11) towards long distances, u goes to 0 logarithmically. **Example:** the $u(\Lambda)=1$ column of Table 4 reads 0.696 at distances 10 orders of magnitude longer than the scale where $u=1$, and 0.222 at 80 orders. The coupling stays small, so this is the side where the renormalisation group (7-05) is easier to control. Physics The 'asymptotically free' in the title of Gawędzki–Kupiainen in Table 3 ('Massless Lattice Φ^4_4 Theory: Rigorous Control of a Renormalizable Asymptotically Free Model') can be read as meaning that in the massless (the 'Massless' of the title) model the coupling goes to 0 on the long-distance side (formula (11) read towards the infrared). Physics Yang–Mills is the exact opposite: the coupling becomes strong at long distances, and the gap is thought to come from that strong side (7-07 § 01). Physics

10

The requirement of non-triviality: what Yang–Mills needs

The problem statement asks that 'a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$', and requires that the correlation functions agree at short distances with the predictions of asymptotic freedom and perturbative renormalisation (JW § 4, 0-03). Known In the prediction of perturbation theory, at short distances the coupling goes to 0 logarithmically, like $g^2(\ell)\approx 1/(2b_0\ln(\ell_0/\ell))$, but is not identically 0 (4-06; an example is the right end of Table 4). Physics The meaning of the requirement can be organised into three points (this chapter's reading).

1. **Non-triviality is a property to be shown separately.** It does not follow from ultraviolet estimates or from the existence of a limit (the level of non-triviality in 7-07). Four-dimensional ϕ^4 is an actual example that, within the family of nearest-neighbour, ferromagnetic, one-component lattice models, becomes Gaussian however the parameters are moved (if $S_2 \to 0$ at large distance).
2. **The theorem for ϕ^4 does not show that Yang–Mills is trivial.** The problem statement contrasts the two by whether they are asymptotically free, and calls Yang–Mills the 'natural candidate' (JW § 6.2). The committee's assessment is that the four-dimensional Yang–Mills construction published on 2026-11-10 also shows non-triviality (6-01; the evidence that it is Yang–Mills is in Part 13). The Gaussian limit of the $SU(2)$ Yang–Mills–Higgs model comes from a way of taking the limit that sends the gauge coupling to 0 in step with the lattice spacing, and non-Gaussian limits are said to be open (Chatterjee 2026).
3. **The tools of the proof cannot be carried over as they are.** The random current representation is a structure of Ising-type models.

The state before 2026 was as in the problem statement's 'at present we do not know any non-trivial relativistic field theory that satisfies the Wightman (or any other reasonable) axioms in four dimensions' (JW § 6), and the survey's 'there is yet no construction of a continuum limit of a lattice gauge theory in any dimension higher than two where Wilson loop variables have been shown to have nontrivial behavior' (Chatterjee 2018 § 6). Known For the construction published on 2026-11-10, the committee's assessment is that non-triviality is also proved (the second item above). The whole of where things stop is seen in 7-10.

What this chapter can say

	Content
Said	That "trivial" means Gaussian, with the four-point Ursell function as the marker. That the limits approaching the critical point of one-component ϕ^4 with a lattice ultraviolet cutoff and of the Ising model were shown Gaussian for $d > 4$ in 1981–82 and for $d = 4$ in 2021 ($d > 4$ from the description in ADC § 1.1 and JW § 6.2; $d = 4$ checked in the text of ADC) Known
Said	That the counting of formula (6) appears in the free-field bubble (Table 1) and the random-walk samples (Table 2). The arithmetic of Table 4 and Figure 2 Computed . The contrast of ϕ^4 and Yang–Mills in the one-loop picture Physics
Not said	Triviality of ϕ^4_4 outside lattice, nearest-neighbour, one-component models. The exact size of the logarithmic correction, the covariance of the limit, the content of the correction. The values for the Ising model in Tables 1 and 2. Non-triviality or triviality of four-dimensional Yang–Mills (neither this chapter nor this book shows either)

Sources and reproduction

Item	Kind	Source / tools
Formulas (1)–(10), Theorems 1.2 and 1.3, Proposition 1.4, the quotations in § 03, § 07, § 09	Text (arXiv version 4)	M. Aizenman, H. Duminil-Copin, Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973
The other literature in Table 3	Bibliographic data only	Aizenman, PRL 47 (1981) 1–4, CMP 86 (1982) 1–48; Fröhlich, NPB 200 (1982) 281–296; Aizenman–Graham, NPB 225 (1983) 261–288; Aragão de Carvalho–Caracciolo–Fröhlich, NPB 215 (1983) 209–248; Gawędzki–Kupiainen, CMP 99 (1985) 197–252; FMRS, CMP 109 (1987) 437–480; Hara–Tasaki, JSP 47 (1987) 99–121; BBS, JSP 157 (2014) 692–742
Newman, the infrared bound, the correction, the extension	Bibliographic data only	Newman, CMP 41 (1975) 1–9; Fröhlich–Simon–Spencer, CMP 50 (1976) 79–95; Glimm–Jaffe, Positivity of the ϕ^4_3 hamiltonian, Fortschr. Phys. 21 (1973) 327–376 (ADC cites these two as [19, 21] as the source of the infrared bound (1.23)); Ann. of Math. 199 (2024) 479; R. Panis, Ann. Probab. 54 (2026) no. 2
The requirement of the problem statement, the contrast with ϕ^4, the present state	Text § 4, § 6, § 6.2	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000)
Non-trivial continuum limits of gauge theories, the Gaussian limit of YMH	§ 6, abstract	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950; A scaling limit of SU(2) lattice Yang–Mills–Higgs theory, Prob. Math. Phys. 7 (2026) 339–381, arXiv:2401.10507
Tables 1, 2, 4, Figures 1 and 2, the Landau pole	Computed	Python on this machine: `bubble.py`, `rw_intersect.py`, `running.py`

Next chapter to read: 7-09 Stochastic quantisation and regularity structures (subcritical and critical: the face, on the side of equations, of four dimensions being the borderline).

Stochastic quantisation and regularity structures — the road of making the measure "by moving it", and why it stops short of four dimensions

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — The idea of making the measure e^{-S} from a stochastic time evolution that leaves it invariant. Why products stop being definable for fields, and the prescriptions in two and three dimensions. Where subcriticality $d < 4$ comes from, and what has been obtained for Yang–Mills.

Prerequisite chapters — 7-01 What constructive field theory is, 1-09 Probability in infinite dimensions. We also refer to 7-03 (Wick products), 1-03 (the Lie bracket) and 1-07 (connections and curvature). The notation for gauge fields and the terms for stochastic processes used in § 07 and § 08 are defined on the spot, one line each.

Said first — This introduces known results and follows the proofs only in outline. The numbers are a one-dimensional model, counting of degrees and counting of symbols, and show nothing about the Yang–Mills measure. Three-dimensional Yang–Mills has reached the dynamics (local solutions and a Markov process), and the measure has not been constructed, even in finite volume. Four dimensions is critical and lies outside the general theory of regularity structures.

The order of this chapter

1. Why make the measure "by moving it" — the balance between the flow down the slope and the noise leaves e^{-S}/Z unmoved. A moving figure of a ball in one dimension
2. The stochastic heat equation and the free field — if linear, the stationary distribution is the Gaussian free field. Accounting for the roughness of the noise
3. Products cannot be defined — products of distributions only when the sum of degrees is positive. A table of the "margin" in each dimension

4. Two dimensions: splitting the solution in two — the rough part is made first with Wick products, and the rest solved by a fixed-point theorem
5. Three dimensions: regularity structures — the Taylor polynomials are replaced by symbols built from the noise
6. Subcritical and critical — the non-linear term vanishes under magnification for $d < 4$. A moving figure of the number of symbols growing with dimension
7. Gauge fields: the DeTurck term and covariance — a term smoothing in the direction of the orbits is added, and the renormalisation constants are made compatible with covariance
8. What has been reached for Yang–Mills — the state space, the Markov process, the invariant measure in two dimensions, and a table of the remaining problems
9. Detour: coming down from infinity — estimates that forget large initial values are the key to making invariant measures
10. What can be carried to four dimensions — the viewpoint and the state space can be carried over; the premise of local subcriticality cannot

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01

Why make the measure "by moving it"

To draw points from a finite-dimensional density $e^{-S(x)}/Z$ (Z is the normalising constant), one keeps shaking the point with noise while letting it go down the slope S.

$$dX_t = -\nabla S(X_t)\,dt + \sqrt{2}\,dW_t \quad (1)$$

W_t is Brownian motion (1-09). Formula (1) means: advance in small steps Δt by the rule

$$X_{t+\Delta t} = X_t - \nabla S(X_t)\,\Delta t + \sqrt{2\Delta t}\;N$$

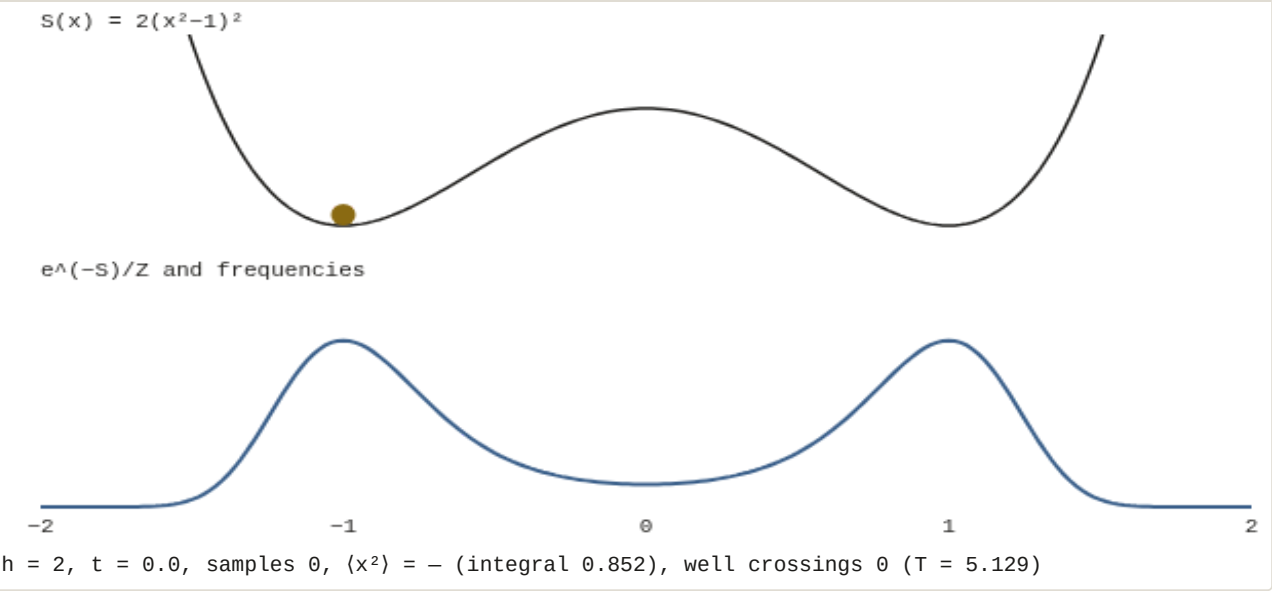
(N is a standard normal random number drawn afresh each time), and take the limit $\Delta t \to 0$. The ball in Figure 1 also moves by this rule (the Euler–Maruyama method). Formula (1) is called the **Langevin equation**, and the time evolution it defines the Langevin dynamics.

If many points run at once, the density $\rho(t,x)$ of the swarm obeys the Fokker–Planck equation $\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla S)$. The right-hand side has the form of the divergence of a flow, so the total number of points is conserved (a conservation law obeyed by the density). For $\rho = e^{-S}/Z$ the inside of the bracket is 0 and nothing moves. This is the

invariant measure. As a metaphor, it is the balance between the flow down the slope and the flow spread by the noise. Precisely, when S grows fast enough at infinity, the law of X_t approaches e^{-S}/Z wherever it starts. Known

Making the measure of a field as the invariant measure of an equation of this form is **stochastic quantisation**, proposed in the context of gauge theory by Parisi–Wu (1981) (Chevyrev 2022 § 1.2). The formula in 7-02 is the same thing with time rescaled, and the Monte Carlo method (5-07) rests on the same idea. The example is $S_h(x) = h(x^2 - 1)^2$, with wells at $x = \pm 1$ and a hill of height h.

Figure 1 The knob changes the height of the hill h (five settings 0.5, 1, 2, 3, 4), and "Play" shakes the ball by formula (1). The bars below are the frequencies of where the ball has been, and the line is e^{-S_h}/Z



h = 2, t = 0.0, samples 0, ⟨x²⟩ = – (integral 0.852), well crossings 0 (T = 5.129)

Computed Z, $\langle x^2 \rangle$ and T in this figure are values computed on this machine. The ball is moved on the spot by the Euler–Maruyama method. The simulation in Table 1 is 2000 chains started from $x = -1$ (step 0.002, times 20–120), and T is the mean hitting time from -1 to $+1$, $\int_{-1}^{1} e^{S(y)} \int_{-\infty}^{y} e^{-S(z)} dz\, dy$.

h	⟨x²⟩ integral	⟨x²⟩ simulation	T
0.5	0.8935	0.8932	3.35
1	0.8327	0.8316	3.57
2	0.8521	0.8497	5.13
3	0.8893	0.8872	8.88
4	0.9177	0.9152	17.3

$\langle x^2 \rangle$ agrees within 0.3%. On the other hand T grows exponentially when h is large (80.9 and 437.5 for $h = 6, 8$), and for $h = 4$ the chain average $\langle x \rangle$ is still −0.33 at time 10, biased towards the left well. Computed

How it grows for large h is given by Kramers' formula: from the curvatures at the bottom of the wells and the top of the hill, $S_h''(\pm 1) = 8h$ and $S_h''(0) = -4h$,

$$T \approx \frac{2\pi}{\sqrt{S_h''(-1)\,|S_h''(0)|}}\, e^{h} = \frac{\pi}{2\sqrt{2}\,h}\, e^{h}$$

Known Forming $T \cdot h/e^{h}$ from the computed T gives 1.27, 1.20, 1.17 for $h = 4, 6, 8$, slowly approaching the coefficient $\pi/(2\sqrt{2}) \approx 1.11$. Computed That an invariant measure exists and that it is approached quickly are separate problems, and in field theory too, estimates uniform in time are the hard part (§ 08).

02

The stochastic heat equation and the free field

For a field on the torus T^d with $S(\phi) = \int \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right)dx$, formula (1) becomes

$$\partial_t \phi = \Delta\phi - m^2\phi - \lambda\phi^3 + \sqrt{2}\,\xi \quad (2)$$

(Δ is the Laplacian, not the Δ of the mass gap). ξ is space-time white noise, a random distribution formally satisfying $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\,\delta(x-y)$ (1-09). The linear **stochastic heat equation** with $\lambda = 0$ splits into independent $d\hat\phi_k = -(|k|^2 + m^2)\hat\phi_k\,dt + \sqrt{2}\,dW_k$ for each Fourier component, and the stationary distribution is the Gaussian free field with covariance $(-\Delta + m^2)^{-1}$ (the starting measure of 7-03). Known

In the parabolic scaling, which counts time as two lengths, white noise has regularity just below degree $-(d+2)/2$, and solving with the heat equation raises it by 2 (Hairer 2014 § 8). The linear solution Ψ lies in

$$\Psi \in \mathcal{C}^{\,1-d/2-\kappa} \qquad (\kappa > 0 \text{ arbitrarily small}) \quad (3)$$

For $0 < \alpha < 1$, $\mathcal{C}^\alpha$ is the smoothness $|f(x) - f(y)| \lesssim |x-y|^\alpha$; for $\alpha < 0$ it is the space of distributions whose values smeared with a bump function of width r stay of order r^α. For $d = 1$ it is $\frac{1}{2} - \kappa$ (the roughness of Brownian motion), for $d = 2$ $-\kappa$ (no longer a function), and for $d = 3$ $-\frac{1}{2} - \kappa$.

03

Products cannot be defined

The trouble is ϕ^3. Products of distributions have the following limit (Hairer 2014 § 1, Chevyrev 2022 § 1.3; α, β are not integers). Known

$$(f, g) \mapsto fg \text{ extends continuously to } \mathcal{C}^\alpha \times \mathcal{C}^\beta \iff \alpha + \beta > 0 \quad (4)$$

For $\Psi \in \mathcal{C}^{-\kappa}$ with $d = 2$, Ψ^2 cannot be defined; the mean of the cut-off Ψ_ε^2 diverges like $\log(1/\varepsilon)$, and subtracting a constant (Wick products; 7-03) lets the limit be taken. The **margin** in Table 2 is "the degree of the non-linear term − the degree of the noise"; if positive, the non-linear term stays a correction smoother than the noise at fine scales.

