Volume 2 Existence — reading the published construction

Part 15 Record of the attempts at ultraviolet control

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2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control entrance, the number of steps, closing in one step · Kiichi, Shiori and Rin

Entrance — what ultraviolet control is

This chapter is a guide. § 01 is established knowledge, § 03 is this book's own investigation, and § 08 is a record of this book's study.

Where this part stands — This part is a record. It keeps the traces of this book's study's attempt at ultraviolet control (existence of the continuum limit) with a design different from that of the published construction — statements, line-by-line computations, elementary lemmas, numbers measured on minimal systems, and rows that do not close — as they are. Existence itself was shown by the construction published on 2026-11-10 (at the level "the committee's assessment is that it is proved in Lean"). How to read it is in Parts 6–13 of Volume 2, and the differences between the two designs are in the immediately preceding Part 14, as a stage-by-stage comparison.

The premises that appear in this part and in the list of premises (28-04) — the small-field step, the large-field estimate and gas, induction uniform in the step, the coupling bracket, full-sequence convergence — are statements about this book's design (measures pushed forward by a fixed block map independent of the cutoff, and matching by quantities determined by the measure). The results of the published construction are statements about correlations of local fields and about that construction's kernel and representation, so these premises do not follow from them. The premises on large fields and the gas are answered by the published construction in a different form (no history of regions, re-counting seeds at each step) (14-04). What can be called specific to this book's design is at most three things: a fixed block map independent of the cutoff (consistency between steps can be written as a push-forward identity), reflection positivity of the pushed-forward measure (Lean), and matching by quantities determined by the measure (at the level of design choices) (14-06).

The theorems machine-checked all the way are in Parts 20–22, and the machine-checked form of the free-field single step is in 15-12 in the second half of this part. The conditional implication that this book's study assembled beyond the components of this part, and its known flaws, are written in § 08.

What this chapter shows — That sending the lattice spacing a to 0 makes the number of renormalisation steps $\log_L(L_0/a)$ grow without bound. That what is then needed is to show "one step returns to the same form" at every step without letting the constants degrade. The form the effective density must keep and the three conditions (the

content of ultraviolet control). Seeing the difference between a one-step estimate and an all-step proof through one number (the product of losses), and the ten patterns of misreading that difference. The distinction between the words of this part (renormalisation step, the staircase of Part 23, the band of Part 26), and the map of chapters, reading order and symbols of the five grades. The conditional implication assembled by this book's study, and its known flaws.

Prerequisite chapters — 7-05 § 01 (the sequence of renormalisation group steps), 7-05 § 04 (small fields and large fields), 7-07 § 03 (four dimensions: Bałaban's main theorem), 26-00 (confluence and the band), 27-02 (the ledger of conditions).

Said first — What this part treats is the ultraviolet-control row of the ledger of conditions, the foundation at the bottom of 16-09 Figure 1, underneath both of the two roads. It is a task separate from the crossover band (Part 26), and deals only with scales finer than the band. Within this part, ultraviolet control is not closed (existence was shown by the published construction with a different design). What this part has are statements of what needs to be shown, line-by-line computations, a few elementary lemmas, and numbers measured on minimal systems; each restates a part of ultraviolet control or splits it into several conditions. This part expands 26-11 (ultraviolet control) by splitting it, and 26-11 keeps only a summary and pointers to each chapter. A note on symbols: in this part L is the blocking factor (usually 2), and L_0 is the side of a fixed box (as in 7-05 § 01).

The order of this chapter

1. What the problem is to show — the number of steps, the same form, the three conditions (the content of ultraviolet control), the form that closes in one step
2. A note on words — the renormalisation step, the staircase of Part 23, the band, the window (Table 0)
3. What it means to close at infinitely many steps — the product of losses, and the ten patterns of misreading (Table 1)
4. The map of this part and the reading order (Table 2)
5. The symbols of the five grades (Table 3)
6. What cannot be said
7. Table of the five grades and the dependency graph (Table 4)
8. The conditional implication assembled by this book's study, and its known flaws — as a record

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00

What this chapter shows

- Ultraviolet control means showing "one step returns to the same form" at infinitely many steps without the constants degrading (§ 01).
- If the constant shrinks by a factor $1-\delta_k$ at each step, what remains after K steps is $\prod_k(1-\delta_k)$, and everything is decided by whether the sum of losses $\sum_k \delta_k$ is finite uniformly in the number of steps (§ 03).
- The misreading of writing a one-step estimate and reading it as closing the induction happens again and again. We divide it into ten patterns and set each beside the correct treatment (§ 03 Table 1).
- The "step" of this part is one renormalisation step (step k); it is different both from the steps of the "staircase" of Part 23 and from the "band" of Part 26 (§ 02).
- The chapters of the first half are arranged in the order: the difficulty (15-01), the sources (15-02), the decomposition statement (15-03), numbers on minimal systems (15-04 to 15-06), background dependence (15-07), the four rows that do not close (15-08), the index (15-09), and the systematic attempts (15-10). The four chapters of the second half (15-11 to 15-14) are theorems completed in Lean for one step of the free field and one scale (§ 04).
- The conditional implication assembled by this book's study has three known flaws (§ 08).

01

What problem is to be shown — the number of steps, the same form, the three conditions

The object is Wilson's lattice for $SU(2)$ on a four-dimensional torus of side L_0, with lattice spacing a. L_0 is fixed at a physical scale, and only a is varied. As in 7-05 § 01, we repeat blocking with factor L and integrate from the finest scale upwards. The number of steps is

$$K = \log_L \frac{L_0}{a} \longrightarrow \infty \qquad (a \to 0) \quad (1)$$

Proved here (from the definition). Example: if $L=2$ and $L_0/a = 2^{20}$, then $K=20$, and each halving of a adds one step. Writing ρ_k for the weight (effective density) after step k, we have $Z = \int \rho_0 = \int \rho_1 = \cdots = \int \rho_K$.

The same form. In Bałaban's method, at each step the lattice is divided into a small-field region Ω and its outside, and one sums over all the ways of dividing it (7-05 § 04). The form the effective density must keep is, in abbreviated notation,

$$\rho_k(V) = \sum_{\Omega} \chi(\Omega, V)\, \zeta(\Omega^c, V)\, \exp\Big(-\frac{1}{g_k^2} A_\Omega(V) - \sum_{X \subset \Omega} E_k(X, V)\Big)\, R_k(\Omega^c, V) \quad (2)$$

Cited (the form follows the exposition for scalar fields, Dimock II § 1, and the wording of the three-dimensional theorem; 7-05 § 07). A_Ω is Wilson's action on the small-field region (written in terms of the background field), $E_k(X, \cdot)$ is a correction localised on a connected lump X, R_k is the factor carried by the large-field region, and g_k is the effective coupling at step k.

The three conditions. Ultraviolet control means keeping the form (2) at every step while satisfying the following three. The constants r, C, κ, c, γ depend neither on the step k nor on the lattice spacing a.

- **Small-field analyticity condition** (locality and analyticity for small fields). Each $E_k(X, \cdot)$ is analytic in a complex neighbourhood of radius r in the background-field variables, and $|E_k(X)| \le C\, e^{-\kappa\, d(X)}$ ($d(X)$ is the length of a tree joining X).
- **Large-field rarity condition.** $|R_k(\Omega^c)| \le e^{-c\, p(g_k)^2\, |\Omega^c|}$ ($|\Omega^c|$ is the number of large-field blocks, and $p(g)$ is the threshold factor of equation (7) in 7-05).
- **Condition that the coupling stays small** (below, "the coupling stays small"). The sequence of effective couplings $g_0, g_1, \dots, g_K$ lies entirely in $(0, \gamma]$.

The three put the content of ultraviolet control into the language of this book. This book's study has not shown any of the three in four dimensions Assumption (the ultraviolet-control row of 27-02). Existence of the four-dimensional continuum limit was shown by the published construction with a different design (one that carries the representation of the density at each step through all steps as the induction hypothesis) (6-03, 14-01).

The form of closing in one step. Let the class $\mathcal{K}(g;\, r, \kappa, c)$ be "the collection of effective densities that have the form (2) and satisfy the small-field analyticity condition and the large-field rarity condition at coupling g with constants r, κ, c". What has to be shown is

$$\rho_k \in \mathcal{K}(g_k;\, r, \kappa, c) \implies \rho_{k+1} \in \mathcal{K}(g_{k+1};\, r, \kappa, c), \qquad \frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - 2b_0 \ln L + r_k \quad (3)$$

The point is that **the constants r, κ, c are the same on both sides, and only the coupling may move**. $b_0 = 22/(48\pi^2)$ is the one-loop coefficient for $SU(2)$ (7-07 § 01), and r_k is the remainder in the flow of the coupling. 15-03 restates (3) by splitting it into "a class that closes with finitely many parameters" and "the one-step remainder".

02

A note on words — the renormalisation step, the staircase of Part 23, the band, the window

In this part we use the word "step" only in the sense of one renormalisation step. This book has other words built on "step", "band" and "window", and when words coincide they read as if they referred to the same thing, so Table 0 separates them.

Table 0 The words of this part, and similar words in other parts

Word	Meaning	Chapters that use it	Relation to this part
The renormalisation step (step k)	One blocking with factor L. The effective density ρ_k after step k, the effective coupling g_k, the number of steps $K = \log_L(L_0/a)$	7-05, Part 15	This is the only "step" of this part
The steps of the staircase of Part 23 (the first step, the second step, ...)	An arrangement that divides the road towards a proof into four steps. The first step is a study on the double well in 0+1 dimensions	Part 23, 26-00	A different word. Unrelated to the number of steps K of this part
The crossover band (the band)	The interval around the scale $L \sim 1/\Lambda$ at which weak coupling passes over to strong coupling	Part 26	This part deals only with scales finer than the band. Inside the band the coupling is not small
The window (the A_W window, the $\hbar$ window, the time window)	Test functions restricted to an interval of finitely many sites, an interval of parameters, or an interval of time	Parts 23 and 24	Not used in this part. A single plaquette of the coarse lattice is written "a coarse plaquette" (2×2, 4×4, side L)
The ultraviolet	All the steps from the lattice spacing a up to the scale L_0 at which the coupling is still small	26-11, Part 15	The ultraviolet-control row of the ledger of conditions. The coupling is small at every step

One more caution: in the smallest systems of this part (15-04 to 15-06), we sometimes call "the marginal distribution of one coarse plaquette" a one-step map. This map looks only at the distribution of the holonomy of one coarse plaquette; it is not a true blocking step, which would produce the joint distribution of all the plaquettes of the coarse lattice. Each chapter says this once where it first appears.

03

> **What it means to close at infinitely many steps — the product of losses, and the ten patterns of misreading**

Let us see the difference between a one-step estimate and an all-step proof in a single number. If a one-step estimate comes only in the form "constant $r_{k+1} \geq r_k(1-\delta_k)$", then what is left after K steps is

$$r_K \;\geq\; r_0 \prod_{k=0}^{K-1} (1-\delta_k), \qquad \prod_{k=0}^{K-1} (1-\delta_k) \;\geq\; \exp\Bigl(-2 \sum_{k=0}^{K-1} \delta_k\Bigr) \quad (0 \leq \delta_k \leq \tfrac{1}{2}) \tag{4}$$

Proved here (the inequality on the right follows from $\ln(1-x) \geq -2x$ for $0 \leq x \leq \frac{1}{2}$). If the sum of losses $\sum_k \delta_k$ is finite uniformly in K, the constant stays positive uniformly in K. If the sum grows with K, the constant degrades with the number of steps. **The content of ultraviolet control is making this sum finite.** Bałaban's three-dimensional theorem (equation (10) in 7-05 § 07) and the four-dimensional theorem of 1989 (7-07 § 03) carry out an induction of this form in finite volume Cited . Why in four dimensions this sum is not naively finite ($\sum_k g_k^2$ grows like $\ln K$) is calculated in 15-01 § 01.

In examining ultraviolet control, the same kind of misreading happens again and again. The common pattern is **writing a one-step estimate and reading it as closing the induction**, which amounts to looking at just one δ_k without looking at the product $\prod(1-\delta_k)$. Table 1 divides it into ten logical patterns. Every row can be avoided by putting the one-step claim and the all-step claim on separate lines.

Table 1 Patterns of misreading one step as all steps (logical cautions; labels only on the rows with numbers)

Pattern	The misreading (one line)	The correct treatment
Circularity	Taking the inputs needed to close the expansion (the lower bounds on the Hessian at each step) from the outputs of that same expansion	Put both inputs and outputs on the side of the induction hypothesis, and bring nothing in from outside
Mistaking the component	Reading the replacement of a one-step estimate and a structural lemma as replacing the whole induction	Write the components replaced and the parts of the induction not replaced on separate lines
Accumulation of constants	Writing "the constants are fixed afresh at each step, so they do not accumulate"	That is a reduction to the radius of analyticity being uniform in the step, and depends on the small-field analyticity condition
Worst-case geometry	Running the large-field induction as "if it is large at the next step, it was large at one of the earlier steps"	A constant is lost at each step, and by type (4) of 15-01 it collapses to 0 over k steps Proved here . What is needed is one-step stability that passes the stiffness on through the integral over the fluctuations

Pattern	The misreading (one line)	The correct treatment
Goodness on the fine lattice	Defining a "good frame" by all plaquettes of the shell being small	With probability $1-q$ at each point ($q>0$ fixed), at n points this gives $(1-q)^n \to 0$ Proved here . Define goodness at the renormalised scale
Direction of the inequality	Using the variational principle for the partition function (which gives only lower bounds) for an upper bound on a difference of free energies	An upper bound needs an upper bound on the partition function, or control of the conditional measure
The cost at the bare coupling	Deriving $e^{-c/g^2(L)}$ at the running coupling from the existence of localised configurations of finite action	Mere existence gives a cost at the bare coupling, which vanishes as $a \to 0$. A cost at the running coupling needs uniform control
Scheme-dependent numbers	Testing the coefficient of F^2 in one step against the universal one-loop coefficient	The one-step coefficient depends on the blocking scheme. What is universal is the coefficient of the logarithm accumulated over many steps
Dropping the assumption	Citing four-dimensional ultraviolet stability as an unconditional theorem	It is conditional on the coupling staying small, and the basic inequality that would remove that condition (Theorem 2 of Bałaban 1989) is unpublished Cited (15-02 § 01)
Overlooking the domain	Settling the continuity of the map from the bare coupling to the coupling at the last step with "expected to follow from the construction"	The effective coupling is defined only while the previous step is small. Write continuity within the domain and the intermediate-value argument separately (15-03 Coupling-Smallness Induction Lemma)

The top four rows of the table have the same form. Getting the all-step version from the one-step version needs the assumption that "the constants do not degrade over the steps", and unless this is made explicit on the side of the rules, a one-step proof looks like an all-step proof. The bundle of rules in 15-03 puts this caution into the way the rules are written.

04

The map of this part and the reading order

Table 2 lists the chapters of Part 15. The first half (15-00 to 15-10) is the record of the study of ultraviolet control, and the second half (15-11 to 15-14) consists of theorems completed in Lean for one step of the free field and one scale. The level marks are: [2nd year] = readable by second-year students of mathematics or physics; [3rd year+] = third year and above (uses estimates of measures, operators and norms); [research] = research level (follows the construction of the study as it stands).

Table 2 The chapters of Part 15

Chapter	Title	One line	Premise	Level
15-00	Entrance — what ultraviolet control is	The number of steps, the same form, the three conditions, the patterns of misreading, a note on words	7-05, 7-07	[2nd year]
15-01	The difficulty of the ultraviolet	With a dimensionless coupling the sum of losses diverges logarithmically; properties of the weight that group-valued fields lack; re-fixing the gauge	15-00	[2nd year]–[3rd year+]
15-02	A map of the sources	The five stages of Bałaban's series, the BBS chain, the ultraviolet statement for a small twisted four-dimensional torus, two rows absent from the sources	15-01	[3rd year+]
15-03	The decomposition statement	A class that closes with finitely many parameters and three remainders, the Coupling-Smallness Induction Lemma, the ledger of the transverse contraction, questions	15-01, 15-02	[3rd year+]
15-04	The one-link heat-kernel class and the hierarchical model	A class that closes with one parameter, the invariant curve of Migdal–Kadanoff and the contraction 1/16	15-03	[2nd year]–[3rd year+]
15-05	The flux-network theorem	The transverse contraction and the parameter flow in one projection, the box table, two steps	15-03, 15-04	[2nd year]–[3rd year+]
15-06	Counting the parameters	Couplings by loop shape, the criterion, the attracting class	15-04, 15-05	[3rd year+]
15-07	Background-dependent fluctuations η	The Background–Fluctuation Decomposition Lemma, the group version of closure under the fluctuation expectation ((iii) of BBS), the Lemma on One Step with Background-Independent Fluctuations, shifting the local couplings	15-02, 15-03	[3rd year+]
15-08	The four rows that do not close, and proposals for them	The remainder including large fields, the source of the observables, η over many steps, gauge fixing	15-03, 15-07	[3rd year+]–[research]
15-09	What cannot be said, the index of questions, the dependency diagram	What cannot be said for the part as a whole, the ultraviolet questions, the table of Lean correspondences	All chapters	[2nd year]
15-10	Systematic attempts	Dominant matrices, the loss from discarding signs, the geometry of representatives, the double valuation, the cone calculus (a study; not the main line)	15-05, 15-08	[3rd year]–[research]

Chapter	Title	One line	Premise	Level
15-11	Entrance — one step and one scale	The scope of the chapters of the second half (linear, vacuum, one scale)	19-04	[2nd year]
15-12	One step and many steps of the free field	The recursion for the symbol obtained by pushing the free field with the path kernel; contraction uniform in the step (Lean)	15-11, 19-04	[3rd year+]
15-13	Linearisation of the block map and the covariance of the push-forward	Locality, gauge covariance, linearisation around the identity, the covariance of the push-forward (Lean)	15-12	[3rd year+]
15-14	What the second half (15-11 to 15-13) says, and what it does not	The theorems of the second half in one table, with their scope and Lean declarations	15-11 to 15-13	[2nd year]

There are three reading orders. **Shortest**: 15-00 → 15-03 → 15-09 (what has to be shown, and what remains). **The computational line**: 15-04 → 15-05 → 15-06 (how the numbers that appear in the decomposition can be measured on the smallest systems). **The line of the wall**: 15-02 → 15-07 → 15-08 (where the sources stop, and what does not close). 15-01 can be read before any of the lines.

Relation to the wall of Part 23. For the core Poincaré uniformity assumption of the first step of Part 23 (the study on the double well in 0+1 dimensions), besides the two existing ways of supporting it, there is a candidate for a third route that supports it by the influence of a coarse chain (the Dobrushin condition), and the column sums of that influence are bounded by $\rho_{\mathrm{col}} \le 0.176 < 1$ at 23 points of $\hbar \in [10^{-5}, 1]$ [computer-assisted (double precision, not interval arithmetic)] (on the assumptions of an outward margin and of the precision of the floating-point functions; between the points and for $\hbar < 10^{-5}$ it is Reading). This is a part of the route on the time side of Part 23 and is independent of the renormalisation steps of this part (for the distinction between the words, see Table 0).

05

The symbols of the five grades

Every statement in this part carries one of the following five labels (the same five grades as the whole book). A label marks "how it was checked", not how weighty the statement is.

Table 3 The labels of the five grades (how they are used in this part)

Label	Meaning	Examples in this part

Label	Meaning	Examples in this part
Proved here	A proof placed on paper (an elementary proof)	The Coupling-Smallness Induction Lemma, the Background–Fluctuation Decomposition Lemma, the rational numbers of the bundle network, the algebra of the one-loop leading term (written "Proved here, at the one-loop level")
Cited	Known results checked in the sources	Bałaban's Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989), Lemma A.6.1 of BBS
Computer-assisted	Computer-assisted (interval arithmetic)	The tube and chain of the hierarchical model in 15-04. An estimate that only adds an outward margin to double-precision floating point is written separately as [computer-assisted (double precision, not interval arithmetic)]
Reading	Numerical readings, physical estimates	Values of the source, the contraction at finite β, agreement with Monte Carlo
Assumption	Assumptions, statements not yet reached	Ultraviolet control, the decomposition statement, the twisted-torus ultraviolet statement

Components checked in Lean are written in a separate column rather than as a label, and are collected in the correspondence table of 15-09. The Lean components of this part are all lemmas about finite sums of real numbers or algebra; the lattice, the group and the measure are not formalised.

06

What cannot be said

(a) Within this part, ultraviolet control is not closed. Neither this chapter nor any chapter of this part shows the three conditions of ultraviolet control, in four dimensions or in two. Existence of the four-dimensional continuum limit was shown by the published construction with a different design (Parts 6–13).

(b) (4) is arithmetic for when the one-step losses act multiplicatively; it is not a theorem about an actual sequence of effective couplings or about actual constants.

