Volume 2 Existence — reading the published construction

Part 12 Infinite volume and reconstruction

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

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Chapter		Page
12-01	Lattice reflection positivity — two kinds of plane	2
12-02	The chessboard, and the root seminorm	11
12-03	The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm	19
12-04	The transfer operator and equicontinuity	27
12-05	Arzelà–Ascoli, and the properties that pass to the limit	35
12-06	Euclidean invariance	44
12-07	Phase decomposition	53
12-08	Reconstruction and the Wightman axioms	62

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction Reflection positivity · Kiichi, Shiori and Rin

Lattice reflection positivity — two kinds of plane

This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

What this chapter shows — For the Wilson measure on a periodic lattice, the published construction states that the expectation of a function times its mirror image is never negative (reflection positivity), for both planes through lattice sites and planes through link midpoints, along all four coordinate axes, for $\beta \geq 0$. This chapter reads the form of that statement (the class of functions), the skeleton of the proof (split into slices, fix what lies on the plane, turn the interaction crossing the plane into a positive kernel), and how it is used in later chapters of this part.

Chapters assumed — 5-03 (Reflection positivity and the transfer matrix), 4-10 (the OS axioms), 8-01 (conventions for the lattice and the Wilson measure in the published construction). The same subject on this book's side is 20-01.

Said first — The results of the published construction written here are at the level "the committee's assessment is that they are proved in Lean". What this book has checked goes only as far as reading the form of the definitions and statements in the source; the correctness of the proofs has not been checked. Reflection positivity is a property on a single finite lattice, and the statement is only about the original Wilson measure. Reflection positivity for the measures at renormalisation steps (push-forward measures) is not in what we have read.

The order of this chapter

1. Legend of labels
2. Statement — planes, class of functions, positivity
3. Two kinds of plane and four axes
4. Outline of the mechanism — split into slices, fix what lies on the plane, turn the crossing interaction into a positive kernel
5. How it is used in this part
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

Statement — planes, class of functions, positivity

The introduction is 5-03. The setting is the periodic lattice $(\mathbb{Z}/n)^4$ of side n (even, $n = 2h$) and the Wilson measure on it for a gauge group G (a closed matrix group $G \subset U(m)$). To the holonomy U_p of a plaquette p assign the "price"

$$\operatorname{price}(U_p) \;=\; 1 - \frac{\operatorname{Re}\operatorname{tr} U_p}{m} \qquad (1)$$

and put the density $Z^{-1}\exp\bigl(-\beta\sum_p \operatorname{price}(U_p)\bigr)$ on the product of Haar measures; this is the Wilson measure μ_β^{W}, and its expectation is $\langle\cdot\rangle_\beta$ Source (construction source `YM/Gauge/MatrixGroup.lean:112` `Gauge.price`, `YM/Wilson/Law.lean:87` `law`, :99 `moment`). It differs from this book's convention only by a positive constant multiple of the weight. In the two-dimensional representation of SU(2), $\operatorname{Re}\operatorname{tr} U_p/2$ is this book's $\operatorname{Re} U_p$ (the real part of the quaternion), so the scale of β is the same Proved here (for the full conventions see 8-01).

A plane C is determined by four data: direction (coordinate axis) a, kind (a plane through lattice sites or a plane through link midpoints), base position b, and half-length h ($n = 2h$) Source (`YM/Cells/Cut.lean:40` `Cut`). A plane comes with a "one-sided band" Λ_C (`sector`). The class of functions is given by the following four conditions.

Definition (the class of functions local to one side of a plane)

A function of configurations $F\colon (\text{link configurations}) \to \mathbb{C}$ is an **inside function** for the band Λ_C if it satisfies four conditions: (i) gauge invariant, (ii) measurable, (iii) depends only on links with both endpoints in Λ_C (local), (iv) bounded.

Source Construction source `YM/Cells/Cut.lean:109` `Interior` (fields `gauge`, `borel`, `localSites`, `cost`). Locality is `YM/Lattice/Path.lean:196` `LocalOn` (the

values are equal on two configurations that agree on links with both endpoints in the set).

Statement (lattice reflection positivity)

Let $\beta \geq 0$. For every plane C (both kinds, all four axes, every base position) and every inside function F of Λ_C,

$$0 \;\leq\; \langle F \cdot F \circ \Theta_C \rangle_\beta . \quad (2)$$

Here Θ_C is the map that pulls back the reflection in the plane C to configurations, and the inequality is in the order on complex numbers (the left-hand side is a non-negative real number).

Source A theorem of the published construction (reflection positivity of the Osterwalder–Seiler type). Lean: `YM.Slice.reflected_positive` (construction source `YM/Slice/Reflection.lean:98`; the only hypothesis is $0 \leq \beta$). The definition of positivity is `YM/Cells/Cut.lean:116` `Positive`.

Source The form of the statement and the four conditions of the class of functions were read in the source. The bundle saying that inequality (2) holds for every C is collected in the field `positive` of `YM/Cells/Certificate.lean:14` `Seam`, and `laws` (same file, :20) produces the whole bundle from $\beta \geq 0$.

Which target it enters. In the record of declaration usage (`verification/declaration-usage.tsv`) the status of `reflected_positive` is `used`. In this record, `used` means "reached from one of the 17 headline theorems listed by the verification record"; it does not mean the chain to the public theorem `JointExistence` has been checked (the first headline the record found is `SharpGram.habitat`). We infer that the road to the public theorem goes through `seamless` and `crops` in § 04 Inferred.

02

Two kinds of plane and four axes

Let $x_a \in \mathbb{Z}/n$ be the coordinate in direction a, and b the base position. For each kind of plane, the reflection and band are as follows Source (`Kind.shift`, `Kind.Strict`, `reflect_here` in `YM/Cells/Cut.lean`).

Kind	Position of the plane	Reflection $x_a \mapsto$	Band ΛC ($r := x_a - b \bmod n$)
Plane through lattice sites (site)	$x_a = b$ and $x_a = b + h$	$2b - x_a$	$0 < r < h$ (does not include points on the plane)
Plane through link midpoints (link)	$x_a = b - \frac{1}{2}$ and $x_a = b + h - \frac{1}{2}$	$2b - 1 - x_a$	$0 \le r < h$

The other three coordinates do not move. The direction a may be any of the four axes, and the base position b is also arbitrary.

Planes through lattice sites. The mirror fixes the layer of sites $x_a = b$ (and, by periodicity, the opposite layer $x_a = b + h$). The band is the **open** part between the two mirrors and does not include the points on the mirrors. Since the function depends only on links with both endpoints in the band, it depends neither on links on the mirror nor on links in direction a leaving the mirror Source (the note at the top of the source, "site tilings do not read transverse links at seams").

Planes through link midpoints. The mirror passes through the middle of links in direction a and fixes no lattice site. The band is the h layers $x_a = b, \dots, b + h - 1$, and reflection moves them to the opposite h layers (`Cut.link_switch`: the image of a band point is not in the band). The links crossing the mirror (from $x_a = b - 1$ to b, and from $b + h - 1$ to $b + h$) are not inside links of either band. The source's note says this includes the degenerate case $h = 1$ Source (top of `YM/Slice/Gate.lean`).

The functions are complex-valued and gauge invariant. The class of the published construction allows complex-valued functions, and in exchange imposes gauge invariance for both kinds. The conjugate on the left-hand side of (2) is there to make pairs of complex-valued functions positive semi-definite Proved here (setting $F = \sum_i c_i F_i$, (2) says that the matrix $\left(\langle F_j F_i \circ \Theta \rangle\right)$ is positive semi-definite).

> 03
>
> **Outline of the mechanism — split into slices, fix what lies on the plane, turn the crossing interaction into a positive kernel**

The idea is the same as in 5-03 § 04. We write how the source assembles it in three stages Source (as far as the statements and lemma names; the content of each lemma's proof has not been read).

Stage 1: split the configuration into slices

Cut the lattice into layers by the coordinate in direction a. The configuration of links within one layer (links of the three-dimensional lattice perpendicular to direction a) is

called a **slice** (`Leaf`), and the set of links in direction a joining adjacent layers a **thread** (`Thread`). The whole configuration is split into four parts.

- Slices on the mirror (`Posts`, for planes through lattice sites)
- Threads crossing the mirror (`Doors`, for planes through link midpoints)
- Slices and threads inside the band (`Open`)
- Their mirror images (the second copy of `Open`; threads read by their inverses)

The lemma is that this split is an isomorphism preserving the product Haar measure Source (`YM/Slice/Independent.lean:165` `board`; that the copy reading threads by inverses preserves the measure is `twisted`, same file :149). It uses the invariance of Haar measure under inversion.

Stage 2: fixing what lies on the mirror makes the two copies independent and identically distributed

Fix the slices on the mirror and the crossing threads. Under the product Haar measure, the inside of the band and its mirror image are independent and identically distributed. The logarithm of the Wilson weight (the action) splits into plaquettes inside the band, plaquettes on the mirror-image side, and plaquettes touching the mirror. The part not touching the mirror has the form $A(U_>) + A(U_<)$ and can be collected on the same side as F.

Stage 3: the interaction crossing the mirror is a positive kernel

For planes through lattice sites, the function reads neither the slices on the mirror nor the threads leaving it, so for each fixed value on the mirror the integrand takes the form $g(U_>)\,g(U_<)$, and integrating over the two copies gives $\left|\int g\right|^2 \geq 0$ (the kernel is the constant 1).

For planes through link midpoints, the threads crossing the mirror join the slices on both sides of the band. Integrating out the crossing threads gives, as a function of the end slices X and Y on the two sides, the transfer-operator kernel

$$K_\beta(X,Y) \;=\; \int \mathrm{weight}_\beta(X,Y,u)\,du \qquad (3)$$

(u is the crossing thread, and weight is the weight of the plaquettes having that thread as an edge) Source (`YM/Slice/Positive.lean:54` `heat`). If $\beta \geq 0$, this kernel is a **positive kernel** (for any finitely many points $X_1, \ldots, X_k$, the matrix $K_\beta(X_i, X_j)$ is positive semi-definite) Source (Lean: `YM.Slice.step_positive`, `YM/Slice/Positive.lean:81`; the definition of a positive kernel is `YM/Analysis/Nonnegative.lean:18` `NNKernel`). For a positive kernel, the paired integral

$$\iint g(X)\, K_\beta(X,Y)\, g(Y)\, d\nu(X)\, d\nu(Y) \;\geq\; 0 \qquad (4)$$

holds; this is the last stage Source (`NNKernel.integrated`, used in `YM/Slice/Reflection.lean:72`; `portPos`, `YM/Slice/Gate.lean:94`, makes the kernels of both kinds of plane positive together: for site planes the kernel is 1, for link-midpoint planes a product of K_β).

The roles of gauge invariance and $\beta \geq 0$. $\beta \geq 0$ is used for the positivity of the kernel in stage 3. Gauge invariance is part of the definition of the class, but where in stages 1–3 it is needed for planes through lattice sites has not been read Inferred (for link-midpoint planes it is needed in handling the crossing threads). In this book's own proof (20-01 § 05), gauge invariance was used for link-midpoint planes to remove the crossing links by setting them to 1 by a gauge. The published construction does not remove the crossing threads; it integrates them into a kernel.

04

How it is used in this part

Within what we have read, reflection positivity is used in the construction only for **the Wilson measure on the original lattice** Source (within the scope of the name search). The family of the construction has, for each cutoff N and volume p, a lattice of side $n = L_p/a_N$ and a coupling $\beta_{N,p}$ (8-02), and the bundle `Seam` is set up from $\beta_{N,p} \geq 0$ for all (N,p) Source (Lean: `YM.Cells.seamless`, `YM/Cells/Packing.lean:81`; $\beta_{N,p}$ is non-negative because the coupling lies in the band $[\hat\beta/2,\, 2\hat\beta]$ and $\hat\beta > 1$).

Use	What it is applied to	Chapter
The chessboard inequality and the root triangle inequality	Positivity on all band planes (`bandCut`)	12-02
Positivity of the transfer-operator kernel	The kernel (3) of planes through link midpoints	12-04
Reflection positivity of the continuum limit	Non-negativity of the Gram matrix for the plane through lattice sites (coordinate 0)	12-05
The OS Hilbert space	Reflection positivity of the limiting table (continuum side)	12-07, 12-08

Not used for push-forward measures. A statement of reflection positivity for the measures at renormalisation step m (the fine-lattice measure pushed forward by a probability kernel, 9-01) is not in what we have read Source (within the scope of the name search). Since the kernel's patches are designed to overlap across planes, we infer that the push-forward does not inherit reflection positivity automatically Inferred.

The estimates on the construction's observables (12-03) are also stated as expectations under the Wilson measure $\mu^{\mathrm{W}}_{N,p}$ at step N Source .

How the assessment puts it. § 5 of the assessment writes that local boundedness and equicontinuity in the infinite-volume limit come from "a general argument based on reflection positivity, the Chessboard Estimate (Fröhlich–Israel–Lieb–Simon), the transfer-operator representation and Cauchy estimates" Assessment .

05

What this book's study had seen

This book's Lean had three forms of reflection positivity for the Wilson measure on the SU(2) lattice $(\mathbb{Z}/2M)^4$ (20-01).

	Published construction (`reflected_positive`)	**This book**
Group	All closed matrix groups G	SU(2) (unit quaternions)
Planes through lattice sites	$\beta \geq 0$, four axes, gauge invariant, complex-valued, local to the band (not including the mirror)	All real β, four axes, no gauge invariance imposed, real-valued, local to the closed positive half (including the mirror) (`TorusRP.wilson_rp`, `PeierlsPerm.wilson_rp_dir`)
Planes through link midpoints	$\beta \geq 0$, four axes, gauge invariant	$\beta \geq 0$, time direction only, gauge invariant, $M \geq 2$ (`MidRP.wilson_rp_mid`)
Handling of crossing links	Integrated into the transfer-operator kernel (a positive kernel)	Removed by setting them to 1 by a gauge; the remaining crossing plaquettes written as sums of products of coordinates

Lean This book's three theorems are formalisations of results of the Osterwalder–Seiler type (Lean: `TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir`; scope as in the table of 20-01).

There are two differences. First, the published construction has link-midpoint planes along all four axes, while this book had only the time direction. Second, the published construction imposes gauge invariance even for site planes and is limited to $\beta \geq 0$, while this book, for site planes, asked neither the sign of β nor gauge invariance. Which class is broader does not go in one direction.

For push-forward measures, this book had, with a different design (side 3, centred blocks, a block map with uniform weights), reflection positivity for site planes in Lean (Lean: `Block3.rp_push`, `Block3.rp_push_prob`; any real β, bounded measurable functions depending only on links strictly between two coarse site planes). The way

steps are made differs from the design of the published construction (the published construction uses probability kernels, this book deterministic maps), so this is not a comparison of the same object. This design is not this book's main line (28-05).

06

What has not been read

- **Where gauge invariance is needed for site planes.** We have read only as far as the names of the lemmas of stages 1–3 (`link_positive`, `insidePos` in `YM/Slice/Reflection.lean`).
- **The content of the proof of kernel positivity.** The proof of `step_positive` (`YM/Slice/Positive.lean:81`) has not been opened. Since the weight is a product over plaquettes of $\exp(\beta \operatorname{Re}\operatorname{tr}(\cdot)/m)$, we infer that it expands this into products of positive kernels Inferred .
- **The chain to the public theorem.** `used` in the record means reached from one of the headline theorems; the chain up to `JointExistence` has not been traced step by step.
- **That reflection positivity of push-forward measures is absent.** This judgement is within the scope of a search by names (reflection, RP, positive); we have not read the whole repository.
- **Rotated planes.** `YM/Euclid/OtherPlanes.lean` seems to treat other planes on the continuum side, but it is a location of declarations not used according to the record Source (record). Its content has not been read.

What this chapter can say

Can say	Cannot say
Source For $\beta \geq 0$, the Wilson measure on a periodic lattice is reflection positive for both kinds of plane and all four axes, on the class of gauge-invariant, measurable, band-local, bounded complex-valued functions (`YM.Slice.reflected_positive`)	$\beta < 0$. Reflection positivity of the measures at renormalisation steps (push-forward). The continuum limit itself
Source The form of the statement, the four conditions of the class of functions, the definitions of plane and band, and the names of the lemmas in the stages of the proof were read in the source	The content of the lemmas' proofs. The role of gauge invariance for site planes
Source The bundle `Seam` is set up from $\beta_{N,p} \geq 0$ for all (N, p) of the construction (`seamless`)	That the chain to the public theorem `JointExistence` has been checked declaration by declaration

Can say	Cannot say
`Lean` Differences from this book's three forms (20-01): the published construction has link-midpoint planes along four axes, and this book, for site planes, does not ask the sign of β or gauge invariance	That either class contains the other (neither does)

Sources and reproduction

Item	Kind	Source / tool
Reflection positivity in lattice gauge theory	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-03, references in A-03
(2) and the definitions of plane and class	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Slice/Reflection.lean:98` `reflected_positive`, `YM/Cells/Cut.lean:40` `Cut`, :109 `Interior`, :116 `Positive`, `YM/Lattice/Path.lean:196` `LocalOn`, `YM/Wilson/Law.lean:87` `law`, `YM/Gauge/MatrixGroup.lean:112` `price`
Lemmas of the stages of the proof	Source	`YM/Slice/Independent.lean:165` `board`, `YM/Slice/Positive.lean:54` `heat`, :81 `step_positive`, `YM/Slice/Gate.lean:94` `portPos`, `YM/Analysis/Nonnegative.lean:18` `NNKernel`
The bundle, and its application to the construction	Source	`YM/Cells/Certificate.lean:14` `Seam`, :20 `laws`, `YM/Cells/Packing.lean:81` `seamless`. Usage in `verification/declaration-usage.tsv`
Description in the assessment	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 5
This book's corresponding theorems	Lean	`TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir` (20-01), `Block3.rp_push`, `Block3.rp_push_prob`. sorry 0, axioms the standard three

Words used in this chapter — reflection positivity, plane through lattice sites, plane through link midpoints, band, inside function, gauge invariance, slice, thread, transfer-operator kernel, positive kernel, push-forward measure. Their meanings are in the A-02 glossary.