Computed Table 2 Accounting of degrees (excluding κ; output in rational numbers by `trees.py`). $d = 4$ is critical, $d = 5$ supercritical.

d	Noise ξ	Ψ	φ³	A∂A	Margin φ⁴	Margin YM
2	−2	0	0	−1	2	1
3	−5/2	−1/2	−3/2	−2	1	1/2
4	−3	−1	−3	−3	0	0
5	−7/2	−3/2	−9/2	−4	−1	−1/2

The margin is $4 - d$ for ϕ^4 and $(4 - d)/2$ for $A\partial A$ (A^3 is the same as ϕ^3), positive for $d < 4$. Hairer takes this as the condition of **local subcriticality**, and checks that for ϕ^4 it is equivalent to $d < 4$ (Hairer 2014 Assumption 8.3, § 8). Known

04

Two dimensions: splitting the solution in two

In two dimensions the solution is split into "the roughest part of the noise" and the rest, setting $\phi = \Psi + v$.

$$\partial_t v = \Delta v - m^2 v - \lambda\left(v^3 + 3v^2\Psi + 3v :\Psi^2: + :\Psi^3:\right) \quad (5)$$

$:\Psi^n:$ is the Wick product in $\mathcal{C}^{-\kappa}$, which can be made first by probabilistic computation. The right-hand side is in $\mathcal{C}^{-\kappa}$, so $v \in \mathcal{C}^{2-\kappa}$, and $v \cdot :\Psi^2:$ has a positive sum of degrees and falls under (4), so it can be solved by an ordinary fixed-point theorem (for short times, the right-hand side of the equation written in integral form is a contraction, and there is exactly one fixed point). Returning to ϕ, it is the limit of the equation with ϕ^3 replaced by $\phi^3 - 3C_\varepsilon\phi$ (C_ε of order $\log(1/\varepsilon)$). Da Prato–Debussche (2003) obtained **strong solutions** in two dimensions in this way (solutions satisfying the equation for each given realisation of the noise). Known

The construction of the process goes back further. Jona-Lasinio–Mitter (1985), on a square with Dirichlet boundary, made a **Markov process** (a stochastic process whose future distribution is determined only by the present state) converging to the ϕ_2^4 measure as time goes to infinity, as a weak solution (one where the noise may be replaced too, satisfying the equation in law) of the equation smoothed by a power of an operator (Summers 2016 § 5). Hairer 2014 § 1 lists Albeverio–Röckner (1991) and Da Prato–Debussche (2003) as earlier work in two dimensions. Known

With $a = 1 - d/2$, $v \in \mathcal{C}^{2+3a}$, so $v \cdot {:}\Psi^2{:}$ needs $5a + 2 > 0$, that is $d < 14/5$. In three dimensions a single split is not enough (Hairer 2014 § 8). Computed

05

Three dimensions: regularity structures

A smooth function can be written near each point by Taylor polynomials. Hairer (2014) replaced these polynomials by a set of distributions built from the noise for each problem (in the words of the abstract, "a model built for the problem at hand").

$$\phi(y) - \phi(x) \approx \sum_{\tau} \phi_\tau(x)\,(\Pi_x \tau)(y) \qquad (\text{error of order } |y-x|^{\gamma}) \quad (6)$$

τ are **symbols** such as Ψ, $\mathcal{I}(\Psi^3)\Psi$ ($\mathcal{I}$ is the operation of solving with the heat equation) and polynomials X, and $\Pi_x\tau$ is the distribution obtained by recentring it at the point x. Products that cannot be defined move into the algebra of symbols, and what the analysis needs is only the finitely many $\Pi_x\tau$ that can be made by probabilistic computation. The operation subtracting their divergences appears as the action of a "renormalisation group" (same abstract; the algebra is Bruned–Hairer–Zambotti 2019). For three-dimensional ϕ^4

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + (3C_1^\varepsilon - 9C_2^\varepsilon)\,u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon \quad (7)$$

with $C_1^\varepsilon \sim \varepsilon^{-1}$ and $C_2^\varepsilon \sim \log \varepsilon$ (ibid. (9.21), Remark 10.23). The symbols of negative degree are the 8 symbols $\Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_i, \mathcal{I}(\Psi^3)\Psi, \mathcal{I}(\Psi^2)\Psi^2, \mathcal{I}(\Psi^3)\Psi^2$ (ibid. § 10.5, the proof of Theorem 10.22). There is also another road, paracontrolled distributions (Gubinelli–Imkeller–Perkowski 2015), and later results on invariant measures and the axioms are in Table 4 and 7-03. Known

06

Subcritical and critical

Magnifying as $\phi_\delta(t,x) = \delta^{(d-2)/2}\phi(\delta^2 t, \delta x)$ (the gauge field A is magnified the same way), the law of the noise and Δ do not change; only the coefficients of the non-linear terms change. The right of formula (8) is the Yang–Mills coupling g. With the action $\frac{1}{g^2}\int |F_A|^2$ and A replaced by gA, the $A\partial A$ term of formula (9) in § 07 gets g and the A^3 term gets g^2.

$$\lambda \longrightarrow \lambda\,\delta^{4-d}, \qquad g^2 \longrightarrow g^2\,\delta^{4-d} \quad (8)$$

For $d < 4$ the non-linear terms vanish at fine scales $\delta \to 0$, and the equation becomes a perturbation of the stochastic heat equation. This is subcriticality, which corresponds to super-renormalisability (Chevyrev 2022 § 1.3: subcritical is equivalent to $d < 4$). The general theory of regularity structures assumes local subcriticality (Hairer 2014 Assumption 8.3). Known A figure setting the margin $4 - d$ beside other quantities that switch at four is the dimension knob in 1-12 § 10.

$d = 4$ is critical, and the general theory covers neither model. For ϕ^4, Hairer 2014 § 1 writes that it is not expected to have a non-trivial meaning in four dimensions, and Aizenman–Duminil-Copin later showed that the limit from the lattice is trivial (7-08). Known Yang–Mills is asymptotically free, and the coupling is thought to become logarithmically small at fine scales. This is a result about the perturbative β function (4-06), not a non-perturbative theorem. Physics

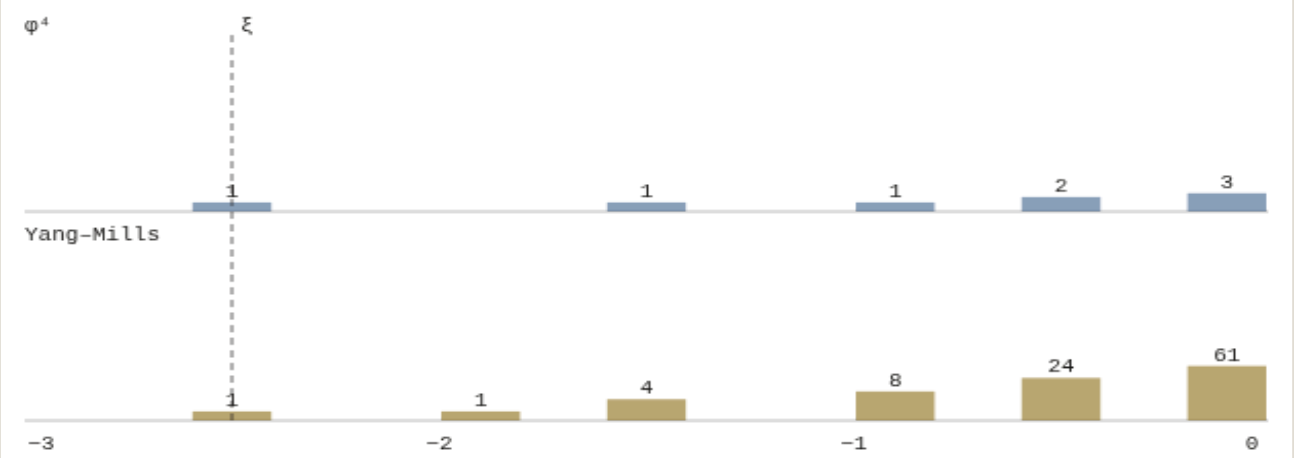
Seen through the number of symbols

The approach to criticality can also be seen in the number of symbols of § 05. Build symbols by the following three rules, and count those with negative degree. κ is treated as a positive infinitesimal, and the directions of X_i and the component indices of the field are not distinguished.

1. **Ingredients.** The noise Ξ (degree $-(d+2)/2 - \kappa$), and for each symbol τ already made, $\mathcal{I}(\tau)$ with $\mathcal{I}$ (the operation of § 05 of solving with the heat equation) applied (degree +2) and its derivative $\partial\mathcal{I}(\tau)$ (degree +1). $\Psi = \mathcal{I}(\Xi)$.
2. **Products.** Ingredients are placed in seats matching the form of the non-linear term and multiplied (degrees add). For ϕ^4, one to three $\mathcal{I}(\cdot)$ in the three seats of ϕ^3. For Yang–Mills, up to one each of $\mathcal{I}(\cdot)$ and $\partial\mathcal{I}(\cdot)$ in the two seats of $A\partial A$, or one to three $\mathcal{I}(\cdot)$ in the three seats of A^3.
3. **Empty seats.** If even one seat is empty, a polynomial X (degree +1, any power) may be placed there.

Example (ϕ^4 for $d = 3$): Ξ is $-\frac{5}{2} - \kappa$ and Ψ is $-\frac{1}{2} - \kappa$. Ψ^3, filling the three seats with Ψ, is $-\frac{3}{2} - 3\kappa$, and $\mathcal{I}(\Psi^3)$, with $\mathcal{I}$ applied to it, is $\frac{1}{2} - 3\kappa$. $\mathcal{I}(\Psi^3)\Psi$, putting $\mathcal{I}(\Psi^3)$ and Ψ in two seats, is -4κ, just below 0, so it is counted. Putting X in the empty third seat gives $1 - 4\kappa$, positive, so it is not counted. This is repeated until no new negative symbols appear.

Figure 2 Moving the dimension d with the knob changes the number of symbols of negative degree at each degree (top ϕ^4, bottom Yang–Mills. Bar heights are logarithms of the counts; the dotted line is the degree of the noise)



d = 3.00, noise degree −2.500, symbols of negative degree: φ⁴ 8, Yang–Mills 99

Computed The numbers in this figure and Table 3 were computed on this machine (`trees.py`; the number of "shapes" by the rules above; — means not computed).

d	2	2.8	3	22/7	10/3	3.5	3.7	3.8
φ⁴	4	5	8	10	22	44	525	38561
YM	11	27	99	163	1918	65197	—	—

ϕ^4 agrees at $d = 3$ with the 8 of § 05. $\mathcal{I}(\Psi^3)\Psi^2$, added at $d = 14/5$, is the borderline where the split of § 04 breaks down, and the two added at $d = 22/7$ are the borderline that Hairer 2014 § 9.2 flags as $\alpha = -18/7$. Yang–Mills, with half the margin, grows much faster, and at $d = 4$ stacking $\mathcal{I}(\tau)\Psi^2$ does not change the degree, so there are infinitely many. Computed For ϕ^4, Chandra–Moinat–Weber take the noise in $\mathcal{C}^{-3+\delta}$ to imitate non-integer dimensions, and give a priori estimates over the whole subcritical range. Known

07

Gauge fields: the DeTurck term and covariance

The components of the gauge field $A = (A_1, \dots, A_d)$ are Lie algebra elements (matrices), and $[X, Y] = XY - YX$ is the Lie bracket (1-03). $d_A = d + [A, \cdot]$ is the covariant derivative, F_A its curvature (the field strength), and d_A^* the adjoint of d_A (the operator corresponding to the transpose with respect to the inner product) (1-07). Applying formula (1) to the action $S(A) = \int |F_A|^2$ and recounting time by a factor 2 gives $\partial_t A = -d_A^* F_A + \xi$. Even without noise, this equation is not **parabolic** (the type that, like the heat equation $\partial_t u = \Delta u$, smooths out fine fluctuations in every direction over time). This is because the action does not change under gauge transformations, so no smoothing force acts in the direction of the orbits (Chevyrev 2022 § 1.2). Adding the term tangent to the orbits $-d_A d^* A$ ($d^* A = -\sum_i \partial_i A_i$) (Zwanziger 1981, Donaldson 1985; for Ricci flow, DeTurck 1983)

$$\partial_t A_i = \Delta A_i + [A_j,\, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i \quad (9)$$

makes it parabolic (ibid. (1.3); in shorthand $\partial_t A = \Delta A + A\partial A + A^3 + \xi$). Formally, $B = A^g$, the solution A moved by a time-dependent gauge transformation g, also satisfies the same equation with the noise $\mathrm{Ad}_g \xi = g\xi g^{-1}$ (rotated by sandwiching with g), and $\mathrm{Ad}_g \xi$ has the same law as ξ. This **gauge covariance in law** is the ground for defining a process on the orbits. Known

In renormalisation, a mass term $(C^\varepsilon_{\mathrm{bphz}} + \mathring{C})A$ is added to the equation with the noise smoothed at width ε. $C^\varepsilon_{\mathrm{bphz}}$ (the BPHZ constant) is the part depending on ε; in two dimensions it converges to a finite value by cancellation, and in three dimensions it diverges like ε^{-1} (Chevyrev 2022 Remark 1.10). $\mathring{C}$ (the bare mass) is the finite part independent of ε, and can be chosen freely. Both this mass term and the smoothing of the noise break covariance. Chandra–Chevyrev–Hairer–Shen (CCHS) showed, in two dimensions (2022) and three dimensions (2024), that choosing $\mathring{C}$ to be a certain value $\check{C}$ makes the two breakings cancel in the limit. Known

Whether $\check{C}$ is fixed uniquely has been answered in two stages. CCHS showed that there is a unique choice in the sense of a certain coupling (a construction that places the solutions from two gauge-equivalent initial values on the same probability space with rotated noise and compares them) (Chevyrev 2022 Theorem 1.14). That there is only one mass renormalisation giving solutions gauge covariant in law was shown in two dimensions by Chevyrev–Shen (Comm. Pure Appl. Math. 2026, Theorem 2.15 of arXiv:2302.12160) and in three dimensions by Chevyrev–Shen (Arch. Ration. Mech. Anal. 2026, arXiv:2503.03060), strengthening the result of CCHS. Known

Example: for $G = U(1)$ the equation is $\partial_t A = \Delta A + CA + \xi$, and on T^d only $C = 0$ is covariant (with $C \neq 0$, equivalent initial values with A_1 shifted by the constant 2π give different expectations of $e^{i\int A_1}$; ibid. § 1.4). Known

08

What has been reached for Yang–Mills

A **state space** for the solutions to live in is also needed. In two dimensions line integrals along segments make sense, and the space of orbits is a Polish space (a complete metric space with a countable dense subset; convenient for handling probability measures) (CCHS 2022). In three dimensions line integrals of the Gaussian free field already diverge ($\mathbb{E}|A(\ell)|^2 \asymp \int_0^1\int_0^1 |\ell|/|t-s|\,ds\,dt = \infty$), so one uses a non-linear space in which gauge equivalence is defined for fields smoothed by the Yang–Mills flow (the deterministic flow obtained by removing the noise from formula (9)) (Chevyrev 2022 § 3.1, 7-06). What was obtained is a Markov process on the space of

orbits (§ 04), in a form that leaves open the possibility of blow-up in finite time. In two dimensions Chevyrev–Shen (Comm. Pure Appl. Math. 2026) showed that the Yang–Mills measure on T^2 is its unique invariant measure, by moment estimates using solvability, gauge fixing on the lattice, and Bourgain's argument (7-02). Gauge fixing, avoided in the dynamics, is used on the lattice in the proof of the invariant measure. Known

Known Table 4 What stochastic quantisation has reached (finite volume is, as a rule, the torus).

Model	Local solutions, process	Invariant measure	Infinite volume, axioms
ϕ^4_2	Da Prato–Debussche 2003 (strong solutions. Processes came earlier: Jona-Lasinio–Mitter 1985, Albeverio–Röckner 1991)	A process converging to the ϕ^4_2 measure (known from constructive field theory): Jona-Lasinio–Mitter 1985 (square, Dirichlet boundary, equation smoothed by a power of an operator)	Known by other methods (7-03)
ϕ^4_3	Hairer 2014	Mourrat–Weber 2017	Gubinelli–Hofmanová 2021 (except rotation invariance and clustering)
YM two dimensions	CCHS 2022 (with possible blow-up)	Chevyrev–Shen (Comm. Pure Appl. Math. 2026): the YM measure is the unique invariant measure	$T^2 \to \mathbb{R}^2$ not yet by this method
YM three dimensions	CCHS 2024 (likewise; also with Higgs. Uniqueness of the mass renormalisation: Chevyrev–Shen, Arch. Ration. Mech. Anal. 2026)	Open	Not yet
YM four dimensions	Critical, outside the general theory of regularity structures		

The open problems of the survey are the existence of the invariant measure (in three dimensions, the construction of a Yang–Mills measure, which does not yet exist even in finite volume), $T^d \to \mathbb{R}^d$ (not trivial even in two dimensions), non-trivial principal bundles, and $N \to \infty$ (Chevyrev 2022 § 4). Chevyrev–Shen (Comm. Pure Appl. Math. 2026) attribute the restriction to two dimensions to the moment estimates required by Bourgain's argument, and leave extension to higher dimensions open. Reflection positivity is another task (7-01). Known

09

Detour: coming down from infinity

A section you may skip.

