Volume 2 Existence — reading the published construction

Part 10 Matching the coupling, and asymptotic freedom

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

> This PDF is a print of the chapters of the web edition. Interactive figures are frozen in the state in which they were printed. Links between chapters lead to the web edition, which holds the latest text.

Chapter		Page
10-01	What "the coupling at step m" is — a coefficient in the representation	2
10-02	The one-step map and the one-loop computation — the shell integral and the coefficient $11C_g/(24\pi^2)$	10
10-03	Rereading — reading the coupling from the log-ratio of the density at two configurations	17
10-04	The shooting method — choosing the bare coupling by nested intervals and the intermediate value theorem	25
10-05	Why the sides grow, and the calendar — dividing the one-step error by log b to remove it	34
10-06	The running theorem and the short-distance effective coupling — inside and outside the dependencies	42

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom What "the coupling at step m" is · Kiichi, Shiori and Rin

What "the coupling at step m" is — a coefficient in the representation

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the definition of the number written $\beta(N, p, m)$ in the published construction: that it is the field `beta` of the representation `Rep`, a coefficient taken after two normalisations (kinetic term, one-loop part) have stopped F^2 from being shifted around; the bare coupling $\beta_{N,p} = \beta(N, p, N)$, the target value $\bar{\beta}$ at step 0, and the decrease per step; and that it is not an invariant readable from the measure alone.

Prerequisite chapters — 9-06 The density representation, 8-01 The lattice, the Wilson measure, conventions. The physics of asymptotic freedom is in 4-06.

Said first — this is a chapter of definitions. That "the coupling at step m" depends on the representation is stated explicitly by the source itself; it is not this book's inference. That the representation is not unique is a separate matter from whether there is another definition of the coupling determined by the measure, and on the latter the source says nothing.

The order of this chapter

1. **The statement — definition of the coupling at step m**: `History.β` and `Rep.beta`
2. **Two normalisations**: the coefficient of the kinetic term, and the leading Fourier coefficients of the one-loop part
3. **The direction of the steps, and the target value at step 0**: the bare coupling $\beta_{N,p}$, $\bar{\beta}$, the decrease per step
4. **Not a number determined by the measure alone**: three places in the source's docstrings
5. **Converting conventions**: $\beta = 1/g_0^2$, C_g, and the relation to this book's β

6. **What this book's study had seen**: matching quantities definable from the measure alone
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — definition of the coupling at step m

For each cutoff N (lattice spacing $a_N = 1/D_N$) and volume index p (physical side L_p), the published construction has a representation of the density at steps $m = N, N-1, \ldots, 0$, `rep N p m` (`YM/RG/History.lean`, lines 204–206). "The coupling at step m" is the field `beta` of that representation Source.

$$\beta(N,p,m) \;:=\; (\texttt{rep}\ N\ p\ m).\texttt{beta} \qquad (\texttt{History.}\beta,\ \text{line 211}).\ (1)$$

In the form of the representation ((1) in 9-06), this number multiplies the main action S_β^{pr}, and is the overall coefficient of the truncated plaquette terms, the squares of transported curvature, and the cubic and quartic atoms. It sits in a different field from the one-loop atoms, which are not multiplied by β, the regular activities, and the gas of forced activities Source.

The value at the finest step is the **bare coupling**. `Family.beta N p` is defined as $\beta(N,p,N)$ (`YM/Source/Construction.lean` line 38), and becomes the field `WilsonFamily.beta` of the family in the public theorem Source. The assessment's glossary describes this field as "the coupling multiplying the Wilson energy; for each N, p, the construction chooses the bare coupling so that the coupling after N renormalisations equals a fixed reference value $\bar{\beta}$" (Appendix A, page 23) Assessment.

02

Two normalisations

To call one coefficient in the representation "the coupling", one must make it impossible to shift F^2 terms between it and the other fields. For example, by adding a term proportional to $\sum_p F_p^2$ to the one-loop atoms and decreasing β, the same density could be written with a different β Inferred . The source stops this with two fields of the path-data condition `PathData.Bounds` (`YM/RG/PathActions.lean`, lines 161–169 and 181–185) Source .

- **Normalisation of the kinetic term** (`metric_zero`): the matrix of quadratic coefficients at zero momentum is the identity matrix ((4) in 9-06). This corresponds to what the assessment calls "the normalisation of the kinetic term" Inferred . With it, the quadratic part of the main action for smooth small configurations is $\frac{\beta}{2}\sum F^2$ Inferred .
- **Normalisation of the one-loop part** (`loop_jets`): the zeroth- and first-order Fourier coefficients (the sum and the first moment) of the coefficients of the β-independent one-loop atoms are 0. No marginal F^2 term remains in the one-loop part Inferred .

On the side of the regular activities too there are the conditions of vanishing at the flat configuration and vanishing log-Hessian in affine directions (`RegularFamily`, 9-06 § 04) Source . Together the three have the effect that all marginal F^2 coefficients are collected into β Inferred . The assessment writes in § 7.4: "integrating out fluctuations shorter than length r changes the coefficient of the square-of-curvature term of the effective action; the inverse of its square root in a fixed normalisation is the running coupling" Assessment .

03

The direction of the steps, and the target value at step 0

The structure of the sequence of steps `History.Selected` (`History.lean`, lines 234–240) imposes the following on the sequence of couplings Source .

Field	Content
`starts`	The representation at step N (the finest) is the bare Wilson weight $e^{-\beta N,p S_{\mathrm{W}}}$, with $v = 0$
`steps`	The one-step identity and $\lvert\beta(N,p,m+1) - \beta(N,p,m) - s\log b_m\rvert \le \mathrm{drift}_{m+1}$ (`Transfer`, lines 223–227)
`represents`	$\beta(N,p,m) \in [\hat\beta_m/2,\ 2\hat\beta_m]$ (`Conv.band`)
`reaches`	$\beta(N,p,0) = \bar\beta$ for all N,p

Field	Content
`running`	$\lvert\beta(N,p,m)-\hat{\beta}_m\rvert \le \varepsilon\log(1/a_m)$ (for m large enough, uniformly in N,p; `Running`, lines 229–230)

Here $s = 11C_g/(24\pi^2)$ (`slopeConstant` in the source, `YM/Gauge/Colors.lean` line 184) and $\hat{\beta}_m = \bar{\beta} + s\log D_m$ Source . Each time one step coarser is taken, the coupling **decreases** by about $s\log b_m$ (the coupling constant $g^2 = 1/\beta$ increases). The finer the step, the larger β; this is the direction of asymptotic freedom Inferred .

At step 0 (lattice spacing 1) the coupling is matched to the same value $\bar{\beta}$ for all N,p. The bare coupling $\beta_{N,p}$ is a number chosen to satisfy this condition (how it is chosen is in 10-04) Source . $\bar{\beta}$ is the calendar field `Calendar.bar` (line 67) and is taken large (table in 9-07: about 10^6 or more for the threshold to be at most 1). At every step the coupling stays in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$, and renormalisation does not pass through the strong-coupling region Inferred .

`History.Selected` is the field `wilson` of the construction theorem's conclusion `Family.Makes` (`Construction.lean` line 45) Source . The public theorem's statement `WilsonRegularity.running` follows from `running` (10-06).

04

Not a number determined by the measure alone

The source's docstrings state in three places that this coupling is not an invariant of the measure Source .

1. `YM/RG/History.lean`, lines 25–33: the definition of the inverse coupling here is the actual scalar in the base part of the representation, under conditions such as the normalisation of the flat coefficients and the kinetic metric, the one-loop kernel with its leading Fourier coefficients removed, and the vanishing of affine remainders. **It is not a number tied independently to the measure.** The formulation is of the form that such a representation and continuation **exist**, and uniqueness is not required. One should not infer claims about the usual beta function beyond `Running`.
2. The docstring of `YM/UVNext/COUPLING_RESET/Reset.lean`: nothing is said about the existence of an intrinsic readout of β on a measure for which paths and activities have not been fixed.
3. The docstring of `YM/Pert/Asymptotics/RawAxis.lean`: it extracts the leading logarithm of the same running inverse-action coefficient, and does not say, beyond the extracted comparison, that this equals a coupling in a physical position-space scheme.

Since the representation is not unique (9-06 § 05), the same density ρ_m can have a different β in a different representation Inferred . So "the coupling at step m" is a number inside the sequence of steps `History` chosen by the construction. The matching condition $\beta(N, p, 0) = \bar{\beta}$ is also a condition matching a coefficient in that representation, not a condition fixing a property of the measure at step 0 Inferred .

Couplings visible from the side of the measure are defined separately, in two ways (10-06) Source .

- $g_W(r)$: starting from the sequence of bare couplings, renormalise down to the step just below length r, take the limit, and take the $-1/2$ power of the coefficient of the representation. This too is a coefficient of a representation.
- A second coupling built from continuum correlations: normalise the derivative of the scalar two-point function and the integrated tensor–scalar–scalar correlation to their free-field values, and measure with the derivative of the tensor–scalar correlation (the assessment § 7.4) Assessment . This is written only in terms of limiting correlations. The assessment writes that the construction proves that the ratio of this coupling to g_W tends to 1 Assessment . In Lean this is the field `calibrationX` of the structure `FieldComparison` (`YM/TraceExtension/Endpoint.lean` line 49); which targets' conclusions it remains in is summarised in 10-06 § 05.

05

Converting conventions

In the assessment's convention, the Wilson measure is $\exp\bigl(-\beta \sum_p (1 - \operatorname{Re}\operatorname{tr} U_p / d_G)\bigr)$, the inner product on the Lie algebra is $\langle X, Y\rangle = -\operatorname{Re}\operatorname{tr}(XY)/d_G$, and the coupling is read as $\beta = 1/g_0^2$ (§ 3, § 4) Assessment . C_g is the Casimir constant of the adjoint representation, which the source fixes by $-\operatorname{tr}(\operatorname{ad} x\ \operatorname{ad} y) = C_g \langle x, y\rangle$ (`YM/Gauge/Casimir.lean` line 140, `Colors.lean` lines 180–182) Source .

For SU(2) ($d_G = 2$), taking the basis $i\sigma_a$, orthonormal for this inner product, gives $C_g = 8$, and $s = 11 \cdot 8/(24\pi^2) = 11/(3\pi^2) \approx 0.3715$ Proved here (numerical check done in floating point). Since the weight $1 - \operatorname{Re}\operatorname{tr} U_p/2$ is the same as this book's $1 - \operatorname{Re} U_p$ (19-01), the β of the published construction and the β of this book are the same number, as coefficients of the same weight Proved here . The value $11/(3\pi^2)$ that Volume 3 of this book uses as the one-step slope agrees with this s (23-01) Proved here . This book's β is $4/g_{\text{std}}^2$ and the published construction's β is $1/g^2$; since $g_{\text{std}}^2 = 4g^2$, they are the same number (8-01 § 04) Proved here .

The value of C_g depends on which $U(dG)$ the group is placed in (the representation). Because the convention divides the inner product by dG, $C_g = 2N^2$ for the defining representation of $SU(N)$ (8 for $SU(2)$, 18 for $SU(3)$), and placing it by another representation gives another value Proved here . For the full conversion ($g^2_{\mathrm{std}} = 2dG\, g^2$; for $SU(2)$, β is the same number in all three conventions), the conversion table in 8-01 § 04 is the reference. `submissions/openai-20261028/checks/ImportedAFGauge.lean` in the repository on the assessment side (`yang-mills-group/YM`) writes b_0 in another convention, but we have read only its docstring.

06

What this book's study had seen

The "matching" in this book's study took the form of matching **a quantity definable from the measure alone** to a fixed value (one of the statements to be shown, 19-07, matching).

- The expectation of the coarse plaquette average of the push-forward measure: the expectation of the average of Re over coarse plaquettes under the measure pushed forward to step m_0 by the deterministic block map. In the side-3 centred design, it is closed in Lean that for each k there is a non-negative coupling making this value equal to a target $\kappa_0 \in [0, 1)$ Lean : `Block3.tuned3_exists` (the assumptions are only the design values and arithmetic; monotonicity, divergence and uniqueness of the coupling are not claimed). For the current design too, continuity of the expectation and existence of a solution between its values are closed (`continuous_Yc`, `exists_tuning`).
- The vortex ratio θ_M: a ratio of twisted partition functions; for every $\kappa_0 \in [0, 1)$ there is a solution of $\theta(\beta) = \kappa_0$ (`tuning_all`, 21-07) Lean .