(c) The ten patterns of Table 1 are logical cautions; they are not a claim that any existing argument is wrong.

(d) The level marks and reading orders of Table 2 are the book's arrangement; they are not a judgement of how hard the contents of the chapters are.

07

Table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Grounds	Scope
[01]	The number of steps $K = \log_L(L_0/a) \to \infty$ ((1))	Proved here	§ 01 (definition)	L_0 fixed
[02]	The form (2) of the effective density (a sum over small fields and large fields)	Cited	§ 01, 7-05 § 04, § 07	The level of the form (Dimock II, the wording of the three-dimensional theorem)
[03]	The three conditions of ultraviolet control and the form (3) that closes in one step	Assumption	§ 01, the ultraviolet-control row of 27-02	From a to L_0; constants independent of k, a
[04]	The lower bound $\prod(1-\delta_k) \geq e^{-2\sum \delta_k}$ on the product of losses ((4))	Proved here	§ 03	$0 \leq \delta_k \leq \frac{1}{2}$
[05]	Bałaban's theorems in three and four dimensions (1989) are inductions of the type (3)	Cited	§ 03, 7-05 § 07, 7-07 § 03	Finite volume; in four dimensions, conditional on the coupling staying small
[06]	The ten patterns of misreading, and the two pieces of arithmetic for worst-case geometry and goodness on the fine lattice	Proved here	§ 03 Table 1	Logical patterns. The rows with numbers are for fixed $q > 0$ and one-step losses that act multiplicatively

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV STEP   number of steps K → ∞                                  (L₀ fix
[02] CIT FORM   the form (2) of the effective density
[03] ASM UV     the three conditions of UV control and the form (3) that close
[04] PRV PROD   lower bound (4) on the product of losses                (0 ≤ δ
[05] CIT BAL    Bałaban's theorems are inductions of type (3) ⟸ [02]    (fini
[06] PRV MIS    the ten patterns of misreading ⟸ [04]                   (logi
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. The row of the three conditions of ultraviolet control is an assumption row, a restatement of ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

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08
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The conditional implication assembled by this book's study, and its known flaws

Beyond the components of this part, this book's study assembled a conditional implication (a chain) about families of pushed-forward measures (28-01 implication of subsequential limits, 28-02 implication of full-sequence convergence). Its form is: push the fine-lattice measure forward to the coarse lattice by a fixed block map, and under the premises of this implication (from the list of premises in 28-04: the small-field step, the large-field estimate and gas, induction uniform in the step, the coupling bracket and so on), subsequential limits of the pushed-forward measures exist, and properties of the finite lattice (rarity, reflection positivity, symmetries, how observables factorise) pass to those limits Assumption . Since four-dimensional existence was shown by the published construction with a different design, we read this implication not as a statement of existence but as a record of what this book's design took as premises. As a record, we write down three known flaws.

(a) **The two parts are not connected inside the theorem.** The Lean theorem of the chain is a juxtaposition of two things: a lemma about the function of the effective coupling (a point matching any target value can be taken inside the bracket), and compactness and passage to the limit for families of measures. Inside the theorem, both the family of fine measures and the sequence of bare couplings can be chosen arbitrarily, and there is no field linking the two. What would link them is the condition "match the sequence of bare couplings with a quantity determined by the measure", and existence of a matched sequence remains a condition inside this implication. That a sequence matching the expectation of the average of coarse plaquettes exists for the side-3 design (14-03) is a separate theorem, and there is no statement connecting it to the chain theorem.

(b) **Without matching, the convergence conclusion has no content.** Subsequential convergence of pushed-forward measures holds for any family of probability measures, by compactness alone. The content of this implication lies in carrying properties of the finite lattice over to the limit. Full-sequence convergence was placed not among the conclusions but on the premise side.

(c) **There are two ways of measuring compatibility in the large-field gas.** In the premise of the large-field gas there is a form that measures compatibility of polymers by "not adjacent" and a form that measures it by "supports do not intersect", and the prose description and the Lean statement took different forms. They are different gases. Which form the premise should be written in has not been decided.

All three are flaws on the side of this book's study that remain regardless of the results of the published construction.

What this chapter can say

Said	Not said
Proved here That the number of steps grows like $\ln(1/a)$, the lower bound (4) on the product of losses, and the arithmetic of the rows with numbers among the patterns of misreading	That the sum of losses is finite for an actual sequence of effective densities
Cited The form (2) of the effective density, and that Bałaban's theorems are inductions of the type (3)	An induction in four dimensions without the coupling-staying-small condition
Assumption The form that states ultraviolet control as three conditions	Any of the three conditions (four-dimensional existence was shown by the published construction with a different design)
Assumption The form of the conditional implication assembled by this book's study, and its three known flaws (§ 08)	Any of the premises of the implication. A statement connecting the two parts of the chain

Sources and reproduction

Item	Kind	Source / tools
The sequence of steps, the form of the effective density, small fields and large fields	Cited	The source table of 7-05 (J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705. T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275)
The Conditional Ultraviolet Stability Theorem in four dimensions	Cited (abstract and introduction, pp. 355–356)	T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (the source table of 7-07)
(1)(4), the rows with numbers in Table 1	Proved here	The arithmetic in the text of this chapter
The ten patterns of Table 1, Table 0, Table 2	This book's study	The record of this book's study (the arrangement of the study of ultraviolet control)

Words used in this chapter — ultraviolet control, the number of steps, the same form, a class that closes in one step, the small-field analyticity condition, the large-field rarity condition, the condition that the coupling stays small, the product of losses, conditional implication, the renormalisation step, the coarse plaquette, the five grades. Their meanings are in the A-02 glossary.

Next chapter: 15-01 The difficulty of the ultraviolet — why in four dimensions the sum of losses is not naively finite.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control dimensionless coupling, group-valued fields, gauge · Kiichi, Shiori and Rin

Why the ultraviolet is hard — dimensionless coupling, group-valued fields, gauge

This chapter organises established knowledge (sources carry the “Cited” label in the text). It is placed in this part as background for this book's own investigation.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — In four dimensions the coupling is dimensionless, so the sum of the one-step losses of size g_k^2 diverges logarithmically in the number of steps, and a class that absorbs the losses exactly into the flow of the coupling has to be built (computed line by line). In three dimensions this sum is finite and there is no need to follow the flow of the coupling. For group-valued fields, among the tools for the weight of flat fields, the property of "closing in one step under the expectation of the fluctuations" is missing. Gauge-invariant functions have no convexity in the orbit directions, the gauge can only be re-fixed scale by scale, and the seams also enter the estimates. The three sources of difficulty, and that even the two-dimensional model without gauge is stuck.

Prerequisite chapters — 15-00 § 01 (the three conditions and the form that closes in one step), 7-05 § 06 (the running of the coupling), 7-05 § 05 (local gauges), 24-07 § 05 (the three sources of difficulty), 24-06 § 07 (the four properties of the weight).

Said first — The computations in this chapter are arithmetic that treats the one-loop flow as an exact equation; they are not theorems about the actual sequence of effective couplings. We split the difficulty into three (the dimensionless coupling, group-valued fields, gauge), but the three are not independent; in 15-07 and 15-08 they gather in one place (the fact that the measure of the fluctuations depends on the background).

The order of this chapter

1. Difficulty 1: the dimensionless coupling — the sum of losses diverges logarithmically (equations (1)–(5)), and the contrast with three dimensions
2. Difficulty 2: a group-valued field — the four properties of the weight, and closure under the fluctuation expectation ((iii) of BBS), which falls short for groups
3. Difficulty 3: gauge symmetry — convexity in the orbit directions, re-fixing at each scale
4. The sources of difficulty, and models without gauge — the three sources, the two-dimensional $O(N)$ (Table 1)
5. What cannot be said
6. Table of the five grades and the dependency graph (Table 2)

00

What this chapter shows

- In four dimensions the sum of the one-step losses g_k^2 diverges like $\ln K$, so that part has to be absorbed exactly into the flow of the coupling, and the rest confined to sizes whose sum is finite (of type $g_k^{2+\delta}$, or a geometric series) (§ 01).
- In three dimensions $\sum_k g_k^2$ is finite, and the one-step losses can be carried as errors (§ 01).
- For a group-valued field, among the four properties of the weight, "closes in one step under the expectation of the fluctuations" (closure under the fluctuation expectation) is missing, and the measure of the fluctuations depends on the background (§ 02).
- At a minimum of a gauge-invariant function the Hessian form in the orbit directions is 0, and the tools of convexity cannot be used unless the gauge is removed. The gauge fixing works only at the size of a block, and the seams where it is re-fixed scale by scale also enter the estimates (§ 03).
- The core of the wall lies in "group-valued field × dimensionless coupling × large fields"; even for the two-dimensional $O(N)$ σ model without gauge ($N \geq 3$), the ultraviolet construction is stuck (§ 04).

01

Difficulty 1: the dimensionless coupling — the sum of losses diverges logarithmically

This section summarises existing knowledge (the arithmetic of the one-loop flow and the contrast with three dimensions). The introduction is in 7-06 § 02.

In four dimensions the coupling g has no dimension of length, and the only thing that weakens the coupling at fine scales is asymptotic freedom (24-07 § 01). Cutting the one-loop flow into steps, with $b := 2b_0 \ln L > 0$,

$$\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - b \quad (1)$$

Cited (equation (9) of 7-05 § 06, keeping only one loop). Below, we treat (1) as an exact equation and count the sum of losses line by line.

(a) The sequence of couplings. Fix the coupling at the coarsest step at $g_K = g_*$, and write in terms of the step $j = K - k$ counted from the coarse end ($j = 1, \dots, K$). Using (1) j times,

$$\frac{1}{g_{K-j}^2} = \frac{1}{g_*^2} + bj, \qquad \frac{1}{(b + g_*^{-2})\, j} \;\le\; g_{K-j}^2 \;\le\; \frac{1}{bj} \quad (j \ge 1) \quad (2)$$

Proved here. The inequality on the left comes from $1/g_*^2 + bj \le (1/g_*^2 + b)j$ ($j \ge 1$), and the one on the right just drops $1/g_*^2 > 0$. The coupling is smaller on the finer side, but only like $1/j$.

(b) The sum. Comparing with the integral of $1/x$, $\ln(K+1) \le \sum_{j=1}^{K} \frac{1}{j} \le 1 + \ln K$, so adding up (2),

$$\frac{\ln(K+1)}{b + g_*^{-2}} \;\le\; \sum_{k=0}^{K-1} g_k^2 \;\le\; \frac{1 + \ln K}{b} \quad (3)$$

Proved here. **If there is a loss of size g_k^2 at each step, its sum keeps growing like the logarithm of the number of steps.** Example: for $SU(3)$ and $L = 2$, $1/b = 1/(2b_0 \ln 2) \approx 10.4$, and each doubling of K raises the upper bound by $(\ln 2)/b \approx 7.2$. This matches the partial sums in 7-05 Table 4, which grow by about 7.2 with each doubling of K (arithmetic).

The measured one-step coefficient (note). The b of (1) treats the one-loop coefficient as an exact formula; if one fixes a blocking rule and measures one step, the decrease at the first step does not agree with the one-loop coefficient. On a periodic lattice that fixes every coarse link (the product along a line of L fine links), running the long-

wavelength coefficient of the coarse action of $SU(2)$ and $\beta = 4/g^2$ by one step ($L = 2$) gives $\beta' - \beta = -0.745$ (the Manton action), -1.078 (heat kernel) and -1.136 (Wilson): the direction is that of asymptotic freedom, and the size is 2.9–4.4 times the one-loop value $8b_0 \ln 2 = \frac{11}{3\pi^2} \ln 2 \approx 0.2575$ in units of β Reading (one loop, $O(1)$ of $\beta \to \infty$, a commuting background, extrapolated to wavenumber 0 from the series of transverse coarse tori 6^3 and coarse lengths ≤ 32 along the fourth axis). Of this, the constants that depend on the action (-0.31 for Manton, -0.70 for Wilson, and for the heat kernel additionally $-\frac{1}{3}$ from the Van Vleck prefactor) take other values for the actions of the second and later steps (which are no longer Wilson or Manton). The change when the blocking factor goes from 2 to 3 is -0.254 in the part that does not depend on the action, 1.7–1.9 times $-\frac{11}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ Reading (wavenumber 0 for factor 3 is a two-point extrapolation). Therefore, if the arithmetic of (1)–(5) is read with the numbers of the first step, the one-step decrease 0.2575 of β must be read as 0.75–1.14, and the coefficient when steps are stacked has not been measured. That the coefficient of $\ln L$ does not match at small blocking factors can be read as due to the rule that fixes the products along lines integrating fluctuations of wavelengths longer than the coarse scale Reading (15-03 § 02). The difference $-\frac{1}{3}$ between heat kernel and Manton and the difference $\frac{5}{3}t$ between Wilson and Manton (t is the Wilson value of $U(1)$) are identities Proved here (a count of the coefficients of s^2; the arithmetic of the coefficients is in Lean: `tadpole_weights`, `vanvleck_ratio`, `shift_bounds` ($0.2575 < \frac{11 \ln 2}{3\pi^2} < 0.2576$), `beta_shift_of_inv_sq` (assumption $1/g'^2 = 1/g^2 - 2b_0\ell$)).

(c) The product. If at each step a constant (the radius of analyticity, the rate of decay, the lower bound of the Hessian, and so on) shrinks by a factor $1 - cg_k^2$, then from $1 - x \leq e^{-x}$ and the left of (3),

$$\prod_{k=0}^{K-1} \left(1 - c\, g_k^2\right) \;\leq\; \exp\!\Big(-c \sum_{k=0}^{K-1} g_k^2\Big) \;\leq\; (K+1)^{-c/(b+g_*^{-2})} \;\longrightarrow\; 0 \qquad (K \to \infty) \tag{4}$$

Proved here (in the range $cg_k^2 \leq 1$). Since $K = \log_L(L_0/a)$, the constant goes to 0 like a power of $\ln(1/a)$. Slowly, but not uniformly.

(d) The size at which the sum is finite. If the loss is of size $g_k^{2+\delta}$ ($\delta > 0$), then from the right of (2)

$$\sum_{k=0}^{K-1} g_k^{2+\delta} \;\leq\; \sum_{j=1}^{\infty} (bj)^{-1-\delta/2} \;=\; b^{-1-\delta/2}\, \zeta\big(1 + \tfrac{\delta}{2}\big) \;<\; \infty \tag{5}$$

and this is finite uniformly in K Proved here. For $\delta = 2$ (a loss of g_k^4, the size next to one loop) it is $\pi^2/(6b^2)$.

Hence what is needed. Putting (4) and (5) together, the part that moves by size g_k^2 after one step cannot be carried as an "error". That part must be made **the motion of the parameter of the class itself**. Concretely, one defines the class $\mathcal{K}$ in a form that has the coupling g as its parameter, reads off the coefficient of the leading term (the Wilson term, the direction of F^2) from the effective action after one step, and sets that to be g_{k+1}. In this way the part of size g_k^2 is absorbed exactly into the flow of the coupling (3) of 15-00. The losses left after absorption (such as the degradation of the constants of the localised corrections E_k) must be confined to sizes whose sum is finite, of the type (5) — typically $O(g_k^4)$, or irrelevant terms that shrink geometrically in the scale Proved here (a requirement that follows from the arithmetic of (4)(5), in the case where the one-step losses act multiplicatively). Rewriting this requirement in the form of a statement gives the decomposition of 15-03.

On the side of the flow of the coupling things are different. The remainder r_k in (3) of 15-00 may be allowed, in the variable $1/g^2$, to be of size $|r_k| \le C g_k^2$ (the size of two loops); its sum grows like $\ln K$ as in (3), but since $1/g_k^2$ itself grows like bK, the sequence of couplings does not collapse. What is needed is that the sequence stays in $(0, \gamma]$, and that is handled by the Coupling-Smallness Induction Lemma of 15-03.

Contrast with three dimensions. In three dimensions g^2 has the dimension of length to the power -1, the dimensionless coupling at step k has the form $g^2 L^k a$ and is geometrically small on the fine side, and $\sum_k g_k^2$ is finite (it stays within 2 in 7-05 Table 4) Cited . Bałaban 1985 writes that the types that can diverge are finitely many, that the counterterm needed is a single constant, and that by semisimplicity there are no variables that grow in the effective action (relevant variables) (7-06 § 05) Cited . In three dimensions the one-step loss g_k^2 can be carried as an error, and there is no need to follow the flow of the coupling. In four dimensions the F^2 term is exactly marginal and must be followed. This is the precise form, on the ultraviolet side, of the source: dimensionless coupling, among the sources of difficulty in 24-07 Table 3.

How "absorption" looks in the smallest setting — that for the one-link heat-kernel class the losses form a telescoping sum — is seen in 15-04 § 01.

02

Difficulty 2: a group-valued field — the four properties of the weight, and "closure under the fluctuation expectation", which falls short for groups

The edge variables U_l of the lattice are elements of the group G. Group elements cannot be scaled, nor added and divided (7-05 § 05 (i)(ii)) Cited . Since the group is compact the field is bounded, but a "large field" is the far side of the group (far from the identity), where the curvature of Wilson's one-link weight is negative (the One-Link

Wilson Weight Proposition of 24-06 § 03) Proved here (proved in 24-06). Large fields can only be paid for with rarity, not with convexity.

The tools for flat fields. In the renormalisation of scalar fields, the rarity of large fields is put into the norm of a weight (regulator) $\tilde{G}(X, \phi)$, and closing under the expectation of one step is derived from the following four properties Cited (Lemma A.6.1 (One-Step Bound on the Activity Norm) in Appendix A.6 of the book by Bauerschmidt–Brydges–Slade; the four properties are those listed in § A.6.2 as the desired properties of a regulator; 24-06 § 07; the chain of the original papers is in 15-02 § 04).

1. It factorises into a product over disjoint sets.
2. $\tilde{G}(X, \phi + \phi') \leq \tilde{G}^2(X, \phi)\, \tilde{G}^2(X, \phi')$ (for a sum of fields).
3. $\mathbb{E}_+ \tilde{G}^t(X) \leq 2^{|B(X)|}$ (closes in one step under the expectation of the fluctuations).
4. $\tilde{G}^t \leq \tilde{G}_+$ (bounded by the weight of the next step).

From these follows the one-step estimate $\|\mathbb{E}_+ \theta K(X)\|_{\tilde{G}_+} \leq 2^{|B(X)|} \|K(X)\|_{\tilde{G}}$ (One-Step Bound on the Activity Norm). Submultiplicativity in sums of fields uses the addition of fields $\phi + \phi'$, and closure under the fluctuation expectation uses the fact that the fluctuations are Gaussian variables independent of the background. In the proof of the lemma, closure under the fluctuation expectation is used as the expectation of a function of the fluctuations alone, and that measure does not depend on the background ϕ Reading (from reading the chain of the proof).

What survives on the group, and what falls short. Submultiplicativity in sums of fields holds in the version where the sum of fields $\phi + \phi'$ is replaced by "background $\times$ fluctuation" $Ul = Wl e^{Al}$, using only the triangle inequality and conjugation invariance (the Background–Fluctuation Decomposition Lemma and its corollary, the Splitting of the Weight; 15-07 § 02) Proved here . What falls short is closure under the fluctuation expectation. For a group, the measure of the fluctuations depends on the background (the propagator in a background field, the title of the foundational paper in 7-07 Table 1). So the estimate of $\mathbb{E}_+ G^t$ needs the regularity of the background, and in the region of large fields, where the background is on the far side of the group, the fluctuations are not a small perturbation of the background. Writing the part due to the background dependence of the measure as η_k, stating the group versions of closure under the fluctuation expectation and of monotonicity to the next step in the form of a statement, and seeing how η_k is absorbed is the job of 15-07. For scalar fields the measure does not depend on the background, so $\eta_k = 0$, and the problem itself does not exist.

> 03

Difficulty 3: gauge symmetry — convexity in the orbit directions, and re-fixing at each scale

There is no convexity in the orbit directions. On a finite lattice the group of gauge transformations is a product of compact groups, and the action V is gauge invariant, that is, constant along orbits.

Orbit-Direction Flatness Proposition. Let V be a smooth gauge-invariant function on the configuration space (a compact manifold), and x_0 a minimum point of V. For a tangent vector $\xi \neq 0$ to the orbit through x_0, $\operatorname{Hess} V(x_0)(\xi,\xi) = 0$. In particular, if there is even one such ξ, then $\operatorname{Hess} V \geq m\, g$ $(m > 0)$ does not hold.

