Volume 2 Existence — reading the published construction

Part 8 From the lattice to the continuum — objects

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The lattice, the Wilson measure, conventions

This chapter is existing knowledge (a reading of the published construction). § 05 is this book's study.

What this chapter shows — the definitions of the objects the published construction takes as its starting point — a group $G \subset U(m)$, a periodic lattice, plaquette holonomies, the Wilson measure — and its conventions ($\beta = 1/g_0^2$, the inner product $\langle X, Y\rangle_{\mathfrak{g}} = -m^{-1}\,\mathrm{Re}\,\mathrm{Tr}(XY)$, the Casimir constant $C_{\mathfrak{g}}$). At the end, the conversion to this book's convention (the standard $\beta = 4/g^2$ for $SU(2)$) is put into one table.

Prerequisite chapters — 5-01 Wilson's lattice (introduction), 19-01 The lattice, link variables and the Wilson measure (this book's conventions), 4-06 Asymptotic freedom (the one-loop coefficient). The citation conventions of this volume are in 6-00.

Said first — this is a chapter of definitions and conventions. The sequence of lattice spacings and the sequence of volumes are treated in the next chapter, 8-02, and the continuum limit in 8-05. Results of the published construction are cited at the level of "the committee's assessment is that it is proved in Lean"; what this book has checked for itself goes no further than reading the definitions and the form of the statements.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — group, lattice, Wilson measure: the objects set up by the published statement
2. Continuum conventions — β = 1/g² and the inner product: how the plaquette term approaches the square of the curvature
3. The Casimir constant C_g: definition, value for SU(N), dependence on the size of the representation

4. Conversion table — the published construction, this book, the standard convention: what the same number means in each convention
5. What this book's study had seen: comparison with the conventions of this book's Lean
6. What we have not read: with locations

01

The statement — group, lattice, Wilson measure

The published construction was made public as the Lean repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the committee's assessment, `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10, hereafter "the assessment"), evaluates it. In this part we read the objects that construction places on the lattice and the tools that carry it from the lattice to the continuum. Below, relative file paths are paths inside this repository.

The group. The input is a closed subgroup G of $U(m)$ for some m, whose real Lie algebra $\mathfrak{g}$ is non-abelian and simple (it has no proper non-zero ideal) Source (`GaugeGroup` and `GaugeGroup.IsSimple` in `PublicTheory.lean`, lines 25–45). m is "the size of the matrices", a number distinct from the rank of the group and from $\dim \mathfrak{g}$. $G = SU(2)$ is given by its two-dimensional defining representation, with $m = 2$ Source (`su2` in `ConcreteSU2.lean`).

The lattice. There are two indices: the spacing index N and the volume index p. A side has $L_p D_N$ points, and the lattice is

$$\Lambda_{N,p} = \big(\mathbb{Z}/(L_p D_N)\mathbb{Z}\big)^4, \qquad a_N = D_N^{-1} \quad (1)$$

A lattice point y is placed at the continuum point $a_N \cdot y$ (each coordinate read through its representative between $-\frac{1}{2}L_pD_N$ and $\frac{1}{2}L_pD_N$) Source (`Site`, `mesh` and `place` in `PublicWilson.lean`, lines 11 and 50–52). The side length in continuum units is $a_N \cdot L_p D_N = L_p$. How D_N and L_p are chosen is read in 8-02.

Plaquettes and the Wilson energy. A link variable $U_{y,\mu} \in G$ is placed on each positively oriented edge, and the plaquette holonomy is

$$U_{\square} = U_{y,\mu}\, U_{y+e_\mu,\nu}\, U_{y+e_\nu,\mu}^{-1}\, U_{y,\nu}^{-1} \qquad (\mu < \nu) \quad (2)$$

Source (`plaquette`, lines 20–23). The Wilson energy is the sum over all lattice points and all pairs of directions $\mu < \nu$

$$E(U) = \sum_{y \in \Lambda_{N,p}} \sum_{\mu<\nu} \left(1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\square}\right) \quad (3)$$

Source (`wilsonEnergy`, lines 25–27; the denominator is `g.n+1`, which is m). Each plaquette is counted exactly once, $6|\Lambda_{N,p}|$ plaquettes in all.

The Wilson measure. The product of Haar probability measures h, multiplied by the density $e^{-\beta E}$ and normalised,

$$d\mu^{\mathrm{W}}_{N,p}(U) = Z^{-1}_{N,p} \exp\bigl(-\beta_{N,p} E(U)\bigr) \prod_{e} dh(U_e) \quad (4)$$

is the lattice theory Source (`wilsonLaw`, lines 29–33) Assessment (the assessment § 4, equation (3)). The public structure `WilsonFamily` (lines 35–46) has as its fields the sequence of volumes `period`, the sequence of spacing denominators `denominator` (both strictly increasing), the coupling `beta N p`, the Haar measure (a probability measure, left-invariant), the type of labels `Token` and the insertions `insertion`. Labels and insertions are the subject of 8-03 and 8-04.

This structure and the conditions it satisfies, `WilsonRegularity` (`ChallengeSpec.lean`, lines 12–22), are part of the statement of the public theorem `JointExistence` (same file, lines 186–188) Source. In other words, "which measure on which lattice to start from" is written into the public statement itself.

The lattice-side definitions have been checked against an independently built Lean formulation of lattice gauge theory (`mrdouglasny/lgt`). According to the committee's record, that check relabels lattice points, links and plaquettes and shows in Lean that the action, the partition function and the probability measure agree, with the coupling conversion $\beta_{\text{submission}} = m\,\beta_{\text{lgt}}$ Assessment (the assessment § 11). The text is in `same_plaquette` and `coupling_conversion` (lines 35 and 55) of the committee's annotated example `LatticeAndScale-20261028.lean` Source.

02

Continuum conventions — β = 1/g² and the inner product

The assessment § 3 sets the conventions of continuum Yang–Mills as follows Assessment. For a smooth $\mathfrak{g}$-valued connection A the curvature is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the invariant inner product on $\mathfrak{g}$ is

$$\langle X, Y\rangle_{\mathfrak{g}} = -\frac{1}{m}\operatorname{Re}\operatorname{Tr}(XY) \tag{5}$$

The formal measure is

$$Z^{-1}\exp\Big(-\frac{1}{2g^2}\int_{\mathbb{R}^4}\sum_{\mu<\nu}\|F_{\mu\nu}(x)\|_{\mathfrak{g}}^2\,d^4x\Big)\,\mathcal{D}A \tag{6}$$

In this convention, the β of equation (4) is read as $\beta = 1/g_0^2$ (g_0 is the coupling at the cutoff) Assessment (the assessment § 4). The assessment states explicitly that constructing a probability measure corresponding to (6) is not claimed. What is constructed are correlation functions of local observables (8-05).

Why $\beta = 1/g^2$ can be seen from the Taylor expansion of the plaquette term Proved here . When the links come from parallel transport by a smooth connection, the plaquette holonomy at spacing a can be written $U_{\mu\nu} = \exp X$, $X = a^2 F_{\mu\nu}(x) + O(a^3)$. Since X is anti-Hermitian, $\operatorname{Re}\operatorname{Tr} X = 0$, and expanding $\exp X$ to second order gives

$$1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu} = -\frac{1}{2m}\operatorname{Re}\operatorname{Tr}(X^2) + O(a^8) = \frac{a^4}{2}\|F_{\mu\nu}\|_{\mathfrak{g}}^2 + O(a^5) \tag{7}$$

The sum over lattice points $\sum_y$ corresponds to $a^{-4}\int d^4x$, so

$$\beta\,E(U) \;\approx\; \frac{\beta}{2}\int\sum_{\mu<\nu}\|F_{\mu\nu}\|_{\mathfrak{g}}^2\,d^4x \tag{8}$$

and comparing with (6) gives $\beta = 1/g^2$. Equation (4) of the assessment § 4.1, $a^{-4}\big(1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}\big) \to -\frac{1}{2m}\operatorname{Tr} F_{\mu\nu}(0)^2$, is the same fact as (7) Assessment .

Measuring lengths with this inner product is not a mere convention. The coefficients that appear later — the slope of the running coupling, the Casimir constant, the coefficient of the trace anomaly — are all written relative to this inner product. Changing the convention changes the numerical values of the coefficients, not their meaning. The table in § 04 is the correspondence between those values.

03

The Casimir constant C_g

What the public statement `WilsonRegularity.running` asks for is only that $\beta_{N,p}/\log(1/a_N)$ approach some positive slope s — that a positive slope **exists** Source (`ChallengeSpec.lean`, lines 13–14). The construction meets this with $s = 11C_{\mathfrak{g}}/(24\pi^2)$ Source (`regular` in `HierarchyBridge.lean`, line 18) Assessment (equation (7) of

Theorem 4.1 in the assessment). Here $C_{\mathfrak{g}}$ is the adjoint Casimir constant relative to the inner product (5), the number determined by the identity

$$-\operatorname{tr}\big(\operatorname{ad} x \circ \operatorname{ad} y\big) = C_{\mathfrak{g}}\,\langle x, y\rangle_{\mathfrak{g}} \qquad (x, y \in \mathfrak{g}) \quad (9)$$

In Lean it is defined, for a basis b_i orthonormal for the inner product (5), as $C_{\mathfrak{g}} = -\sum_i \operatorname{tr}(\operatorname{ad} b_i \circ \operatorname{ad} b_i)/\dim \mathfrak{g}$ (`casimir` in `YM/Gauge/Colors.lean`, line 180), and the fact that (9) holds for simple $\mathfrak{g}$ is `trace_adjoint` (`YM/Gauge/Casimir.lean`, line 140) Source . That the inner product is (5) can be read from `trace_color` (`Colors.lean`, line 71; $-\operatorname{Re}\operatorname{Tr}(cd) = m\langle c, d\rangle$) Source . The slope constant is `slopeConstant` $= 11C_{\mathfrak{g}}/(24\pi^2)$ (`Colors.lean`, line 184), and $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$ (`YM/Pert/Asymptotics/RawAxis.lean`, line 13); $2b_0$ is the slope Source .