Both are formalisations of arguments of a known type (intermediate values of continuous functions). We tabulate the differences.

	Published construction	This book's study
Quantity matched	The F^2 coefficient of the representation at step 0	Expectation of the coarse plaquette average; the vortex ratio
Writable from the measure alone	No (a representation is needed first)	Yes
Relation to the F^2 coefficient	It is the definition itself	Needs to be shown separately (not shown)
Range of the coupling	Within the weak-coupling band at all steps	All of $\beta \geq 0$ (including strong coupling)

What they share is the idea of fixing the scale by matching a quantity determined by the unit length to a fixed value Inferred . The published construction's rereading tool (log-ratios at two configurations and strong monotonicity, 10-03) could serve as a model for linking "the value of an observable" and "the F^2 coefficient" step by step Inferred . The comparison is summarised in 14-03.

07

What we have not read

- The source gives no concrete example of non-uniqueness of the representation (two representations of the same density), and this book has not built one either. The shifting example in § 02 is an inference.
- How the limiting correlations change when $\bar{\beta}$ is changed (whether they map into each other by a change of length). The shooting toward nearby targets in the source's `YM/Extension/UV/` lies outside the dependencies (outside), and we have not read what it means.
- The content of `submissions/openai-20261028/checks/CommonScaleAudit.lean` (`calibrated_reset`, line 67) on the assessment side (only the form of the statement).

What this chapter can say

Can say	Cannot say
Source The coupling at step m is the field `beta` of the representation (the coefficient of the main action), and the bare coupling is its value at step N	A reading as a coupling determined by the measure alone (denied by the source)
Source The two normalisations, of the kinetic term and of the one-loop part, stop F^2 from being shifted	That the representation is unique (not claimed)
Source At step 0, $\beta = \bar{\beta}$ for all N, p; at all steps it lies in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$; it decreases by about $s \log b_m$ per step (fields of the construction theorem's conclusion)	How things depend on $\bar{\beta}$ when it is changed
Proved here For SU(2), $C_g = 8$ and $s = 11/(3\pi^2)$; the coefficient of the same weight as this book's β	Comparison of values in a different inner-product convention (on the assessment side we read only the docstring)
Lean This book's matching quantities (expectation of the coarse plaquette average, the vortex ratio) can be written from the measure alone, and existence of solutions is closed in Lean	Their relation to the F^2 coefficient

Sources and reproduction

Item	Kind	Source / tools
Step coupling, sequence of steps, docstrings	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean` lines 25–33, 67, 79, 204–240
Normalisations	Source	Same repository, `YM/RG/PathActions.lean` lines 161–185
Bare coupling, fields of the construction theorem	Source	Same repository, `YM/Source/Construction.lean` lines 38 and 44–45
Slope and Casimir	Source, Proved here	Same repository, `YM/Gauge/Colors.lean` lines 180–184, `YM/Gauge/Casimir.lean` line 140. SU(2) values by hand computation and floating-point check
Docstrings (rereading, leading logarithm)	Source	Same repository, head of `YM/UVNext/COUPLING_RESET/Reset.lean`, head of `YM/Pert/Asymptotics/RawAxis.lean`
The assessment's conventions and description of the coupling	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 3, § 4, § 7.4, Appendix A
This book's matching quantities	Lean	This book's Lean bundle (`Block3.tuned3_exists`, `continuous_Yc`, `exists_tuning`, `tuning_all`), a bundle that passed the check by the Lean kernel. 19-07, 21-07

Terms used in this chapter — step coupling, bare coupling, representation, normalisation of the kinetic term, normalisation of the one-loop part, matching, reference value $\bar{\beta}$, band, running coupling, Casimir constant. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-02 The one-step map and the one-loop computation — we read where the main part $s \log b$ of how much the coupling decreases in one step is computed as a shell integral.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom The one-step map and the one-loop computation · Kiichi, Shiori and Rin

The one-step map and the one-loop computation — the shell integral and the coefficient 11C_g/(24π²)

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — the condition $|\alpha + s \log b_m| + 1 \le \mathrm{drift}_{m+1}$ on the one-step change of the coupling $\alpha(N, m)$; the computation (with formulas) of the shell integral $(2\pi)^{-4} \int_{\mu < |k| \le 1}$ that produces its main part; the estimate bounding the difference between the continuum shell integral and the lattice Brillouin-zone integral by a constant independent of L; and an entry point to why the two-loop coefficient is not needed.

Prerequisite chapters — 10-01 The coupling at step m, 4-06 Asymptotic freedom. Four-dimensional polar coordinates and averages over the sphere are in 1-11.

Said first — we read the statement of the shell integral and the statement of the difference from the lattice, but have not traced the proof linking them to the one-step change α (the link between `alphaPay` and `Kin.alpha`). The whole of the "one-loop" part of the one-step map (the Gaussian integral around the stationary point) is read in 11-09.

The order of this chapter

1. **The statement — the condition on the one-step change**: `Kin.Planned.kinetic` and the one-step map of the coupling
2. **The one-loop effective action and second derivatives**: `FiniteBound`
3. **The shell integral**: the main part of the integrand and the angular average, `poleSpin`
4. **The difference from the lattice**: `absoluteGeode`, splitting into inner ball, shell and outside; the side-independent constant `HorizCalc`
5. **Two loops are not used**: the one-step error stays a constant
6. **What this book's study had seen**: the one-step coefficient for side 2, $11 \ln 2/(3\pi^2)$

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — the condition on the one-step change

In one step (from step $m+1$ to step m), the coupling changes in the following form Source .

$$\beta(N,p,m) = \beta(N,p,m+1) + \alpha(N,m) + d, \qquad |d| \le 1. \tag{1}$$

$\alpha(N,m)$ is the **one-loop change**, `Kin.alpha` in the source (`YM/PCalc/Loop.lean` line 75). It does not take the volume index p as an argument Source . d is the correction made when the representation is reread, read in 10-03. $|d| \le 1$ is the field `δ` of the one-step structure `Directed.Move` (`YM/PCalc/Band.lean` line 55: $|u.\beta - v.\beta - \alpha| \le 1$), supplied by `CouplingReset.advance` from $|d| \le$ width and width ≤ 1 (`ResetPrices.unit`, `YM/UVNext/COUPLING_RESET/Prices.lean` line 43) Source .

The one-loop change is subject to the condition `Kin.Planned.kinetic` (same file, line 110) Source .

$$\left|\alpha(N,m) + s\log b_m\right| + 1 \;\le\; \mathrm{drift}_{m+1}. \tag{2}$$

$s = 11C_g/(24\pi^2)$, b_m is the side at step m, and drift_{m+1} is a step constant. Combining (1) and (2) gives $|\beta(N,p,m+1) - \beta(N,p,m) - s\log b_m| \le \mathrm{drift}_{m+1}$, which is the second formula of the one-step condition of the sequence of steps, `History.Transfer` (`YM/RG/History.lean`, lines 226–227) Proved here . The docstring describes drift as "the

per-step size of the remainder from the universal one-loop main part, not an identification of a particular one-loop computation" (lines 220–222) Source .

That α does not depend on the volume is one of the reasons the running estimate is uniform in p (10-06) Inferred .

02

The one-loop effective action and second derivatives

The meaning of the one-loop change can be read from the condition `FiniteBound` (`Loop.lean`, lines 62–71) Source . Fix the coarse field u and do the Gaussian integral over the fine field around the stationary point (the stationary point of the fine action plus the penalty term, 9-04 § 05); this gives in the effective action of the coarse field

$$\Gamma^{1\mathrm{loop}}(u) = \text{one-loop atoms}(u) + \tfrac{1}{2}\log\det \mathrm{Hess}(u) - \log(\text{Jacobian of the coordinates}) \quad (3)$$

(`thermal`, lines 46–49). The field `wave_price` of `FiniteBound` requires that the second derivative of $\Gamma^{1\mathrm{loop}}$ taken in two plane-wave directions agree with the second derivative of $\alpha \times (\text{quadratic part of the kinetic term}) + (\text{one-loop atoms})$ (`draft`, line 51) to within (number of points) $\times$ constant $\times\, e^{-\gamma\cdot(\mathrm{side})}$ Source . That is, α is the change in the coefficient of the kinetic term of the one-loop effective action Inferred .

Since the second derivative is taken with plane waves, α can be written as an integral over momentum space. In units of the fine lattice, taking the second-order coefficient in the external momentum, it becomes an integral over the Brillouin zone with an infrared cutoff $\mu = 1/L$ that cuts off wavelengths longer than the coarse scale Inferred . The source calls this coefficient `alphaPay` (`YM/Next/STATIONARY/Films/LeadingRay.lean` line 699) Source .

03

The shell integral

The main part of the integrand, homogeneous of degree -4 in the momentum $k \in \mathbb{R}^4$, is (`poleVec`, `YM/Next/STATIONARY/OneLoop/Bubbles.lean` line 125) Source

$$\Pi(k) = 4C_g\Big(\frac{k_1^2}{|k|^6} - \frac{4k_1^2k_0^2}{|k|^8} - \frac{1}{|k|^4}\Big). \quad (4)$$

k_0 is the direction of the external momentum (the direction in which the second derivative is taken), and k_1 is the component corresponding to the colour/polarisation direction of the field Inferred . `poleSpin` (`YM/Next/STATIONARY/OneLoop/MomentumIntegral.lean` line 326) says that for $0 < \mu \le 1$

$$\frac{1}{(2\pi)^4}\int_{\mu<|k|\le 1}\Pi(k)\,d^4k=-\,s\,\log\frac{1}{\mu},\qquad s=\frac{11C_g}{24\pi^2}\quad(5)$$

Source (the factor $(2\pi)^{-4}$ is `cartWeight`, line 28).

We check this by hand Proved here . Writing $\langle\cdot\rangle$ for the average over the unit sphere in four dimensions, symmetry gives

$$\langle\hat k_1^2\rangle=\frac14,\qquad \langle\hat k_1^2\hat k_0^2\rangle=\frac{1}{d(d+2)}=\frac{1}{24}\quad(d=4).\quad(6)$$

So the average of Π over the sphere $|k|=r$ is $\frac{4C_g}{r^4}\left(\frac14-\frac{4}{24}-1\right)=-\frac{11C_g}{3r^4}$. The volume element of a four-dimensional spherical shell is $2\pi^2r^3\,dr$, so

$$\int_{\mu<|k|\le1}\Pi\,d^4k=-\frac{11C_g}{3}\cdot2\pi^2\int_\mu^1\frac{dr}{r}=-\frac{22\pi^2C_g}{3}\log\frac{1}{\mu},\qquad \frac{1}{16\pi^4}\cdot\left(-\frac{22\pi^2C_g}{3}\right)=-\frac{11C_g}{24\pi^2}=-s.$$

This book also checked (6) and (5) by averaging over 400,000 uniform points on the unit sphere ($\langle\hat k_1^2\rangle\approx0.2503$, $\langle\hat k_1^2\hat k_0^2\rangle\approx0.0417$; at $\mu=0.05$ the difference in the integral is of order 10^{-4}; $C_g=8$ for SU(2)) Proved here .

Setting $\mu=1/L$, the shell integral is $-s\log L$. In one step with side L, the main part decreases the coupling by $s\log L$. This is the $-s\log b_m$ in (2) Inferred . Of the three terms, $-1/|k|^4$ is the largest (coefficient -1), and the first two ($+1/4$ and $-1/6$) are corrections Proved here . That the integral of $1/|k|^4$ in four dimensions is logarithmic is the computational form of the coupling being dimensionless and running logarithmically Inferred .

04

The difference from the lattice

The actual α is an integral over the lattice Brillouin zone, and its integrand is not the main part Π itself. `absoluteGeode` (`MomentumIntegral.lean` line 346) splits the zone into three and bounds the difference Source .

- Inner ball $|k|\le\mu$: by the size assumption on the integrand g, `Waves1`: $|g(k)|\le A/(|k|+\mu)^4$ (`Bubbles.lean` line 276), the product with the volume $\sim\mu^4$ is $O(A)$.
- Shell $\mu<|k|\le1$: by the assumption on the difference from the main part, `Waves2`: $|g(k)-\Pi(k)|\le D/|k|^3+D\mu^2/|k|^6$ (line 277), $\int_\mu^1(r^{-3}+\mu^2r^{-6})r^3dr=O(1)$.
- Outside (the rest of the Brillouin zone): $O(A)$ by `Waves1`.

