Volume 2 Existence — reading the published construction

Part 9 One step of renormalisation

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The block kernel — one step is a probability kernel, not a map

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — what mathematical object "one renormalisation step" is defined to be in the published construction: the formula of the probability kernel $T_\beta(U, V)$ from fine configurations U to coarse configurations V, the role of the penalty coefficient c_{pen}, the fact that the β entering the kernel is the value at the fine step, and which targets these declarations are used for.

Prerequisite chapters — 8-01 The lattice, the Wilson measure, conventions, 8-02 The set-up, 7-05 The sequence of renormalisation group steps. For comparison, 19-04 The block map and one step of renormalisation.

Said first — results of the published construction are written at the level of "the committee's assessment is that it is proved in Lean". What this book has checked goes no further than reading the definitions and the form of the statements in the source. The content of the predictor, the gauge fixing and the weights that enter the kernel is read in 9-02 to 9-04. Whether the kernel is of positive type with respect to reflections is not found in the statements of the source, and this book has not examined it either.

The order of this chapter

1. **The statement — one step is a probability kernel**: `StepData.density` and `push`, and the theorem that it is a kernel
2. **Reading the kernel's formula as three factors**: the probability of the choices, the predictor, Gaussian-type noise
3. **The penalty coefficient**: any positive number, and the main theorem quantifies over each
4. **The β of the kernel is the β of the fine step**: the one-step identity and preservation of the partition function
5. **Which targets it belongs to**: a table of declarations and records
6. **What this book's study had seen**: the difference from the push-forward by a deterministic block map

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — one step is a probability kernel

The published construction starts from the Wilson measure at lattice spacing $a_N = 1/D_N$ and coarsens it down to the step with spacing 1 by applying the block transformation N times (step numbering as in 8-02: $m = N$ is the finest and $m = 0$ the coarsest). This "one step" is not a map that determines a single coarse configuration V from a fine configuration U. It is defined as **a kernel that draws the coarse configuration at random** Source .

We set notation. For an element u of the gauge group $G \subset U(d_G)$, write

$$\mathrm{price}(u) = 1 - \frac{\mathrm{Re}\,\mathrm{tr}\,u}{d_G}, \qquad \mathrm{chord}(u) = \frac{\|u-1\|_F}{\sqrt{d_G}}, \qquad \mathrm{price}(u) = \tfrac{1}{2}\,\mathrm{chord}(u)^2 \tag{1}$$

($\|\cdot\|_F$ is the Frobenius norm; the last identity is `chord_price` in `YM/Gauge/MatrixGroup.lean`, line 117) Source . price is a function of the same form as a single plaquette term of the Wilson action, $1 - \mathrm{Re}\,\mathrm{tr}\,U_p/d_G$; it is 0 at $u = 1$ and non-negative on the group. For SU(2), $d_G = 2$, and it agrees with this book's $1 - \mathrm{Re}\,U_p$ (19-01) Proved here .

The one-step kernel is defined in `StepData.density` (line 71) of `YM/RG/Block.lean` in the source, in the following form Source .

$$T_\beta(U,V)=\sum_{\tau} P(\tau\mid U)\prod_{b}\frac{1}{Z_\beta}\exp\!\Big(-c_{\mathrm{pen}}\,\beta\,\operatorname{price}\big(V_b\,Q_b(U,\tau)^{-1}\big)\Big),\qquad Z_\beta=\int_G e^{-c_{\mathrm{pen}}}$$

The sum over τ runs over finitely many discrete choices (pairs of "frame labels" and "path labels", read in 9-03 and 9-04), $P(\tau\mid U)$ is the probability of that choice, the product over b runs over all positively oriented links of the coarse lattice, $Q_b(U,\tau)\in G$ is the **predictor** of a coarse link built from the fine configuration (9-04), and h is the Haar measure normalised to a probability. The transport of functions by the kernel (push-forward) is

$$(\mathcal{T}_\beta F)(V)=\int T_\beta(U,V)\,F(U)\prod_{\ell} dh(U_\ell)\qquad (3)$$

whose name in the source is `StepData.push` (same file, line 77) Source . The output is, by convention, read as "a density with respect to the product of Haar measures on the coarse lattice" (docstring, lines 74–76).

That it is a kernel is stated as a theorem. `StepData.isKernel` (same file, line 203) assumes the three obligations `Patch.Branches` of 9-03 and 9-04 (uniqueness of the minimiser of the frame, uniqueness of the predictor, non-emptiness of the labels drawn at large field) and concludes, for every β, the following five things Source : $T_\beta\geq 0$; measurable in (U,V); bounded above; integrable in V for each U; $\int T_\beta(U,V)\,dV=1$. It is shown separately that, for each fixed penalty coefficient, the three obligations can be met for all patches (`Axial.viable` in § 03) Source .

02 Reading the kernel's formula as three factors

Formula (2) writes, in the language of probability, a procedure that draws the coarse configuration in the following order Inferred .

1. **Draw the choices.** Looking at the fine configuration U, draw the discrete choice τ with probability $P(\tau\mid U)$. `likelihood` in the source (line 49) is the product of the probabilities of the frame labels at each coarse point, `frameMass`, and of the path labels on each coarse link, `predictorMass`. At small field the labels are not used, and a uniform probability is put in formally (9-03, 9-04).
2. **Build the predictor.** From U and τ, build one group element $Q_b(U,\tau)$ for each coarse link (`prediction`, line 44).
3. **Add noise.** Draw each coarse link V_b independently, link by link, with the weight $e^{-c_{\mathrm{pen}}\beta\,\operatorname{price}(V_bQ_b^{-1})}$ centred at Q_b (`noise`, line 68).

The third factor is a probability measure in V for each Q, because by right invariance of the Haar measure $\int Z_\beta^{-1} e^{-c_{\mathrm{pen}}\beta\,\mathrm{price}(vQ^{-1})}\,dh(v) = 1$ (`noise_mass`, line 146) Source. Together with the sum of the first factor being 1 (`chosen`, line 123), this gives $\int T\,dV = 1$ Proved here.

Let us look at the width of the noise. Writing $V_b Q_b^{-1} = e^X$ (X a small element of the Lie algebra), with the assessment's inner product $\langle X, Y\rangle = -\,\mathrm{Re}\,\mathrm{tr}(XY)/dG$ (the assessment § 3), $\mathrm{price}(e^X) = \frac{1}{2}\langle X, X\rangle + O(|X|^3)$ Proved here. So the weight is roughly $\exp(-\frac{1}{2}c_{\mathrm{pen}}\beta\langle X, X\rangle)$, and V_b is scattered around Q_b with width about $(c_{\mathrm{pen}}\beta)^{-1/2}$ Inferred. At weak-coupling steps where β is large, the kernel concentrates strongly near the predictor, but does not collapse to a point.

A deterministic map (for comparison)

The coarse configuration is fixed as a single $V = B(U)$. The push-forward is $\int F(B(U))\,d\mu(U)$; written as a kernel, it is the point measure $\delta_{B(U)}$ in V.

A probability kernel (the published construction)

V fluctuates around the predictor with width $(c_{\mathrm{pen}}\beta)^{-1/2}$. No δ-constraint is imposed. The docstring of `Block.lean` also states that no division by a density is done outside the kernel.

03

The penalty coefficient — any positive number, and the main theorem quantifies over each

The c_{pen} in formula (2) is `KernelChoices.penalty` in the source (`YM/RG/Patch.lean`, lines 37–45). `KernelChoices` is a structure bundling the penalty coefficient and three thresholds (the frame threshold `frame_cut`, the predictor threshold `predictor_cut`, the axial-gauge factor `axial_factor`) as positive numbers, and its docstring says it is "chosen once" Source. This book writes the penalty coefficient as c_{pen}.

The penalty coefficient can be taken to be any positive number. `Axial.viable` (`YM/Branch/Pick.lean`, line 137) says that for any $c_{\mathrm{pen}} > 0$ (the argument is named κ in the source) there is a tuple whose `penalty` is that value, whose frame and predictor thresholds are determined by the gauge group alone, and whose axial-gauge factor is 16, such that the three obligations `Branches` hold for **all patches** (independently of the side L and the smoothing exponent J) Source.

The construction's theorem `YangMills.constructed` (`Construction.lean`, lines 15–16) takes $b > 0$ as an argument and builds a family whose penalty coefficient is exactly b. That is, the main conclusion is quantified over each penalty coefficient Source . The public theorem `YangMills.jointExistence` (`Solution.lean`, line 4) uses it with 1 put in for this b Source . The docstring says "the penalty coefficient is fixed before the family is chosen" (`Construction.lean`, line 13), and the step constants are chosen after the penalty coefficient is fixed Source .

The penalty coefficient also matters outside the kernel. The precision matrix of the one-step free field is defined as $K + c_{\mathrm{pen}} Q^{\mathsf{T}} Q$ (`baseMatrix`, `YM/Push/FREE_RESOLVENT/Flat.lean` lines 80–81), and the lower bound on the precision on the tree-gauge slice has the form $\min(a, c_{\mathrm{pen}})/(1 + \beta_L + \gamma_L)$ (`treeFare`, `tree_gap`, same file, lines 189–198) Source . If the penalty coefficient is small, this lower bound and the decay constants of the propagator worsen in proportion Inferred . The detailed form of the small-field step is read in 11-09. We find no estimate in the source's statements that is uniform as $c_{\mathrm{pen}} \to 0$ or $c_{\mathrm{pen}} \to \infty$ Source .

04

The β of the kernel is the β of the fine step

Which value is put into the β of formula (2) is determined on the side of the one-step identity. `History.Transfer` in `YM/RG/History.lean` (lines 223–227) imposes, for the density representation ρ_m at step m (9-06),

$$\rho_m = \mathcal{T}_{\beta_{m+1}}\, \rho_{m+1}, \qquad \left|\beta_{m+1} - \beta_m - s \log b_m\right| \le \mathrm{drift}_{m+1} \quad (4)$$

Source . What enters the kernel is the coupling of the **fine step**, $\beta_{m+1} = \beta(N, p, m+1)$ (line 225). Here b_m is the side at step m, $s = 11 C_g/(24\pi^2)$ is the slope of 10-02, and drift is a step constant.

Two things can be read from this choice.

- The kernel differs from step to step. Since $\beta(N, p, m+1)$ depends on the cutoff N and the volume index p (Part 10), the kernel changes with N even for one step at the same step m Inferred . The "push-forward measure" is an object whose map itself changes with N (11-10).
- The partition function is preserved. Since the total mass of the kernel is 1, $\int \mathcal{T}_\beta F = \int F$ (`push_integral`, `YM/RG/Push.lean` line 37) Source . Repeating this from step N, `History.partition_eq` (`History.lean`, line 242) says that integrating the representation at any step gives the Wilson partition function $Z_N(\beta_N)$ at step N Source .

The first formula of the one-step identity (4) is an **equality**, not an approximation Source . The measure does not change in one step; it is the representation (9-06) that is rewritten.

05

Which targets it belongs to

The source's `verification/declaration-usage.tsv` records as `used` the declarations reached from any of the 17 headline theorems listed in the verification record, and as `outside` those not reached. `used` does not mean "used by the public theorem" (6-02) Source .

Declaration (location)	Content	Record	Relation to the targets
`StepData.density` (`RG/Block.lean`:71)	The kernel (2)	used	Enters the one-step identity of `History.Selected`, a field of the construction theorem's conclusion `C.Makes` Source
`StepData.isKernel` (:203)	That it is a probability kernel	used	Reached from one of the headline theorems. We have not traced the chain of the existence route
`Axial.viable` (`Branch/Pick.lean`:137)	The obligations hold for any penalty coefficient	used	As above
`History.Transfer`, `Selected` (`RG/History.lean`:223, 234)	The one-step identity and the sequence of steps	used	The field `wilson` of `Family.Makes` (`Source/Construction.lean`:45); the conclusion of the construction's theorem Source
`History.partition_eq` (:242)	Preservation of the partition function	used	Reached from one of the headline theorems

Neither the kernel nor the sequence of steps appears in the statement of the public theorem `JointExistence` (the assessment's glossary, page 23) Assessment . The kernel is an object inside the construction that builds the **witness** of the public theorem.

06

What this book's study had seen

One step in Volume 3 and Part 15 of this book is a **deterministic** block map B (19-04). It fixes one coarse link by pulling a weighted average of path products back to the sphere, and carries the measure by the push-forward $B_{*}\mu$ (19-04, push-forward). In the language of kernels, this book's step is a point measure in V, whereas a step of the

published construction is a fluctuation of width $(c_{\mathrm{pen}}\beta)^{-1/2}$ around the predictor Inferred .

This book's study had two block designs. One is the current design with side 2 and overlapping weights $(1/8, 3/4, 1/8)$ (28-05); the other has side 3, blocks centred on coarse points, and uniform weights. For the latter, it has been machine-checked in Lean that the measure obtained by pushing the SU(2) Wilson measure forward m steps is reflection positive with respect to reflections in planes of lattice points (for bounded measurable functions depending only on coarse links lying strictly between two planes of coarse lattice points) Lean : `Block3.rp_push`, `Block3.rp_push_prob` (periodic lattice with even side, any real β). This is a formalisation of a result of a known type (a local map commuting with reflections preserves reflection positivity under push-forward).

This argument uses that the deterministic map commutes with reflections, and that the map is local. Since a step of the published construction is a kernel, the same argument does not transfer as it is Inferred . A separate question arises: whether the kernel is of positive type with respect to reflections. On the published construction's side too, we found no statement of reflection positivity for the one-step kernel within what we read Source . Reflection positivity is stated for the original Wilson measure (12-01). The comparison of designs is summarised in 14-02.