The value for the defining representation of $SU(N)$. The Killing form of $\mathfrak{su}(N)$ is $\operatorname{tr}(\operatorname{ad} x\ \operatorname{ad} y) = 2N\operatorname{Tr}(xy)$. With $m = N$ in (5), $-2N\operatorname{Tr}(xy) = C_{\mathfrak{g}} \cdot (-\operatorname{Tr}(xy)/N)$, so

$$C_{\mathfrak{su}(N)} = 2N^2, \qquad C_{\mathfrak{su}(2)} = 8 \quad (10)$$

Proved here (we also checked this numerically for $N = 2, 3, 4$: $8, 18, 32$). The slope for $SU(2)$ is $11 \cdot 8/(24\pi^2) = 11/(3\pi^2) \approx 0.3715$.

Dependence on the size of the representation. Since (5) divides the matrix trace by m, the inner product, and with it the value of $C_{\mathfrak{g}}$, changes with the choice of $U(m)$ into which the same Lie algebra is embedded. For example, if the same $\mathfrak{su}(2)$ is carried by the three-dimensional adjoint representation ($SO(3) \subset U(3)$, $m = 3$), (9) gives $C_{\mathfrak{g}} = 3$ Proved here (checked numerically). The meanings of β and g then also change to match (5). $C_{\mathfrak{g}} = 8$ **is the value in the convention "$SU(2)$ carried by its two-dimensional representation".** Do not carry the number alone into formulas written in another convention.

Comparison with the standard convention. In the standard convention for $SU(N)$, $\beta_{\mathrm{std}} = 2N/g_{\mathrm{std}}^2$ and the one-loop coefficient is $b_0^{\mathrm{std}} = 11N/(48\pi^2)$ (Gross–Wilczek, Politzer; 4-06). Rewriting the one-loop running of asymptotic freedom, $1/g_{\mathrm{std}}^2 \approx 2b_0^{\mathrm{std}} \log(1/a)$, in terms of β gives $\beta_{\mathrm{std}}/\log(1/a) \to 2N \cdot 2b_0^{\mathrm{std}} = 11N^2/(12\pi^2)$, which agrees with $11C_{\mathfrak{g}}/(24\pi^2) = 11 \cdot 2N^2/(24\pi^2)$ after inserting (10) Proved here .

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04
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Conversion table — the published construction, this book, the standard convention

The object of Volume 3 of this book is $SU(2)$, and its weight is written $\exp\left(\beta \sum_p \operatorname{Re} U_p\right)$ ($\operatorname{Re} U_p = \frac{1}{2}\operatorname{tr} U_p$, the real part of a quaternion) (19-01 § 05). Taking $G = SU(2)$, $m = 2$ in the published construction, $\operatorname{Re}\operatorname{Tr} U/m = \frac{1}{2}\operatorname{tr} U = \operatorname{Re} U$, so

$$e^{-\beta E(U)} = e^{-6|\Lambda|\beta} \exp\left(\beta \sum_p \operatorname{Re} U_p\right) \quad (11)$$

and the constant disappears under normalisation Proved here . **For $SU(2)$, the β of the published construction and the β of this book are the same number.** Only the convention for writing that number in terms of g differs.

Quantity	Published construction ($G \subset U(m)$)	This book ($SU(2)$)	Standard convention ($SU(N)$, definir
Plaquette weight	$\exp\left(-\beta \sum (1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_\square)\right)$	$\exp\left(\beta \sum \operatorname{Re} U_p\right)$ (same as the left, up to a constant factor)	$\exp\left(-\beta_{\text{std}} \sum (1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p)\right)$
β and g	$\beta = 1/g^2$ (inner product (5))	$\beta = 4/g^2_{\text{std}}$	$\beta_{\text{std}} = 2N/g^2_{\text{std}}$
Between the g's	$g^2_{\text{std}} = 2m\,g^2$. For $SU(2)$, $g^2_{\text{std}} = 4g^2$, and the number β is the same in all three Proved here		
Casimir	$C_{\mathfrak{g}}$ ((9), relative to the inner product (5)). $2N^2$ for the defining representation of $SU(N)$, 8 for $SU(2)$	(not used)	$C_A = N$ (in the convention $\operatorname{tr} T^a T^b$
One-loop coefficient	$b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$. $11/(6\pi^2)$ for $SU(2)$	(written in units of β; next row)	$b_0^{\text{std}} = 11N/(48\pi^2)$. $11/(24\pi^2)$ for SU
Slope of the running of β	$\beta/\log(1/a) \to 2b_0 = 11C_{\mathfrak{g}}/(24\pi^2)$. $11/(3\pi^2) \approx 0.3715$ for $SU(2)$	One step with side 2: $\frac{11}{3\pi^2}\ln 2 \approx 0.2575$ (15-01)	$2N \cdot 2b_0^{\text{std}} = 11N^2/(12\pi^2)$
Independent lattice formulation	$\beta = m\,\beta_{\text{lgt}}$ Assessment	—	—
Coefficient of the trace anomaly (13-02)	$3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4$. $9/\pi^4 \approx 0.0924$ for $SU(2)$	(no counterpart)	—

The ratio of the two b_0's is 4, which comes from $g^2_{\text{std}} = 4g^2$. The slope written in terms of β agrees in all three conventions (because β is the same number). This book's one-step value with side 2 is slope $\times \ln 2$; since the sides in the published construction are

odd numbers of at least 3 (8-02), the one step with side 2 itself is not treated there. Only the coefficient of $\log b$ can be compared; the $O(1)$ constant of one step cannot Inferred .

The unit of volume also needs care. The unit of length in the published construction is "the spacing at step 0", with $a_0 = 1$ (8-02). The lattice of Volume 3 of this book is a single torus with spacing 1. The "1" is the same, but in the published construction the physical unit is fixed by matching β to a value $\bar{\beta}$ determined at step 0 (10-01). No such matching enters this book's lattice spacing 1.

05

What this book's study had seen

This book's Lean treated only $SU(2)$, as unit quaternions (19-01). The form of the weight, the meaning of β and the product of Haar measures are, as in § 04, the same as in the case $m = 2$ of the published construction. The differences are the following three.

- This book's lattice is a single torus with the spacing fixed at 1; there is neither a sequence of spacings nor a sequence of volumes. The published construction has a family with two indices (N, p) from the start.
- This book's Lean defines neither the inner product on the Lie algebra nor the Casimir constant. The one-loop value $\frac{11}{3\pi^2} \ln 2$ was used only on the side of paper and numerics (15-01 and elsewhere).
- The published construction states the same statement for any simple closed matrix group, and $SU(2)$ is a special case (`su2Theory` in `ConcreteSU2.lean`).

We found no disagreement at the level of conventions. Numbers written in this book's β (the strong-coupling window $|\beta| < 0.0556$, Part 20, and others) can be read as the same numbers in the β of the published construction. The published construction, however, keeps β large at every step (10-01) and does not pass through the strong-coupling window.

06

What we have not read

- We have not read the content of the transfer from `GaugeGroup` to the construction's internal type `Gauge` (`g.core` , `GaugeGroup.coreSimple`). Nor have we read the proof of $C_{\mathfrak{g}} > 0$ (`casimir_pos`).
- The expansion (7) is this book's hand computation. We read the Taylor conditions used when the construction chooses representatives of labels (8-03), but we have not

looked for a declaration stating (7) for the plaquette term itself.

- On the committee's side, `submissions/openai-20261028/checks/ImportedAFGauge.lean` (lines 81–86) holds the one-loop coefficient in another convention, $\mu\, dg/d\mu = -b_0 g^3/(16\pi^2)$, $b_0 = \frac{11}{3} C_2(\mathrm{adj})$ Source . For $SU(N)$ this is $11N/3$, and dividing by $16\pi^2$ gives the same as the standard $b_0^{\mathrm{std}} = 11N/(48\pi^2)$ Inferred . We have not read where in the committee's checks the coefficient in this convention is used.
- For the check against the independent lattice formulation (end of § 01), we read only the statements in the annotated example, not the proof of the check.

What this chapter can say

Can say	Cannot say
Source The public statement starts from $G \subset U(m)$ (with simple Lie algebra), the lattice (1), the plaquettes (2), the energy (3) and the family of measures (4) (`PublicWilson.lean`, lines 20–46)	Which $\beta_{N,p}$ to choose (that belongs to the construction; 10-04)
Assessment $\beta = 1/g_0^2$, and the inner product is (5) (the assessment § 3, § 4)	The existence of a continuum probability measure corresponding to (6) (the assessment states explicitly that it is not claimed)
Source The definition of $C_{\mathfrak{g}}$ and (9) (`casimir`, `trace_adjoint`)	That the value of $C_{\mathfrak{g}}$ does not depend on the representation (it does; § 03)
Proved here $C_{\mathfrak{g}} = 2N^2$ for the defining representation of $SU(N)$, 8 for $SU(2)$. The slope is $11/(3\pi^2)$	A list of values for other embeddings and other groups
Proved here For $SU(2)$, the β of the published construction and the β of this book are the same number ((11))	A comparison of the $O(1)$ constants of one step (the sides differ)

Sources and reproduction

Item	Kind	Source / tools
Lattice, plaquettes, energy, measure, `WilsonFamily`	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 10–46, `ChallengeSpec.lean` lines 12–22 and 186–188, `PublicTheory.lean` lines 25–45, `ConcreteSU2.lean`
Conventions (5)(6), $\beta = 1/g_0^2$, equations (3)(4)	Assessment	The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 3, § 4, § 4.1

Item	Kind	Source / tools
$C_{\mathfrak{g}}$, the slope, b_0	Source	`YM/Gauge/Colors.lean` lines 71, 180 and 184, `YM/Gauge/Casimir.lean` line 140, `YM/Pert/Asymptotics/RawAxis.lean` line 13
Check against the independent lattice formulation	Assessment	The assessment § 11
Check against the independent lattice formulation (annotated example)	Source	On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 35 and 55
The Killing form of $\mathfrak{su}(N)$, the one-loop coefficient	Cited	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346 (4-06)
Values of $C_{\mathfrak{g}}$ ($N = 2, 3,$ 4, adjoint embedding), the bookkeeping of 24 versus 48/16	Proved here	Numerical record of this book's study (`numcheck.py`; builds matrix traces and the adjoint action numerically and computes the ratio in (9))

Terms used in this chapter — Wilson measure, plaquette holonomy, Casimir constant, one-loop coefficient, convention. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-02 The set-up — the sequence of spacings, the sequence of volumes, growing sides — how the two indices of the family of lattices are chosen.