The conclusion is $|\alpha_{\mathrm{pay}} + s \log \mu^{-1}| \le (2\pi)^{-4}\, c(A, D)$, whose right side does not depend on μ Source . `HorizCalc` (`YM/Next/STATIONARY/Level/Transition.lean` line 20) is a statement restating this in terms of the side: there are $L_{\min}$ and $D \ge 0$ such that $|\alpha_{\mathrm{pay}} + s \log L| \le D$ for every odd L at least $L_{\min}$ Source (the statement further contains "for all" over sequences of sides, path data, floor conditions and so on, and "there exists" for the quantities estimated; we have omitted these here). And `pouring` (`YM/Next/STATIONARY/FullPlanned.lean` line 17) says that if the gauge group is simple, the bundle of estimates of this form, `Fluent G`, holds Source .

What matters is that the difference constant D **does not depend on the side** L Source . The one-step error is $O(1)$, small compared with $\log L$.

05

Two loops are not used

In standard perturbation theory the running of the coupling is $\mu\, dg/d\mu = -b_0 g^3 - b_1 g^5 - \cdots$ (4-06), and the $O(1)$ part of the one-step change contains two-loop and higher contributions and finite lattice effects Cited . The published construction does not compute this $O(1)$ Source . (2) only places α within the constant drift of the main part $-s \log b$; it does not estimate the content of drift.

If the side were fixed, the error over n steps would be $n \cdot O(1)$, of the same size as the main part $s\, n \log b$, and the slope s would not be determined. The published construction lets the side grow with the step, making $\mathrm{drift}/\log b_m \to 0$, so that the sum of errors is $o(\cdot)$ of the main part (10-05) Source . It is a design that determines the one-loop slope without using the two-loop coefficient. In exchange, the sequence of spacings becomes a special one (the assessment § 4.3: there is no universality with respect to the type of the sequence of spacings) Assessment .

The running theorem too states only the main part. The two-loop coefficient, $\log \log$ corrections, and boundedness of $\beta - (\bar\beta + s \log D)$ are not stated (10-06) Source .

06

What this book's study had seen

Part 15 and Volume 3 of this book took the main part of the decrease of the coupling in one step with side 2 to be $\frac{11}{3\pi^2} \ln 2$ (SU(2)) (15-01, 23-01). Numerically, $11 \ln 2/(3\pi^2) \approx 0.2575$.

With conventions aligned, the slope s of the published construction is $11/(3\pi^2)$ for SU(2) (10-01 § 05; the conversion is in 8-01 § 04), and the amount for one step with side 2, $s \ln 2$, agrees with this book's value Proved here . The premise is that this book's β is the coefficient of $\sum_p (1 - \operatorname{Re} U_p)$ (19-01). The agreement is a hand computation of this book's study, not a Lean statement.

- The published construction uses only odd sides of at least 3, so side 2 is not treated Source .
- Only the coefficient of $\log b$ agrees; the $O(1)$ constant of one step cannot be compared Inferred .
- The ultraviolet attempts of this book's study (Part 15) fixed the side at 2 and looked for a form paying the net bracket of one step (main part $\pm$ what one step misses) uniformly over steps. They did not take the route of growing the side and removing the $O(1)$ by dividing it by $\log b$ (14-05).

07

What we have not read

- The proof linking `alphaPay` and `Kin.alpha` (beyond `YM/Next/STATIONARY/FullPlanned.lean`, from `under_contract` on).
- The exact definition of `alphaPay` (the content of `leapTip` and `profitable`), and the meaning of k_0, k_1 in (4) (the reading in § 03 is an inference).
- The supply of the constants in the assumptions `Waves1` and `Waves2` (`astronomy`, `saunter`, `vintageWilson`).
- Whether these declarations lie on the existence route of the public theorem. According to the record, `poleSpin`, `absoluteGeode`, `HorizCalc`, `pouring` and `Kin.Planned.kinetic` are used (witness `SharpGram.habitat`); we have not traced the chain of the existence route.

What this chapter can say

Can say	Cannot say
Source The one-step change is $\beta_m = \beta_{m+1} + \alpha + d$, $\lvert d\rvert \le 1$, $\lvert\alpha + s \log b_m\rvert + 1 \le \mathrm{drift}_{m+1}$, and α does not depend on the volume	The content of drift (by design, it is not estimated)
Source $(2\pi)^{-4} \int_{\mu<\lvert k\rvert\le 1} \Pi = -s \log(1/\mu)$ (`poleSpin`)	The two-loop coefficient

Can say	Cannot say
Proved here The angular averages $1/4$, $1/24$ and the shell volume $2\pi^2 r^3\,dr$ give the coefficient $11C_g/(24\pi^2)$ (also checked numerically)	—
Source The difference between the lattice integral and the main part is bounded by a constant independent of the side (`absoluteGeode`, `pouring`)	The link between `alphaPay` and `Kin.alpha` (not traced)
Proved here For SU(2), the amount for one step with side 2, $s \ln 2 = 11 \ln 2/(3\pi^2)$, agrees with this book's value (hand computation with aligned conventions)	A comparison of the $O(1)$ constants of one step

Sources and reproduction

Item	Kind	Source / tools
Condition on the one-step change, effective action	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/PCalc/Loop.lean` lines 46–51 (`thermal`, `draft`), lines 62–71 `FiniteBound`, line 75 `Kin.alpha`, line 110 `kinetic`
Main part and shell integral	Source	Same repository, `YM/Next/STATIONARY/OneLoop/Bubbles.lean` lines 125, 276, 277, `MomentumIntegral.lean` lines 28, 174, 326, 346
Side-independent constant	Source	Same repository, `YM/Next/STATIONARY/Level/Transition.lean` line 20 `HorizCalc`, `FullPlanned.lean` line 17 `pouring`, `Films/LeadingRay.lean` line 699 `alphaPay`
One-step condition of the sequence of steps	Source	Same repository, `YM/RG/History.lean` lines 220–227
(6)(7) and the numerical check	Proved here	Hand computation in the text; averages over 400,000 uniform points on the sphere (record of floating-point computation)
Absence of universality	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4.3
This book's one-step value for side 2	Proved here	15-01, 23-01

Terms used in this chapter — one-loop change, drift, one-loop effective action, shell integral, angular average, Brillouin zone, infrared cutoff, two loops, slope $2b_0$. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-03 Rereading — we read the procedure that, after one step, rereads the β of the representation from the log-ratio of the density at two configurations.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom Rereading · Kiichi, Shiori and Rin

Rereading — reading the coupling from the log-ratio of the density at two configurations

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

What this chapter shows — how the correction d in the one-step change $\beta_m = \beta_{m+1} + \alpha + d$ ((1) in 10-02) is determined: a single equation matching the log-ratio of the values of the push-forward density H at the flat configuration $\mathbf{1}$ and at a test configuration V to the formula of the representation; and, from the fact that its left side increases in d with slope at least $K(V)/2$ (strong monotonicity), that the root is unique, depends continuously on the input, and satisfies $|d| \le 1$.

Prerequisite chapters — 10-01 The coupling at step m, 10-02 The one-step map and the one-loop computation, 9-06 The density representation.

Said first — rereading determines only two scalars (β and the vacuum term v). That the representation equals the density **at every configuration** comes from a separate remainder estimate (`Remainder`). The content of that estimate (the expansion of the small-field step) is within the scope of 11-09; in this chapter we read only the form of the statement. We have not read the concrete form of the test configuration V.

The order of this chapter

1. **The statement — rereading two scalars**: `Template`, `residual`, `correct`, `advance`
2. **Two configurations**: v at the flat configuration, β at the test configuration
3. **Strong monotonicity and a unique root**: `Response` and `solve`
4. **Bound on the shift and continuity**: `posterior`, `balance`, $|d| \le 1$
5. **From equality at two points to equality everywhere**: `ShapeControl` and `realizes`
6. **What this book's study had seen**: a tool linking the values of observables and the F^2 coefficient

7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

The statement — rereading two scalars

After one step, the density at step m is the function pushed forward by the kernel, $H = \mathcal{T}_{\beta_{m+1}}\rho_{m+1}$ (9-01). When rewriting it in the representation at step m (9-06), the source first fixes the rest of the representation (path data, truncation parameters, budget point, family of activities), and rereads the two scalars β and v last (`YM/UVNext/COUPLING_RESET/Reset.lean`, `Advance.lean`) Source .

The exponential part of the representation is written split into a form linear in β (`Template`, `Reset.lean` lines 43–47 and 60–62) Source .

$$-\log\frac{\rho(U)}{e^{v|\Lambda|}} = \beta\,K(U) + B(U) + A_d(U) \qquad (\text{flag-free } U). \tag{1}$$

K is the part of the main action multiplied by β (`leg`; `kinetic` in the source), B is the main action at $\beta = 0$, that is, the one-loop atoms (`background`), and A_d is the sum of regular activities (`add`). The activities are allowed to depend on the correction d (`terms : ℝ → ℝ → Activities`). With no flags the gas is 1 (9-06).

The rereading equation is `residual` in the source (lines 64–65) Source :

$$\Phi_x(d) = \big(\beta_x^{\mathrm{pred}} + d\big)K(V) + B(V) + A_{x,d}(V) - \big[\log H_x(\mathbf{1}) - \log H_x(V)\big] = 0. \tag{2}$$

x is the parameter of the curve (the shooting bare coupling, 10-04), $\beta_x^{\mathrm{pred}} = \beta_{m+1}(x) + \alpha(N,m)$ is the one-loop prediction (`predictor`, `Advance.lean` line 51), and the bracket

is the log-ratio of the density at the two configurations (`contrast`, `Reset.lean` line 50).

The theorem `CouplingReset.correct` (`Advance.lean` line 157) says that, under its assumptions (the condition on the test configuration `Usable`, the funding of the input `Income`, and that the input is a curve at step $m+1$), the root $d(x)$ of (2) exists, is continuous in x, and the reread representation equals H_x at the two points $\mathbf{1}$ and V Source . `advance` (line 175) says further that, under the remainder assumption `Remainder`, the reread representation satisfies the one-step identity (equality at every configuration), and $|d(x)| \le \text{width} \le 1$ Source .

02

Two configurations

Rereading uses only two configurations Source .

- **The flat configuration $\mathbf{1}$** (every link the identity): every plaquette value is 1, the main action and the one-loop atoms are 0, the regular activities are 0 by the condition $E_i(\mathbf{1}) = 0$, there are no flags, and the gas is 1. The value of the representation is just $e^{v|\Lambda|}$, so $v = \log H(\mathbf{1})/|\Lambda|$ can be read off (`scalar`, `scalar_eq`, lines 51 and 85).
- **The test configuration V**: a configuration with no flags in which the β part of the main action is large compared with the number of points. The condition `Usable` (`Advance.lean`, lines 55–58) requires $\text{flags}_{t_m}(V) = \varnothing$ and $K(V) \ge |\Lambda|\,\eta\,t_m^2$ (η a step constant, t_m the threshold of 9-07). That such a V exists is supplied by `usable` (line 60).

The two equalities matching the values of the representation at the two configurations to H have, for the two unknowns v, β, a separated form: v is determined at the flat configuration, and β by the log-ratio (2) (`contrast_eq`, line 88) Source . Since a log-ratio is used, multiplying the density by a positive constant only moves v and leaves β unchanged (docstring, `rescale_*`) Source .

Since only flag-free small-field configurations are read, β is read as "the coefficient of the kinetic term of the effective action at small field" Inferred . The docstring says that agreement at other configurations requires, besides the separation, the price of a remainder estimate (head of `Reset.lean`) Source .

03

Strong monotonicity and a unique root

The source's `Response` (`Root.lean`, lines 27–31) is a structure bundling the following four properties of a two-variable function $\Phi(x, d)$ Source .

$$\mu > 0, \qquad a \le b \;\Rightarrow\; \Phi(x,b) - \Phi(x,a) \ge \mu\,(b-a), \qquad d \mapsto \Phi(x,d) \text{ continuous}, \qquad x \mapsto \Phi(x$$

The second is **strong monotonicity** (slope at least μ). `correct` builds a `Response` for (2) with $\mu = K(V)/2$ Source . The slope of (2) in d is the sum of $K(V)$ and the dependence of the activities on d, $\partial_d A_{x,d}(V)$. The source's `response` (`Reset.lean` line 131) is a sufficient condition of the form: if the change in the activities is at least $-(K(V) - \mu)(b - a)$, the slope is at least μ; its price is paid as a field of `Income` Source .