Next chapter to read: 12-02 The chessboard, and the root seminorm — stacking reflection positivity over all band planes to build a seminorm that measures size box by box.

The chessboard, and the root seminorm

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

What this chapter shows — Tile the torus with boxes of equal size, carry a function local to one box to every box by reflections, and multiply: the size of the expectation is then bounded by the product of "the expectation when each function is copied to every box, raised to the power one over the number of boxes (the root)" (the chessboard inequality). The published construction also has that the root satisfies the triangle inequality for sums of functions, that is, the root is a seminorm. This chapter reads those two statements, the skeleton of the proof, and their uses in the construction.

Chapters assumed — 12-01 (lattice reflection positivity and the class of functions). The idea of the Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) is also in 22-01.

Said first — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean". What this book has checked goes as far as the form of the definitions and statements. The chessboard inequality is an inequality on a single finite lattice, and its only hypothesis is "reflection positivity on all band planes". A quantity uniform in the volume comes out because the root is a per-box quantity; bounding the size of the root itself is separate work (12-03).

The order of this chapter

1. Legend of labels
2. Before the statements — tiling, copy, root
3. Statement (1) — the chessboard inequality
4. Statement (2) — the root triangle inequality (the root is a seminorm)
5. Outline of the mechanism — maximisation and reflection, expansion and root
6. Uses in the construction
7. What this book's study had seen
8. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

Before the statements — tiling, copy, root

Tile the periodic lattice $(\mathbb{Z}/n)^4$ of side n with boxes of side s. The boxes are indexed by $(\mathbb{Z}/P)^4$ (P even, $n = Ps$), and the kind of box matches the kind of reflection plane (through lattice sites or through link midpoints) Source (construction source `YM/Cells/Tiling.lean:19` `Tiling`: fields `period` = P, `units`, `widthHalf` = $P/2$, `kind`, `base` = the corner of the base box). The **inside** of box v (`chamber v`) is, by the same rule as the bands of 12-01, the set of points that excludes the box's boundary (site boxes) or includes one side of the boundary (link-midpoint boxes).

Copies. A function F local to the inside of the base box $\mathbf{0}$ is carried to box v. The way to carry it is: for each axis j, if v_j is odd insert one reflection in direction j, then translate Source (same file, :58 `omega`, :63 `translateBy`, :65 `copyMap`). In boxes with an odd number of reflections, complex conjugation is applied:

$$\operatorname{copy}_v F(U) \;=\; \begin{cases} F(\vartheta_v^{*}U) & \text{(odd number of reflections)} \\ F(\vartheta_v^{*}U) & \text{(even)} \end{cases} \tag{1}$$

Here ϑ_v^{*} is the pull-back to configurations of the map taking box $\mathbf{0}$ to box v Source (same file, :220 `hand`, :222 `copy`). The copies in two adjacent boxes are mirror images of each other across the plane between them. Like the white and black squares of a chessboard, the orientation flips every other box.

Root. The size of the expectation of the product of copies of a single function F over all boxes, raised to the power one over the number of boxes, is written

$$\|F\|_{\square} \;:=\; \Big|\Big\langle \prod_{v} \operatorname{copy}_v F \Big\rangle_{\beta}\Big|^{1/P^4} \tag{2}$$

and called the **chessboard root** Source (same file, :226 `rootTest`). The subscript $\square$ indicates which tiling it is measured with. The root is a per-box quantity. Since the same function is placed in every box, even when the volume grows and the number of boxes increases, it can be compared as a per-box contribution.

Example Proved here . If $F \equiv c$ (a constant), then $\|F\|_\square = |c|$. If F is the indicator $\mathbf{1}_E$ of some event in box 0, then $\|F\|_\square^{P^4}$ is "the probability that the (copied) event occurs in all boxes simultaneously", and $\|F\|_\square$ is the per-box rate of that probability.

02

Statement (1) — the chessboard inequality

Statement (the chessboard inequality)

Suppose reflection positivity holds on every band plane of the tiling (planes through box boundaries, one for each direction a and position p). Choosing for each box an inside function F_v of box 0 (the class of 12-01: gauge invariant, measurable, local, bounded),

$$\left|\left\langle \prod_v \mathrm{copy}_v F_v \right\rangle_\beta\right| \;\le\; \prod_v \|F_v\|_\square. \quad (3)$$

Source A theorem of the published construction (the Chessboard Estimate of the Fröhlich–Israel–Lieb–Simon type). Lean: `YM.Cells.Tiling.tiles` (construction source `YM/Cells/Chess.lean:80`; its name in the source's docstring is WFS1; the hypotheses are `Positive` on every `bandCut`, and that each F_v is an inside function of box 0).

Source The band planes are `YM/Cells/Tiling.lean:40` `bandCut` (half-length $s \cdot P/2$, direction a, base at the corner of box p). The size of the expectation on the left-hand side is written as `YM/Cells/Polar.lean:91` `rootPower` ($\|\langle \prod_v \mathrm{copy}_v F_v \rangle\|$).

Meaning. The right-hand side is the product of "the per-box size when each function is replicated in every box". A correlation with different functions in different boxes is bounded by correlations with the same function in all boxes. This is the result of stacking the Cauchy–Schwarz inequality that comes from reflection positivity many times, over all planes (§ 04).

03

Statement (2) — the root triangle inequality (the root is a seminorm)

Statement (the root triangle inequality)

Under the same hypotheses, for two inside functions F, F' of box 0,

$$\|F + F'\|_\square \;\le\; \|F\|_\square + \|F'\|_\square. \quad (4)$$

Source A theorem of the published construction. Lean: `YM.Cells.Tiling.adds` (construction source `YM/Cells/Chess.lean:108`; its name in the docstring is WFS2).

The root satisfies $\|cF\|_\square = |c|\,\|F\|_\square$ for scalar multiples Proved here (the source's `rootPower_scale`, `YM/Cells/Chess.lean:70`, states only non-negative real multiples; it extends to complex c because, by the definition (1) of copies taking conjugates in boxes with an odd number of reflections, the product is multiplied by a factor of modulus $|c|$ P^4). Together with (4), the root is a **seminorm** on the space of inside functions of box 0 (it need not be a norm, since $F \neq 0$ is possible with $\|F\|_\square = 0$) Proved here .

The bundle. Reflection positivity, the chessboard inequality and the root triangle inequality are collected in one bundle `Seam` (`YM/Cells/Certificate.lean:14`: fields `positive`, `inequality`, `semi`), and `laws` produces all three at once from $\beta \geq 0$ Source . This bundle holds for every cutoff N and volume p of the construction (`seamless`, `YM/Cells/Packing.lean:81`; 12-01 § 04).

Why a seminorm is useful. When an observable splits into "a main part + a small remainder", (4) lets the roots be added part by part. In the ultraviolet estimates, when a renormalised observable is held as a sum of many terms, bounding the root of each term and adding them is enough to bound the root of the whole (12-03) Inferred (a reading from the names of the places where it is used).

04

Outline of the mechanism — maximisation and reflection, expansion and root

We write the source's line as an argument of the Fröhlich–Israel–Lieb–Simon type Source (lemma names and statements; details are inference).

Stage 1: Cauchy–Schwarz on a single plane

A single band plane C divides the boxes into two halves. Splitting the product of an assignment of boxes $F_\bullet$ into the part Π_+ on the positive side of the plane and the part on the negative side, the negative side becomes the mirror image of another assignment Π'_+ on the positive side Source (`YM/Cells/Copies.lean:124` `reflect_equation`: the product with the assignment rearranged into "half F, the other half the mirror image of Q" becomes $\Pi_+(F)\cdot\Pi_+(Q)\circ\Theta_C$). Reflection positivity makes the pairing $(X,Y)\mapsto\langle X\cdot Y\circ\Theta_C\rangle$ positive semi-definite, so by Cauchy–Schwarz

$$\left|\langle \Pi_+(F)\,\Pi_+(Q)\circ\Theta_C\rangle\right|^2 \;\leq\; \langle \Pi_+(F)\,\Pi_+(F)\circ\Theta_C\rangle\,\langle \Pi_+(Q)\,\Pi_+(Q)\circ\Theta_C\rangle. \quad (5)$$

Both factors on the right-hand side are expectations of assignments in which "one side's assignment is replicated to the other side by the mirror".

Stage 2: a maximising assignment stays maximal when symmetrised by reflection

Consider assignments A that choose one of finitely many candidate functions $f_1, \dots, f_k$ for each box, and take an A maximising $\|\langle \prod_v \mathrm{copy}_v\, f_{A(v)} \rangle\|$. By (5), replicating a maximal assignment from one side across any plane keeps it maximal. Repeating this over planes of every direction and position, the assignment that places the same candidate f_c in every box becomes one of the maxima Source (`YM/Cells/Chess.lean:32` `maximize`: there is a c such that, for every assignment A, the value is at most the value with f_c placed in every box).

Stage 3: normalise to get (3)

Applying stage 2 to the functions F_v divided by $\|F_v\|_\square + \varepsilon$ makes the left-hand side ≤ 1, and $\varepsilon \to 0$ gives (3) Source (the shape of the proof of `tiles`: add ε, divide, take the limit).

Stage 4: expand and take the root

Expanding $\prod_v \mathrm{copy}_v (F + F')$ gives a sum over the 2^{P^4} assignments choosing F or F' for each box. Applying (3) to each term,

$$\|F + F'\|_\square^{P^4} \;\le\; \sum_{\text{assignments}} \prod_v \|F_{\text{chosen}}\|_\square \;=\; \left(\|F\|_\square + \|F'\|_\square\right)^{P^4} \tag{6}$$

and taking the P^4-th root gives (4) Proved here (the source's `adds` has as a lemma the expansion into a sum over assignments $i : \text{boxes} \to \{\text{true}, \text{false}\}$).

05 Uses in the construction

The centred tiling. The tiling the construction uses has boxes of side equal to the lattice spacing a_m at step m, with the base box placed with the observation point o at its **centre** Source (`YM/Cells/Packing.lean:34` `center`: the kind is link-midpoint boxes, a box side is an odd number of fine lattice points, the physical side is a_m (`trueLength`), and the reciprocal of the number of boxes is $(a_m / L_p)^4$ (`root_fraction`)). The root for this tiling is written $\|\cdot\|_{m,o}$.

Use	What it does	Location	Chapter
Form of the input from the ultraviolet	Imposes upper bounds on the roots $\|\cdot\|_{m,o}$ of renormalised observables	`YM/Extraction/Center.lean:21` `centeredBudget`, :38 `Wcmain`	12-03
Local boundedness	Encloses observables at separated points in separate boxes, and bounds the correlation by a product of roots via (3)	`YM/Extraction/Held.lean:120` `Around.size_done`	12-03, 12-05
Equicontinuity	Bounds the norm of the "column" of the transfer-operator representation by roots	`YM/Extraction/Difference.lean:51` `reading`	12-04
Estimates on the ultraviolet side	Adds the roots of terms by the triangle inequality (field `semi`)	`YM/FullVolume/Rent.lean` and other places using `.semi` Source (search)	Part 11

How the assessment puts it. § 5 of the assessment writes that boundedness and equicontinuity in the infinite-volume limit come from "general mechanisms not specific to Yang–Mills" — reflection positivity, the Chessboard Estimate, the transfer-operator representation and Cauchy estimates — and that the input specific to the construction is control of renormalised lattice observables Assessment .

06

What this book's study had seen

This book's Lean also had a Chessboard Estimate for the Wilson measure on the SU(2) lattice $(\mathbb{Z}/2m)^4$.

Lean For the expectation V of the product of a family of bounded measurable functions depending only on the links of a unit cube, carried to each block: $|V(F)|^{N^4} \le \prod_w V(\text{constant family } F_w)$. All even sides, all real β. A formalisation of the Fröhlich–Israel–Lieb–Simon type. Lean: `FoldWilson.wilson_chessboard_even` (22-01).

There are three differences.

- **The use was different.** This book used the chessboard for **the rarity of large plaquettes** (the probability that $\mathrm{Re}\,U_p < 0$ on every plaquette of a set X is at most $q^{\lfloor |X|/6 \rfloor}$, with q independent of the volume; Lean: `FoldCap.rarity_explicit`, $\beta \ge 81$). The published construction uses the chessboard only for estimating the size of observables, and a statement about rarity of plaquettes is not in what we have read Inferred (within the scope of the search).

- **The root triangle inequality was not in this book's study.** This book applied its estimate only to one form, products of indicator functions, and did not use the root as a seminorm on a space of functions.
- **How the boxes are made.** This book's boxes are unit cubes (boxes of lattice spacing 1); the published construction places boxes of the step-m spacing with the observation point at the centre.

This book's statement of rarity has a form that can be restated in the language of roots Proved here : setting $F = \mathbf{1}\{\text{at a chosen plaquette in the box},\ \operatorname{Re} U_p < 0\}$, the rarity estimate states an upper bound on the root $\|F\|_{\square}$. The details of this restatement (box size and choice of plaquettes) have not been checked. The restatement is treated in Part 22 of Volume 3.

07

What has not been read

- **Details of the maximisation argument.** We read the statement of the type "stays maximal under reflection" (`T.Reflects`) inside `maximize` (`Chess.lean:32`), but not how the induction that walks over all band planes is organised.
- **The exact shape of the inside of link-midpoint boxes.** We read the definition of `chamber` (`Tiling.lean:36`), but have not checked the treatment of the boundaries of both kinds of box in every case.
- **How `semi` is used on the ultraviolet side.** We only found the places using it by name; we have not read in which stage of which estimate it is used.
- **The chain to the public theorem.** `tiles` and `adds` are `used` in the record (reached from one of the headline theorems). The chain to the public theorem has not been traced declaration by declaration.

What this chapter can say

Can say	Cannot say
Source The chessboard inequality (3), from reflection positivity on all band planes (`Tiling.tiles`)	Application to measures without reflection positivity (push-forward measures and so on)
Source The root triangle inequality (4); the root is a seminorm (`Tiling.adds`)	That the root is a norm (the root of a non-zero function can be 0)
Source The definitions of tiling, copy, root and centred tiling were read in the source	Details of the induction in the maximisation

Can say	Cannot say
`Proved here` (4) follows from expanding (3) and taking roots ((6))	—
`Lean` This book's Chessboard Estimate (`FoldWilson.wilson_chessboard_even`) and rarity of large plaquettes (`FoldCap.rarity_explicit`)	Details of the form restating rarity in the language of roots. A root triangle inequality on this book's side

Sources and reproduction

Item	Kind	Source / tool
The Chessboard Estimate (from reflection positivity)	Cited	Reflection positivity and the Chessboard Estimate of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (bibliographic details as in 22-01 and A-03)
(3), (4) and definitions	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Cells/Chess.lean:80` `tiles`, :108 `adds`, :23 `powering`, :32 `maximize`, `YM/Cells/Tiling.lean:19` `Tiling`, :40 `bandCut`, :222 `copy`, :226 `rootTest`, `YM/Cells/Polar.lean:91` `rootPower`, `YM/Cells/Copies.lean:124` `reflect_equation`
The bundle and the centred tiling	Source	`YM/Cells/Certificate.lean:14` `Seam`, `YM/Cells/Packing.lean:34` `center`, :81 `seamless`
Uses	Source	`YM/Extraction/Center.lean:38` `Wcmain`, `YM/Extraction/Held.lean:120` `size_done`, `YM/Extraction/Difference.lean:51` `reading`
Description in the assessment	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` § 5
This book's chessboard and rarity	Lean	`FoldWilson.wilson_chessboard_even`, `FoldCap.rarity_explicit` (22-01). sorry 0, axioms the standard three

Words used in this chapter — Chessboard Estimate, tiling, inside of a box, copy, chessboard root, seminorm, band plane, centred tiling, rarity of large plaquettes. Their meanings are in the A-02 glossary.

Next chapter to read: 12-03 The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm — we read the form of the input that imposes volume-uniform upper bounds on the roots of renormalised observables.