Proved here Proof. Let $\gamma(t) = e^{t\eta} \cdot x_0$ (η an element of the Lie algebra of gauge transformations with $\gamma'(0) = \xi$). Then $V(\gamma(t)) = V(x_0)$ holds for all t. Since $dV(x_0) = 0$, $\frac{d^2}{dt^2} V(\gamma(t))|_{t=0} = \operatorname{Hess} V(x_0)(\xi,\xi) + dV(x_0)(\nabla_{\gamma'}\gamma') = \operatorname{Hess} V(x_0)(\xi,\xi)$, and the left-hand side is 0. ∎

Tools that rely on convexity (Bakry–Émery, Brascamp–Lieb) cannot be used unless the orbit directions are removed Reading (a physical estimate; the same as the "theory" paragraph of 24-07 § 05). To remove them one fixes the gauge, but there is no continuous global gauge fixing that works for all configurations at once (Gribov 1978, Singer 1978) Cited (bibliographic only; 7-05 § 05).

A gauge works only at the size of a block. Taking the complete axial gauge on a two-dimensional block of side ℓ, the plaquette condition $\|U_{\partial p} - 1\| < \varepsilon$ gives for edges only $\|U_b - 1\| < \ell\,\varepsilon$ (7-05 § 05) Proved here (proved in 7-05). Once $\ell\varepsilon$ exceeds 2 this says nothing, so the gauge is chosen anew at each scale, within the size of a block. Bałaban's three-dimensional theorem introduces the axial gauge block by block by the Faddeev–Popov method (7-06 § 05) Cited . We have not checked in the text how the gauge is chosen concretely in the four-dimensional papers (7-07 Table 3).

The seams of re-fixing. The gauge at step k and the gauge at step $k+1$ are different, and the re-fixing is a field-dependent gauge transformation. Its Jacobian (the Faddeev–Popov determinant) and the terms that appear at the seams between neighbouring blocks also enter the effective action, and within the class of (3) of 15-00 they must be estimated as losses whose sum is finite in the sense of § 01 Reading (a physical estimate; it states the form of the estimate that is needed). MRS, constructed with a continuum cutoff, also joins the axial gauge of the large-field region with the background-field-dependent gauge of the small-field region, and has seams of the same type (7-07 § 06) Cited . The organisation in which, in a tree gauge, the Jacobian of re-fixing becomes a constant and the cost moves elsewhere (to the ghost determinant and the coupling at the seams) is in the row of the boundary of gauge re-fixing in 15-08.

04

The sources of difficulty, and models without gauge

Transcribing the three difficulties into the language of the sources of difficulty of 24-07 Table 3 gives Table 1. In this part, which deals only with the ultraviolet, the coupling is small at every step, so the source: no expansion linking scales does not appear Reading (a physical estimate).

Table 1 The difficulties and the three sources of difficulty (a reading; the names of the sources are from 24-07 Table 3)

Source of difficulty	Content (one line)	Form on the ultraviolet side	Chapters in this part
Source: dimensionless coupling	The coupling is dimensionless	The F^2 term is exactly marginal, and the sum of the losses g_k^2 diverges like $\ln K$ ((3)(4) of § 01). A class that absorbs the losses into the flow of the coupling is needed	15-03, 15-04–15-06
Source: no expansion linking scales	There is no expansion connecting the ultraviolet and the infrared	Does not appear (the coupling is small at every step). The task of the band in Part 26	—
Source: no global gauge fixing	There is no global gauge fixing, and large fields cannot be controlled by convexity	Broken down within one step: closure under the fluctuation expectation for group-valued fields (§ 02), convexity in the orbit directions (§ 03, Orbit-Direction Flatness Proposition), and the seams of re-fixing (§ 03)	15-07, 15-08

Models without gauge are also stuck. The two-dimensional $O(N)$ sigma model ($N \geq 3$) has no gauge and is a model with a dimensionless coupling and asymptotic freedom, yet the construction of its ultraviolet limit is listed as an open problem Cited (§ II.1.1 of the text of Rivasseau's 2000 review, which says that many experts have tried and not succeeded). The same place says that a hierarchical version of this model has been constructed (Gawędzki–Kupiainen, Comm. Math. Phys. 106 (1986) 533) Cited (the review's bibliography). That there is no construction even in finite volume, and that the point is restricted to asymptotically free $N \geq 3$, are this book's reading, not the words of the original Reading . The case $O(4)$ is the principal chiral model, whose field takes values in $S^3 = SU(2)$. As of 2024, transplanting Bałaban's method has reached the variational problem and the background-field stage (Goswami, a family of two-dimensional nonlinear sigma models; Dybalski–Stottmeister–Tanimoto, two-dimensional $O(4)$: the equation for the critical point is derived by the calculus of variations on the Lie group and solved by a contraction mapping) Cited (both from abstracts). On the other hand, the fermionic two-dimensional Gross–Neveu model (the

massive case) has been constructed (24-07 Table 3) Cited . The proposal to use this model as a test bench is in 15-02 § 05.

Reading from the contrast Reading (a physical estimate): the core of the wall can be read as lying in the combination "a field that takes values in a group (a nonlinear manifold)", "a dimensionless coupling (source: dimensionless coupling)" and "large fields", with gauge invariance a layer sitting on top of it. We do not say that gauge invariance is not an essential difficulty.

05

What cannot be said

(a) (1)–(5) of § 01 are arithmetic that treats the one-loop flow (1) as an exact equation, not theorems about the actual sequence of effective couplings. "A class that absorbs the losses into the flow is needed" is a requirement for the case where the losses act multiplicatively at each step.

(b) § 02 only states that, among the four properties of the weight, closure under the fluctuation expectation falls short for groups; it does not prove the group version of closure under the fluctuation expectation.

(c) "The core of the wall is group-valued field × dimensionless coupling × large fields" is an estimate from the known situation of two-dimensional models; we do not say that gauge invariance is not an essential difficulty.

(d) We do not show that the seams of re-fixing can be estimated as losses whose sum is finite (we only state the form of the estimate that is needed).

(e) The measured one-step coefficient of § 01 is a one-loop number at the first step of a single rule (a periodic lattice that fixes every coarse link by the product along a line); the coefficient when steps are stacked, finite β and non-commuting backgrounds have not been measured. The value for factor 3 is a two-point extrapolation.

06

Table of the five grades and the dependency graph

Table 2 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Grounds	Scope
[01]	The one-loop flow (1)	Cited	§ 01, 7-05 § 06	The equation keeping only one loop
[02]	Upper and lower bounds (2) on the sequence of couplings, and the logarithmic growth of the sum (3)	Proved here	§ 01	Under the one-loop flow (1)

Row	Statement (one line)	Grade	Grounds	Scope
[03]	The product of the losses $c g_k^2$ is $(K+1)^{-c'} \to 0$ ((4)); the sum of the losses $g_k^{2+\delta}$ is finite ((5))	Proved here	§ 01	Under (1), $c g_k^2 \le 1$, $\delta > 0$
[04]	In three dimensions $\sum g_k^2$ is finite and the counterterm is a single constant	Cited	§ 01, 7-05 § 06, 7-06 § 05	Three dimensions
[05]	The four properties of the weight and the one-step estimate (One-Step Bound on the Activity Norm)	Cited	§ 02, 24-06 § 07 (the text of Appendix A.6.2)	Flat fields
[06]	On the far side of the group, the curvature of the one-link Wilson weight is negative	Proved here	§ 02, 24-06 § 03 (One-Link Wilson Weight Proposition)	One link
[07]	Orbit-Direction Flatness Proposition: at a minimum point of a gauge-invariant function, the Hessian in the orbit directions is 0	Proved here	§ 03	Tangent vectors to the orbit $\xi \not= 0$
[08]	There is no continuous global gauge fixing	Cited	§ 03 (bibliography only)	—
[09]	The terms at the seams of re-fixing at each scale also need estimating as losses whose sum is finite	Reading	§ 03 (a physical estimate)	—
[10]	The ultraviolet construction of the two-dimensional $O(N)$ sigma model is listed as an open problem; the hierarchical version has been constructed. Transplantation has reached the background-field stage	Cited	§ 04 (the text of the review § II.1.1, its bibliography and abstracts)	Within our search
[11]	The core of the wall is group-valued field × dimensionless coupling × large fields	Reading	§ 04 (a physical estimate)	—
[12]	The measured one-step coefficient: $\beta' - \beta = -0.745 -- -1.136$ at the first step on the periodic lattice (2.9–4.4 times the one-loop value); the change for factor 2→3 is 1.7–1.9 times. The differences of the actions $-\frac{1}{3}$ and $\frac{5}{3}t$ are identities	Reading, Proved here (identities)	§ 01	$SU(2)$, one loop, commuting background, transverse 6^3, wavenumber 0 extrapolated, factor 3 from two points, first step only

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] CIT FLOW   the one-loop flow (1)
[02] PRV RUN    couplings (2), log sum (3) ⟸ [01]  (one loop)
[03] PRV LOSS   product → 0 (4), sum finite (5) ⟸ [02]  (cg² ≤ 1, δ > 0)
[04] CIT 3D     in three dimensions the sum is finite
[05] CIT BBS    the four properties of the weight
[06] PRV CURV   negative curvature on the far side  (one link)
[07] PRV ORB    Orbit-Direction Flatness Proposition
[08] CIT GRIB   no global gauge fixing
[09] RDG SEAM   the seams of re-fixing ⟸ [03][07][08]
[10] CIT SIGMA  two-dimensional O(N) not constructed
[11] RDG CORE   the core of the wall ⟸ [03][04][06][10]
[12] RDG MEAS   the measured one-step coefficient ⟸ [01][02]  (one rule, firs
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The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

What this chapter can say

Said	Not said
Proved here Under the one-loop flow, the sum of the one-step losses g_k^2 diverges like $\ln K$, and the sum of losses $g_k^{2+\delta}$ is finite ((2)–(5)). Hence a class that absorbs the losses into the flow is needed. Orbit-Direction Flatness Proposition	Theorems about the actual sequence of effective couplings. The construction of the absorbing class
Cited The finiteness of the sum in three dimensions, the four properties of the weight, the absence of a global gauge fixing, that the construction of the two-dimensional sigma model is listed as an open problem, and the stage its transplantation has reached	The group version of closure under the fluctuation expectation. The choice of gauge in the four-dimensional papers
Reading The form of the estimate of the seams of re-fixing; the reading of the core of the wall	The estimate of the seams itself
Reading The measured one-step coefficient (one loop at the first step on the periodic lattice; direction and order of magnitude agree with the one-loop value, the coefficient is 2.9–4.4 times it)	The coefficient when steps are stacked; comparison with averaging-type rules

Sources and reproduction

Item	Kind	Source / tools
The one-loop flow (1); the sum of the couplings in three dimensions	Cited	The source table of 7-05 (J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705. T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275)

Item	Kind	Source / tools
The four properties of the weight, One-Step Bound on the Activity Norm	Cited (the text of Appendix A.6.2 and of the One-Step Bound on the Activity Norm)	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474 (the source table of 24-06)
The situation of the two-dimensional sigma model	Cited (the text of the review § II.1 and § II.1.1 and its bibliography; abstracts)	V. Rivasseau, Constructive field theory and applications: perspectives and open problems, arXiv:math-ph/0006017 (the source table of 24-07). P. Goswami, arXiv:2204.08252, Ann. Henri Poincaré 25 (2024) 2065–2085. W. Dybalski, A. Stottmeister, Y. Tanimoto, arXiv:2403.09800. Hierarchical version: K. Gawędzki, A. Kupiainen, Comm. Math. Phys. 106 (1986) 533 (according to the review's bibliography)
Global gauge fixing	Cited (bibliographic data only)	I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12 (the source table of 7-05)
(2)–(5), Orbit-Direction Flatness Proposition	Proved here	Proofs in the text of this chapter. The only numbers are the arithmetic of $1/(2b_0 \ln 2)$ and $\pi^2/6$
The measured one-step coefficient	Reading (the identities for the differences of the actions are Proved here)	Numerical records of this book's study (the two routes, the Hessian of the real-space analysis and the transverse Bloch sum, agree to 10^{-10}; the tree eigenvalues are reproduced; a negative control giving 0 for Manton at $U(1)$). The arithmetic of the coefficients is in the Lean bundle (15-09 Table 3, re-verified)

Words used in this chapter — the dimensionless coupling, the sum of losses, absorption, marginal, the four properties of the weight, closure under the fluctuation expectation, Orbit-Direction Flatness Proposition, the seams of re-fixing, the three sources of difficulty. Their meanings are in A-02 Glossary.

Next chapter to read: 15-02 A map of the sources — how far Bałaban's series went, and where it stopped.

A map of the sources — Bałaban's programme, the BBS chain, two rows absent from the sources

This chapter organises established knowledge. § 05 and § 06 are a map of open questions.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — A map dividing Bałaban's series on gauge theories into five stages by the work each takes on. That the Conditional Ultraviolet Stability Theorem of the last four-dimensional paper (Theorem 1 of Bałaban 1989) assumes "the effective coupling stays small", and that the basic inequality which removes it (Theorem 2 of Bałaban 1989) is unpublished (from the text of the source). How far MRS and the rewriting in an easier model (three-dimensional QED) have come. By what chain the one-step estimate for flat fields (Lemma A.6.1 in Appendix A.6 of Bauerschmidt–Brydges–Slade, the One-Step Bound on the Activity Norm) is obtained. A proposal for stating ultraviolet control on a small twisted T^4 with the zero modes removed, and where each of its rows falls in the sources. That what is absent from the sources are two rows — the one-step remainder including large fields, and the source for observables.

Prerequisite chapters — 15-01 § 02 (the four properties of the weight), 7-07 § 03 (the main theorem), 7-07 Table 1 (the bibliography of the series), 25-01 (the small twisted T^2).

Said first — The texts we have checked are the opening of the three-dimensional paper, the abstract and introduction of the last paper, and the text of Appendix A.6.2 of BBS and of the One-Step Bound on the Activity Norm. For the other papers of the series we have gone only as far as the abstracts or the titles, and the label column states the range that was checked. The ultraviolet statement for a small twisted four-dimensional torus is a proposal and has not been proved.

The order of this chapter

1. Bałaban's programme — the five stages of the series (Table 1)
2. Where it stopped — the assumption of the Conditional Ultraviolet Stability Theorem, and the unpublished basic inequality
3. MRS, and rewriting in easier models
4. The chain of Appendix A.6 of BBS — from submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step ((ii)–(iv) of BBS) to the One-Step Bound on the Activity Norm
5. The twisted-torus ultraviolet statement and the test bench — the small twisted T^4 and two-dimensional $O(4)$
6. Correspondence with the sources — two rows absent from the sources (Table 2)
7. What cannot be said
8. The table of five grades and the dependency graph (Table 3)

00

What this chapter shows

- Bałaban's four-dimensional theorem assumes "the effective coupling stays in $(0, \gamma]$", and the theorem that removes this is unpublished. What was reached is: on a finite-volume lattice, upper and lower bounds on the effective densities and the partition function uniform in the lattice spacing, assuming the condition that the coupling stays small (§ 01, § 02).
- Rewriting in easier models has reached three-dimensional ϕ^4 and three-dimensional QED, but the later stages for three-dimensional Yang–Mills and the two-dimensional sigma model are not to be found (§ 03).
- The one-step estimate for flat fields follows from the chain: interchange of expectation and seminorm → the definition of the norm → submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step; and the measure of the expectation taken in closure under the fluctuation expectation does not depend on the background (§ 04).
- On a twisted T^4 with the zero modes removed, ultraviolet control can be stated in five rows, and only two rows are absent from the sources: the content of the coupling staying small in the twisted box (the one-step remainder including large fields, that is, the net one-step bracket of the coupling) and the observable-source condition (§ 05, § 06).

01

Bałaban's programme — the five stages of the series

This section summarises existing knowledge (Bałaban's series). The introduction is in 7-07 § 02.

Dividing Bałaban's series on gauge theories (Commun. Math. Phys., 1983–1989) into five stages by the work each takes on gives Table 1. The bibliography is the same as in 7-07 Table 1. The only text we have checked is the opening of the three-dimensional paper and the abstract and introduction of the last paper; stages 4 and 5 and the row for 119 were confirmed from abstracts. The "work it takes on" for stages 1–3 is read from the titles, not checked against the content. The "three conditions" are the ultraviolet control of 15-00 § 01.

Table 1 The five stages of Bałaban's series (labels are for the range checked; the one-line content is from the titles for stages 1–3, and from abstracts for stages 4 and 5 and 119)

Step	Papers (CMP volume, year, page)	Work it takes on	Correspondence with the three conditions	Label and range checked
1 Foundation	89 (1983) 571, 95 (1984) 17, 96 (1984) 223	Regularity and decay of lattice Green functions. The block-averaging transformation and the propagators under it	Components for every step	Cited Bibliography and titles only
2 Averaging of group-valued fields, and gauge	98 (1985) 17, 99 (1985) 75	Gauge-covariant averaging operations. The space of regular (small-field) configurations, and gauge fixing that works locally on it	The handling of the source: no global gauge fixing in 15-01 § 02, § 03	Cited Bibliography and titles only
3 Background field	99 (1985) 389, 102 (1985) 277	Propagators in a background field. The variational problem with the averages fixed, and the background field	The entrance to the small-field analyticity condition	Cited Bibliography and titles only

Step	Papers (CMP volume, year, page)	Work it takes on	Correspondence with the three conditions	Label and range checked
4 Small-field expansion and the coupling	109 (1987) 249, 116 (1988) 1	In the small-field approximation, a sequence of localised effective actions by a cluster expansion at each step. Constructs the β function and defines the renormalisation of the coupling by recursive renormalisation-group equations (109). The exponentiated cluster expansion satisfies the induction hypotheses, and the construction in the small-field approximation is completed (116)	The small-field analyticity condition, and the perturbative part of the coupling staying small	Cited Abstracts
(Induction including large fields)	119 (1988) 243	A description of the complete induction for the effective densities including large-field regions, and the renormalisation transformation preserving that form. Completes here the analysis of superrenormalisable models (three dimensions) and gives a convergent expansion	Completes the small-field analyticity condition and the large-field rarity condition in three dimensions. In four dimensions, the foundation of stage 5	Cited Abstracts
5 Large fields	122 (1989) 175, 122 (1989) 355	The basic step of the R operation. Localisation, exponentiation, estimates. The Conditional Ultraviolet Stability Theorem	The large-field rarity condition. The coupling staying small is assumed	Cited 175 from the abstract (described as an operation that removes the main obstacle to the ultraviolet stability of four-dimensional gauge theories). 355 from the abstract and introduction, pp. 355–356
(Three dimensions)	102 (1985) 255	Ultraviolet stability in three dimensions (equation (10) of 7-05). The author writes 'the proof is very sketchy'	The small-field analyticity condition and the large-field rarity condition in three dimensions. The coupling staying small is not needed (15-01 § 01). The complete induction including large fields is completed in 119	Cited Text pp. 255–257 (the relation to 119 from the abstract of 119)

02

Where it stopped — the assumption of the Conditional Ultraviolet Stability Theorem, and the unpublished basic inequality

The theorem in the introduction of the last paper is as follows Cited (7-07 § 03).

'Theorem 1. If the sequence of the effective coupling constants is contained in an interval]0, γ] with a sufficiently small positive γ, then the effective densities ρk have the form, and satisfy all the conditions and bounds, described in Sect. 2.'

The assumption is the condition that the coupling stays small itself, and the conclusion is the form of the effective densities ((2) of 15-00) and the estimates of the small-field analyticity condition and the large-field rarity condition. The author says that the assumption can be removed because it follows from a basic inequality derived from Theorem 2 of an earlier paper, and continues: 'The proof of Theorem 2, which is based on second order perturbative calculations, is very awkward and long ... and has not been published yet' Cited . Within our search of the bibliography of the author's papers after 1989, we found none that contains the proof of the basic inequality (unpublished) (the source table of 7-07). So the four-dimensional achievement is: **on a finite-volume lattice, upper and lower bounds on the effective densities and the partition function uniform in the lattice spacing, assuming that the coupling stays small** Cited (the level of uniform estimates in 7-07 Table 4). No source includes estimates of correlation functions, uniqueness of the limit, the OS axioms, infinite volume or the gap in what was reached. The review (Rivasseau 2000 § II.1) summarises Bałaban's result as the ultraviolet stability of the effective action after arbitrarily many steps in finite volume, and does not mention the assumption that the coupling stays small Cited (the text of the review). This is a difference in the level of summary; the statement in this section of what was reached rests not on the review but on the introduction of the last paper.

In one line, where it stopped is **the estimate of the one-step remainder that removes the assumption that the coupling stays small**. In the Coupling-Smallness Induction Lemma of 15-03 we see that the coupling staying small splits into that estimate (the net one-step bracket of the coupling) and an elementary part, continuity within the domain (definition and continuity of the effective coupling).

03

MRS, and rewriting in easier models

Magnen–Rivasseau–Sénéor (1993), with a continuum momentum cutoff, stated the ultraviolet limit of the Schwinger functions for four-dimensional $SU(2)$ in the trivial topological sector, with a fixed infrared cutoff and a regularised axial gauge, but wrote 'we do not provide a detailed proof' and said the details would run to 'several hundred pages' Cited (7-07 § 06).