By strong monotonicity there is at most one root. Existence comes from the intermediate value theorem: `solve` (`Root.lean` line 60) says that if at each x there is a bracket with $\Phi(x, \mathrm{lo}) \le 0 \le \Phi(x, \mathrm{hi})$, then a root $d(x) \in [\mathrm{lo}, \mathrm{hi}]$ exists and is continuous in x Source .

Note that strong monotonicity is a property in the **correction** d. The docstring (above `response`, lines 127–130) says that since strong monotonicity includes the contributions of paths already fixed, **monotonicity in the incoming bare coupling is not required** Source . On the shooting side (10-04) too, monotonicity of the map from the bare coupling to the coupling at step 0 is not used.

04

Bound on the shift and continuity

Strong monotonicity bounds the shift of the root by the value of the equation Source (`Response.inverse`, `posterior`, `Root.lean` lines 42 and 49).

$$|b - a| \le \frac{|\Phi(x,b) - \Phi(x,a)|}{\mu}, \qquad |d(x) - a| \le \frac{|\Phi(x,a)|}{\mu}. \quad (4)$$

Setting $a = 0$, the size of the correction is bounded by $|\Phi(x, 0)|$, how far the equation is off at the one-loop prediction itself Proved here . `balance` (line 72) says that if $|\Phi(x,0)| \le \mu r$, then $\pm r$ is a bracket and the root lies in $[-r, r]$ Source . `correct` uses this with the width r (`width`; `ResetPrices.radius` in the source's price table), giving the conclusion of `advance`, $|d| \le$ width, and further width ≤ 1 (`q.unit`) Source .

From the first part of (4), the root depends continuously on the input (`Response.continuous`, `Root.lean` line 54) Source . If $x \mapsto \Phi(x, a)$ is continuous, so is $x \mapsto d(x)$. This is one of the components that make the couplings at each step a

continuous curve (`Curve`) when the bare coupling is varied in the shooting method (10-04) Inferred .

Tool	Property used	What comes out
Intermediate value theorem (`solve`)	Bracket and continuity	Existence of the root
Strong monotonicity	Slope $\geq \mu$	Uniqueness of the root
`posterior`	Strong monotonicity	Bound on the correction $\lvert\Phi(x, 0)\rvert/\mu$
`Response.continuous`	Strong monotonicity + continuity in x	Continuity of the root

05

From equality at two points to equality everywhere

Matching at two configurations alone does not show that the representation equals the density at every configuration. The source fills this gap with a separate assumption Source .

`ShapeControl` (`Reset.lean`, lines 115–117) is the assumption that, for each x and each configuration U, the difference between the shape part of the representation (without $e^{v|\Lambda|}$) and $H_x(U)/H_x(\mathbf{1})$ is bounded by a constant times the value $|\Phi_x(a)|$ of (2). `realizes` (line 119) concludes that, since the right side is 0 at the root $d(x)$, the representation **equals H_x at every configuration** Source .

That is, the content of the one-step identity (an equality with no approximation, 9-01 § 04) is divided as: (i) showing separately, by the expansion of the small-field step, that "the shape of the representation agrees except for the scalars" (the field `shape` of `Remainder`), and (ii) matching the two remaining scalars by rereading Inferred . The expansion of the small-field step consists of three identities that carry both the error of the Gaussian approximation and the non-Gaussian remainder inside the integrand, and yields `Move.ident` (11-09) Source .

Which targets it belongs to: according to the record, `correct`, `advance`, `residual` and `solve` are used (witness `SharpGram.habitat`), and the direct user of `advance` is `AnalyticBridge.VerifiedReset.Return.reset` Source . This book's study traced the source and confirmed through the chain that `reset` lies on the existence route of the public theorem (from `rodB`) Source .

06

What this book's study had seen

The matching in this book's study (10-01 § 06) viewed the expectation of the coarse plaquette average of the push-forward measure, or the vortex ratio, as a continuous function of the bare coupling β, and matched it to a target by the intermediate value theorem. What is closed in Lean goes only as far as existence of a solution; monotonicity and uniqueness in β are not claimed (`Block3.tuned3_exists`, `tuning_all`) Lean .

The published construction's rereading gets uniqueness in a different place.

- What is varied is not the bare coupling but **the correction d inside the representation**. With the other parts of the representation fixed, the log-ratio equation increases in d with slope at least $K(V)/2$, and the root is unique.
- No monotonicity in the bare coupling is required, and the shooting (10-04) uses only existence.

As a tool for carrying, step by step, the relation "value of an observable $\leftrightarrow$ F^2 coefficient" that was missing in this book's study (the table in 10-01 § 06), rereading by a log-ratio at two configurations and strong monotonicity could serve as a model Inferred . However, the published construction's log-ratio is a quantity inside the representation, a different thing from this book's matching quantities (expectations writable from the measure alone). There is no statement linking the two, either on this book's side or on the published construction's side Inferred . The comparison is summarised in 14-03.

07

What we have not read

- The concrete form of the test configuration V (`YM/UVNext/COUPLING_RESET/Probe.lean`; we read only the docstring), and the content of the proof of `usable`.
- How the funding of the input `Income` and the remainder assumption `Remainder` are supplied on the existence route (on the `AnalyticBridge/VERIFIED_RESET_STEP` side).
- The location and form of the statement of the uniqueness `scalars_unique` mentioned in the docstring (head of `Advance.lean`).
- The value of the width `ResetPrices.radius`, and where among the step constants it comes from.

What this chapter can say

Can say	Cannot say
Source v is reread at the flat configuration, and β from the log-ratio of the density at the flat configuration and a flag-free test configuration (equation (2))	The concrete form of the test configuration
Source Under the assumptions, (2) increases in d with slope at least $K(V)/2$, and the root is unique and continuous in the input (`correct`, `solve`)	Monotonicity in the bare coupling (not required)
Source Under the remainder assumption, the reread representation equals the density at every configuration, and $\lvert d\rvert \le 1$ (`advance`, `realizes`)	Supply of the remainder assumption (within the scope of 11-09)
Source That rereading lies on the existence route was confirmed in this book's study by tracing the chain in the source	—
Inferred Rereading could serve as a model for a tool linking the values of observables and the F^2 coefficient step by step	A statement linking this book's matching quantities and the coefficients of the representation (on neither side)

Sources and reproduction

Item	Kind	Source / tools
Shape, equation, equality at two points, equality everywhere	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/UVNext/COUPLING_RESET/Reset.lean`: `Template` line 43, `contrast` line 50, `scalar` line 51, `leg`, `background`, `add` lines 60–62, `residual` line 64, `scalar_eq` line 85, `contrast_eq` line 88, `passes` line 99, `ShapeControl` line 115, `realizes` line 119, `response` line 131
Strong monotonicity and roots	Source	Same repository, `Root.lean`: `Response` line 27, `inverse` line 42, `posterior` line 49, `continuous` line 54, `solve` line 60, `balance` line 72, `balance_solve` line 80
One-step rereading	Source	Same repository, `Advance.lean`: `predictor` line 51, `Usable` line 55, `usable` line 60, `Income` line 78, `Remainder` line 97, `correct` line 157, `advance` line 175
Dependency record	Source	Same repository, `verification/declaration-usage.tsv`
Existence of this book's matching	Lean	This book's Lean bundle (`Block3.tuned3_exists`, `tuning_all`). 21-07

Terms used in this chapter — rereading, log-ratio, flat configuration, test configuration, strong monotonicity, root, bracket, bound on the shift, remainder estimate, one-step identity. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-04 The shooting method — we read how the bare coupling that makes the coupling at step 0 equal to $\bar{\beta}$ is chosen, by nested intervals of the per-step

bands and the intermediate value theorem.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom The shooting method · Kiichi, Shiori and Rin

The shooting method — choosing the bare coupling by nested intervals and the intermediate value theorem

> This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study, and § 08 is a map of what is unresolved.

What this chapter shows — the procedure that, for each cutoff N and volume p, views the couplings at each step as a continuous curve $x \mapsto \beta_m(x)$ parametrised by the bare coupling x at step N, builds nested intervals from the per-step bands, and at step 0 uses the intermediate value theorem to take x with $\beta_0(x) = \bar{\beta}$; that it uses only the continuity of the curves and the one-step error inequality $\mathrm{drift} \le (s/4)\log b$; and where uniqueness is stated and where it is not.

Prerequisite chapters — 10-01 The coupling at step m, 10-03 Rereading. The intermediate value theorem is within the prerequisites of Part 1.

Said first — the source has two kinds of shooting band. The one on the existence route of the public theorem is the band in `YM/FullVolume/Arc.lean`; the band in `YM/Shoot/` (which collapses to a point at step 0) belongs to another headline theorem. This chapter mainly describes the former, and describes the latter in § 05 to distinguish them. Shooting gives existence only; uniqueness of the bare coupling is not stated on the shooting route.

The order of this chapter

1. **The statement — a bare coupling making the coupling at step 0 equal to $\bar{\beta}$ exists**: `Arc.bought`, `covers`, `rodB`
2. **Bands, and the inequality for crossing a band in one step**: `low`, `high`, `crosses`
3. **Nested intervals**: `narrow` (the intermediate value theorem twice), and induction from step N to step 0
4. **Taking a point at step 0**: `nearby` and `covers`, choice

5. **Not confusing the two shooting bands**: how it differs from the band in `Shoot/`
6. **On uniqueness**: not stated on the shooting route; a separate field of uniqueness within a fixed recursion
7. **What this book's study had seen**: matching by global endpoints
8. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — a bare coupling making the coupling at step 0 equal to $\bar{\beta}$ exists

The field `reaches` of the structure of the sequence of steps `History.Selected` requires that the coupling at step 0 be $\bar{\beta}$ for all N, p (10-01). The shooting method is what chooses the bare coupling $\beta_{N,p}$ for this. On the existence route of the public theorem, it is done by the following three Source .

1. `FullVolume.Arc.bought` (`YM/FullVolume/Arc.lean` line 95): under its assumptions (the recursion materials `Issues` , a simple group, the one-step error condition `Drifting`), there is an "arc" `Arc` for each $m \le N$ Source .
2. `Arc.covers` (line 71): for the arc at step 0, the interval $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ is contained in the image of "bare coupling $x \mapsto$ coupling at step 0" Source .
3. `FullVolume.rodB` (`YM/FullVolume/Stage/Rods.lean` line 26): using `bought` and `covers` , and the fact that $\bar{\beta}$ lies in the interior of the interval (`centers` , `Arc.lean`

line 25), it **chooses** (`choose`) for each N, p one bare coupling that makes the coupling at step 0 equal to $\bar{\beta}$, and assembles the sequence of steps `Selected` Source .

`rodB` is one of the declarations that this book's study confirmed, by tracing the chain, to lie on the existence route of the public theorem, and according to the record it is the direct user of `Arc.bought` and `covers` Source . `Issues` does not remain an assumption: on the existence route it is supplied by `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean` line 76) Source .

02

Bands, and the inequality for crossing a band in one step

The band at step m is set as follows (`low`, `high`, `Arc.lean` lines 21–22) Source .

$$\mathrm{low}_m = \tfrac{3}{4}\bar{\beta} + \tfrac{5}{8}\cdot s\log D_m, \qquad \mathrm{high}_m = \tfrac{5}{4}\bar{\beta} + \tfrac{3}{2}\cdot s\log D_m. \quad (1)$$

At step 0, $D_0 = 1$, so the band is $[\tfrac{3}{4}\bar{\beta}, \tfrac{5}{4}\bar{\beta}]$ and does not collapse to a point. At every step the band lies within the band of the representation $[\hat{\beta}_m/2, 2\hat{\beta}_m]$ ($\hat{\beta}_m = \bar{\beta} + s\log D_m$) (`fits`, line 31) Source . Comparing coefficients: $\tfrac{3}{4} \ge \tfrac{1}{2}$, $\tfrac{5}{8} \ge \tfrac{1}{2}$, $\tfrac{5}{4} \le 2$, $\tfrac{3}{2} \le 2$ Proved here .

The one-step error condition `Drifting` (`YM/Shoot/Reach.lean`, lines 70–71) is $\mathrm{drift}_{i+1}/\log b_i \le s/4$ Source . Combined with the one-step relation $|\beta_{i+1} - \beta_i - s\log b_i| \le \mathrm{drift}_{i+1}$ (10-02), it gives the following (`crosses`, `Arc.lean` line 75) Source .