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The set-up — the sequence of spacings, the sequence of volumes, growing sides

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study.

What this chapter shows — how the two indices of the family of lattices $\Lambda_{N,p}$ — the spacing index N and the volume index p — are determined, in the public statement and on the construction's side. The spacing is $a_N = 1/D_N$, $D_N = \prod_{k<N} b_k$, where the ratios b_k are odd numbers of at least 3 that do not decrease with the index and grow without bound. The side of the volume, L_p, is measured in units in which the spacing at step 0 is 1.

Prerequisite chapters — 8-01 The lattice, the Wilson measure, conventions. The idea of renormalisation steps is in 7-05, and this book's block map in 19-04.

Said first — the sequence of spacings is chosen by the construction, and the published construction does not say that "every sequence of spacings gives the same limit". The detailed accounting of why the sides grow is in 10-05, and the order in which constants are chosen is in 11-08.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — what the public statement asks for: two strictly increasing sequences, and the running condition
2. The direction of the steps and the size of the lattice: step N is the finest, step 0 has spacing 1
3. The sequence of ratios — odd, non-decreasing, growing: what can be read from the conditions of the calendar

4. Why the sides grow (outline): dividing the one-step error by log b
5. The sequence of volumes and the unit: L_p = 4(M+p+1), where test functions are placed
6. What this book's study had seen: fixed blocks with side 2 and side 3
7. What we have not read

01

The statement — what the public statement asks for

The public structure `WilsonFamily` (8-01) has a sequence of volumes $p \mapsto L_p$ (`period`) and a sequence of spacing denominators $N \mapsto D_N$ (`denominator`), and asks only that both be positive integers that are **strictly increasing** Source (`PublicWilson.lean`, lines 36–39). The spacing is $a_N = 1/D_N$ (`mesh`, line 50).

The first of the conditions `WilsonRegularity` imposed on this family is the running of the coupling Source (`ChallengeSpec.lean`, lines 13–14).

$$\exists\, s > 0 \;\; \forall \varepsilon > 0 \;\; \exists N_0 \;\; \forall N \geq N_0 \;\; \forall p : \qquad \left| \frac{\beta_{N,p}}{\log(1/a_N)} - s \right| < \varepsilon \quad (1)$$

Note the order of the quantifiers. N_0 is determined by ε alone and does not depend on the volume index p. That is, the running is **uniform in the volume**. The construction meets it with the slope $s = 11C_{\mathfrak{g}}/(24\pi^2)$ (the value the assessment writes as twice the one-loop coefficient; 8-01 § 03) Source (`regular` in `HierarchyBridge.lean`, lines 16–25) Assessment (equation (7) of Theorem 4.1 in the assessment). The proof of the running is read in 10-06.

The assessment states the following plainly about this family Assessment (the assessment § 4, § 4.3).

- The construction **chooses** the two sequences. The ultraviolet convergence theorem does not assert convergence for every possible sequence of spacings.
- Convergence holds along the **whole** of the chosen sequence of spacings (not along a subsequence). There is, however, no universality result about the type of the sequence of spacings.

The appendix of the assessment (a glossary of the Lean statements) describes the construction's choice as $L_p = 4(M + p + 1)$ (M fixed large enough to satisfy the size conditions), $D_N = \prod_{k<N} b_k$, $b_N \geq 3$ odd with $b_{N+1} \geq b_N$, and $\log b_N / \log D_N \to 0$

Assessment (`WilsonFamily.period` and `denominator` in Appendix A of the assessment). In § 02– § 05 we compare this with the definitions on the construction's side.

02

The direction of the steps and the size of the lattice

The set-up on the construction's side is collected in the structure `Schedule` Source (`YM/Lattice/Scale.lean`, lines 28–34). Its fields are the sequence of ratios `block` (each b_m odd with $b_m \geq 3$) and the sequence of volumes `period` (divisible by 4, strictly increasing). From these,

$$D_0 = 1, \quad D_{m+1} = D_m\, b_m, \qquad a_m = \frac{1}{D_m}, \qquad n_{m,p} = L_p\, D_m \quad (2)$$

are defined Source (`denominator` lines 39–41, `a` line 99, `side` line 67). $n_{m,p}$ is the number of points on a side of the lattice at step m, and the lattice at step m is $(\mathbb{Z}/n_{m,p})^4$. The physical side length is

$$a_m \cdot n_{m,p} = L_p \qquad (\text{for all } m) \quad (3)$$

and does not depend on the step Proved here (also checked in the committee's example, `side_length_in_physical_units`, `LatticeAndScale-20261028.lean` line 92 Source).

The direction of the steps. The theory with cutoff N starts from the Wilson measure at step N (the finest, spacing a_N) and is coarsened down to step 0 (spacing $a_0 = 1$) by applying one renormalisation step (9-01) N times. In one step from step m to step $m-1$, the number of points on a side is divided by b_{m-1}. Since D_m is a product of odd numbers, a coarse point can be placed at the **centre** of a fine block (9-02).

Symbol	Meaning	Lean side
N	Cutoff index (the finest step)	The index `N` of the public `WilsonFamily`
$m \in \{0, \dots, N\}$	Renormalisation step ($m = N$ is bare, $m = 0$ has spacing 1)	The step index of `History`
b_m	Ratio from step $m+1$ to step m (side of the block)	`Schedule.block m`
$D_m,\ a_m$	Denominator and spacing	`Schedule.denominator`, `Schedule.a`
L_p	Physical side length (in units of the spacing at step 0)	`Schedule.period p` (the public `period`)
$\log(1/a_m) = \sum_{k<m} \log b_k$	Scale of the steps	`Schedule.scale` (`scale_succ`, line 124)

The denominator D_N of the public statement is exactly `Schedule.denominator` here Source (`wilsonFamily` in `WilsonBridge.lean`, lines 20–30).

03

The sequence of ratios — odd, non-decreasing, growing

`Schedule` itself asks only for "odd and at least 3". The other properties come from the calendar `Calendar` and its condition `Calendar.Fits` Source (from `YM/RG/History.lean`, lines 64–77 and 97). A calendar has a sequence of tuples of step constants $j \mapsto \mathrm{stage}(j)$, a sequence of candidate sides $j \mapsto \mathrm{factor}(j)$, and step depths $J(m) \geq 4$, and `Fits` asks for the following.

- `sameFactor`: $b_m = \mathrm{factor}(J(m+1))$. `monotoneL`: factor is monotone. `mono`: J is monotone. `stepSlow`: $J(m+1) \leq J(m) + 1$. `deep`: $J(m) \to \infty$.
- `mesh` (`Gentle`, `Scale.lean` lines 129–130): $\log b_m / \log(1/a_m) \to 0$.
- `driftSmall`: for the drift of the tuple of constants at step $m+1$, $\mathrm{drift} / \log b_m \leq J(m+1)^{-2}$.
- `logCost`: $\log\bigl(1 + \mathrm{paths} + \mathrm{width}^{-1} + \mathrm{columns} + \mathrm{relaxation} + \mathrm{supportExponent}\bigr) / \log b_m \leq J(m+1)^{-2}$.
- Size conditions (`sizeT`, `fitsPatch`): the side of the lattice exceeds 100 times the size of the cover and the radius of the patches, and so on.

We list what can be read from these.

Non-decreasing. $b_m = \mathrm{factor}(J(m+1))$, and since both factor and J are monotone, $b_{m+1} \geq b_m$ Proved here.

Growing without bound. The field columns of a tuple of constants is at least 1 (`columns_ge`), so the numerator of `logCost` is at least $\log 2$. Then $\log b_m \geq J(m+1)^2 \log 2$, and $J \to \infty$ gives $b_m \to \infty$ Inferred (a derivation from the definitions, using that the other terms in the numerator are non-negative). The appendix of the assessment says "b_N is chosen to satisfy complicated but explicit inequalities written in terms of the constants appearing in the RG transformation" Assessment.

Growing slowly. `Gentle` says that b_m is small compared with D_m ($\log b_m = o(\log D_m)$). `mesh_inequality` in `Scale.lean` (line 133) derives from it that there is $c > 0$ with $c\, a_m^2 \leq a_{m+1}$ Source. This means the spacing does not shrink too abruptly from one step to the next.

The short-distance analysis of the trace (13-03) imposes a further condition `PostT` on the calendar. One part of it, `bloom`, is $b_i \le \log \hat\beta_i$ ($\hat\beta_i$ is the one-loop running at step i), and asks that the sequence of ratios grow more slowly than a logarithm Source (`YM/TraceExtension/Terrace/Base.lean`, lines 29–34).

In summary, the sequence of ratios is "odd and at least 3, non-decreasing, growing without bound, but slowly compared with $\log D_m$". No explicit formula for b_m appears in the statement (§ 07).

04

Why the sides grow (outline)

The details are read in 10-05. Here we give only the skeleton.