07

What we have not read

- Reflection positivity of the kernel: we have not searched the whole repository for a statement on whether the one-step kernel is of positive type with respect to reflections (we read within `YM/RG/` and `YM/Branch/`).
- Whether `isKernel` and `Axial.viable` lie on the existence route of the public theorem: we have not traced the chain. We know only that, according to the record, they are reached from one of the headline theorems (with `SharpGram.habitat` as the witness).
- How the step constants change when the penalty coefficient is varied (we go only as far as reading the docstring saying that the constants are chosen after the penalty coefficient is fixed).
- We found no explanation in the source of why the kernel's β is taken on the fine side; the reading in § 04 is an inference from the definitions.

What this chapter can say

Can say	Cannot say
Source One step is defined as the probability kernel (2), and the push-forward is (3) (`StepData.density`, `push`)	A statement treating the kernel as a limit of deterministic maps (not in the source)
Source Under the three obligations the kernel is a probability kernel (`isKernel`), and the obligations hold for all patches for any penalty coefficient (`Axial.viable`)	Estimates uniform in the penalty coefficient
Source The construction's theorem is quantified over each penalty coefficient $b > 0$, and the public theorem uses $b = 1$	A conclusion in a form independent of the penalty coefficient
Source The β entering the kernel is the value at the fine step (`History.lean`:225)	The reason for that choice (not in the source's docstrings)
Source The partition function is preserved by one step (`push_integral`, `partition_eq`)	Reflection positivity of the one-step kernel (no statement found)
Lean In this book's side-3 centred design, the push-forward measure is reflection positive for planes of lattice points (`Block3.rp_push`)	A claim of the same form for the probability kernel (this book has not examined it)

Sources and reproduction

Item	Kind	Source / tools
Kernel, push-forward, being a kernel	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Block.lean`: `StepData.density` line 71, `push` line 77, `noise_mass` line 146, `isKernel` line 203
price and chord	Source	Same repository, `YM/Gauge/MatrixGroup.lean` lines 112, 115 and 117
Penalty coefficient and three thresholds, obligations	Source	Same repository, `YM/RG/Patch.lean` lines 37–45 and 182–185, `YM/Branch/Pick.lean` line 137 `Axial.viable`
Quantification over the penalty coefficient	Source	Same repository, `Construction.lean` lines 13–16 `YangMills.constructed`, `Solution.lean` lines 4–6
Precision matrix and the lower bound on the tree slice	Source	Same repository, `YM/Push/FREE_RESOLVENT/Flat.lean` lines 80–81 and 189–198
One-step identity, partition function	Source	Same repository, `YM/RG/History.lean` lines 223–227 and 242, `YM/RG/Push.lean` line 37
Dependency record	Source	Same repository, `verification/declaration-usage.tsv`, `verification/Usage.lean`
Conventions and glossary	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 3, Appendix A

Item	Kind	Source / tools
This book's deterministic block map and push-forward	Lean	The source table of 19-04
Reflection positivity of the side-3 centred design	Lean	This book's Lean bundle (`Block3.rp_push`, `Block3.rp_push_prob`; components `Block3.down_refl`, `down_local`), a bundle that passed the check by the Lean kernel

Terms used in this chapter — block kernel, probability kernel, push-forward, predictor, penalty coefficient, price, chord, one-step identity, preservation of the partition function, headline theorem, used and outside. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-02 Patches and weights — we read the patches centred at coarse points used to build the predictor, and the weights given by convolutions of the uniform distribution.

Patches and weights — centred blocks and convolutions of the uniform distribution

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the definitions of the box of fine points attached to each coarse point (the patch) and of the weights w_J on it, used inside the one-step kernel (9-01): that the coarse point sits at the centre of its block, that the weights are the J-fold convolution of a uniform distribution, that the support overlaps heavily with neighbouring blocks, and that J grows to infinity with the step.

Prerequisite chapters — 9-01 The block kernel, 8-02 The set-up. Convolution of probabilities is in 1-08.

Said first — the weights and patches are definitions, and there are almost no theorems here. The reasons for the choice of weights (why convolutions, why $J \geq 4$) are not written in the source's docstrings, and the reading in § 04 includes inference.

The order of this chapter

1. **The statement — definitions of patches and weights**: `Patch`, `radius`, `weight`
2. **The coarse point sits at the centre of its block**: odd side and the embedding $x \mapsto Lx$
3. **Convolutions of the uniform distribution**: the form along one axis, and the example $L = 3,\ J = 4$
4. **The radius of the support and the overlap**: $J(L-1)/2$ versus the distance L to the neighbouring coarse point
5. **J grows with the step**: `Calendar.op` and `Fits`
6. **What this book's study had seen**: side 2 with transverse-shift weights; side 3, centred, uniform
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment.

01

The statement — definitions of patches and weights

The shape of one step is determined by the structure `Patch` (lines 29–34) in `YM/RG/Patch.lean` of the source. It has four ingredients Source.

- The side L: odd, with $L \geq 3$ (`odd_side`, `coarse`). The spacing of the coarse lattice is L times that of the fine lattice, and the side of the fine torus is L times the side of the coarse torus (`StepData.sizes`).
- The smoothing exponent J: $J \geq 4$ (`order_cut`).
- The patch radius $R = (J+3)L$ (`radius`, line 51). The patch is the box of the fine lattice centred at the origin with side $2R+1$, $\{z \in \mathbb{Z}^4 : |z|_\infty \leq R\}$ (`sites`, line 53).
- The weights w_J (`weight`, line 87): a probability on the points of the patch.

We write the definition of the weights as a formula. Take the small box $C = \{z : |z|_\infty \leq (L-1)/2\}$ (`Small` in the source is `box (L/2)`, and since L is odd the radius is $(L-1)/2$), and let w_J be the distribution of the sum $z_1 + \cdots + z_J$ when a J-tuple $(z_1, \ldots, z_J)$ of points of C is drawn uniformly Source.

$$w_J(x) = \frac{\#\{(z_1,\ldots,z_J) \in C^J : z_1 + \cdots + z_J = x\}}{|C|^J} = \big(u_C^{*J}\big)(x), \qquad u_C = \frac{1}{|C|}\mathbf{1}_C. \tag{1}$$

u_C is the uniform distribution on C, and $*$ is convolution. The source shows $\sum_x w_J(x) = 1$ in `weight_sum` (line 90) and non-negativity in `weight_nonneg` (line 96) Source. The sum $z_1 + \cdots + z_J$ lies in the box of radius $J(L-1)/2$, which is inside the patch (radius $(J+3)L$), so the weights are defined on the points of the patch (`roll_sites`, line 63) Source.

The docstring says of the shape of the patch that it is "a single shape, also used for analytic conventions before the ratio of sides is fixed" (line 28) Source.

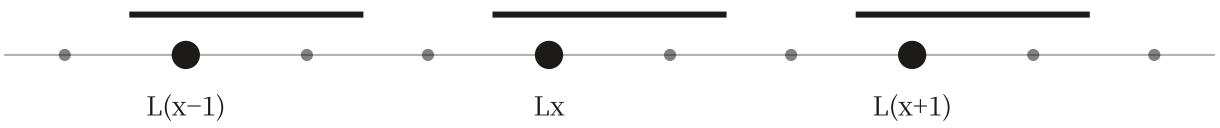
02

The coarse point sits at the centre of its block

A coarse point x is mapped to the fine point Lx. The source's `Ratio.embed` (`YM/Lattice/Subdivision.lean`, line 29) takes the representative `lift x` of the coarse point, centred, multiplies it by L, and wraps it back onto the fine torus Source . The patch is then placed, centred at the origin, on the configuration seen from the point Lx (`expandAt`, `YM/Lattice/Path.lean` line 224) (`StepData.frames`, `YM/RG/Block.lean` lines 36–37) Source .

Since L is odd, the fine lattice splits, with neither overlaps nor gaps, into cubes $Lx + C$ of side L centred at the coarse points Proved here . Each coarse point sits right in the middle of its own cube. With an even side this kind of splitting is impossible, and the coarse point falls at a corner of the block or at a position with no fine point.

Figure 1 Blocks of side $L = 3$ (thick intervals) and coarse points (●) seen along one axis. The coarse point Lx sits at the centre of its own block.



The figure is a cross-section along one axis. In four dimensions, L^4 fine points form one block.

03

Convolutions of the uniform distribution

Since C is the product of the intervals $\{-(L-1)/2, \dots, (L-1)/2\}$ along the four axes, u_C is a product of one-axis uniform distributions, and the convolution also splits axis by axis Proved here .

$$w_J(x) = \prod_{i=1}^{4} v_J(x_i), \qquad v_J = \left(\tfrac{1}{L}\mathbf{1}_{\{-(L-1)/2,\dots,(L-1)/2\}}\right)^{*J}. \tag{2}$$

Take $L = 3,\ J = 4$ as an example. The one-axis distribution is the 4-fold convolution of the uniform distribution on $\{-1, 0, 1\}$:

$$v_4(-4), \dots, v_4(4) = \frac{1,\ 4,\ 10,\ 16,\ 19,\ 16,\ 10,\ 4,\ 1}{81} \tag{3}$$

Proved here . It is a hill, largest at the centre and decreasing towards the ends. The variance along one axis is $J(L^2-1)/12$, and for large J, v_J approaches a discrete

Gaussian by the central limit theorem Proved here .

The weights are invariant under the hypercubic group (permutations of axes and sign flips). The source's `weight_invariant` (`YM/Branch/Transport.lean`, line 135) shows $w_J(s \cdot x) = w_J(x)$ for signed permutations s Source . The proof has the form: the uniform distribution u_C is invariant, so its convolutions are too. This invariance is one of the origins of the symmetries of the one-step kernel (9-05).

The weights are used on the predictor side (9-04) Source .

- Weighted average: the predictor of a coarse link is built from the w_J-weighted average of the transports $Y(z)$ at the points z of the patch (`bondCost`, `Patch.lean` line 159).
- Probability of drawing a single one: when the average cannot be used, a point z is drawn with probability $w_J(z)$ and $Y(z)$ is used as it is (`predictorMass`, line 177).

The average in the gauge fixing (9-03) does not use the weights; it is a uniform average over all pairs of points of the patch Source .

04

The radius of the support and the overlap

The support of w_J is the box of radius $J(L-1)/2$ Proved here . The distance to the neighbouring coarse point is L, so if $J(L-1)/2 \geq L$ the support reaches the centre of the neighbouring block. For $J \geq 4,\ L \geq 3$, $J(L-1)/2 \geq 2(L-1) \geq L$, so it always reaches Proved here .

L	J	Support radius $J(L-1)/2$	Patch radius $(J+3)L$	Layers of coarse points reached by the support
3	4	4	21	1
3	10	10	39	3
5	4	8	35	1
9	6	24	81	2

The last column of the table is $\lfloor J(L-1)/(2L) \rfloor$: how many layers of coarse points, counting from the neighbour, the support reaches Proved here . As J grows, the fine region seen by the predictor of one coarse link spreads to about $J/2$ layers in units of the coarse lattice. The patch radius $(J+3)L$ is $J+3$ layers in units of the coarse lattice, taken about $J/2+3$ layers wider than the support (about $J/2$ layers) (`Patch.radius`, `YM/RG/Patch.lean` line 51) Source . Since the frame (9-03) is defined on all points of the patch, there is a margin outside the points used for the predictor's average Inferred .

The overlap is large. A single fine link enters the averages of the predictors of many coarse links. Even though the kernel's formula ((2) in 9-01) is a product of independent noise for each coarse link, the predictors $Q_b(U,\tau)$ are related to one another through shared fine links Inferred . How distant coarse links become independent (or do not) in the push-forward measure cannot be read from the definition of the kernel alone.

That the lattice is large enough is imposed by the conditions of the calendar. The field `fitsPatch` of `Calendar.Fits` (`YM/RG/History.lean`, line 110) requires $100 \cdot (\text{patch radius} + 1) \le$ the side of the fine lattice Source . A patch never wraps around the torus and overlaps itself.

05

J grows with the step

One step from step $m+1$ to step m, `Calendar.op m p` (`History.lean`, lines 88–91), uses the patch with side $L = b_m$ (the side at step m) and smoothing exponent $J = J(m+1)$ Source . J is the calendar field `J : ℕ → ℕ`, and `Calendar.Fits` (lines 97–118) imposes the following Source .

- `mono`: J is non-decreasing. `stepSlow`: it increases by at most 1 per step. `deep`: $J \to \infty$.
- `sameFactor`: the side b_m is the value `factor (J (m+1))`, determined by $J(m+1)$ alone. `monotoneL`: that value is non-decreasing in J.
- `deepOrder` and others: the step constants `Stage`, such as the Taylor order, are indexed by J and monotone in J.

So J is also "the index of the tuple of step constants", and the side b_m, the number of convolutions in the weights, and the order of the expansion of the action (9-06) all grow together with the same index Source . From $J \to \infty$ and the `logCost` condition, the side $b_m \to \infty$ follows (10-05) Inferred . Why the sides grow is read in 10-05.

As the number of convolutions increases, the weights become smoother, and the step in the weights at the edge of a block becomes smaller Inferred . The source's docstrings do not give the reason for choosing convolutions.

06

What this book's study had seen

We set it beside the two designs of this book's study (this book's designs are in 19-04, weights and 28-05).