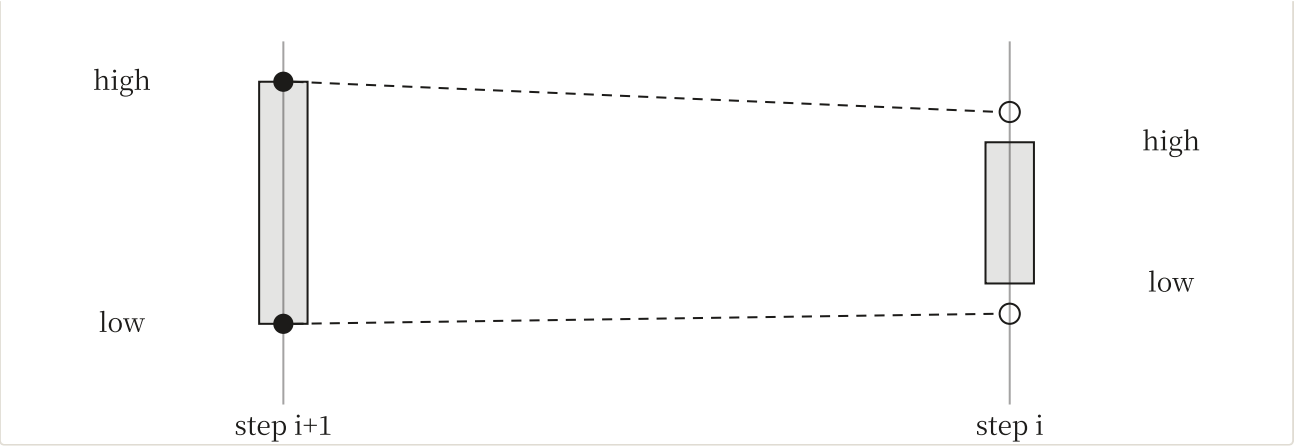
$$\beta_{i+1} = \mathrm{low}_{i+1} \;\Rightarrow\; \beta_i < \mathrm{low}_i, \qquad \beta_{i+1} = \mathrm{high}_{i+1} \;\Rightarrow\; \beta_i > \mathrm{high}_i. \quad (2)$$

We check it Proved here . Writing $\ell = \log b_i$, $\mathrm{low}_{i+1} = \mathrm{low}_i + \tfrac{5}{8}s\ell$ and $\mathrm{high}_{i+1} = \mathrm{high}_i + \tfrac{3}{2}s\ell$. In one step β decreases by $s\ell \pm \tfrac{1}{4}s\ell$, so

$$\beta_i \le \mathrm{low}_i + \tfrac{5}{8}s\ell - s\ell + \tfrac{1}{4}s\ell = \mathrm{low}_i - \tfrac{1}{8}s\ell, \qquad \beta_i \ge \mathrm{high}_i + \tfrac{3}{2}s\ell - s\ell - \tfrac{1}{4}s\ell = \mathrm{high}_i + \tfrac{1}{4}s\ell. \quad (3)$$

The floor of the band drops by only $\tfrac{5}{8}s\ell$ in one step, while the coupling drops by at least $\tfrac{3}{4}s\ell$. The ceiling of the band drops by $\tfrac{3}{2}s\ell$, while the coupling drops by at most $\tfrac{5}{4}s\ell$. So a trajectory at the edge of the band at a fine step leaves the band at the step one coarser.

Figure 1 The band and trajectories in one step (step $i+1 \to$ step i). Trajectories at the floor and ceiling of the left step (●) leave the band at the right step (○). By the intermediate value theorem, the values of the trajectories in between cover the band.



The figure redraws the vertical positions step by step so that the shape of the band is visible; the scale is not exact.

03

Nested intervals

`Arc` (`Arc.lean`, lines 58–67) is, for step m, the following tuple Source : an interval $[l, u]$, and for each $x \in [l, u]$ a trajectory from step N to step 0, $x \mapsto$ (representation at each step). The conditions are: the coupling at step N is x (x is the bare coupling); at each step from m to N, $x \mapsto \beta_i(x)$ is a continuous curve (`Curve`); at each step it is in the band and the one-step relation holds; and at step m, $\beta_m(l) = \mathrm{low}_m$ and $\beta_m(u) = \mathrm{high}_m$.

`bought` builds arcs by induction downward from step N Source .

1. **Step N**: using the continuous curve of `Issues.initial` (`YM/Shoot/Criteria.lean`, lines 100–101) (the representation parametrised by the bare coupling), take the interval $[\mathrm{low}_N, \mathrm{high}_N]$. The coupling at step N is x itself.
2. **From step $k+1$ to step k**: `Issues.advances` (lines 102–106) builds a continuous curve at step k from the continuous curve at step $k+1$ (one renormalisation step and rereading, 10-03). By (2), the trajectories at the ends of the interval leave the band at step k. `narrow` (`Reach.lean` line 18) then uses the intermediate value theorem twice to choose a subinterval $[l', u'] \subset [l, u]$ with $\beta_k(l') = \mathrm{low}_k$, $\beta_k(u') = \mathrm{high}_k$, and β_k taking values in the band $[\mathrm{low}_k, \mathrm{high}_k]$ on $[l', u']$.

`narrow` chooses by taking the infimum w of the set of points where $\beta_k = \mathrm{high}_k$, and then the supremum z of the set of points in $[l, w]$ where $\beta_k = \mathrm{low}_k$ (`sInf`, `sSup`) Source . Then in the interior of $[z, w]$, β_k stays between the two end values Proved here .

The intervals narrow in a nested way as one goes down the steps, and only trajectories inside the band at every step remain. What is used is **only the continuity of the**

curves and the one-step error inequality Source . No estimate on derivatives of the one-step map is used. The docstring (head of `Arc.lean`) describes it as "shooting before the one-step derivative estimates" Source .

04

Taking a point at step 0

For the arc at step 0, $\beta_0(l) = \mathrm{low}_0 = \frac{3}{4}\bar{\beta}$ and $\beta_0(u) = \mathrm{high}_0 = \frac{5}{4}\bar{\beta}$. Applying the intermediate value theorem to the continuous curve $x \mapsto \beta_0(x)$, every value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ is in the image (`covers`) Source . Since $\bar{\beta} > 0$, $\bar{\beta}$ is in the interior of this interval (`centers`) Source . So there is a bare coupling x with $\beta_0(x) = \bar{\beta}$, and `rodB` takes one by the axiom of choice Source .

From the trajectory chosen in this way, all the fields of the sequence of steps `Selected` (`starts`, `steps`, `represents`, `reaches`, `running`) are assembled (`Rods.lean`, lines 39–46). `running` comes from the sum of the one-step errors (`Shoot.trajectory`, 10-05) Source .

Since the band is not collapsed at step 0 and a width around $\bar{\beta}$ is kept, shooting can be done with the same calendar even if the target is replaced by another value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ Inferred . The docstring says "whether one can shoot at a slightly moved target is not a question of prescribing another calendar" (head of `Arc.lean`) Source . However, the arc states only a **forward curve**; it does not say that the inverse map (value at step 0 → bare coupling) is a function (same place) Source .

05

Not confusing the two shooting bands

The source contains another shooting, `YM/Shoot/Reach.lean` Source . Its band has floor $\bar{\beta} + \frac{1}{2} \cdot s \log D_m$ and ceiling $\bar{\beta} + 2 \cdot s \log D_m$ (`floorBeta`, `ceilBeta`), and collapses to the single point $\bar{\beta}$ at step 0. `gates` (line 81) plays the same role as (2), and `steps` (line 112) and `successful` (line 156) take the left end of the interval. `Recipe.races` (`YM/Shoot/Race.lean` line 67) assembles `Selected` from this. The docstring says "it is not claimed that `Issues` exists for every family" (`Race.lean` line 5).

	`FullVolume/Arc.lean`	`Shoot/Reach.lean`
Floor	$\frac{3}{4}\bar{\beta} + \frac{5}{8} \cdot s \log D_m$	$\bar{\beta} + \frac{1}{2} \cdot s \log D_m$
Ceiling	$\frac{5}{4}\bar{\beta} + \frac{3}{2} \cdot s \log D_m$	$\bar{\beta} + 2 \cdot s \log D_m$

	`FullVolume/Arc.lean`	`Shoot/Reach.lean`
Step 0	$[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ (not collapsed); $\bar{\beta}$ taken via `covers`	The single point $\bar{\beta}$; left end of the interval
One-step condition	The same `Drifting` ($\text{drift} \le \frac{1}{4} \cdot s \log b$)	
Record	used; direct user `rodB` (witness: a headline on the existence side)	used; witness `parityEndpoint` (on the `Graces.winnow` side)

The existence route does not use `Shoot.successful`; it uses `Arc.bought` (according to the record, the direct user of `Arc.bought` is `rodB`) Source . The two share the interval-narrowing tool `narrow`. What we cite as the shooting of the existence route is the band of `Arc.lean`.

06

On uniqueness

The shooting route (`Arc.bought`, `covers`, `rodB`) gives only the **existence** of the bare coupling Source . `rodB` takes one by choice and says nothing about whether there are other solutions. The docstring of `Rods.lean` also says "this first purchase is only a surjection; uniqueness is something later analyticity estimates should impose" Source .

On the other hand, the source also has fields that state uniqueness. The field `unique` (line 142) of `FullVolume.Final.Jewel` (`YM/FullVolume/Finale/Program.lean` line 133) says that for each p, N and $\zeta \in [-1, 1]$ there is **exactly one** "successful" bare coupling making the coupling at step 0 equal to $\bar{\beta} + \zeta \cdot w$ (w is the width `beamWidth`). The field `tuning` (line 143) says that this bare coupling is given by an analytic function of ζ (on the unit disc) Source . They are supplied by `wins` (line 60) and `bareTrial` (line 70), with the structure `Winter` as assumption Source . The reading we confirmed in the source, free of contradiction, is as follows.

- "Successful" (`successful`, lines 57–58) means that within a **fixed recursion** (`T.L.tunes`), there is a competing trajectory from step N to step 0. The docstring (lines 17–20) states explicitly that "this is uniqueness only within this recursion, not uniqueness among all conceivable representations" Source .
- The core of uniqueness is `Winter.unique` (`YM/FullVolume/Glide/Vac.lean` line 40): two competing trajectories with equal couplings at step 0 agree at every step. This is a claim of the same type as the uniqueness in 11-10 (anchoring at step 0) Inferred .
- According to the record, `wins` is used (it is used to build `Jewel`, with a headline on the existence side as witness), while the fields `Jewel.unique` and `Jewel.tuning` themselves are outside (read from no headline theorem) Source .

In summary, the place on the existence route of the public theorem where the bare coupling is chosen uses only existence and choice, not uniqueness. Uniqueness and analyticity within a fixed recursion exist as fields of a separate structure `Jewel`, and there are lemmas `wins` and `bareTrial` (`Program.lean` lines 60 and 70, used) that fill those fields. The fields themselves (`unique`, `tuning`) are read from no headline theorem (outside) `Source` . Neither says "unique among all representations".

07

What this book's study had seen

The "existence of a matched sequence" closed in Lean in this book's study (10-01 § 06) derives existence from the **global endpoints** of a single continuous function `Lean` : the expectation of the coarse plaquette average is 0 at $\beta = 0$ and approaches 1 as $\beta \to \infty$ (`Block3.Y3_zero`, `Block3.tendsto_Y3_one`), and a value κ_0 in between can be reached by the intermediate value theorem (`Block3.tuned3_exists`). The vortex ratio has the same form (`tuning_all`).

The published construction's shooting builds nested intervals from the **local edges of the band** at each step and the one-step error inequality. There are three differences `Inferred` .

- Since this book knows the endpoint values globally, no shooting induction is needed. Instead, it passes through all $\beta \geq 0$ (including strong coupling).
- The published construction carries every step within the weak-coupling band, and obtains per-step continuity (`Issues`) from the analysis of renormalisation.
- Both give existence only, and use neither uniqueness nor monotonicity (uniqueness within the published construction's fixed recursion is in § 06).

That the crux of the shooting is a single inequality, "one-step error $\leq \frac{1}{4} \cdot s \log b$", gives an axis of comparison with where the sum of errors across steps got stuck in this book's study (Part 15) (14-03).

08

What we have not read

- The content of how `Roof.Lyrics.supplier` closes `Issues` (we read as far as the names of the materials `Graces.hops`, `Chord.spores`, `moves`).
- The assumptions of `offers` (`Glide/Vac.lean` line 65), which supplies `Winter`, and whether the recursion of `Jewel` is the same as that of `rodB`.