The change of the coupling in one step carries a deviation (drift) from the one-loop value. The one-step formula has the form $|\beta_{m+1} - \beta_m - s \log b_m| \le \mathrm{drift}$ Source (`History.Transfer`, `History.lean` line 223), where drift is a real-valued field of the tuple of step constants. This book reads the construction as bounding this deviation only by a step constant ($O(1)$) Inferred. If the side were fixed at a constant b, the sum over N steps would be $N \cdot O(1) = \frac{\log D_N}{\log b} \cdot O(1)$, of the same size as $\log D_N$, and the slope in (1) would not be determined Inferred. Since `driftSmall` asks that $\mathrm{drift}_m \le \log b_m / J(m+1)^2$, the sum of deviations is

$$\sum_{m<N} \mathrm{drift}_m \le \sum_{m<N} \frac{\log b_m}{J(m+1)^2} = o\Big(\sum_{m<N} \log b_m\Big) = o(\log D_N) \quad (4)$$

Inferred (from $J \to \infty$ and $\sum \log b_m = \log D_N \to \infty$; the precise form is in 10-05). The design keeps the one-step error at $O(1)$ and removes it by dividing by $\log b$. The two-loop coefficient is not used.

A second reason is that the step constants themselves increase with the step. To handle local fields of degree d (8-03), the Taylor order of the steps beyond a reference step must be at least $d+1$ Source (`Makes.stages` in `YM/Source/Construction.lean`, lines 47–49: $d+3 \le J(r(d))$, order $\ge d+1$ from step $r(d)$ on). Since fields of every degree are obtained, the step depth J grows without bound, and the sides grow along with it Inferred.

The price is that **the sequence of spacings is a special one, adapted to the construction**. This is the content of "no universality is claimed" in § 01.

05

The sequence of volumes and the unit

The side of the volume is $L_p = \mathrm{page}(M + p) = 4(M + p + 1)$ Source (`YM/Price/Waits.lean`, lines 87 and 102). M is a natural number taken large enough to satisfy the calendar, and the assessment says it is "fixed so as to satisfy the minimum size and the initial volume estimates" Assessment .

The unit. The unit of length is the spacing at step 0, with $a_0 = 1$. The committee has checked that every cutoff N and every volume p use the same yardstick, as the following four facts Assessment (the assessment § 11, part of the committee's combined target).

1. The reference spacing is $a(0) = 1$.
2. For all N, p, the RG trajectory arrives at step 0 at the same reference coupling $\beta_{N,p}(0) = \bar{\beta} > 0$ (10-04).
3. The quadratic term of the effective action is normalised to the identity matrix (10-01).
4. The coefficient of a label of degree d equals the label itself at the reference step $r(d)$, independently of the cutoff and the volume (8-04).

So the length "1" is not just a lattice graduation; it is fixed as a physical unit by matching the coupling to $\bar{\beta}$ at step 0.

Where test functions are placed. At volume p, only test functions whose support lies **strictly** inside $\big((-L_p/2, L_p/2)^4\big)^n$ can be used Source (`suitable` in `PublicWilson.lean`, lines 75–77). A fixed test function always satisfies this condition for p large enough.

Distances longer than spacing 1. Renormalisation stops at step 0 and does not proceed to scales coarser than spacing 1. The volume limit is handled by reflection positivity, the Chessboard Estimate, the transfer operator and compactness (Part 12).

06

What this book's study had seen

This book's block map has two designs, with side 2 and side 3, both with a fixed side (19-04). The box at step m has the form $(\mathbb{Z}/2^m L)^4$, and the direction of the steps is the same as in the published construction (the fine side has the larger index).

- The side-3 design (odd side, centred) is in some respects close to how the published construction is built. The difference is that in the published construction the side

grows.

- In this book's chain, the sequence of bare couplings β_N was given from outside, and an estimate across steps was placed as a premise (28-04). The circumstance that one-step errors accumulate when the side is fixed is the same as in § 04; this book had placed it on the side of the premises.
- The idea of fixing the unit of volume by a matching at step 0 points in the same direction as this book's "matching" (matching a quantity determined by the unit length to a fixed value). The quantity used for the matching differs (14-03).

07

What we have not read

- The explicit formula for b_m. We read only the statements of `Wait.ready` (`YM/Price/Waits.lean` line 137), which builds the calendar, and of `Program.exhaust` and `Rolls`.
- The size of M, and the content of the "initial volume estimates" that determine it.
- The values of the fields of the tuple of step constants `Stage` (paths, width, columns, relaxation and so on): which of them increase with J, and how fast.
- (4) is this book's inference. The conditions `slow` (drift/log L $\le$ 1/j^2) and `kick` (drift/log L $\le$ s/4) of the same accounting inside the construction, `YM.RG.Supply.Large` (`YM/Price/Intent.lean` lines 39–47), have the same form as `driftSmall` and do not contradict (4). We have not matched the whole accounting.

What this chapter can say

Can say	Cannot say
Source The public statement asks for strictly increasing L_p, D_N and a running (1) uniform in the volume	How $\beta_{N,p}$ depends on p, beyond the slope of the running
Source b_m is odd and at least 3, $D_m = \prod b_k$, $a_m = 1/D_m$, and L_p is a multiple of 4, $4(M+p+1)$	The explicit values of b_m and M
Proved here From the conditions of the calendar, b_m is non-decreasing. Inferred $b_m \to \infty$	An explicit form of the growth rate
Assessment Convergence holds along the whole of the chosen sequence of spacings	Universality with respect to the choice of the sequence of spacings (the assessment states explicitly that it is open)

Can say	Cannot say
Assessment The same yardstick for all N, p (reference spacing, reference coupling, kinetic term, calibration of labels)	How things depend on $\bar{\beta}$ when it is changed

Sources and reproduction

Item	Kind	Source / tools
Public sequences and the running condition	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 36–39, 50, 75–77, `ChallengeSpec.lean` lines 13–14, `HierarchyBridge.lean` lines 16–25, `WilsonBridge.lean` lines 20–30
The set-up and the calendar	Source	`YM/Lattice/Scale.lean` lines 28–34, 39–41, 67, 99, 124, 129–133, `YM/RG/History.lean` lines 64–77, 97–118, 223, `YM/Price/Waits.lean` lines 87 and 102, `YM/Source/Construction.lean` lines 47–49
Absence of universality, choice of the sequences, common yardstick	Assessment	The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4, § 4.3, § 11, Appendix A (entry for `WilsonFamily`)
Step-independent side, common yardstick (annotated example)	Source	On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 90–105

Terms used in this chapter — renormalisation step, side of a block, calendar, universality, physical unit. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-03 Labels of local fields and their lattice representatives — what is called a "field" on this lattice.

2026-11-13 · chapter Yang–Mills and the mass gap Part 8 From the lattice to the continuum — objects Local fields · Kiichi, Shiori and Rin

Labels of local fields and their lattice representatives

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study.

What this chapter shows — that in the published construction local fields are given in three stages: labels (algebraic objects), lattice representatives (polynomials in finitely many links), and fields (classes of labels distinguishable by correlations). The space of fields $F_d = V_d/N_d$ is defined intrinsically as a quotient by the null space of the correlations, and in each degree it is linearly isomorphic to the space of polynomials of the free Maxwell field.

Prerequisite chapters — 8-01 (the lattice and conventions), 8-02 (the sequence of spacings). Classical curvature and covariant derivatives are in 3-02, and fields in the axioms in 4-09.

Said first — this chapter goes only as far as the **definitions** of labels and representatives. The renormalisation coefficients multiplying the representatives are in 8-04, and how correlations are built is in 8-05. The linear isomorphism with the Maxwell field means "the number of fields is the same"; it does not mean that the correlations are the same as those of the free field.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — fields in the public statement: an abstract type of labels, and finite-dimensional spaces of fields
2. Labels — curvature polynomials up to degree d: how weights are assigned, and the basis
3. Lattice representatives — polynomials that match in the Taylor expansion: a finite stencil and seven conditions

4. The space of fields — dividing by the null space of the correlations: an intrinsic definition, unchanged across phases
5. The linear isomorphism with the free Maxwell field: what is the same and what is not
6. What this book's study had seen: the plaquette average is a naive representative of degree 4
7. What we have not read

01

The statement — fields in the public statement

In the public theorem `JointExistence`, fields appear in the following form Source (`PublicWilson.lean` lines 44–46 and `ChallengeSpec.lean` lines 28–33 in `yang-mills-group/OpenAI-YM-20261028`).

- The family of lattices `WilsonFamily` has a type `Token` (labels) and insertions `insertion` that return a complex number for a label, a lattice point and a configuration.
- The continuum-side hierarchy `Hierarchy` has, for each degree d, a finite-dimensional complex normed space `Field d` ($= F_d$), a linear map `toLabel` into the space of labels, and `project`, which sends labels to fields.
- The reconstruction field (`Reconstructed`, same file, lines 120–123) asks that `toLabel` be injective, that every field be the image of some label (`represents`), and that the space of labels be exhausted by the images of fields (`covers`).

Reading the public statement alone, `Token` is a type whose content is unknown. **Which curvature polynomial a label stands for is not written in the public structure** Source (`PublicWilson.lean` line 44 has only `Token : Type`). It is written on the construction's side (`YM.Register.Token`, `YM.Labels.Frame`) and in the committee's checks Assessment (the assessment § 3, and the entry `WilsonFamily.Token` in Appendix A). This chapter reads the content on the construction's side.

The summary in the assessment § 3 is as follows Assessment . Local observables are gauge-invariant polynomials in the curvature and its covariant derivatives at a point, defined not as functions or distributions but as **abstract labels**, elements of a vector space. Labels have a weight (degree), and the labels of weight at most d form a finite-dimensional complex vector space V_d. There are linear relations among labels coming from the Bianchi identity and elsewhere, and these relations are formulated through

correlation functions. The fields of the main theorem are elements of the sum over all degrees of the quotients $F_d = V_d/N_d$.

02

Labels — curvature polynomials up to degree d

Weight. Give the curvature $F_{\mu\nu}$ weight 2 and each covariant derivative weight 1, and let the weight of a product be the sum. The curvature with k covariant derivatives applied, $D_{\lambda_1}\cdots D_{\lambda_k}F_{\mu\nu}$, has weight $k+2$. V_d is the set of all gauge-invariant polynomials in which every monomial has weight at most d. Examples Assessment (the assessment § 3):

$$-\operatorname{Tr}(F_{\mu\nu}F_{\rho\sigma}) \in V_4, \qquad -\operatorname{Tr}\big((D_\lambda F_{\mu\nu})(D_\kappa F_{\rho\sigma})\big) \in V_6 \quad (1)$$

The weight corresponds to the "engineering dimension" of a field (it scales like length to the power $-d$). Since $V_d \subset V_{d+1}$, the degree is an upper bound.