The standard way to make an invariant measure is to take a limit from uniform estimates on time averages (Krylov–Bogoliubov), and what works is an estimate that "from any initial value however large, it comes back to a fixed size in a fixed time". The simplest example is $\dot{x} = -x^3$.

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2\, t}} \longrightarrow \frac{1}{\sqrt{2t}} \qquad (x_0 \to \infty) \quad (10)$$

The size does not exceed $1/\sqrt{2t}$, and the initial value is forgotten. At $t = 0.01$ the solutions from $x_0 = 10, 10^3, 10^6$ are 5.7735, 7.0709, 7.0711, lining up at the limit 7.0711. The solution $x_0 e^{-t}$ of the linear $\dot{x} = -x$ keeps remembering its initial value (for $x_0 = 10^6$, 9.90×10^5). Computed

Mourrat–Weber (2017) showed an estimate of this type (the "coming down from infinity" of the title) for three-dimensional ϕ^4 and made the invariant measure (abstract). For Yang–Mills, even for abelian $U(1)$ the spatial average $\int A_i$ of the solution of formula (9) moves like Brownian motion and there is no invariant probability measure; an invariant measure appears only after passing to orbits (Chevyrev 2022 Remark 1.17). The estimate needed in the non-abelian case in three dimensions is the content of the open problems of § 08. Known

10

What can be carried to four dimensions

What can be carried over: the viewpoint of seeing the measure as the invariant measure of a dynamics compatible with gauge. The state space that compares after smoothing by the Yang–Mills flow (on the lattice side too, Lüscher 2010 writes that theory and numerics suggest the flowed field becomes a smooth renormalised field Physics). That the DeTurck term was originally a device of Ricci flow is a clue for thinking, in 16-06, about "what corresponds to Ricci flow".

What cannot be carried over: the premise of local subcriticality. In four dimensions the margin is 0, and the symbols do not stay finite in number (Table 3). Even in three dimensions, the existence of the invariant measure, infinite volume and reflection positivity remain. The mass gap is a property of the spectrum of the infinite-volume Hamiltonian (4-11), and does not come directly from an invariant measure in finite volume. How four dimensions stops is summarised in 7-10.

What this chapter can say

	Content
Said	The simulation of formula (1) agrees with the integral within 0.3%, and the time to cross between wells grows exponentially with h ($T \cdot h/e^{h}$ is 1.17 at $h = 8$, approaching the Kramers coefficient $\pi/(2\sqrt{2}) \approx 1.11$). The margin is $4 - d$ for ϕ^4 and $(4-d)/2$ for $A\partial A$. The symbols of negative degree number 8 for ϕ^4 at $d = 3$ (agreeing with Hairer 2014), 38561 at $d = 3.8$, and 65197 for Yang–Mills at $d = 3.5$ Computed
Said	The Yang–Mills measure on T^2 is the unique invariant measure of the Langevin dynamics. Three-dimensional Yang–Mills(–Higgs) has reached local solutions and a Markov process on the orbit space. The mass renormalisation that makes it gauge covariant in law is unique in both two and three dimensions Known
Not said	The number of symbols is a number of "shapes", defined differently from the "symbols that can contribute to renormalisation" counted in the literature. The three-dimensional Yang–Mills measure and the extension to four dimensions (open). The content of the proofs of regularity structures. Complete coverage of the literature from 2025 on

Sources and reproduction

Item	Kind	Source / tools
Degrees, products, formula (7), symbols	Text checked (§ 1, § 8, § 9, § 10)	M. Hairer, Invent. Math. 198 (2014) 269–504, arXiv:1303.5113
Quotations, formula (9), Remark 1.10, Theorem 1.14, Remark 1.17	Survey, text checked	I. Chevyrev, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359
The invariant measure in two dimensions, uniqueness of the mass renormalisation	Text checked (§ 1, Theorems 2.15 and 2.16, the title of § 9.1)	I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026), arXiv:2302.12160
Results in § 04– § 09	Checked in the abstracts	CCHS: arXiv:2006.04987 (Publ. IHÉS 2022), 2201.03487 (Invent. Math. 2024). Chevyrev–Shen: 2503.03060 (Arch. Ration. Mech. Anal. 250 (2026) Paper No. 11). Da Prato–Debussche, Ann. Probab. 31 (2003). Mourrat–Weber: 1601.01234. Gubinelli–Hofmanová: 1810.01700. Chandra–Moinat–Weber: 1910.13854
Jona-Lasinio–Mitter	From the survey (original papers from bibliographic data only)	S. J. Summers, arXiv:1203.3991v2 (2016) § 5. Jona-Lasinio–Mitter, CMP 101 (1985) 409–436
Albeverio–Röckner	Bibliographic data only (reference list of Hairer 2014)	S. Albeverio, M. Röckner, Probab. Theory Related Fields 89 (1991) 347–386

Item	Kind	Source / tools
§ 01, § 05, § 07, § 10	Bibliographic data only (reference list of Chevyrev 2022)	Parisi–Wu, Sci. Sinica 24 (1981). Zwanziger, Nucl. Phys. B 192 (1981). DeTurck, J. Diff. Geom. 18 (1983). Donaldson, PLMS 50 (1985). Gubinelli–Imkeller–Perkowski, Forum Math. Pi 3 (2015). Bruned–Hairer–Zambotti, Invent. Math. 215 (2019)
Lüscher's "suggest" in § 10	Checked in the abstracts	M. Lüscher, JHEP 2010(8) 071
Kramers' formula in § 01	Standard result (original paper not checked)	An approximation to the mean time to cross a hill, fixed by the second derivatives at the bottom of the well and the top of the hill. The ratio to the T of Table 1 is computed by `kramers.py`
Tables 1–3, the figures, the numbers in § 09	Computed	Python on this machine: `langevin.py` (about 1 minute), `kramers.py` (T and $T \cdot h/e^h$ for $h = 0.5$ to 8), `trees.py` (counting in rational numbers; checked against the 8 for ϕ^4 at $d = 3$ and the borderline at $d = 22/7$; about 30 seconds), `fig2_pack.py`, `comedown.py`. The figures are canvas and plain JavaScript

Next chapter to read: 7-10 Where four dimensions stops. Stochastic quantisation as a candidate for a new concept is looked at in 16-06.

Where four dimensions stops — the ultraviolet is a prospect in finite volume; what was stuck was infinite volume and a uniform gap (the map before the 2026 publication)

This chapter is a map of open questions. § 01, § 03, § 06, § 08 and § 09 are established knowledge.

What this chapter shows — That the setters (Jaffe–Witten 2000) distinguished in their wording between the construction on T^4, a "prospect", and $T^4 \to \mathbb{R}^4$ and a uniform gap, for which "no present ideas point the direction". What has been reached in four dimensions at each of the six stages, and the words of the literature. Reading the clustering formula of the problem statement with the numbers of a free field.

Prerequisite chapters — 7-07 Four dimensions: Bałaban and MRS. The six stages are in 7-01, non-triviality in 7-08, the definition of the gap in 4-11.

Said first — This chapter rearranges the words of known literature step by step, and proves no theorem. In the literature before the 2026 publication, the construction of four-dimensional Yang–Mills is not complete, even on T^4. Literature after 2000 is limited to what we checked in the text or the abstract. For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet-limit step, the infinite-volume step along subsequences of volumes, and the axiom step are proved in Lean. Where things are stuck has moved to uniqueness of the infinite-volume limit and the uniform-gap step (6-04; in Volume 3, 16-02 and 16-03).

The order of this chapter

1. The setters' two sentences — the construction on T^4 has "ample indication"; for a uniform gap and $T^4 \to \mathbb{R}^4$ "no present ideas point the direction"
2. A map by stage — overlaying the six stages with what has been reached in four dimensions and the words of the literature (a moving figure)
3. Regularisation and finite volume: closed on the lattice — on the lattice it becomes a finite-dimensional integral, but rotation symmetry is broken

4. The ultraviolet-limit step: a "prospect" in finite volume — the closed form is the limit of renormalised Wilson loops; what remains is uniqueness and non-triviality
5. The infinite-volume step: it was stuck here — there is no step that crosses, independently of the volume, the scales where the coupling grows to $O(1)$
6. The axiom step: the axioms and the additional conditions of the problem statement — reflection positivity is inherited; the problem statement adds local fields, asymptotic freedom, non-triviality and $m < \infty$
7. The uniform-gap step: a gap uniform in the volume — the setters write that a uniform gap could open the road to infinite volume
8. Reading the clustering formula JW (2) — seeing $C < \Delta$ and "sufficiently far" with the numbers of a free field (computation, a moving figure)
9. Detour: a gap not visible classically — in Debye screening, a change of variables makes the mass visible

01

The setters' two sentences

§ 6 of the problem statement (Jaffe–Witten 2000, JW below) divides the state of four dimensions into two sentences. Known

'Some results are known for Yang–Mills theory on a 4-torus T^4 approximating R^4, and, while the construction is not complete, there is ample indication that known methods could be extended to construct Yang–Mills theory on T^4.' / 'Yet, even if this were accomplished, no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume. Nor do present methods suggest how to obtain the existence of the infinite volume limit $T^4 \to R^4$.'

The construction on the torus T^4 (a box with opposite faces glued together) is not complete, but there are ample signs that it can be made by extending known methods. For a gap uniform in the volume and for $T^4 \to \mathbb{R}^4$, no present idea points the direction. Writing μ_L for the measure on a box of side L and Δ_L for the gap of the Hamiltonian, the objects are

$$\underbrace{\mu_L = \lim_{a \to 0} \mu_{a,L} \quad (L \text{ fixed})}_{\text{ample indication}} \qquad \underbrace{\inf_L \Delta_L > 0, \qquad \lim_{L \to \infty} \mu_L}_{\text{no ideas point the direction}} \tag{1}$$

($\mu_{a,L}$ is the measure at lattice spacing a; formula (6) of 7-01). **Example:** the setters distinguish the strength of their assessments in words (Table 1). This chapter maps that difference onto the stages.

Known Table 1 Words of strength in the problem statement (checked in the text of the Clay PDF).

Words	Gloss	About what	Section
'ample indication'	enough signs	The construction on T^4	§ 6
'no present ideas point the direction'	no idea points the way	A gap uniform in the volume	§ 6
'New ideas are needed'	new ideas required	A gap uniform in the volume	§ 6.5
'may play a fundamental role'	can be fundamental	Uniform gap → infinite-volume limit	§ 5

02

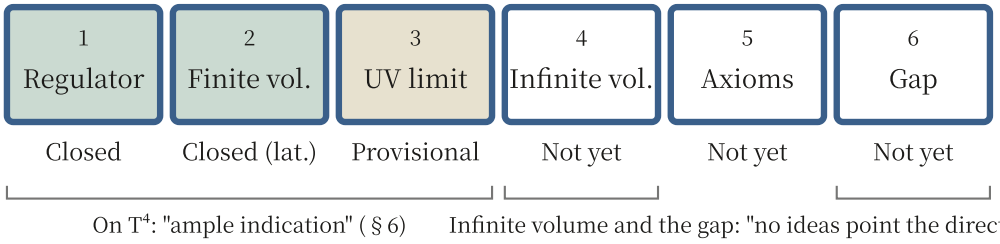
A map by stage

We overlay the six steps of 7-01 with what has been reached in four dimensions and the words of the literature. What has been reached is one of three: **Closed** (there is a theorem), **Provisional** (a theorem under a hypothesis, or a sketch without a detailed proof) and **Not yet** (no theorem). The steps from the regularisation step to the ultraviolet-limit step are inside the double limit

$$\mu = \lim_{L\to\infty} \lim_{a\to 0} \mu_{a,L} \quad (2)$$

the infinite-volume step is outside it, and the axiom step and the uniform-gap step are properties of μ. The first sentence before formula (1) points to the inside, the second to the outside and the uniform-gap step.

Figure 1 Choosing a work outlines the stages it mentions and shows the points (pressing a stage box or a row of Table 2 shows the words for that stage). "Play" cycles through the works




```
Problem statement 2000
Regulator step: Wilson defined a gauge-invariant action on the lattice (JW §6.5).
Finite vol. step: For a compact group and space-time, the lattice reduces the functional integral
to a finite-dimensional integral (JW §6.5).
UV limit step: The construction on T⁴ is not complete but there is ample indication (JW §6).
Infinite vol. step: Present methods do not suggest T⁴ → R⁴ (JW §6).
Axioms step: Reflection positivity holds in the Wilson approximation, is hard to recover once
lost, and setting up the Hilbert space is part of the problem (JW §6.5).
Gap step: For a gap uniform in the volume no present ideas point the direction, and new ideas are
needed (JW §6, §6.5).
```

Known The sentences in this figure are the same as in Table 2 (from the texts of the problem statement, the report, the papers and the surveys). The brackets below show that the first sentence of §6 points to the steps from regularisation to the

ultraviolet limit, and the second to the infinite-volume step and the uniform-gap step.

Known Table 2 What has been reached at each step for four-dimensional Yang–Mills (as a continuum theory; from the literature before the 2026-11-10 publication; the published construction is in the paragraph after the table).

Stage	Reached	Words of the literature (key points)	Chapter
1 Regulator	Closed	Wilson defined a gauge-invariant action on the lattice (JW § 6.5)	5-01
2 Finite vol.	Closed (lat.)	For a compact group and space-time, the lattice reduces the functional integral to a finite-dimensional integral (JW § 6.5). The continuum measure is not constructed on T^3 even in three dimensions (Chevyrev 2022)	7-06
3 UV limit	Provisional	The construction on T^4 is not complete but there is ample indication (JW § 6). The ultraviolet limit is a sketch, with no detailed proof (MRS 1993). As far as Douglas knows there has been no breakthrough in recent years, and the state of the art is still Bałaban and MRS (Douglas 2004). There is no construction of a non-trivial continuum limit above two dimensions (Chatterjee 2018). In four dimensions, "a form of ultraviolet stability" (Chevyrev 2022)	7-07
4 Infinite vol.	Not yet	Present methods do not suggest $T^4 \to R^4$ (JW § 6). The infrared cutoff is not removed; removing it makes the coupling large, and there is no solvable model to serve as a foothold (MRS 1993). Uniqueness at large β is generally unknown for d > 2 (Chatterjee 2018)	5-04
5 Axioms	Not yet	Reflection positivity holds in the Wilson approximation, is hard to recover once lost, and setting up the Hilbert space is part of the problem (JW § 6.5). With an infrared cutoff the OS axioms cannot be treated as a whole (MRS 1993)	4-10
6 Gap	Not yet	For a gap uniform in the volume no present ideas point the direction, and new ideas are needed (JW § 6, § 6.5). The lattice correlation-length problem (Chatterjee's Problem 5.1): finite correlation length at every β, infinite as $\beta \to \infty$ (Chatterjee 2018).	5-09

Within what this chapter has read, the four-dimensional results after 2000 are of two kinds. One is on the side of a fixed lattice spacing (§ 05). The other is a special scaling limit of an $SU(2)$ model with a Higgs field added (one of the matter fields that couple to the gauge field; 3-07), taken by moving the lattice spacing, the coupling and the length of the Higgs field together (in any dimension $d \geq 2$). That limit is Gaussian (a free field), and the author writes 'The question of constructing a non-Gaussian scaling limit remains open' (Chatterjee 2026). Known Within what this chapter has read, there is no literature before the 2026 publication that claims to close the steps from the continuum ultraviolet-limit step to the uniform-gap step for pure Yang–Mills.

For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet-limit step (convergence along the whole sequence of spacings chosen by the family), the infinite-volume step along subsequences of volumes, and the axiom step (each component of the phases satisfies a variant of the OS axioms, from which a

Wightman theory is reconstructed) are proved in Lean. The uniform-gap step and uniqueness of the infinite-volume limit are not shown (6-01, 6-04; for reconstruction, 12-08).