Item	Published construction	This book's current design	This book's side-3 design
Side	Odd ≥ 3, growing with the step	2 (fixed)	3 (fixed)
Position of the coarse point	Centre of the block	On a fine point (corner of the block)	Centre of the block
What the weights are on	Points of the patch (all translations in four dimensions)	Transverse shifts of the path $-1, 0, +1$	Uniform average within the block
Shape of the weights	J-fold convolution of a uniform distribution ($J \geq 4$, growing)	$(1/8,\ 3/4,\ 1/8)$	Uniform
Overlap with neighbours	Large (support radius $J(L-1)/2 \geq L$)	Present (shift of 1 sideways)	—

Centring the point goes in the same direction as this book's side-3 design. In the side-3 design, using that centring the coarse points fits with reflections in planes of lattice points, reflection positivity of the push-forward measure is closed in Lean (9-01 § 06, `Block3.rp_push`) Lean . The weights of the published construction have a growing support that overlaps heavily with neighbours, so they do not fit the premises of this book's reflection-positivity argument (locality of the map, and not crossing the reflection plane) Inferred . The idea of smoothing the weights by convolution and letting J grow is one this book's study did not have.

07

What we have not read

- Why $J \geq 4$, and why the patch radius is $J + 3$ layers. The source's docstrings give no reason, and we have not traced where they are used in downstream estimates (the Taylor order, estimates on supports).
- In which estimates the smoothness of the weights (the number of convolutions) matters. We infer that it is used in the small-field step (11-09), but have not checked.
- How the dependence between heavily overlapping predictors is reflected in the locality of the activities (the supports of the regular activities in 9-06).
- For the explicit sequence of sides b_m (the construction in `YM/Price/Waits.lean`), we read only the statements.

What this chapter can say

Can say	Cannot say
Source The patch is a box with odd side $L \geq 3$, $J \geq 4$ and radius $(J+3)L$, and the coarse point x sits at the centre, at the fine point Lx	The reasons for $J \geq 4$ and $J+3$
Source The weights are the J-fold convolution of a uniform distribution: total 1, non-negative, invariant under the hypercubic group (`weight_sum`, `weight_invariant`)	The reason for choosing convolutions (not in the docstrings)
Proved here The support radius $J(L-1)/2$ is always at least L for $J \geq 4$, so it overlaps the neighbouring block	How the overlap affects independence in the push-forward measure
Source One step at step m uses $L = b_m$, $J = J(m+1)$; J is non-decreasing, increases by at most 1 per step, and grows to infinity	The explicit growth of J (we read only the statements of the calendar's construction)

Sources and reproduction

Item	Kind	Source / tools
Patch, radius, weights	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Patch.lean`: `Patch` line 29, `radius` line 51, `roll_sites` line 63, `weight` line 87, `weight_sum` line 90, `bondCost` line 159, `predictorMass` line 177
Embedding and view of configurations	Source	Same repository, `YM/Lattice/Subdivision.lean` line 29 `Ratio.embed`, `YM/Lattice/Path.lean` line 224 `expandAt`, `YM/Lattice/Box.lean` line 13 `box`
Invariance of the weights	Source	Same repository, `YM/Branch/Transport.lean` line 135 `weight_invariant`
Patches per step and the calendar	Source	Same repository, `YM/RG/History.lean` lines 88–91 `Calendar.op`, lines 97–118 `Calendar.Fits`
The numbers in (3) and in the table	Proved here	Hand computation in the text (coefficients of $(1+x+x^2)^4$)
This book's two designs	Lean	The source table of 19-04, and this book's Lean bundle (`Block3.paths`, `Block3.wts`, `Block3.rp_push`)

Terms used in this chapter — patch, smoothing exponent, convolution of the uniform distribution, support radius, centre of the block, hypercubic group, calendar. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-03 Two kinds of gauge fixing — we read in which frame the fine configuration is viewed on the patch (symmetric minimisation at small field, and a randomly drawn axial gauge at large field).

2026-11-13 · chapter Yang–Mills and the mass gap Part 9 One step of renormalisation Two kinds of gauge fixing · Kiichi, Shiori and Rin

Two kinds of gauge fixing — symmetric minimisation, and a randomly drawn axial gauge

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the definition of the "frame" that enters the one-step kernel (9-01). A frame is a group element at each point of the patch; the fine configuration is re-viewed through it before the predictor of a coarse link is built. We read the small-field frame (symmetric minimisation, Landau type) and the large-field frame (an axial gauge whose root and axis order are drawn at random), the condition separating the two, and how, at large field, drawing the symmetry-breaking choices at random and summing over them keeps the discrete symmetries of the kernel.

Prerequisite chapters — 9-01 The block kernel, 9-02 Patches and weights. Gauge fixing in general is in 4-08.

Said first — we have not read the content of the proofs of uniqueness of the minimiser (`Branch.align`) and of the existence of labels that may be drawn (`Axial.existence`). Nor have we read the concrete values of the thresholds. What we can say goes as far as the definitions and the form of the statement that uniqueness holds if the threshold is taken small enough.

The order of this chapter

1. **The statement — frames and their two kinds**: `Patch.frame`, `usesMin`, `frameMass`
2. **The small-field frame — symmetric minimisation**: an average over all pairs of points and the 24 axis orders
3. **The large-field frame — a randomly drawn axial gauge**: labels for root and axis order, the condition for labels that may be drawn
4. **The branching condition**: the chord of transports in the central axial gauge, and a threshold
5. **Why two kinds**: uniqueness, continuity, discrete symmetries
6. **What this book's study had seen**: symmetric averages over families of paths

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

The statement — frames and their two kinds

A **frame** on a patch B (9-02) is a collection of group elements, one per point, $h : B \to G$ (`Patch.Frame` in the source, `YM/RG/Patch.lean` line 99; the type itself contains no normalisation). The normalisation at the centre, $h(0) = 1$, is part of the minimisation condition `minimizing` (line 121) at small field, and the axial-gauge frame at large field satisfies it automatically by formula (3). A frame is a gauge transformation on the patch; the fine configuration is re-viewed through it before the predictor is built (9-04) Source .

We set notation. For an axis order l (an ordering of the four axes, 24 in all; `Order` is a permutation of $\{0,1,2,3\}$, `YM/Lattice/Box.lean` line 39) and two points z, z', let $\Gamma_l(z \to z')$ be "the coordinate path from z to z' going straight along one axis at a time in the order l" (`route`, same file, line 44). The product of a configuration U along a path (transport) is written $U(\Gamma)$ Source .

The frame is defined in two kinds (`Patch.frame`, lines 153–155) Source .

$$h = \begin{cases} h^{\mathrm{sym}}(U) & \text{(small field: } \texttt{usesMin} \text{ holds)}, \\ h^{\mathrm{ax}}_{(r,l)}(U) \ \text{ with the label } (r,l) \text{ drawn at random} & \text{(large field)}. \end{cases} \tag{1}$$

The probability of the label (r, l) is `frameMass` (lines 148–151). At small field a uniform probability $1/|B \times \mathrm{Order}|$ is put on all labels (the labels are not used); at large field a uniform probability $1/|E(U)|$ is put on the set $E(U)$ of "labels that may be drawn"

Source . This probability is the frame part of $P(\tau \mid U)$ in the kernel's formula ((2) in 9-01).

That the two frames are well defined is two of the three obligations `Patch.Branches` (lines 182–185): that the small-field minimiser is unique (`frame_unique`), and that the set of labels that may be drawn is non-empty (`axial_exists`) Source . For the former, `RG.frame_unique` (`YM/Branch/Uniform.lean` line 18) says "if the frame threshold is at most some $\delta > 0$ determined by the group alone, it is unique for all patches"; the latter is met by `Axial.viable` (`YM/Branch/Pick.lean` line 137) with factor 16 Source .

02

The small-field frame — symmetric minimisation

The small-field frame is the frame minimising the following function (`score`, lines 114–115; `minimizing`, lines 121–122) Source .

$$\mathcal{S}U(h) = \frac{1}{|B|^2 \cdot 24} \sum_{z \in B} \sum_{z' \in B} \sum_{l} \operatorname{price}\Big(h(z)\, U\big(\Gamma l(z \to z')\big)\, h(z')^{-1}\Big), \qquad h(0) = 1. \tag{2}$$

Here $\operatorname{price}(u) = 1 - \operatorname{Re}\operatorname{tr} u/dG$ ((1) in 9-01). The sum runs over **all pairs of points** of the patch and **all axis orders**. A minimiser exists by compactness of the set of frames fixed to 1 at the centre (the group is compact) (`min_exists`, line 124), and the chosen one is called `selected` (line 129) Source .

This minimisation is one discrete form of the Landau gauge. The Landau gauge looks only at links between neighbouring points and minimises $\sum \operatorname{price}(h(z) U_{zz'} h(z')^{-1})$, whereas (2) joins every pair of points by coordinate paths, so it is a Landau type on the complete graph over the patch Inferred . By looking at all pairs of points uniformly and averaging over the 24 orders, (2) becomes a function invariant under the hypercubic group (permutations and reflections of axes) Inferred .

The minimiser satisfies a self-consistency equation. The source's `Branch.self_consistent` (`YM/Branch/Synchronize.lean` line 100) says that at each point z, $h(z)$ is the group element closest to "the average of the values of h carried from all other points" Source . The docstring of the lemma that pins the minimiser down to one, `Branch.align` (same file, line 232), says it uses that the difference of averages lies in the normal slice, together with an estimate on the derivative of the nearest-point map, to show that the difference of two minimisers contracts in the sup norm Source . We have not read the content of the proof.

03

The large-field frame — a randomly drawn axial gauge

At large field, one label (r, l) (a root $r \in B$ and an axis order l) is drawn, and the frame is built from values transported from the root along coordinate paths (`axial`, lines 102–103, and the second branch of `frame`) Source .

$$A_{r,l}(z) = U\big(\Gamma_l(r \to z)\big), \qquad h^{\mathrm{ax}}_{(r,l)}(z) = A_{r,l}(0)^{-1} A_{r,l}(z). \quad (3)$$

This is the axial gauge from the root r (one kind of tree gauge), normalised from the left so that $h = 1$ at the centre 0 Inferred . In the axial gauge, every link on the coordinate paths leaving the root becomes 1, and the values of the remaining links can be written as products of curvatures (plaquette values).

The labels that may be drawn are those for which the sum of the chords of the links in the axial gauge is at most a fixed multiple of the sum of the chords of the plaquettes (`eligible`, lines 143–146) Source .

$$E(U) = \Big\{ (r,l) : \sum_{\text{links } e \text{ in the patch}} \mathrm{chord}\big(U_e^{h^{\mathrm{ax}}_{(r,l)}}\big) \;\le\; c_{\mathrm{ax}}\,(2R+1) \sum_{\text{plaquettes } p \text{ in the pat}}$$

c_{ax} is `axial_factor`, and R is the patch radius. `Axial.existence` (`Pick.lean` line 80) shows that for every configuration there is a root whose sum is bounded by (number of axes) × (number of layers) × 4 times the plaquette sum, and with four axes, taking the factor 16 gives that $E(U)$ is non-empty (`Axial.eligible`, same file, line 119; if the factor is at least 16, for every axis order there is a root that may be drawn) Source . The meaning is "choose an axial gauge in which the size of the links is bounded by the size of the curvature". At large field some plaquette values are large; including the places where they are not, a frame is taken in which the total size of the link values can be measured by the total size of the curvature Inferred .

The choice of label breaks the symmetry. Once a root position and an axis order are chosen, the action of the hypercubic group moves them to another label. So instead of fixing one label, **the labels that may be drawn are drawn uniformly, and the kernel sums over them**. A group element s maps the set of labels that may be drawn one-to-one onto the set of labels (`orient` in 9-05), so the summed kernel is symmetric Source .

04

The branching condition

Which frame is used is decided by `usesMin` (lines 109–110) Source .

$$\mathtt{usesMin}(U) \iff \forall\, l,\ k,\ z,\ z' : \quad \mathrm{chord}\Big(A_{0,l}(z)\, U\big(\Gamma_k(z \to z')\big)\, A_{0,l}(z')^{-1}\Big) < \delta_F. \quad (5)$$

That is, the small-field frame is used when, seen in **the axial gauge rooted at the centre** 0 (for every axis order l), the transport along every coordinate path joining every pair of points of the patch lies at distance less than δ_F from the identity. δ_F is `frame_cut` Source .

The condition is written not in terms of the size of plaquettes but in terms of the **chord of transports**. Even if every plaquette value is small, the transport can become large if curvature accumulates over the size of the patch (radius $(J+3)L$) Inferred . (5) is a condition rewritten in gauge-invariant form via the axial gauge, and does not change under gauge transformations of the configuration (gauge invariance is in 9-05).

	Small field (`usesMin`)	**Large field (otherwise)**
Frame	The minimiser of (2) (unique)	The axial gauge (3) for each label (r, l)
Probability of labels	Uniform (not used)	Uniform on the labels that may be drawn, $E(U)$
How symmetry is kept	(2) is invariant, and the minimiser is unique	Sum over labels
Threshold used	δ_F (small enough for uniqueness of the minimiser)	Factor $c_{\mathrm{ax}} = 16$

05

Why two kinds

From the source's docstrings and definitions, we read the roles of the two kinds as follows Inferred .

- **Why minimisation is used at small field.** The minimiser is unique (§ 01), and as the unique minimiser of a symmetric function it commutes with the action of the symmetries. Moreover, it is expected to move continuously (analytically) with the configuration, which suits the expansion of the small-field step (11-09).
- **Why minimisation is not used at large field.** When transports are large, the minimiser need not be unique, and the choice can be discontinuous in the configuration. The axial gauge is the explicit formula (3) and has no uniqueness problem.
- **Why draw at random.** The axial gauge breaks symmetry through the choice of root and order. Choosing by a single rule would make the kernel non-invariant under the hypercubic group. Drawing labels uniformly and summing restores the symmetry at the level of the kernel.