Work rewriting the method in easier models has reached three-dimensional ϕ^4 (Dimock I–III, up to convergence of the expansion and the stability estimates) and three-dimensional QED (Dimock's three papers arXiv:1512.04373, 1712.10029, 2009.01156: the stage of tuning the bare quantities in the bounded-field approximation to fix the endpoint of the flow; multiscale propagators and the local regularity of minimisers; and upper and lower bounds for ultraviolet stability in a fixed finite volume, with the flow fully controlled at weak coupling) Cited (the three QED papers from abstracts; the source table of 7-05, 24-07 § 05). For three-dimensional Yang–Mills, no series of the same form was found within our search by author name. Transplanting it to the two-dimensional sigma model has reached the point corresponding to stage 3 of Table 1 (15-01 § 04), and work corresponding to stages 4 and 5 has not yet been written even for easier models (within our search) Cited (abstracts). For fields valued in a sphere, Bałaban's series on the low-temperature expansion of the classical N vector model gives a description of the complete induction including large-field regions (abstract of Comm. Math. Phys. 196 (1998) 485) Cited (abstract). This is an infrared (low-temperature) problem, not the two-dimensional ultraviolet Reading .

04

The chain of Appendix A.6 of BBS — from three properties of the weight to the one-step estimate

In the renormalisation of flat fields (scalar fields), the estimate of the expectation in one step follows by a short chain from the properties of the weight (regulator) $\tilde{G}$ alone. Of the four properties of 15-01 § 02, the lemma uses

three: submultiplicativity in sums of fields, closure under the fluctuation expectation (closing one step with the expectation over the fluctuation) and monotonicity to the next step (bounding by the weight of the next step). The text of the source is as follows Cited (Bauerschmidt–Brydges–Slade, arXiv:1907.05474v2, Appendix A § A.6.2, the text on pp. 214–215).

'Lemma A.6.1. Suppose that $\tilde{G}$ obeys (ii), (iii) and (iv) above. Then $\|\mathbb{E}_+\theta K(X)\|_{\tilde{G}_+} \le 2^{|B(X)|}\|K(X)\|_{\tilde{G}}$.'

The proof is a chain of three steps Cited (the text). θ is the map that shifts the field by $\phi \mapsto \phi + \zeta$, $\mathbb{E}_+$ is the expectation over the fluctuation ζ, and $T_\phi(h)$ is the Taylor seminorm at scale h at the point ϕ.

$$\|\mathbb{E}_+\theta F\|_{T_\phi(h_+)} \;\le\; \mathbb{E}_+\|F\|_{T_{\phi+\zeta}(h)} \;(1)$$

$$\|K(X)\|_{T_\phi(h)} \;\le\; \|K(X)\|_{\tilde{G}}\,\tilde{G}(X,\phi) \;(2)$$

$$\mathbb{E}_+\tilde{G}(X,\phi+\zeta) \overset{\text{(ii)}}{\le} \tilde{G}^2(X,\phi)\,\mathbb{E}_+\tilde{G}^2(X,\zeta) \overset{\text{(iii)}}{\le} 2^{|B(X)|}\,\tilde{G}^2(X,\phi) \overset{\text{(iv)}}{\le} 2^{|B(X)|}\,\tilde{G}_+(X,\phi) \;(3)$$

(1) is the interchange of expectation and seminorm ((A.6.5) of the source, which takes it from Prop. 3.19 of reference [54]), (2) is the definition of the norm ((A.6.7) of the source), and (3) is the application of the properties ((A.6.8) of the source; the power convention in (3) is written with the notation abbreviated, and in the source it is in the form of the power t). Putting (2) into (1) and paying for the expectation with (3) gives the lemma.

One thing is learned from reading this chain Reading (a reading of the chain of the proof). The expectation in closure under the fluctuation expectation is the expectation of a function of the fluctuation ζ alone, and **its measure does not depend on the background** ϕ. The interchange in (1) also holds with factor 1 because the law of the fluctuation does not depend on the background. For group-valued fields, a background-dependent factor $1 + \eta_k$ enters here. What becomes of each step of this chain on the group — that submultiplicativity in sums of fields holds by the Background–Fluctuation Decomposition Lemma, and that if the law of the fluctuation does not depend on the background every step goes through and gives $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations) — is in 15-07 § 02 and 15-07 § 05.

05

The twisted-torus ultraviolet statement and the test bench — the small twisted T^4 and two-dimensional $O(4)$

On a periodic T^4 the classical minima at the last step (flat connections) form a continuous family, and the integrals of the last few steps degenerate. We put forward, as a proposal, a statement in which this incidental difficulty is removed by 't Hooft's twist. It moves the small twisted T^2 of the spatial road (25-01) to the whole four-dimensional torus.

Setting. $G = SU(2)$, a four-dimensional torus of side L_0, blocking factor 2, the number of points on a side of the last lattice M (fixed), the number of steps n, and lattice spacing $a_n = L_0/(M \cdot 2^n)$. Put the twist $n_{12} = 1$ into Wilson's action (multiply one stack of plaquettes in the $(1,2)$ plane by the central element $z = -1$):

$$S_{\mathrm{tw}}(U) = \beta \sum_p \Big(1 - \tfrac{1}{2}\operatorname{tr}\big(z_p U_{\partial p}\big)\Big), \qquad \beta = \frac{4}{g_0^2} \;(4)$$

Define the blocking transformation T by an averaging that is gauge covariant and commutes with central shifts, and set $\rho_0 = e^{-S_{\mathrm{tw}}}$, $\rho_{k+1} = T\rho_k$.

The twisted-torus ultraviolet statement (proposal, unproved). There are a $\gamma > 0$ and constants $C, \kappa, c, r > 0$ depending only on M such that, for every $\gamma_0 \in (0, \gamma)$ and every n, choosing the bare coupling $g_0 = g_0(n, \gamma_0)$, the following hold Assumption.

- **The coupling stays small (twisted box)**: the sequence of effective couplings $g_0 < g_1 < \cdots < g_n = \gamma_0$ is defined throughout.

- **Small-field analyticity (twisted box)**: ρ_n has the form (2) of 15-00, and each $E_n(X,\cdot)$ is analytic in a complex neighbourhood of radius r in the background-field variables, with $|E_n(X)| \le C\gamma_0\, e^{-\kappa d(X)}$.
- **Large-field rarity (twisted box)**: $|R_n(\Omega^c)| \le \exp\bigl(-c\,p(\gamma_0)^2|\Omega^c|\bigr)$ (p is the same threshold factor as in the large-field rarity condition of 15-00).
- **Gaussian comparison condition for the partition function**: $e^{-CM^4} \le Z_n/Z_n^{\mathrm{G}} \le e^{CM^4}$ (Z_n^{G} is the sum of the Gaussian integrals around the isolated flat connections).
- **Observable-source condition**: the three above hold with the same constants even when sources (of strength $\le s_0$) are inserted for smoothed observables whose support is larger than the scale of the last lattice L_0/M.

None of the constants depends on n or on a_n. The coupling staying small in the twisted box has the form of the conclusion of the Coupling-Smallness Induction Lemma.

Where the twist works. The flat connections on the last lattice are only those whose holonomies in directions 1 and 2 anticommute and whose holonomies in directions 3 and 4 are central elements, and up to gauge equivalence there are finitely many of them. The adjoint fluctuations around them are antiperiodic in direction 1 or 2, and there are no zero modes Assumption (a reading that extends to four dimensions the facts about the twisted part of T^2 in 25-01; the four-dimensional classification has not been checked against the sources). At fine steps, on the other hand, the twist has no effect, because the twist's plane can be moved by central gauge transformations and is not visible locally Reading (a physical estimate). The Background–Fluctuation Decomposition Lemma (15-07 § 02) holds as it is for the twisted plaquette $z_p U_{\partial p}$: since z_p is a central element, one only replaces $W_{\partial p}$ in the proof by $z_p W_{\partial p}$ Proved here.

What the statement says and does not say. If shown, what it says is an estimate of the effective action uniform in the lattice spacing, and non-degeneracy at the last step; this amounts to Bałaban's Conditional Ultraviolet Stability Theorem + the unpublished basic inequality (the coupling staying small in the twisted box) + an extension to observables (the observable-source condition) + non-degeneracy at the last step from the twist. Subsequential limits of the smoothed observables follow from it by compactness. What it does not say is uniqueness of the limit, the OS axioms, recovery of rotations, non-triviality, or a gap uniform in the volume. **The difficulty of ultraviolet control at the fine steps is exactly the same as for the periodic T^4**; what the twist removes is only the incidental difficulty of the degeneracy of the last few steps Reading (a physical estimate). The role of the twisted-torus ultraviolet statement is not to reduce the difficulty but to state ultraviolet control in a pure form without zero modes.

A test bench with the gauge taken out. The two-dimensional $O(4)$ sigma model (the principal chiral model of $SU(2)$) has a field taking values in $SU(2)$, a dimensionless and asymptotically free coupling, and large fields, and no gauge. Writing ultraviolet control on the twisted cylinder $\mathbb{R}\times S^1_L$ (no zero modes because of the twist) gives the four-dimensional ultraviolet control with the gauge half of the source of difficulty "source: no global gauge fixing" taken out Reading (a physical estimate). As in 15-01 § 04, the ultraviolet limit has not been constructed for this model either, and transplantation has reached the variational problem and the background-field stage Cited. Writing stage 4 of Table 1 (one step of the small-field expansion) for this model is cheaper than rereading the second half of the four-dimensional series, and is a test that cuts out the "group-valued field × dimensionless coupling" part of ultraviolet control. Two-dimensional results imply nothing about four dimensions.

06

Correspondence with the sources — two rows absent from the sources

Table 2 sets out, for each row of the twisted-torus ultraviolet statement, where it falls in Bałaban's series, whether it changes with the twist, and whether it is absent from the sources. The last column gives the correspondence in three-dimensional QED, where the method has been rewritten for an easier model (Dimock's three papers in § 03).

Table 2 Each row of the twisted-torus ultraviolet statement and its correspondence with the sources (labels per cell; the labels for the series are at the level of abstracts, or of the introduction via 7-07 § 03)

Row	Where it falls in the series	With the twist ($n_{12}=1$)	Absent from the original sources?	Correspondence in three-dimensional QED

Row	Where it falls in the series	With the twist ($n_{12} = 1$)	Absent from the original sources?	Correspondence in three-dimensional QED
The coupling stays small (twisted box)	109: constructs the β function and defines the renormalisation of the coupling by recursive renormalisation-group equations (within the small-field approximation) Cited (abstract). 122, 355: with large fields it is an assumption (Conditional Ultraviolet Stability Theorem), and the basic inequality, which removes it, is unpublished Cited (7-07 § 03)	Unchanged Reading (a local quantity of the fine steps; the twist's plane is moved by central gauge transformations)	**Absent**: the one-step remainder including large fields (the net one-step bracket of the coupling). The rest is elementary by the Coupling-Smallness Induction Lemma	1512.04373: shows, in the bounded-field approximation, the pattern of making the bare quantities depend on the lattice spacing so that they take predetermined values at the unit scale Cited (abstract). What is tuned are relevant quantities (the vacuum energy density, the fermion mass), not a marginal coupling
Small-field analyticity (twisted box)	109 (localised effective actions by a cluster expansion at each step) and 116 (completion of the construction in the small-field approximation) Cited (abstracts). The abstracts contain no wording that the radius of analyticity is uniform in the step	The fine steps are unchanged. In the last few steps, the centre of the expansion becomes an isolated twisted flat connection Reading	The form exists (in four dimensions, under the assumption that the coupling stays small). The expansion around the twisted configuration at the last step is absent from the sources Reading	1712.10029: multiscale propagators and the local regularity of minimisers Cited (abstract)
Large-field rarity (twisted box)	119 (a description of the complete induction including large-field regions), 122, 175 (the basic step of the R operation), 122, 355 (localisation, exponentiation, estimates) Cited (abstracts)	Unchanged Reading	The form exists (in four dimensions, under the assumption that the coupling stays small)	2009.01156: closes including large fields Reading (from "the flow is fully controlled at weak coupling" in the abstract)
Gaussian comparison condition for the partition function	Upper and lower bounds on the partition function from the Conditional Ultraviolet Stability Theorem. The constants do not depend on the step, the lattice or the field Cited (7-07 § 03)	**Changes**: the integral at the last step becomes non-degenerate around the isolated flat connections Assumption (a reading that extends the fact about T^2 in 25-01 to four dimensions)	The form of the upper and lower bounds exists. The normalisation that makes the denominator a sum of Gaussian integrals around the isolated flat connections is absent from the sources, but it is a problem of finite-dimensional integrals Reading	2009.01156: upper and lower bounds on the renormalised partition function, uniform in the lattice spacing Cited (abstract)
Observable-source condition	The introduction of 122, 355 leaves the analysis of the expectations of loop variables as future work Cited (7-07 § 03). MRS: Schwinger functions in the axial gauge (with an infrared cutoff, no detailed proof) Cited	The fine steps are unchanged. The last step becomes non-degenerate, as in the Gaussian comparison condition for the partition function Reading	**Absent**	Correlation functions are left as the task of the next stage Cited (abstract of the overview arXiv:2204.07201)

What to read from Table 2 Reading . What is absent from the sources are two rows, the content of the coupling staying small in the twisted box (the one-step remainder including large fields, that is, the net one-step bracket of the coupling) and the observable-source condition; these two are where new mathematics is needed. The

normalisation of the Gaussian comparison condition for the partition function and the non-degeneracy of the last step are specific to the twist, but they are problems of finite-dimensional integrals, and the difficulty is of a different quality. Three-dimensional QED in 1512.04373 has, in the bounded-field approximation, a statement of the type that tunes the bare relevant quantities to fix the endpoint of the flow, and this is the form of the conclusion of the Coupling-Smallness Induction Lemma. Writing the exponent of the three ultraviolet conditions (twisted-box version) with $p(\gamma_0)^2$ matches the form of the large-field factor that comes from the original threshold $\varepsilon = g\,p(g)$ (equation (7) of 7-05); writing it with a negative power of γ_0 would give a statement stronger than the original threshold. A one-line statement of what the two rows must show to close, and proposals for them, are in 15-08.

07

What cannot be said

(a) The content of the series in Table 1, apart from the opening of the three-dimensional paper and the abstract and introduction of the last paper, is from abstracts for stages 4 and 5 and 119, and a reading from the titles for stages 1–3.

(b) We do not know the content of the basic inequality (unpublished). The judgement that it is unpublished rests on the sentence in the introduction and on the bibliography within the range we searched.

(c) The chain of § 04 is the text of a source on flat fields and says nothing about the group. "The measure of the expectation taken in closure under the fluctuation expectation does not depend on the background" is a reading of the chain of the proof.

(d) The twisted-torus ultraviolet statement is a proposal and has not been proved. Even if proved, it says nothing about uniqueness, the OS axioms, non-triviality, infinite volume, or a gap uniform in the volume. The classification of twisted flat connections in four dimensions has not been checked against the sources.

(e) The correspondence in Table 2 is at the level of abstracts and of introductions (via 7-07); it does not reach the section and theorem numbers in the text of the series.

08

Table of the five grades and the dependency graph

Table 3 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Grounds	Scope
[01]	Abstracts of 109, 116, 119, 122 (175) of the series, and of the three three-dimensional QED papers	Cited	§ 01, § 03, Table 1 (abstracts)	Section and theorem numbers in the text not checked
[02]	The Conditional Ultraviolet Stability Theorem assumes that the coupling stays small; the basic inequality is unpublished	Cited	§ 02 (abstract and introduction)	Finite-volume lattice
[03]	MRS has an infrared cutoff and gives no detailed proof	Cited	§ 03 (introduction)	$SU(2)$, trivial sector, axial gauge
[04]	The One-Step Bound on the Activity Norm and the chain (1)–(3)	Cited	§ 04 (the text of Appendix A.6.2, pp. 214–215)	Flat fields. The source takes the inequality exchanging the expectation and the seminorm from reference [54]
[05]	In the chain, the measure of the expectation taken in closure under the fluctuation expectation does not depend on the background	Reading	§ 04 (a reading of the chain of the proof)	Flat fields
[06]	Ultraviolet statement for a small twisted four-dimensional torus	Assumption	§ 05 (proposal, unproved)	$SU(2)$, $n_{12} = 1$, finite volume; the three ultraviolet conditions (twisted-box version) in the form of the original threshold

Row	Statement (one line)	Grade	Grounds	Scope
[07]	The Background–Fluctuation Decomposition Lemma also holds for the twisted plaquette $z_p U_{\partial p}$	Proved here	§ 05 ($W_{\partial p}$ replaced by $z_p W_{\partial p}$ in the proof of 15-07 § 02)	z_p is a central element
[08]	What the twist removes is only the degeneracy of the last few steps	Reading	§ 05 (a physical estimate)	—
[09]	The twisted cylinder for two-dimensional $O(4)$ is a test bench that takes the gauge out of ultraviolet control	Reading	§ 05 (a physical estimate)	Two-dimensional results imply nothing for four dimensions
[10]	Correspondence of each row of the twisted-torus ultraviolet statement with the sources. Absent from them: the net one-step bracket of the coupling and the observable-source condition; the Gaussian comparison condition for the partition function is a finite-dimensional problem	Reading	§ 06 Table 2	Level of abstracts and introductions

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] CIT ABS    abstracts of the series and of three-dimensional QED
[02] CIT THM    Conditional Ultraviolet Stability Theorem, basic inequality unpublished
[03] CIT MRS    what MRS reached
[04] CIT BBS    One-Step Bound on the Activity Norm and the chain           flat fields
[05] RDG FLAT   measure of closure under fluctuation expectation independent of background ⟸ [04]
[06] ASM T4     twisted-torus ultraviolet statement ⟸ [02]                  proposal
[07] PRV RTW    Background–Fluctuation Decomposition Lemma unchanged by the twist   z_p central
[08] RDG TW     the twist only at the last step ⟸ [06]
[09] RDG O4     the two-dimensional O(4) test bench ⟸ [06]
[10] RDG MAP    twisted-torus ultraviolet statement and the sources ⟸ [01][02][03][06][07]   level of a
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The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. The row of the twisted-torus ultraviolet statement is an assumption row, in the form of a question that restates ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

What this chapter can say

Said	Not said
Cited The assumption of Bałaban's Conditional Ultraviolet Stability Theorem and the unpublished basic inequality, the abstracts of stages 4 and 5 and 119 of the series and of the three three-dimensional QED papers, the reservations of MRS, the One-Step Bound on the Activity Norm of BBS and its chain	Section and theorem numbers in the text of the series (stages 4 and 5 and 119 only up to abstracts, stages 1–3 titles only). The content of the basic inequality (unpublished)
Proved here That the Background–Fluctuation Decomposition Lemma is unchanged for the twisted plaquette	The non-degeneracy of the last step around the twist
Reading The correspondence of each row of the twisted-torus ultraviolet statement with the sources (Table 2), and that the measure in the chain does not depend on the background	Checking against the text of the series
Assumption The twisted-torus ultraviolet statement	Its proof. Any prospect that ultraviolet control closes

Sources and reproduction

Item	Kind	Source / tools
Conditional Ultraviolet Stability Theorem, basic inequality (unpublished)	Cited (abstract and introduction, pp. 355–356)	T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (the source table of 7-07)

Item	Kind	Source / tools
Stages 1–3 of the series (Table 1)	Cited (bibliography and titles only)	Comm. Math. Phys. 89 (1983); 95, 96 (1984); 98, 99 (two papers), 102 (277–309) (1985)
Stages 4 and 5 of the series and 119 (Tables 1 and 2)	Cited (abstracts)	T. Bałaban, Comm. Math. Phys. 109 (1987) 249–301; 116 (1988) 1–22; 119 (1988) 243–285; 122 (1989) 175–202
Three-dimensional ultraviolet stability	Cited (the text, pp. 255–257)	T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275 (the source table of 7-05)
Three-dimensional QED (§ 03, Table 2)	Cited (abstracts)	J. Dimock, Ultraviolet regularity for QED in d=3, arXiv:1512.04373; Multiscale block averaging for QED in d=3, arXiv:1712.10029; Ultraviolet stability for QED in d=3, arXiv:2009.01156; overview arXiv:2204.07201
Low-temperature expansion of the N vector model (§ 03)	Cited (abstracts)	T. Bałaban, Comm. Math. Phys. 196 (1998) 485
What MRS reached, and its reservations	Cited (pp. 325–328)	J. Magnen, V. Rivasseau, R. Sénéor, Construction of YM4 with an infrared cutoff, Comm. Math. Phys. 155 (1993) 325–383
The One-Step Bound on the Activity Norm and the chain (§ 04)	Cited (the text of Appendix A.6.2 and of the One-Step Bound on the Activity Norm, pp. 214–215)	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474v2 (the source table of 24-06)
Table 2, the twisted-torus ultraviolet statement	This book's study	The record of this book's study (organising the study of ultraviolet control)

Terms used in this chapter — the five stages of Bałaban's series, the Conditional Ultraviolet Stability Theorem, the basic inequality (unpublished), MRS, the One-Step Bound on the Activity Norm, the interchange of expectation and seminorm, the twisted-torus ultraviolet statement, 't Hooft's twist, the test bench, the two rows absent from the sources. Their meanings are in the A-02 glossary.