03

Regularisation and finite volume: closed on the lattice

The regularisation step and the finite-volume step are closed even in four dimensions, on the lattice. JW § 6.5 writes 'the lattice approximation reduces the functional integration to a finite-dimensional integral', and says that reflection positivity also holds (Osterwalder–Seiler 1978). Known In the notation of Chatterjee (2018), with P the set of plaquettes (the smallest squares) and U_p the product of group elements around p,

$$S(U) = \sum_{p \in P} \operatorname{Re}\operatorname{tr}\left(I - U_p\right), \qquad 0 \le \operatorname{Re}\operatorname{tr}(I - U_p) \le 2N, \qquad e^{-2N\beta|P|} \le Z = \int e^{-\beta S(U)}$$

This is because the eigenvalues of $U_p \in SU(N)$ have absolute value 1, and the Haar measure $\mathrm{d}U_\ell$ has total weight 1. **Example:** a four-dimensional periodic lattice with 4 points on a side has $6 \times 4^4 = 1536$ plaquettes, so for $SU(2)$, $e^{-6144\beta} \le Z \le 1$. Computed

In exchange, the lattice breaks rotation symmetry, and a continuum cutoff breaks gauge invariance (7-07 § 06). On reflection positivity, JW § 6.5 writes that if it is lost in regularisation there are few ways of recovering it (quoted in 0-03 § 03). Known A four-dimensional regularisation keeping all four of gauge invariance, reflection positivity, Euclidean invariance and ultraviolet control at once was not found in the literature this chapter read (this chapter's reading).

04

The ultraviolet-limit step: a "prospect" in finite volume

We read the content in 7-07 (Bałaban 1989 is estimates with a hypothesis, MRS 1993 a sketch). The setters write in § 6.5: Known

'One presumably needs to revisit known results at a deep level, simplify the methods, and extend them.'

Review, simplify, extend. What is asked for at the ultraviolet-limit step is not new ideas but this work. The closed form can be written with Wilson loops W_γ (the $\frac{1}{N}\operatorname{tr}$ of the product of group elements along a closed path γ; 5-01). For lattice paths $\gamma_a \to \gamma$

$$\langle W_{\gamma_a} \rangle^{\mathrm{ren}}_{a,L} \xrightarrow[a \to 0]{} \langle W_\gamma \rangle_L \qquad \text{(for the whole sequence, not a subsequence, to a limit that}$$

(ren is the renormalisation of the next paragraph; the form follows Chatterjee 2018 § 5, § 6).

Why renormalisation is needed can be seen in the continuum string-tension problem of Chatterjee 2018 (Problem 5.2; the continuum limit). On an infinite four-dimensional lattice, writing $\gamma_{R/a,T/a}$ for the rectangular path of physical size $R \times T$ (R/a and T/a steps on the lattice), the problem is to find sequences, as $\beta \to \infty$, such that

$$\log \left\langle W_{\gamma_{R/a,\,T/a}} \right\rangle = -c\,(R+T) - \sigma\,RT + o(1), \qquad a = a(\beta) \to 0, \quad c = c(\beta) \to \infty, \quad \sigma \neq$$

(σ is the string tension of the continuum theory; the original's symbol is d). $R+T$ is proportional to the perimeter, and RT is the area. Since $c \to \infty$, as it stands $\log\langle W\rangle \to -\infty$, that is, the expectation collapses to 0. Adding back the term $c(R+T)$ proportional to the perimeter before taking the limit is the renormalisation here. Known But Chatterjee himself writes 'it is not clear to me whether the renormalization term c(R + T) is indeed necessary', so whether this term is really needed is also an open question.

Example: for the abelian $U(1)$ four-dimensional lattice there is a construction of the continuum limit (Driver 1987). Chatterjee 2018 § 6 describes Driver as using in four dimensions the method Gross showed for three-dimensional $U(1)$ as convergence in a sense different from the usual one. Known For an abelian group the terms of the lattice action higher than quadratic (self-interactions of the field) do not survive in the continuum limit, and the limit is expected to be the free electromagnetic field: that is this chapter's explanation (Driver's text not read).

For the non-abelian case there are two steps before reaching formula (4). (i) Ultraviolet stability only gives convergence of subsequences (Chatterjee 2018 § 6). Convergence of the whole sequence, that is, uniqueness of the limit, is separate work (7-07 § 05). (ii) Even if the limit exists, it may be Gaussian (a free field). Not being a free field, that is, non-triviality, is also separate work (7-08 § 10). Footnote 2 does not accept weak existence by compactness as a solution, unless the properties of the limit are shown separately (end of § 6.5). The committee's assessment is that the construction published on 2026-11-10, for correlations of local observables, gets past (i) with convergence along the whole sequence of spacings chosen by the family, and also shows the non-triviality of (ii). It does not, however, build the Wilson-loop observables of formula (4) themselves (6-04).

05

The infinite-volume step: it was stuck here

The setters' words are a single sentence: 'Nor do present methods suggest how to obtain the existence of the infinite volume limit' (§ 6). From the inside, MRS 1993 § I says removing the infrared cutoff makes the coupling large, and writes 'there is no easy solvable model of these phenomenons around which to expand'. Known Laying out the sequence of renormalisation group steps (7-05) along the axis of scales,

$$\underbrace{a \to 2a \to \cdots \to a0}_{g^2 \le \gamma\text{: stage 3 (provisional)}} \quad \longrightarrow \quad \underbrace{a0 \to \cdots \to \ell_*}_{g^2 \text{ grows to } O(1)} \quad \longrightarrow \quad \underbrace{\ell_* \to \cdots \to L \to \infty}_{\text{stage 4}} \tag{5}$$

a_0 is the coarsest scale the hypothesis reaches ($g^2(a_0) \le \gamma$; as in 7-07, γ is used as an upper bound on g^2), and ℓ_* is the rough-guide length where extending the running gives $g^2 = 1$; for $SU(2)$ with $\gamma = 0.3$, $\ell_*/a_0 \approx 10^{10.7}$ (two loops; Table 2 of 7-07 § 04). Computed Reading ℓ_* as the scale of confinement is a physics estimate. Physics In the organisation of this book, what is missing is the middle: one step that crosses the scales where the coupling grows to $O(1)$ with estimates independent of the volume (7-04 § 10). The construction published on 2026-11-10 takes the infinite-volume limit along subsequences of volumes from estimates independent of the volume (Part 12, 12-03, 12-05). Uniqueness of the limit and exponential decay uniform in the volume are not shown (6-04).

Example: with the lattice spacing fixed, the infinite-volume step is closed on the strong-coupling side (Osterwalder–Seiler 1978). Shen–Zhu–Zhu 2023 showed, for $SU(N)$ with $|\beta| < 1/(16(d-1))$ ($1/48 \approx 0.0208$ when $d = 4$), uniqueness of the infinite-volume measure and exponential decay of correlations. This β is the value in 't Hooft's convention. In this convention the inverse coupling in front of the action is written βN, so converting to the β of formula (3) multiplies it by N, and the condition is $|\beta| < N/48$ ($N^2/48$ in this book's Wilson β; the conversions among the five conventions are in Table 4 of 5-01). On the weak-coupling side even uniqueness is generally unknown for non-abelian groups (Chatterjee 2018 § 3), and exponential decay has been shown only for finite groups (Adhikari–Cao 2025, 5-10). $U(1)$ has a non-confining phase at large β (Guth 1980, Fröhlich–Spencer 1982), and that there is no phase transition in the non-abelian case is only believed Physics (5-06).

06

The axiom step: the axioms and the additional conditions of the problem statement

The axiom step is the Osterwalder–Schrader axioms (4-10). Reflection positivity holds at each lattice spacing, and being an inequality it is inherited by the limit (7-01 § 07).

Recovering rotation symmetry is separate work, and setting up the Hilbert space is itself part of the problem (JW § 6.5). Known How the published construction satisfies these is read in Part 12 (12-01 Lattice reflection positivity, 12-06 Euclidean invariance, 12-08 Reconstruction and the Wightman axioms).

§ 4 adds conditions. (i) Local fields corresponding to gauge-invariant local polynomials in the curvature such as $\mathrm{Tr}\, F_{ij}F_{kl}(x)$ (footnote 1: there is no natural one-to-one correspondence, because of renormalisation). (ii) Agreement at short distances with the predictions of asymptotic freedom and perturbative renormalisation (including the stress tensor and the operator product expansion). (iii) Non-triviality (7-08). And

$$\mathrm{spec}\, H \cap (0, \Delta) = \varnothing \quad (\Delta > 0), \qquad m = \sup\{\Delta\}, \qquad m < \infty \ (6)$$

($H \geq 0$ is the Hamiltonian, m the mass). **Example:** since $H \geq 0$, $m < \infty$ is the same as the spectrum of H containing a positive point, that is, $H \neq 0$. If $m = \infty$, there is no spectrum in $(0, \infty)$, and $H = 0$. Further, using the form that states uniqueness of the vacuum with respect to translations (exactly one translation-invariant unit vector; the wording of Axiom 3 in nLab's "Wightman axioms"), the spectral condition (energy and momentum lie in the forward cone) makes the momentum 0 as well, every vector becomes translation invariant, and the Hilbert space consists of the vacuum alone. Under axioms of this form, $m < \infty$ can be read as a condition excluding the empty theory (this chapter's reading). The summary in § 3 of the problem statement takes the form "a Poincaré-invariant vacuum vector unique up to a phase", from which the same conclusion does not follow immediately. The example that fails (iii) is four-dimensional ϕ^4, whose limit from the lattice is Gaussian (Aizenman–Duminil-Copin 2021).

07

The uniform-gap step: a gap uniform in the volume

On a lattice in a finite box the gap exists automatically (7-01 § 08), and the question is its independence of the side L. JW § 6.5 writes 'New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time. Such a result presumably would enable the study of the limit as $\mathrm{T}^4 \to \mathrm{R}^{4}$' (§ 5 points the same way; Table 1). Known

The setters write that a uniform gap in finite volume (the uniform-gap step) could open the road to infinite volume (the infinite-volume step) (the 'may play a fundamental role' of § 5, the 'presumably would enable' of § 6.5). In 7-01 the infinite-volume step was placed first because the gap is defined for the theory on $\mathbb{R}^4$, not because of the order of proof. The mechanism is a one-line estimate. For the box

Hamiltonian H_L, vacuum Ω_L and gap Δ_L, separating the vacuum component from the rest by the spectral decomposition,

$$\left|\langle \Omega_L,\, A\, e^{-tH_L} B\, \Omega_L\rangle - \langle \Omega_L, A\,\Omega_L\rangle\langle \Omega_L, B\,\Omega_L\rangle\right| \;\le\; \|A^*\Omega_L\|\,\|B\,\Omega_L\|\, e^{-\Delta_L t} \tag{7}$$

(correlations separated by imaginary time t), and if $\Delta_L \ge \Delta_* > 0$ independently of L, the rate of decay does not depend on L either. Known **Example:** on the strip of the two-dimensional Ising model, $E_1 - E_0$ closes like $1/L$ at the critical point and closes exponentially at low temperature (2-09 § 07, § 08). For weakly coupled ϕ^4_2 and ϕ^4_3 the expansion in a small parameter gave the infinite-volume step and the uniform-gap step at once (7-04 § 08), but the long distances of four-dimensional Yang–Mills have no such parameter.

08

Reading the clustering formula JW (2)

JW § 5 writes, as a consequence of the gap, 'For any positive constant C < Δ and for any local quantum field operator O … such that ⟨Ω, OΩ⟩ = 0, one has (2) |⟨Ω, O(x)O(y)Ω⟩| ≤ exp(−C|x − y|), as long as |x − y| is sufficiently large' (x, y are points of space. Below this inequality is written JW (2), to distinguish it from formula (2) of this chapter). Known The point is $C < \Delta$ and a "sufficiently large" distance. The right-hand side needs no constant because C is taken strictly smaller than Δ, so constant factors and power factors alike lose, far enough away, to $e^{-(\Delta - C)r}$. The equal-time correlation of a free scalar field of mass Δ is

$$\langle \Omega, \phi(\underset{\sim}{x})\phi(\underset{\sim}{y})\Omega\rangle = \frac{\Delta\, K_1(\Delta r)}{4\pi^2 r} \;\sim\; \frac{\sqrt{\pi/2}}{4\pi^2}\,\frac{\Delta^{1/2}}{r^{3/2}}\, e^{-\Delta r} \qquad (r = |\underset{\sim}{x} - \underset{\sim}{y}| \to \infty) \tag{8}$$

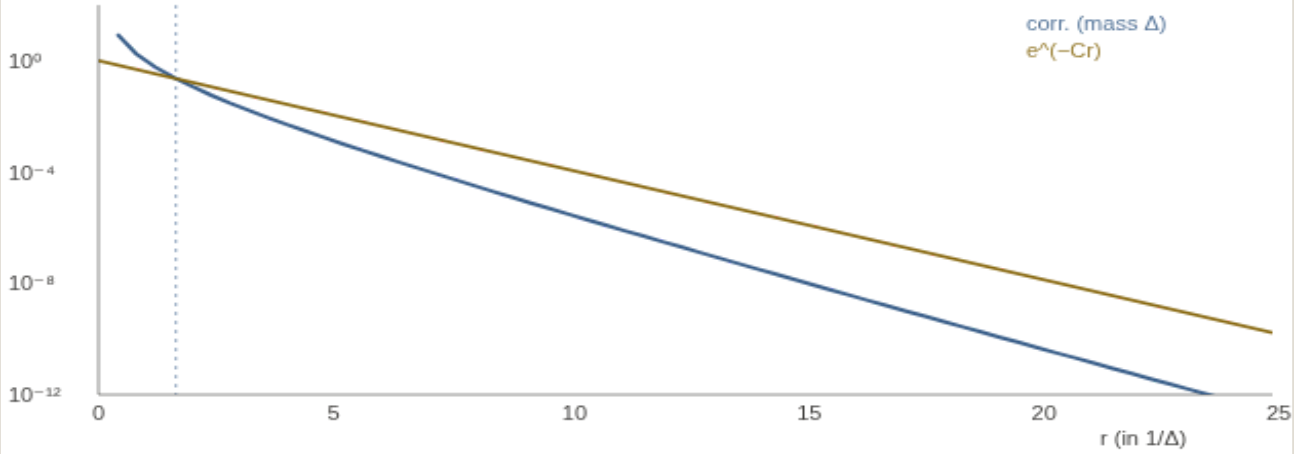
which has the same form as the Euclidean two-point function (K_1 is the modified Bessel function of the second kind. The point field is a stand-in for a local operator; smearing changes the form far away only by a constant factor). Normalising to 1 at $r = 1/\Delta$, we compare with e^{-Cr}, and also set mass 0 (correlation proportional to $1/r^2$) beside it.

Computed Table 3 The range of r in which the normalised correlation is at most e^{-Cr} (r in units of $1/\Delta$, examined for $r \le 5000$).

C/Δ	Mass Δ	Mass 0
0.10	$r \ge 1.04$	$1.05 \le r \le 90$
0.50	$r \ge 1.25$	$1.43 \le r \le 8.6$
0.90	$r \ge 1.65$	never holds

C/Δ	Mass Δ	Mass 0
1.00	$r \geq 1.82$	never holds
1.05	$1.93 \leq r \leq 132$	never holds
1.20	$2.47 \leq r \leq 17.8$	never holds

Figure 2 Moving C/Δ changes the distance at which the line e^{-Cr} and the correlation (logarithmic vertical axis) swap places. "Compare with Δ = 0" overlays the massless correlation



C/Δ = 0.90: with mass Δ, r ≥ 1.65 is where it holds

Computed The numbers in this figure were computed on this machine (formula (8). The vertical dotted lines are the distances where JW (2) starts to hold and starts to fail, for mass Δ).

How to read it: (a) for $C < \Delta$ it holds beyond r of about 2. (b) For $C = \Delta$ it still holds thanks to the power $r^{-3/2}$, but the problem statement only speaks of $C < \Delta$. (c) For $C > \Delta$ it always fails far away. (d) For mass 0, every $C > 0$ fails far away. The slope far away, $-\frac{\mathrm{d}}{\mathrm{d}r}\log$, is 1.153Δ at $r = 10$ and 1.015Δ at $r = 100$, approaching Δ from above.

Computed Lattice measurements sum over space to extract the zero-momentum component, remove the same kind of power factor from correlations in the time direction, and look at sums of exponentials e^{-mt} (5-08). For a free field, the correlation in the time direction obtained by integrating the Euclidean two-point function over space is exactly $e^{-\Delta|t|}/(2\Delta)$. Known To cross from the uniform-gap step to the infinite-volume step, JW (2) and formula (7) must hold independently of the size of the box.

09

Detour: a gap not visible classically

A section you may skip.