On the construction's side, polynomials are classified by a pair of numbers (a bidegree) Source (`weight_slot` in `YM/Labels/Grading.lean`, lines 76–79): the first is the weight above, the second the number of curvature factors (one curvature variable with k derivatives has $(k+2, 1)$).

Basis and labels. The structure `Frame G d` chooses a basis of V_d Source (`YM/Labels/Basis.lean`, lines 50–57). There are three conditions.

- Basis element 0 is the identity field $\mathbf{1}$ (the constant polynomial 1).
- Each basis element is real (unchanged by complex conjugation).
- Each basis element belongs to a single bidegree, with weight at most d.

A label is a pair (d, c), where c is the vector of coefficients in this basis Assessment (`WilsonFamily.Token` in Appendix A of the assessment). Having the identity field at the head of the basis matters when renormalisation deals with mixing with the identity field (8-04).

The committee's annotated example shows that every gauge-invariant polynomial appears as a label of some degree d (no upper bound on the degree is imposed overall) Source (`every_registered_polynomial_appears`, line 41 of `submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` in `yang-mills-group/YM`, revision 55ba9fc, on the assessment side). The same example writes that this is exhaustiveness "with respect to the submitted algebraic dictionary", that identifying the dictionary with the intended local fields is a matter of human review, and that

completeness for non-local observables such as line defects is not claimed (lines 35–38).

03

Lattice representatives — polynomials that match in the Taylor expansion

Labels are algebraic objects and have no correlations. To measure on the lattice, a **lattice representative** is chosen for each label. The idea can be seen in the example in the assessment § 4.1 Assessment . For smooth links, $U_{\mu\nu} = 1 + a^2 F_{\mu\nu}(0) + O(a^3)$, so

$$a^{-4}\Big(1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}\Big) \;\longrightarrow\; -\frac{1}{2m}\operatorname{Tr} F_{\mu\nu}(0)^2 \quad (2)$$

(the same computation as (7) in 8-01). Hence $1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$ is a **candidate representative** of the label $-\frac{1}{2m}\operatorname{Tr} F_{\mu\nu}^2$. The assessment notes, however, that the construction does not choose this naive plaquette term itself as the representative, because of additional symmetries.

The definition on the construction's side has the following two stages.

Stencil and polynomial. For each degree d, choose a finite stencil (a finite set of lattice offsets) $S_d \subset \mathbb{Z}^4$ and a linear map from coefficient vectors to polynomials. The variables of the polynomial are the matrix entries of the links on the edges in the stencil and their complex conjugates Source (`SectionData` in `YM/Source/Local.lean`, lines 103–105; the variable index `Index` is at lines 92–93). The stencil and the polynomial do not depend on N, p Assessment (`WilsonFamily.insertion` in Appendix A). The representative at a point x is the stencil translated to x (`places`).

Seven soundness conditions. `SectionData.Sound` asks for the following Source (`Local.lean`, lines 140–147).

1. The stencil contains the origin.
2. The stencil is invariant under the hypercubic group (signed permutations of coordinates).
3. The polynomial is gauge invariant.
4. The coefficient vector of the identity field is mapped to the constant 1.
5. It is compatible with the action of the hypercubic group (rotating the configuration is the same as rotating the coefficient vector).
6. Complex conjugation of the coefficients is mapped to complex conjugation of the values (in particular, a real coefficient vector has real values; `ordinary_real`).

7. **The Taylor condition** (`Projects`, lines 85–90): for every "jet" J (the coefficients of a polynomial gauge potential), the value $f(h)$ of the polynomial on the links built from parallel transport at scale h, and the value $g(h)$ of the continuum label at J shrunk by scale h (the coefficient of order i multiplied by h^{i+2}), have the same derivatives at $h = 0$ up to order d.

Condition 7 is the precise form of "the Taylor expansion matches up to order d". In the words of the assessment, "on links obtained by parallel transport of a polynomial gauge potential, the Taylor expansion in the spacing agrees up to order d with the rescaled continuum label" Assessment . This book reads conditions 2 and 5 as keeping the lattice symmetry (the hypercubic group) at the level of representatives, corresponding to the "additional symmetries" in the assessment Inferred .

On the side of the public statement, only three properties of the insertions are asked for Source (`ChallengeSpec.lean`, lines 15–22): continuity in the configuration, invariance under gauge transformations, and, for each label, a finite stencil independent of N, p, x such that the insertion depends only on the links in it (`localSupport`).

04

The space of fields — dividing by the null space of the correlations

Different labels can have the same correlations (the Bianchi identity, the equations of motion and so on). The construction defines this identification **through the correlation functions themselves** Source (`YM/Euclid/Null.lean`).

For a table of continuum correlations S (the S^{mix} of 8-05), a label v of degree d is **null** if, for any number of spectator labels $B_1, \dots, B_n$ and any mutually distinct points $x_0, \dots, x_n$,

$$S\big(v(x_0)\, B_1(x_1) \cdots B_n(x_n)\big) = 0 \quad (3)$$

(`criterion`, lines 36–38; the null space `null` at line 34 is the kernel of the linear map listing all correlations at separated points). The space of fields is the quotient

$$F_d = V_d / N_d \quad (4)$$

(`Field`, line 74). Its norm is the quotient norm coming from the maximum norm of the basis coefficients Assessment (`Hierarchy.Field` in Appendix A).

Two injections appear, so we separate them. One is the map from the quotient F_d to the totality of correlations at separated points, induced by the map whose kernel is the null space (`spectra`, `Null.lean` line 93); it is injective by the definition of the quotient

Source . The other is `toLabel` in the public statement, a linear map from the space of fields into the space of labels of the reconstructed theory (`ChallengeSpec.lean` line 32). Its injectivity is asked for as the public field `Reconstructed.injective` (line 120), and the construction sets `toLabel := S.pin` Source (`HierarchyBridge.lean` line 32). The committee's example notes about the former that "injectivity alone does not show that too many fields have not been killed", and says that the substantive result is the comparison of dimensions in § 05 below Source (`OperatorContent-20261028.lean` lines 71–86).

Choosing a phase loses no fields. In 8-05 the mixed theory S^{mix} is split into an average of components (phases) S^{ξ}. The committee's example shows that in every phase ξ the null space is the same as that of S^{mix} Source (`each_phase_retains_the_dictionary`, line 103). The quantifier is "every phase", not "almost every phase", which is a stronger statement than the mixed theory being an average of phases.

Defining fields by a quotient is the natural choice of treating as the same whatever the correlations cannot distinguish. The space of fields does not depend on the choice of lattice representatives; it is determined by the continuum correlations alone.

05

The linear isomorphism with the free Maxwell field

The assessment § 3 says that the construction has shown that, in each degree d, Fd is **linearly isomorphic** to the space of gauge-invariant polynomials coming from classical Maxwell theory Assessment . The space compared with is the polynomials in the free curvature and its derivatives of weight at most d, with two identified when they agree on formal curvature data satisfying the source-free Maxwell equations and the Bianchi identity (`Maxwell.shellSector G d`).

$$Fd \;\cong_{\mathbb{C}\text{-linear}}\; (\text{polynomials on the Maxwell shell, weight} \le d) \quad (5)$$

The committee's example shows (5) as a Lean statement Source (`interacting_field_dictionary`, `OperatorContent-20261028.lean` lines 89–91; from `local_field_content` at line 45 of `submissions/openai-20261028/checks/CompletenessAudit.lean` on the assessment side). On the free side, it is also shown that the space compared with has no extra null elements (`free_residual_vanishes`, line 62).

What is the same and what is not. The same example states the following explicitly Source (lines 84–86).

- (5) is an **isomorphism of vector spaces**, not an equality of interacting and free correlations.
- It is neither an isomorphism of algebras nor a unitary equivalence of quantum field theories.
- Nor is it a comparison with an independently constructed Hilbert space of standard Yang–Mills.

What (5) says is that "in each degree, the number of independent fields is the same as for the free Maxwell field". Put differently, there are exactly as many relations among labels that vanish in the correlations as there are relations of the free field (the equations of motion and the Bianchi identity) Inferred . That the number of fields is neither too large nor too small confirms, at the level of counting, the requirement of the problem statement that "the space of fields corresponds to curvature polynomials" (item 2 of the assessment § 1).

This comparison is part of the committee's combined target (`combined_acceptance`) Assessment (the field-content check in the assessment § 9.3). On the construction's side, the source of the comparison is the witness field `F.phase.transport.comparator` (`Maxwell.Ranks`) Source (the field `comparator` at line 40 of `YM/Spec/Closed.lean`).

06

What this book's study had seen

This book's Lean has **no** labels of local fields, no space of fields, and no comparison with the Maxwell field. This book's observables were functions of the links of a coarse lattice — the plaquette average, Wilson loops, the ratio of twisted partition functions θ_M (Parts 19–21).

- The plaquette average $\frac{1}{6}\sum_{\mu<\nu}\operatorname{Re}U_{\mu\nu}$ is, in view of (2), a **naive representative** (plus a constant) of a label of degree 4 (of the form $\operatorname{Tr}F^2$) Inferred . In the published construction, however, correlations converge only for the renormalised insertions of the next chapter, 8-04, whose coefficients should include mixing with the identity field (a term of order a^{-4}). To speak of the continuum limit of this book's plaquette average, this mixing would have to be reread in this book's ledger (14-06).
- Wilson loops and twisted partition functions do not belong to the space of local labels. The committee's example also explicitly excludes line defects.
- The pattern of defining an object intrinsically by dividing by the null space of the correlations is one that could also be used to define continuum objects for this book's non-local observables Inferred .