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction From the ultraviolet to infinite volume · Kiichi, Shiori and Rin

The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

What this chapter shows — The published construction copies renormalised local observables with a tiling of boxes centred on the observation point, and has, in a form uniform over all volumes, the estimate that their chessboard root (12-02) is at most $A\,a^{d-\varepsilon}_{m}$ (the Lean name is `Wcmain`). This estimate is the input to the compactness argument that takes the infinite-volume limit. This chapter reads the form of that statement, the mechanism by which it turns into local boundedness, and the fact that "this is one of the inputs received from the ultraviolet, and there are others".

Chapters assumed — 12-02 (roots and centred tilings), 8-04 (renormalised insertions), 8-02 (the sequence of spacings a_m, the sequence of volumes L_p).

Said first — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean"; what this book has checked goes only as far as reading the form of the statements. `Wcmain` is the input to infinite-volume compactness (`crops`), not **all** of the inputs passed from the ultraviolet to the infrared side. Other estimates (`Scrim` and so on) are passed at the same time from the same run of the construction. How `Wcmain` itself is proved (the content of the estimates on the ultraviolet side) is not read in this chapter.

The order of this chapter

1. Legend of labels
2. Statement — the size of renormalised observables measured by the root
3. How to read the statement — what it is uniform in, and what it depends on
4. Outline of the mechanism (1) — from upper bounds on roots to local boundedness

5. Outline of the mechanism (2) — the hand-over to compactness, and other inputs
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

Statement — the size of renormalised observables measured by the root

We fix notation (8-02). Let a_N be the lattice spacing at cutoff N, L_p the physical side of the box with volume index p, and $\mu^{\mathrm{W}}_{N,p}$ the Wilson measure on the lattice with cutoff N and volume p (coupling $\beta_{N,p}$). The spacing a_m at step $m \le N$ is the scale on the coarse side. To a label v of a local field of degree d (a vector of coefficients, 8-03) corresponds a renormalised insertion $O_{N,p}(v;o)$ (a polynomial in finitely many links placed at the lattice point o; 8-04) Source (`C.seen v o` , `C.O N p v x` in the construction source).

Write $\|\cdot\|_{m,o}$ for the root of the tiling by boxes of physical side a_m with the lattice point o at the centre (12-02 § 05, `YM/Cells/Packing.lean:34` `center`).

Form of the statement (estimate of renormalised observables measured by the root)

For every degree d, radius $R > 0$ and $\varepsilon > 0$, there is a constant $A \ge 0$ such that for every step m at or above the reference step $r(d)$, for all sufficiently large cutoffs N, for every volume p, every label v with $\|v\| \le R$ and every lattice point o,

$$O_{N,p}(v;o) \text{ is an inside function of the centre box, and } \quad \big\|O_{N,p}(v;o)\big\|_{m,o} \;\le\; A\, a_m^{-d-\varepsilon}.$$

The root is taken with $\mu^{\mathrm{W}}_{N,p}$ (coupling $\beta_{N,p}$).

Source This is the content of the predicate `YM.Extraction.Wcmain` in the construction source (`YM/Extraction/Center.lean:38` (lines 38–40)). The estimate

for one box is `centeredBudget`, same file :21 ($m \le N$, $r(d) \le N$, and for every o, "is an inside function" and "root ≤ bound").

`Wcmain` is a predicate (the name of a property); that the construction's family satisfies it is given by other theorems. For example, `Wcmain C` appears in the conclusions of `oboes` in `YM/FullVolume/Dusk/Graff.lean:20` and `noel` in `YM/MergeProof/Noel.lean:24` Source . Which theorem lies on the road to the public theorem `JointExistence` has not been checked.

02

How to read the statement — what it is uniform in, and what it depends on

Quantity	Treatment in (1)	Meaning
Volume p	Uniform ($\forall$ p inside $\exists$ A)	The key uniformity for passing to infinite volume
Observation point o	Uniform	The same bound under translation
Cutoff N	For sufficiently large N (with a threshold for each m)	A form that can be stated on the side of the ultraviolet limit
Step m	Depends through the power $a_m^{-d-\varepsilon}$	The naive dimension a^{-d} of a field of degree d, with a loss of ε
Label v	Uniform over $\|v\| \le R$	On a bounded set of coefficients

Why measuring by the root can give a quantity uniform in the volume. The root is the per-box quantity of "the expectation with the same function copied to every box" (12-02 (2)). As the volume grows the number of boxes increases, but the per-box quantity does not. That is why a volume-uniform upper bound like (1) is meaningful Proved here . However, that the bound actually holds uniformly in the volume is the **content** of (1), and the work of proving it lies on the ultraviolet side (Part 11). The one-step ledgers on the ultraviolet side (norms of activities, contraction constants) are written as pointwise sums and do not depend on the volume, but estimates at the level of densities and partition functions include an exponential in the volume Source (the activity ledgers `EBound`, `KBound` in `YM/RG/Activities.lean:98-104` are sums per lattice point; `refreeze` in `YM/FullVolume/Fields/Flight.lean:28` contains the number of lattice points in an exponent; 11-10). The chain showing from which estimate the volume uniformity of (1) comes has not been read (§ 06).

Reading a_m^{-d}. A field of degree d (weight 2 for curvature, weight 1 for derivatives) has the dimension of length to the power $-d$. The right-hand side of (1) is the dimensional

count that "the typical size of a field placed in a box of side a_m is a_m^{-d}", with a margin of ε added Inferred .

How the assessment puts it. § 5 of the assessment writes, about the argument for boundedness and equicontinuity in infinite volume, that "the input specific to the construction is control of renormalised lattice observables uniform in the spacing" Assessment . (1) is one form of that input.

03

Outline of the mechanism (1) — from upper bounds on roots to local boundedness

Boundedness of n-point functions at separated points follows from (1). The line is as follows Source (statements and proof stages of `YM/Extraction/Held.lean:74` `fits_budget` and :120 `Around.size_done`).

1. Suppose the continuum points $x_1, \dots, x_n$ are pairwise at distance at least r. Choose a step m with $a_m < r$, and tile with boxes of spacing a_m.
2. Shift the position of the tiling so that each point falls inside a different box (sufficiently far from its boundary). Since the points are separated, no two points fall in the same box (the injection D in `fits_budget`).
3. Each observable $O(v_i; x_i)$ is the copy to box $D(i)$ of an inside function f_i of the base box. Boxes without an observable get the constant 1.
4. By the chessboard inequality ((3) of 12-02), the size of the n-point function is at most the product of the roots of the f_i. The root of the constant 1 is 1.

$$\left| \left\langle \prod_{i=1}^{n} O_{N,p}(v_i; x_i) \right\rangle_{\mu_{N,p}} \right| \;\le\; \prod_{i=1}^{n} \|f_i\|_{\square} \;\le\; A^n. \quad (2)$$

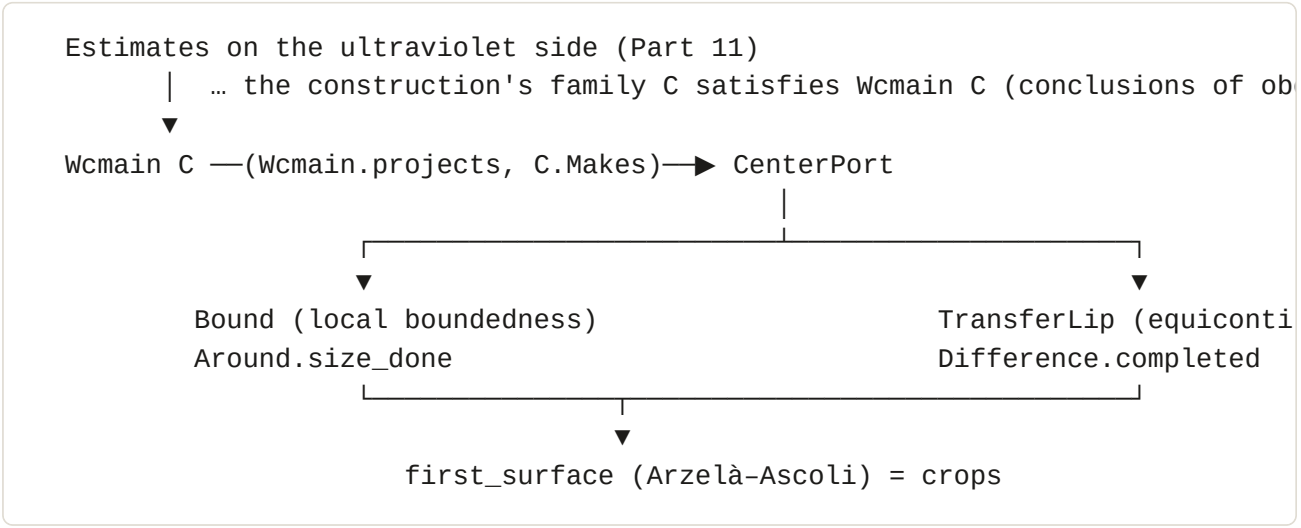
Here A is the per-box bound coming from (1) Source (the conclusion $\|\cdot\| \le A^n$ of `fits_budget`). This bound depends neither on the volume p nor on the positions of the points. This is `Extraction.Bound` (`YM/Extraction/Integrate.lean:25`), corresponding to the first part of equation (11) of the assessment (uniform boundedness on compact sets) Source . `Bound` has two fields: `cyclic` (volume fixed, uniform in the cutoff) and `volume` (uniform in both volume and cutoff, for sufficiently late volumes).

Equicontinuity (the second part of equation (11) of the assessment) is obtained by combining (1) with the transfer-operator representation. That is read in 12-04.

04

Outline of the mechanism (2) — the hand-over to compactness, and other inputs

The form of the hand-over. (1) is converted into the form `CenterPort` along an extraction sequence (a pair of a subsequence of cutoffs and a subsequence of volumes) (`Center.lean:30`: fields `per` and `cells`) Source (`Wcmain.projects`, same file :42; it takes the construction's certificate `C.Makes` as a hypothesis). The field `cells`, `CellHyp` (`YM/Cells/Packing.lean:68`), holds the bundle `Seam` for all (N, p) (12-01, 12-02) and a statement of the root bound with a margin η for sufficiently late volumes. From `CenterPort` come the boundedness `Bound` and the equicontinuity `TransferLip`, which are passed to the Arzelà–Ascoli argument `first_surface` (`crops`, `YM/Extraction/Held.lean:136`; 12-05).



This is not the only input from the ultraviolet. The lemma using `Wcmain`, `MergeProof.Accent.rated` (`YM/MergeProof/Powder.lean:48`), takes, together with `Wcmain C`, a bundle of estimates `Scrim` and hypotheses about rank and metric Source. This is the main basis for "there are others". As supporting evidence, the bundle `Run.Ties` (`YM/FullVolume/Cross.lean:65-71`) that the construction's run carries out of the ultraviolet side at once also has, alongside `control : Wcmain`, the fields `tabs`, `scrim`, `fridge`, `receipts` and `detectors` Source (in the record of declaration usage, `Run.Ties` and the field `detectors` are `used`, and the field `control` is `outside`). So, precisely, we read that **`Wcmain` is the input to compactness (`crops`), and there are other inputs from the ultraviolet**. What the other inputs are used for (quotients of the space of fields, the stress tensor, the operator product expansion and so on) concerns chapters outside this part (Part 13) and Part 11 Inferred.

`Wcmain` also appears as a hypothesis of lemmas on the ultraviolet side (for example `YM/FullVolume/Rent.lean:120` `primaries`) Source. It is not a one-way "output" but an estimate referred to many times within the construction.

05

What this book's study had seen

A statement of this form did not exist in this book's study. This book's results on the ultraviolet side (Part 15) were statements at a single scale, such as two-sided bounds on the covariance of one step of the free field and the linearisation of the block map. There is no estimate stating the size of renormalised local observables in a volume-uniform seminorm. This book's conditional chain theorem (28-01, Lean: `BoxWalk.chain_end_to_end_box`) was an implication that put rarity for the push-forward measures, the splitting of observables, reflection positivity and so on in its hypothesis fields, and concluded the existence and properties of subsequential limits. The quantities placed in the hypotheses there are not defined from the lattice.

(1) in the published construction is one answer to "in what form is it enough to hand ultraviolet results over to the infrared argument". It can be read as the position where this book's ultraviolet output should be placed Inferred . This book's rarity of large plaquettes (22-01, Lean: `FoldCap.rarity_explicit`) can be restated as bounding the root of an indicator function of a unit-size box under the original Wilson measure (12-02 § 06). That, however, is not an estimate of renormalised observables, and cannot replace (1).

On the source of volume-uniform estimates. This book's infinite-volume limit inside the strong-coupling window (20-03) obtained volume-uniform decay from Dobrushin's uniqueness argument. (1) in the published construction expresses uniformity by imposing an upper bound on a per-box quantity, the chessboard root. The ways of expressing uniformity differ (from which estimate the volume uniformity of (1) comes has not been read, as in § 06).

06

What has not been read

- **The proof of (1).** The content of the theorems having `Wcmain C` as conclusion (and how `oboes`, `noel`, `gardena` are used) has not been read. The chain showing from which ultraviolet estimate the volume uniformity comes has not been checked either.
- **Which theorem lies on the road to the public theorem.** There are several theorems with `Wcmain` as conclusion, and which one the proof of `JointExistence` goes through has not been traced. The headline for `used` in the record is `SharpGram.habitat`.
- **The content of the other inputs.** What the fields `Scrim`, `tabs`, `fridge` and `receipts` say, we have only seen by name.

- **Where the ε loss comes from.** Where the ε in $a_{\overline{m}}^{d-\varepsilon}$ comes from has not been read.
- **How the reference step $r(d)$ is fixed.** The definition of the reference step for each degree (`C.ref d`) has not been traced (material for 8-04).

What this chapter can say

Can say	Cannot say
Source The form of the estimate (1) of renormalised observables measured by the root: uniform in volume and observation point, $a_{\overline{m}}^{d-\varepsilon}$ in the step m, for sufficiently large cutoffs (`Wcmain`)	The content of the proof of (1). From which estimate the volume uniformity comes
Source From (1) and the chessboard inequality, n-point functions at separated points are bounded uniformly in the volume (2) (`fits_budget`, `size_done`)	Estimates where points coincide (not treated)
Source `Wcmain` is the input to compactness (`crops`). Other inputs such as `Scrim` are passed from the same run (`rated`, `Run.Ties`)	It cannot be said that "there is a single input from the ultraviolet". The content of the other inputs
Inferred $a_{\overline{m}}^{d}$ is the dimensional count for a field of degree d	Where the ε comes from
This book has no estimate of the same form	Restating this book's ultraviolet results in the form of (1)

Sources and reproduction

Item	Kind	Source / tool
The predicate of (1) and the box estimate	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Extraction/Center.lean:21` `centeredBudget`, :30 `CenterPort`, :38 `Wcmain`, :42 `Wcmain.projects`, `YM/Cells/Packing.lean:34` `center`, :68 `CellHyp`
Local boundedness	Source	`YM/Extraction/Held.lean:74` `fits_budget`, :120 `Around.size_done`, :136 `crops`, `YM/Extraction/Integrate.lean:25` `Bound`
Other inputs	Source	`YM/MergeProof/Powder.lean:48` `Accent.rated`, `YM/FullVolume/Cross.lean:65-71` `Run.Ties`, `YM/FullVolume/Rent.lean:120` `primaries`
Theorems with (1) as conclusion	Source (statements only)	`YM/FullVolume/Dusk/Graff.lean:20` `oboes`, `YM/MergeProof/Noel.lean:24` `noel`
Description in the assessment	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` § 5 and equation (11)
Related results in this book	Lean	`BoxWalk.chain_end_to_end_box` (implication, 28-01), `FoldCap.rarity_explicit` (22-01)

Words used in this chapter — renormalised insertion, chessboard root, centred tiling, uniform in the volume, local boundedness, equicontinuity, extraction sequence, compactness. Their meanings are in the A-02 glossary.

Next chapter to read: 12-04 The transfer operator and equicontinuity — the difference of correlations when an insertion is moved is bounded by the trace representation of the transfer operator and Hölder's inequality.

The transfer operator and equicontinuity

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

What this chapter shows — When one insertion is moved ℓ steps on the lattice, how much does a correlation at separated points change? The published construction bounds this in the form $\left(\prod_i c_i\right) \ell/r$ (r is the margin between points, c_i the upper bound on the chessboard root of each insertion). The tools are positivity of the lattice transfer operator, the representation of expectations as traces of products of powers of the transfer operator and insertion operators, re-assigning gaps, and Hölder's inequality. This chapter reads the form of that statement and the mechanism. Volume-uniform equicontinuity becomes the second input to the Arzelà–Ascoli argument (12-05).

Chapters assumed — 12-01 (positivity of the transfer-operator kernel), 12-02 (roots), 12-03 (upper bounds on roots). The introduction to the transfer matrix is 5-03.