- The content of the proof of uniqueness at step 0, `one` (`YM/FullVolume/Gain/Resolve.lean` line 40).
- How the band coefficients $\frac{3}{4}, \frac{5}{8}, \frac{5}{4}, \frac{3}{2}$ of `Arc` were re-chosen from those of `Shoot/` (the docstring gives no reason).

What this chapter can say

Can say	Cannot say
Source Under the assumptions, for each N, p there are bare couplings making the coupling at step 0 equal to every value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ (`Arc.bought`, `covers`)	That the inverse map is a function (not claimed)
Proved here If the one-step error is $\leq \frac{1}{4} \cdot s \log b$, trajectories at the edge of the band leave the band in one step ((3))	—
Source The existence route (`rodB`) uses the band of `Arc.lean` and takes one by choice; the band of `Shoot/` belongs to another headline	—
Source There are fields of uniqueness and analyticity within a fixed recursion (`Jewel.unique`, `tuning`); their supply `wins` is used, the fields themselves are outside	Uniqueness among all representations (denied by the source)
Lean Existence of this book's matching comes from global endpoints and the intermediate value theorem (`Block3.tuned3_exists`, `tuning_all`)	Uniqueness and monotonicity on this book's side

Sources and reproduction

Item	Kind	Source / tools
Shooting on the existence route	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/FullVolume/Arc.lean`: `low`, `high` lines 21–22, `nearby` line 24, `centers` line 25, `fits` line 31, `Arc` line 58, `covers` line 71, `crosses` line 75, `bought` line 95
Choice and assembly of the sequence of steps	Source	Same repository, `YM/FullVolume/Stage/Rods.lean` line 26 `rodB` (docstring lines 5–14), `YM/FullVolume/C/Songbook.lean` line 76 `Roof.Lyrics.supplier`
Recursion materials and one-step condition	Source	Same repository, `YM/Shoot/Criteria.lean` lines 63, 88, 99 (`band`, `Relays`, `Issues`), `YM/Shoot/Reach.lean` line 18 `narrow`, line 70 `Drifting`
The other shooting	Source	Same repository, `YM/Shoot/Reach.lean` lines 81, 112, 156, `YM/Shoot/Race.lean` lines 5 and 67

Item	Kind	Source / tools
Uniqueness within a fixed recursion	Source	Same repository, `YM/FullVolume/Finale/Program.lean` docstring lines 17–20, `successful` line 57, `wins` line 60, `bareTrial` line 70, `Jewel` line 133 (`unique` line 142, `tuning` line 143); `YM/FullVolume/Glide/Vac.lean` lines 31–40 `Winter`; `YM/FullVolume/Ray/Gain.lean` line 22 `sun`; `YM/FullVolume/Gain/Resolve.lean` line 72 `ask`
Dependency record	Source	Same repository, `verification/declaration-usage.tsv` (`Arc.bought`, `covers`, `rodB`, `wins` are used; `Jewel.unique`, `tuning`, `july` are outside; `Shoot.successful` has witness `parityEndpoint`)
(3) and the figure	Proved here	Computation in the text
Existence of this book's matching	Lean	This book's Lean bundle (`Block3.Y3_zero`, `tendsto_Y3_one`, `tuned3_exists`, `tuning_all`), a bundle that passed the check by the Lean kernel

Terms used in this chapter — shooting method, band, arc, nested intervals, intermediate value theorem, choice, one-step error, uniqueness within a fixed recursion, forward curve. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-05 Why the sides grow, and the calendar — we read the design of growing the sides so that the one-step error, kept a constant, is removed by dividing by $\log b$, and the order in which constants are chosen.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom Why the sides grow, and the calendar · Kiichi, Shiori and Rin

Why the sides grow, and the calendar — dividing the one-step error by log b to remove it

> This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study, and § 08 is a map of what is unresolved.

What this chapter shows — that the sides b_m are odd numbers of at least 3, non-decreasing, and grow to infinity; the reason (keeping the one-step error drift at $O(1)$ while making $\text{drift}/\log b_m \to 0$, so that the sum of errors is $o(\log D_m)$) and the computation by which the running estimate `Running` follows; the calendar conditions and the order in which constants are chosen; and, as the price, that the sequence of spacings becomes a special one.

Prerequisite chapters — 10-02 The one-step map and the one-loop computation, 10-04 The shooting method, 8-02 The set-up.

Said first — growing the sides does not remove the need for conditions uniform in the number of steps. Conditions uniform over steps are paid for on the calendar's side (the entry coupling of each step, lower bounds on the sides, margins and so on). The whole calendar (including large-field constants) is read in 11-08. This chapter is limited to what concerns the running of the coupling.

The order of this chapter

1. **The statement — the sequence of sides and the calendar conditions**: `Schedule`, `Calendar.Fits`, `Supply.Large`
2. **Dividing the one-step error by log b to remove it**: the computation in `Shoot.trajectory`
3. **The sides grow to infinity**: `logCost` and the smoothing exponent
4. **Gentle — one step is small compared with the whole**: $\log b_m / \log D_m \to 0$
5. **The order in which constants are chosen**: `Sheet` → `Rolls` → `Protection` → `Delays` → `Bar` → `Wait` → `Paid`
6. **The price — the sequence of spacings becomes special**
7. **What this book's study had seen**: the net one-step bracket with a fixed side

8. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

The statement — the sequence of sides and the calendar conditions

How the steps are taken is `Schedule` (`YM/Lattice/Scale.lean`, lines 28–34) Source : the sides b_m are odd with $b_m \geq 3$, $D_m = \prod_{k<m} b_k$, the lattice spacing is $a_m = 1/D_m$, $\mathrm{scale}(m) = \log D_m$ (`scale`, line 115), and the physical sides L_p are multiples of 4, strictly increasing. The condition `Gentle` (lines 129–130) is $\log b_m / \log D_m \to 0$. The assessment's glossary explains that the construction takes $L_p = 4(M + p + 1)$ (M fixed large enough) (Appendix A, page 23) Assessment .

Of the calendar conditions `Calendar.Fits` (`YM/RG/History.lean`, lines 97–118), those concerning the running of the coupling are the following Source .

Field	Content
`mesh`	`Gentle`
`sameFactor`, `monotoneL`	b_m is a value determined by the smoothing exponent $J(m+1)$ alone, non-decreasing in J
`mono`, `stepSlow`, `deep`	J is non-decreasing, increases by at most 1 per step, and $J \to \infty$
`driftSmall`	$\mathrm{drift}_{m+1} / \log b_m \leq 1/J(m+1)^2$
`logCost`	$\log(1 + \text{sum of step constants}) / \log b_m \leq 1/J(m+1)^2$
`delayed`, `smallField`	$1 < \hat{\beta}_m$, threshold $t_m \leq 1$ (9-07)

The condition on the side of candidate sides, `Supply.Large` (`YM/Price/Intent.lean`, lines 39–48), imposes, for a smoothing exponent j and a side L, $\mathrm{drift}/\log L \le s/4$ (`kick`: the shooting condition `Drifting`, 10-04), $\mathrm{drift}/\log L \le 1/j^2$ (`slow`), and $\log(1+\text{step constants})/\log L \le 1/j^2$ (`fits`) Source. `Rolls.eventually_large` (line 55) says that for each j every sufficiently large odd L satisfies these, and `Rolls.exists_rolls` (line 79) says that a sequence of sides exists Source. The assessment describes this as "the block ratios are chosen to satisfy complicated but explicit inequalities using constants that appear in the renormalisation transformation" (Appendix A) Assessment.

02

Dividing the one-step error by log b to remove it

The one-step relation (10-02) is $|\beta_{m+1}-\beta_m - s\log b_m| \le \mathrm{drift}_{m+1}$, and the matching condition is $\beta_0 = \bar\beta$. Since $\hat\beta_m = \bar\beta + s\sum_{k<m}\log b_k$, the triangle inequality gives

$$\left|\beta(N,p,m) - \hat\beta_m\right| \;\le\; \sum_{k=1}^{m} \mathrm{drift}_k \qquad (1)$$

Proved here. The source's `Shoot.trajectory` (`YM/Shoot/Race.lean` line 23) shows this as the lemma `hd`, and calls the sum `βCost` Source.

If the side were fixed ($b_k = b$, with drift also a constant δ), the right side would be $m\delta$ and the main part $s\log D_m$ would be $s\,m\log b$; their ratio stays at $\delta/(s\log b)$ and does not go to 0 Proved here. The slope s would not be determined.

The published construction sets $\mathrm{drift}_{k+1} \le \log b_k / J(k+1)^2$ by `driftSmall`, and uses $J \to \infty$. The proof of `trajectory` (`Race.lean`, lines 32–52) takes, for $\varepsilon > 0$, a step j beyond which $1/J(k+1)^2 \le \varepsilon/2$, writes

$$\sum_{k\le i}\mathrm{drift}_k \;\le\; \sum_{k\le j}\mathrm{drift}_k + \frac{\varepsilon}{2}\sum_{j\le k<i}\log b_k = \mathrm{const}_j + \frac{\varepsilon}{2}\left(\log D_i - \log D_j\right) \qquad (2)$$

and, since $\log D_i \to \infty$, concludes that from the step where $\mathrm{const}_j \le \frac{\varepsilon}{2}\log D_i$ on, the sum is at most $\varepsilon \log D_i$ Source. This is `History.Running`: $|\beta(N,p,m) - \hat\beta_m| \le \varepsilon\log(1/a_m)$ (for m large enough, uniformly in N,p) Source. The sum of errors becomes $o(\cdot)$ of the main part, and the slope s is determined without ever computing the two-loop coefficient Inferred.

The crux of (2) is that the one-step error drift **may be large** ($O(1)$, and may even increase with the step). What is made small is the ratio $\mathrm{drift}/\log b$, and that is paid for by growing the sides Inferred.

03

The sides grow to infinity

The source's `Schedule` does not directly require the sides to grow (only $b_m \geq 3$ and `Gentle`). But $b_m \to \infty$ follows from the calendar conditions Inferred : by `logCost`, $\log(1 + \cdots + \text{columns} + \cdots)/\log b_m \leq 1/J^2$, and the step constant `columns` is at least 1 (`Stage.columns_ge`, `History.lean` line 60), so the numerator is at least $\log 2$, hence $\log b_m \geq J(m+1)^2 \log 2 \to \infty$ Proved here . This derivation presupposes that a calendar satisfying the calendar conditions exists (`Wait.ready`) Inferred .

The step constants (the Taylor order `order`, the power `power`, the support exponent and so on) are indexed by the smoothing exponent J and increase monotonically in J (`deepOrder` and others) Source . The field `stages` of the construction theorem's conclusion `Family.Makes` requires that, from the reference step for local fields of degree d on, the Taylor order be at least $d+1$ (`YM/Source/Construction.lean`, lines 47–49) Source . The form is: grow the step constants to handle local fields of every degree, and absorb their logarithmic loss $\log(1 + \text{step constants})$ with the logarithm of the side (`logCost`) Inferred .

04

Gentle — one step is small compared with the whole

`Gentle`, $\log b_m / \log D_m \to 0$, is the condition that even as the sides grow, the logarithm of one step is negligible compared with the logarithm of the whole Source . With it, even if a length r is read at "the step just below r", the ratio of $\log(1/r)$ to $\log D_m$ tends to 1 Inferred . The lemma `pace_log` of the theorem on the short-distance effective coupling $g_W(r)$ (10-06) uses not `Gentle` but a stronger condition in the same direction, the field `bloom` of the additional calendar condition `PostT` on the trace side: $b_i \leq \log \hat{\beta}_i$ (`YM/TraceExtension/Terrace/Base.lean` line 31) Source . With it, the logarithm of one step, $\log b_{i-1}$, is negligible compared with $\hat{\beta}_i$.

From `Gentle` also follows a lower bound on the ratio of lattice spacings, $c\, a_m^2 \leq a_{m+1}$ (`mesh_inequality`, `Scale.lean` line 133; the docstring says "it need not be assumed separately") Source . The sides grow slowly, in a way bounded by a ratio of logarithms.