The docstring calls the small-field branch "the open branch" (`Patch.lean` line 108), and says that the branching threshold and the existence of uniqueness are "proof

obligations, not assumed for any gauge group" (lines 10–12) Source . At the branching boundary the frame switches discontinuously, but the kernel is defined measurably as a probability sum over the two branches (`density_meas` in 9-01), and the design is that this is enough Inferred .

06

What this book's study had seen

This book's deterministic block map (19-04, families of paths) does no gauge fixing. It builds a coarse link as a weighted average of products along fine paths, and makes the map commute with the discrete symmetries by taking the family of paths itself symmetric under permutations and reflections of the transverse directions. A path product is gauge transformed only at its two ends, so gauge covariance follows directly from the path products.

The published construction fixes the gauge with a frame and then averages (9-04). The aim is the same (commuting with discrete symmetries, and gauge covariance); the route is different.

- This book "averages paths symmetrically"; the published construction "minimises the frame symmetrically (small field) / draws the symmetry-breaking choices at random and sums (large field)".
- This book's block map uses the same formula at large field, only pulling back inside the sphere when the average becomes small (19-04). A design that treats large field as a separate branch inside the definition of the map is something this book's study did not have.
- The step-by-step gauges in this book's study mixed the Landau slice, the axial gauge and the tree gauge depending on the place (28-05 § 05). The published construction puts them in at once inside the definition of the one-step kernel, in the form of two frames switched by a condition.

This book's study did not have the idea of drawing the large-field frame at random. The comparison is summarised in 14-02 and 14-05.

07

What we have not read

- The content of the proof of `Branch.align` (`YM/Branch/Synchronize.lean` line 232), and the value of the frame threshold δ_F.
- The details of the proof of `Axial.existence` (`YM/Branch/Pick.lean` line 80) (we read as far as its form: choosing, layer by layer, the slice with the smallest plaquette sum,

and inducting).

- Where the statement that the frame moves analytically with the configuration within small field is located (we saw only the name `YM/PREC/FrameResponse` on the side of derivatives of the predictor).
- How the discontinuity of the frame at the branching boundary is kept from affecting the estimates of the small-field step.

What this chapter can say

Can say	Cannot say
Source The frame comes in two kinds: if the chord of the transport along every coordinate path in the central axial gauge is below the threshold, the symmetric minimisation (2); otherwise the randomly drawn axial gauge (3)	The value of the threshold δ_F
Source If the frame threshold is at most a value determined by the group alone, the small-field minimiser is unique for all patches (`RG.frame_unique`)	The content of that proof (not read)
Source With factor 16, the set (4) of labels that may be drawn is non-empty (`Axial.viable`, `Axial.eligible`)	Whether the factor 16 is necessary
Inferred At large field, drawing the symmetry-breaking choices at random and summing keeps the symmetry of the kernel (the statement is in 9-05)	The mechanism by which the discontinuity of the frame does not affect the estimates

Sources and reproduction

Item	Kind	Source / tools
Frames, labels, score, branching, labels that may be drawn	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Patch.lean`: lines 99–155 (`Frame`, `axial`, `via`, `usesMin`, `score`, `minimizing`, `selected`, `facesum`, `axialCost`, `eligible`, `frameMass`, `frame`), obligations lines 182–185
Coordinate paths and axis orders	Source	Same repository, `YM/Lattice/Box.lean` lines 30–44 (`axisRoute`, `Order`, `route`)
Uniqueness of the minimiser	Source	Same repository, `YM/Branch/Uniform.lean` line 18 `RG.frame_unique`, `YM/Branch/Synchronize.lean` line 100 `self_consistent`, line 232 `align`
Existence of labels that may be drawn	Source	Same repository, `YM/Branch/Pick.lean` line 80 `Axial.existence`, line 119 `Axial.eligible`, line 137 `Axial.viable`
This book's symmetric averages over families of paths	Lean	The source table of 19-04

Terms used in this chapter — frame, symmetric minimisation, Landau gauge, axial gauge, label, labels that may be drawn, branching condition, chord, hypercubic group. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-04 The predictor and projection to the group — we read the procedure that averages the transports re-viewed through the frame with weights and projects onto the group to build the predictor of a coarse link.

2026-11-13 · chapter Yang–Mills and the mass gap Part 9 One step of renormalisation The predictor and projection to the group · Kiichi, Shiori and Rin

The predictor and projection to the group — a weighted average of transports, onto the group

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — how the centre $Q_b(U,\tau)$ of the one-step kernel (9-01) is built: the transports along parallel straight lines joining corresponding points of two neighbouring patches, their weighted average, the projection onto the group (nearest point), and the condition for switching to drawing a single one at random (both the spread and the distance from the average to the group). And that the predictor is a different thing from the background field of a Bałaban-type constrained minimisation.

Prerequisite chapters — 9-01 The block kernel, 9-02 Patches and weights, 9-03 Two kinds of gauge fixing.

Said first — we have not read the value of the predictor threshold δ_P, nor the estimate that the predictor moves smoothly with the configuration. "Not a background field" in § 05 is a comparison of definitions; it says nothing about estimates between the two objects.

The order of this chapter

1. **The statement — definition of the predictor**: `connector`, `bondCost`, `best`, `predict`
2. **Transports along parallel straight lines**: joining corresponding points of neighbouring patches
3. **Minimising the weighted average is projection onto the group**: `Branch.min_iff`, and the identity spread + distance
4. **The condition for switching to drawing a single one**: both the spread and the distance from the average to the group
5. **The predictor is not a background field**: an explicitly built central value, versus a stationary point inside the analysis
6. **What this book's study had seen**: pull-back and projection coincide for SU(2)

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

The statement — definition of the predictor

The **predictor** $Q_b(U,\tau) \in G$ of a coarse link $b=(x,i)$ (from the coarse point x in direction i) is built in `StepData.prediction` in the source (`YM/RG/Block.lean` line 44) in the following three stages Source .

1. **The collection of transports** (`connector`, lines 39–42): for each point z of the patch,

$$Y_b(z) = h_x(z)\, U\big(Lx+z \;\to\; Lx+z+Le_i \text{ straight line}\big)\, h_{x+e_i}(z)^{-1}. \quad (1)$$

   h_x is the frame of the patch at the coarse point x (9-03), and L is the side.

2. **The weighted cost** (`bondCost`, `YM/RG/Patch.lean` line 159): for a group element u,

$$C_b(u) = \sum_z w_J(z)\ \mathrm{price}\big(Y_b(z)\,u^{-1}\big). \quad (2)$$

   One of its minimisers is u_b^* (`best`, line 169). A minimiser exists by compactness (`bond_min`, line 164).

3. **Branching** (`predict`, lines 173–175):

$$Q_b = \begin{cases} u_b^* & C_b(u_b^*) < \delta_P, \\ Y_b(r)\ \ (r \text{ drawn with probability } w_J(r)\ ) & \text{otherwise.} \end{cases} \quad (3)$$

   δ_P is the predictor threshold `predictor_cut`. The probability of drawing a single one is `predictorMass` (lines 177–179); on the branch that uses the average, the label r is not used and a uniform probability is put.

That the minimiser is unique on the branch using the average is the second of the three obligations `Branches`, `predictor_unique`. `RG.predictor_unique` (`YM/Branch/Uniform.lean` line 58) says that if the predictor threshold is at most some value determined by the group alone, it is unique for all patches Source .

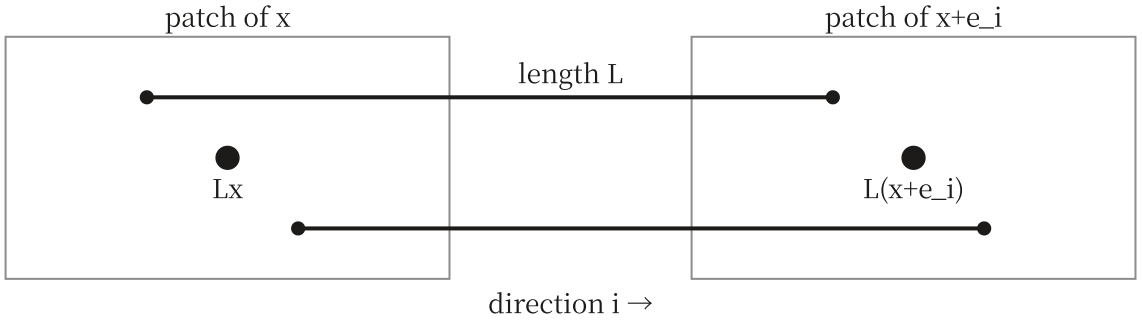
02

Transports along parallel straight lines

The straight line in formula (1) goes from the point $Lx + z$ of the patch at the start of the coarse link, exactly L steps in direction i, to the **corresponding point** $L(x + e_i) + z$ of the patch at the end Source . Since z ranges over all points of the patch (a four-dimensional box of radius $(J + 3)L$), the lines include all translations in four dimensions. Both shifts along direction i (longitudinal shifts) and transverse shifts are included Source .

The frames of the respective patches are multiplied at the two ends of the transport. Since a frame is a gauge transformation on the patch, $Y_b(z)$ is "the transport from the point z seen in the frame at the start to the point z seen in the frame at the end" Inferred . Applying a gauge transformation g to the fine configuration moves the frame as $h_x(z) \mapsto g(Lx)\, h_x(z)\, g(Lx + z)^{-1}$ (since it is normalised to $h = 1$ at the centre, the value $g(Lx)$ at the coarse point attaches on the left), and $Y_b(z)$ changes to $g(Lx)\, Y_b(z)\, g(L(x + e_i))^{-1}$, only by end elements **independent of** z (`bond_gauge` in 9-05) Inferred . So averaging over z does not break gauge covariance.

Figure 1 The collection of transports for a coarse link $b = (x, i)$. A point z of the starting patch and the same point z of the ending patch are joined by a straight line of length L in direction i (only two are drawn here).



Since all points of the patch are used, the predictor of one coarse link depends on the fine links in the union of the two patches Inferred . The weights w_J act on the central part of radius $J(L - 1)/2$ (9-02), but the frame is determined by the whole patch.

03

Minimising the weighted average is projection onto the group

Viewing a group element as a $dG \times dG$ matrix $\mathrm{mat}(u)$, set the average matrix

$$M_b = \sum_z w_J(z)\ \mathrm{mat}\big(Y_b(z)\big) \tag{4}$$

In general M_b is not a group element. The source's `Branch.min_iff` (`YM/Branch/Center.lean` line 72) says that when the weights sum to 1,

$$u \text{ is a minimiser of } (2) \iff \mathrm{mat}(u) \text{ is the group element nearest to } M_b \text{ in the Frobenius nor}$$

Source . "Nearest" is `Gauge.Nearest` (`YM/Gauge/Nearest.lean` line 26), meaning $\|M - \mathrm{mat}\,u\|_F \le \|M - \mathrm{mat}\,v\|_F$ for all v Source . So the predictor on the branch using the average is the **projection onto the group** of the average matrix, $\Pi_G(M_b)$.

The core of the proof is a single identity (`hk` inside `min_iff`, lines 76–77) Source . Expanding with $\|\mathrm{mat}(u) - \mathrm{mat}(v)\|_F^2 = 2dG\,\mathrm{price}(uv^{-1})$ (`price_diff`, line 63) gives

$$2dG\,\mathcal{C}_b(v) = \underbrace{\sum_z w_J(z)\,\big\|\mathrm{mat}\,Y_b(z) - M_b\big\|_F^2}_{\text{weighted spread}} \;+\; \big\|M_b - \mathrm{mat}(v)\big\|_F^2 \tag{6}$$

Proved here . The first term does not depend on v, so minimising $\mathcal{C}_b$ is the same as minimising the second term, that is, the distance from M_b to the group. Uniqueness of the minimiser is derived by `Branch.weighted_unique` (`Center.lean` line 95) from the fact that if $\mathcal{C}_b$ is small enough at some group element, M_b lies in a tube near the group, where the nearest point is unique (`nearest_tube`, `Nearest.lean` line 141) Source .

04

The condition for switching to drawing a single one

The switching condition in (3) is $\mathcal{C}_b(u_b^*) \ge \delta P$ (the negation of `usesBest`, `Patch.lean` line 171) Source . Putting the minimiser into (6),

$$2dG\,\mathcal{C}_b(u_b^*) = \sum_z w_J(z)\big\|\mathrm{mat}\,Y_b(z) - M_b\big\|_F^2 + \mathrm{dist}_F\big(M_b, G\big)^2 \tag{7}$$

Proved here , so the switch to drawing a single one happens when

$$\text{weighted spread} + \text{distance from the average to the group}^2 \;\ge\; 2dG\,\delta P \tag{8}$$

Proved here . **Both terms matter.** Writing only "when the average is far from the group" drops the contribution of the spread. For example, if the transport values split into two

distant places in the group, the spread is large even if the average happens to be close to the group, and the branch drawing a single one is entered Inferred .

On this branch, a point r is drawn with weight $w_J(r)$, and that single transport $Y_b(r)$ is used as the predictor as it is. It is a group element, so no projection is needed. The drawing probability enters the sum in the kernel ($P(\tau \mid U)$ in 9-01), and since the weights are invariant under the hypercubic group, the summed kernel is symmetric (9-05) Source .

Branch	Condition	Predictor	Probability of labels
Average	Left side of (8) $< 2d_G\delta_P$	$\Pi_G(M_b)$ (unique)	Uniform (not used)
Single	Left side of (8) $\geq 2d_G\delta_P$	$Y_b(r)$	$w_J(r)$

05

The predictor is not a background field

In Bałaban's programme (7-07), one builds the configuration minimising the fine action with the coarse variables held fixed (the constrained minimiser, the background field), and integrates the fluctuations around it. The predictor of the published construction is a different object Source .