Said first — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean", and what this book has read goes as far as the form of the statements. The proof of the body of the gap re-assignment estimate (`TimeKernel.passage`) has not been read. In the construction, the lattice transfer operator is used only for equicontinuity; analytic continuation in the time direction and the spectral condition are obtained separately from the semigroup on the continuum side of the limit (12-07).

The order of this chapter

1. Legend of labels
2. Statement (1) — the transfer-operator kernel and its positivity
3. Statement (2) — estimate of the difference when gaps are re-assigned
4. Outline of the mechanism — trace representation, re-assignment, Hölder
5. The form that moves one insertion, and assembling equicontinuity
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

Statement (1) — the transfer-operator kernel and its positivity

The introduction is 5-03. Choose one time direction a (any of the four axes) and cut the lattice into layers perpendicular to direction a. The configuration of links within one layer is called a **slice** X, and the set of links in direction a joining adjacent layers a **thread** u (12-01 § 03). Let $\mathrm{weight}_\beta(X, Y, u)$ be the plaquette weight of two adjacent slices X, Y and the thread u between them; integrating the thread over Haar measure,

$$K_\beta(X, Y) \;=\; \int \mathrm{weight}_\beta(X, Y, u)\, du \qquad (1)$$

is the kernel of the one-step transfer operator Source (construction source `YM/Slice/Positive.lean:54` `heat`). The weight is positive (`weight_pos` in `YM/Slice/Hamilton.lean`).

Statement (properties of the transfer-operator kernel)

(i) If $\beta \geq 0$, the kernel K_β is a positive kernel: for any finitely many slices $X_1, \ldots, X_k$, the matrix $\big(K_\beta(X_i, X_j)\big)$ is positive semi-definite. (ii) Reversing the direction of the thread swaps the arguments of the kernel (adjoint under time reversal).

Source Theorems of the published construction. Lean: `YM.Slice.step_positive` (`YM/Slice/Positive.lean:81`, hypothesis $0 \leq \beta$), `YM.Slice.transition_reflect` (`YM/Slice/Hamilton.lean:121`).

From (i) and (ii), the operator T that integrates against the kernel is a positive self-adjoint operator on L^2 of the space of slices Proved here (symmetric by (ii), positive by (i)). The operators of insertions (observables) can also be written as multiplication by functions of a slice, or as "blocks" spanning several layers (`Brick`: an operator that inserts a function local to a band of layers of width w between products of transfer operators) Source (the name and usage of `Brick`; its full definition has not been read).

02

Statement (2) — estimate of the difference when gaps are re-assigned

On the time circle (length n), place blocks $B_1, \dots, B_k$ of width w in this order, and let the gaps between blocks be $Y_1, \dots, Y_k$ (numbers of layers) ($kw + \sum_i Y_i = n$). Write the expectation of the product for this arrangement as $\langle B; Y \rangle_\beta$.

The **column** of a block B_i is the arrangement in which B_i and its mirror image ϑB_i alternate, p pairs, filling the time circle ($n = 2pw$) Source (`YM/Slice/Calendar.lean:60` `column`). A column is the chessboard copy in one direction, the time direction.

Statement (Lipschitz estimate for re-assigning gaps)

Let $\beta \geq 0$, $n = 2pw$, all gaps larger than $2r$ ($Y_i > 2r$), and let the gap tuple Z be obtained from Y by moving ℓ layers between two gaps $i \neq j$ ($Z_i = Y_i + \ell$, $Z_j = Y_j - \ell$, others unchanged), with $\ell \leq r$. If the power of the size of the column expectation $\| \langle \mathrm{column}(B_i) \rangle \|^{\rho_p}$ ($\rho_p = 1/(2p)$; `Holder.rate p` in the source, `YM/Complex/Gaps.lean:105`) is at most K_i, then

$$\left| \langle B; Z \rangle_\beta - \langle B; Y \rangle_\beta \right| \;\leq\; \frac{\prod_i K_i}{r}\, \ell. \quad (2)$$

Where on the time circle the sequence of blocks starts to be read (the base position) may differ between the two sides (the source reads the two words with separate base positions `b`, `c`).

Source A theorem of the published construction. Lean: `YM.Slice.Time.powering` (`YM/Slice/Calendar.lean:72`). The body of the estimate is `YM.TimeKernel.passage` (`YM/TimeKernel/Unit.lean:76`).

Meaning. Shortening the distance between two insertions by ℓ and widening the distance between two other insertions by the same amount changes the correlation only by an amount proportional to ℓ/r. The constant of proportionality is the product of the per-box sizes when each insertion is replicated in the time direction (powers of columns). r is the margin between insertions; the larger the margin, the smaller the difference.

03

Outline of the mechanism — trace representation, re-assignment, Hölder

Stage 1: write the expectation as a trace

The expectation on a periodic lattice is the trace of the product of operators going once around the time circle, divided by the partition function Source (the source's docstring and the definitions of `closed` and `along`; `YM/TimeKernel/Unit.lean:17`):

$$\langle B;Y\rangle_\beta \;=\; \frac{\operatorname{Tr}\bigl(B_1 T^{Y1} B_2 T^{Y2}\cdots B_k T^{Yk}\bigr)}{\operatorname{Tr} T^{\,n}}. \tag{3}$$

Numerator and denominator change in the same ratio when T is multiplied by a positive constant, so one may normalise $\|T\| = 1$ Proved here .

Stage 2: the difference from re-assigning gaps is of order ℓ/r

Since T is positive and $\|T\| = 1$, for an eigenvalue $\lambda \in [0,1]$ and $s \geq Y > 2r$,

$$\Bigl|\frac{d}{ds}\lambda^s\Bigr| \;=\; \lambda^s\,|\log\lambda| \;\leq\; \frac{1}{e\,s} \;\leq\; \frac{1}{2e\,r} \tag{4}$$

holds Proved here (the maximum of $t \mapsto te^{-t}$ is $1/e$). Moving a gap by ℓ replaces the two powers T^{Yi}, T^{Yj} by $T^{Yi+\ell}$, $T^{Yj-\ell}$, and by (4) the change per power is of order ℓ/r. We infer this is where the ℓ/r of (2) comes from Inferred .

Stage 3: split the remaining factors into columns by Hölder

The difference in the numerator of (3) is still a trace of the product of all blocks and powers. The trace Hölder inequality splits it into a product of "Schatten-type norms" of each block. Raising the Schatten norm of block B_i together with the gap powers to the power $2p$ gives the trace of the product with B_i and its adjoint alternating, that is, the column expectation (times the partition function) Inferred (from the source's docstring, "Read replica column strengths $\operatorname{Tr}((B^*B)^p)$"). So the constant becomes the product of the powers of the column expectations, K_i.

According to the source's docstring, the kernel is approximated by finite matrices (`Quad.shrink`), the estimate is done with the approximating matrices, and then the limit is taken. The partition function is bounded below by its actual value, and no assumption about the ground state is made Source (docstring).

Stage 4: columns are bounded by roots

Since a column is a copy in one direction, the time direction, the column expectation is bounded by the chessboard root (12-02) Source (`YM/Extraction/Difference.lean:51`

`reading`: if the root of an observable and its mirror image in direction a is at most c, a block whose column power is at most c can be built). In this way the K_i of (2) can be replaced by the root bound $A\,a_m^{-d-\varepsilon}$ of 12-03.

04

The form that moves one insertion, and assembling equicontinuity

Statement (estimate for moving one insertion in one direction)

On the lattice of the construction's family with cutoff N and volume p, place $m+1$ renormalised insertions at points $x_0,\dots,x_m$, whose coordinates in direction a are pairwise more than $2h\square + 2r$ apart ($h\square$ is the number of lattice points along a side of the centred box; the source's letter is `d`, a different quantity from the degree of a field). If the root of each insertion and its mirror image in direction a is at most c_i, then the difference of the $(m+1)$-point function when one point x_u is moved $\ell \le r$ steps in direction a is at most $\left(\prod_i c_i\right)\ell/r$.

Source A theorem of the published construction. Lean: `YM.Extraction.Difference.primitive` (`YM/Extraction/Difference.lean:90`). An intermediate form is `YM.Slice.Time.organized` (`YM/Slice/Ordered.lean:68`).

Why the coordinates in that direction must be separated. The representation (3) uses the fact that the insertions are lined up in order in the time direction a. If two insertions have close coordinates in direction a, they fall in the same band of layers and cannot be split into blocks. So the equicontinuity statement is made on configurations whose coordinates differ pairwise along every coordinate axis (`generic`)Source. By the hypercubic symmetry of the lattice, the time direction can be taken along any of the four axes, so combining moves in the four directions gives a Lipschitz estimate in all directionsInferred.

Definition (the equicontinuity input `TransferLip`)

(i) With the volume p fixed: on a compact set K of configurations with no coincident points, including periodic images, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for sufficiently fine cutoffs, the difference of lattice correlations at two configurations in K at distance less than δ is less than ε. (ii) With the volume also moving: on a compact set K of configurations whose coordinates differ along every axis, the same holds uniformly for sufficiently late volumes and cutoffs.

Source Construction source `YM/Extraction/Lipschitz.lean:26` `TransferLip` (fields `grid`, `expanding`).

Source That `TransferLip` follows from the bundle of root bounds (`CenteredBook`) and the construction's certificate is a theorem of the published construction (Lean: `YM.Extraction.Difference.completed`, `YM/Extraction/Equicontinuity.lean:181`). This corresponds to the second part of equation (11) of the assessment (a uniform modulus of continuity ω_K on compact sets)Assessment .

The role of the lattice transfer operator ends here. In the construction, the lattice transfer operator is used for equicontinuity. The time evolution e^{-tH} of the limiting theory, the spectral condition and analytic continuation are handled in the continuum-side OS space built from reflection positivity of the limiting table (12-07, 12-08)Inferred (a reading from names and usage).

05

What this book's study had seen

Equicontinuity via the transfer-operator representation did not exist in this book's study. This book's transfer operator is the abstract operator T in the vessel of reconstruction (the vessel of 19-03), built by OS reconstruction from the infinite-volume limiting measure inside the strong-coupling window (20-04). What was shown there is $T \geq 0$ and $\|T - P_\Omega\| \leq e^{-m}$ (the form of the mass gap); it was never used for the modulus of continuity of correlations.

Lean This book's $T \geq 0$: for $\beta \geq 0$ and in the Dobrushin window, the transfer operator of the vessel at a limit point is positive (Lean: `FullT.wilson_Tpos_full`; its source is `FullT.full_rp_torus`, which extends reflection positivity for link-midpoint planes by averaging over gauge transformations). It is a formalisation of the Osterwalder–Seiler type, with the lattice spacing fixed at 1 and only inside the strong-coupling window.

(2) in the published construction is an estimate for the continuum-limit problem in which point positions move continuously; in this book's setting with lattice spacing 1, point positions are discrete and the question of equicontinuity did not even arise. In this book's conditional chain theorem (28-01), the modulus of continuity of observables was placed as an **assumption** called "the splitting of observables" (Lean fields `split`, `lattice_vanishing`, `shift_vanishing`). The published construction **derives** it from the transfer operator and the root bounds.

06

What has not been read

- **The proof of `TimeKernel.passage`** (`YM/TimeKernel/Unit.lean:76`). Stages 2 and 3 of § 03 are inferences from the docstrings and the form of the statements. We checked as far as the Hölder exponent `Holder.rate p` = $1/(2p)$ (`YM/Complex/Gaps.lean:105`).
- **The full definition of the block `Brick`**, and how block operators enter the trace.
- **Approximation by finite matrices** (`Quad.shrink`): how it is organised, and how it passes to the limit.
- **How the four directions are combined.** The assembly from `primitive` to `completed` (`stepTotal` in `Equicontinuity.lean` and so on) has been read only as far as the statement names.
- **The chain to the public theorem.** In the record, the direct user of `completed` is `crops`. The road from `crops` to the public theorem has been traced only as far as written in 12-05.

What this chapter can say

Can say	Cannot say
Source The lattice transfer-operator kernel is positive for $\beta \geq 0$ and adjoint under time reversal (`step_positive`, `transition_reflect`)	$\beta < 0$. Quantitative estimates on the spectrum of the kernel
Source Difference from re-assigning gaps by $\ell \leq (\prod K_i)\,\ell/r$ (`Time.powering`)	When there is no margin r (points close together)
Source The estimate for moving one insertion in one direction (`Difference.primitive`) and the equicontinuity input `TransferLip` (`Difference.completed`)	Estimates for configurations whose coordinates in a direction coincide
Proved here Normalisation, and $\lvert d\lambda^{s}/ds\rvert \leq 1/(es)$	—
Inferred The difference estimate has two stages: "derivative of a power $\sim 1/r$" and "trace Hölder down to columns"	How the proof of `passage` is actually organised

Sources and reproduction

Item	Kind	Source / tool
Transfer matrix and reflection positivity	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. 5-03

Item	Kind	Source / tool
Kernel and positivity	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Slice/Positive.lean:54` `heat`, :81 `step_positive`, `YM/Slice/Hamilton.lean:121` `transition_reflect`
(2) and its body	Source	`YM/Slice/Calendar.lean:60` `column`, :72 `Time.powering`, `YM/TimeKernel/Unit.lean:17` `along`, :76 `passage` (statements and docstrings only)
Estimate for moving one insertion, and assembly	Source	`YM/Slice/Ordered.lean:68` `organized`, `YM/Extraction/Difference.lean:51` `reading`, :90 `primitive`, `YM/Extraction/Equicontinuity.lean:181` `completed`, `YM/Extraction/Lipschitz.lean:26` `TransferLip`
Description in the assessment	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` § 5, equation (11)
This book's $T \geq 0$	Lean	`FullT.wilson_Tpos_full`, `FullT.full_rp_torus` (20-04)

Words used in this chapter — transfer operator, slice, thread, positive kernel, column, trace representation, re-assigning gaps, Hölder's inequality, equicontinuity, configurations whose coordinates differ along every axis. Their meanings are in the A-02 glossary.

Next chapter to read: 12-05 Arzelà–Ascoli, and the properties that pass to the limit — from boundedness and equicontinuity, limiting correlations are extracted by a two-stage subsequence.

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction The infinite-volume limit · Kiichi, Shiori and Rin

Arzelà–Ascoli, and the properties that pass to the limit

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 05 and § 07 are a map of open questions.

What this chapter shows — From local boundedness (12-03) and equicontinuity (12-04), the published construction extracts limiting correlations by a two-stage subsequence: the first step is a subsequence of cutoffs common to all volumes, the second step a subsequence of volumes. The cutoff subsequence is finally returned to the whole sequence using convergence on the ultraviolet side. Reflection positivity, the symmetry of the lattice hyperoctahedral group and translation invariance pass to the limit. This chapter reads that assembly, and why uniqueness of the limit and convergence along the whole sequence cannot be stated.

Chapters assumed — 12-03, 12-04, 8-05 (Correlation functions and the order of limits). The Arzelà–Ascoli theorem is functional analysis within the scope of 1-09.

Said first — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean", and what this book has checked goes only as far as reading the form of the statements. The infinite-volume limit is taken **only along a subsequence of volumes**; it has not been shown that the limit is independent of the choice of subsequence (uniqueness), nor that it converges along the whole sequence of volumes. The assessment also lists this as the first of its open questions.

The order of this chapter

1. Legend of labels
2. Statement — the form of the subsequential limit
3. Outline of the mechanism (1) — two-stage Arzelà–Ascoli
4. Outline of the mechanism (2) — returning the cutoff subsequence to the whole sequence
5. Properties that pass to the limit
6. Why uniqueness and convergence cannot be stated
7. What this book's study had seen
8. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

Statement — the form of the subsequential limit

Write $S_p(v;x)$ for the kernel of correlations with the volume p fixed and the cutoff $N \to \infty$ (8-05). $x = (x_1, \dots, x_n)$ is a configuration of mutually distinct points, and v a sequence of field labels.

Statement (infinite-volume limit along a subsequence, Theorem 5.1 of the assessment)

For any strictly increasing sequence of volume indices $p(k)$, there are a strictly increasing $q(k)$ and a kernel $S^{\mathrm{mix}}(v;x)$, smooth on configurations with no coincident points, such that for every sequence of labels v and every smooth compactly supported test function f supported away from coincident points,

$$\lim_{k\to\infty} S_{p(q(k))}(v;f) \;=\; \int f(x)\, S^{\mathrm{mix}}(v;x)\, dx. \quad (1)$$

A single subsequence q works for all correlations in common.

Assessment Theorem 5.1 in § 5 of the assessment. Source This corresponds to the field `wilson` (`C.Converges (p∘q) …`) of the public theorem `JointExistence` (construction source `ChallengeSpec.lean:186`), "for any strictly increasing p there are a strictly increasing q and a table T with $T.\mathrm{Properties}(p \circ q)$". `Converges` (`PublicWilson.lean:85-90`) has the form "for every volume in which the test function fits, the limit $\mathrm{row}(p)$ along the **whole sequence** of cutoffs exists, and $\mathrm{row}(p(q(k)))$ converges to $\int f\, S^{\mathrm{mix}}$".

The "mix" in the limit symbol S^{mix} is a name anticipating that it will later be decomposed as a mixture of phases (12-07) Assessment .