Next chapter: 15-03 The decomposition statement — splitting ultraviolet control into a class that closes with finitely many parameters and three remainders.

The decomposition statement — a class that closes with finitely many parameters, and three remainders

This chapter is a map of open questions. § 03 and § 04 are this book's own investigation.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — That ultraviolet control can be stated in a form that splits it into "a class that closes with finitely many parameters" and three remainders — the coupling-flow remainder condition, the shape remainder condition, and continuity within the domain (continuity of the coupling) (a statement only). That, among these, the condition that the coupling stays small follows in an elementary way from the estimate of the one-step remainder (the net one-step bracket of the coupling) and the definition and continuity of the effective coupling (the Coupling-Smallness Induction Lemma, with proof). That if the one-step map contracts transversely around the class, the departure from the class is bounded by a supremum rather than by a sum over steps, uniformly in the number of steps (the bookkeeping of the transverse contraction). That the class is better taken not as a family of constant multiples of the heat kernel or the Wilson action, but as the one-parameter class into which the one-step map attracts. That compared with the decomposition of the band, the form of the split is the same, and the difference lies in whether there is something to pay the one-step loss against. The three ultraviolet questions that come out of this decomposition (transverse contraction on a one-parameter class, minimal model of the remainder including large fields, smallest adjoint eigenvalue on the twisted lattice). In a revision we added a proposal for the final form of the coordinates of the class in the class-closure condition, and how the shape remainder condition closes when the background is given to the class as a parameter (conditionally). In the third revision we added the final form of the four-dimensional class coordinates (one parameter and ratio coordinates whose values are fixed by the flow), the blocking conditions needed for the shape remainder condition over many steps, the one-loop measurement of the flow of the coupling, and a one-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling.

Prerequisite chapters — 15-00 § 01 (ultraviolet control and the form that closes in one step), 15-01 § 01 (that the sum of losses diverges logarithmically), 15-02 § 01 (the assumption of Bałaban's Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989)), 26-06 § 01 (the decomposition of the band).

Said first — This chapter does not show ultraviolet control. What it shows are a statement of what would need to be shown, two elementary components (the Coupling-Smallness Induction Lemma and the bookkeeping of the transverse contraction), candidate rules, and questions. Of the implications of the decomposition, what we have checked in an elementary way is only the Coupling-Smallness Induction Lemma and the arithmetic of products and geometric series; none of the class-closure condition, the coupling-flow remainder condition or the shape remainder condition is shown in four dimensions. Unless stated otherwise, a "step" in this chapter means one step of the real blocking (with

many links, gauge re-fixing and large fields), and the numbers for the smallest systems (15-04 to 15-06) are not a substitute for it.

The order of this chapter

1. The class and the one-step map — the flow of the parameters Φ and the remainder r_θ
2. The decomposition statement — the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling
3. The Coupling-Smallness Induction Lemma — splitting the coupling staying small into the net one-step bracket of the coupling and continuity of the coupling (with proof)
4. The bookkeeping of the transverse contraction — the departure by a supremum, not a sum
5. The bundle of rules (Table 1)
6. Contrast with the decomposition of the band (Table 2)
7. Three ultraviolet questions (Table 3)
8. What cannot be said
9. The table of the five grades and the dependency graph (Table 4)

00

What this chapter shows

- The ultraviolet wall can be stated in a form that splits it into "a class that closes with finitely many parameters" (the class-closure condition) and "the remainder of the flow, the remainder of the shape, and continuity" (§ 02, a statement only).
- The condition that the coupling stays small, "the effective coupling stays in $(0, \gamma]$", follows from the estimate of the one-step remainder (the net one-step bracket of the coupling) and continuity within the domain (continuity of the coupling) by monotonicity and an intermediate-value argument alone (the Coupling-Smallness Induction Lemma, § 03). The real content is the net one-step bracket of the coupling.
- If the one-step map contracts transversely around the class by $\lambda < 1$, the departure from the class is bounded by $\sup_j \varepsilon(\beta_j)/(1-\lambda)$, uniformly in the number of steps (§ 04).
- The class is taken as "the one-parameter attracting class", and the deviation of the form $O(1/B)$ is carried by the shape remainder condition. We do not state it as a class that keeps the ratio of the quadratic couplings at 2 (§ 02; grounds in 15-06).
- The proposal is that the coordinates of the class in the class-closure condition can be written out as the coupling matrix, the shape, the background, the one-link coupling, the corner triple product and the multi-face quartic. If the background is given to the class as a parameter, $\sum_k \eta_k$ of the shape remainder condition becomes finite under conditions, and the difficulty gathers in the class-closure condition (§ 02).
- The four-dimensional class has the form "one parameter B + ratio coordinates whose values are fixed by the flow (2 quadratic, 3 cubic, 65 quartic)"; the coordinate of the skew neighbour is nonzero whatever the rule, but its value depends on the rule. The "local" of the class-closure condition is written with the rule attached. The shape remainder condition over many steps needs blocking

conditions, and the flow of the coupling agrees in direction and order of magnitude at one loop (§ 02; all at tree level or within the range measured at one loop).

- The way of splitting is the same as in the decomposition of the band, with the difference that in the ultraviolet there is nothing to pay the one-step loss with (§ 06).

01

The class and the one-step map

The goal of ultraviolet control was to show at every step the formula (3) of 15-00 § 01 — if the effective density at step k lies in the class $\mathcal{K}(g_k; r, \kappa, c)$, then the effective density at step $k+1$ lies in the class $\mathcal{K}(g_{k+1}; r, \kappa, c)$ with the same constants. As we saw in 15-01 § 01, in four dimensions the part that moves by an amount of size g_k^2 in one step cannot be carried as an error, and must be made a motion of the parameters of the class. This section sets up the frame in which that "motion of the parameters" and "the rest" are written separately.

Take a family of weights $\mathcal{C} = \{w_\theta : \theta \in \Theta\}$ indexed by finitely many parameters $\theta = (\theta^1, \dots, \theta^m) \in \Theta \subset \mathbb{R}^m$, and suppose that one step of blocking T takes the form

$$T\big(w_\theta\,(1+\rho)\big) = w_{\Phi(\theta)}\,\big(1 + r_\theta[\rho]\big) \quad (1)$$

Here $\Phi : \Theta \to \Theta$ is the **flow of the parameters** and $r_\theta[\rho]$ the **remainder** (gathering the localised irrelevant terms and the large-field factors); the size of the remainder is measured by the norm of the weight G_k of 15-07. (1) is a form of definition: for any T it can be written once the choice of Φ and r is fixed. The content lies in "whether Φ and r can be chosen so as to have estimates uniform in the step".

A rule of thumb for counting the parameters. For pure gauge theory in four dimensions, the local terms that remain under power counting, given gauge invariance and the lattice symmetries, are two, the vacuum energy (a constant) and the coefficient of F^2, and we read $m = 2$, $\theta = (E, \beta)$, $\beta = 4/g^2$ Reading (power counting; the four-dimensional version of the sentence in 15-01 § 01 that in three dimensions "the counterterm is a single constant"). In the smallest systems, however, the action after one step does not fit into a constant multiple of a single quadratic form, and "shape coordinates" such as the ratio of the couplings by loop shape move (15-06). Under power counting these coordinates correspond to irrelevant directions that act at the square of the lattice spacing; if the one-step map contracts them, then there is effectively one parameter, β, and the shape deviation is paid for by the transverse contraction λ — that is the reading Reading . The class-closure condition of § 02 puts this reading into the form of a statement.

02

The decomposition statement — the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling

From the following four, we set down a statement of the form that ultraviolet control follows uniformly in the number of steps Assumption (the form of the statement only; none of the four is proved in four dimensions).

- **Class-closure condition.** The number of parameters m is finite independently of the step, and the leading part of T returns into the family (every change of the leading part goes into Φ). The class is taken not as the family of constant multiples of the heat kernel or the Wilson action, but as **the one-parameter attracting class of the one-step map** — a single coupling B (in the direction of β) and a

fixed shape of $O(1/B)$ determined by the curvature of the group. The shape coordinates (the coefficients of the action of order ψ^4 and higher, the ratio of the couplings by loop shape, and so on) are directions that contract at each step, and the shape deviation from the class is carried by the shape remainder condition.

- **Coupling-flow remainder condition.** The β component of Φ is $\beta_{k+1} = \beta_k - 8b_0 \ln L + O(1/\beta_k)$. This is the net one-step bracket of the coupling of § 03 written in the variable β. The values measured at one loop for one step with one rule are in "the measured flow of the coupling" below: the direction and order of magnitude agree, and the coefficient of $\ln L$ does not agree at small blocking factors Reading .
- **Shape remainder condition.** $\sum_k \eta_k < \infty$ of the type $\|r\theta_k[\rho_k]\|_{G_{k+1}} \le (1+\eta_k)\, 2^{|B|}\, \|\rho_k\|_{G_k}$, and ρ_k itself is either bounded by a supremum through the transverse contraction (§ 04) or stays at a finite size of the sum. $\sum_k \eta_k < \infty$ can be expected only for the background-dependent **shape part** (the part whose coefficients are proportional to the values of the shape coordinates and which enters a geometric series); the **group-geometry part** of $SU(2)$ (the BCH cubic) has a divergent sum of the type $g_k^2 p$, so the field strength of the background is given to the class as a parameter and moved to the side of the parameters Reading (15-07 § 07). If the logarithm a of the background holonomy is given to the class as a parameter, in the smallest system the rest drops to $\theta^0 \beta^{-3/2}$, and $\sum_k \eta_k$ is bounded uniformly in the number of steps (conditionally, 15-07 § 06). The difficulty has not disappeared, however, but moved to the class-closure condition — whether a class carrying a closes in one step of faces and multiple loops Reading . The contraction of the cone ℓ^4 at tree level for the shape remainder is at most 0.823 in the first step (blocking that takes a coarse edge as the sum of two fine ones, with the normalisation that divides by the long-wavelength coupling), but **it is not uniform in the number of steps, and blocking conditions are needed** Reading : the covariance-flow fixed-point condition: the flow of the covariance of the blocked field has a scale-invariant fixed point (with blocking by edge sums the coarse action becomes half as soft at each step and there is no fixed point, the factor being 1.26 or more at the third step; with gauge-covariant transverse averaging the direction is towards a fixed point). The cone-contraction condition at the fixed point: $\sup \mu_\infty < 1$ at that fixed point (with transverse averaging the lower bound 0.57–0.64 does not grow with the step; there is no upper bound). Conditions: restricted to the abelian case, tree level, periodic, 2^4 blocks, linearised; the numbers come from the symbol flow ($k \le 5$) and local optimisation over periodic configurations on the torus, with no interval estimates (15-05 § 07).
- **Continuity of the coupling** (continuity within the domain; § 03).

The coordinates of the class in the class-closure condition (a proposal for the final form). Putting the results for the smallest systems side by side, we read that the class of the class-closure condition is best stated as a family with the following coordinates Assumption (only the names and origins of the coordinates; we do not show that it closes in four dimensions).

- **The coupling matrix** B_{pq}: the coupling B_p per face, and the coupling of pairs of adjacent faces (placed at the ends of the shared edge). In one step of three-dimensional $2 \times 2 \times 2$, the quadratic part closes exactly with these two by the symmetry of the coarse box, and the tree-level ratio is $1/10$ (15-07 § 07).
- **The shape** q_p: the shape coordinates of $O(1/B)$ per face. They contract at each step (at most $17/125$ for four in a row, $1/25$ for the cross).
- **The background** a_p: the logarithm of the background holonomy per face. In one step of a single loop it is updated exactly by the ordered product in the group (non-abelian Stokes), and the BCH

cubic enters this coordinate at all orders (15-07 § 07).

- **The one-link coupling** b: the size of the sum of staples $\times\ \beta$. Re-taking the centre moves it only by cg^2 p^2, and it is carried as an update of the parameters (15-07 § 05).
- **The corner triple product** κ: $\kappa \sum_c a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of the cube. It is the only cubic non-abelian term in $2 \times 2 \times 2$ ($3/1024$ at tree level), is larger than the shape truncation in the range $\theta \sim \beta^{-1/2}$, and cannot be omitted. In genuine multi-step blocking that fixes only the edges of the coarse cube (three dimensions, free boundary L^3, $L \leq 20$, tree level), the ratio κ/c_2 does not contract at the second step but grows as 0.019, 0.033, 0.050, while the step-to-step ratios 1.78, 1.30, 1.13, 1.10 approach 1, so it looks like a shape approaching a value of 0.055–0.066 Reading (in the form of a fit; the existence of the limit is not shown; one loop ($\beta = 32$–128, $L \leq 4$) moves the ratio by no more than 3%). The first-step ratio 1.78 is not a wall but the shape of an approach to the fixed point from below (15-07 § 07).
- **The multi-face quartic**: couplings of three and four faces (counted by orbits of the symmetry of the coarse box).

This writes out the above "one-parameter attracting class + a contracting shape" down to the coordinates of the background and of multiple loops. The view is that there is still a single marginal direction (the direction of the coupling β), and that the other coordinates either contract like the shape coordinates or are carried by the group product as the background itself Reading .

The final form of the coordinates of the four-dimensional class. Measuring the blocking that fixes only the 32 edges of the coarse hypercube ($32L$ fine links) in the four-dimensional free-boundary box $\{0, \dots, L\}^4$, from one step ($L = 2$) up to genuine multi-step blocking ($L \leq 10$ at tree level), the class of coarse weights takes the form of **one parameter** B ($\beta' = \beta b$) **and ratio coordinates whose values are fixed by the flow** Reading ($SU(2)$, heat kernel, the leading saddle-point term and one loop ($L \leq 4$, $\beta = 32$–128, $O(1/\beta)$ not included), the face angle $\lesssim 0.5$). The number of ratio coordinates is counted by the representation theory of the hyperoctahedral group B_4: 2 quadratic (pairs of neighbours sharing an edge, and pairs of "skew neighbours" sharing no vertex), 3 cubic (the corner, path and parallel types of a cell), and 65 quartic Proved here (the space of linearised coarse face angles is 17-dimensional and splits into $3 + 6 + 8$; the counts are computed by two routes, the average of characters and the numerical rank; the Bianchi rank 7 of a cell is in Lean: `relation_is_multiple_of_sigma`, assuming 24 equalities per face). Unlike in three dimensions, the per-face class and the neighbour pairs alone do not close even at quadratic order (0.6% remains). The quadratic and cubic ratios do not go to 0 at the second step; the step-to-step ratios approach 1 (1.55, 1.12, 1.08 for the skew neighbour), and they approach values around -0.145 for the neighbour, -0.021 for the skew neighbour, and 0.064, 0.013, -0.021 for the cubic ones Reading (fit $r_\infty - A/L - B/L^2$ over $L = 4$–10; the existence of the limit is not shown; one loop moves the two-step ratios by less than 1%). Many of the first-step values lie on the 0 side of the fixed point, and a ratio exceeding 1 at the second step is the shape of an approach Proved here (Lean: `ratio_gt_one_below_fixpt`, assuming $0 \leq \lambda < 1$ and $0 < r < s/(1 - \lambda)$). For the quartic, the size of the multi-face part at the second step is 1.6 times its ratio to c_2, the direction is nearly kept (cosine 0.99), and we have not measured whether it is a fixed point.

The skew neighbour is not 0 — but its value and "local" depend on the rule. Within one coarse hypercube, the rule that fixes edges and the rule that fixes all faces give the same quadratic form Proved here ($32 - 15 = 24 - 7 = 17$ and (iii) of the Bundle Network Theorem; the count of degrees of freedom is in Lean: `edge_face_dof_agree`, assuming exactness of the augmented cochain complex). A genuinely different rule fixes all coarse faces on a periodic network in which coarse cells share fine fields; there too the skew neighbour approaches a nonzero negative value, but the value is about twice

as large (-0.045), or -0.06–-0.10 depending on the convention for projecting onto cell types, and the part that does not fit in one cell grows with the step up to about 20% of the quadratic form Reading (tree level of the abelian linearisation, quadratic, infinite coarse lattice, $L \leq 16$). **The sign and the non-vanishing depend neither on the rule nor on the projection convention; the value does.** The "local" of the class-closure condition is therefore written with the rule attached: for rules that separate cells, "closes within one coarse hypercube (reach one side of the coarse lattice)"; for rules that share fields, "quasi-local with a tail" Reading . The only independent parameter is B, and the ratio coordinates are quantities that carry the values of the fixed point. Note that at tree level, with either rule, b of the cell part falls off as $1/L$ ($bL \to 0.39, 0.20$), the long-wavelength eigenvalue stays at 1, and **there is no running in** $\ln L$ Reading . The logarithmic running comes only from one loop (next paragraph).

The measured flow of the coupling (one loop). Measuring the flow of the class parameter B at one loop, on the periodic network that fixes all coarse links (products along lines of b fine links), the long-wavelength coefficient of the coarse action for $SU(2)$ goes down in one step (block 2) in the direction of asymptotic freedom for every action: $\beta' - \beta = -0.745$ (Manton), -1.078 (heat kernel), -1.136 (Wilson) Reading ($\beta = 4/g^2$, one loop of $\beta \to \infty$, abelian background, transverse coarse torus 6^3, extrapolated to wavenumber 0 from length ≤ 32). For $U(1)$ it is 0 for Manton and only the constant -0.234 of the non-quadratic nature of the action for Wilson, depending hardly at all on the blocking factor. The non-abelian part with the action-dependent constants removed (the tadpole; for the heat kernel also $-\frac{1}{3}$ of the Van Vleck prefactor) is -0.43, and its change -0.25 from blocking factor 2 to 3 is 1.7–1.9 times the one-loop coefficient $-\frac{11}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ Reading (wavenumber 0 for blocking factor 3 is a two-point extrapolation). With this rule one can say as far as "the direction and order of magnitude of the flow agree with asymptotic freedom, but the coefficient of $\ln b$ does not agree at small blocking factors". The difference $-\frac{1}{3}$ between heat kernel and Manton, and the difference $\frac{5}{3}t$ between Wilson and Manton (t being the Wilson value of $U(1)$), are identities Proved here (bookkeeping of the coefficients of s^2; the arithmetic is in Lean: `tadpole_weights`, `vanvleck_ratio`). A rule fixing the products along lines is likely to integrate fluctuations up to wavelengths longer than the coarse scale (the $1/L$ of the tree level above); comparing the coefficients needs an averaging rule or larger blocking factors (the question "one-loop coefficient of the long-wavelength network"). The coefficient b_0 follows the convention of equation (9) of 7-05.

Of the implications, that the coupling stays small follows from the coupling-flow remainder condition and continuity of the coupling is the Coupling-Smallness Induction Lemma Proved here ; that the product of constants $\prod_k (1 + \eta_k)$ is bounded by the shape remainder condition is the arithmetic of products Proved here ($1 + x \leq e^x$); that the departure is uniform in the number of steps under the transverse contraction is a geometric series Proved here (§ 04, assuming the one-step inequality); and the part in which localisation and rarity follow from the norm of the weight is a reading from the framework of Appendix A for flat fields Reading (15-02 § 04). As for the correspondence between components and conditions: the parameter β of the class-closure condition is the "coupling" of the coupling staying small itself (the rule of projection onto the leading term in § 05), the coupling-flow remainder condition is the coupling staying small (the rule of the flow of the coupling), the localised part of the shape remainder condition is the small-field analyticity condition (the rule of the activity norm), and its large-field part is the large-field rarity condition (the rule of large-field rarity).

Why we do not state it as "a class that keeps the ratio at 2". In the smallest system (four plaquettes in a row, 15-06) one finds a curve of actions that keeps the ratio of the two quadratic couplings at its tree value 2 (actions mixing the fundamental and adjoint representations, with adjoint coupling $\rho \approx -1/4$). But the one-step map acts in the direction that contracts the shape coordinates, and sends that curve in

one step to a place $O(1)$ away from it, so the curve does not close under the flow (15-06; SU(2), four in a row, the one step defined on the diagonal, at the level of the leading term) Reading . On the attracting class of the flow, the ratio is $2 + \frac{2}{3B} + O(B^{-2})$, and the leading term of the deviation is a one-loop quantity coming from the curvature of the group Proved here (at the one-loop level, the same system). So the ratio 2 is not a coordinate that can be tuned by a parameter, but a quantity that remains in the attracting class. Stating the class-closure condition not as "a class that keeps the ratio at 2" but as "the one-parameter attracting class + a contracting shape" is consistent with the smallest systems we measured Reading (limited to this system).