Condition	Role	Chapter
$\text{drift}/\log b \leq s/4$	Nesting of the shooting bands (leaving the band in one step)	10-04
$\text{drift}/\log b \leq 1/J^2 \to 0$	Making the sum of errors $o(\log D)$ (running)	This chapter § 02

Condition	Role	Chapter
$\log(1+\text{constants})/\log b \le 1/J$ 2	Absorbing the loss from growing step constants; $b \to \infty$	This chapter § 03
$\log b_m/\log D_m \to 0$	The logarithm of one step is small compared with the whole	This chapter § 04
$b_i \le \log \hat{\beta}_i$ (`PostT.bloom`, trace side)	The ratio does not break down when the coupling is read at the step just below r	10-06 § 04

05

The order in which constants are chosen

`YM/Price/Intent.lean` defines the materials for building the calendar as a sequence of structures arranged in the order in which they are chosen Source .

1. **`Sheet`** (lines 26–36): the kernel choices (penalty coefficient and three thresholds), the table of step constants `profile`, the initial exponent, and the lower bound on the sides `floorL`. The docstring says "orders and constants are chosen **before** the block ratios" (head of file).
2. **`Rolls`** (lines 50–53): a side L_j for each smoothing exponent j (satisfying `Large`, non-decreasing in j).
3. **`Protection`** (lines 88–92): margins $\theta_j \le 1/j$ (decreasing), fixed after L is chosen. The docstring says "θ that accommodates the powers after L".
4. **`Delays`** (lines 115–120): deadlines after the margins, such as the lower bound on the entry coupling of each step, `betaEntry`.
5. **`Bar`** (lines 123–127): $\bar{\beta}$. $\bar{\beta} > 1$, at least the entry of the first step, and threshold ≤ 1 for $x \ge \bar{\beta}$.
6. **`Wait`** (from line 134): the working window (the width of the $o(\cdot)$).
7. **`Paid`** (lines 145–159): that the calendar fits these: `betaEntry` $\le \hat{\beta}_m$ at each step (`pays`), lower bounds on the volume, and so on.

`Wait.ready` (`YM/Price/Waits.lean` line 137) says that a sequence of steps `Levels` satisfying all the requirements exists Source . The docstring says that this existence result "works for any constants already chosen, so after all the assignments and ratios before L, the delays after L can be chosen arbitrarily large", and at the same time says "it does not check whether the renormalisation argument actually fits this order of dependence" (head of `Intent.lean`) Source . That the order fits is paid for as an assumption on the side of each step's argument (that the step constants do not depend on L) Inferred .

Conditions uniform over steps are paid for within this order Inferred . The constants of the one-step certificate (Part 11) may grow depending on the step constant J; in exchange, the calendar guarantees that the entry coupling $\hat{\beta}_m$ at step m is at least `betaEntry` (`pays`). One step is performed only where the coupling is large enough.

06

The price — the sequence of spacings becomes special

The sequence of sides is a special sequence chosen to fit the renormalisation constants. The assessment writes in § 4.3 that "convergence holds along the whole sequence of spacings used in the proof, but there is no universality result about the type of the sequence of spacings", and at the start of § 4 that "since the construction chooses the two sequences (spacings and volumes), the ultraviolet convergence theorem does not assert convergence for every sequence of spacings" Assessment .

The design of growing the sides is a trade: in exchange for determining the one-loop slope without a two-loop estimate, it does not say whether the limiting theory is the same for another sequence of spacings (for example, halving with side 2) Inferred . This universality relates to the questions of 16-02.

07

What this book's study had seen

The ultraviolet attempts of this book's study (Part 15) fixed the side at 2 and looked for a form paying, uniformly over steps, the decrease of the coupling in one step by the main part $s \ln 2$ (with $s = 11/(3\pi^2)$ for SU(2)) $\pm$ what one step misses (the net one-step bracket of the coupling, 15-03). With a fixed side, as in § 02, the sum of the $O(1)$ one-step errors is of the same size as the main part, and to determine the slope the one-step error itself had to be estimated in a form summable over steps (for example, a form decreasing with the step) Inferred .

The published construction takes a different route: it leaves the one-step error at $O(1)$, grows the sides, and divides by $\log b$. The price is the universality of § 06, and the conditions uniform over steps move to the calendar's side. This route is an idea this book's study did not have (14-05). Both this book's current design (side 2) and the side-3 design fix the side, and this book has no Lean for a design with growing sides.

08

What we have not read

- How the sequence of sides is concretely built (`Program.exhaust` in `YM/Price/Waits.lean`, line 41; we read only the statement), and how slowly the sides grow.
- Where in the table of step constants the value of drift comes from (`Stage.drift` is a field of the table; we have not read how it is computed).
- The roles of `scaleEntry`, `shells` and `growth` in `Delays` (on the large-field side; 11-08).
- Which declaration guarantees that the order of dependence fits the actual argument (the docstring says it is not checked).

What this chapter can say

Can say	Cannot say
Source The sides are odd numbers of at least 3, non-decreasing in J; `Gentle`; drift$/\log b \le 1/J^2$; $\log(1+\text{constants})/\log b \le 1/J^2$	The explicit form of the sequence of sides
Proved here The difference between the coupling and the one-loop running is bounded by the sum of one-step errors; with a fixed side, the sum is of the same size as the main part	—
Source Since the error ratio goes to 0, the running `Running` holds uniformly in N, p (`Shoot.trajectory`)	The two-loop coefficient; log log corrections
Proved here From the calendar conditions, $b_m \to \infty$ ($\log b_m \ge J(m+1)^2 \log 2$)	Premises other than the existence of the calendar itself (supplied by `Wait.ready`)
Source Constants are chosen in the order step constants → sides → margins → delays → $\bar\beta$	A check that this order fits the actual argument (the docstring says it is not checked)
Assessment There is no universality with respect to the type of the sequence of spacings	Agreement of the limit for another sequence of spacings

Sources and reproduction

Item	Kind	Source / tools
How the steps are taken	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/Lattice/Scale.lean` lines 28–34 `Schedule`, line 115 `scale`, line 129 `Gentle`, line 133 `mesh_inequality`

Item	Kind	Source / tools
Calendar conditions and step constants	Source	Same repository, `YM/RG/History.lean` lines 45–61 `Stage`, lines 97–118 `Calendar.Fits`
Derivation of the running	Source	Same repository, `YM/Shoot/Race.lean` lines 19–52 (`βCost`, `trajectory`)
Order of choice	Source	Same repository, `YM/Price/Intent.lean`: docstring at the head, `Sheet` line 26, `Large` line 39, `Rolls` line 50, `eventually_large` line 55, `exists_rolls` line 79, `Protection` line 88, `Delays` line 115, `Bar` line 123, `Wait` line 134, `Paid` line 145; `YM/Price/Waits.lean` line 137 `Wait.ready`
Degree of local fields and depth of steps	Source	Same repository, `YM/Source/Construction.lean` lines 47–49 `Makes.stages`
Universality, description of block ratios	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4, § 4.3, Appendix A
This book's one step with side 2	Proved here	The source table of 15-03

Terms used in this chapter — side, calendar, drift, smoothing exponent, Gentle, running, sum of errors, step constants, order of dependence, universality. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-06 The running theorem and the short-distance effective coupling — we read where the public statement `WilsonRegularity.running` follows from the running of the sequence of steps, and the theorem on the short-distance effective coupling $g_W(r)$, separating what is inside and outside the dependencies.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom The running theorem and the short-distance effective coupling · Kiichi, Shiori and Rin

The running theorem and the short-distance effective coupling — inside and outside the dependencies

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

What this chapter shows — The route from the running of the sequence of steps (`History.Running`) to `WilsonRegularity.running`, which enters the statement of the public theorem: $\beta_{N,p}/\log(1/a_N) \to s$ (uniformly in p). The headline theorem of the verification record, `Pert.Asymptotics.comparison`. The definition of the short-distance effective coupling $g_W(r)$ and its theorem, and its ratio to a second coupling built from the continuum correlations. And which target each of these enters.

Chapters assumed — 10-01, 10-05. For the physics of asymptotic freedom see 4-06; for how the limits are taken, 8-05.

Said first — The running theorem states only the one-loop leading part; it does not state the two-loop coefficient, the $\log\log$ correction, or the boundedness of $\beta - (\bar{\beta} + s \log D)$. It has not been shown that the short-distance effective coupling g_W is independent of how the limit is taken (the infinite-volume subsequence). The theorem on g_W does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target (§ 05 sets out the three places separately).

The order of this chapter

1. **Statement — the running theorem**: the public statement `WilsonRegularity.running`
2. **The route of the running**: `Running` → `zeroUp_bound` → `regular`
3. **The headline theorem comparison**: the $-1/2$ power of the coupling at step i, and $\sqrt{\log(1/a_i)}$
4. **The short-distance effective coupling g_W(r)**: the coefficient at the limiting step, and the step just below r

5. **Inside and outside the dependencies**: two Lean forms, and the statement of the public theorem
6. **What this book's study had seen**
7. **What has not been read**

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0); the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For the theorems of the published construction, the wording of the statements is given under Source, and the level "it is proved" under Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

Statement — the running theorem

The public theorem `JointExistence` says that for each simple gauge group there is a Wilson family C satisfying the condition `WilsonRegularity C` (glossary of the assessment, p. 23) Assessment. The field `running` of that condition (`ChallengeSpec.lean`, lines 13–15) is Source

$$\exists\, s_* > 0 \;\; \forall \varepsilon > 0 \;\; \exists N_0 \;\; \forall N \geq N_0 \;\; \forall p : \qquad \left| \frac{\beta_{N,p}}{\log(1/aN)} - s_* \right| < \varepsilon \quad (1)$$

The ratio of the bare coupling to the logarithm of the inverse lattice spacing approaches a positive slope s_* **uniformly in the volume index** p. The construction uses $s_* = s = 11C_g/(24\pi^2)$ (the assessment writes this slope as twice the one-loop coefficient) (glossary, p. 25) Assessment. Equation (7) of Theorem 4.1 in the assessment writes the same content as $\lim_N \sup_p |\beta_{N,p}/\log a_N^{-1} - 11C_g/(24\pi^2)| = 0$ Assessment.

This is the leading form of asymptotic freedom. Reading $\beta = 1/g_0^2$, we get $g_0^2 \log(1/aN) \to 1/s$; that is, the bare coupling constant approaches 0 at the rate $1/\sqrt{s \log(1/a)}$ Proved here. Converting the standard-convention formula of 4-06, $1/g_{\mathrm{std}}^2(\mu) = 1/g_{\mathrm{std}}^2(\mu_0) + c_{\mathrm{std}} \ln(\mu/\mu_0)$ (with $c_{\mathrm{std}} = 11N/(24\pi^2)$), into units of β gives the same slope: for

$SU(2)$, $\beta = 4/g^2_{\mathrm{std}}$ and $4c_{\mathrm{std}} = 4 \cdot 22/(24\pi^2) = 11/(3\pi^2) = s$ Proved here (for the conversion of conventions see 8-01 § 04).

The running theorem **enters the statement of the public theorem**: `Solution.lean` takes the family and `WilsonRegularity` from the construction theorem and proves `JointExistence`, and the construction theorem obtains it from `YangMills.regular` (`Construction.lean`, line 33) Source .

02

The route of the running

(1) comes out in three stages Source .

1. **Running of the sequence of steps** `History.Running` (`YM/RG/History.lean`, lines 229–230): $\forall \varepsilon > 0$, for sufficiently large m, and for all $N \geq m$ and p, $|\beta(N, p, m) - \hat{\beta}_m| \leq \varepsilon \log(1/a_m)$. For the sequence chosen by shooting, `Shoot.trajectory` derives this from the sum of the one-step errors (10-05 § 02).
2. **Ratio form** `Pert.Asymptotics.zeroUp_bound` (`YM/Pert/Asymptotics/RawAxis.lean`, line 15): for sufficiently large m and all $N \geq m$ and p, $|\beta(N, p, m)/\log(1/a_m) - s| < \varepsilon$. It uses the fact that the $\bar{\beta}$ in $\hat{\beta}_m = \bar{\beta} + s \log(1/a_m)$ becomes negligible as $\log(1/a_m) \to \infty$.
3. **Public statement** `YangMills.regular` (`HierarchyBridge.lean`, line 16): sets $m = N$ in `zeroUp_bound` to obtain (1).

Only properties of the sequence of steps are used; neither the limiting correlations nor infinite volume is used Source . The uniformity in the volume comes from three facts: the one-step transition α does not depend on the volume (10-02), the shooting aims at the same $\bar{\beta}$ for every p, and the upper bound drift on the one-step error is a constant of the step, independent of p Inferred .