- The predictor is a **central value** built from the fine configuration by explicit rules (frame, parallel lines, weighted average, projection, or drawing a single one). No minimisation of the action enters (the minimisation of the frame is a minimisation of an average of price, not of the action).
- The coarse configuration is not constrained to the predictor; it fluctuates around it in Gaussian fashion (9-01).

What corresponds to a stationary point appears separately, inside the **analysis** of the small-field step. The field `stationary` of `Kin` (lines 74–78) in `YM/PCalc/Loop.lean` is a stationary point of the "coupled action", the fine action plus the penalty term, and its properties (real analyticity, a lower bound on the Hessian, agreement of jets up to order 4 with the coarse action) are carried as fields of `Kin.Planned` (lines 93–116) Source . This is an object for setting up the integration variables, and does not enter the definition of the kernel. It is read in detail in 11-09.

06

What this book's study had seen

This book's block map (19-04, the non-linear block map) builds a weighted average of path products $q_b(U)$ (a quaternion), and returns it to the sphere by the pull-back with

threshold $r_0 \in (0, 1]$, $\rho_{r_0}(q) = q/\max(|q|, r_0)$. For SU(2), there is a relation between these two constructions Proved here .

A real linear combination of elements of SU(2) is a real linear combination of unit quaternions, hence a quaternion q (the 2×2 matrix $q_0 \mathbf{1} + i\underset{\sim}{q} \cdot \underset{\sim}{\sigma}$). The Frobenius norm is $\|q\|_F^2 = 2|q|^2$, and SU(2) is the sphere $|q| = 1$. So the nearest point to $q \neq 0$ is $q/|q|$ Proved here . That is,

$$\Pi_{SU(2)}(q) = \frac{q}{|q|} = \rho_{r_0}(q) \qquad (|q| \geq r_0). \quad (9)$$

Where the average is at least the threshold, this book's pull-back and the published construction's projection are the same map Proved here . They differ in three points.

- **What is averaged**: this book uses products along a fixed family of paths (paths shifted by ± 1 sideways); the published construction uses transports along parallel straight lines re-viewed through the frame (all points of the patch, including longitudinal shifts).
- **Where the average cannot be used**: for $|q| < r_0$ this book returns the value inside the sphere, q/r_0, widening the configuration space to the ball. The published construction switches branch by spread + distance^2 and stays with a group value by drawing a single one at random.
- **After the central value**: this book makes the central value itself the coarse link (deterministic); the published construction adds Gaussian-type fluctuations around the central value (9-01).

On the analysis side of this book's study (Part 15, 28-05 § 05), a constrained minimiser of the action was used as the background. Defining the kernel with an explicitly built central value and using stationary points only as an analytic tool is a division this book's study did not have (14-05).

07

What we have not read

- The value of the predictor threshold δP (we read as far as `weighted_unique` building it from the radius of the nearest-point tube of the group).
- The estimate that the predictor moves smoothly with the configuration within small field (we saw only the name `YM/PREC/FrameResponse`).
- How the branch drawing a single one is paid for in the large-field estimate (11-04).
- What kernel the inclusion of longitudinal shifts in the parallel lines produces in the first-order part (linearisation) of the predictor. We have not read the formula for the

counterpart of this book's path kernel (19-04) (a candidate is `probe` in `YM/Push/FREE_RESOLVENT/Flat.lean`).

What this chapter can say

Can say	Cannot say
Source The predictor is built from the weighted average of the transports (1) along parallel straight lines re-viewed through the frame; if the minimum of the cost (2) is at least the threshold, a single one is drawn with probability wJ	The value of the threshold δ P
Source The minimiser of (2) is the nearest point on the group to the average matrix (4) (`Branch.min_iff`), unique if the threshold is small enough (`RG.predictor_unique`)	An estimate that the minimiser moves smoothly with the configuration
Proved here The switch happens when "weighted spread + distance from the average to the group^2" is at least $2dG\delta P$ (both terms matter)	—
Source The predictor is not a constrained minimiser of the action; a stationary point enters separately, inside the analysis, as `Kin.stationary`	An estimate of the difference between the predictor and the stationary point
Proved here For SU(2), the projection onto the group is $q/\|q\|$, which agrees with this book's pull-back for $\|q\| \geq r0$	How the difference in the families of averaged transports matters

Sources and reproduction

Item	Kind	Source / tools
The collection of transports, the predictor	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Block.lean` lines 39–45 (`connector`, `prediction`)
Cost, minimiser, branching	Source	Same repository, `YM/RG/Patch.lean` lines 159–179 (`bondCost`, `bond_min`, `best`, `usesBest`, `predict`, `predictorMass`)
Being a projection, the identity	Source	Same repository, `YM/Branch/Center.lean` line 63 `price_diff`, line 72 `min_iff` (identity at lines 76–77), `weighted_unique`; `YM/Gauge/Nearest.lean` line 26 `Nearest`, line 141 `nearest_tube`
Uniqueness of the predictor	Source	Same repository, `YM/Branch/Uniform.lean` line 58 `RG.predictor_unique`
Stationary point (inside the analysis)	Source	Same repository, `YM/PCalc/Loop.lean` lines 74–116 (`Kin`, `Kin.Planned`)
(6)–(9)	Proved here	Computations in the text

Item	Kind	Source / tools
This book's block map and pull-back	Lean	The source table of 19-04

Terms used in this chapter — predictor, transport, parallel straight lines, weighted cost, projection onto the group, nearest point, spread, background field, stationary point, pull-back. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-05 Symmetries — from how the frame, weights and predictor are built, we read that the one-step kernel is invariant under the hypercubic group and coarse translations, and covariant under gauge transformations.

Symmetries — invariance under the hypercubic group and coarse translations, gauge covariance

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the exact form of the two equalities satisfied by the one-step kernel $T_\beta(U, V)$ (invariance under discrete spacetime symmetries, and covariance under gauge transformations), and which components they come from. The mechanism of drawing the symmetry-breaking choices at large field at random and summing over them does its work here.

Prerequisite chapters — 9-01 to 9-04. The hypercubic group is in 1-02, and gauge transformations in 19-01.

Said first — the symmetry group is the set of all maps of coarse lattice points $x \mapsto s \cdot x + b$ (s a signed permutation, b a coarse point). Both reflections in planes through coarse lattice points and reflections in planes through the midpoints of coarse links are included (§ 02). This is an equality of **covariance as a symmetry** of the kernel, not a statement of reflection positivity. Continuous rotations are not stated at the lattice level; they are raised separately in the limit (12-06).

The order of this chapter

1. **The statement — two equalities**: `StepData.channel` and `StepData.covary`
2. **The form of the group action**: the 384 signed permutations, shifts by coarse points only, and reflections in two kinds of planes
3. **Origin (1) — invariance of the weights and patches**
4. **Origin (2) — uniqueness of the minimiser, and the sum over labels**
5. **Origin (3) — orientation reversal and inverses, gauge covariance**
6. **What this book's study had seen**: symmetric families of paths, and commuting with reflections in the side-3 design
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

The statement — two equalities

The source's `YM/Branch/Covariant.lean` assumes the three obligations `Branches` (9-03, 9-04) and shows two equalities for the one-step kernel Source .

$$T\beta\big(\mathrm{pull}_{s,\,Lb}\,U,\ \mathrm{pull}_{s,\,b}\,V\big) = T\beta(U,V) \qquad (\texttt{StepData.channel},\ \text{line } 50\ )\ (1)$$

$$T\beta\big(g\cdot U,\ (g\circ\iota)\cdot V\big) = T\beta(U,V) \qquad (\texttt{StepData.covary},\ \text{line } 56\ )\ (2)$$

(1) holds for every signed permutation s, every coarse point b and every β; (2) holds for every gauge transformation g of the fine lattice (a group element at each point) and every β. ι is the embedding sending a coarse point to the fine point Lx (9-02), and $g\cdot U$ is the action $U_{z,z+e}\mapsto g(z)U_{z,z+e}g(z+e)^{-1}$ on each link Source .

(2) can be read as "if the fine configuration is gauge transformed and the coarse configuration is gauge transformed by the values $g(Lx)$ at the coarse points, the kernel does not change". A gauge transformation of the coarse lattice is the restriction of a fine gauge transformation to the coarse points Inferred .

These two are required at each step as the fields `symmetric` and `gauge` of the calendar structure `Calendar.BlockControl` (`YM/RG/History.lean` , lines 120–128), and `Calendar.blocks_go` (`Covariant.lean` line 70) supplies them from (1)(2) Source . Since `BlockControl` is a field of the structure `Levels` of the sequence of steps, the sequence of steps of `History` under the construction theorem's conclusion `Family.Makes` uses kernels with these symmetries at every step Source .

02

The form of the group action

A signed permutation s is a pair of a permutation of the four axes and a sign flip on each axis (`Signed` in the source, `YM/Space/Index.lean` line 23). There are $4! \times 2^4 = 384$ elements, and this is the symmetry group of the four-dimensional hypercube (the hyperoctahedral group) Proved here .

The action on configurations, $\mathrm{pull}_{s,b}$ (`YM/Lattice/Path.lean` line 147), pulls a configuration back by the affine map of points $x \mapsto s \cdot x + b$ (`affine`, line 145). On links whose orientation is reversed, the value traversed backwards, that is the inverse, is read (`Step.pull_read`) Source .

s acts about the origin, and then the shift by b follows. It is not a rotation **about** the coarse point b Source . Since b in (1) ranges over all coarse points, the group contains all of the following maps Proved here .

- **Coarse translations**: with $s = 1$, $x \mapsto x + b$. On the fine side this is a translation by Lb.
- **Reflections in planes through lattice points**: taking s to be the sign flip of axis i (`Signed.mirror` in the source, `YM/Space/Index.lean` line 55) and $b = 0$ gives $x_i \mapsto -x_i$, the reflection in the plane $x_i = 0$. With $b = 2c\,e_i$ it is the reflection in the plane $x_i = c$ (a plane through coarse lattice points).
- **Reflections in planes through the midpoints of coarse links**: with the same s and $b = e_i$, $x_i \mapsto 1 - x_i$. This is the reflection in the plane $x_i = \frac{1}{2}$, which passes through the midpoint of the coarse link $(0, e_i)$. On the fine lattice it is the plane $x_i = L/2$; since L is odd, this is a block boundary, a plane through the midpoints of fine links. With $b = (2c + 1)e_i$ it is the plane $x_i = c + \frac{1}{2}$.
- **Permutations of axes, and their combinations with translations**: the case where s contains a permutation. For example, with $b = 0$ it is a hypercubic rotation about the origin.

So (1) contains reflections in **both** kinds of planes of the coarse lattice (planes through lattice points and planes through the midpoints of links) Proved here . These two kinds have the same form as the two kinds of planes for which reflection positivity of the original Wilson measure is stated (12-01). However, (1) is an equality of **covariance**, saying that the kernel commutes with symmetries; it does not state reflection positivity of the push-forward measure Source . We found no statement of reflection positivity of the one-step kernel within what we read (9-01 § 07).

03

Origin (1) — invariance of the weights and patches

The patch is a box centred at the origin, and the weights are convolutions of a uniform distribution (9-02). Both are invariant under signed permutations. The source builds the permutation of the points of the box `offset` (`YM/Branch/Transport.lean` line 18), and shows $w_J(s \cdot z) = w_J(z)$ in `weight_invariant` (line 135) Source .

This invariance acts on the predictor side. The cost $C_b(u) = \sum_z w_J(z)\,\mathrm{price}(Y_b(z)u^{-1})$ ((2) in 9-04) does not change value when the collection of transports is reordered by $z \mapsto s \cdot z$, so the minimiser and the probability of drawing a single one commute with the reordering (`bond_pull`, `YM/Branch/Predict.lean` line 37) Source .

04

Origin (2) — uniqueness of the minimiser, and the sum over labels

The symmetry arises differently on the two branches of the frame (9-03) Source .

- **Small field**: the function minimised, $\mathcal{S}_U(h)$ ((2) in 9-03), is an average over all pairs of points and all axis orders, so it is invariant under signed permutations (`via_channel`, `Transport.lean` line 26). The branching condition is also invariant (`min_channel`, line 34). **Since the minimiser is unique**, the minimiser of the rotated configuration equals the rotated minimiser (`selected_channel`, line 56). Without uniqueness, there would be no guarantee that the chosen minimiser commutes with the symmetries Inferred .
- **Large field**: a label (r, l) of the axial gauge moves under a group element s to $(s \cdot r,\ s \circ l)$ (`orient`, line 20). The set of labels that may be drawn also moves one-to-one (inside the proof of `frames_channel`, lines 77–90). Since the probability of labels is the uniform distribution on the set, the probability of a label for the rotated configuration equals the probability of the moved label. Changing the variable of the kernel's sum $\sum_\tau$ as $\tau \mapsto s \cdot \tau$ (`withinChoice`, `Predict.lean` line 88; `Equiv.sum_comp`) then recovers the value of the kernel (`mass_channel`, `Covariant.lean` line 22).

An axial gauge fixed by a single label breaks the symmetry, but by **putting the probability of drawing the labels into the kernel and summing**, the kernel becomes invariant as an equality Inferred . The branch of the predictor that draws a single one (9-04) is recovered in the same way, since the weights w_J are invariant.

05

Origin (3) — orientation reversal and inverses, gauge covariance

A sign flip can move a coarse link b to a negatively oriented link. In that case one uses that the predictor of the oppositely oriented coarse link is the inverse. `bond_inv` (`Predict.lean` line 44) says that inverting the collection of transports inverts the predictor and leaves the probability of drawing a single one unchanged Source . The proof is $\mathbf{price}(k^{-1}u) = \mathbf{price}(ku^{-1})$ (invariance of $\mathbf{price}$ under inverses and conjugation, `price_inv`, `price_conj`) Source . The noise factor is recovered by the same invariance (`noise_channel`, `Covariant.lean` line 37).