An example with remainder 0. In the one-link heat-kernel class $\{p_\tau\}$ of 15-04 § 01 (one parameter), $T(p_\tau) = p_{\tau+t}$ exactly and the remainder is 0 Proved here (the semigroup property). That the losses form a telescoping sum is a restatement of this "remainder 0", and can be read as another face of the same fact as the background-independent fluctuations (the Lemma on One Step with Background-Independent Fluctuations of 15-07) Reading .

03

The Coupling-Smallness Induction Lemma — splitting the condition that the coupling stays small in two

Bałaban's assumption "the effective coupling stays in $(0, \gamma]$" (15-02 § 02) is a sentence of the form "it can be satisfied by choosing the bare coupling g_0 well". But the effective coupling g_{k+1} is defined only while the earlier steps are small (since it is a coefficient read off from the effective action). Making this explicit through the domain, the coupling staying small splits into two conditions.

Coupling-Smallness Induction Lemma. Suppose $b > 0$, $C \geq 0$, $\gamma_1 > 0$ satisfy $C\gamma_1^2 \leq b/2$ and $\gamma_1^2 < 2/(3b)$, and fix $n \geq 1$. Suppose that for each $g_0 \in (0, \gamma_1]$ the sequence $g_0, g_1, \dots$ is defined inductively in the form "if $g_k \in (0, \gamma_1]$ then g_{k+1} is determined", and satisfies the following two.

- **Net one-step bracket of the coupling** (the remainder of one step). $\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - b + r_k$, $|r_k| \leq C\, g_k^2$.
- **Continuity of the coupling** (continuity within the domain). On $D_k := \{g_0 :\ g_0, \dots, g_k \in (0, \gamma_1]\}$, $g_0 \mapsto g_{k+1}(g_0)$ is continuous.

Then for every $\gamma_0 \in (0, \gamma_1)$ there is a g_0 with $g_0 < g_1 < \cdots < g_n = \gamma_0$ (all defined).

Proved here Proof. (i) If $g_k \leq \gamma_1$, then from the net one-step bracket of the coupling and $Cg_k^2 \leq b/2$

$$\frac{1}{g_k^2} - \frac{3b}{2} \;\leq\; \frac{1}{g_{k+1}^2} \;\leq\; \frac{1}{g_k^2} - \frac{b}{2} \quad (2)$$

The left-hand side is at least $1/\gamma_1^2 - 3b/2 > 0$, so g_{k+1} is a positive finite number, and from the right, $g_{k+1} > g_k$. (ii) Set $J := \{g_0 \in (0, \gamma_0] :\ g_0, \dots, g_n$ all defined and $\leq \gamma_0\}$. By the monotonicity in (i), $J = \{g_0 \leq \gamma_0 :\ g_n$ is defined and $g_n \leq \gamma_0\}$. If $g_0^2 \leq (1/\gamma_0^2 + 3bn/2)^{-1}$, then repeating the left of (2), $1/g_k^2 \geq 1/g_0^2 - 3bk/2 \geq 1/\gamma_0^2$ $(k \leq n)$, so J is non-empty. (iii) Let $s := \sup J$ and take a sequence $g_0^{(j)} \to s$ of points of J. Since $s \leq \gamma_0 < \gamma_1$, $g_1(s)$ is determined, and by continuity of the coupling $g_1(s) = \lim_j g_1(g_0^{(j)}) \leq \gamma_0$. Repeating the same for $k = 2, \dots, n$ gives $s \in J$. (iv) If $g_n(s) < \gamma_0$, then from $n \geq 1$ and (i), all $g_k(s) < \gamma_0 < \gamma_1$. By continuity of the coupling, $g_0, \dots, g_n$ are also defined in a right neighbourhood of s and are $< \gamma_0$, contradicting $s = \sup J$. Hence $g_n(s) = \gamma_0$, and by monotonicity $g_0 < \cdots < g_n = \gamma_0$. ∎

A variant: if the net one-step bracket of the coupling is replaced by $|r_k| \le C g_k^2 + \eta_k$ ($0 \le \eta_k \le b/4$), the same proof goes through with the constants of (2) replaced by $7b/4$ and $b/4$ and the positivity condition by $\gamma_1^2 < 4/(7b)$ Proved here . Non-universal scale-dependent terms are absorbed as a constant shift in $1/g^2$.

What the Coupling-Smallness Induction Lemma says. The coupling staying small splits into **the estimate of the one-step remainder (the net one-step bracket of the coupling)** and **continuity within the domain (continuity of the coupling)**, and the rest (monotonicity and the intermediate-value argument) is elementary. The net one-step bracket of the coupling is the statement, on the side of the flow of the coupling, of "absorb the part of size g_k^2 into the flow, and confine the remainder to $O(g_k^2)$ in the variable $1/g^2$".

What the Coupling-Smallness Induction Lemma does not say. The net one-step bracket of the coupling has not been shown in this book's design for four dimensions with the non-abelian large-field contributions included (the published construction, by a different design, carries the coupling through nested bands step by step; 14-03). This corresponds to the content of the unpublished basic inequality (Theorem 2 of Bałaban 1989) Assumption (since the text of the basic inequality (unpublished) does not exist, the correspondence is a reading of one sentence in the introduction). Continuity of the coupling is the statement that "the effective action at each step depends continuously on the bare coupling, and the effective coupling is extracted as a continuous functional of it". Since these are integrals over finitely many compact groups, the line of argument goes through by the dominated convergence theorem, but since the way the effective coupling is extracted is fixed inside the small-field expansion, it depends on the analyticity of that expansion (the small-field analyticity condition) Assumption (outline only). We do not say that the Coupling-Smallness Induction Lemma has made the coupling staying small easier. It only separates the bookkeeping part from the main part. A proposal for estimating the net one-step bracket of the coupling with large fields included is in 15-08.

Splitting the net one-step bracket of the coupling further. By an identity the remainder of the net one-step bracket of the coupling splits into $r = r^{\mathrm{small}} + r^{\mathrm{large}}$, and if the fluctuation is a product of heat kernels link by link, the large-field part r^{large} is bounded by the Lipschitz constant of the read-out $\times$ the rarity Proved here (one link, one step, background-independent fluctuations; the Large-Field Remainder Bound Lemma and the Heat-Kernel Fluctuation Rarity Lemma of 15-08 § 02). The small-field part (the small-field contribution to the coupling) splits into three: the Taylor truncation ($(h_0/r_0)^{2k+2}$, of the same form as Lemma 7.5.3 of the source for flat fields), the small-field cut-off ($e^{-c(1-q)^2 h^2 \beta}$), and the explicit quadratic terms between blocks together with their local cancellation; what is absent from the sources is the group version of the third — the place the source itself calls a sketch for flat fields Reading (15-08 § 06; the first two are Proved here under the analyticity condition at complex backgrounds and the margin q). This writes, conditionally and on the group, the same form as the structure known in the sources; it does not show the net one-step bracket of the coupling. In the smallest system of two blocks, the explicit quadratic terms between blocks appear at one loop as $O(1)$ (the local part goes into β_0', the rest into the shape coordinates), and the local cancellation is needed for the part $O(1/\beta)$. The type of its non-local tail (flat, exponential, power) is decided by the definition of the coarse variables, and a sufficient way to obtain an exponential one is to avoid sharing that creates conserved quantities, to promote local constraints to coarse variables, and to separate global constraints as finitely many tags Reading (numbers for one-, two- and three-dimensional models; four dimensions not measured). Of the rows of the net one-step bracket of the coupling, what has the same form as in the sources is the projection of $\mathbf{Loc}$, the way of writing W, and the form of the Taylor remainder; four points — taking $\mathbf{Loc}$ as the projection onto components, the reason for the zero sum,

how the cut is paid for, and how the sum over steps is taken — may be cut differently from the sources Reading (15-08 § 06).

The recursion of the inter-block terms of the net one-step bracket of the coupling — its form in the one-row model. Carrying the inter-block terms to the second step, in the one-row model it can be written in the form "the image of the first-step non-local term closes in the same type, contracts by the ratio of couplings to at most $5/8$ at quadratic and $5/48$ at quartic order, and the terms newly born at that step do not depend on the step", and the non-local term is drawn geometrically not to 0 but to a definite value Proved here (a first-order Taylor formula writing the image as a conditional expectation, and rational numbers at tree level), Reading (the numbers: $SU(2)$ heat kernel, $U(1)$ Wilson as a control, one row, the diagonal definition, one step of two blocks of seven plaquettes taken twice, four types of adjacent pairs, first order in the non-local terms, up to quartic in the coarse field, $\beta = 32$–512 and extrapolation). When three or more superblocks are lined up, the quartic image contains terms with three coarse fields at once. For coarse variables that give half of each boundary plaquette to each side (with no conserved quantity), their coefficient decreases with the width d of the triple as $e^{-\kappa d}$, $\kappa = \ln(5 + 2\sqrt{6}) \approx 2.29$ (the rate of the background map), the sum over terms containing one block is uniform in the volume, and its own image contracts to at most 5×10^{-3} Reading (one step, tree and one loop, $L \leq 8$, $d \leq 7$; the prefactor is constant at tree level and first order in d at one loop). The three-body coefficient for $d = 2$, however, is of the same order as the coefficient of adjacent pairs; only the tail is small. Hence in the one-row model, the inter-block terms of the net one-step bracket of the coupling close up to quartic order as "the pair type (fixed-point value + contracting deviation) + a three-body type decreasing exponentially in the width" Reading (the bookkeeping is in Lean: `affine_bound_uniform` ($0 \leq \lambda < 1$, $0 \leq C$), `pair_tail_uniform` ($0 \leq \lambda, \mu < 1$ and two recursive inequalities), `three_body_sum_le` ($0 \leq \rho < 1$, $|q(d)| \leq C\rho^d$)). Going back to a definition with a conserved quantity, the three-body coefficient is flat in the distance. For continuum fields in the sources this is the part written by the additivity "the non-local terms of the previous step remain without changing the kernel, and the part of the new step is added" (BBS (A.1.4)–(A.1.5) and Exercise 2.2.4); in one row on the group it becomes "a contracting image + new terms independent of the step" Cited (the formulas of the sources), Reading (the correspondence). Four-body terms, the self-image of the three-body term at one-loop level, and the transfer to four dimensions remain (15-08 § 06).

04

The bookkeeping of the transverse contraction — the departure by a supremum, not a sum

We write the requirement of the shape remainder condition that "ρ_k stays uniform in the number of steps" in the language of measured quantities. Set up the space of weights X, one step $T : X \to X$, and a measured quantity $\Phi_{\mathrm{obs}} : X \to \mathbb{R}$ (for example the curvature outside the layer κ/β, or the total variation distance); take a class $\{w_\beta\}$ and a read-out $b : X \to \mathbb{R}$ ($b(w_\beta) = \beta$), and define the function of the class $F(\beta) := \Phi_{\mathrm{obs}}(w_\beta)$. With $u_k := T^k(u_0)$, $\beta_k := b(u_k)$, and the departure from the class $\delta_k := \Phi_{\mathrm{obs}}(u_k) - F(\beta_k)$, the identity

$$\Phi_{\mathrm{obs}}(u_K) - \Phi_{\mathrm{obs}}(u_0) = \big[F(\beta_K) - F(\beta_0)\big] + \big[\delta_K - \delta_0\big] \quad (3)$$

holds Proved here (from the definitions). The first term is the part absorbed into the movement of the parameters, and only the difference of the end values remains (the telescoping sum). The second term is the loss, and if $r_k \geq |\delta_{k+1} - \delta_k|$ then $|\delta_K - \delta_0| \leq \sum_{k<K} r_k$ Proved here .

The case of transverse contraction. Let $T(w_\beta) = w_{\tau(\beta)}(1 + r_\beta)$, $\|r_\beta\| \le \varepsilon(\beta)$, and assume transverse stability $T(w_\beta(1+\rho)) = T(w_\beta)(1+\rho')$, $\|\rho'\| \le \lambda\|\rho\|$ (a linear inequality with the quadratic terms dropped). Writing $u_k = w_{\beta_k}(1+\rho_k)$, we have $\|\rho_{k+1}\| \le \lambda\|\rho_k\| + \varepsilon(\beta_k)$, and

$$\|\rho_k\| \;\le\; \lambda^k\|\rho_0\| + \frac{\sup_j \varepsilon(\beta_j)}{1-\lambda} \qquad (0 \le \lambda < 1) \quad (4)$$

Proved here (a geometric series, under the linear inequality above). **Even if the remainder is $O(1/\beta)$, the departure is bounded by a supremum rather than a sum, uniformly in the number of steps.** If $\lambda = 1$ then $\|\rho_K\| \le \|\rho_0\| + \sum_{k<K}\varepsilon(\beta_k)$, and if $\varepsilon(\beta) = C/\beta^2$ and $\beta_k \ge \beta_* + ck$ (growing linearly at one loop towards the fine side, 15-01 § 01), then

$$\sum_{k<K} \frac{C}{(\beta_* + ck)^2} \;\le\; C\Big(\frac{1}{\beta_*^2} + \frac{1}{c\beta_*}\Big) \quad (5)$$

is finite Proved here (the telescoping sum of $1/(u+c)^2 \le 1/(cu) - 1/(c(u+c))$). For $\varepsilon(\beta) = C/\beta$ the sum grows like $\ln K$ and is not uniform (the same type as the logarithmic divergence of 15-01). So there are two routes: "uniform if the one-step loss is $O(1/\beta^2)$" and "uniform even with a remainder of $O(1/\beta)$ if things contract geometrically to a fixed shape", and the latter goes through on a weaker assumption.

The shape deviation when the flow makes the coupling smaller. In one step of the smallest system (15-06), the tree coupling changes in one step as $B_{n+1} = B_n/5$, and the departure r_n of the shape coordinates (in units of q/B) satisfies

$$r_{n+1} \le \lambda\, r_n + \frac{C}{B_n}, \quad B_n = \frac{B_0}{5^n},\ 0 \le \lambda < 5 \qquad \Longrightarrow \qquad r_n \;\le\; \lambda^n r_0 + \frac{C}{(5-\lambda)\,B_n} \quad (6)$$

holds Proved here (by induction; the one-step inequality is assumed). For the projected model of four in a row there is a form that gives the constants of this inequality as numbers: at the scale of the actual steps, $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, $C = 0.0108$ (boxes of $B \ge 32$ and $r \le 0.05$) Proved here , Reading (the line of the estimate is on paper; the constants are numerical values of explicit integrals and series; the Shape-Remainder Theorem for Four Plaquettes in a Row of 15-06). If $\lambda < 1$, the initial deviation dies out geometrically, and what remains is always the $O(1/B_n)$ of that step. In the smallest system, λ was at most $17/125$ in the shape coordinates (15-06, four in a row, leading term) Reading . That (6) is written with B_n decreasing towards the fine side is a convention of that system; in the ultraviolet direction (where B is larger on the finer side), C/B_n becomes smaller at finer steps.

Implication for the decomposition: if the class of the class-closure condition is taken to be the attracting curve, then what must be asked of one step in four dimensions becomes two numbers, "the size of the source ε" and "the transverse contraction λ" Reading . How these two can be measured in the smallest systems is in 15-04 (the hierarchical model) and 15-05 (the bundle network).

Contraction-map window for the fixed point of the class — the fixed point of the class in the class-closure condition in the form of a contraction map. Under asymptotic freedom the only fixed point of the coupling g is $g = 0$, and it repels towards the infrared. What should be stated is therefore not a fixed point of the pair (g, W), but that when the one-step maps $W \mapsto T(g_k)W + R(g_k, W)$ are composed while g flows, there is exactly one two-sided infinite orbit that stays in a certain closed ball. That orbit is the one-parameter attracting class $W = h(g)$. If the map of each step preserves the closed ball and contracts on it, this existence and uniqueness are the arithmetic of the contraction mapping principle Lean (ClassFixedPoint: `exists_unique_entire_orbit` , `tracking` , `norm_iterate_sub_le` . Even if one step

does not contract, it suffices that $\mu_n + (\tau+\ell)^n - \tau^n < 1$, with μ_n the linear part of n steps, τ the linear part of one step and ℓ the Lipschitz constant of the remainder. The deviation from the fixed point at frozen g stays within $\delta/(1-K)$, the same form as $\sup \varepsilon/(1-\lambda)$ in (4)). What remains, then, are the numbers on the side of the assumptions.

The remainder is measured relative to the action of the tree fixed point, with a weight of dimension 6. As the yardstick we use a gauge-invariant action S^*, obtained by turning the quadratic part of the fixed-point action back into a kernel on pairs of plaquettes and wrapping it with plaquette holonomies and parallel transport (its quadratic part is within 15% of the fixed point). In this yardstick the linear part of one step contracts by $\tau = 0.695$ at every momentum, and the nonlinear coefficient is $a = 4.335$. The one-loop source is assembled with the same action, and its F^2 part is moved into β along the direction of the action itself. When the quartic vertex of the action is tuned by one number so that the shift of β takes the universal value, the remaining source is $c_0 = 0.02$–0.04 per g^2 (on a 12^4 lattice). With a three-step estimate ($\mu_3 = \tau^3$) the window of the contraction map opens from β of about 28–54 Computed (the Gaussian part of the tree and the one-loop quadratic term). There are four conditions. (i) Taking the direction of the shift into β as the direction of the action itself (the A-type **Loc**) is an argument on paper; a formal check has not been done Paper . (ii) a is the agreement of upper and lower values at a point found by a search over momenta (the upper value from a semidefinite relaxation, the lower value from a concrete field), not the supremum itself over continuous momenta. (iii) S^* is not a fixed point (its quadratic part moves by 23% in one step). (iv) The Lipschitz constant of the remainder is set to $\ell = 0$.

In the original yardstick (A_4), the maximum of the nonlinear coefficient over momenta is at most 4.3313 by a computation covering every box of momenta with the upper bound of the second-order expansion, and it agrees with the value 4.3308 from a concrete field at the maximising point to within 5×10^{-4} Computed (a floating-point computation; it was confirmed computer-assisted with interval arithmetic only on the small box $|q - q^*|_\infty \le 3 \times 10^{-4}$ around the maximising point. On the other hand, the positive definiteness of the action of the step, needed to use this upper bound, was confirmed with interval arithmetic at every momentum except in a small box around the origin Computer-assisted ; around the origin it was looked at by sampling).

05

The bundle of rules — what to make into rules so that one step closes

We rewrite the form that closes in one step of 15-00 § 01 as a bundle of rules. Each row of Table 1 is a candidate for a rule needed to build "a class that closes in one step", set against which of the rule candidates in 24-08 Table 2 (the rule of axiomatising scale, the rule of the rarity class, the rule of an asymptotic-freedom axiom, the curvature rule) it fills. Every row is a candidate; we have checked neither consistency nor that they can restate existing results Reading (a physical estimate).

Table 1 The bundle of rules for one step to close (candidates)

Rule	Content (one line)	Candidate in 24-08	Components at hand	What is missing
Gauge-covariant averaging	The averaging map is covariant under gauge transformations and central shifts; its linearisation is the scalar average, and the nonlinear part goes up to second order	Rule of axiomatising scale	Stage 2 of the series (titles)	Estimates of the constants of the second-order remainder

Rule	Content (one line)	Candidate in 24-08	Components at hand	What is missing
The background-field map	The configuration that minimises the action for given averages is unique, analytic and local within a radius independent of scale	Rule of axiomatising scale	Stage 3 of the series (titles). The contraction mapping in two dimensions $O(4)$	Writing it out in four dimensions, local gauge fixing included
The activity norm	The background $\times$ fluctuation weight has the group version of the four properties	Rule of the rarity class (a gauge-covariant activity norm)	The group version of submultiplicativity in sums of fields ((ii) of BBS) (the Background–Fluctuation Decomposition Lemma). One step with background-independent fluctuations (the Lemma on One Step with Background-Independent Fluctuations). A statement of the group versions of closure under the fluctuation expectation and monotonicity to the next step (15-07)	A proof of the group versions of closure under the fluctuation expectation and monotonicity to the next step, and the absorption of η_k into the parameters
Projection onto the leading term	Split the effective action into "$\frac{1}{g_k^2}\times$ Wilson's action + localised irrelevant terms", and send only the marginal F^2 to the leading term	Rule of axiomatising scale (contracting directions and marginal directions)	The telescoping-sum type of the heat kernel. The attracting class of the smallest system (15-06)	Defining this projection on the group (up to what order in gauge-invariant local functions)
Rarity of large fields	The weight of the large-field region is $z_k^{\lvert Z\rvert}$, $z_k \le e^{-cp(g_k)^2}$, and c does not degrade over the steps	Rule of the rarity class	Stage 5 of the series (with assumptions). The negative one-link curvature (15-01)	One step of the induction (in a form that avoids "worst-case geometry")
The flow of the coupling	The remainder of one step $\lvert r_k\rvert \le Cg_k^2 + \eta_k$	Rule of an asymptotic-freedom axiom	Coupling-Smallness Induction Lemma (§ 03)	The net one-step bracket of the coupling including large fields
Making the constants explicit	All the constants above as explicit numbers or explicit functions	New candidate	A record of floating-point numbers	Interval arithmetic, analytic estimates

Correspondence with the sources of difficulty Reading (a physical estimate): the source: dimensionless coupling goes with the rules of projection onto the leading term and of the flow of the coupling, and the source: no global gauge fixing (large fields cannot be controlled by convexity) with the rules of gauge-covariant averaging, the background-field map, the activity norm and the rarity of large fields (15-01 § 03). In ultraviolet control, which deals with the ultraviolet alone, the source: no expansion linking scales (no expansion connecting the ultraviolet and the infrared) does not appear.