What it does not say Source : the two-loop coefficient, the $\log\log(1/a)$ correction, or that $\beta_{N,p} - (\bar{\beta} + s \log D_N)$ is bounded. (1) is a limit of ratios and says the difference only up to $o(\log D_N)$. The docstring of `History.lean` also says that it does "not infer the usual beta-function claims beyond `Running`" (lines 32–33).

03

The headline theorem comparison

Among the 17 headline theorems listed by the verification record (`verification/joint-headlines.txt`) is `YM.Pert.Asymptotics.comparison` (`RawAxis.lean`, line 36) Source . For

the quantity built from the coupling at step i, $g_{\mathrm{reg}}(N,p,i)=\beta(N,p,i)^{-1/2}$ (`regulatorRun`, line 35), and for sequences of cutoffs and volumes $N(i)\ge i$ (for sufficiently large i) and arbitrary $k(i)$,

$$g_{\mathrm{reg}}\big(N(i),k(i),i\big)\,\sqrt{\log(1/a_i)}\;\longrightarrow\;\frac{1}{\sqrt{s}}\qquad(i\to\infty)\tag{2}$$

holds Source . It says that the coupling at step i, read from any cutoff and any volume, lies on the leading part of the one-loop running. The docstring says that, "beyond the extracted comparison, it does not say this equals a physical position-space scheme coupling" (at the top) Source . (2) is a headline theorem, but it is not in the statement of the public theorem `JointExistence` Source .

04

The short-distance effective coupling g_W(r)

§ 7.4 of the assessment describes the short-distance effective coupling as follows Assessment . Starting from the family of bare couplings of Theorem 4.1, renormalise down to the **step of lattice spacing just below** the length r, send the cutoff to 0, take the limit along an infinite-volume subsequence, and define $g_W(r)$ as the $-1/2$ power of the coefficient of the normalised action. Theorem 7.2: for small r, and for any chosen infinite-volume subsequential limit, $g_W(r)>0$ and

$$\log(1/r)\,g_W(r)^2\;\longrightarrow\;\frac{1}{s}\qquad(r\downarrow 0).\tag{3}$$

The assessment states that "it has not been shown that $g_W(r)$ is independent of the choice of infinite-volume subsequential limit" Assessment .

We read the form in the source Source .

- **Coefficient at the limiting step**: `Terrace.Arch` and `steep` (`YM/TraceExtension/Terrace/Dusk.lean`, lines 65–71). Along an ultrafilter $\mathcal{L}$ on (N,p), the coupling $\beta(N,p,i)$ at each step i has a limit $Z.\beta_i$. `holds` (line 79) gives $Z.\beta_i\in[\hat\beta_i/2,2\hat\beta_i]$, and `gamut` (line 89) gives $Z.\beta_i/\hat\beta_i\to 1$.
- **The step just below** r: `glen` (`YM/TraceExtension/Atlas/Sink.lean`, line 14) is the first step i at which the lattice spacing a_i falls below r.
- g_W: `Pilot.wilson` (line 44), $g_W(r)=Z.\beta_{\mathrm{glen}(r)}^{-1/2}$. `Coupling` (lines 45–47) has the form "positive for sufficiently small r as $r\downarrow 0$, and $\log(1/r)g^2\to 1/s$". `wilson_link` (line 50) proves `Coupling` for g_W Source .

The proof is short Inferred : `gamut` gives $Z.\beta/\hat{\beta} \to 1$, and `pace_log` (line 21) gives $\log(1/r)/\hat{\beta}_{\mathrm{glen}(r)} \to 1/s$. The latter comes from $\hat{\beta}_i = \bar{\beta} + s \log D_i$ and from the field `bloom` of the additional calendar condition on the trace side, `PostT` : $b_i \le \log \hat{\beta}_i$ (`YM/TraceExtension/Terrace/Base.lean`, line 31). This makes $\log D_i - \log D_{i-1} = \log b_{i-1}$ negligible compared with $\hat{\beta}_i$, so the ratio does not break when read at the step just below r Source (`H.bloom` inside the proof of `pace_log`). `PostT` is also among the hypotheses of `wilson_link` (line 50). Multiplying the two gives (3).

For a second coupling built from the continuum correlations, `Sundial.glo` (`YM/TraceExtension/Sundial/Noon.lean`, line 28), the field `calibrationX` (`YM/TraceExtension/Endpoint.lean`, line 49) states $\mathrm{glo}/gW \to 1$ Source . The assessment writes that "the running law is visible not only in the effective action but also in continuum observables" Assessment .

05

Inside and outside the dependencies

The assessment does not give the Lean name for Theorem 7.2 Assessment . Within what we read, statements close to (3) appear in two places; together with the headline theorem (2), we set out the three places separately Source (the dependency record is `verification/declaration-usage.tsv`).

Place	Content	Record	Relation to the t
(a) `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean`:145)	Ratio form: $g(\mu)(s\log\mu)^{1/2} \to 1$ as $\mu \to \infty$. Here g is the inverse square root of the value of the step limit at the first step with $a_m \le 1/\mu$ (`gLedger`, line 130)	outside	**Outside the dependencies** reached from a headline theor

Place	Content	Record	Relation to the t
(b) `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean`:50)	$\log(1/r)\, g_W(r)^2 \to 1/s$ ($r \downarrow 0$, at the first spacing step below r). Hypothesis includes `PostT` ($b_i \le \log \hat{\beta}_i$)	used (first headline: `continuum_with_position_tests`; direct user: `crest_ready`)	**An intermedia inside the dependencies.** conclusion ent `Crest.charts` `FieldComparisc` and is dropped passing to the trace tests
(c) `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean`:36)	(2): the running of the cutoff coupling at step i, $\beta(N_i, k_i, i) - 1/2$. A different quantity from g_W	headline	Itself a headlin of the verificat record. Outsid statement of th theorem

We checked in the source where the conclusion of (b) appears Source : in the field `charts` (lines 20–22) of the intermediate structure `Crest` of the short-distance analysis of the trace (`YM/TraceExtension/Belfry.lean`, line 16), and in the field `wilson` of the comparison structure `FieldComparison` (`YM/TraceExtension/Endpoint.lean`, line 50). This reads as the proof of the trace law (13-02) using the running of the coupling as an intermediate stage Inferred . In `Integration.ManuscriptTraceTests.of_result` (`YM/Integration/ManuscriptTraceTests.lean`, line 77), which passes from these structures to the public trace tests `Tests`, the fields of `Tests` (lines 62–75) contain no comparison of couplings, and the comparison fields are dropped Source .

Summary for the targets.

- The public theorem `JointExistence`: the only running that enters its statement is the running of the bare coupling in (1), `WilsonRegularity.running` Source . The public statement asks only that "some positive slope exists"; $s = 11C_g/(24\pi^2)$ is the value the construction satisfies Source .
- The construction theorem `constructed`: its conclusion fields (`Construction.lean`, lines 15–31) contain no statement about the effective coupling Source .
- The committee's combined target: it does not include a check of the coupling (13-03).
- The verification record: (a) is outside the dependencies, (b) is an intermediate stage inside them, and (c) is itself a headline theorem. **The running of g_W does not**

survive in the conclusion of any headline theorem, but the running of the bare step coupling in (c) is a headline conclusion.

The one whose wording is close to the statement of Theorem 7.2 of the assessment ($\log(1/r) g_W^2 \to 1/s$; the assessment writes the right-hand side as the reciprocal of twice the one-loop coefficient; $r \downarrow 0$, at the first spacing below r) is (b) Inferred. This book treats the short-distance effective coupling as "a result that is in Lean and that the committee states as Theorem 7.2 of the assessment, but that does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target". 13-03 has the same shape.

06

What this book's study had seen

This book's study took the form of supplying the sequence of bare couplings from outside. The matching in the statements to be shown (19-07) says, in terms of quantities of the measure, that a sequence of couplings exists; that the sequence grows with slope s against $\log(1/a)$ (the running) is neither in the statements nor in Lean. The one-step slope of this book's Volume 3 (value $11/(3\pi^2)$) is a number set as the leading part of the one-step decrease of the coupling (23-01), not a running theorem.

The published construction obtains the running as a theorem about "the sequence of coefficients in the representation", from the matching ($\bar{\beta}$ at step 0), the one-loop one-step estimate, and the calendar of growing sides. The matching quantity of this book's study can be written in terms of the measure alone, but there is no tool linking it to the coefficients of the representation, so a running theorem of the same form cannot be stated on this book's side Inferred. This difference is treated in 14-03 and in the remaining questions of Volume 3 (16-03).

One more point: the published construction carries every step inside the weak-coupling band $[\hat{\beta}/2, 2\hat{\beta}]$ and stops at step 0 (spacing 1). It does not pass through the strong-coupling region by renormalisation. The firm theorems of this book's Volume 3 are on the side of the strong-coupling window (Part 20), and do not overlap with the infrared edge of the published construction (still weak coupling at spacing 1) Inferred (16-04).

07

What has not been read

- The chain from `wilson_link` to the headline theorem (`crest_ready` → ... → `continuum_with_position_tests`), and whether `Crest` appears in the statement of

the headline theorem or only inside its proof.

- The name of the declaration the assessment takes as the location of Theorem 7.2 (no name is given in the text).
- The content of the definition of the second coupling `glo` (`lampX`), and the proof of `calibrationX`.
- Whether g_W is independent of the choice of subsequential limit (the assessment says this has not been shown).

What this chapter can say

Can say	Cannot say
Source The running of the bare coupling, $\beta_{N,p}/\log(1/a_N) \to s$ (uniformly in p), enters the statement of the public theorem	The two-loop coefficient, the log log correction, boundedness of the difference
Source Running of the sequence of steps → ratio form → public statement (`trajectory`, `zeroUp_bound`, `regular`)	—
Source The headline theorem `comparison`: the $-1/2$ power of the coupling at step $i \times \sqrt{\log(1/a_i)} \to s^{-1/2}$	Identification with a physical-scheme coupling (denied by the source)
Assessment The short-distance effective coupling: $\log(1/r)g_W(r)^2 \to 1/s$, and its ratio to the second coupling $\to 1$	Independence from the choice of subsequential limit
Source There are three places: `Running.ratio` is outside the dependencies, `wilson_link` is an intermediate stage inside them (its conclusion is dropped on passing to the public trace tests), and `comparison` is a headline theorem (a different quantity). The running of g_W does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target	The declaration where Theorem 7.2 of the assessment lives (the assessment gives no name; the closest wording is `wilson_link`, Inferred)

Sources and reproduction

Item	Kind	Source / tool
Public statement and its supply	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `ChallengeSpec.lean` lines 13–15, `HierarchyBridge.lean` line 16 `regular`, `Construction.lean` line 33, `Solution.lean` lines 4–10
Running of the sequence of steps and the ratio form	Source	Same, `YM/RG/History.lean` line 229 `Running`, `YM/Shoot/Race.lean` line 23 `trajectory`, `YM/Pert/Asymptotics/RawAxis.lean` lines 13–36 (`b0`, `zeroUp_bound`, `regulatorRun`, `comparison`)

Item	Kind	Source / tool
Short-distance effective coupling	Source	Same, `YM/TraceExtension/Terrace/Dusk.lean` lines 65–95, `YM/TraceExtension/Atlas/Sink.lean` lines 11–50, `YM/TraceExtension/Belfry.lean` lines 16, 20–22, 31, `YM/TraceExtension/Terrace/Base.lean` lines 29–34 (`PostT`), `YM/Integration/ManuscriptTraceTests.lean` lines 62–77 (`Tests`, `of_result`), `Construction.lean` lines 15–31, `YM/TraceExtension/Sundial/Noon.lean` line 28, `YM/TraceExtension/Endpoint.lean` lines 49–50, `YM/PertExtension/Running.lean` line 145
Dependency record and headlines	Source	Same, `verification/declaration-usage.tsv`, `verification/joint-headlines.txt`
Theorem 4.1, Theorem 7.2, glossary	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4.3, § 7.4, Appendix A

Words used in this chapter — running theorem, asymptotic freedom, slope $2b_0$, uniform in the volume, headline theorem, short-distance effective coupling, second coupling, ultrafilter, subsequential limit, outside the dependencies. Their meanings are in the A-02 glossary.

Next chapter to read: 11-00 Entrance — the induction hypothesis and the one-step certificate — on to Part 11, which shows with two ledgers, the induction hypothesis and the one-step certificate, that the one-step representation is preserved from step to step.