07

What we have not read

- The explicit form of the representative polynomials (the size of the stencil, which polynomial). We have not read where concrete instances of `SectionData` are built.
- The details of the parallel-transport flow in the Taylor condition (`Hol.flow`, `links`) and of the definition of reading the continuum label (`read`).
- The proof of (5) (`Maxwell.Ranks` and its source), including where the basis of the space compared with is counted.
- Where the second component of the bidegree (the number of curvature factors) is used, other than in the renormalisation scaling (`scaled` in 8-04).

What this chapter can say

Can say	Cannot say
Source In the public statement, labels are an abstract type and fields are finite-dimensional spaces `Field d`	The meaning of labels from the public statement alone (it lies on the construction's side and in the committee's checks)
Source Representatives are polynomials in the links on a finite stencil, satisfying seven conditions (including the Taylor condition)	The explicit form of the representatives
Source $F_d = V_d/N_d$, where N_d consists of the labels all of whose correlations at separated points vanish	Relations where points coincide (correlations are only at separated points; 8-05)
Source In each degree, F_d is linearly isomorphic to the space of polynomials on the Maxwell shell, the same in every phase	Agreement of correlations, isomorphism of algebras, unitary equivalence; completeness for line defects

Sources and reproduction

Item	Kind	Source / tools
Public labels, fields, conditions on insertions	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 44–46, `ChallengeSpec.lean` lines 15–33 and 117–123
Weight, basis, representatives, the Taylor condition	Source	`YM/Labels/Grading.lean` lines 76–79, `YM/Labels/Basis.lean` lines 50–57, `YM/Source/Local.lean` lines 85–105 and 140–147
The null space and the quotient	Source	`YM/Euclid/Null.lean` lines 34–38 and 74

Item	Kind	Source / tools
Description of labels, comparison with Maxwell, examples (1)(2)	Assessment	The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 3, § 4.1, § 9.3, Appendix A
The committee's examples (exhaustiveness, linear isomorphism, retention across phases)	Source	On the assessment side, `submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` lines 35–38, 41, 62, 71–91, 103, and `submissions/openai-20261028/checks/CompletenessAudit.lean` line 45

Terms used in this chapter — label, lattice representative, stencil, Taylor condition, null space of the correlations, space of fields, Maxwell shell. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-04 Renormalised insertions — what coefficients the representatives must be multiplied by for the correlations to converge.

Renormalised insertions

This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study.

What this chapter shows — that in the published construction it is not the naive representatives of labels but the **renormalised insertions** whose correlations converge, and how their coefficients are determined. The insertion is carried exactly to the coarser step one step at a time, and the coefficients at the fine step are chosen through the inverse of the transport so that at the reference step the coefficient is the label itself.

Prerequisite chapters — 8-03 Labels of local fields and their lattice representatives. The idea of divergences and renormalisation is in 4-04, and one renormalisation step in 9-01 and 9-06.

Said first — there is no explicit formula for the coefficients (the assessment says so too). How mixing with the identity field enters can be read only as far as the types in the source allow; we have not been able to read the form of the mixing matrix. The line of proof that the coefficients converge in N is treated in 11-10.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — the form of the insertion: a sum of representatives multiplied by a^{d}_{N} and coefficients
2. Why coefficients are needed — mixing: why the correlations of naive representatives diverge
3. Transport and its inverse — calibration at the reference step: carry the insertion exactly, and read it at the reference step

4. Where mixing with the identity field and lower-degree fields sits: within the coordinates of the space of coefficients
5. Stability in the cutoff (outline): finite-dimensional linear maps and shrinking remainders
6. What is not said
7. What this book's study had seen
8. What we have not read

01

The statement — the form of the insertion

The public statement asks only three things of an insertion $I_{N,p}(v,x;U)\in\mathbb{C}$: continuity, gauge invariance, and locality on a finite stencil (8-03 § 03) Source . What an insertion concretely is lies on the construction's side.

The assessment § 4.2 writes as follows Assessment . For a label $v\in V_d$ of weight d and a basis $v1,\dots,vM$ of Vd, the ultraviolet convergence statement associates to v

$$I_{N,p}(v,x;U)=\sum_{j=1}^{M} c_j(N,p;v)\,L_{vj}(x;U) \tag{1}$$

a **renormalised lattice observable**. The coefficients are linear in v, contain powers of the spacing, and may diverge in the limit. The convergence theorem says that renormalisation coefficients $c_j(N,p;v)$ making the correlations converge **exist**. The submission gives formulas expressing the coefficients through lattice renormalisation transformations, but does not express them explicitly as sums or integrals.

The entry `WilsonFamily.insertion` in Appendix A goes one step further Assessment . At sufficiently fine spacing, the renormalisation transport gives an **invertible linear map** from coefficients at spacing a_N to coefficients at the reference spacing $a_{r(d)}$ ($r(d)\le N$, with $r(d)$ depending only on d). The construction applies its inverse to c (so that the transported coefficients equal c) and multiplies by a scale factor coordinate by coordinate. Writing the resulting coefficients as $\tilde c_{N,p,j}$,

$$I_{N,p}\big((d,c),x;U\big)=a_N^{-d}\sum_{j=1}^{k}\tilde c_{N,p,j}\,P_{d,j}(x;U) \tag{2}$$

where $P_{d,j}$ are the lattice representatives of 8-03. The code defines this map and shows that it is invertible, but gives no explicit formula for the matrix entries of the inverse.

02

Why coefficients are needed — mixing

Consider the plaquette example of 8-03. The expectation of $P = 1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$ is not determined by the Taylor expansion of a smooth configuration, $a^4 \cdot \frac{1}{2}\|F\|^2$, alone. Quantum fluctuations occur at the scale of the spacing a, and their contribution enters the expectation of P as a quantity far larger than a^4. At weak coupling $\langle P \rangle$ is of order g_0^2, and since $g_0^2 \sim 1/(2b_0 \log(1/a))$, it becomes small only logarithmically in the spacing Inferred. The expectation of $a^{-4}P$ then diverges like $a^{-4}/\log(1/a)$. This is a manifestation of the degree-4 field $\operatorname{Tr} F^2$ **mixing with the identity field 1 (degree 0)**. For the same reason, in general a field of degree d mixes with lower-degree fields of the same symmetry with coefficients $a^{-(d-d')}$ (renormalisation of composite operators; 4-04).

So the coefficients of a linear combination of naive representatives must be adjusted along with a to cancel the mixing. To write the cancellation explicitly, one would need to know the mixing coefficients in every degree. Instead of writing them explicitly, the published construction takes the approach of **letting the renormalisation transformation itself carry the coefficients**.

03

Transport and its inverse — calibration at the reference step

Density with an insertion. The construction treats the insertion as a first-order perturbation of the density. At step m, the information about the insertion is held in a tuple (`Coefficient`) Source (`YM/Source/Insertion.lean`, lines 26–30):

- local coefficients $x \in \mathbb{C}^k$ (coordinates in the basis of 8-03; coordinate 0 is the identity field),
- a regular remainder (a sequence of functions with supports),
- a large-field remainder (the part attached to forced activities).

The density with an insertion is the density ρ_m at step m (9-06) multiplied by "x_0 + the non-identity coefficients placed on local representatives + the regular remainder", with the large-field part added Source (`density`, lines 47–49; the docstring says "the formula written with $\lambda^2 = 0$ for the source λ"). The non-identity coefficients are placed with scaling by powers of the small-field threshold t_m at step m (9-07)

(`exponentialEntry`, lines 37–39; the scaling map `scaled` has the number of curvature factors as its exponent; `YM/Labels/Reference.lean` line 58).

Transport. For each label of degree d, cutoff N and volume p, the construction has a family of transports `TransferData` Source (`Insertion.lean`, lines 73–75): `entries m v`, which at each step m returns the tuple at step m for an input v, and a linear isomorphism of the space of coefficients `terminal`. The main fields of the condition `Based` (lines 89–114) are as follows.

Field	Content
`bare_coord`, `bare_germ`	At step N the local coefficients are the input v itself, and the remainders are 0
`encode`	The density with an insertion at step N is the bare Wilson weight multiplied by the initial insertion (with a factor a_N^d)
`unnormalized`	The density with an insertion at step $m = b_m^d \cdot$ (the density with an insertion at step $m+1$ pushed forward by the one-step kernel). **An equality, not an approximation**
`reads`	The local coefficients at the reference step r are $\mathbf{terminal}(v)$
`unit_coord`	The identity field goes to the identity field: $a_N^d \mathbf{1}$ at step N is mapped to $a_m^d \mathbf{1}$ at step m
`back`, `front`	Estimates in both directions on the size of the coefficients at adjacent steps
`linear`	Coefficients and remainders are linear in v

That is, `terminal` is the linear map "place coefficients v at step N, carry them by exact push-forward down to step r, and the local coefficients at step r become $\mathbf{terminal}(v)$".

Calibrating with the inverse. The insertion for a label v uses $\mathbf{terminal}^{-1}(v)$ as its input Source (`cylinder`, line 87; `Family.O`, `YM/Source/Construction.lean` line 42):

$$I_{N,p}(v) = \mathbf{initial}\big(\mathbf{terminal}^{-1}_{N,p}\, v\big), \qquad \mathbf{initial}(w) = a_N^{-d} \cdot P_d\big(\mathbf{scaled}_{t_N^{-1}}\, w\big) \quad (3)$$

(`initial`, lines 77–79). The reading at the reference step is then $\mathbf{terminal}(\mathbf{terminal}^{-1} v) = v$, **the label itself** (`normalized`, `Construction.lean` lines 65–68; the hypotheses are `Makes` and $r(d) \le N$) Source. The construction's witness condition `Makes.calibrated` (line 62) asks for this `Based` for all d, N, p (with $r(d) \le N$).

The committee checks this as one of the normalisations common to all volumes Assessment (the fourth item of the assessment § 11: "the normalised coefficient of a label v of degree d equals v at the reference step $r(d)$, independently of the cutoff and the volume"). The text is in `insertion_has_the_prescribed_reference_value` (line 105) of the committee's annotated example `LatticeAndScale-20261028.lean` Source.