Cautions in turning these into rules (from the patterns of misreading in 15-00).

- Write a "one-step" version and an "all-step" version of each rule separately, and make explicit on the side of the rules the assumption that gets the all-step version from the one-step version (the constants do not degrade over the steps).
- Do not take the input of a rule from the output of the same rule (the "Circularity" pattern).
- Write one direction of inequality (upper bound or lower bound) for each rule.

06

Contrast with the decomposition of the band — what pays the one-step loss

In 26-06 § 01 the crossover band was split into β-monotonicity (the vortex free energy is monotone in β) and the renormalisation comparison (the comparison with the doubled lattice) Assumption . Table 2 sets the two decompositions side by side.

Table 2 The decomposition of the band and the decomposition of the ultraviolet (a reading)

	Band (26-06)	Ultraviolet (this chapter)
One-step comparison	The renormalisation comparison: compares the doubled lattice with the original lattice at a strengthened coupling, with constant K	(1): $T(w\theta) = w\Phi(\theta)(1 + r)$
Where asymptotic freedom enters	Only the direction of $\beta' < \beta$	Only that the β component of Φ points in the direction of $\beta_{k+1} < \beta_k$ (the main term of the coupling-flow remainder condition)
Components that do not distinguish the group	β-monotonicity: tuning (choosing β' on the small side costs nothing)	Continuity of the coupling and monotonicity: the intermediate-value argument of the Coupling-Smallness Induction Lemma (tuning the bare coupling)
Handling the one-step loss	Pays the loss K^k with the smallness of the area exponent at the strong-coupling end (doubly exponential as steps accumulate)	There is nothing to pay $\prod_k(1 + \eta_k)$ with (the weak-coupling end is Gaussian and has no smallness to cancel the loss). Hence $\sum_k \eta_k < \infty$ or a transverse contraction $\lambda < 1$ is needed
Allowed size of the loss	A constant K per step (it may exceed 1)	$1 + \eta_k$ per step, with η_k of type $g_k^{2+\delta}$ (type g_k^2 is ruled out because the product tends to 0). In a transversely contracting direction, even a remainder of $O(g_k^2)$ can be carried by a supremum (§ 04)
Undecided core	β-monotonicity carries the same weight as the β-monotonicity of the string tension. The renormalisation comparison is open within the band	The net one-step bracket of the coupling including large fields. Absorbing η_k into the parameters (15-07)

What to read from Table 2 Reading (a physical estimate). The two decompositions are of the same type (a one-step comparison that distinguishes the group, and a tuning that does not), and they split in the same way. The difference lies in **whether there is something to pay the one-step loss with**. The band has a doubly exponential smallness at the strong-coupling end, so the one-step constant K may be crude; the ultraviolet has nothing to pay with, so the one-step loss itself can only be confined to the finite size of a sum, to the movement of the parameters, or to the supremum given by the transverse contraction.

07

Three ultraviolet questions

Table 3 lists the questions that come out of this decomposition, in the form of 24-08 Table 1. The numbering carries over from Table 4″ of 26-11. The ultraviolet questions of the other chapters of Part 15 and the overall index are in 15-09.

Table 3 Questions arising from the decomposition of the ultraviolet wall (status: New = a question added in 26-11; Updated = rewritten in this chapter)

Q.	Question	Size	Status	What moves if settled	What does not move when settled	From which chapter

Q.	Question	Size	Status	What moves if settled	What does not move when settled	From which chapter
1	Does it contract transversely on the one-parameter attracting class? In the smallest system there is an attracting class (a single coupling B + a fixed shape of $O(1/B)$), and the shape coordinates contract in one step (at most $17/125$ for four in a row, leading term). Questions: (i) for a block embedded in the four-dimensional lattice (including the escape routes of the surrounding flux and the re-fixing of the gauge), do the transverse contraction λ and the source stay finite? (In a four-dimensional box with a 2×2 coarse face inside, $\lambda \approx 0.027$ and source $\approx 5 \times 10^{-4}$; in the network with all coarse faces fixed, λ is a matrix and exceeds $1/16$ in the coarse directions. 15-05.) (ii) Does the constant C of the one-step inequality (6) exist independently of the step?	Paper and small computations	Updated	The number and coordinates of the parameters of the class-closure condition. The dimension of the attracting class	The class-closure condition and the net one-step bracket of the coupling on the four-dimensional lattice	15-03 § 02, § 04, 15-05, 15-06

Q.	Question	Size	Status	What moves if settled	What does not move when settled	From which chapter
	(For the projected model of four in a row, the Shape-Remainder Theorem for Four Plaquettes in a Row gave it as numbers; 15-06.) (iii) Is the weight of the sign of the central $\mathbb{Z}_2$ (the vortex weight) needed as a separate parameter?					

Q.	Question	Size	Status	What moves if settled	What does not move when settled	From which chapter
2	Which is the smallest model in which the net one-step bracket of the coupling can be shown including large fields? (The large-field part is settled by one link of heat-kernel fluctuations, and the core that remains is the small-field contribution to the coupling; 15-08 § 02, § 06.) Candidate: the hierarchical $SU(2)$ model: does $\lvert r_k \rvert \le C g_k^2$ hold including the contribution of the far hemisphere of one link (of weight of order $e^{-\beta}$)? Also, can the absorption under a constraint $\mu_W^\beta = \mu_1^{\beta\cos(\theta/2)}$ for one plaquette be restated at the true constrained minimum of many plaquettes?	Paper	New	One line of the net one-step bracket of the coupling in the rule of the flow of the coupling, at the hierarchical level	The basic inequality (unpublished) on the four-dimensional lattice	15-03 § 03, 15-07

Q.	Question	Size	Status	What moves if settled	What does not move when settled	From which chapter
3	The role of the twist: on the M^4 lattice of n $12 = 1$, is the smallest eigenvalue of the adjoint fluctuation operator around the isolated twisted flat connections at least c/M^2 (finite-dimensional linear algebra)? And, beyond the Background–Fluctuation Decomposition Lemma being unchanged by the twist, is the group version of closure under the fluctuation expectation unchanged by the twist?	Paper (partly small computations)	New	The denominator of the partition function in the ultraviolet statement for a small twisted four-dimensional torus, and the non-degeneracy of the last step	Ultraviolet control at the fine steps. The net one-step bracket of the coupling	15-02 § 05

08

What cannot be said

(a) Within this part, ultraviolet control is not closed. This book's study has shown none of the pieces of the decomposition statement (the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling) in four dimensions. Existence in four dimensions was shown by the published construction, with a different design (Parts 6–13).

(b) The Coupling-Smallness Induction Lemma only splits the coupling staying small into the net one-step bracket of the coupling and continuity of the coupling; it does not show the net one-step bracket of the coupling (the content of the unpublished basic inequality).

(c) The bookkeeping of § 04 is arithmetic under the assumption of a linear inequality with the quadratic terms dropped and of the one-step inequality. We do not show that $\lambda < 1$ and the estimate of the remainder hold for one step in four dimensions.

(d) $m = 2$ is a reading from power counting, not shown for a non-perturbative class. "The one-parameter attracting class" is a reading of the smallest system — four in a row, the diagonal definition, the leading term (the limit of large coupling) — and says nothing about blocking in four dimensions.

(e) The bundle of rules is a set of candidates; we have checked neither consistency nor that they can restate existing results. The contrast with the band is an estimate, and shows none of the band's β-

monotonicity, the renormalisation comparison, or the ultraviolet remainders.

(f) The splitting of the net one-step bracket of the coupling in § 03 is a summary of the statements and estimates of 15-08 § 06, and does not show the small-field contribution to the coupling. The same form as in the sources holds under the analyticity condition at complex backgrounds and the margin q, and we do not show the group version of the explicit quadratic terms between blocks. The "geometry part to the side of the parameters" of § 02 is a reading that carries the numbers of the two-link model over to the four-dimensional steps. The final form of the coordinates of the class in § 02 is a proposal from the saddle-point expansion of one three-dimensional step and the numbers for two links and the cross; we do not show that it closes in one four-dimensional step.

(g) The final form of the four-dimensional class in § 02 consists of tree-level fits for the rule that fixes only the edges in the free-boundary box (and for the rule that fixes all faces and the quadratic order of the periodic network), and one-loop numbers ($L \le 4$). We show neither the existence of fixed points of the ratio coordinates nor the values of the 65 quartic ones. On the periodic network the class does not fit within one cell. The measured flow of the coupling is a one-loop number for the first step of one rule; we have not measured the coefficient over repeated steps. Of the multi-step conditions for the shape remainder condition, the cone-contraction condition at the fixed point is at the level of a lower bound; there is no upper bound. The recursion of the net one-step bracket of the coupling in § 03 is a statement for one row, a definition with no conserved quantity, and first order in the non-local terms; we have not measured four dimensions, four-body terms, or the three-body image at one-loop level.

09

The table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Grounds	Scope
[01]	The class and the form (1) of the one-step map; $m = 2$ by power counting	Reading	§ 01 (power counting)	Four-dimensional pure gauge, local terms
[02]	Coupling-Smallness Induction Lemma: net one-step bracket of the coupling + continuity of the coupling $\Rightarrow g_0 < \cdots < g_n = \gamma_0$ (with a variant)	Proved here	§ 03	$C\gamma_1^2 \le b/2$, $\gamma_1^2 < 2/(3b)$, n fixed
[03]	The content of the net one-step bracket of the coupling is the one-step remainder including large fields, corresponding to the content of the basic inequality (unpublished)	Assumption	§ 03 (a reading of one sentence in the introduction)	Four-dimensional non-abelian
[04]	The identity (3); the telescoping-sum upper bound on the loss	Proved here	§ 04	By definition
[05]	The bookkeeping of the transverse contraction (4); the sum of $\lambda = 1$ (5); the shape deviation (6)	Proved here	§ 04 (geometric series, telescoping sum, induction)	Assuming the linear inequality and the one-step inequality
[06]	The curve that keeps the ratio at 2 does not close under the flow, and the ratio on the attracting class is $2 + \frac{2}{3B}$	Reading , Proved here	§ 02 (the results of 15-06)	SU(2), four in a row, the diagonal definition, leading term / one-loop level

Row	Statement (one line)	Grade	Grounds	Scope
[07]	The decomposition of the ultraviolet wall: class-closure condition, coupling-flow remainder condition, shape remainder condition, continuity of the coupling ⇒ ultraviolet control. The class-closure condition is the one-parameter attracting class + a contracting shape	Assumption	§ 02 (the form of the statement)	Statement only
[08]	The bundle of rules and its correspondence with the roots	Reading	§ 05 (physical estimate)	Candidate
[09]	Contrast with the decomposition of the band: they split in the same way, and differ in whether there is something to pay the one-step loss with	Reading	§ 06 (physical estimate)	—
[10]	Three ultraviolet questions	Assumption	§ 07	The form of questions
[11]	The splitting of the net one-step bracket of the coupling: $r^{\mathrm{small}} + r^{\mathrm{large}}$; the small-field contribution to the coupling splits into the Taylor truncation (of the same form as in the sources), the cut-off, and the quadratic terms between blocks. The background-dependent geometry part goes to the side of the parameters	Assumption , Reading	§ 02, § 03 (summaries of 15-07 and 15-08)	Analyticity condition at complex backgrounds, margin q, a reading of the two-link model
[12]	The final form of the coordinates of the class in the class-closure condition (the coupling matrix, the shape, the background, the one-link coupling, the corner triple product, the multi-face quartic)	Assumption	§ 02 (summary of 15-07)	A reading of three-dimensional $2 \times 2 \times 2$, two links and the cross
[13]	$\sum_k \eta_k$ of the shape remainder condition is uniform in the number of steps if the background is given as a parameter; the difficulty moves to the class-closure condition	Reading , Proved here (arithmetic of the budget)	§ 02 (15-07 § 06)	The smallest system, conditional
[14]	The final form of the four-dimensional class: one parameter B + ratio coordinates whose values are fixed by the flow (2 quadratic, 3 cubic, 65 quartic). The skew neighbour is nonzero whatever the rule; the value and "local" depend on the rule. No running in $\ln L$ at tree level	Reading , Proved here (counting, agreement of degrees of freedom)	§ 02 (15-07 § 07)	$SU(2)$, heat kernel, tree $L \leq 10$, one loop $L \leq 4$, periodic network at abelian tree level and quadratic order $L \leq 16$, extrapolation
[15]	The measured flow of the coupling: $\beta' - \beta = -0.745 - -1.136$ in one step of the periodic network; the change from blocking factor 2 to 3 is 1.7–1.9 times the one-loop coefficient	Reading , Proved here (identities for the differences of actions)	§ 02	$SU(2)$, one loop, abelian background, transverse 6^3, wavenumber 0 extrapolated
[16]	The contraction of the cone of the shape remainder condition is not uniform in the number of steps; blocking conditions (the covariance-flow fixed-point condition and the cone-contraction condition at the fixed point) are needed	Reading	§ 02 (15-05 § 07)	Abelian, tree level, periodic, 2^4, linearised, no interval estimates, the cone-contraction condition at the fixed point a lower bound only
[17]	One-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling: closes up to quartic order with a contracting image of the pair type + new terms independent of the step + three-body terms decreasing exponentially in the width	Reading , Proved here (first-order Taylor formula and bookkeeping)	§ 03 (15-08 § 06)	One row, definition with no conserved quantity, heat kernel, one step, first order in the non-local terms, up to quartic

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] RDG CLS    the class and the one-step map, m = 2                  power counting
[02] PRV IND    Coupling-Smallness Induction Lemma                     under the net bracket a
[03] ASM S      the net one-step bracket of the coupling is the main part ⟸ [02]   four-dime
[04] PRV TEL    the identity and the telescoping sum
[05] PRV LEDG   bookkeeping of the transverse contraction ⟸ [04]     one-step inequality ass
[06] RDG ATTR   ratio-2 curve does not close, attracting ratio ⟸ [01]  four in a row, leadin
[07] ASM DEC    decomposition of the UV wall ⟸ [01][02][05][06]      form of the statement
[08] RDG RULE   bundle of rules ⟸ [02][07]
[09] RDG BAND   contrast with the band ⟸ [07]
[10] ASM Q      three ultraviolet questions ⟸ [03][05][07]
[11] ASM SPLIT  splitting of the net bracket and the small-field contribution, geometry part t
[12] ASM COORD  final form of the class coordinates ⟸ [07][11]       three-dimensional readi
[13] RDG MOVE   sum of η to parameters, difficulty to the class-closure condition ⟸ [07][11]
[14] RDG CLS4   final form of the 4D class, local with the rule ⟸ [07][12]  tree L ≤ 10, one
[15] RDG FLOWM  measured flow of the coupling ⟸ [01][07]            first step, one loop
[16] RDG CONE   multi-step conditions of the shape remainder (fixed point, cone contraction) ⟸
[17] RDG SREC   one-row version of the recursion of the net bracket ⟸ [03][11]   one row, up
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. [03], [07], [10], [11] and [12] are assumption rows, and all are in the form of questions that split ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

What this chapter can say

Said	Not said
Proved here The Coupling-Smallness Induction Lemma (the coupling stays small ⇐ net one-step bracket of the coupling + continuity of the coupling, with a variant), and the bookkeeping of the transverse contraction (3)–(6) (under the assumption of the linear inequality and the one-step inequality)	Neither the net one-step bracket of the coupling nor continuity of the coupling. $\lambda < 1$ and the estimate of the remainder for one step in four dimensions
Assumption The decomposition of the ultraviolet wall and the form of the class-closure condition (the one-parameter attracting class + a contracting shape), the three ultraviolet questions, the splitting of the net one-step bracket of the coupling and of the small-field contribution to the coupling (the part of the same form as in the sources, and the group version of the quadratic terms between blocks)	A proof of any of them. Any prospect of ultraviolet control closing
Assumption A proposal for the final form of the coordinates of the class in the class-closure condition. Reading How $\sum_k \eta_k$ closes when the background is given as a parameter (conditionally)	That the class closes in one four-dimensional step. The second step for faces and multiple loops
Reading Power counting for $m = 2$, the bundle of rules, the contrast with the band, the reason for not adopting a class that keeps the ratio at 2 (the smallest system)	The number of parameters for a non-perturbative class. The attracting class for blocking in four dimensions
Reading The final form of the four-dimensional class (one parameter + ratio coordinates; the skew neighbour is nonzero), the one-loop measurement of the flow of the coupling, the multi-step conditions for the shape remainder condition, the one-row version of the recursion of the net one-step bracket of the coupling	The existence of fixed points of the ratio coordinates. That the class closes within one cell on the periodic network. The coefficient of the coupling over repeated steps. An upper bound on $\sup \mu_\infty < 1$. The recursion of the net one-step bracket of the coupling in four dimensions and for four-body terms

Sources and reproduction

Item	Kind	Source / tools
The assumption of the Conditional Ultraviolet Stability Theorem (the coupling stays small) and the unpublished basic inequality	Cited (abstract and introduction)	T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (7-07 § 03, 15-02)
The decomposition of the band	Assumption (statement)	26-06 § 01
Rule candidates (axiomatising scale, the rarity class, an asymptotic-freedom axiom, curvature)	Reading	24-08 Table 2
Coupling-Smallness Induction Lemma, bookkeeping (3)–(6)	Proved here	Proofs in the text of this chapter
The attracting class, the ratio-2 curve, the contraction **17/125**	Reading (the tree map and the one-loop formula are proved here)	This book's study (15-06; the four-in-a-row model, numbers by two routes)
Tables 1–3, the decomposition statement	This book's study	The record of this book's study (an organisation of the study of ultraviolet control)
The coordinates of the class, how $\sum_k \eta_k$ closes	Reading (the arithmetic of the budget is Proved here)	This book's study (the numbers of 15-07 § 06 and § 07)
The coordinates of the four-dimensional class, comparison of rules, the flow of the coupling, the cone over many steps, the one-row version of the recursion of the net one-step bracket of the coupling	Reading (counting, agreement of degrees of freedom, the first-order Taylor formula and the bookkeeping are Proved here)	The record of this book's study (the numbers of 15-07 § 07, 15-05 § 07 and 15-08 § 06). Additivity in the sources: (A.1.4)–(A.1.5) and Exercise 2.2.4 of BBS arXiv:1907.05474v2. The bookkeeping is in the Lean bundle (15-09 Table 3, re-verified)

Words used in this chapter — class, the flow of the parameters Φ, remainder, class-closure condition, coupling-flow remainder condition, shape remainder condition, continuity of the coupling, net one-step bracket of the coupling, Coupling-Smallness Induction Lemma, the bookkeeping of the transverse contraction, the transverse contraction λ, the source, the attracting class, the bundle of rules, something to pay against. Their meanings are in A-02 Glossary.

Next chapter to read: 15-04 The one-link heat-kernel class and the hierarchical model — how the decomposition of this chapter looks in the smallest setting.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control heat-kernel class, Migdal–Kadanoff, the invariant curve · Kiichi, Shiori and Rin

The one-link heat-kernel class and the hierarchical model — the class that closes with one parameter, and the attracting curve

This chapter is this book's own investigation. The established knowledge it builds on is in 15-01 and 15-02.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — How the decomposition of 15-03 (a class that closes with finitely many parameters × a remainder) looks in the smallest setting. The one-link heat-kernel class is a semigroup and closes with one parameter, its remainder is 0, and the loss becomes a telescoping sum. In one step of the Migdal–Kadanoff hierarchical model the heat-kernel class does not close exactly, but the image is attracted, step by step at the rate $1/16$, to a one-parameter invariant curve, and the position of the curve is fixed by the balance $1/3600$ between "the source poured in at every step" and "the contraction" (a saddle-point estimate). In this model, over a finite range of couplings, the invariance of the tube and the attracting curve have been confirmed by interval arithmetic. One step on the two-dimensional open strip is exactly a one-link convolution and is no test of many links.

Chapters assumed — 15-03 § 04 (the ledger of the transverse contraction), 18-04 (the conventions of one Migdal–Kadanoff step, and $(1-b^{-2})/3$), 24-06 § 07 (the class of measures paid for by rarity), 15-01 § 01 (the logarithmic divergence of the sum of losses).