Gauge covariance (2) comes from three components Source .

1. The frame: under a fine gauge transformation, the frame moves by having the value at the coarse point attached on the left, and the probability of labels does not change (`frame_gauged`, `Transport.lean` line 111). The branching condition and the set of labels that may be drawn are gauge invariant.
2. The predictor: sandwiching the collection of transports between constants u, v on both sides sandwiches the predictor between the same u, v, and leaves the probability of drawing a single one unchanged (`bond_gauge`, `Predict.lean` line 54).
3. The noise: $\mathbf{price}\big(uxv^{-1}(uyv^{-1})^{-1}\big) = \mathbf{price}(xy^{-1})$ (invariance under conjugation, at the end of the proof of `covary`).

As seen in 9-04 § 02, the collection of transports $Y_b(z)$ is sandwiched between end elements $g(Lx)$ and $g(L(x+e_i))^{-1}$ that do not depend on z, so the form of the predictor component (bond_gauge) can be used as it is Proved here .

Component of the kernel	Origin of the spacetime symmetry (1)	Origin of the gauge covariance (2)
Weights w_J, patch	`weight_invariant`, `offset`	—
Small-field frame	Invariance of score + uniqueness of the minimiser (`selected_channel`)	`frame_gauged`
Large-field frame	Permutation of labels `orient` + sum over labels	`frame_gauged`
Predictor (average, single)	`bond_pull`, `bond_inv`	`bond_gauge`
Noise	`noise_channel` (invariance of price under inverses and conjugation)	Invariance of price under conjugation

This symmetry is carried over to the representation side as well. The main action is written in a form averaged over signed permutations (`inChannels` in 9-06), and that the families of activities are moved into one another by permutations (`Activities.Moves`) is part of the conditions on the representation Source . It is also

from this symmetry that, in the small-field step, the sum of the log-Hessians in the curvature directions takes the form $\delta_{ab}\Delta$ (becomes isotropic) (11-09) Inferred .

06

What this book's study had seen

This book's block map was made to commute with the discrete symmetries by taking the family of paths symmetric under permutations and reflections of the transverse directions (19-04, families of paths). Gauge covariance follows from path products being gauge transformed only at their two ends.

The symmetry statements closed in this book's Lean are the following Lean .

- The current design (side 2): when the fine step is large enough, the push-forward Wilson measure is invariant under coarse translations (for expectations of bounded continuous functions; Lean: `hsym_translation`, no assumption on the coupling).
- The side-3 centred design: commuting with reflections in planes of lattice points (all steps, no gauge transformation; Lean: `Block3.blockMap_refl_cfg`, `Block3.down_refl`), coarse translations (`Block3.blockMap_trans`, `down_trans`), sign flips (`Block3.blockMap_flip`), and boundary reflections (`Block3.down_reflB`).

Since this book's map is deterministic, its symmetries can be stated as equalities of maps (of the form $B(\Theta U) = \Theta B(U)$). Since the published construction uses a kernel, its symmetries are stated as the kernel equality (1), and the configuration-by-configuration symmetry-breaking choices (axial-gauge labels, the point drawn singly) are put into the kernel together with their probabilities and recovered by the sum. The aim is the same; the route is different. This book's study did not have the idea of recovering symmetry by averaging the symmetry-breaking choices over a probability (14-05).

07

What we have not read

- Whether reflection positivity (positive type with respect to reflections) is stated for the one-step kernel. (1) is covariance as a symmetry, and we found no positivity statement within what we read. Reflection positivity is stated for the original Wilson measure, for two kinds of planes (12-01).
- The details of the proofs of `selected_channel`, `min_channel` and `good_channel` (we read only the statements and how they are used).
- Where (1)(2) are used in the proof that carries the symmetry of the representation (`Activities.Moves`) from step to step (we go as far as the record that `push_columns`

in `Push/SectorMeasure` uses them).

What this chapter can say

Can say	Cannot say
Source Under the obligations `Branches`, the kernel is invariant under maps of coarse lattice points $x \mapsto s \cdot x + b$ (the 384 signed permutations and coarse points b), which include coarse translations and reflections in both planes through lattice points and planes through the midpoints of coarse links (`StepData.channel`)	Reflection positivity of the one-step kernel (this equality is covariance as a symmetry, not positivity)
Source Under the same assumptions, the kernel is covariant under fine gauge transformations and their restriction to coarse points (`StepData.covary`)	—
Source `BlockControl` at each step requires the two equalities, and `blocks_go` supplies them	The details of the proof carrying the symmetry of the representation from step to step
Inferred The origins are the invariance of the weights and patches, the uniqueness of the small-field minimiser, the sum over large-field labels, and the invariance of price under inverses and conjugation	—

Sources and reproduction

Item	Kind	Source / tools
The two equalities and their supply	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/Branch/Covariant.lean`: `prediction_channel` line 13, `mass_channel` line 22, `noise_channel` line 37, `channel` line 50, `covary` line 56, `Calendar.blocks_go` line 70
Per-step requirements	Source	Same repository, `YM/RG/History.lean` lines 120–134 (`BlockControl`, `Levels`)
Signed permutations and affine pull-back	Source	Same repository, `YM/Space/Index.lean` line 23 `Signed`, `YM/Lattice/Path.lean` lines 145–147 `affine`, `pull`
Permutations of patches and labels, symmetry and gauge of the frame	Source	Same repository, `YM/Branch/Transport.lean`: `orient` line 20, `offset` line 18, `via_channel` line 26, `min_channel` line 34, `selected_channel` line 56, `frames_channel` line 77, `frame_gauged` line 111, `weight_invariant` line 135
Symmetry and gauge of the predictor	Source	Same repository, `YM/Branch/Predict.lean`: `bond_pull` line 37, `bond_inv` line 44, `bond_gauge` line 54, `withinChoice` line 88

Item	Kind	Source / tools
This book's symmetry statements	Lean	This book's Lean bundle (`hsym_translation`, `Block3.blockMap_refl_cfg`, `down_refl`, `blockMap_trans`, `down_trans`, `blockMap_flip`, `down_reflB`), a bundle that passed the check by the Lean kernel

Terms used in this chapter — signed permutation, hyperoctahedral group, coarse translation, gauge covariance, uniqueness of the minimiser, sum over labels, orientation reversal. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-06 The density representation — we read the "representation" that writes the density carried by one step as a vacuum term, a main action, regular activities and a gas of forced activities.

The density representation — the vacuum term, the main action, regular activities, the gas of forced activities

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the four parts of the form `Rep` in which the density ρ_m at step m is written, and the conditions imposed on it (normalisation, budgets, symmetry); that β is defined as the coefficient of the kinetic term in this representation; that at the finest step the representation agrees with the bare Wilson weight; and that the representation is not said to be unique.

Prerequisite chapters — 9-01 The block kernel, 9-05 Symmetries. Gases of polymers (clusters) are in 7-04.

Said first — this chapter is about the **definition** of the representation and the form of its conditions. The proof that this form is preserved from step to step is read in Part 11 (11-01 to 11-09). The details of the small-field threshold and the truncated action are in 9-07.

The order of this chapter

1. **The statement — the form of the representation**: `Rep` and `Rep.eval`
2. **The main action**: truncated plaquette terms, squares of transported curvature, cubic and quartic atoms, one-loop atoms
3. **Two normalisations**: the coefficient of the kinetic term `metric_zero` and the one-loop part `loop_jets`
4. **Regular activities and the gas of forced activities**: `Activities` and `gas`
5. **Conditions imposed on the representation, and what is not said**: `Conv`, `Lawful`, no uniqueness
6. **What this book's study had seen**: the push-forward measure itself, versus coefficients inside a representation

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — the form of the representation

When a density is carried by the one-step kernel (9-01), it can become an arbitrary function of the configuration. The published construction requires, as a field of the structure of the sequence of steps, that the density at each step can be written in a **fixed form**. That form is `Rep` (lines 165–171) in `YM/RG/History.lean`, a tuple of six fields (path data, truncation parameters, activities, budget point, β, v) Source . `Rep.eval` (line 179) is

$$\rho(U) = \exp\Big(v\,|\Lambda| \;-\; S^{\mathrm{ch}}_{\beta}(U) \;-\; \sum_{i} \mathbf{1}\big[\text{support } X_i \text{ is healthy}\big]\, E_i(U)\Big) \cdot \mathrm{gas}(U) \tag{1}$$

Source . We name the four parts.

Part	Name	Source	Chapter in this book
$v\|\Lambda\|$	Vacuum term ($\|\Lambda\|$ is the number of points)	`Rep.v`	This chapter
S^{ch}_{β}	Main action (averaged over signed permutations)	`PathData.inChannels`	§ 02, § 03, 9-07
$\sum_i \mathbf{1}[\cdot]E_i$	Sum of regular activities (healthy places only)	`Activities.additive`	§ 04, 11-09
gas	Gas of forced activities	`Activities.gas`	§ 04, 11-06

"Healthy" means that no flag is raised at the points of the support. A flag is a point touching a plaquette whose chord exceeds the step threshold t_m (9-07) Source .

The sequence of steps `History` has one representation for each (N, p, m) (lines 204–206). In the structure `History.Selected` (lines 234–240), the field `represents` requires that the representation satisfy the condition `Conv` at every step $m \le N$; `starts` requires that at the finest step $m = N$, $v = 0$ and the representation equal the bare Wilson weight $e^{-\beta N S_{\mathrm{W}}(U)}$; and `steps` requires the one-step identity ((4) in 9-01) `Source` . `Selected` is the field `wilson` of the construction theorem's conclusion `Family.Makes` (`YM/Source/Construction.lean` line 45) `Source` .

02

The main action

The main action is `PathData.principal` (`YM/RG/PathActions.lean`, lines 236–238) averaged over the signed permutations R (384 elements, 9-05) (`inChannels`, lines 241–242) `Source` .

$$S_\beta^{\mathrm{ch}}(U) = \frac{1}{384} \sum_R S_\beta^{\mathrm{pr}}(\mathrm{pull}_{R,0}\, U), \tag{2}$$

$$S_\beta^{\mathrm{pr}}(U) = \beta \Big[\sum_p \Big(\mathrm{clip}_t\big(\mathrm{chord}\, U_p\big) + \mathbf{1}[\mathrm{kept}]\, \tfrac{1}{2} \|\ell_p(U)\|^2 \Big) + \sum_x \big(\text{cubic atoms} + \text{quartic atoms} \big)$$

We read the four ingredients in turn `Source` .

- **The truncated plaquette term** $\mathrm{clip}_t(\sigma)$ (`clipped`, line 218): if the plaquette chord $\sigma = \mathrm{chord}\, U_p$ is at most the threshold t, it is $c_{\mathrm{dir}} \sigma^2/2$; beyond it, the linear $c_{\mathrm{lin}}\, t\, \sigma$. The form and the condition at the kink, $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$, are read in 9-07.
- **The square of the transported curvature** $\frac{1}{2}\|\ell_p\|^2$: a Lie-algebra value for each plaquette p, $\ell_p(U) = \sum_k r_{p,k}\ \mathrm{Ad}\big(U(\gamma_k)\big) F(U_{p_k})$ (`ell`, line 190). $F(u)$ is the projection of $(u - u^{-1})/2$ onto the Lie algebra (`Gauge.curvature`, `YM/Gauge/Colors.lean` line 173), γ_k are paths, p_k are other plaquettes, and the list of coefficients $r_{p,k}$ is a "row" of the path data (`PathData.row`, line 151; `Slot`, lines 38–56). It is a term that carries the curvature of distant plaquettes along paths and adds them up. This term enters only when the configuration satisfies the condition kept (the chord of every plaquette is at most $t\, e^{\mathrm{dist}/\text{relaxation length}}$, lines 210–211).
- **Cubic and quartic atoms**: sums, with coefficients, of products of three or four Lie-algebra values (`Atom`, lines 84–105). They are set to 0 where flags are raised (`placed` uses `masked`, lines 107–108).
- **One-loop atoms**: atoms with two legs, **not multiplied by** β. This is the part carrying the one-loop contribution, which does not depend on β.

The coefficients of the main action (rows, atoms) are held in the path data `PathData` for each step m and cutoff N, and do not depend on the volume index p (`Selected.path_shared`, line 235) Source .

03

Two normalisations

The β in representation (1) is **a single real number** multiplying the main action. As it stands, F^2 terms could be shifted between β and the one-loop atoms or the regular activities, so two normalisations stop such shifting (the fields `metric` and `loop_normal` of the path-data condition `PathData.Bounds`, lines 173–185) Source .

The coefficient of the kinetic term (`metric_zero`, lines 161–162). Writing the zero-momentum sum of the row coefficients as $S^{(0)}_{k,i} = \sum_{\text{rows of plaquette type } i} r$ (`symbol0`, line 158), for all plaquette types i, j,

$$c_{\mathrm{dir}}\,\delta_{ij} + \sum_{k} S^{(0)}_{k,i}\,S^{(0)}_{k,j} = \delta_{ij}. \quad (4)$$

For smooth small configurations, $\operatorname{chord} U_p \approx |F_p|$ and $\ell_p \approx \sum_j S^{(0)}_{p,j} F_j$ (zero momentum), so the quadratic part of the main action at zero momentum becomes $\frac{\beta}{2}$ $\sum_{ij}\big(c_{\mathrm{dir}}\delta_{ij} + (S^{(0)\mathsf{T}} S^{(0)})_{ij}\big)F_iF_j = \frac{\beta}{2}\sum_i F_i^2$ Inferred . (4) is the normalisation for "reading β exactly as the coefficient of the kinetic term".