JW § 6.6 writes '... a theory in which the mass gap is not classically visible, to demonstrate it may require a non-classical change of variables or "duality transformation."' Linearising the

classical Yang–Mills equations for small fields gives the wave equation, with frequency $\omega(k) = |k|$: massless (3-03). The example given is Debye screening in the Coulomb gas (a gas of positive and negative point charges exerting on each other a potential $1/r$ inversely proportional to the distance r). That the force between test charges decreases exponentially is shown by rewriting the partition function as that of sine-Gordon field theory (the theory of a scalar field ϕ with a $\cos(\lambda\phi)$ term in the action) and approximating the bottom of the well by a parabola (Brydges–Federbush 1980). Known

$$-z\cos(\lambda\phi) = -z + \tfrac{1}{2}\, z\lambda^2\, \phi^2 + O(\phi^4) \quad (9)$$

The coefficient $z\lambda^2$ of ϕ^2 plays the role of the mass squared (z is the activity = how readily particles appear; with chemical potential μ and temperature T, $z \propto e^{\mu/k\mathrm{B}T}$), and a mass invisible in the language of $1/r$ becomes visible as the curvature of the well. Precisely, (9) is an approximate form, and the proof estimates the conditions under which it works. The same section also lists the quartic term $(A \wedge A)^2$ (Feynman 1981), bound states in quantum mechanics with flat directions (Simon 1983; 2-05), and large-N duality (17-04).

What this chapter can say

	Content
Said	The sentences of the problem statement § 3– § 6.6 and its footnotes, the 2004 report (text of the Clay PDF), MRS 1993, Chatterjee 2018 (Problems 5.1 and 5.2, § 6) and Chevyrev 2022. Shen–Zhu–Zhu and Adhikari–Cao from the text of § 1 (formulas (1.2)(1.3), introduction), Chatterjee 2026 from the sentences of the abstract Known
Said	In the literature before the 2026 publication, in the four-dimensional continuum, the ultraviolet-limit step is provisional (with a hypothesis, or a sketch) in finite volume, and the infinite-volume, axiom and uniform-gap steps are not yet. With the lattice spacing fixed, the infinite-volume step and the uniform-gap step are closed on the strong-coupling side. The one-line estimates of formulas (3) and (7) Known
Said	JW (2) read with a free field: Table 3, Figure 2 and the slope far away. $\ell_*/a_0 \approx 10^{10.7}$ (the computation of 7-07) Computed
Not said	That the road from a uniform gap to the infinite-volume limit actually goes through for four-dimensional Yang–Mills (the setters too write 'may', 'presumably')
Not said	"The missing step", reading $m < \infty$ as a condition excluding the empty theory (depends on the form of uniqueness of the vacuum), the view that no regularisation keeps all four (this book's readings). That the limit of Driver 1987 is the free electromagnetic field (text not read). Complete coverage of the literature after 2000 (Douglas's 2026 survey from bibliographic data only). For none of the steps from the ultraviolet-limit step to the uniform-gap step in four dimensions does this book show anything of its own (the reading of the published construction is in Volume 2)

Sources and reproduction

Item	Kind	Source / tools
Problem statement (§ 4–§ 6.6, footnotes 1 and 2, reference list)	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000)
The 2004 report	Text	M. R. Douglas, *Report on the Status of the Yang–Mills Millenium Prize Problem*, Clay Mathematics Institute (2004)
Reservations on the infrared cutoff and the axioms	Text pp. 325–327	J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383
The action, uniqueness, Problems 5.1 and 5.2, ultraviolet stability and the description of Driver (§ 6)	Text	S. Chatterjee, *Yang–Mills for probabilists*, arXiv:1803.01950 (2018)
The form stating uniqueness of the vacuum with respect to translations (§ 06)	Text	nLab, *Wightman axioms* (Axiom 2, Axiom 3)
Ultraviolet stability, three dimensions	Text §1.1, §4	I. Chevyrev, arXiv:2202.13359, J. Math. Phys. 63 (2022) 091101
The theorem of Bałaban 1989	Abstract and introduction (7-07)	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392
Results at strong coupling and for finite groups	Text §1 (formulas (1.2)(1.3), introduction; cross-checked in 16-01)	H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 / A. Adhikari, S. Cao, arXiv:2202.10375, Ann. Probab. 53 (2025) 140–174
The result for Yang–Mills–Higgs	Abstract only	S. Chatterjee, arXiv:2401.10507, Probab. Math. Phys. 7 (2026) 339–381
Other literature in §03–§06 and §09	Bibliographic data only	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 / Guth, Phys. Rev. D 21 (1980) 2291–2307 / Fröhlich–Spencer, CMP 83 (1982) 411–454 / Driver, CMP 110 (1987) 479–501 / Aizenman–Duminil-Copin, Ann. of Math. 194 (2021) 163–235 / Brydges–Federbush, CMP 73 (1980) 197–246 / Feynman, Nucl. Phys. B188 (1981) 479–512 / Simon, Ann. Phys. 146 (1983) 209–220
Douglas's 2026 survey (the table of scope)	Bibliographic data only	M. R. Douglas, Nature Reviews Physics 8 (2026) 86–97 (bibliographic data from search results)

Item	Kind	Source / tools
Table 3, Figure 2, the slope far away, the example of formula (3)	Computed	Python on this machine (SciPy's K_1; agrees with the integral representation via the heat kernel to a relative difference within 2×10^{-8}. Step 10^{-4} for $r \le 50$, step 10^{-2} for $50 < r \le 5000$). The example of formula (3) is arithmetic
$\ell_*/a_0 \approx 10^{10.7}$ (§ 05)	Computed	The computation of 7-07 (Table 2; $SU(2)$, $\gamma = 0.3$, two loops)

Next chapter to read: 7-11 Obstacles other than divergence, which lists the reasons the infinite-volume, axiom and uniform-gap steps were stuck as obstacles distinct from divergence. A chronology and a table by dimension are in 16-01.

Obstacles other than divergence — problems remain even on a lattice that contains not a single infinity

> This chapter organises established knowledge. § 02 and § 08 are a map of open questions.

What this chapter shows — Six obstacles that remain even on a lattice with the ultraviolet divergences removed, and how much of each is a problem of divergence. Seeing three of them by computing the smallest examples (orders of perturbation theory and $e^{-c/\alpha}$, a sum of signs equal to 0 in gauge fixing, a vacuum overlap that goes to 0 through volume alone).

Prerequisite chapters — 7-10 Where four dimensions stops. The tools are 2-06 (Fock space), 3-04 (the gauge orbit space), 4-08 (gauge fixing), 4-10 (reflection positivity) and 5-01 (the lattice).

Said first — This chapter rearranges obstacles found in the literature, and removes none of them. Counting them as six, and judging whether each is a divergence, is this book's organisation. The computations are smallest examples, not computations for four-dimensional Yang–Mills.

The order of this chapter

1. The test — fix the lattice spacing
2. No small parameter in the infrared — change the group
3. Invisible to perturbation theory — Table 1, Figure 1
4. Not collapsing to a free field — triviality and 0/0
5. The state space is curved — the part reducible to divergence, and the 0/0 that remains (Table 2)
6. Reflection positivity — an example where it fails
7. The structure of infinitely many degrees of freedom — vacuum overlaps (Figure 2)
8. The six side by side — Table 4
9. Detour: topological sectors and θ

01

The test: does it remain with divergence removed?

The explanation that "the difficulty is handling infinities" is one side of it. This chapter puts each obstacle to the following test. **Does the obstacle remain even with the lattice spacing a kept fixed (with the ultraviolet cutoff left in place)?** If it remains, no handling of divergences will dispose of it.

Divergences were quantities that go to infinity when the cutoff is removed, like the variance of the free field at a point, $G_a(0) \approx 0.1549/a^2$ (in four dimensions) (7-01 § 02). Lattice gauge theory in a finite box with fixed lattice spacing is a finite-dimensional integral.

$$\langle F \rangle_{\beta,\Lambda} = \frac{1}{Z_{\beta,\Lambda}} \int F(U)\, e^{-\beta S(U)} \prod_{\ell \in \Lambda} \mathrm{d}U_\ell, \qquad 0 < Z_{\beta,\Lambda} \leq 1 \quad (1)$$

Λ is the set of edges of the box, $\mathrm{d}U_\ell$ the Haar measure of total weight 1, S the Wilson action, and β the inverse lattice coupling (not an inverse temperature; large β is weak coupling) (formula (3) of 7-10). There is not a single infinity.

Example: even so, for the theory with only the box extended to $\mathbb{Z}^4$, Chatterjee's talk slides (no date given; the PDF was created in September 2020) write 'Even showing any kind of correlation decay at large β for non-Abelian theories in 3D and 4D is open.' Known Since then, Adhikari–Cao have shown decay of correlations at weak coupling for finite groups. Known Within the literature this chapter has read, what is open now is the case of continuous non-abelian Lie groups ($SU(N)$ and so on). The results are summarised in Table 4.

02

No small parameter in the infrared

Many mathematical estimates are expansions in a small number, but the coupling of four-dimensional Yang–Mills is weak at short distances (asymptotic freedom) and strong at long distances, and no foothold for an expansion is found on the infrared side (7-10 § 05). The small β (strong coupling) where the expansion converges is the shore opposite to the continuum limit $\beta \to \infty$. More precisely, the range of β where the strong-coupling expansion converges is limited to near 0, and does not overlap with $\beta \to \infty$, which the continuum limit needs.

The target on the lattice is the lattice correlation-length problem of Chatterjee (2018) (Problem 5.1): for the length $\xi(\beta)$ (in lattice units) over which correlations of gauge-

invariant local quantities decrease like $e^{-|x-y|/\xi}$,

$$0 < \xi(\beta) < \infty \quad \text{(for all } \beta > 0), \qquad \lim_{\beta \to \infty} \xi(\beta) = \infty \quad (2)$$

is to be shown (four dimensions, compact non-abelian Lie group). Known The first half is that there is a gap at every lattice spacing; the second half is the prospect of a continuum limit (5-09).

Example: changing the group breaks (2). $U(1)$ enters a massless phase at large β, and the area law fails (Guth 1980; a complete proof is Fröhlich–Spencer 1982). Known For finite groups correlations decrease exponentially at weak coupling (Adhikari–Cao) Known , but they are thought not to confine and to have no continuum limit. Physics $SU(N)$ is believed to be in the confining phase at every β. Physics

An argument that works equally for every group goes wrong for $U(1)$, so what ought to work is "for a continuous non-abelian group, the gap survives even with the transition point at $\beta = \infty$" (5-06, 5-10). What is missing is an estimate that crosses, independently of the volume, the scales where the coupling grows to $O(1)$.

03

Invisible to perturbation theory

In the physics picture, the mass gap is a constant multiple of the single scale Λ of the theory (4-07), and at one loop in the renormalisation group it is related to the coupling g at scale μ as follows. Physics

$$\frac{\Lambda}{\mu} = \exp\Big(-\frac{1}{2b_0 g^2}\Big) = \exp\Big(-\frac{cN}{\alpha}\Big), \qquad b_0 = \frac{11N}{48\pi^2}, \quad \alpha = \frac{g^2}{4\pi}, \quad cN = \frac{6\pi}{11N} \quad (3)$$

b_0 is the coefficient of the running of the coupling $\mu\,\mathrm{d}g/\mathrm{d}\mu = -b_0 g^3 + \cdots$ (4-06). The right-hand side approaches 0 faster than any power.

$$\lim_{\alpha \to 0+} \alpha^{-n}\, e^{-c/\alpha} = 0 \qquad (n = 0, 1, 2, \dots) \quad (4)$$

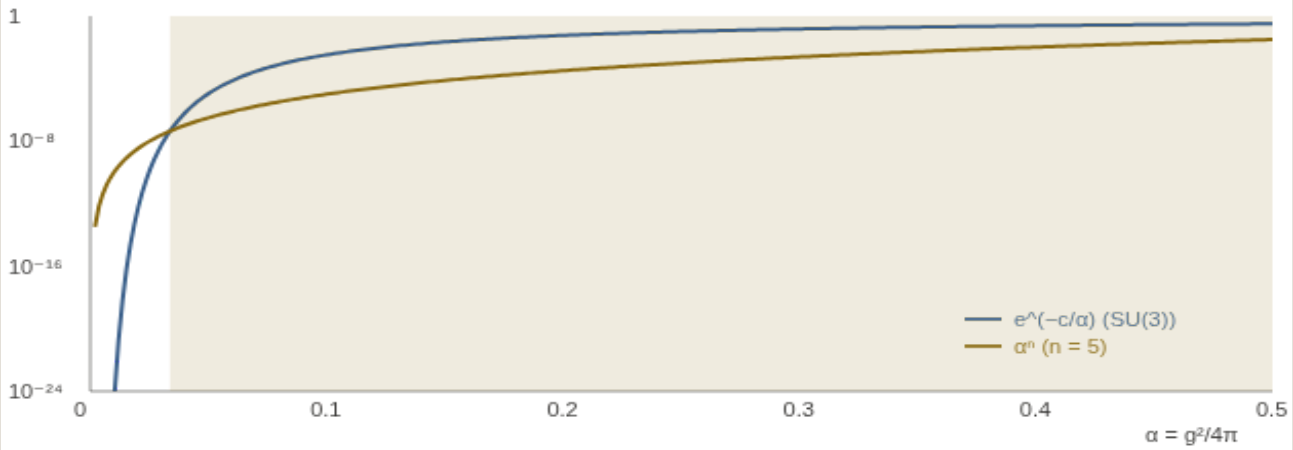
So all Taylor coefficients at $\alpha = 0$ are 0, and perturbation theory answers $\Lambda = 0$ at every order (for quantum mechanics, 2-06). Known

Example: the order $n_* = cN/\big(\alpha \ln(1/\alpha)\big)$ at which α^n equals $e^{-cN/\alpha}$ grows as the coupling gets smaller.

Computed Table 1 $e^{-cN/\alpha}$ and the order n_* ($c_2 = 0.8568$, $c_3 = 0.5712$).

α	$SU(2)$	n_*	$SU(3)$	n_*
0.30	5.75e−2	2.37	1.49e−1	1.58
0.10	1.90e−4	3.72	3.31e−3	2.48
0.05	3.61e−8	5.72	1.09e−5	3.81
0.02	2.48e−19	10.95	3.95e−13	7.30
0.01	6.16e−38	18.61	1.56e−25	12.40

Figure 1 Moving n moves the band where $e^{-c/\alpha}$ exceeds α^n (vertical axis in common logarithms). To the left of the band there always remains a range where α^n is larger



SU(3), n = 5: 0.0337 < α < 0.878 is the only range where e^(−c/α) exceeds αⁿ. For α < 0.0337, αⁿ is larger

Computed The numbers in this figure were computed on this machine (formula (3) and bisection for the crossing points).

What is missing is not the value of an infinity but information that does not appear in the series. Divergence of the series is a separate phenomenon. The problem statement (Jaffe–Witten 2000, JW below) writes in § 6.1 that perturbation series are in many examples 'divergent expansions'. Beneke (1999) writes 'The sum of a convergent series may still differ from R by exponentially small terms exp(−1/α)'. Known Even a convergent series can miss $e^{-c/\alpha}$ (4-12, 7-12).

04

Not collapsing to a free field

Even if divergences are handled and a limit is taken, a free field (a Gaussian measure) is not an answer (JW § 4 asks for a non-trivial theory). From the two- and four-point correlations S_2, S_4 of a field with mean 0, form

$$u_4(x_1,x_2,x_3,x_4) = S_4 - \big[S_2(x_1,x_2)S_2(x_3,x_4) + S_2(x_1,x_3)S_2(x_2,x_4) + S_2(x_1,x_4)S_2(x_2,$$

and for a Gaussian $u_4 \equiv 0$ (Wick's rule; 7-08 § 01).

Example: for four-dimensional ϕ^4, the infinite-volume scaling limit with the ultraviolet cut on a lattice and approaching the critical point is Gaussian (Aizenman–Duminil-Copin 2021). Known The divergences have been handled, and still the limit is a free field. A special scaling limit of $SU(2)$ Yang–Mills–Higgs is also Gaussian, and a non-Gaussian limit is said to 'remain open' (Chatterjee 2026). Known

What to hold fixed is a problem of the same kind. The mass in physical units, $m_{\text{lat}}(\beta)/a(\beta)$, has the form 0/0, and what has meaning is the limit of dimensionless ratios such as $m/\sqrt{\sigma}$, that is, ratios of rates of approach to 0 (5-09). The requirement is in 7-08 § 10.

05

The state space is curved

The place to put the measure is the orbit space, the space of connections divided by gauge transformations, which is not a linear space (3-04). An orbit is a set of connections carried into one another by gauge transformations. Singer (1978) showed that on S^4 ($\mathbb{R}^4$ with one point at infinity added), for non-abelian compact groups, one cannot continuously choose exactly one connection from each orbit ('No gauge fixing is possible.'). Known "Curved" in the title is a metaphor; precisely, it refers to these two things (not being a linear space, and representatives not being choosable globally and continuously).