Comparing (2) and (3), the $\tilde{c}_{N,p}$ of the assessment corresponds to $\mathrm{scaled}_{t_N^{-1}}\left(\mathrm{terminal}_{N,p}^{-1} c\right)$ Inferred . The "scale factor coordinate by coordinate" is the threshold t_N at step N raised to the power of the number of curvature factors.

04

Where mixing with the identity field and lower-degree fields sits

The space of coefficients consists of coordinates in the basis of labels of degree at most d (the `Frame` of 8-03). Basis element 0 is the identity field, and the other basis elements have weight at most d, so **lower-degree fields are in the same space** Source (`YM/Labels/Basis.lean`, lines 50–57). The mixing of § 02 can therefore be read as appearing as **off-diagonal entries** of the matrices of the step-by-step linear maps (`leanings` in § 05) Inferred .

- `unit_coord` fixes the column of the identity field (the identity field goes to the identity field, multiplied only by the ratio of the step scales a^d) Source .
- Components flowing from other fields into the identity field (of the type a^{-4}) are allowed as coordinate 0 Inferred . The mixing of $\mathrm{Tr}\, F^2$ with $\mathbf{1}$ in § 02 should enter here.
- The estimates on the remainders are written in terms of the size of the part with the identity-field component removed, `core` (`Reference.lean` line 123) Source .

Within what we read, we found no statement writing the mixing as a triangular matrix, and no statement subtracting the identity-field component. The inverse map $\mathrm{terminal}^{-1}$ works by cancelling the mixing all at once.

05

Stability in the cutoff (outline)

The coefficients in (3) differ for each N. For the correlations to converge as $N \to \infty$, the transport maps must be stable in N. The construction's line of argument is as follows Source (details in 11-10).

1. The map at each step (`Club`, `YM/FullVolume/Mark/Nursery.lean` lines 11–16): a linear isomorphism `leanings` from the coefficients at step $m+1$ to those at step m, depending on the Gram of the remainder at step $m+1$. Its inverse is uniformly bounded (`hail`). The remainder map `flow` halves differences (`trail`).
2. Fix the value at the reference step r and carry it to the fine side by inverse maps (`Gear.spends`, `Mark/Sages.lean` line 66): if it is Cauchy at the reference step, the

map at each step converges in N, and the inverses are bounded, then it is Cauchy at every step above.

3. As a result, the coefficients and the values of the insertions at steps $m \geq r$ are Cauchy in N (`Branch.winds`, `stays`, `Gables/Master.lean` lines 44 and 50).

The maps are finite-dimensional linear maps, and the remainders shrink from above, so no compactness is needed in this part. There is no statement of a rate; the convergence is qualitative Source .

06

What is not said

- An explicit formula for the coefficients $\tilde{c}_{N,p,j}$ (stated explicitly by the assessment) Assessment .
- That the mixing matrix is triangular, or that the identity-field component is written as a subtraction (not within what we read).
- That the coefficients and the limit do not depend on the choice of the reference step $r(d)$.
- That the coefficients agree with the renormalisation of composite operators in perturbation theory.

07

What this book's study had seen

This book's Lean has **no** renormalisation of composite operators.

- This book's observables (the plaquette average, the ratio of twisted partition functions) were used without renormalisation. In the chain that matches the plaquette average at a coarse step to a fixed value at step m_0 (Part 28), the design was such that no coefficient needed to multiply the observable itself.
- This book treated the linearisation of the block map and the covariance of the push-forward (15-13). There the linearisation is how field fluctuations are carried; it was not used as a map on the space of coefficients of local observables.
- The view "give the renormalisation of composite fields as the inverse of the linearisation of the renormalisation transport" is an idea this book did not have (14-05). It is also an entry point for speaking of the continuum limit of this book's plaquette average.

08

What we have not read

- The content of the step-by-step linear isomorphisms (`Gram.reads`, `Mark/Hinge.lean`) and the form of their matrices.
- How `terminal` is actually built (concrete instances of `Family.factory`).
- The value of the reference step $r(d)$. We have read nothing beyond `Makes.stages` asking for $d + 3 \leq J(r(d))$.
- How the large-field remainder of the density with an insertion (`marked`) is reorganised under one step of push-forward.

What this chapter can say

Can say	Cannot say
Assessment The insertion is a linear combination of representatives; the coefficients are determined by the inverse of the renormalisation transport and a scale factor, with no explicit formula	An explicit formula for the coefficients
Source The transport is the exact push-forward by the one-step kernel; the reading at the reference step is $\mathrm{terminal}(v)$, and the insertion takes $\mathrm{terminal}^{-1}(v)$ as its input	The explicit form of terminal
Source At the reference step the coefficient is the label itself (independent of cutoff and volume)	Independence of the choice of reference step
Inferred Mixing with the identity field enters as components into coordinate 0 of the linear maps	The form of the mixing matrix (whether triangular)
Source The coefficients at steps $m \geq r$ are Cauchy in N	The rate of convergence

Sources and reproduction

Item	Kind	Source / tools
The insertion tuple, density, transport, calibration	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `YM/Source/Insertion.lean` lines 26–49 and 73–114, `YM/Source/Construction.lean` lines 24–30, 42, 62, 65–68, `YM/Labels/Reference.lean` lines 58 and 123
Stability of the coefficients	Source	`YM/FullVolume/Mark/Nursery.lean` lines 11–16, `YM/FullVolume/Mark/Sages.lean` line 66, `YM/FullVolume/Gables/Master.lean` lines 44 and 50

Item	Kind	Source / tools
Equations (1)(2), absence of an explicit formula, common normalisation	Assessment	The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4.2, § 11, Appendix A (`WilsonFamily.insertion`)
Common normalisation (annotated example)	Source	On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` line 105
Mixing of composite operators	Cited	Introduction in 4-04

Terms used in this chapter — renormalised insertion, renormalisation transport, reference step, calibration, identity field, mixing. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-05 Correlation functions and the order of limits — in what order of limits the correlations of renormalised insertions are carried to the continuum.

Correlation functions and the order of limits

This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study.

What this chapter shows — the form of the convergence statement `Converges` inside the public statement `JointExistence`. Limits of the smeared lattice correlations $C_{N,p}(v;f)$ are taken in the order (i) $N \to \infty$ at fixed volume p (along the whole chosen sequence of spacings), then (ii) along a subsequence of volumes, giving a mixed theory S^{mix}. S^{mix} is a probability average of components S^{ξ}.

Prerequisite chapters — 8-02 (spacings and volumes), 8-03 (labels and fields), 8-04 (renormalised insertions). Distributions and test functions are in 1-11, and the OS axioms in 4-10.

Said first — all correlations are defined only where the points are mutually distinct. The volume limit is along a subsequence, and it has not been shown that the limit is independent of the choice of subsequence. Nor is it said that the order of the two limits may be exchanged. The content of the phase decomposition and of reconstruction is treated in Part 12.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Smeared lattice correlations: the formula and the admissible test functions
2. The convergence statement — two limits and their order: the three fields of `Converges`
3. The quantifiers of the public theorem — a subsequence and one hierarchy: $\forall p$ $\exists q$ $\exists T$
4. Why test functions at separated points: singularities and contact terms

5. The mixed theory and its components: $S^{\mathrm{mix}} = \int S^{\xi}\, d\mu$
6. What is not said
7. What this book's study had seen
8. What we have not read

01

Smeared lattice correlations

For a sequence of labels $v = (v_1, \dots, v_n)$ and a test function f, the lattice correlation is defined as

$$C_{N,p}(v; f) = a_N^{4n} \sum_{x_1,\dots,x_n \in \Lambda_{N,p}} f(a_N x_1, \dots, a_N x_n)\, \mathbb{E}_{\mu^{\mathrm{W}}_{N,p}}\Big[\prod_{j=1}^{n} I_{N,p}(v_j, x_j; U)\Big] \quad (1)$$

Source (`PublicWilson.lean` in `yang-mills-group/OpenAI-YM-20261028`: `finiteMoment` lines 54–57, `smear` lines 79–81) Assessment (equation (6) of the assessment § 4.3). The sum runs over all ordered n-tuples of lattice points, and a_N^{4n} is the factor that matches the lattice sum to the integral $\int d^{4n}x$. $I_{N,p}$ is the renormalised insertion of 8-04, and $\mu^{\mathrm{W}}_{N,p}$ the Wilson measure of 8-01.

Admissible test functions Source (lines 58–77):

- Smooth, compactly supported complex functions (`SeparatedTest n`) on the open set $\{(x_1, \dots, x_n) : x_i \neq x_j \ (i \neq j)\}$ (`separated n`). The support stays away from where points coincide.
- They are not restricted to product form $f_1(x_1) \cdots f_n(x_n)$.
- When used at volume p, the support must lie inside $\big((-L_p/2, L_p/2)^4\big)^n$ (`suitable p`). A fixed f always satisfies this for p large enough.

02

The convergence statement — two limits and their order

For a strictly increasing sequence of volume indices $k \mapsto p_k$ (`volumeOrder`) and continuum correlations S, `Converges` asks for the following three things Source (`PublicWilson.lean`, lines 85–90). For every n, every sequence of labels v and every test function f:

1. For k large enough, f fits in volume p_k.
2. There is a sequence $\mathrm{row}(p)$ such that, for **every** volume p in which f fits, $C_{N,p}(v;f) \to \mathrm{row}(p)$ $(N \to \infty$, over **all** $N)$.
3. $\mathrm{row}(p_k) \to \int f(x)\, S(v;x)\, dx$ $(k \to \infty)$.

In the assessment's notation Assessment (`Properties.wilson` in Appendix A)

$$\lim_{k\to\infty} \lim_{N\to\infty} C_{N,p_k}(v;f) = \int_{(\mathbb{R}^4)^n} f(x)\, S^{\mathrm{mix}}(v;x)\, dx \quad (2)$$

The cutoff is inside and the volume outside, and this order is fixed.