The one-loop part (`loop_jets`, lines 167–169). For the coefficients of the one-loop atoms, it requires, for each pair of plaquette type and colour, that the sum of the coefficients be 0 and that the first moment in each direction (the sum weighted by the difference of the positions of the two legs) also be 0 Source . This means that no F^2 terms at zero and first order in momentum remain in the one-loop part, that is, all marginal F^2 terms are collected into β Inferred .

The source's docstring (`History.lean`, lines 25–33) describes β as "the actual scalar in the base part of the representation, under the flat coefficient arrays and the normalisation of the kinetic metric, the one-loop kernel with its leading Fourier coefficients removed, and the condition that affine remainders vanish; not a number tied independently to the measure" Source . The discussion of "the coupling at step m" that starts from this definition is read in 10-01.

04

Regular activities and the gas of forced activities

The family of activities `Activities` (`YM/RG/Activities.lean`, lines 62–66) is two kinds of sequences of functions, addressed by natural numbers Source .

Regular activities E_i: the address is an anchor point and a support (`RegularAddress`, lines 37–40). In the representation they enter only when the support is healthy (`regularOccurrence` is `masked`, line 131). The condition `RegularFamily` (lines 70–75) requires locality on the support, gauge invariance, C^{order} smoothness at healthy places, **vanishing at the flat configuration** ($E_i(1) = 0$), and **vanishing log-Hessian in affine directions**. The last two express that this is the form **after** the constant has been moved into the vacuum term v and the marginal quadratic term into β (the rereading in the small-field step, 11-09) Inferred .

Forced activities K_i: the address is a support and a "stock" (a non-empty set of points responsible for flags; `ForcedAddress`, lines 42–47). The field `permit` of the condition `ForcedFamily` (lines 77–84) requires that if $K_i \neq 0$, the flags in the support coincide exactly with the stock Source . The gas is

$$\mathrm{gas}(U) = \sum_{\substack{b\text{: disjoint supports,} \\ \bigcup_{i\in b} \mathrm{stock}_i = \mathrm{flags}(U)}} \prod_{i\in b} K_i(U) \tag{5}$$

(`admissible`, `gasOccurrence`, `gas`, lines 114–129) Source . It is a sum only over collections of forced activities with non-overlapping supports that **exactly cover** all the flags. If there are no flags, only the empty collection remains, and $\mathrm{gas} = 1$ (`gas_one`, line 135) Source . All large-field contributions go into this gas, and at small field with no flags the representation reduces to the exponential part of (1).

The size budget is measured by point-by-point weighted sums (`EBound`, `KBound`, lines 98–104) Source . For regular activities per anchor point, and for forced activities per point contained in the support, $e^{c_{\mathrm{sup}}\cdot(\text{cover number})} \times (\text{size})$ is summed and kept within the budget. c_{sup} is the step constant `supportExponent`; the cover number is the smallest number of balls of radius $B_m = \lceil (\log \hat{\beta}_m)^2 \rceil$ in the lattice maximum distance (centred cubes of side $2B_m + 1$), arranged so that their union is connected, needed to cover the support (`coverNumber`, `YM/Lattice/Support.lean` line 71; `BudgetPoint.mass`, `YM/RG/Masks.lean` line 144); and the "size" of a regular activity is the supremum, over healthy configurations, of finite-order derivatives in directions of size t_m (`supBounds`, lines 164–166). Since these are point-by-point sums, the budget has a volume-independent form Inferred .

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05
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Conditions imposed on the representation, and what is not said

The condition `History.Conv` (`History.lean`, lines 213–218) satisfied by the representation at step m has four parts Source .

Field	Content
`vac`	The budget point (threshold t_m, cover scale B_m, order) is the fixed value for step m, `budget m`
`clip`	The truncation parameters are step constants
`good`	The budget of the path data (`Bounds`) and the budget of the activities (`Valid`). The budget for regular activities is $\hat{g}_m^{11/5}$ (`allowance`), that for forced activities $e^{-P_m^{1/5}}$ (`forcedAllowance`)
`band`	$\beta(N,p,m) \in [\hat{\beta}_m/2,\ 2\hat{\beta}_m]$

Here $\hat{\beta}_m = \bar{\beta} + s\log D_m$, $\hat{g}_m = \hat{\beta}_m^{-1/2}$, $P_m = \exp((\log\log\hat{\beta}_m)^2)$ (definition of `Calendar`, lines 79–86) Source . The budget of activities also includes a symmetry condition (`Activities.Moves`, lines 87–96: activities are moved into one another, addresses included, by signed permutations).

The representation at the finest step is the bare Wilson weight itself (`starts`): the jet up to order 4 of the main action of the path data agrees with the Wilson energy, and the coefficients of the one-loop atoms are 0 (the fields `peak` and `initial` of `Kin.Planned`, `YM/PCalc/Loop.lean` lines 100 and 102) Source . The main action at a coarse step is tied to the fine step by the condition that "its jet up to order 4 at the flat configuration agrees with the stationary value of the fine main action plus the penalty term" (`ClassicalMatch`, the field at line 108 of the same file), but this is a condition; it does not define the main action as an expansion of the stationary value Source .

What is not said. The source's docstrings say in two places that the representation is not claimed to be unique Source .

- `PathActions.lean`, lines 25–26: uniqueness of the fitted representation as a function is neither claimed nor used.
- `History.lean`, lines 29–31: the formulation is of the form that such a representation and continuation **exist**; neither uniqueness nor reproduction of the implementation's choice of paths is required.

Since the same measure (the same density ρ_m) can have different representations, the β in the representation is not determined by the measure alone. This point is the subject of 10-01.

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06
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What this book's study had seen

One renormalisation step in this book's study took the push-forward measure $B_*\mu$ itself as its object (19-04, push-forward). It did not write the theory at a coarse step in a representation of fixed form. The statements to be shown (19-07) are also written only in terms of the push-forward measure and expectations of observables.

On the side of the list of premises (28-04), there were two proposals on paper for the definition of the effective coupling — "read it as a coefficient of the blocked action" and "read it as a renormalised coupling" — and the choice was open (28-05 § 06). The published construction fixes the former form, with the two normalisations (4) and `loop_jets`, and requires the existence of a representation at each step. In exchange, it does not say that the representation is unique, and β becomes a number that depends on the representation.

Holding forced activities in the representation as a gas that "exactly covers the flags" gives an axis of comparison with this book's study's treatment of large fields (the rows for the large-field estimate and the gas in the list of premises). The comparison is in 14-04.

07

What we have not read

- How the rows $r_{p,k}$ and paths γ_k of the path data are concretely built (the body of the "explicit renormalisation map" for the main terms). Candidate locations are `YM/Push/BETA_TRANSPORT`, `LOOP_COEFFICIENT`, `FREE_RESOLVENT` and `STATIONARY_UNIFORM`; we have not read them.
- The value of the relaxation length in the condition kept (`Clip.relaxation`), and where among the step constants it comes from.
- The construction that brings the forced activities back within the budget $e^{-P_m^{1/5}}$ after one step (the subject of 11-06; this book's study has not checked the balance either).
- The proof that path data satisfying the normalisation (4) exist at each step.

What this chapter can say

Can say	Cannot say
Source That the density at each step is written in a representation of the form (1) is required as a field of the structure of the sequence of steps (and is part of the construction theorem's conclusion)	Uniqueness of the representation (the source states that it does not claim it)
Source The main action is the average over signed permutations of the truncated plaquette terms, squares of transported curvature (only when kept), cubic and quartic atoms, and one-loop atoms not multiplied by β	How the path data are concretely built
Inferred The normalisation (4) makes β read as the coefficient of the kinetic term at zero momentum, and `loop_jets` leaves no marginal F^2 in the one-loop part	The proof of existence of path data satisfying the normalisations (not read)
Source With no flags the gas is 1 (`gas_one`); the gas is a sum over collections with non-overlapping supports that exactly cover the flags	Bringing the forced activities back within budget after one step
Source At the finest step the representation is the bare Wilson weight; the budgets are $\hat{g}^{11/5}$ and $e^{-P^{1/5}}$; β lies in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$	The proof that the budgets are preserved from step to step (Part 11)

Sources and reproduction

Item	Kind	Source / tools
Representation, sequence of steps, conditions	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean`: docstring lines 25–33, `Calendar` lines 79–86, `Rep` lines 165–179, `History` line 204, `Conv` lines 213–218, `Selected` lines 234–240
Main action and normalisations	Source	Same repository, `YM/RG/PathActions.lean`: docstring lines 25–26, `Slot` line 38, `Atom` line 84, `Clip` line 139, `PathData` line 150, `symbol0` line 158, `metric_zero` line 161, `loop_jets` line 167, `Bounds` line 173, `ell` line 190, `kept` line 210, `clipped` line 218, `principal` line 236, `inChannels` line 241
Curvature	Source	Same repository, `YM/Gauge/Colors.lean` line 173 `curvature`
Activities, gas, budgets	Source	Same repository, `YM/RG/Activities.lean` lines 37–135; `YM/RG/Masks.lean` lines 125–166 (`BudgetPoint`, `mass`, `Regular`, `supBounds`)
Conditions at the finest step, conditions on the coarse main action	Source	Same repository, `YM/PCalc/Loop.lean` lines 100, 102 and 108 (fields of `Kin.Planned`)
Relation to the construction theorem's conclusion	Source	Same repository, `YM/Source/Construction.lean` lines 44–45 `Family.Makes.wilson`

Item	Kind	Source / tools
This book's push-forward measure and statements to be shown	Lean	The source tables of 19-04 and 19-07

Terms used in this chapter — representation, vacuum term, main action, truncated plaquette term, transported curvature, atom, one-loop atom, normalisation of the kinetic term, regular activity, forced activity, stock, flag, healthy, gas, budget, cover number. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-07 The small-field threshold and the truncated action — we read the step-dependent threshold that separates large and small fields in the representation, and the form that truncates the plaquette term to linear.

The small-field threshold and the truncated action — a step-dependent threshold

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the formula and properties of the threshold t_m that decides, inside the representation at step m (9-06), whether a plaquette is "large"; the flags raised at points touching plaquettes beyond the threshold, and "healthy" supports with no flags; the form that truncates the plaquette term of the main action to linear above the threshold, with the kink condition $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$; and that the threshold goes to 0 with the step while staying much larger than the fluctuation size $\hat{\beta}_m^{-1/2}$.

Prerequisite chapters — 9-06 The density representation, 8-02 The set-up.

Said first — the truncated action is the **form** of the main part of the representation; it does not change the measure. The one-step identity remains an equality, and the truncated difference goes into the activities. It cannot be read as saying that truncation makes the difficulties of large fields disappear. The large-field estimate itself is read in Part 11 (11-03, 11-04).

The order of this chapter

1. **The statement — the step threshold**: `Calendar.threshold` and the conditions of the calendar
2. **The size of the threshold**: it goes to 0, but stays larger than the fluctuation size by the factor P_m
3. **Flags and healthy configurations**: `flags`, `Healthy`, `masked`
4. **The truncated action**: the form of `clipped` and the kink condition
5. **The threshold depends on the step**: two thresholds within one step
6. **What this book's study had seen**: a fixed threshold versus a step threshold
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — the step threshold

The calendar `Calendar` in `YM/RG/History.lean` of the source (lines 79–86) defines the following quantities for each step m Source.

$$\hat\beta_m = \bar\beta + s\log D_m, \qquad \hat g_m = \hat\beta_m^{-1/2}, \qquad P_m = \exp\big((\log\log\hat\beta_m)^2\big), \qquad t_m = \hat g_m\, P_m. \tag{1}$$

$\bar\beta$ is the value matched at step 0 (10-01), D_m is the reciprocal of the lattice spacing at step m, and $s = 11C_g/(24\pi^2)$ is the one-loop slope (`hatBeta`, `hatCoupling`, `monitor`, `threshold`). $\hat\beta_m$ is the coupling at step m expected from the one-loop running, and $\hat g_m$ is the size of that coupling. t_m is the **small-field threshold** at step m, used as the threshold of the budget point of the representation (`budget m`) (`Levels.budget`, lines 141–156) Source.

The calendar condition `Calendar.Fits` requires $1 < \hat\beta_m$ (`delayed`) and $t_m \le 1$ (`smallField`) at every step (lines 107–108) Source. The latter is supplied by the condition on $\bar\beta$, `Supply.Bar.threshold` (`YM/Price/Intent.lean` line 127): if $x \ge \bar\beta$ then $x^{-1/2}e^{(\log\log x)^2} \le 1$ Source. Since $\hat\beta_m \ge \bar\beta$, the threshold is at most 1 at every step Proved here.

The same calendar also defines the cover scale $B_m = \lceil(\log\hat\beta_m)^2\rceil$ (`coverSize`), the budget for regular activities $\hat g_m^{11/5}$ (`allowance`), and the budget for forced activities $e^{-P_m^{1/5}}$ (`forcedAllowance`), all as functions of the same $\hat\beta_m$ (lines 82–85) Source.

02

The size of the threshold

$P(x) = e^{(\log\log x)^2}$ grows faster than any power of $\log x$ and slower than any positive power of x Proved here , because $\log P = (\log\log x)^2$ grows faster than $\log(\log x)^k = k \log\log x$ and slower than $\log x^\epsilon = \epsilon \log x$. `cover_pay` and `inactive_price` in the source's `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean` state forms of this property ($b^n / P^a \to 0$ and so on) Source .