02

Outline of the mechanism (1) — two-stage Arzelà–Ascoli

Extraction sequences. The construction holds a pair of a cutoff subsequence mesh and a volume subsequence per as an **extraction sequence** (`Extraction.Order`, `YM/Extraction/Orders.lean:34`), and refinement (`refine`, same file :40) composes each of the two subsequences Source . The meaning of the limit (`limits`, same file :61) is two ordinary limits: "for sufficiently late volumes, with each volume fixed, the limit $v(p)$ along mesh exists, and $v(\mathrm{per}(k))$ converges to the limit".

Statement (limit along a common subsequence)

For an extraction sequence f, if local boundedness (`Bound`) and equicontinuity (`TransferLip`) hold, then there are a refinement g, a limiting table S and per-volume tables P_p such that: (i) the readings of S are continuous on configurations whose coordinates differ along every axis; (ii) for each volume, the lattice correlations approach P_p along the cutoff subsequence of $f \circ g$, uniformly on compact sets with no coincident points (including periodic images); (iii) $P_{\mathrm{per}(k)}$ approaches S uniformly on compact sets; (iv) the smeared readings of S are double limits of the smeared lattice correlations.

Source A theorem of the published construction (of Arzelà–Ascoli type). Lean: `YM.Extraction.first_surface` (`YM/Extraction/Lipschitz.lean:166`; conclusion `RawLimits`, same file :156). What calls it from the construction's inputs is `YM.Extraction.crops` (`YM/Extraction/Held.lean:136`).

The proof has two stages Source (statements of the lemmas and stages of the proof).

1. **Cutoff subsequence (common to all volumes).** The pairs of a volume p and a sequence of labels are countably many. For each pair, the family of lattice correlations in volume p is bounded (`Bound.cyclic`) and equicontinuous (`TransferLip.grid`) on compact sets with no coincident points. Applying Arzelà–Ascoli simultaneously to the countably many families gives a cutoff subsequence t common to all volumes and per-volume limits P_p (`setMeshes`, `Lipschitz.lean:78`).
2. **Volume subsequence.** Now consider the family of P_p (with p varying) on configurations whose coordinates differ along every axis. Boundedness (`Bound.volume`) and equicontinuity (`TransferLip.expanding`) are uniform for sufficiently late volumes, so applying Arzelà–Ascoli once more gives a volume subsequence u and a limit S (`localAscoli`, `YM/Extraction/Tame.lean:100`).

`localAscoli` is a lemma that, for a family of functions with countably many indices, chooses a subsequence and limit common to all indices from pointwise boundedness and control of "oscillation" (the local form of equicontinuity, `oscillation`). Its content is the classical line of taking a diagonal subsequence on a countable dense set and extending to everything by equicontinuity Cited (the Arzelà–Ascoli theorem). What we read in the source is the statement of the lemma and the shape of the proof using `simultaneous` and `growFrom` Source .

Note that the limiting kernel is taken **as a table of correlation functions, not as a measure**. The assessment lists defining a continuum probability measure as open question 2 Assessment .

03

Outline of the mechanism (2) — returning the cutoff subsequence to the whole sequence

A cutoff subsequence was taken in the first step of § 02, but the public statement's `Converges` requires, for each volume, a limit along the **whole sequence** of cutoffs (§ 01). The construction pays for this with convergence on the ultraviolet side Source .

- The ultraviolet side shows, with the volume fixed, that the lattice kernels converge along the whole sequence of cutoffs (including the sequences of differences), from the Cauchy property of the kernels (`PositionsUV C e`, `YM/FullVolume/Profiles.lean:25`; `Sequence.full` when the sequence e is the whole sequence; Part 11, 11-10).
- From the final certificate built with the subsequence f and `PositionsUV C full`, it follows that, for the extraction sequence "cutoffs: the whole sequence; volumes: the volume subsequence of f", the limiting table can be read as a double limit (`from_spec`, `YM/FullVolume/Finale/Program.lean:117`).

For a sequence with a unique limit, a limit taken along a subsequence agrees with the limit of the whole sequence. Since the ultraviolet limit at fixed volume exists along the whole sequence, the cutoff subsequence can be removed at the end Proved here . What cannot be removed is the volume subsequence.

Which target it enters. In the field `wilson` of the public theorem too, the ultraviolet (cutoff) is the whole sequence and infinite volume is the subsequence $p \circ q$. The strong form with kernels and differences, `PositionsUV … Sequence.full`, is only in the conclusion of the construction theorem `constructed`, not in the public theorem Source (6-02).

04

Properties that pass to the limit

Properties that hold exactly on finite lattices are preserved in the limit as non-negativity or equalities. The construction transfers the following four Source .

Property	Lattice side	Limit side	Lean (construction source)
Reflection positivity	Reflection positivity for planes through lattice sites (coordinate 0, four axes) (12-01)	The Gram matrix of smeared words supported at positive time is positive semi-definite (`RPSmear`)	`Order.realPositive` (`YM/Extraction/Harvest.lean:46`). The lattice plane is `Banks.plane` (`Banks.lean:19`: kind site, base 0)
Discrete symmetry	Covariance and reality under the lattice hyperoctahedral group (permutations and signs of coordinates)	`RealSpin`	`Order.spin` (`YM/Extraction/Inherited.lean:124`)
Translation invariance	Invariance under lattice translations	Invariance under continuum translations (`Homogeneous`)	`Order.homogeneous` (`Inherited.lean:83`)
Basic properties	Symmetry, the 0-point function is 1, the unit field can be removed	`Basic`	`Order.limits_basic` (`Inherited.lean:151`)

All four take smoothness of the limiting table (`S.Smooth`) as a hypothesis (`realPositive` also symmetry, `S.Symmetric`) Source .

How reflection positivity is transferred. Reflection positivity on the lattice side is the same as positive semi-definiteness of the Gram matrix $(\langle F_j F_i \circ \Theta \rangle)$ for linear combinations $\sum_i c_i F_i$ of finitely many words (12-01 § 02). Since each entry converges under the double limit in cutoff and volume, the limiting Gram matrix is also positive semi-definite, as a limit of positive semi-definite matrices Proved here (the source's `posVectors`, and the limit of sums). The continuum-side definition of reflection positivity, `Table.RP` (`YM/Euclid/Words.lean:123`), is that for every coordinate axis a, the Gram matrix of families of positive-time words is positive semi-definite; together with translation invariance it holds for every plane parallel to a coordinate plane Source .

The bundle of the limiting table. Adding smoothness away from coincident points (`Smooth`) and a growth condition (`Capped`) to the four properties gives the published construction's version of the bundle of Euclidean axioms, `Table.Hierarchy` (`Words.lean:141`) (`Order.assembles`, `Harvest.lean:62`) Source . `assembles` takes smoothness and the growth condition as **hypotheses**. Where they come from has been

read only as far as written in 12-08. Invariance under continuum rotations (O(4)) is not in this bundle (12-06).

05

Why uniqueness and convergence cannot be stated

The inputs of § 02 are **only upper bounds**: boundedness and equicontinuity. Compactness gives existence of subsequential limits, but not that the limit is unique Proved here . To obtain uniqueness of the limit or convergence along the whole sequence, statements of kinds such as the following are needed: the difference of correlations when the volume changes becomes small (a Cauchy property in the volume), volume-uniform decay of correlations, disappearance of dependence on boundary conditions, uniqueness of DLR type. Within what we read, none of these is among the inputs Source (the types of the inputs, and within the scope of the name search).

The source's `Order.unique` (`YM/Extraction/Orders.lean:70`) says only that "two limits along the same extraction sequence are equal" (uniqueness of limits in a Hausdorff space), not independence from the choice of subsequence. In the record of declaration usage, the status of this declaration is `outside` (not reached from any of the 17 headline theorems) Source .

In § 5 the assessment states that "whether the subsequential limit depends on the chosen subsequence is unresolved", and lists as open question 1 in § 8 "convergence rather than a subsequence; showing uniqueness of the infinite-volume limit" Assessment . The same § 5 also writes that the Jaffe–Witten problem statement regards a subsequential limit obtained by compactness as insufficient unless there is a separate argument establishing the properties of the limit Assessment .

06

What this book's study had seen

(1) The implication from subsequential limits and transfer of properties. This book's conditional chain theorem (28-01) included a part that takes subsequential limits by compactness for a family of push-forward measures and transfers to the limit reflection positivity, invariance, lower bounds on the mass of closed sets, and the modulus of continuity of observables.

Lean Machine-checked as an implication (hypothesis fields as in the table of 28-01). Lean: `BoxWalk.chain_end_to_end_box` , `ChainBlockMap.chain_end_to_end_blockMap` .

The **skeleton points in the same direction**: subsequential limits and transfer of properties. The difference is that this book took the limit **as measures** (a family of

probability measures on a compact configuration space always has a convergent subsequence) and put boundedness and the modulus of continuity in **hypothesis fields**, whereas the published construction takes the limit **as tables of correlation functions** and **derives** boundedness and equicontinuity from root bounds and the transfer operator (12-03, 12-04).

(2) A limit without subsequences inside the strong-coupling window. This book has the infinite-volume limit without subsequences inside the strong-coupling window.

Lean For $0 < \beta$, $18 \tanh \beta < 1$, the sequence of periodic lifts of the SU(2) Wilson measures on $(\mathbb{Z}/2(n+1))^4$ converges weakly, without subsequences, to a single probability measure ν_β on $\mathbb{Z}^4$, and the sequence over all sizes has the same limit. A formalisation of the Osterwalder–Seiler strong-coupling and Dobrushin-uniqueness type. Lean: `InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN` (20-03). The statement bundling translation invariance, reflection positivity and the mass gap into one is `InfMain2.wilson_infvol_main2` (20-04).

This is a stronger form than the published construction's subsequential limit, in that it has convergence and uniqueness. But it concerns **a different regime**: the lattice spacing is fixed at 1, the coupling is only inside the strong-coupling window (about $|\beta| < 0.0556$), and no continuum limit is taken. The published construction carries every step at weak coupling, takes the continuum limit with lattice spacing $a_N \to 0$, and then goes to infinite volume. The two differ in object and in order of limits; what can be compared is only the pattern of "what kind of input gives uniqueness" (in this book, volume-uniform decay from the Dobrushin condition).

(3) Twisted boundary conditions. The lattices of the published construction have only periodic boundary conditions Source . This book has partition functions with twisted boundary conditions and upper and lower bounds on the vortex free energy (Part 21). These are statements on lattices with fixed spacing, and do not overlap with the infrared components of the published construction.

07

What has not been read

- **Details of the proof of the Arzelà–Ascoli lemmas** (`simultaneous`, `growFrom`, `components`).
- **The chain to the public theorem.** In the record, the direct user of `crops` is `MergeProof.Accent.rated`, and the record's headline is `SharpGram.habitat`. Literally, `fromFunded` on the construction's chain (`YM/R2I/Campaign/Accepted.lean:150`) calls `Run.extractible17` (`YM/RG17/Joint.lean:61`) at :157, which uses the field `reception` of the structure `Chateau` (`YM/MergeProof/Almandine.lean:31`), and `rated` is called

inside the proof (same file :79) of the lemma `grantor` (same file :40) that fills that field; so it is likely on the road to the public theorem Inferred . The chain has not been checked declaration by declaration.

- **Where smoothness and the growth condition come from.** Who pays for the hypotheses `Smooth` and `Capped` of `assembles` has not been traced.
- **That there is no Cauchy property in the volume.** This is a judgement from the types of the inputs and a name search; we have not read the whole repository.

What this chapter can say

Can say	Cannot say
Source From boundedness and equicontinuity, a limit along a common subsequence by two-stage Arzelà–Ascoli (`first_surface`, `crops`)	Uniqueness of the limit. Convergence along the whole sequence of volumes
Source The cutoff subsequence is removed at the end by convergence along the whole sequence on the ultraviolet side (`from_spec`, `PositionsUV … full`)	Removing the volume subsequence
Source Reflection positivity, hyperoctahedral symmetry, translation invariance and the basic properties pass to the limit (`realPositive`, `spin`, `homogeneous`, `limits_basic`)	Invariance under continuum rotations (separately in 12-06). A limiting probability measure
Assessment The assessment: subsequential limit; dependence on the choice is unresolved (§ 5, item 1 of § 8)	—
Lean This book's limit without subsequences inside the strong-coupling window (`wilson_infvol_unique`, `wilson_infvol_allN`)	Varying the lattice spacing. Weak coupling. The continuum limit

Sources and reproduction

Item	Kind	Source / tool
The Arzelà–Ascoli theorem	Cited	A standard theorem of functional analysis (1-09)
Extraction sequences and the two-stage limit	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Extraction/Orders.lean:34` `Order`, :40 `refine`, :61 `limits`, :70 `unique`, `YM/Extraction/Lipschitz.lean:78` `setMeshes`, :156 `RawLimits`, :166 `first_surface`, `YM/Extraction/Tame.lean:100` `localAscoli`, `YM/Extraction/Held.lean:136` `crops`
Returning to the whole sequence	Source	`YM/FullVolume/Profiles.lean:25` `PositionsUV`, `YM/FullVolume/Finale/Program.lean:117` `from_spec`, `PublicWilson.lean:85-90` `Converges`, `ChallengeSpec.lean:186` `JointExistence`

Item	Kind	Source / tool
Properties that pass to the limit	Source	`YM/Extraction/Harvest.lean:46` `realPositive`, :62 `assembles`, `YM/Extraction/Banks.lean:19` `plane`, `YM/Extraction/Inherited.lean:83` `homogeneous`, :124 `spin`, :151 `limits_basic`, `YM/Euclid/Words.lean:123` `Table.RP`, :141 `Table.Hierarchy`
Description in the assessment	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` § 5 (Theorem 5.1, equations (10) and (11)), § 8
This book's counterparts	Lean	`BoxWalk.chain_end_to_end_box`, `ChainBlockMap.chain_end_to_end_blockMap` (implication, 28-01), `InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN` (20-03), `InfMain2.wilson_infvol_main2` (20-04)

Words used in this chapter — extraction sequence, cutoff subsequence, volume subsequence, Arzelà–Ascoli theorem, equicontinuity, subsequential limit, reflection positivity, hyperoctahedral group, translation invariance, uniqueness. Their meanings are in the A-02 glossary.

Next chapter to read: 12-06 Euclidean invariance — how the discrete symmetries of the lattice are lifted to invariance under continuum rotations.

Euclidean invariance

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 07 is a map of open questions.

What this chapter shows — The lattice breaks rotational symmetry and keeps only a finite group made of coordinate permutations and sign flips (the hyperoctahedral group, with 384 elements). The committee's assessment is that the published construction shows, with three components, that the limiting theory is invariant under O(4), the group of continuous rotations and reflections: (1) keep the discrete symmetries of the lattice, gauge fixing included, and carry them to the limit; (2) build a divergence-free stress tensor and set up the Ward identities for translations and rotations; (3) show that the rotational flux vanishes far away, so that rotational invariance is not spontaneously broken. This chapter reads the **statements** of the three components and writes out why O(4) invariance follows from them. We have not read the **proof** of (3).

Prerequisite chapters — 12-05 § 04 (properties that pass to the limit), 13-01 (uniqueness and normalisation of the stress tensor), 9-05 (symmetries of the single step).

Said first — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean". This book has read the **definitions and the form of the statements** for the stress tensor, the Ward identities and rotational invariance. The proof of the vanishing of the flux (`Outer.outer`) is split across about 40 files, one group per coordinate axis, and we have not read its content. The steps of the argument in § 05 that cannot be read directly off the statements carry Inferred .

The order of this chapter

1. Key to the labels
2. What the problem is — the symmetry left on the lattice is a finite group
3. Component (1) — keep the discrete symmetries and carry them to the limit
4. Component (2) — the stress tensor and the Ward identities
5. Component (3) — the local rotational flux vanishes for configurations on a line

6. Outline of the mechanism — from the three components to O(4) invariance
7. What this book's investigation had seen
8. What we have not read

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

What the problem is — the symmetry left on the lattice is a finite group

The OS axioms (4-10) require the Euclidean correlation functions to be invariant under the group of translations, rotations and reflections $E(4) = \mathbb{R}^4 \rtimes O(4)$. The lattice $a\mathbb{Z}^4$ preserves only the lattice translations and the hyperoctahedral group of coordinate permutations and sign flips (of order $2^4 \cdot 4! = 384$). Translation invariance returns in the continuum limit (12-05 § 04), but rotational invariance does not return automatically. That the limit is rotation invariant is a property that has to be proved, and one must also rule out that rotational invariance is **spontaneously broken** in the limiting state.

Assessment Assessment § 6.3: the proof has three main parts. The first is that the gauge fixing preserves the lattice symmetries (axial gauge fixing, for example, does not), which is achieved by averaging over the symmetry-breaking choices. The second is to show that there is a divergence-free stress tensor field and to use it to set up local Ward identities. Finally, one shows that Euclidean invariance is not spontaneously broken in the limiting state. This follows if the stress-energy flux falls off fast enough at infinity, and it is shown by an involved argument using reflection positivity in several orthogonal planes.