The part reducible to divergence

With the formal $\mathrm{d}A$ (meant as Lebesgue measure in infinitely many dimensions), $Z = \infty$ (Chatterjee 2018), and the volume of the orbits is also expected to be infinite (Singer 1978). The lattice disposes of this. Formula (1) is finite without fixing the gauge.

The 0/0 that remains

What remains is when gauge fixing is needed in weak-coupling analysis. Bałaban uses local gauges that change from cell to cell (7-05 § 05). Fixing a single covariant (Landau-type) gauge on the whole lattice BRST-symmetrically produces the following 0/0 (Neuberger 1987; the formula is (2) of Mehta–Daleo–Hauenstein–Seaton 2014). Known We define the words used, one line each (details in 4-08).

- **Landau-type gauge**: a fixing that builds a function F (a function of the gauge transformation) measuring how far the transformed edge variables are from 1, and chooses its critical points as representatives of the orbit. The F of the example below has this form.

- **Gribov copies**: when there are several critical points of F on one orbit, each one of them.
- **Faddeev–Popov matrix** M_{FP}: the matrix of second derivatives of F at a critical point. The gauge-fixed integral has $\det M_{\mathrm{FP}}$ as its weight.
- **BRST-symmetric fixing**: a way of fixing so that a formal symmetry (BRST) remains after gauge fixing. Then the contribution of one orbit is the sum of $\operatorname{sign}\det M_{\mathrm{FP}}$ over all the copies.
- **Euler characteristic** χ: the number obtained by adding up the critical points of a smooth function (with non-degenerate critical points) on a compact manifold without boundary (1-05), each with the sign of the determinant of the matrix of second derivatives ($+1$ if the number of negative eigenvalues is even, -1 if odd). It does not depend on the choice of function (the Poincaré–Hopf theorem). For a surface it is "the number of minima and maxima − the number of saddles", and for the sphere it is 2. On a circle a minimum counts $+1$ and a maximum -1, so it is 0.

$$Z_{\mathrm{GF}} = \sum_{\text{Gribov copies}} \operatorname{sign}\det M_{\mathrm{FP}} = \chi(G)^N = 0 \tag{6}$$

N is the number of lattice points, and a gauge transformation places one element of the group G at each point, so the space moved over is the product of N copies of G. The sum of signs is its Euler characteristic $\chi(G)^N$, and for compact connected Lie groups of positive dimension $\chi(G) = 0$. Gauge-invariant quantities are constant on orbits, so for gauge-fixed expectation values both numerator and denominator become 0: 0/0.

Example: for a smooth function on the circle $S^1 = U(1)$, maxima and minima alternate and the sum of signs is $\chi(S^1) = 0$. On a ring of N points we counted all the critical points of $F = \sum_j \left(1 - \cos(\phi_j + \theta_{j+1} - \theta_j)\right)$ in formula (3) of Mehta et al. (θ_j is the angle of the gauge transformation at point j, with $\theta_0 = 0$ to remove the freedom of rotating everything uniformly; ϕ_j are the edge angles, uniform random numbers, 300 orbits).

Computed Table 2 Gauge fixing on a $U(1)$ ring.

N	Number of copies	Sum of signs	Number of minima
2	2	0	1
3	6	0	1–2
4	12	0	2
8	280	0	4
12	5544	0	6

The number of copies was the same on every orbit examined, growing exponentially as $\frac{N}{2}\binom{N}{N/2}$ for even N, and the sum of signs was always 0. Computed Restricting to minima avoids $0/0$, but that the number of minima can vary from orbit to orbit ($N = 3$) is another difficulty (Mehta et al. § I). The 0 comes from topology. Singer himself named as a way round it localisation, gluing local covers together with a partition of unity (4-08, 7-12).

06

Few regularisations keep reflection positivity

The road back from Euclidean to space-time depends on reflection positivity $\langle \Theta F\, F\rangle \geq 0$ (Θ is time reflection, F a quantity on the positive-time side) (4-10). The Wilson action keeps it (Osterwalder–Seiler 1978) and breaks rotation symmetry. Continuum regularisations are the reverse: the problem statement writes that there are few ways to recover reflection positivity lost in regularisation, giving dimensional regularisation and Pauli–Villars as examples (JW § 6.5; quoted in 0-03 § 03). Known

Example: fixing the spatial momentum p of a free field, the time dependence of the two-point function is $G(t) = e^{-\omega|t|}/(2\omega)$ ($\omega = \sqrt{p^2 + m^2}$). For a Gaussian field, reflection positivity is, for positive times t_i and real numbers f_i,

$$Q(f) = \sum_{i,j} f_i f_j\, G(t_i + t_j) \;\geq\; 0 \qquad (7)$$

If G is a sum of exponentials with positive weights $\rho(E)$, then $Q = \sum_E \rho(E)\big(\sum_i f_i e^{-Et_i}\big)^2 \geq 0$. Pauli–Villars subtracts a term with a heavy mass M, giving $G_{\mathrm{PV}}(t) = e^{-\omega t}/(2\omega) - e^{-\Omega t}/(2\Omega)$ ($\Omega = \sqrt{p^2 + M^2}$), so for f with $\sum_i f_i e^{-\omega t i} = 0$, $Q = -\frac{1}{2\Omega}\big(\sum_i f_i e^{-\Omega t_i}\big)^2 < 0$. With $p = 0$, $m = 1$, $M = 5$, $t = (0.1, 0.2)$ and $f = (1, -e^{0.1})$, $Q = -4.0 \times 10^{-3}$. Computed As $M \to \infty$ the violation shrinks, but it cannot be used at the stage where the cutoff is kept.

That Wilson loops must be smoothed by the heat flow above two dimensions (Cao–Chatterjee) is an ultraviolet problem, but regularisations that keep positivity, recovery of rotation symmetry, and putting observables valued on curves rather than points (Chatterjee 2018) into the axioms are problems of structure. Known

07

The structure of infinitely many degrees of freedom

With finitely many oscillators the irreducible representation of the canonical commutation relations is essentially unique (Stone–von Neumann; 2-06), but with

infinitely many, countless inequivalent representations appear. Known One consequence is Haag's theorem, by which an interacting theory cannot be put on the Fock space of the free field (in the interaction picture) (Haag 1955, Hall–Wightman 1957; 4-09).

The mechanism can be seen without infinities. The overlap of the ground states of oscillators with frequencies $\omega \neq \omega'$ is $\left(2\sqrt{\omega\omega'}/(\omega+\omega')\right)^{1/2} < 1$, and the overlap of the vacua of free fields of masses m and m' on a three-dimensional periodic lattice with L points on a side and spacing a is the product over modes.

$$\langle \Omega_m, \Omega_{m'} \rangle = \prod_{k} \left(\frac{2\sqrt{\omega_k \omega_k'}}{\omega_k + \omega_k'} \right)^{1/2}, \qquad (a\omega_k)^2 = (am)^2 + 4\sum_{j=1}^{3} \sin^2 \frac{\pi n_j}{L} \tag{8}$$

The product runs over the L^3 modes $k = (n_1, n_2, n_3)$, $n_j = 0, \dots, L-1$ (ω_k' is the same with m'). The logarithm for each mode is negative.

Example: we compare two limits. A keeps the lattice spacing fixed ($am = 0.5$) and enlarges the box; B keeps the box fixed (side $2/m$) and refines the lattice. Both increase the number of points on a side, L.

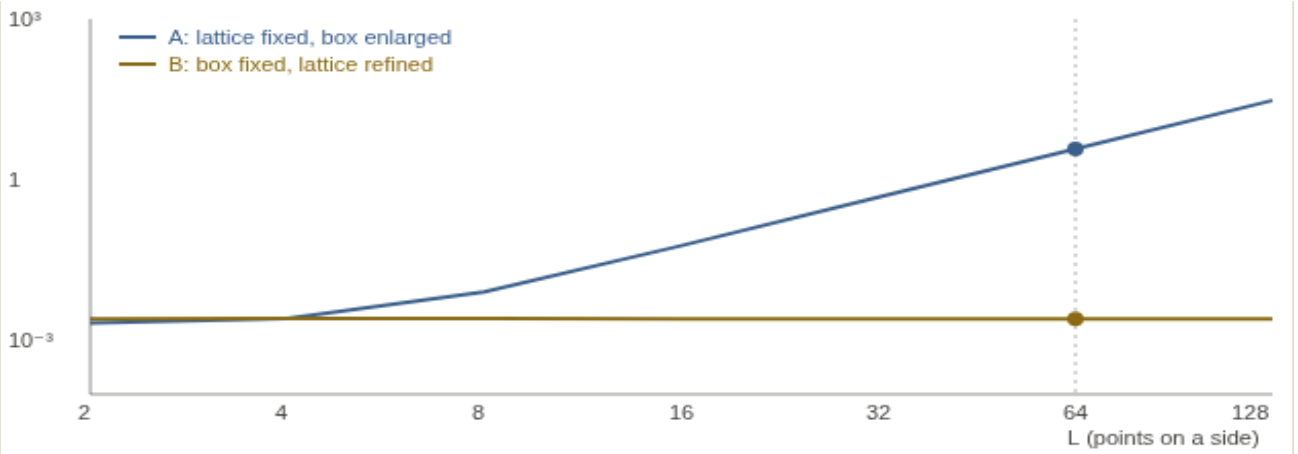
Computed Table 3 The vacuum overlap $\langle \Omega_m, \Omega_{m'} \rangle$ for $m' = 1.2m$.

L	Overlap in A (box side $0.5L/m$)	Overlap in B (lattice spacing $2/(Lm)$)
4	0.9974	0.99741
32	0.6237	0.99745
64	0.02289	0.99746
128	7.53e−14	0.99747

In A, $-\ln\langle \Omega_m, \Omega_{m'} \rangle$ is proportional to L^3 (1.44×10^{-5} per point), and the overlap goes to 0. Computed

The reason the overlap settles to a positive value in B lies in the high-momentum modes. When ω_k is large, the contribution of one mode to the logarithm is about $-(m'^2 - m^2)^2/(64\omega_k^4)$, and ω_k grows nearly in proportion to $|k|$, so the whole is a sum of the form $\sum_k |k|^{-4}$. In three dimensions the shell $|k| \approx R$ has about R^2 modes, and the contribution of each shell decreases like R^{-2}, so this sum converges. Computed

Figure 2 Choosing m'/m changes $-\ln\langle \Omega_m, \Omega_{m'} \rangle$ (log–log). "Play" increases L. A has slope 3 (proportional to the volume), B is flat




```
m'/m = 1.2, L = 64: overlap in A 0.0229 (box side 32/m) / overlap in B 0.99746 (lattice spacing 0.0313/m)
```

Computed The numbers in this figure were computed on this machine (formula (8); m′/m = 1.1, 1.2, 1.5, 2).

The overlap goes to 0 because of the volume, not the ultraviolet (that free fields of different masses give inequivalent representations is said to be a generalisation of Haag's theorem; Reed–Simon). Constructive field theory makes the measure as a density in a finite box on a lattice, and reconstructs the state space from its limit (4-10). The state space is an output, not an input (Chatterjee 2018's 'we do not even know for certain the right spaces').

We contrast this with a related phenomenon. The vacua of this section (at a fixed time, on a lattice in three space dimensions) have positive overlap in finite volume (B of Table 3). Comparing two free fields of different masses as measures on Euclidean four-dimensional space-time, they are mutually singular even in a finite box (torus), because of the ultraviolet (the detour in 1-09, formula (11)). How it acts divides between space dimension 3 and space-time dimension 4.

08

The six side by side: how much is divergence

Table 4 The six obstacles and their divergent part (the part that only arises when the ultraviolet cutoff is removed; the judgements are this book's organisation).

Obstacle	With a fixed	Divergent part	What is missing
1 Infrared parameter	Remains	None	A uniform estimate crossing the scales where the coupling is $O(1)$
2 Invisible to perturbation theory	Remains	None	Handling the $e^{-c/\alpha}$ that does not appear in the series
3 Non-triviality	A question about the continuum limit	None	A proof that it is not a free field

Obstacle	With a fixed	Divergent part	What is missing
4 State space	Local gauges and 0/0	The formal $Z = \infty$, the volume of the orbits	Something to replace global gauge fixing
5 Reflection positivity	Recovering rotations	Observables of rough fields	Regularisations that keep positivity, axioms for observables on curves
6 Infinitely many degrees of freedom	Remains (orthogonal through volume)	None in this example (free fields, different masses). For measures on four-dimensional space-time the ultraviolet acts too (1-09)	Making the state space from the theory

In the literature before the 2026 publication, the ultraviolet side was not complete either (Bałaban's four-dimensional result went only as far as estimates with a hypothesis, and subsequential limits were a prospect; 7-07 § 05). For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet side and the infinite-volume limit along subsequences of volumes are proved in Lean (6-01). Even so, the side of a gap uniform in the volume, for which the setters asked for new ideas, is not shown (7-10 § 01, 6-04). Among the obstacles of Table 4, the obstacle of the infrared parameter lies on the side that the published construction has not crossed either (this book's reading). Outside the six are the correspondence with local fields (JW footnote 1) and topological sectors (detour). The seams are in 16-05 (a glossary of names in the bilingual table of 16-05), the tasks in 16-07.

09

Detour: topological sectors and θ

A section you may skip.

Smooth connections divide into sectors by an integer topological charge (the second Chern number) (3-05). On the lattice things change. For two-dimensional $U(1)$, from the angle $\theta_p \in (-\pi, \pi]$ of the plaquette p (the principal value of the argument of U_p), form

$$Q = \frac{1}{2\pi} \sum_p \theta_p \ \in \ \mathbb{Z} \quad (9)$$

which is an integer on a periodic lattice. The sum of the edge angles as they are is 0, since each edge appears twice with opposite orientations, and the shifts to principal values are integer multiples of 2π. Paper But when the configuration is moved continuously, Q jumps when θ_p crosses $\pm\pi$, and it does not jump if one restricts to configurations in which every plaquette is close to 1 (the action density is small). Lüscher (1982) showed this for non-abelian lattice fields in four dimensions, and also showed that Q returns to the Chern number in the classical continuum limit. Known Typical configurations are rough, so sectors need a condition of this kind. The construction of Magnen–Rivasseau–Sénéor (1993, MRS) is restricted to the trivial sector (7-07 § 06).

A θ term multiplying the weight by $e^{i\theta Q}$ is invisible to perturbation theory. At $\theta = \pi$, it has been argued that because of an anomaly the vacuum cannot be a trivial, non-degenerate gapped state (Gaiotto–Kapustin–Komargodski–Seiberg 2017). Physics The problem statement does not mention θ, and implicitly refers to $\theta = 0$.