The inner limit. Theorem 4.1 of the assessment states that the ultraviolet limit at fixed volume, $S_p(v;f) = \lim_N C_{N,p}(v;f)$, exists along the whole chosen sequence of spacings, and moreover that the kernel $S_p(v;x)$ is C^∞ away from coincident points (modulo the period L_p) Assessment . The assessment summarises the proof as using the construction's renormalisation normalisation to compare sufficiently fine cutoffs with each other and show that the sequence is Cauchy in N. That line of argument is read in 11-10.

The outer limit. Theorem 5.1 of the assessment states that the volume limit can be taken along a subsequence, and that **a single subsequence works for all correlations at once** Assessment . The tool is Arzelà–Ascoli compactness: for each compact set K of configurations at separated points, boundedness and equicontinuity uniform in the volume,

$$\sup_{p \ge p_0(K)} \sup_{x\in K} |S_p(v;x)| \le M_K, \qquad |S_p(v;x) - S_p(v;y)| \le \omega_K(|x-y|) \quad (3)$$

are shown (equation (11) in the assessment § 5). The assessment writes that these two estimates come from general tools — reflection positivity, the Chessboard Estimate, the transfer-operator representation and Cauchy estimates — and that the input specific to the construction is control of the renormalised observables uniform in the spacing Assessment . The content is in 12-04 and 12-05.

The committee's annotated example states explicitly that this statement "makes no pointwise claim at coincident points" and "does not say that the ultraviolet and infrared limits can be exchanged" Source (`submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` lines 113–116 in `yang-mills-group/YM`, revision 55ba9fc, on the assessment side).

> 03

The quantifiers of the public theorem — a subsequence and one hierarchy

The public theorem has the following form Source (`ChallengeSpec.lean`, lines 186–188).

```
∀ g (simple), ∃ C : WilsonFamily g, WilsonRegularity C ∧
  ∀ p (strictly increasing sequence of volumes), ∃ q (strictly increasing),
    ∃ T : Hierarchy C, T.Properties (p ∘ q)
```

We split the reading into three points.

- **The family comes first.** The family of lattices C (spacings, volumes, couplings, insertions) is fixed once, before the sequence of volumes is asked for.
- **A subsequence for every sequence of volumes.** For each requested sequence of volumes p, taking a further subsequence q, (2) holds along $p \circ q$. This is the form of a compactness statement.
- **One hierarchy.** On that subsequence, one hierarchy T (fields, correlations, the reconstructed theory, the OPE, the trace field) satisfies four properties at once — convergence `wilson`, reconstruction `reconstruction`, the OPE `ope`, and non-freeness `nonfree` (`Properties`, lines 177–181).

T is chosen for each p, and it is not said that different p give the same T. The assessment writes that whether the limiting correlations depend on the chosen subsequence is unresolved, and notes that the Jaffe–Witten problem statement regards subsequential limits obtained by compactness alone as insufficient (unless arguments showing properties of the limit are added) Assessment (the assessment § 5).

The conclusion of the construction's theorem `constructed` contains more than the public statement, and among it is the statement `FullVolume.PositionsUV C Sequence.full` Source (`Construction.lean` line 24). The definition (`YM/FullVolume/Profiles.lean`, lines 25–36) has the form: for each volume p, there is a kernel smooth at separated points, the smeared correlations converge to that kernel along the whole sequence of cutoffs, and the differenced correlations converge to the derivatives of the kernel Source (we read only the form of the definition). In the public statement too, the cutoff runs along the whole sequence and only the volume is a subsequence. The stronger form with kernels and differences exists only in the construction's theorem.

04

Why test functions at separated points

Correlations of local fields become singular as points approach each other. For example, the two-point function of a degree-4 field is naively of size $|x-y|^{-8}$, with logarithmic corrections on top (13-02). This is not locally integrable on $\mathbb{R}^4 \times \mathbb{R}^4$, so to smear it with test functions that include coincident points, terms supported on the coincidence set (contact terms) must be determined. Contact terms are not uniquely determined by the theory (the ambiguity of renormalisation; 4-04).

By restricting the convergence statement from the start to test functions at separated points, the published construction removes this problem from the convergence statement. That a distribution extending to coincident points **exists** is included in the construction's theorem as a separate statement (extension of contact terms, existence only) (13-04).

The committee's independent check of the OS axioms uses a different space of test functions — Schwartz functions all of whose derivatives of every order vanish on all diagonals Assessment (equation (21) in the assessment § 10). This contains the compactly supported functions at separated points, but contact terms on the diagonals are not visible.

05

The mixed theory and its components

The limit S^{mix} in (2) is called a "mixed" theory, because the property that correlations of two groups moved far apart factorise into a product (clustering) has not been shown Assessment (the assessment § 6). Without clustering, the theory obtained by OS reconstruction need not have a unique vacuum.

The construction gets around this by a **phase decomposition** Assessment (the assessment § 6.1, and `Hierarchy.Character` in Appendix A). In the OS Hilbert space built from S^{mix}, it shows that the C*-algebra generated by operators compressed to the zero-energy subspace H_0 is commutative, and takes its Gelfand spectrum X (a compact metric space) as the space of phases. For a probability measure μ on X representing the vacuum state (`phaseMeasure` in the source; the assessment writes ν),

$$S^{\mathrm{mix}}(v;x) = \int_X S^{\xi}(v;x)\, d\mu(\xi) \qquad (x \text{ a configuration of separated points}) \quad (4)$$

In the public statement, correlations come in two kinds, `correlation none` (S^{mix}) and `correlation (some ξ)` (S^{ξ}), and (4) is the reconstruction field `mixture` Source (`ChallengeSpec.lean` lines 41 and 133–134). The same collection of fields includes that

μ is a probability measure (`normalized`) with full support (`fullSupport`), and that $\xi \mapsto S^{\xi}(v;x)$ is continuous (`characterContinuity`) (lines 124–125 and 131–132).

Each component S^{ξ} satisfies a variant of the OS axioms that includes clustering, and one phase ξ^* (`selected`) is chosen to reconstruct a Wightman theory. The content is in 12-07 and 12-08.

The limit is non-zero. The committee has checked that, for a fixed test function f at separated points, the two-point lattice correlation of the renormalised trace field has a non-zero limit in the same order of limits Assessment (equation (23) in the assessment § 11):

$$\lim_{k\to\infty}\ \lim_{N\to\infty} \left|C_{N,p(q(k))}(\Theta,\Theta;f)\right|^{1/8} = \left|S_2^{\mathrm{mix}}(\Theta,\Theta;f)\right|^{1/8} > 0 \quad (5)$$

The committee's example writes that this is neither a spectral mass, nor a mass gap, nor a rate of convergence, and, since the test function and its normalisation may depend on the witness, not an intrinsic phase-independent scale either (`LatticeAndScale-20261028.lean` lines 136–157).

06

What is not said

- Universality with respect to the choice of the sequence of spacings Assessment .
- Uniqueness of the volume limit (independence of the choice of subsequence). The first of the open questions in the assessment § 8 Assessment (16-02).
- Clustering of S^{mix}. Equality of the components with one another. That each component is a subsequential limit of states in the original periodic volumes Source (`LeadingTrace-20261028.lean` lines 151–153).
- Exchanging the order of the two limits. Values at coincident points.
- A continuum probability measure (there are correlations, but no measure on field configurations is constructed; the second item of the assessment § 8) Assessment .

07

What this book's study had seen

The chain of Volume 3 of this book took as its object the **distribution of block variables** at a coarse step (the push-forward measure) (28-01), not correlations of local fields at separated points.

- What is shared: the form of first considering the limit that removes the cutoff in a box of fixed physical size, and taking the limit along a subsequence by compactness.

- What differs: this book's implications obtained the existence of subsequential limits from compactness alone and carried the properties in the fields of assumptions. The published construction obtains the ultraviolet limit at fixed volume **along the whole sequence, not a subsequence** (its source is the uniqueness from "anchor and contraction" in 11-10).
- This book has nothing corresponding to the phase decomposition; its non-degeneracy condition (matching the expectation of the plaquette average to a fixed value) was designed to play the same role as the non-zero limit (5), with a different observable Inferred .

08

What we have not read

- How the differences inside `FullVolume.PositionsUV` are built (`squall`, `Cluster`, `Meets`), and their proofs.
- How $p_0(K)$ in (3) is determined, and the details of why the lower bound on the volume depends on K.
- The proof of `fromWilson`, beyond `wilsonLimit` (`HierarchyBridge.lean` lines 44–56), which derives the public `Converges` from the construction.

What this chapter can say

Can say	Cannot say
Source Correlations have the form (1) and are smeared with smooth compactly supported test functions at separated points (not restricted to product form)	Convergence with test functions that include coincident points
Source The order of limits is "fix the volume and remove the cutoff along the whole sequence → subsequence of volumes"	Exchanging the order
Source The family is fixed first, and for every sequence of volumes there are a subsequence and one hierarchy	That the hierarchy does not depend on the subsequence
Source $S^{\mathrm{mix}} = \int S^{\xi} d\mu$, where μ is a probability measure with full support	Clustering of S^{mix}, agreement of the components
Assessment The two-point lattice correlation of the trace field has a non-zero limit (5)	That (5) is an intrinsic value of a mass or a scale

Sources and reproduction

Item	Kind	Source / tools
Correlations, test functions, the convergence statement, the public theorem	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 54–90, `ChallengeSpec.lean` lines 41, 117–134, 177–188, `Construction.lean` lines 15–31, `HierarchyBridge.lean` lines 44–56
Theorems 4.1 and 5.1, equations (6)(11)(23), the phase decomposition, open questions	Assessment	The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4.3, § 5, § 6, § 6.1, § 8, § 10, § 11, Appendix A
The committee's annotated examples	Source	On the assessment side, `submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` lines 113–116 and 151–153, `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 136–157
The Jaffe–Witten problem statement	Cited	A. Jaffe, E. Witten (official problem statement of the Clay Mathematics Institute, 2000). 0-03

Terms used in this chapter — smeared correlation, test function at separated points, order of limits, subsequence, mixed theory, phase (component), clustering. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-01 The block kernel — in Part 9 we read the content of what was called "one renormalisation step" here.