02

Component (1) — keep the discrete symmetries and carry them to the limit

Symmetries of the single step. The kernel of one renormalisation step is invariant (covariant) under the signed permutations of the hyperoctahedral group (384 of them), the translations of the coarse lattice and the gauge transformations of the fine lattice.

The choice of gauge for large fields breaks the symmetry, but by drawing the choice at random and averaging, the symmetry of the kernel as a whole is preserved (9-03, 9-05). This part reads on that basis.

Carrying to the limit. The hyperoctahedral covariance and the reality of the lattice pass to the property `RealSpin` of the limiting table Source (construction source `YM/Extraction/Inherited.lean:124` `Order.spin`). `RealSpin` is the field stating that the limiting correlations do not change when an element of the hyperoctahedral group moves the points and the field labels together (a representation of the group acts on the field labels), and stating reality Source (from its name and how it is used; it is defined as the structure `RealSpin` in `YM/Euclid/Table.lean:80`, which we have not read in full).

03

Component (2) — the stress tensor and the Ward identities

We use the field spaces F_d (local fields of degree d, 8-03). A **stress tensor** is a family that assigns a field $T_{ab} \in F_4$ of degree 4 to each pair of indices $a, b \in \{0,1,2,3\}$ Source (`Stress S` in `YM/Euclid/Stress.lean` $= \mathrm{Axis} \to \mathrm{Axis} \to F_4$).

Definition (conserved stress tensor)

(i) symmetric, $T_{ab} = T_{ba}$; (ii) real (invariant under conjugation of fields); (iii) divergence zero: $\sum_a \partial_a T_{ab} = 0$ (∂_a is the derivative on the field spaces, $F_4 \to F_5$). In addition, it transforms as a rank-two tensor under the rotation action R (`Tensorial`).

Source Construction source `YM/Euclid/Stress.lean:59` `Conserved` (fields `symmetric`, `real`, `divergence`), :64 `Tensorial`. The action of rotations on fields, `Rotation` (:22, fields `act`, `lie`), and its laws `Rotation.Laws` (:31: continuity, group action, agreement with the lattice reflections for signed permutations, reality, preserving the unit field, compatibility with derivatives, the infinitesimal generator `lie`).

The uniqueness of the stress tensor (the space of conserved symmetric tensors is one-dimensional over the reals) and its normalisation are treated in 13-01 (Lean: `YM.Pert.one_line`, `YM/Pert/Weak.lean:68`) Source.

Windows. For a configuration of points $x = (x_0, x_1, \ldots, x_n)$ (pairwise distinct), a function χ is a **window** if it is smooth with compact support, equal to 1 near x_0, and its support contains none of the other points $x_1, \ldots, x_n$ Source (`Window` in `Stress.lean`).

$\partial\chi$ is supported away from all the insertion points. A window is a smoothed version of a surface enclosing x_0.

Form of the statement (Ward identities for translations and rotations)

Put a central field $B \in F_d$ at x_0 and a list q of spectator fields at $x_1, \dots, x_n$, and let χ be a window.

$$-\sum_a \int \partial_a \chi(y)\, \langle T_{ab}(y)\, B(x_0)\, q \rangle \, dy \;=\; \langle (\partial_b B)(x_0)\, q \rangle, \quad (1)$$

$$-\sum_{a,b} \int \partial_a \chi(y)\, (g\,(y - x_0))_b \, \langle T_{ab}(y)\, B(x_0)\, q \rangle \, dy \;=\; \langle (\mathrm{lie}_g B)(x_0)\, q \rangle. \quad (2)$$

(1) is the identity for translations and (2) the identity for an infinitesimal rotation g (an antisymmetric matrix). lie_g is the derivative of the rotation action on fields.

Source Construction source `YM/Euclid/Stress.lean:90` `StressLaws` (fields `translation`, `rotation`); the left-hand sides are `shellGrad` and `shellSpin` in the same file. As properties of the limiting table S they enter the construction's bundle of certificates `Spec.Locals` (`YM/Spec/Closed.lean`, fields `conserved`, `spin`, `charges`), and for the tables of all phases, including the chosen one, they enter the certificate `OS.Selection.Certificate` (`YM/Phase/Mixture.lean:136`, field `flux`).

(1) is a form of Gauss's theorem in which the step of χ plays the role of a surface enclosing x_0. Since $\partial\chi$ vanishes near the insertion points, the integral does not touch coinciding points. The committee's example check confirms that the right-hand side of (1) agrees with the ordinary coordinate derivative, that it holds in every phase and for every field, that the integrand is integrable, that it does not depend on the choice of χ, and that admissible χ actually exist Assessment (on the assessment side, `submissions/openai-20261028/docs/examples/TranslationWard-20261028.lean`). The same check states that it does not confirm that all improvement terms vanish, nor the contact terms, nor the dilation Ward identity Source .

04

Component (3) — the local rotational flux vanishes for configurations on a line

The predicate stating rotational invariance says that, for every $K \in O(4)$, the correlations do not change when the points are moved by K and the field labels by the

rotation action Source (`YM/Euclid/Stress.lean:46` `Rotation.Oriented`: $S(Kx;\, R_K v) = S(x; v)$ for every configuration of non-coinciding points).

Form of the statement (for configurations on a line, the local rotational flux vanishes)

Suppose that the limiting table satisfies the Euclidean bundle (`Table.Hierarchy`), and that a derivative plan and an orientation condition, a stress tensor J, and further hypotheses `Pre` hold. Then the following two statements hold. (i) The sum condition `Bid.LineSum` (`YM/Bid/Angular.lean:35`: for configurations on a line, the value of a certain diagram is zero; we have not read its content). (ii) For a configuration x whose points all lie on one line parallel to some coordinate axis a (points pairwise distinct), the local rotational flux `localTurn` is zero for every rotation K and every list of fields q.

Source A theorem of the published construction. Lean: `YM.Outer.outer` (construction source `YM/Next/ROTATIONS/Route.lean:114`). Its form applied to every object of the construction is `YM.Outer.miles` (same file, :123), which satisfies the predicate `YM.Bid.Miles` (`YM/Bid/Town.lean:13`).

The link that (ii) corresponds to "the stress-energy flux falls off fast enough at infinity" in Assessment § 6.3 is Inferred (because we have not read the definition of `localTurn`).

Source We have read the form of the statement. We have not read the content of the hypothesis `Pre` (`Climb.Pre`) of `outer`, nor the definition of the local rotational flux `localTurn`. The notes in the source say "the analytic estimates come before the continuous-rotation bootstrap and do not use it" and "include every spectator on the line and every empty tuple" Source (docstrings).