Hence, as $\hat\beta \to \infty$,

$$t = \hat g\, P \longrightarrow 0, \qquad \frac{t}{\hat g} = P \longrightarrow \infty, \qquad \hat\beta\, t^2 = P^2 \gg (\log\hat\beta)^k \ \text{(for all } k\text{)} \quad (2)$$

Proved here . At weak-coupling steps, the typical fluctuation size of a plaquette is of order $\hat g$ Inferred . The threshold is larger than that by the factor P, so a plaquette beyond the threshold is "a plaquette far from typical". On the other hand, since $t \to 0$, the small-field expansion of the action can be used below the threshold Inferred . The action weight of one plaquette beyond the threshold is of order $\beta t^2 \approx P^2$, and the form is chosen so that this outweighs the loss from counting covers (of the order of a power of $\log\hat\beta$) Inferred .

$\hat\beta$	P	$t = \hat\beta^{-1/2} P$	$\hat\beta t^2 = P^2$	$B = \lceil(\log\hat\beta)^2\rceil$	$e^{-P^{1/5}}$
10^5	392	1.24	1.5e5	133	3.7e-2
10^6	987	0.987	9.7e5	191	1.9e-2
10^8	4.86e3	0.486	2.4e7	340	4.2e-3
10^{12}	6.08e4	0.0608	3.7e9	764	1.2e-4
10^{24}	9.79e6	9.8e-6	9.6e13	3054	1.4e-11

The table gives values of (1) evaluated with a calculator (2–3 significant figures) Proved here . $t \le 1$ requires $\hat\beta$ of roughly 10^6 or more. This fits the calendar's design of taking $\bar\beta$ large (10-05). No concrete value of $\bar\beta$ is in the source; it is chosen in existence form Source .

03

Flags and healthy configurations

A **flag** is raised at each point touching a plaquette beyond the threshold t. The source's `flags` (`YM/RG/Masks.lean`, lines 34–36) is

$$\mathrm{flags}_t(U) = \{x : \exists p\ (\text{plaquette}),\ x \in \partial p \text{ and } \mathrm{chord}(U_p) > t\} \quad (3)$$

where ∂p is the four vertices of the plaquette Source . Plaquette values only undergo conjugation under gauge transformations of the configuration, so flags are gauge invariant (`gauge_flags`, line 43) Source . A set of points X is **healthy** if there are no flags in X (`Healthy`, line 41) Source .

Flags are used in three places in the representation Source .

- Regular activities E_i and cubic, quartic and one-loop atoms enter only when their support (for atoms, their footprint) is healthy; otherwise they are set to 0 (`masked`, lines 70–72; `Atom.placed`, `Activities.regularOccurrence`).
- The gas of forced activities is a sum over collections that exactly cover all the flags ((5) in 9-06). With no flags the gas is 1.
- The transported-curvature term $\frac{1}{2}\|\ell_p\|^2$ enters only when $\operatorname{chord} U_q \le t\, e^{\operatorname{dist}(p,q)/\text{relaxation length}}$ for every plaquette q (kept) (`PathActions.lean`, lines 210–211). The threshold is looser for more distant plaquettes; it is a condition that looks at the whole configuration.

The size of a regular activity is measured by the supremum, over healthy configurations, of finite-order derivatives in directions of size t (`dirBound`: the size in the direction of link e is at most $t(1+\operatorname{dist}(e,o))^{\text{power}}$) (`Masks.lean`, lines 141–166) Source . The threshold t enters both the decision about flags and the yardstick for the size of activities.

04

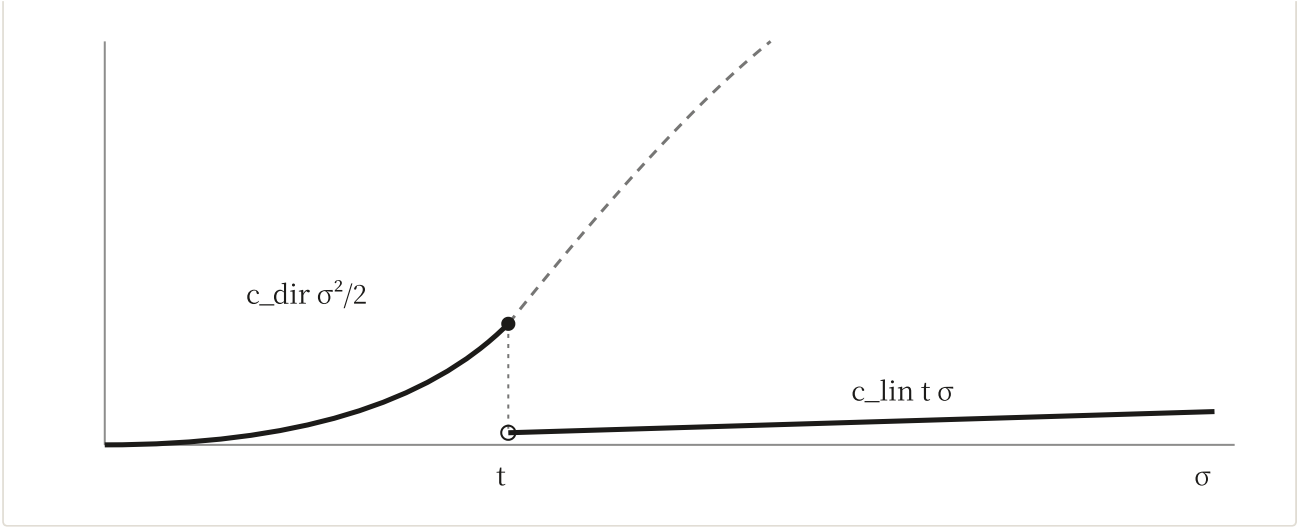
The truncated action

The plaquette term of the main action, as a function of the chord $\sigma = \operatorname{chord} U_p$, is truncated to linear above the threshold t (`clipped`, `PathActions.lean` lines 218–219) Source .

$$\operatorname{clip}_t(\sigma) = \begin{cases} \dfrac{c_{\text{dir}}}{2}\,\sigma^2 & \sigma \le t, \\ c_{\text{lin}}\, t\, \sigma & \sigma > t, \end{cases} \qquad 0 < c_{\text{lin}} < \frac{c_{\text{dir}}}{16}. \tag{4}$$

$c_{\text{dir}}, c_{\text{lin}}$ and the relaxation length are the step constants `Clip` (lines 139–146), and the kink condition `kink` is $c_{\text{lin}} < c_{\text{dir}}/16$ Source . c_{dir} also enters the normalisation of the kinetic term ((4) in 9-06).

Figure 1 The form of the truncated plaquette term (4) (example with $c_{\text{lin}} = c_{\text{dir}}/20$). At $\sigma = t$ the value jumps down, and beyond it is a straight line of slope $c_{\text{lin}}t$. The dashed curve is the untruncated quadratic term.



(4) is not continuous at $\sigma = t$. It jumps down from the left value $c_{\mathrm{dir}} t^2/2$ to the right value $c_{\mathrm{lin}} t^2 < c_{\mathrm{dir}} t^2/16$ Proved here . The action is neither monotone nor convex in σ Proved here . The term of a plaquette beyond the threshold is bounded below by $c_{\mathrm{lin}} t\,\sigma \geq c_{\mathrm{lin}} t^2$, and multiplied by β it is of order $\beta c_{\mathrm{lin}} t^2 \approx c_{\mathrm{lin}} P^2$ (under the band $\beta \in [\hat{\beta}/2, 2\hat{\beta}]$) Inferred .

We write what the truncated action means, following the source's definitions Source .

- Only **the main part of the representation** is truncated. The one-step identity $\rho_m = \mathcal{T}_{\beta_{m+1}} \rho_{m+1}$ remains an equality, and the measure does not change. The truncated difference goes to the side of the regular and forced activities.
- The large-field estimate is written using the truncated action, in the form of an inequality uniform over configurations satisfying the gauge slice and region conditions (`Teller.Grants` in 11-04, with a funding assumption).

The linear tail is a form that keeps the plaquette term from growing quadratically above the threshold. If a quadratic action were used in the large-field region, negative terms in the decomposition of the action within one step (the balance of fine plaquettes, penalty and coarse plaquettes) could grow quadratically, and this can be read as holding them down to linear Inferred . We have not read where the number 16 in the kink comes from.

05

The threshold depends on the step

Since $\hat{\beta}_m$ increases with m (larger at finer steps), t_m becomes smaller at finer steps Proved here . The threshold is a threshold on chords measured in lattice units, not a quantity in physical units Inferred . At each step, the threshold, the cover scale, the budgets and the truncation parameters (`Conv.clip` is the step constant `(params m).clip`) are determined afresh (`Conv` in 9-06) Source .

Within one step (from step $m+1$ to step m), **two thresholds** appear at once Source .

Threshold	Value	Where it is used
Fine side	t_{m+1} (smaller)	Flags of the incoming representation. In the large-field estimate, deciding seeds (marks of large field) and the left side of the inequality on the fine-plaquette side (`fiery` in the source)
Coarse side	t_m (larger)	Flags of the outgoing representation. The right side of the inequality on the coarse-plaquette side, and the region margin $5\lceil(\log \hat{\beta}_m)^2\rceil$

Which threshold is used in which inequality was checked statement by statement against the source in this book's study (cited in 11-03 and 11-04; the threshold symbols chapter by chapter in Part 11 are in the table correspondence of threshold symbols in 11-01). The two thresholds are linked in the form that wherever a large-field activity coming from the previous step sits always becomes a seed of this step (`flagged` in `Allocation`) Source .

Because the threshold depends on the step, the budgets are redefined at each step as functions of that step's coupling. The one-step certificate has the form of showing only that "the output fits into the fixed budget of the next step" (11-07) Inferred . Conditions uniform in the number of steps are paid for on the calendar's side (the order in which sides and constants are chosen) (10-05) Source .

06

What this book's study had seen

In this book's study, the threshold separating "large plaquettes" was a fixed value independent of the step Proved here .

- The theorem on the rarity of large plaquettes (22-01) counts plaquettes with $\operatorname{Re} U_p < 0$. It can be written in terms of the Wilson measure alone, and the threshold is fixed.
- The pull-back threshold $r_0 \in (0,1]$ of the block map (19-04) is also a fixed parameter.
- On the side of the list of premises, there was a row placing a step-dependent threshold, but its form had not been settled (Table 3 of 28-05 § 08).

The threshold (1) of the published construction is determined explicitly as a function of the step coupling, in a form that leaves a margin P, absorbing polylogarithmic losses, on both sides between the fluctuation size $\hat{g}$ and 1 Inferred . The budget exponent $11/5$ (between the quadratic 2 and the first non-marginal degree 3) and $e^{-P^{1/5}}$ are also functions of the same $\hat{\beta}_m$. The design of resetting thresholds and budgets at each step as functions of the coupling is something this book's study did not have (14-05). Its relation to where this book's study got stuck in the large-field estimate is treated in 14-04.

07

What we have not read

- The relation between the 16 of the kink condition and the axial-gauge factor 16 (9-03). It is the same number, but we have not checked whether they are connected.
- The value of the relaxation length (`Clip.relaxation`), and the role of the condition `slow_selector` ($12 \le$ relaxation length $\times$ width, `History.lean` line 59).
- In which estimate the exponent 2 in $(\log\log)^2$ and the $1/5$ of the budget are first needed. We read as far as the statement of `dictionary_fit` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean` line 78), which states an upper bound of the form $e^{-AP^{\delta}}$ for sufficiently large coupling, for any exponent $0 \le \delta < 2$.
- An explicit form of the lower bound on $\bar{\beta}$ (it is chosen in existence form, and the source has no number).

What this chapter can say

Can say	Cannot say
Source The threshold at step m is $t_m = \hat{\beta}_m^{-1/2} e^{(\log\log\hat{\beta}_m)^2}$, and by the calendar conditions $t_m \le 1$ at every step	An explicit lower bound on $\bar{\beta}$
Proved here $t_m \to 0$, $t_m/\hat{g}_m = P_m \to \infty$, and $\hat{\beta} t^2 = P^2$ exceeds any power of $\log\hat{\beta}$	In which estimates the exponents 2 and $1/5$ are needed
Source Flags are points touching plaquettes beyond the threshold (gauge invariant). Regular activities and atoms enter only on healthy supports	—
Source The plaquette term is truncated to linear above the threshold, with $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$. It jumps down at $\sigma = t$ and is neither monotone nor convex	Where the 16 comes from
Source Only the main part of the representation is truncated; the one-step identity remains an equality, and the difference goes into the activities	The reading that truncation makes the difficulties of large fields disappear (this cannot be written)

Sources and reproduction

Item	Kind	Source / tools
Step quantities and threshold, calendar conditions	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean` lines 79–86 (`hatBeta`, `hatCoupling`, `monitor`, `coverSize`, `threshold`, `allowance`, `forcedAllowance`), lines 107–108 (`delayed`, `smallField`), lines 141–156 `Levels.budget`
Threshold condition on $\bar{\beta}$	Source	Same repository, `YM/Price/Intent.lean` lines 123–127 `Supply.Bar`
Growth of P	Source	Same repository, `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean` (`cover_pay`, `inactive_price`; statements only)
Flags, healthy, masking	Source	Same repository, `YM/RG/Masks.lean` lines 34–72 (`flags`, `Healthy`, `gauge_flags`, `masked`), lines 141–166
Truncated action and kept	Source	Same repository, `YM/RG/PathActions.lean` lines 139–146 `Clip`, lines 210–219 `kept`, `clipped`
Numbers in Table 1	Proved here	Values of (1) evaluated in floating point
This book's fixed thresholds	Lean	The source tables of 22-01 and 19-04

Terms used in this chapter — small-field threshold, flag, healthy, truncated action, kink condition, cover scale, budget, polylogarithmic margin, fine-side and coarse-side thresholds. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-01 What "the coupling at step m" is — we read that β, defined as a coefficient in the representation together with its two normalisations, is a number not determined by the measure alone.