What this chapter can say

	Content
Said	The words of the primary sources for the six obstacles, and that they remain even with the lattice spacing fixed Known
Said	Tables 1–3, Figures 1–2, the Pauli–Villars example Computed , that the Q of formula (9) is an integer Paper
Not said	Counting six and the judgements of Table 4 (this book's organisation). Formula (3), $\Delta \propto \Lambda$, $\theta = \pi$ (at the level of physics). The computations are smallest examples, and formula (8) is not a proof of Haag's theorem. This book has removed none of the obstacles

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement, the lattice correlation-length problem, the talk slides	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Math. Inst. (2000) / S. Chatterjee, *Yang–Mills for probabilists*, arXiv:1803.01950 (2018) / same author, *Yang–Mills on the lattice: New results and open problems* (talk slides; PDF created 2020-09; no date given)
Exponentially small terms, impossibility of gauge fixing	Text	M. Beneke, *Renormalons*, Phys. Rept. 317 (1999) 1–142 / I. M. Singer, CMP 60 (1978) 7–12
Formula (6), the function on the ring, the caution about restricting to minima	Text § I–II	D. Mehta, N. S. Daleo, J. D. Hauenstein, C. Seaton, Phys. Rev. D 90 (2014) 054504, arXiv:1406.6678
Finite groups, $U(1)$, triviality, Gaussian limits, Wilson loops, topological charge, $\theta = \pi$	Abstracts	A. Adhikari, S. Cao, Ann. Probab. 53 (2025) 140–174, arXiv:2202.10375 / J. Fröhlich, T. Spencer, CMP 83 (1982) 411–454 / M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973 / S. Chatterjee, arXiv:2401.10507, Probab. Math. Phys. 7 (2026) 339–381 / S. Cao, S. Chatterjee, arXiv:2111.12813 / M. Lüscher, CMP 85 (1982) 39–48 / J. Magnen, V. Rivasseau, R. Sénéor, CMP 155 (1993) 325–383 / D. Gaiotto, A. Kapustin, Z. Komargodski, N. Seiberg, JHEP 05 (2017) 091

Item	Kind	Source / tools
Original sources for 0/0, Gribov, the phases of $U(1)$, reflection positivity	Bibliographic data only	H. Neuberger, Phys. Lett. B 183 (1987) 337 / M. Schaden, Phys. Rev. D 59 (1999) 014508 / V. N. Gribov, Nucl. Phys. B 139 (1978) 1–19 / A. H. Guth, Phys. Rev. D 21 (1980) 2291–2307 / K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471
Infinitely many degrees of freedom and uniqueness of representations	Text § 4.3	Stanford Encyclopedia of Philosophy, “Quantum Field Theory”
Haag's theorem and the generalisation to different masses	Bibliographic data only (checked via a secondary source)	R. Haag, Mat.-Fys. Medd. 29 (1955) no. 12 / D. Hall, A. S. Wightman, Mat.-Fys. Medd. 31 (1957) no. 5 / M. Reed, B. Simon, *Methods of Modern Mathematical Physics* II (1975), Theorem X.46 (from a secondary source; text not checked)
Table 1, Figure 1	Computed	Python on this machine. Formula (3) and n_*, bisection for the crossing points
Table 2	Computed	Critical points enumerated combinatorially, with signs and minima decided by eigenvalues ($N = 3$ agrees with Newton's method)
The Pauli–Villars example	Computed	Formula (7) evaluated at two points (negative for $M/m = 2, 5, 10, 100$)
Table 3, Figure 2	Computed	Formula (8) computed over all modes ($L \leq 180$). The oscillator formula agrees with numerical integration

Next chapter to read: 7-12 Detour: "if 1/0 is given a symbol", which looks again at the $e^{-c/\alpha}$ of § 03 and the 0/0 of § 05 from the side of symbols. From the physics side, 17-01.

Detour: "if 1/0 is given a symbol" — assigning a value erases the manner of approach, and the scale expected for the gap has the form e^{-1/g^2}

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — What systems that assign a value to $1/0$ give up. The manner of approach carried by $0/0$, and the blow-up that keeps it. e^{-1/g^2}, which looks like 0 at every order, and its tie to the poles of the Borel plane.

Prerequisite chapters — 4-05 The algebra of divergence, 4-12 Detour: perturbation series diverge. What is used is defined on the spot.

Said first — This is an organisation of known mathematics and literature, not a chapter that shows anything about Yang–Mills. "Making $1/0$ a symbol very probably does not change the core" is an assessment, not a theorem. $\Delta \propto \Lambda$ and renormalons are expectations at the level of physics.

The order of this chapter

1. Three readings of the question
2. Systems that assign a value to 1/0, and their price
3. Fixing the path fixes the value
4. 0/0 has a direction: the blow-up
5. Where replacements that keep the direction have worked
6. The algebra of divergence carries more than 1/0
7. A quantity that looks like 0 at every order
8. 1/0 appears in the Borel plane
9. Recording the discrepancy: transseries and renormalons
10. What a symbol should carry — Detour: from the side of logic

01

Three readings of the question

"If $1/0$ were built into mathematics as a symbol, wouldn't the situation of field theory, troubled by divergences, change?" The question can be read in three ways. (i) **Assign a value**: give $1/0$ or $0/0$ a single value (∞, 0, a new symbol $\perp$ and so on). (ii) **Replace**: exchange the singular point for the set of directions of approach, or a boundary. (iii) **Change the axioms**.

(iii) becomes a different problem, because the problem statement names the properties: 'Existence includes establishing axiomatic properties at least as strong as those cited in [45, 35]' (the Wightman and Osterwalder–Schrader axioms) (0-03). Known This chapter examines (i) and (ii), and arrives at the following.

$$\text{assigning a value} \Rightarrow \text{the manner of approach disappears}$$
$$\text{replacement keeping direction} \Rightarrow \text{worked in geometry and perturbation theory} \quad (1)$$

Example: $\sin x/x$ has the form $0/0$ at $x = 0$, yet the limit is fixed at 1. On a line there are two ways of approach, from the left and from the right, and both give the same value. In the plane it is not so (§ 04).

02

Systems that assign a value to 1/0, and their price

There are several systems that assign a value to $1/0$, all built without contradiction, but each gives something up (Table 1).

Known Table 1 Systems that assign a value to $1/0$ (IEEE 754 computed; transreal from bibliographic data only).

System	$1/0$, $0/0$	What it gives up
The Riemann sphere	$1/0 = \infty$	$0 \cdot \infty$ and $\infty - \infty$ undefined. Order breaks at ∞
Wheel (Carlström)	$0/0$ is also an element	$0x = 0$ does not hold in general. $x - x = 0x^2$, $x/x = 1 + 0x/x$
Meadow (Bergstra et al.)	$0^{-1} = 0$	Axiomatised just by adding formula (2) to the axioms of a commutative ring. $x\,x^{-1} \;/= 1$ is possible even with $x \;/= 0$
Transreal (Anderson et al.)	$\pm\infty$, $\Phi = 0/0$	Not a field. The distributive law is said to fail in part
IEEE 754 floating point	$1/\pm 0 = \pm\infty$, $0/0 = \mathrm{NaN}$	Reflexivity of equality and "$x = y \Rightarrow 1/x = 1/y$"
Division-by-zero calculus (Saitoh et al.)	$b/0 = 0$	Fixes the path of regularisation to one (§ 03)

A meadow adds to a commutative ring (a set in which addition, subtraction and multiplication obey the usual rules; division is not assumed) the operation of inverse $x \mapsto x^{-1}$, and imposes only

$$(x^{-1})^{-1} = x, \qquad x \cdot (x \cdot x^{-1}) = x \quad (2)$$

From this follows $0^{-1} = 0$. A field (a commutative ring in which one can divide by elements other than 0; $\mathbb{Q}$ and so on) with the convention $0^{-1} = 0$ is a meadow, and so is its direct product (componentwise operations). For example, for $x = (1, 0)$ in $\mathbb{Q} \times \mathbb{Q}$, $x^{-1} = (1, 0)$ and $x \cdot x^{-1} = (1, 0) \;/=\; (1, 1)$. Computed

Checking IEEE 754 double precision on this machine: in NumPy 1.26, `1/0`, `1/(-0)`, `0/0` give `inf`, `-inf`, `nan`; `0.0 == -0.0` is `True` and `nan == nan` is `False`; and Python's `1.0/0.0` raises the exception `ZeroDivisionError`. Computed The same numbers give different answers. "The value of $1/0$" is a convention of the system. The signed zero is a symbol that keeps one bit of "which side it came from", and in exchange the reciprocals differ although $0 = -0$. The tools of Yang–Mills (Hilbert spaces, spectra; 1-10) rely on $0 \cdot x = 0$, subtraction and equality. Nearby are non-standard analysis, which has $1/\varepsilon$ but does not define $1/0$, and Colombeau's algebras, which allow products of distributions at the cost of weakening equality (1-11).

03

Fixing the path fixes the value

§ 2.1 of Saitoh's draft (2023) regards b/a as the solution of the equation $ax = b$, and fixes it by Tikhonov regularisation. Known

$$x\lambda(a, b) = \arg\min_x \left[\lambda x^2 + (ax - b)^2\right] = \frac{ab}{\lambda + a^2}, \qquad \frac{b}{a} := \lim_{\lambda \to +0} x\lambda(a, b) \quad (3)$$

If $a \;/=\; 0$ it is the ordinary b/a; if $a = 0$, $x\lambda = 0$, so $b/0 = 0$. Replacing the penalty by $\lambda(x - c)^2$ gives

$$x\lambda = \frac{\lambda c + ab}{\lambda + a^2} \xrightarrow[\lambda \to +0]{} \begin{cases} b/a & (a \;/=\; 0) \\ c & (a = 0) \end{cases} \quad (4)$$

and the answer at $a = 0$ is c itself. **Example:** for $(a, b) = (0, 1)$, always $x\lambda = c$ from $\lambda = 1$ down to 10^{-8}. For $(a, b, c) = (2, 1, 5)$, $1.4,\ 0.511,\ 0.5001,\ 0.5$, approaching b/a. Computed

Formula (3) is also the trivial case of the solution of the equation $ax = b$ by the Moore–Penrose generalised inverse (draft § 2.1), and is a consistent definition. What can be said is that the value of $b/0$ comes not from the equation but from the added penalty:

the path of regularisation. What renormalisation in field theory tracks is this dependence on the path (§ 06).

04

0/0 has a direction: the blow-up

In the plane the limit changes with direction.

$$f(x,y) = \frac{2xy}{x^2+y^2} \qquad \big((x,y) \;/=\; (0,0)\big) \quad (5)$$

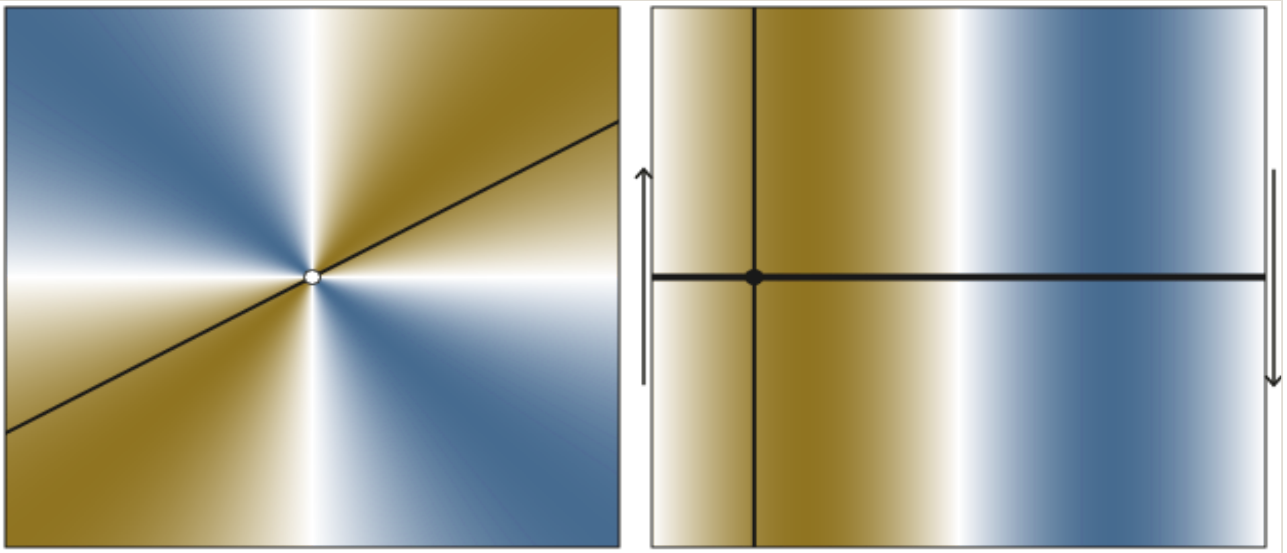
On the half-line $(r\cos\theta,\ r\sin\theta)$ at angle θ, $f = \sin 2\theta$, independent of r. Fixing one value at the origin does not make f continuous. Known

The **blow-up** is the operation of replacing the origin by the set of directions (the projective line $\mathbb{RP}^1$: the circle with θ and $\theta+\pi$ regarded as the same). A direction is written $[u:v]$. $[u:v]$ is $(u,v) \;/=\; (0,0)$ viewed up to non-zero multiples, that is, the name of a line through the origin ($[1:2] = [-2:-4]$). Precisely, one makes the set of pairs of a point and a direction

$$\widetilde{\mathbb{R}^2} = \big\{\, ((x,y),[u:v]) \;:\; xv = yu \,\big\} \quad (6)$$

Away from the origin the direction is fixed uniquely, so it is the same as the original plane; only over the origin does a circle sit, and the whole is an open Möbius band (1-05). On it, $f = 2uv/(u^2+v^2)$ is continuous, including on the circle over the origin.

Figure 1 Turning the direction θ with the knob shows the value $\sin 2\theta$ on approaching the origin from that direction, at once in the plane on the left and in the blow-up on the right. On the right the horizontal axis is $\theta \in [0, \pi)$ and the vertical the signed distance r; the central horizontal line $r = 0$ is the circle that replaces the origin. The left and right edges are glued upside down, as the arrows indicate. "Play" turns the direction



```
θ = 30°: approaching along the line in this direction, f → sin 2θ = 0.866 (the black dot on the central horizontal line of the blow-up)
```

Computed The values in this figure compute formula (5) within the figure (shade is $|f|$, the two colours the sign).

The picture on the right represents the point $(r\cos\theta,\ r\sin\theta)$ by $\theta \in [0, \pi)$ and signed r. Points with $r < 0$ lie on the side of direction $\theta + \pi$, but $\sin 2\theta$ does not change under $\theta \to \theta + \pi$, so the values form vertical stripes and do not break when crossing the central horizontal line $r = 0$. $\theta = \pi$ is the same direction as $\theta = 0$, and there the sign of r is exchanged, so (π, r) on the right edge is glued to $(0, -r)$ on the left edge. Since it is glued upside down, the band becomes a Möbius band. A symbol that assigns one value to $0/0$ has no place to carry these stripes.

05

Where replacements that keep the direction have worked

The examples where replacing the singular point has worked all keep the manner of approach. Known

- **Adding one point at infinity.** By the removable singularity theorem of Uhlenbeck (1982), solutions of the Yang–Mills equations with finite action on $\mathbb{R}^4$ come, up to gauge transformations, from smooth solutions on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ (3-05). Intuitively, the finite action constrains the manner of approach to infinity.
- **Replacing collisions by directions.** Fulton–MacPherson (1994) replace the diagonals (where two or more points coincide) of configuration space (all tuples of positions of n points) by a boundary that records the direction and scale of collision. Part II of Axelrod–Singer (1994) used a compactification of this type to show topological invariance of the perturbation theory of Chern–Simons theory, one of the gauge theories in three dimensions. Bergbauer–Brunetti–Kreimer (2009) showed that extension to wonderful models (De Concini–Procesi; a generalisation of the Fulton–MacPherson type of compactification) agrees with Epstein–Glaser renormalisation.

The same holds within Yang–Mills. Singer (1978) noted that, because there is no continuous global gauge fixing, 'an infinity in the determinant of the Jacobian' arises in the Faddeev–Popov method, and named the road of gluing local gauge fixings together with a partition of unity (3-04, 4-08). The lattice $\Delta = m_{\mathrm{lat}}/a$ has the form $0/0$ as $a \to 0$, and what has meaning is the limit of the dimensionless ratio $m_{\mathrm{lat}}/\sqrt{\sigma_{\mathrm{lat}}}$ (5-09). What has worked is on the side of geometry and perturbation theory, and it has not yet reached non-Gaussian measures and the mass gap.

06

The algebra of divergence carries more than 1/0

Ultraviolet divergences come as families of regularised quantities. A small model:

$$I(p;\eta) = \int_{\eta}^{\infty} \frac{e^{-px}}{x}\,dx,$$
$$I(p;\delta) = \int_{0}^{\infty} x^{\delta-1} e^{-px}\,dx = \Gamma(\delta)\,p^{-\delta} \quad (7)$$

The left diverges like $\ln(1/\eta)$ by cutting the lower end, the right like $1/\delta$ by shifting the power. Written as "1/0" they are the same, but their rates and forms differ, and still their difference goes to the same value.

$$I(1;\cdot) - I(2;\cdot) \;\longrightarrow\; \ln 2 = 0.693147\ldots \quad (8)$$

Computed Table 2 Two regularisations (SciPy on this machine).

η	$I(1;\eta)$	Difference	δ	$I(1;\delta)$	Difference
10^{-2}	4.037930	0.683222	10^{-2}	99.432585	0.686831
10^{-4}	8.633225	0.693047	10^{-4}	9999.422883	0.693083
10^{-8}	17.843465	0.693147	10^{-6}	999999.422785	0.693147

Choosing a scale μ and subtracting as $I_R(p;\mu) = \lim_{\eta\to 0}[I(p;\eta) + \ln(\eta\mu)] = -\gamma - \ln(p/\mu)$ (γ is Euler's constant), the finite part depends on μ but the difference does not. Computed This freedom, and how the finite part moves when μ is moved, are the seed of the renormalisation group (4-04).

In field theory, symbolisations that carry "how it diverges" have already been built mathematically (4-05): the BPHZ subtraction, the Epstein–Glaser method in which no infinity ever appears, and the Hopf algebra of Connes–Kreimer, which Birkhoff-decomposes the pole $1/\varepsilon$ of dimensional regularisation as a formal symbol. Perturbative renormalisability of Yang–Mills was established in 1971–72 by 't Hooft, 't Hooft–Veltman, Lee–Zinn-Justin and others (followed by an organisation through BRS symmetry), and Costello gave another proof in the BV formalism. Known Symbolisation has been realised in a form that carries more than $1/0$, and the problem still remains. What remains is not a problem of assigning values.