Where rotational invariance becomes a conclusion. Rotational invariance of the limiting table appears as the field `motion` (`Rotation.Oriented S`) of the construction's bundle of certificates `Spec.Locals`, and rotational invariance of each phase's table appears as the field `sym` of the certificate `Certificate` (`YM/Phase/Mixture.lean:142`) Source. We have not traced the chain from `outer` to the field `motion`. O(4) invariance is required separately, **outside** the published construction's bundle of Euclidean axioms `Table.Hierarchy` (12-08). This invariance is then used as a hypothesis in the proof of commutativity in the phase decomposition (12-07). The order is "Euclidean invariance → phase decomposition".

```
05
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Outline of the mechanism — from the three components to O(4) invariance

We write out, as the standard argument, how O(4) invariance follows from the three components. We have not checked whether the source assembles it in this order Inferred .

1. **Enlarge the window.** In (2) one cannot take the window χ equal to 1 on a large ball containing all the points $x0, \ldots, x_n$ (a window does not contain the spectators in its support). Instead, add up (2) with a small window around each point, over all the points, and compare the sum with the formula for a window on a large ball. For the large-ball window, $\partial\chi$ is supported on a spherical shell far from all the points, and the left-hand side is **the rotational flux through the distant shell** Inferred .
2. **If the flux vanishes, infinitesimal invariance.** If the flux tends to zero as the shell is moved far away, then the sum over the points of the action of the rotation generator (lie$_g$ on the field labels, together with the derivative terms from the translation Ward identity (1) for rotating the positions of the points) is zero. This says that the correlations do not change under infinitesimal rotations Inferred .
3. **From infinitesimal to the group.** Since $SO(4)$ is connected, integrating along one-parameter subgroups gives $SO(4)$ invariance. Reflections (determinant -1) are contained in the hyperoctahedral group of component (1), so together this gives $O(4)$ Proved here (using that the correlations are smooth in the points, 12-05).

The conclusion of `outer` in the source is "the local flux is zero for configurations on a line", not for general configurations. How to extend from configurations on a line to general configurations, and how to derive the vanishing of the flux far away from reflection positivity (the assessment's "reflection positivity in several orthogonal planes"), is the content of about 40 files in `Next/ROTATIONS/` (with names such as spectral kernels per coordinate axis and Cauchy estimates), which we have not read.

Why the flux is the issue. The infinite-volume limit is taken along a subsequence, and decay of correlations at infinity has not been shown uniformly (12-05 § 05). Without decay, the flux through distant shells need not vanish, and there remains room for rotational invariance to be spontaneously broken. The vanishing of the flux is a statement that, in place of decay, controls only the quantity needed for rotational invariance Inferred .

06

What this book's investigation had seen

Nothing. This book's Lean has none of the stress tensor, the Ward identities or invariance under continuous rotations. This book's lattice results are statements at

fixed lattice spacing, and the only symmetries are those of the lattice.

- The infinite-volume limit $\nu\beta$ inside the strong-coupling window is invariant under unit translations in the four directions (Lean: `TransI.wilson_infvol_translation`, 20-03). Rotations are not treated.
- The measure of the vessel with fermions is invariant under part of the hypercubic symmetry (90° rotations in spatial planes, axis reflections, time reflection) (20-05).
- Among the symmetries of the pushforward measure, commutation with the coarse translations of the side-3 design and with reflection in a boundary face is in Lean, but that stays within the lattice symmetries (`Block3.down_trans`, `Block3.down_reflB`, 28-05). Commutation with exchange of axes was only checked numerically (14-02).

Lifting Euclidean invariance from the Ward identities of the stress tensor and the vanishing of the flux is an idea that was not in this book's plan (the step in the table of construction steps that heads towards reconstruction) (14-05).

07

What we have not read

- **The content of the proof of `outer`** (the 43 files in `Next/ROTATIONS/`). The content of the hypothesis `Pre` and the definition of `localTurn`.
- **The chain from `outer` and `miles` to rotational invariance (`Locals.motion`, `Certificate.sym`).** According to the records, the direct user of `miles` is `MergeProof.Acme.glistens`, and the headline is `SharpGram.habitat`.
- **The assessment's "reflection positivity in several orthogonal planes"**: which lemma in the source it corresponds to.
- **Where the symmetry-preserving gauge fixing lives** (the material of 9-03), and how it reaches `RealSpin` in the limit.
- **The full definition of `RealSpin`** (`YM/Euclid/Table.lean:80`).
- **Reflection positivity in rotated planes** (`YM/Euclid/OtherPlanes.lean`): it is imported, but according to the records it is a place for declarations that are not used Source (records).

What this chapter can say

Can say	Cannot say
Source The hyperoctahedral covariance and reality of the lattice pass to the limit (`Order.spin`)	The full definition of `RealSpin` (`Euclid/Table.lean:80`)
Source The definition of a conserved stress tensor, and the form (1)(2) of the translation and rotation Ward identities using windows (`Conserved`, `StressLaws`). They enter the construction's certificates and the certificates of each phase	That all improvement terms vanish; contact terms; the dilation Ward identity
Source The local rotational flux is zero for configurations on a line (`Outer.outer`)	The content of the proof; how to extend to general configurations
Assessment The three parts of Assessment § 6.3 and "reflection positivity in several orthogonal planes"	The correspondence with lemmas in the source
Inferred The standard argument: vanishing flux + Ward ⇒ infinitesimal invariance ⇒ O(4) invariance	That the source assembles it in this order
This book has no stress tensor, Ward identities or rotational invariance	—

Sources and reproduction

Item	Kind	Source / tool
Euclidean invariance in the OS axioms	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10
Stress tensor, Ward, rotations	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Euclid/Stress.lean:22` `Rotation`, :31 `Rotation.Laws`, :46 `Rotation.Oriented`, :59 `Conserved`, :64 `Tensorial`, :90 `StressLaws`; `YM/Spec/Closed.lean` `Locals`; `YM/Phase/Mixture.lean:136` `Selection.Certificate`
Carrying the discrete symmetries over	Source	`YM/Extraction/Inherited.lean:124` `Order.spin`
Vanishing of the flux	Source (statement only)	`YM/Next/ROTATIONS/Route.lean:114` `outer`, :123 `miles`; `YM/Bid/Town.lean:13` `Miles`
Statement of uniqueness	Source	`YM/Pert/Weak.lean:68` `YM.Pert.one_line` (13-01)
The assessment and the committee's examples	Assessment	`yang-mills-group/YM` `docs/assessment-1.pdf` § 6.3; `submissions/openai-20261028/docs/examples/TranslationWard-20261028.lean`

Item	Kind	Source / tool
This book's lattice symmetries	Lean	`TransI.wilson_infvol_translation` (20-03), `Block3.down_trans`, `Block3.down_reflB`

Terms used in this chapter — hyperoctahedral group, Euclidean invariance, stress tensor, conservation, Ward identity, window, rotational flux, spontaneous breaking, infinitesimal generator. Their meanings are in the A-02 glossary.

Next chapter: 12-07 Phase decomposition — using Euclidean invariance, we make the algebra of operators on the zero-energy space commutative and split a mixed theory into phases.

Phase decomposition

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 07 is a map of open questions.

What this chapter shows — The limiting theory S^{mix}, taken along a subsequence, has not been shown to have clustering (correlations of two groups moved far apart factorise into a product). Without clustering, the vacuum produced by OS reconstruction need not be unique. The published construction goes around this by building a commutative C*-algebra of operators compressed to the zero-energy space and taking its Gelfand spectrum as the space of "phases": it decomposes $S^{\mathrm{mix}} = \int S^{\xi}\, d\mu(\xi)$, and in each component S^{ξ} the vacuum is unique. This chapter reads how this is assembled, and what it says and does not say about the mixed theory and the components.

Prerequisite chapters — 12-05 (the limiting table and the properties that pass to it), 12-06 (Euclidean invariance), 4-10 (OS reconstruction), 1-10 (Hilbert spaces and spectra).

Said first — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean"; this book has read the form of the statements and the names of the lemmas in the steps of the proofs. Uniqueness of the vacuum in each component **comes with no rate** (it is not exponential decay). Clustering of the mixed theory itself, that the space of phases is a single point, and that the components are equal to one another have none of them been shown.

The order of this chapter

1. Key to the labels
2. Why a decomposition is needed
3. Statement (1) — the algebra of operators compressed to the zero-energy space
4. Statement (2) — commutativity, the Gelfand representation, a probability measure
5. Statement (3) — correlations of the components, and uniqueness of the vacuum in each component
6. What is said and not said about the mixed theory and the components

7. What this book's investigation had seen
8. What we have not read

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

Why a decomposition is needed

Since the limiting table S^{mix} is reflection positive, the OS Hilbert space $\mathcal{H}$ and the time semigroup $E(t) = e^{-tH}$ (self-adjoint, contractive) are available Source (construction source `YM/OS/Dynamics.lean:57` `Chamber`: the GNS space built from positive-time words, the time semigroup `flow.E`, the spatial translations `flow.U`). The subspace fixed by $E(t)$ for all t is called the **zero-energy space** $\mathcal{H}_0 = \ker H$ (`YM/OS/Core.lean:14` `ground`), and its orthogonal projection is P_0. As $t \to \infty$, $E(t)x \to P_0 x$ (same file, :38 `emerges`) Source .

In standard OS reconstruction, clustering makes $\mathcal{H}_0$ consist only of multiples of the vacuum Ω, and the vacuum is unique (4-10). Clustering has not been shown for a limit taken along a subsequence, so $\mathcal{H}_0$ need not be one-dimensional. In physical terms, the possibility remains that the limit is a mixture of two or more "pure phases".

Assessment Assessment § 6: rather than applying OS reconstruction directly to this Euclidean theory, it takes the detour of decomposing it into components that satisfy clustering and applying reconstruction to each component. § 6.2: OS reconstruction can also be applied to the mixed limiting theory, but the vacuum of the resulting Wightman theory has not been shown to be unique.

02

Statement (1) — the algebra of operators compressed to the zero-energy space

For a word A (a finite list of field labels and insertion points) with points in the time slab $0 < t < s$, let $B_{s,A}$ be the operator that shifts the insertions of a vector

represented by positive-time words by s into the future and puts the insertions of A in front, in the slab thus freed Assessment (the assessment's glossary, `YangMills.Hierarchy.Character`) Source (`place` of `Spread`). Consider its compressions to the zero-energy space,

$$M_A \;=\; P_0\, B_{s,A}\big|_{\mathcal{H}_0} :\; \mathcal{H}_0 \to \mathcal{H}_0 \quad (1)$$

and let $\mathcal{A}$ be the norm-closed unital *-algebra that they generate.

Source Construction source `YM/Phase/Mixture.lean`: `Spread.atom` ($P_0 \circ \mathrm{place}\, d\,w$ restricted to $\mathcal{H}_0$), `generators`, :54 `Spread.algebra` (the closure of the generated *-subalgebra).

Meaning. $\langle \Omega_0, M_A \Omega_0 \rangle = \langle \Omega_0, B_{s,A} \Omega_0 \rangle = S^{\mathrm{mix}}(A)$ (Ω_0 is the vacuum vector of the mixed theory) Source (`insertValue`). M_A is the operator that looks, on the zero-energy space, at "what remains after inserting the word A and waiting an infinitely long time". If the mixed theory is a mixture of pure phases, then in each phase M_A should act as a scalar (the expectation of A in that phase), and $\mathcal{A}$ should be an algebra of functions on the space of phases Inferred (the motivation).

03

Statement (2) — commutativity, the Gelfand representation, a probability measure

Statement (commutativity of the zero-energy algebra)

If the limiting table satisfies the Euclidean bundle (`Table.Hierarchy`), is O(4) invariant under the rotation action R (`R.Oriented S`, 12-06 § 04), and $\mathcal{H}_0$ is invariant under spatial translations (`Steady`), then $\mathcal{A}$ is commutative: $xy = yx$ for all $x, y \in \mathcal{A}$.

Source A theorem of the published construction. Lean: `YM.OS.Ground.adjunction` (`YM/Ground/Permute.lean:149`; it uses the section variables `Steady V`, `S.Hierarchy`, `Rotation S`, `R.Oriented S`: same file, :12-13).

Why rotational invariance is needed. To swap the order of the product of two generators M_A and $M_{A'}$, one has to swap the time order of two words placed in the time slab. We infer the argument to be that, using O(4) rotations (for example a rotation exchanging the time axis with a space axis, together with spatial translations), one can lay the two words that are lined up in time out in space instead and bring them back into time, thereby swapping their order Inferred (read from the lemma

names `orbit` and `gens`). Either way, the statement in the source takes O(4) invariance as a **hypothesis**. Hence the order is "Euclidean invariance (12-06) → phase decomposition".

Statement (Gelfand representation and the measure on phases)

Under the hypotheses above and given a dense sequence in $\mathcal{A}$, there are a compact metric space X, a Borel measure μ on it, and for each point $\xi \in X$ a character (*-homomorphism) $\xi\colon \mathcal{A} \to \mathbb{C}$, such that: $\mathcal{A}$ is commutative; μ is a probability measure with full support (every non-empty open set has positive measure); the map from X to the set of all characters is a bijection; $\xi \mapsto \xi(a)$ is continuous; and

$$\int_X \xi(a)\, d\mu(\xi) \;=\; \langle \Omega_0,\, a\,\Omega_0 \rangle \qquad (a \in \mathcal{A}). \tag{2}$$

Source A theorem of the published construction (an application of the Gelfand representation of commutative C*-algebras and the Riesz–Markov representation theorem; the measure on phases is written ν in the assessment and μ in this part). Lean: `YM.OS.Ground.resolution` (`YM/Ground/Spectrum.lean:127`), with the conclusion type `Spread.Resolution.Out` (`YM/Phase/Mixture.lean:78`: fields `abelian`, `normalized`, `support`, `onto`, `values`, `weak`, `ground`).

The tools are Mathlib's Gelfand transform for commutative C*-algebras and the Riesz–Markov–Kakutani representation theorem Source (the assessment's glossary and the imports of the source). $\mathcal{A} \cong C(X)$, and the vacuum state $a \mapsto \langle \Omega_0, a\Omega_0 \rangle$ is represented as a probability measure μ on X. Full support follows from the fact that no non-zero element of $\mathcal{A}$ annihilates Ω_0 Assessment (the assessment's glossary, `fullSupport`).

04

Statement (3) — correlations of the components, and uniqueness of the vacuum in each component

Correlations of the components. The correlations of the **phase** at a point $\xi \in X$ are defined, after translating the word A into a positive-time slab, by

$$S^{\xi}(A) \;=\; \xi(M_A) \tag{3}$$

The value does not depend on the translation or on the width of the slab Assessment (the assessment's glossary, `Character`) Source (`Spread.Resolution.receives`, `Mixture.lean:89`). Applying (2) to $a = M_A$, the mixed theory is the average of the phases:

$$\int_X S^{\xi}(A)\, d\mu(\xi) \;=\; S^{\mathrm{mix}}(A). \quad (4)$$

Source Lean: `YM.OS.Spread.Resolution.barycenter` (`Mixture.lean:113`; translation invariance `S.Homogeneous` among the hypotheses). It corresponds to the field `Reconstructed.mixture` of the published statement (`ChallengeSpec.lean:133`).

Each component satisfies the OS bundle. The fields of the phase-selection certificate `OS.Selection.Certificate` (`Mixture.lean:136`) include: the table of each phase satisfies the Euclidean bundle (`states`), it is O(4) invariant (`sym`), its null space of fields is the same as for the mixed theory (`retains`), the laws of the stress tensor (`flux`), and the vacuum is unique in the chosen phase (`heat`) Source .

Statement (uniqueness of the vacuum in each component)

Write `UniqueHeat` (`Mixture.lean:35`) for the statement that the zero-energy space $\mathcal{H}_0^{\xi}$ of the OS space of the phase ξ is one-dimensional (only multiples of the vacuum Ω^{ξ}). (i) `UniqueHeat` holds for μ-almost every ξ. (ii) In the construction's bundle of phases, `UniqueHeat` holds for every ξ.

Source Lean: (i) `YM.OS.Cluster.lifts` (`YM/Phase/Unique.lean:91`), (ii) `YM.EveryPhase.every` (`YM/EveryPhase/Vacuum.lean:146`). (ii) enters the field `euclid` of the construction's conclusion `EveryPhase.Completed` (a field of the structure in `YM/EveryPhase/Extension.lean:51`).

The argument for (i) Source (lemma names and statements). For a positive-time word b, the integral over ξ of $\|P_0^{\xi} b\|^2$ and $\int |\xi(M_b)|^2\, d\mu$ are both equal to $\|P_0 b\|^2$ (`trail`, `at_M`, `almost`). Since $\|P_0^{\xi} b\|^2 \geq |S^{\xi}(b)|^2 = |\xi(M_b)|^2$ (the vacuum component is part of $P_0^{\xi} b$), the integral of the non-negative difference $\|P_0^{\xi} b - S^{\xi}(b)\Omega^{\xi}\|^2$ is 0, so this difference is 0 for almost every ξ (`exp_real`). Combining over a countable family of words spanning the whole space, $\mathcal{H}_0^{\xi}$ consists only of multiples of Ω^{ξ} for almost every ξ Inferred (the details of the direction of the inequality and the choice of the countable family). We have not read how (ii) extends "almost every" to "every" Inferred .

No rate is attached. Uniqueness of the vacuum is connected with $E^{\xi}(t)x \to \langle \Omega^{\xi}, x\rangle \Omega^{\xi}$ under time evolution (the committee's check derives clustering of words separated in the time direction without any assumption on the rate: on the assessment side, `submissions/openai-20261028/checks/ClusterCore.lean` `chamber_word_cluster`), but the rate of approach is not stated Source . An exponential rate would be a mass gap, but that is not in the statement; what is there is a polynomial estimate (12-08 § 05).

The chosen phase. The construction picks one phase ξ^* from a set E with $\mu(E) = 1$ (the set of phases whose zero-energy space is one-dimensional and in which the short-distance limit of the two-point function of the trace holds), and builds the Wightman theory from that phase Assessment (the assessment's glossary, `selected`).

05

What is said and not said about the mixed theory and the components

What is said (at the level of the assessment)	What is not said
$S^{\mathrm{mix}} = \int S^{\xi}\, d\mu$ (where points do not coincide)	Clustering of S^{mix} itself
μ is a probability measure with full support	That X is a single point (one phase)
For each x, $\xi \mapsto S^{\xi}(x)$ is continuous	That the components are equal to one another
The vacuum is unique for every ξ (no rate)	The rate of clustering (the decay rate)
Each component satisfies the OS bundle and O(4) invariance	Comparison of limits along different subsequences
—	Approximating the components directly by lattice measures

Source The left column is in the fields of the published statement (`Reconstructed` in `ChallengeSpec.lean:117-135`: `normalized`, `fullSupport`, `characterContinuity`, `mixture`) and in the fields of the construction's conclusion `Completed`.

If the mixed theory had clustering. Assessment § 6.2 writes that if S^{mix} satisfies clustering, a short calculation gives $\int |S^{\xi}(A) - S^{\mathrm{mix}}(A)|^2\, d\mu = 0$, and continuity and full support then make all the components agree. It also writes that even with clustering, this would not amount to the infrared control needed for uniqueness of the infinite-volume limit Assessment . That all components are equal means "one phase within the limit along one subsequence", which is a different question from whether the limits along different subsequences are equal Proved here .

06

What this book's investigation had seen

The phase decomposition existed only in the plan of this book's investigation. This book's table of construction steps (16-09) places reconstruction and uniqueness of the vacuum as plans for later steps, and there are no Lean statements for them. Separating phases with a commutative algebra compressed to the zero-energy space is an idea that was not in this book's investigation (14-05).

Inside the strong-coupling window, we had it in a form that needs no decomposition. The limit ν_β inside the window is a single measure without subsequences, and for the transfer operator T of the reconstruction vessel it has $\|T - P_\Omega\| \le (18 \tanh \beta)^{1/2}$ and clustering at an exponential rate in the four directions.

Lean `InfMain2.wilson_infvol_main2` (the form of the mass gap and non-degeneracy, 20-04), `WilsonLimit.cluster_limit` (clustering in the four directions at limit points). A formalisation of the Osterwalder–Seiler strong-coupling, Dobrushin-type argument, with the lattice spacing fixed at 1, only inside the strong-coupling window, and with no continuum limit.

If clustering comes with a rate, the vacuum is unique and the decomposition is a single point Proved here . This book's results inside the window have that form, but the regime is different (fixed lattice spacing, strong coupling). The phase decomposition of the published construction is a tool for the weak-coupling continuum limit, where clustering cannot be shown.

One can ask whether the "phases" of the published construction can be distinguished by this book's twisted boundary conditions or vortex free energies (Part 21). That is a question for Volume 3 (16-03).

07

What we have not read

- **How O(4) is used in the proof of commutativity.** We have read only as far as `orbit` and `gens` inside `adjunction`.
- **The proof of `EveryPhase.every`.** How it extends from almost every phase to every phase.
- **That $\mathcal{H}_0$ is invariant under spatial translations** (`Steady`) is derived inside `resolution` from `inert`, but we have not read its content.
- **The construction of the set E of chosen phases** (`YM.Trace.submits` and others listed in the assessment's glossary).
- **The chain to the published theorem.** We read that `EveryPhase.all_characters` is called directly, at :158, inside `R2IGoals.fromFunded` (`YM/R2I/Campaign/Accepted.lean:150`) in the construction's chain Source . We have not traced, declaration by declaration, the chain from `resolution` and `adjunction` to `all_characters` (the headline in the records is `SharpGram.habitat`).

What this chapter can say

Can say	Cannot say
Source The definition of the algebra $\mathcal{A}$ of the operators M_A compressed to the zero-energy space (`Spread.algebra`)	—
Source Assuming O(4) invariance and more, $\mathcal{A}$ is commutative (`Ground.adjunction`)	How O(4) is used in the proof of commutativity
Source The Gelfand representation and a probability measure with full support, $S^{\mathrm{mix}} = \int S^{\xi} d\mu$ (`Ground.resolution`, `barycenter`)	That X is a single point; agreement of the components
Source The vacuum is unique in almost every phase, and moreover in every phase (`Cluster.lifts`, `EveryPhase.every`)	The rate of clustering; the proof of `every`
Assessment If the mixed theory clusters, all components agree, but even then uniqueness of the infinite-volume limit does not follow (Assessment § 6.2)	Clustering of the mixed theory
Lean This book's limit inside the window needs no subsequence and has clustering at an exponential rate (`wilson_infvol_main2`, `cluster_limit`)	Varying the lattice spacing; weak coupling

Sources and reproduction

Item	Kind	Source / tool
Gelfand representation, Riesz–Markov representation theorem	Cited	Standard theorems on commutative C*-algebras and measure theory (1-10). In Lean, Mathlib
The OS space and zero energy	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/OS/Dynamics.lean:57` `Chamber`, `YM/OS/Core.lean:14` `ground`, :38 `emerges`
Algebra, commutativity, decomposition	Source	`YM/Phase/Mixture.lean:35` `UniqueHeat`, :54 `algebra`, :78 `Resolution.Out`, :89 `receives`, :113 `barycenter`, :127 `Selection`, :136 `Selection.Certificate`; `YM/Ground/Permute.lean:149` `adjunction`; `YM/Ground/Spectrum.lean:127` `resolution`
Uniqueness of the vacuum	Source	`YM/Phase/Unique.lean:91` `lifts`; `YM/EveryPhase/Vacuum.lean:146` `every`; `YM/EveryPhase/Extension.lean:51` `Completed`, :75 `all_characters`; `ChallengeSpec.lean:117-135` `Reconstructed`
The committee's checks	Source	`yang-mills-group/YM` `submissions/openai-20261028/checks/ClusterCore.lean` `chamber_word_cluster`
Text of the assessment	Assessment	`docs/assessment-1.pdf` § 6, § 6.1, § 6.2; glossary `Character`, `phaseMeasure`, `selected`, `fullSupport`
Clustering inside this book's window	Lean	`InfMain2.wilson_infvol_main2`, `WilsonLimit.cluster_limit` (20-03, 20-04)

Terms used in this chapter — clustering, zero-energy space, compression, commutative C*-algebra, Gelfand representation, character, phase, measure on phases, full support, uniqueness of the vacuum. Their meanings are in the A-02 glossary.

Next chapter: 12-08 Reconstruction and the Wightman axioms — we build the Wightman theory from the chosen phase and read which axioms come out and what is not said (the mass gap).

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction OS reconstruction · Kiichi, Shiori and Rin

Reconstruction and the Wightman axioms

This chapter is established knowledge (a reading of the published construction). § 05 is a map of open questions (the bridge to Volume 3), § 06 is this book's own investigation, and § 07 is a map of open questions.

What this chapter shows — The published construction defines its own bundle of the Euclidean axioms that the limiting table satisfies (with a growth condition of the form "r^{-A} for point separation $r \leq 1$"), builds a relativistic field theory that satisfies the Wightman axioms in every phase (translation spectrum and uniqueness of the vacuum, a common domain and cyclicity, Poincaré covariance, locality), and states the additional properties — analytic continuation, agreement at purely imaginary times, boundary values, faithfulness — for the chosen phase. The committee checked the standard, independently formulated OS axioms and growth condition for the correlations of **a single scalar field** in the chosen phase, the centred trace of the stress tensor. This chapter reads the bundle of axioms, the list of axioms that come out, and the scope of the check, and at the end confirms that there is no statement of a mass gap, only a polynomial estimate uniform over the phases.

Prerequisite chapters — 4-09 (the Wightman axioms), 4-10 (the OS axioms and reconstruction), 12-07 (phase decomposition).

Said first — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean". What the committee's independent check confirmed against the standard OS axioms concerns the correlations of **the centred trace scalar in the chosen phase**; it did not separately translate and check all the tensor fields. The construction says nothing about the mass gap (Assessment § 2, item 4).

The order of this chapter

1. Key to the labels
2. Statement (1) — the published construction's version of the bundle of Euclidean axioms
3. Statement (2) — reconstruction from phases — every phase, and the chosen phase

4. List of the Wightman axioms and additional properties that come out
5. The scope of the committee's independent check
6. There is no statement of a mass gap — a polynomial estimate uniform over the phases (the bridge to Volume 3)
7. What this book's investigation had seen
8. What we have not read

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

Statement (1) — the published construction's version of the bundle of Euclidean axioms

The introduction is in 4-10. The construction defines the properties satisfied by the table S of limiting correlations (functions on configurations of non-coinciding points) as a bundle of the following seven fields Source (construction source `YM/Euclid/Words.lean:141` `Table.Hierarchy`).

Field	Content	Correspondence with the standard OS axioms
`laws` (`Basic`)	Symmetric under simultaneous permutation of points and labels; the 0-point correlation is 1; the unit field can be removed	Symmetry (E3) and normalisation
`spin` (`RealSpin`)	Covariance under the hyperoctahedral group (coordinate permutations and signs) and reality	The discrete part of Euclidean covariance (E1), and reality
`translations`	Invariant under continuous translations	The translations of E1
`positive`, `onTests`	Reflection positivity for each coordinate axis (in the form with words and in the form smeared with test functions)	Reflection positivity (E2)
`smooth`	C^∞ where points do not coincide	(regularity)

Field	Content	Correspondence with the standard OS axioms
`order` (`Capped`)	Growth condition ((1) below)	(one form of a growth condition)

The form of the growth condition. When a configuration of points x has "separation r" (every pair of points at distance at least r, in the cube norm) with $0 < r \leq 1$, then for the j-th derivative of the correlations

$$\left\| \nabla^j S(v; x) \right\| \;\leq\; C\, r^{-A} \quad (1)$$

holds for some A and $C \geq 0$ (for each label and order) Source (`YM/Euclid/Table.lean:97` `Capped`). Because r is capped at 1, the right-hand side stays C even when the points are far apart, so **no growth at infinity is allowed**. Only the singularity as points approach each other is allowed, as a power.

Properties outside the bundle. Invariance under continuous rotations (O(4)) is not in the bundle and is required separately in the field `sym` of the phase certificate (12-06 § 04). Clustering (uniqueness of the vacuum) is also outside the bundle and is required separately as `UniqueHeat` for each phase (12-07 § 04) Source . That the limiting table satisfies the bundle is assembled from the properties that pass to the limit (12-05 § 04) plus smoothness and the growth condition (`Order.assembles`, `YM/Extraction/Harvest.lean:62`).

02

Statement (2) — reconstruction from phases — every phase, and the chosen phase

The correlations S^ξ of a phase ξ (12-07) are continued analytically to complex times, and the real-time Wightman distributions W^ξ are obtained as boundary values as the imaginary parts tend to 0 (after smearing with Schwartz test functions). On the space $\mathcal{P}$ of linear combinations of formal products of field labels v_i and Schwartz test functions f_i one puts $\langle u, w \rangle_\xi = W^\xi(u^* w)$; quotienting by the null-norm part and completing gives the Hilbert space $\mathcal{H}^\xi$, the empty word is the vacuum Ω^ξ, and the operation of putting (v, f) in front of a word is the smeared field operator $\Phi^\xi_v(f)$ Assessment (Assessment § 6.1).

Statement (reconstruction of the component theories, Theorem 6.1 of the assessment)

In the assessment's words: for every character $\xi \in X$, the component correlations S^ξ reconstruct a Wightman theory on $\mathcal{H}^\xi$ (vacuum Ω^ξ, fields Φ^ξ on a domain $\mathcal{D}^\xi$) satisfying: 1. translation spectrum, positive energy, uniqueness of the vacuum; 2. common domain, regularity, adjoints, cyclicity; 3. Poincaré covariance; 4. locality;

5. analytic continuation and recovery of the correlations; 6. faithfulness of the fields.

Assessment Theorem 6.1 in Assessment § 6.1 (attaching items 1–6 to every ξ is the assessment's wording).

Source Split according to the Lean statements, it is as follows. The Wightman axioms hold **in every phase** (the field `characters` of the construction's conclusion `Headline.Reconstruction`, `YM/Nonperturbative/HeadlineSpec.lean:75`; the underlying structure `WightmanAxioms`, same file :52-71, covers normalisation of the vacuum, density and cyclicity of the domain, continuity in the test functions, adjoints, agreement of correlations, positivity, locality, symmetry, strong continuity, the spectrum, the forward light cone and uniqueness of the vacuum, and has no fields for faithfulness or analytic continuation). The **additional properties** (faithfulness `faithful`, analytic continuation `analytic`, agreement at purely imaginary times `root`, boundary values `boundary`) are stated only for **the chosen phase**, inside the field `Hierarchy.Reconstructed` of the published statement (`ChallengeSpec.lean:117-143`), and `YangMills.publicReconstruction` on the construction's side (`HierarchyBridge.lean:79`) satisfies it.

Which goal it is in. The conclusion field `reconstruction : T.Reconstructed` of the published theorem `JointExistence` is a statement about the Wightman theory of **the chosen phase** ξ^* (the fields `axioms`, `analytic`, `root`, `boundary` concern the continuation $T.\mathrm{continuation}$ and $T.\mathrm{theory}$ of the chosen phase). The Wightman axioms for every phase are in the conclusion of the construction's theorem `constructed` (`Headline.Reconstruction.characters`) Source . In the records `publicReconstruction` is `used` (reached from the headline theorem `su2Theory`), and its direct user is `constructed` Source .

03

List of the Wightman axioms and additional properties that come out

We list the content of the field `Reconstructed` of the published statement (the Wightman-axiom field `axioms` and the additional fields are statements about the chosen phase; the Wightman axioms for every phase are a separate conclusion, as in § 02), following the assessment's glossary Source (field names and types) Assessment (explanations of meaning).

Field	Content
`axioms`: translation spectrum and uniqueness of the vacuum	A strongly continuous unitary representation of translations and a projection-valued energy–momentum measure; the support of the measure is the closed forward light cone; every vector invariant under all translations is a multiple of the vacuum
`axioms`: common domain, regularity, adjoints, cyclicity	The domain is dense, contains the vacuum and is preserved by the fields; $f \mapsto \Phi_v(f)\psi$ is continuous in the Schwartz topology; the adjoint is given by conjugating the label; the domain is the span of field words applied to the vacuum
`axioms`: Poincaré covariance	A strongly continuous unitary representation of the proper orthochronous Poincaré group (with the same translations as above), an action $\rho(g)$ on labels, and the field transformation law $U(g)\Phi_v(f)\psi = \Phi$ $\rho(g)v(fg)U(g)\psi$
`axioms`: locality	Two fields whose supports are spacelike separated commute
`faithful`, `acts`, `injective`, `covers`, `represents`	A non-zero label gives a non-zero field; at least one field is non-zero (faithfulness is not vacuous); the map to labels is injective and surjective; every field is the image of a lattice label
`normalized`, `fullSupport`, `characterContinuity`, `mixture`	The measure on phases is a probability measure with full support; the correlations are continuous in the phase; the average over phases is the mixed correlation (12-07 § 05)
`permutation`, `smooth`	Permutation symmetry and smoothness of the correlations of the mixed phase and of each phase
`analytic`, `root`, `boundary`	The analytic continuation of the chosen phase is holomorphic on $\mathrm{Im}\, z_1 < \cdots < \mathrm{Im}\, z_n$; at purely imaginary times it agrees with the Euclidean correlations; its boundary values at real times agree with the vacuum expectation values (the Wightman distributions)

According to the assessment's glossary, uniqueness of the vacuum goes as follows: an isometry from the Hilbert space on the Wightman side to the OS space of the chosen phase maps the vacuum to the vacuum and commutes with translations; non-negative energy on the OS space and Lorentz covariance give the spectral condition on the forward light cone; a translation-invariant vector is mapped by the isometry into the zero-energy space on the OS side, which is one-dimensional (`UniqueHeat`), so it is a multiple of the vacuum[Assessment] .

04

The scope of the committee's independent check

The committee checks the output of the construction against OS axioms formulated independently of the construction's definitions (reference [DMY] of the assessment, repository `xiyin137/OSreconstruction`)[Assessment] (Assessment § 10).

- **Object.** The hierarchy of n-point functions of **a single scalar field**, the centred trace Θ of the stress tensor, in the chosen phase ξ^*. It does not separately translate all the tensor fields into the upstream formulation Assessment (§ 10.1).
- **Test functions.** `ZeroDiagonalSchwartz`: Schwartz functions all of whose derivatives of all orders vanish on every diagonal ($x_i = x_j$). They need not be 0 near the diagonals, nor have compact support. They include the compactly supported test functions supported away from coinciding points that are used in the statements of convergence from the lattice Assessment (§ 10, equation (21)).
- **Axioms checked.** E0 (temperedness, linearity), E1 (translation and rotation invariance), E2 (reflection positivity), E3 (symmetry), E4 (clustering in spatial directions, in the chosen phase), and the linear growth condition E0′ of OS-II type Source (on the assessment side, `verifiedOS`, `verifiedOS_growth`, `constructed_OS_family` in `submissions/openai-20261028/checks/CompleteOS.lean`; `completeOS`, `E0_prime` in `checks/RemainingObligations.lean`; `E2_reflection_positive` in `checks/ReflectionPositivity.lean`).
- **What is not said.** Test functions that are flat on the diagonals do not see contact terms on the diagonals, so uniqueness of the extension of contact terms and OS positivity on the unrestricted space of test functions are not checked Assessment (§ 10). E4 is a statement about the chosen phase, not a claim that the mixed table clusters Source (the docstring of `submissions/openai-20261028/docs/examples/OSAxioms-20261028.lean` on the assessment side).

Source On the assessment side, `constructed_OS_family` has the form: for a run of the construction (penalty coefficient $b > 0$) and any extraction sequence p, there are a refinement q, a table S and a final certificate F such that the limit can be read off along the extraction "cutoffs along the whole sequence, volumes along the subsequence of volumes of $(p \circ q)$", and E4 clustering and E0′ hold.

05

There is no statement of a mass gap — a polynomial estimate uniform over the phases (the bridge to Volume 3)

A mass gap means that the spectrum of the Hamiltonian $H \geq 0$ lies in $\{0\} \cup [\Delta, \infty)$ (with $\Delta > 0$) (4-11). The assessment writes that "the construction gives no evidence either for or against a mass gap" (§ 2, item 4), and lists as open question 3 "a uniform mass gap (exponential decay of the corresponding correlations and a spectral gap of the OS Hamiltonian)" Assessment.

What the construction has is **a polynomial estimate uniform over all phases** Source.

Statement (low-energy and heat estimates uniform over the phases)

For a finite sum of words w, there is for each of (i) and (ii) a constant $M \geq 0$ (taking the larger makes it one constant Proved here) such that in every phase i: (i) if $0 < u \leq 1$, the spectral weight of the state of w at energies in $(0, u]$ is $\leq M u^2$; (ii) if $\tau \geq 1$, then

$$0 \;\leq\; \langle A,\; (e^{-\tau H} - P_0)\, A \rangle \;\leq\; M\, \tau^{-2} \qquad (2)$$

(A is the state of w, and P_0 is the projection onto energy 0).

Source Theorems of the published construction. Lean: `YM.EveryPhase.light_bound` (`YM/EveryPhase/Light.lean:221`), `YM.EveryPhase.inverse_square` (`YM/EveryPhase/Decay.lean:173`). They enter the fields `light` and `heat` of the construction's conclusion `EveryPhase.Completed`.

How to read it. (ii) follows from (i) by elementary means Proved here : $\langle A, (e^{-\tau H} - P_0)A \rangle = \int_{(0,\infty)} e^{-\tau E}\, d\sigma_A(E)$; the part on $(0, 1]$ is split into dyadic intervals and gives $O(\tau^{-2})$ from $\sigma_A((0, u]) \leq M u^2$, and the part on $(1, \infty)$ is bounded by $e^{-\tau}$. Here σ_A is the spectral measure of the state A (not the measure on phases μ). What can be read in the source is that the proof of `light_bound` uses `late` (the average of the commutator is 0 far away in space, `Light.lean:208`) and the dyadic splitting `dyadic` Source . The reading that this is a general consequence of locality and support in the light cone is Inferred . This form is also compatible with a massless theory Inferred . It is a different thing from an exponential rate $e^{-\Delta\tau}$.

The bridge to Volume 3. The mass-gap question can be restated in these terms: "strengthen the low-energy weight estimate $\sigma_A((0, u]) \leq M u^2$ to $\sigma_A((0, \Delta)) = 0$ below some $\Delta > 0$". This is the same as strengthening the right-hand side of the heat estimate (2) from $M\tau^{-2}$ to $M e^{-\Delta\tau}$ Proved here . This is open question 3 of the assessment, treated in Volume 3 in 16-02 (Five open questions) and 16-03 (The infrared edge of the published construction).

06

What this book's investigation had seen

Reconstruction of a Wightman theory was not in this book's investigation. This book's Lean reconstruction goes only as far as the vessel on the Euclidean side (the reconstruction vessel in 19-03: a positive semidefinite bilinear form on reflection pairs, time translations, a vacuum, a transfer operator T), and treats neither analytic continuation nor Wightman distributions.

Inside the strong-coupling window, we had the form of a mass gap. In the limiting vessel inside the window, $\|T - P_\Omega\| \le e^{-m}$ with $m = -\log(18 \tanh \beta)/2$, the complement of the vacuum is non-zero, and $T \ge 0$.

Lean `InfMain2.wilson_infvol_main2` (20-04). A formalisation of the Osterwalder–Seiler strong-coupling argument. The lattice spacing is fixed at 1, only inside a window of about $|\beta| < 0.0556$, only a lower bound on the mass, and no continuum limit.

The published construction reaches a Wightman theory in the weak-coupling continuum limit and has no statement of a mass gap. This book has the form of a mass gap in a strong-coupling window at fixed lattice spacing, and has no continuum limit. The two results lie in different regimes and are not connected (14-06).

07

What we have not read

- **The proof of the published construction's version of OS reconstruction** (`YM/Boundary/*`, `YM/EveryPhase/*`). On how the analytic continuation is built and how the boundary values are taken, we have only the explanations in the assessment's glossary.
- **Where the growth condition (1) comes from.** We have not traced who discharges the hypothesis `Capped` of `assembles`.
- **The inputs from the construction's estimates in the committee's check.** Assessment § 10.1 writes that "the analytic inputs come from the submitted proof". We have not read which estimates were used.
- **The proof of `light_bound`.** We have only the names `late`, `annulus`, `dyadic`, `fee`.
- **Continuum Wilson loops.** Continuum Wilson-loop observables and an area law are not among the statements in the range we read (the range of a search by name) Source . Parallel transport along lattice paths, `travel` (`YM/Lattice/Path.lean:105`), is there. Assessment § 8 lists as open question 4 constructing Wilson loops, relating them to local observables and proving an area law Assessment .
- **Reconstruction of the mixed theory.** Assessment § 6.2 writes that OS reconstruction can also be applied to the mixed limiting theory (uniqueness of the vacuum is not shown). We have not checked which declaration in the source this is.

What this chapter can say

Can say	Cannot say
Source The published construction's version of the bundle of Euclidean axioms (seven fields) and the form of the growth condition (1) (r^{-A} for separation $r \leq 1$, no growth at infinity)	Where (1) comes from. O(4) and clustering are outside the bundle
Source The Wightman axioms in every phase (`Headline.Reconstruction.characters`); the additional properties (faithfulness, analytic continuation, agreement at purely imaginary times, boundary values) for the chosen phase (`publicReconstruction`). At the level of the committee's assessment	Stating the additional properties in every phase
Assessment The committee's check: the standard OS axioms E0–E4 and E0′ (with Schwartz test functions flat on the diagonals) for the centred trace scalar in the chosen phase	A check of all the tensor fields. Uniqueness of the extension of contact terms
Source Polynomial estimates uniform over the phases: low-energy weight $\leq Mu^2$, heat $\leq M\tau^{-2}$ (`light_bound`, `inverse_square`)	A mass gap. Exponential decay. One-particle states
Proved here The heat estimate follows from the low-energy estimate. The restatement of the mass gap	—
Source (within the range of a search by name) Parallel transport along lattice paths, `travel` (`YM/Lattice/Path.lean:105`), is there	Continuum Wilson-loop observables and an area law (not among the statements; open question 4 in Assessment § 8)
Lean The form of a mass gap inside this book's window (`wilson_infvol_main2`)	The continuum limit. A Wightman theory

Sources and reproduction

Item	Kind	Source / tool
The Wightman axioms and OS reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-09, 4-10; references in A-03
Bundle of axioms and growth condition	Source	Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Euclid/Words.lean:123` `Table.RP`, :141 `Table.Hierarchy`; `YM/Euclid/Table.lean:97` `Capped`; `YM/Extraction/Harvest.lean:62` `assembles`
Statements of reconstruction	Source	`ChallengeSpec.lean:117-143` `Hierarchy.Reconstructed`, :186 `JointExistence`; `HierarchyBridge.lean:79` `publicReconstruction`; `PublicView.lean:102` `view_axioms`; `YM/Nonperturbative/HeadlineSpec.lean:75` `Headline.Reconstruction`
Polynomial estimates	Source	`YM/EveryPhase/Light.lean:208` `late`, :221 `light_bound`; `YM/EveryPhase/Decay.lean:173` `inverse_square`; `YM/EveryPhase/Extension.lean:51` `Completed`

Item	Kind	Source / tool
The committee's checks	Source, Assessment	`yang-mills-group/YM` `submissions/openai-20261028/checks/CompleteOS.lean`, `RemainingObligations.lean`, `ReflectionPositivity.lean`, `ClusterCore.lean`; `submissions/openai-20261028/docs/examples/OSAxioms-20261028.lean`. Assessment § 6.1, § 10, § 10.1; glossary `Reconstructed.*`
Open questions of the assessment	Assessment	`docs/assessment-1.pdf` § 2, § 8
This book's vessel and the form of a mass gap	Lean	`InfMain2.wilson_infvol_main2` (20-04)

Terms used in this chapter — OS axioms, Wightman axioms, growth condition, analytic continuation, boundary value, spectral condition, locality, phase, centred trace, test function flat on the diagonals, mass gap, polynomial estimate. Their meanings are in the A-02 glossary.

Next chapter: 13-01 The stress tensor lies on a single line — we begin reading the evidence that the reconstructed theory is Yang–Mills, starting from the uniqueness of the stress tensor.