07

A quantity that looks like 0 at every order

$$h(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x \le 0) \end{cases} \quad (9)$$

is a calculus staple. For $x > 0$, $h^{(n)}(x) = P_n(1/x)\, e^{-1/x}$ with $P_{n+1}(u) = u^2(P_n(u) - P_n'(u))$ (for example $P_2 = u^4 - 2u^3$), and since $u^k e^{-u} \to 0$ as $u \to \infty$, $h^{(n)}(0) = 0$ for every n. The Taylor series is identically 0, yet h is not 0 for $x > 0$. Known

$e^{-1/x}$ goes to 0 faster than any power, but it starts winning late: the maximum of $e^{-1/x}/x^k$ is $k^k e^{-k}$ at $x = 1/k$. **Example:** $e^{-1/x}/x^{10}$ is 4.5×10^5 at $x = 0.1$, 2.1×10^4 at $x = 0.05$, and only at $x = 0.02$ does it reach 1.9×10^{-5}. Computed

In physics this form appears in the scale Λ. The one-loop running coupling is $\alpha_s(\mu^2) = (b_0 \ln(\mu^2/\Lambda^2))^{-1}$, and the review of the Particle Data Group (PDG) explains Λ as 'the scale where the perturbatively-defined coupling would diverge' (the "1/0" of the one-loop running coupling). Solving the other way,

$$\begin{aligned} \Lambda &= \mu\, \exp\!\Big(-\frac{1}{2b_0\, \alpha_s(\mu^2)}\Big) \\ &= \mu\, \exp\!\Big(-\frac{8\pi^2}{\beta_0\, g^2}\Big), \quad \beta_0 = \frac{11N}{3} \end{aligned} \tag{10}$$

(pure $SU(N)$, $\alpha_s = g^2/4\pi$; PDG's $b_0 = (33 - 2n_f)/12\pi$ with $N = 3$, $n_f = 0$; 4-07). (10) has the form of (9) as a function of g^2, so quantities proportional to Λ look like 0 at every order of a power series in g. The mass gap is expected to be proportional to Λ. Physics **Example:** for $SU(3)$, $g^2 = 1$ gives $\Lambda/\mu = 7.6 \times 10^{-4}$ and $g^2 = 2$ gives 2.8×10^{-2} (g^2 is hypothetical). Computed § 6.6 of the problem statement also writes 'the mass gap is not classically visible' (7-10).

08

1/0 appears in the Borel plane

The series with coefficients $n!$

$$F(x) \sim \sum_{n=0}^{\infty} n!\, x^n \tag{11}$$

does not converge for any $x > 0$, even though each term is finite. The terms decrease until n reaches $1/x$, and increase beyond. By Stirling's formula the smallest term is of order $\sqrt{2\pi/x}\, e^{-1/x}$, and the accuracy of truncation levels off there (Figure 2). The **Borel sum** forms $B(t) = \sum t^n = 1/(1-t)$ by dividing the coefficients by $n!$, and returns via $\int_0^\infty e^{-t} t^n\, dt = n!$.

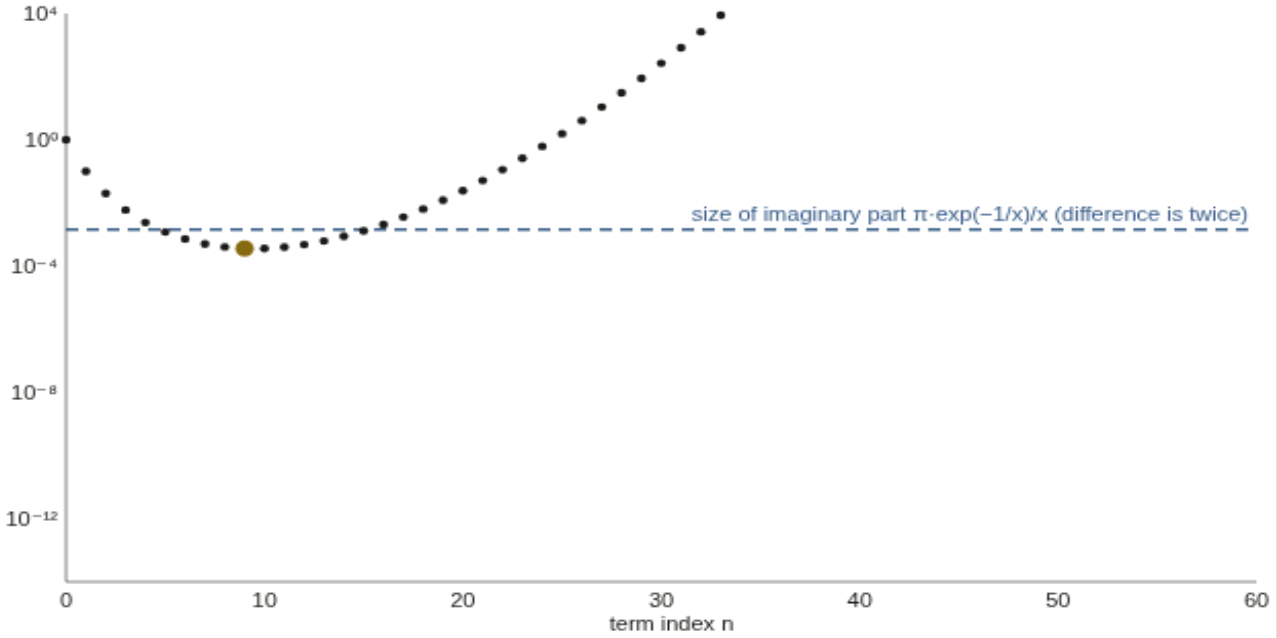
$$S(x) = \int_0^\infty e^{-t}\, B(xt)\, dt = \int_0^\infty \frac{e^{-t}}{1 - xt}\, dt \tag{12}$$

On the path of integration, at $t = 1/x$, there is a pole. The "1/0" of this chapter appears here: in the Borel plane. Avoiding the pole by a small semicircle $t = 1/x + \rho e^{i\phi}$, going over the top moves ϕ from π to 0, and going underneath from $-\pi$ to 0, so $\int dt/(t - 1/x) = \int i\, d\phi$ is $-i\pi$ above and $+i\pi$ below. The integrand is $\frac{1}{1-xt} = -\frac{1}{x} \cdot \frac{1}{t-1/x}$, so the sign is reversed, and putting $e^{-t} \approx e^{-1/x}$ near the pole, the contribution of the semicircle is $+i\pi e^{-1/x}/x$ above and $-i\pi e^{-1/x}/x$ below. Writing S_+ for the sum going over the top and S_- for the sum going underneath,

$$S_\pm(x) = \mathrm{PV}\int_0^\infty \frac{e^{-t}\,dt}{1-xt} \pm i\pi\,\frac{e^{-1/x}}{x},$$
$$\mathrm{PV}\int_0^\infty \frac{e^{-t}\,dt}{1-xt} = \frac{e^{-1/x}\,\mathrm{Ei}(1/x)}{x} \tag{13}$$

and they disagree. PV is the principal value, approaching symmetrically from both sides of the pole, and $\mathrm{Ei}(y) = \mathrm{PV}\int_{-\infty}^{y} e^s\, ds/s$ is the exponential integral. The two differ only in the sign of the imaginary part, whose size is $\pi e^{-1/x}/x$. The difference between above and below is $S_+ - S_- = 2\pi i\, e^{-1/x}/x$, twice that in size. Both the error of optimal truncation and the discrepancy are of the size of $e^{-1/x}$ times a power of x (Table 3). Computed

Figure 2 Changing x with the knob moves the valley of the terms $n!\,x^n$ (log scale), set beside the size of the imaginary part of the Borel sum $\pi e^{-1/x}/x$ (dashed; the difference between above and below is twice that). "Play" moves x back and forth




```
x = 0.100: the terms are smallest at n = 9 (3.63e-4). Size of imaginary part π·exp(−1/x)/x =
1.43e-3 (the difference between the sums above and below is twice that).
```

Computed The values in this figure compute the terms $n!\,x^n$ and $\pi e^{-1/x}/x$ within the figure.

Computed Table 3 The numbers in the table were computed on this machine (SciPy). N^* is the index of the smallest term. The rightmost column is the size of the imaginary part of the Borel sum; the difference between above and below is twice that.

x	Principal value PV	N^*	Smallest term	$\min_N \lvert S_N - \mathrm{PV}\rvert$	Size of imaginary part $\pi e^$
0.05	1.055956	19, 20	2.32e−8	7.70e−9	1
0.10	1.131470	9, 10	3.63e−4	1.20e−4	1
0.15	1.239251	6	8.20e−3	2.74e−3	2
0.20	1.353831	4, 5	3.84e−2	1.26e−2	1

09

Recording the discrepancy: transseries and renormalons

To keep the discrepancy, add $e^{-1/x}$ as an independent unit.

$$F(x) = \sum_{n \geq 0} n!\, x^n \;+\; \sigma\, \frac{e^{-1/x}}{x} \;+\; \cdots \tag{14}$$

The choice of how to avoid the pole moves to the choice of the coefficient σ, and the information about the manner of approach is recorded not in the value of $1/0$ but in the coefficient of $e^{-1/x}$. Such formal sums are called **transseries**, and research in this direction resurgence (4-12). Known

In Yang–Mills the perturbation series is thought to diverge like $\alpha_s^n\, n!$ (renormalons). The $n!$ of (infrared) renormalons is thought to be a contribution from the region of the loop-momentum integral where the momentum is close to Λ, near the point where the one-loop running coupling diverges. There are said to be separate ultraviolet renormalons, coming from the side of large momenta. The PDG writes 'closely connected with non-perturbative contributions', while saying they are 'not usually visible' at the orders available. Physics Results of resurgence at the level of physics exist for the two-dimensional $CP(N-1)$ model on $\mathbb{R} \times S^1$ (twisted boundary conditions; Dunne–Ünsal 2012) and for four-dimensional gauge theories with adjoint fermions compactified on $\mathbb{R}^3 \times S^1$ (Argyres–Ünsal 2012), but not for pure Yang–Mills on $\mathbb{R}^4$.

10

What a symbol should carry

Systems that assign values hold together by weakening the rules of operations or fixing the path to one, and throw away the information about the manner of approach. What has worked are replacements that keep the manner of approach, and symbols that carry the rate of divergence (the Birkhoff decomposition of $1/\varepsilon$, the renormalisation group), both on the side of geometry and perturbation theory. Known The scale of the gap and the expected Λ have the form e^{-1/g^2}, invisible at every order. Physics The obstacles that remained before 2026 — estimates uniform in the volume, the infinite-volume limit, non-triviality, reflection positivity, the gauge orbit space — are not problems of assigning values (7-11). The present state, after the published construction (2026), is in 7-11 and 6-04.

Detour: from the side of logic

A subsection you may skip.

An extension that produces no new theorems about sentences of the original language is a **conservative extension**. Theorem 8.8 of Nelson's (1977) internal set theory IST is 'Every internal theorem of IST is a theorem of ZFC' (a result of Powell). ZFC is the axiom system of ordinary mathematics, and an internal theorem is a theorem written in sentences that do not use the predicate "standard" newly added by IST. IST handles $1/\varepsilon$ and does not define $1/0$. Known But "conservative symbols add no theorems, so they are powerless" is not decisive. The Hopf algebra of graphs, regularity structures and Perelman's entropy are all extensions by definition within ordinary mathematics. Even so, regularity structures were used to construct solutions of singular stochastic partial differential equations, Perelman's entropy in the proof of the geometrisation conjecture, and the Hopf algebra gave a clear view of the combinatorics of renormalisation. Symbols can only be assessed at the level of "what they carry".

This book's reading (unverified)

"The mass gap is not a problem of $1/0$ but a problem of e^{-1/g^2}." Confidence is moderate. Transseries too are formal objects, and by themselves give neither a measure nor estimates uniform in the volume. For pure Yang–Mills on $\mathbb{R}^4$, resurgence is not established even at the level of physics.

What this chapter can say

	Content
Said	The systems of Table 1 weaken the rules of operations or fix the path to one (wheel, meadow and division-by-zero calculus from the texts, IEEE 754 within what was checked on this machine; transreal from bibliographic data only) Known Computed
Said	The blow-up of formula (5), that the Taylor coefficients of formula (9) are 0, the known examples of § 05 and § 06 Known
Said	The numbers in Tables 2 and 3 and in § 02, § 03 and § 07. That the discrepancy of the Borel sum and the error of optimal truncation are both of the size of $e^{-1/x}$ times a power Computed
Not said	The divergence of the perturbation series of four-dimensional Yang–Mills, renormalons, $\Delta \propto \Lambda$ (expectations of physics). That symbolising $1/0$ is powerless in principle (an assessment). That a symbol carrying e^{-1/g^2} reaches the gap (this book's reading). This book shows nothing about Yang–Mills

Sources and reproduction

Item	Kind	Source / tools
Problem statement (§ 6.6, the specification of the axioms)	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000)
Running coupling, b_0, renormalons	Text § 9.1.1, footnote 1	Particle Data Group, *Review of Particle Physics* (2024), § 9
Wheel, meadow, division-by-zero calculus	Text	J. Carlström, Research Reports in Math. 2001:11, Stockholm Univ. (journal version Math. Struct. Comput. Sci. 14 (2004) 143–184) / Bergstra–Hirshfeld–Tucker, arXiv:0901.0823 § 1 / S. Saitoh, *Division by Zero Calculus; an extended version* (draft 2023) § 2.1
IST, Singer's remark	Text	E. Nelson, Bull. Amer. Math. Soc. 83 (1977) 1165–1198 (p. 1196) / I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12
Others	Abstracts	Axelrod–Singer, J. Differential Geom. 39 (1994) 173–213 / Bergbauer–Brunetti–Kreimer, arXiv:0908.0633 / Connes–Kreimer, Comm. Math. Phys. 210 (2000) 249–273 / Epstein–Glaser, Ann. IHP A 19 (1973) 211–295 / Dunne–Ünsal, arXiv:1210.2423, JHEP 11 (2012) 170 ($\mathbb{R} \times S^1$ and twisted boundary conditions checked in the section headings of the text) / Argyres–Ünsal, arXiv:1206.1890, JHEP 08 (2012) 063
Original sources of division-by-zero calculus	Bibliographic data only	M. Kuroda, H. Michiwaki, S. Saitoh, M. Yamane, Int. J. Appl. Math. 27 (2014) 191–198 / S. Saitoh, Springer Proc. Math. Stat. 177 (2016) 151–182 (proceedings of ISAAC 2015). Both from references [58] and [115] of the draft

Item	Kind	Source / tools
Renormalisability	Bibliographic data only	G. 't Hooft, Nucl. Phys. B 33 (1971) 173–199 and B 35 (1971) 167–188 / 't Hooft–Veltman, Nucl. Phys. B 50 (1972) 318–353 / B. W. Lee, J. Zinn-Justin, Phys. Rev. D 5 (1972) 3121–3137 (sequels 3137, 3155) / Becchi–Rouet–Stora, Ann. Phys. 98 (1976) 287–321 / Costello, AMS (2011)
Others	Bibliographic data only	Anderson–Völker–Adams, Proc. SPIE 6499 (2007) / IEEE Std 754-2019 / Colombeau, North-Holland (1984) / Uhlenbeck, Comm. Math. Phys. 83 (1982) 11–29 / Fulton–MacPherson, Ann. of Math. 139 (1994) 183–225 / 't Hooft–Veltman, Nucl. Phys. B 44 (1972) 189–213 (dimensional regularisation) / M. Beneke, Phys. Rep. 317 (1999) 1–142, arXiv:hep-ph/9807443 (renormalons) / M. Hairer, Invent. Math. 198 (2014) 269–504 (regularity structures) / G. Perelman, arXiv:math/0211159 (entropy)
Tables 2 and 3, the examples in § 02, § 03 and § 07, Figures 1 and 2	Computed	Python 3, NumPy 1.26 and SciPy 1.11 on this machine (E_1, Γ, Ei; the principal value agrees to 12 digits with numerical integration with a Cauchy weight; the meadow checked with rational numbers). The figures compute formula (5) and $n!\,x^n$ within the figure

Next chapter to read: 8-01 The lattice, the Wilson measure, conventions. From Part 8, Volume 2 turns to the definitions of the objects of the published construction. The view from the side of physics is in Volume 3, 17-01 How physics sees it; Part 17 begins with how physics handles the scale Λ.