Said first — Every statement in this chapter is a statement about **a map applied to the weight of a single plaquette (a class function)**. The convolution of the one-link heat kernel and the map on the hierarchical lattice (Migdal–Kadanoff) are maps that return the weight of one plaquette to the weight of one plaquette in one step; structurally they have no multi-link couplings, no re-fixing of the gauge and no large-field remainder. So what closes here implies nothing about the ultraviolet control of one step on the four-dimensional hypercubic lattice. Moreover the hierarchical model confines even at $U(1)$, so it is no test of non-commutativity either.

The order of this chapter

1. The one-link heat-kernel class — semigroup, the telescoping sum, curvature $A + B/\beta$, three reservations
2. One Migdal–Kadanoff step — the invariant curve, contraction $1/16$, the flow of the coupling
3. The balance of source and contraction — a saddle-point estimate, $1/3600$, b^{d-4m}
4. The two-dimensional open strip — independent plaquettes, $\lambda = 1/4$, no source
5. The ultraviolet theorem for the hierarchical model — the interval-arithmetic tube, the chain from the ultraviolet to strong coupling, and the range not reached (Tables 1 and 2)
6. What can be said in this class (Table 3)
7. What cannot be said
8. The table of the five grades and the dependency graph (Table 4)

00

What this chapter shows

- The one-link heat-kernel class is a semigroup and closes with the single parameter τ; the remainder of one step is 0, and the change of a quantity over the steps is a telescoping sum in which only the difference of the end values remains (§ 01).
- In one Migdal–Kadanoff step the image is attracted at $1/16$ per step to the curve of the form of the heat kernel $\times \exp(\theta^4/3600 + \cdots)$, and on the curve the coupling flows as $\beta \mapsto \beta - \frac{1}{4} - c/\beta$ (§ 02, § 03).
- The contraction of the coefficient of θ^{2m} is b^{d-4m} (with $d = 4$, $b = 2$, $m = 2$ is $1/16$), the same form as dimensional counting (§ 03, the leading saddle-point term).
- In this model it has been confirmed by interval arithmetic that, for $\beta \in [2.46, 128]$, the tube around the heat-kernel class is invariant under the map, and that, for $\beta \in [15.26, 64]$, all orbits in the tube approach a single curve (§ 05). $\beta > 128$ (the limit of lattice spacing 0) is not closed.

01

The one-link heat-kernel class — the loss becomes a telescoping sum

We look at how "absorbing the loss into the flow of the coupling" (15-01 § 01) appears in the smallest setting, taking as the weight a one-link class function of $SU(2)$ (a function only of the angle θ from the identity) and following the flow that convolves it with the heat kernel one step at a time (the candidate of 24-06 § 07, and the question "Hessian of the one-link weight" of 24-08). Every statement in this section is a statement about **one link, the heat-kernel class, and the range written below**; it says nothing about one four-dimensional step, which contains many links and couplings between plaquettes.

The heat kernel p_τ is a semigroup,

$$p_\tau * p_t = p_{\tau+t} \quad (1)$$

so within the family of heat kernels a step yields a heat kernel again, and the parameter τ (the inverse of the effective coupling) merely moves by t Proved here (the semigroup property alone). Hence every quantity computed from a heat-kernel weight is **a function of the parameter at that moment only**, and when the changes over the steps are added up only the difference of the end values remains (the telescoping sum). It does not become a product Proved here . In terms of the identity of 15-03 § 04, this is the case where the deviation from the class δ_k is identically 0.

In the coordinates divided by the centre $\mathbb{Z}_2$ ($SO(3)$, $\theta \in [0, \pi/2]$, the branch set $\operatorname{tr} U = 0$ is $\theta = \pi/2$), write κ for the minimum of the Bakry–Émery curvature outside the layer $|\theta - \pi/2| < h$. For weights obtained from the Wilson weight by convolving with the heat kernel, the curvature outside the layer was positive at every point of $\beta \in \{2, 3, 4, 6, 8\}$, $t \in \{0, 0.05, 0.1, 0.2, 0.5\}$, half-width of the layer $h = \beta'^{-1/2}$ (without dividing, $SU(2)$, it was negative at every point of the same range) Reading (two independent implementations, in the character series and in position space, agree). The curvature outside the layer takes the form

$$\frac{\kappa}{\beta} = A(h) + \frac{B(h)}{\beta} \qquad \text{(excluding exponentially small terms)} \quad (2)$$

and for the Wilson weight $A = \sin h$, $B = 2$ Reading (for the heat-kernel weight it agrees with the numerics).

Applying this form to one four-dimensional step $\beta' = \beta - 8b_0 \ln L$ (the one-loop flow written in $\beta = 4/g^2$), the leading term $A\beta$ moves with the coupling (the part that is absorbed), and the one-step change of κ/β is $B\,(1/\beta' - 1/\beta) = O(1/\beta^2) = O(g^4)$, a loss with a finite sum Reading (a reading obtained by fitting the form above). The $\kappa_{j+1} \geq \kappa_j(1 - c/\beta_j)$ that 24-06 § 07 put forward as a candidate is of product type and goes to 0 as a power of the number of steps, but within this class it is merely pessimistic.

There are, however, three reservations. (i) The rarity rate of the layer is proportional to β, falls by $O(1)$ per step and runs out at $\beta = O(1)$ Reading . This is the one-link form of what 24-06 § 07 described as the limit of the range of couplings that can be reached, not a new obstacle on the ultraviolet side. (ii) The convexity of the divided coordinates can pay only for the part invariant under the centre (the adjoint part); the sign of $\mathbb{Z}_2$ on each link needs another currency. With many links this sign part corresponds to centre vortices Reading (a physical estimate). (iii) One step of four-dimensional renormalisation is not a convolution of the one-link heat kernel; multi-link couplings between plaquettes enter. Whether the form of $A + B/\beta$ is preserved there has not been examined.

This class has no direction leading off the curve, so neither the transverse contraction nor the source appears. In the language of the decomposition, the heat-kernel class is a class with "remainder 0", the case in which the shape remainder condition of 15-03 is empty.

02

One Migdal–Kadanoff step — the invariant curve and the contraction 1/16

The definition of the single step in this section summarises existing knowledge. The introduction is in 18-04 § 04.

The map. To the weight of one plaquette of $SU(2)$ (a class function) $w = \sum_n c_n \chi_n$ (χ_n is the character of spin $n/2$, $a_n = c_n/(n+1)$) we apply one Migdal–Kadanoff step ($d = 4$, $b = 2$). The conventions are the same as in 18-04:

$$T = (\text{decimation}) \circ (\text{bond moving}) : \qquad w \;\mapsto\; \frac{w^4}{\int w^4} \;\; (\text{exponent } b^{d-2} = 4), \qquad a_n \;\mapsto\; a_n^4 \;\; (\text{convolu}$$

Bond moving is the operation that bundles four parallel plaquettes into one (the product of the weights), and decimation is the operation that convolves four plaquettes in series. On the hierarchical lattice this map is not an approximation but the map itself, yet it is not one step of renormalisation on the four-dimensional hypercubic lattice Cited (the definition of the Migdal–Kadanoff approximation; 18-04).

The invariant curve. The heat-kernel class does not close exactly under this step, but the image is attracted, step by step at the rate $1/16$, to a one-parameter curve (the heat kernel $\times \exp(\theta^4/3600 + \cdots)$). The ratio of the differences (total-variation distance) between weights that arrive at the same β' after different numbers of steps was 16.06, 17.00 at $j = 1:2:3$ (the prediction of a contraction by $1/16$ each: 16.0, 17.0), and it was the same for every starting $\beta T \in \{3.5, 7, 15, 30\}$ (starting from Wilson, 15.8, 16.7) Reading (numerical values from this book's study; one class function, the map on the hierarchical lattice, $\beta \le 32$, half-width of the layer $h \in \{\pi/4, \pi/8\}$). On the curve, κ/β outside the layer is a function only of β at that moment, and the telescoping sum holds.

The flow of the coupling. In one step starting from the heat kernel,

$$\beta \;\mapsto\; \beta - \frac{1}{4} - \frac{0.0215}{\beta} \quad (4)$$

was found Reading (coefficients 0.022, 0.022, 0.021 at the three points $\beta = 4, 8, 16$). $\frac{1}{4}$ agrees with $(1 - b^{-2})/3$ of 18-04. Written in the form of the net one-step bracket of the coupling in the Coupling-Smallness Induction Lemma (15-03 § 03), in this model $b = 1/16$ (in units of $1/g^2$), $|rk| \approx 0.0054/\beta_k \approx 0.0013\, g_k^2$ Reading ($\beta \in [2, 32]$). The remainder relative to the heat-kernel class is $O(1/\beta)$ at κ/β (a deviation of shape that does not depend on β), but it does not accumulate as the steps are repeated Reading. As a negative control, replacing the coefficients by Wilson's at each step keeps the deviation at $\delta \approx -0.04$ ($\beta \approx 16$) without decreasing it (the test has detecting power) Reading.

Fitting into the ledger. In the language of 15-03 § 04, transverse stability holds with $\lambda \approx 1/16$, and the remainder is a correction of the form $O(1)$ relative to the heat-kernel class. It is therefore uniform in the form of "the transversely contracting case", and the deviation from the heat-kernel class was $|\delta| \le 1.6 \times 10^{-3}$ ($\beta \ge 2$, $h = \pi/4$), and $\le 6.0 \times 10^{-5}$ at $\beta \ge 8$ Reading (over the range measured).

03

The balance of source and contraction — why $1/16$, and why $1/3600$

At the saddle point for large β we estimate one step of the map as a map of the coefficients of the action Proved here (the leading saddle-point term; it drops the higher orders of the fluctuations and the other terms of the sum over images, of order $e^{-\pi\beta h}$; the constants of the error are not given).

- Heat kernel (taking only one image): $\mathrm{HK}_\beta(\theta) \propto (\theta/\sin\theta)\, e^{-\beta\theta^2/2}$. $\log(\theta/\sin\theta) = \theta^2/6 + \theta^4/180 + \theta^6/2835 + \cdots$.
- Put the weight $w = \mathrm{HK}_\beta \cdot e^{R(\theta)}$, $R = \sigma\theta^4 + \tau\theta^6 + \cdots$. **Bond moving** gives

$$w^4 = \mathrm{HK}_{4\beta}\Big(\frac{\theta}{\sin\theta}\Big)^3 e^{4R} = \mathrm{HK}_{4\beta-1}\, e^{\rho+4R}, \qquad \rho := 3\log\frac{\theta}{\sin\theta} - \frac{\theta^2}{2} = \frac{\theta^4}{60} + \frac{\theta^6}{945} + \cdots \quad (5)$$

  where $\beta \mapsto 4\beta - 1$ is $b^2\beta - (b^2-1)/3$ of 18-04. ρ is **the source**: what is left over from the prefactor $\theta/\sin\theta$ (the remnant of the Haar weight) when four plaquettes are bundled.
- **Decimation** (the convolution of four plaquettes): for large β the dominant configuration is the one in which the four group elements are spaced equally along the geodesic from the identity to U (each $\theta/4$), and the correction S is mapped to $4S(\theta/4)$. The heat-kernel part is a semigroup: $\mathrm{HK}_{(4\beta-1)/4} = \mathrm{HK}_{\beta-1/4}$.

As a consequence, the map of the coefficients of θ^4 and θ^6 is

$$\sigma \;\mapsto\; \frac{1/60 + 4\sigma}{64} = \frac{\sigma}{16} + \frac{1}{3840}, \qquad \tau \;\mapsto\; \frac{1/945 + 4\tau}{1024} = \frac{\tau}{256} + \frac{1}{967680} \quad (6)$$

The contraction is $1/16$, the source from the heat kernel in one step is $1/3840$, and the fixed point is fixed by the balance of source and contraction,

$$\sigma_* = \frac{\sigma源}{1-\lambda_\sigma} = \frac{1/3840}{1-1/16} = \frac{1}{3600}, \qquad \tau_* = \frac{1/967680}{1-1/256} = \frac{1}{963900} \quad (7)$$

Proved here (the arithmetic of the geometric series on (6)). In general the contraction of the coefficient of θ^{2m} is $b^{d-2}\cdot b^2 \cdot b^{-4m} = b^{d-4m}$ (with $d=4$, $b=2$, 4^{2-2m}): $m=1$ gives 1 (marginal, β itself), $m=2$ gives $1/16$, and $m=3$ gives $1/256$ Proved here (the leading saddle-point term). Reading $\theta \sim a^2 F$, θ^{2m} corresponds to an operator of dimension $4m$, and b^{d-4m} is the same as the dimensional counting for that dimension Reading (an estimate).

Check. The coefficient of θ^4 in one step is 2.615×10^{-4} ($\beta=60$) against $1/3840 = 2.604\times10^{-4}$, and on the attracting curve 2.789×10^{-4} against $1/3600 = 2.778\times10^{-4}$. The difference has the form $6.5\times10^{-5}/\beta$ (the correction from the fluctuations). θ^6 was 1.06×10^{-6} against 1.04×10^{-6} Reading (numerics for the same system). The ratios 16.06, 17.00 of § 02 are a direct measurement of the contraction $1/16$.

Effect on the curvature. Since $-\sigma_*\theta^4$ enters $V = -\log w$, the curvature in the tangential direction at the edge of the layer $\theta = \pi/2 - h$ moves by $-4\sigma_*\theta^3\cot\theta$. With $h = \pi/4$, $-4(\pi/4)^3$

$/3600 = -5.4 \times 10^{-4}$ Proved here (arithmetic). The measured deviation $\delta \cdot \beta$ of κ/β approaches this value: -2.6×10^{-4} ($\beta = 16$), -3.6×10^{-4} (24), -4.1×10^{-4} (32) Reading .

Reading Reading : in this model the source and the contraction come from separate mechanisms. The source is the $\theta^4/60$ that **bond moving** (the operation of bundling in parallel) pours in from the prefactor, and it exists only at $d > 2$. The contraction is what **decimation** (the operation of combining in series) does in dividing the coefficient of θ^4 by $1/64$, the contraction of the scale of the coupling, to give $1/16$. In the language of 15-03 § 04, the two numbers to ask about (the size of the source and λ) are $1/3840$ and $1/16$ in this model. Values of the same two numbers measured in a small four-dimensional box are in 15-05 (λ in the band 0.026–0.049, the source in the band 4–6×10^{-4}).

04

The two-dimensional open strip — no test of many links

Consider on the two-dimensional open strip (a narrow strip-shaped lattice) one step that blocks plaquettes two at a time. In axial gauge the plaquette variables on the open strip are independent, so the step combining two plaquettes is **exactly the one-link convolution** $w * w$, and the two coarse plaquettes are also independent Proved here (a known structure; with 200,000 independent samples the averages of the characters agree to around 1σ, correlation 0.003 ± 0.002 Reading). So this setting is no test of many links (no new couplings are born).

On that basis, decimation combining m plaquettes multiplies the coefficient of θ^4 by $1/m^3$ and the deviation measured at $\beta' = \beta/m$ by $1/m^2$. For two plaquettes this is $\lambda = 1/4$; since there is no bond moving there is no source term either, and the fixed form is the heat kernel itself Proved here (the leading saddle-point term). The path dependence starting from Wilson has an order that differs with the observable: $O(1/\beta^3)$ for the coefficients of low representations, $O(1/\beta)$ for the total-variation distance, and $O(1)$ for κ/β' outside the layer (all $1/m^2$ in m) Reading ($\beta T = 8$–64, $m = 1, 2, 4$). The heat kernel and the Villain weight at $U(1)$ do not depend on the path; the deviation is at most 10^{-8} Reading .

The smallest example with many links is the three-dimensional $2 \times 2 \times 1$ slab, in which the plaquettes are not independent because of the cube constraint. The numbers beyond that are in 15-05.

05

The ultraviolet theorem for the hierarchical model — the interval-arithmetic tube, and the range it does not reach

§ 02 and § 03 were numerics and a saddle-point estimate. Within this model, we list, together with their conditions, the results of checking each row of the ultraviolet decomposition (15-03 § 02) over a finite range of couplings. All are statements about **the MK map T ((3)), the weight of one plaquette of $SU(2)$, $d = 4$, $b = 2$.**

Coordinates and the tube. Let the reading of the coupling be $s(w) := 1/b(w) := -\frac{2}{3}\log a_1$ (for the heat kernel HK_β, $a_n = e^{-q_n/(2\beta)}$ and $q_n := n(n+2)$, so $b(\mathrm{HK}_\beta) = \beta$). Put the transverse coordinate

$$e_n(w) := \log a_n - \frac{q_n}{3}\log a_1 \qquad (n \geq 2) \quad (8)$$

Then in the heat-kernel class $e \equiv 0$, and decimation gives $e \mapsto 4e$, $s \mapsto 4s$ exactly Proved here (from the definition). **The tube** U is the set of weights such that "$b(w) \in [\beta_{\min}, \beta_{\max}]$, $|e_n(w)| \leq R\,(q_n^2 s^4 + 10\,q_n s^3)$ $(2 \leq n \leq N,\ R = 10^{-3})$, and the tail of high n is bounded by a weighted sum of the coefficients". $q_n^2 s^4$ corresponds to the size of a shape of type θ^4 (for $w = \mathrm{HK} \cdot e^{\sigma\theta^4}$, $e_n \approx \sigma q_n(q_n - 3)s^4$).

Table 1 Each ultraviolet row of the hierarchical model, and the range confirmed (the MK map, the weight of one plaquette of $SU(2)$, $d = 4$, $b = 2$, the tube is $R = 10^{-3}$)

Row	Content	Scope	Label
Invariance of the tube	$T(U) \subset U$. At $[64, 128]$ the maximum of the ratio of the images goes from 0.494 to 0.492; no term gets tighter as β is raised	All 6277 intervals of $\beta \in [2.46, 128]$ (the 3077 intervals of $[2.46, 64]$ and the 3200 intervals of $[64, 128]$; the 23 intervals of $\beta \in [2, 2.46)$ fail)	Computer-assisted (interval arithmetic, with positive and negative controls)
Pointwise transverse contraction	The norm of the transverse linearisation on the tube $\lambda_{\mathrm{tube}} < 1$ (0.81 at $\beta = 8$, 0.46 at 16, 0.27 at 64, 0.26 at 128)	$\beta \in [5.88, 128]$	Computer-assisted
Attracting curve	All orbits in the tube approach a single curve with ratio $\Lambda \leq 0.735$ per step ($\Lambda \leq 0.384$ at $\beta \in [23.62, 64]$). The accuracy of the curve is $\lesssim 2.2 \times 10^{-4}$ in the normalisation, from the bound on the tail	$\beta \in [15.26, 64]$	Computer-assisted (input) + Proved here (an elementary lemma on graph transforms)
The flow of the coupling (at heat-kernel points)	$(\beta - \beta' - \frac{1}{4})\beta \in [0.020928,\ 0.023690]$ ($[0.020876,\ 0.020929]$ at $[64, 128]$), monotone towards $1/48 = 0.020833$ (agreeing with $1/48 + 0.0062/\beta$)	$\beta \in [2.5, 128]$	Computer-assisted
The flow of the shape coefficient	$\sigma_k \mapsto (16\sigma_k + 12\ell_k)/4^{2k}$ (ℓ_k is the coefficient of θ^{2k} in $\log(\theta/\sin\theta)$), contraction 16^{1-k}, fixed point $\sigma_k^* = 12\ell_k/(4^{2k} - 16)$ $(1/3600,\ 1/963900)$	Leading term for large β	Proved here (the leading saddle-point term, no constants for the error)
The flow of the coupling on the curve	$\beta' = \beta - \frac{1}{4} - c(\sigma)/\beta$, $c(\sigma) = \frac{1}{48} - 15\sigma$, on the curve $c_* = \frac{1}{60}$ (on the heat-kernel class $\frac{1}{48}$)	Leading term for large β	Proved here (leading term). In floating point $1/60 + 0.0078/\beta$ Reading

Row	Content	Scope	Label
The seam to the band and the chain	The tube $\hat{U}$, with radius and weights narrowed as functions of β, closes under T; the image of its lower end enters the tube D_T on the band side, and from there the orbit goes to the strong-coupling criterion region. Points with $\beta \le 64$ reach the criterion region within 846 steps, points with $\beta \le 128$ within 1137 steps	The 755 intervals of $\beta \in [0.95, 2.46]$, the 27 intervals of the seam, and the 139 intervals of the exit; $[64, 128]$ for the tube above	Computer-assisted (interval arithmetic; the band side uses the earlier computer-assisted record as it is) + Proved here (assembling the step counts)
The envelope of c_* (the recentred tube)	The tube U_σ, centred on the transverse leading term of $\mathrm{HK} \cdot e^{\theta^4/3600}$ and narrowed to radii $R = 10^{-4}$, $A = 0$ and interval width 0.002, is invariant, and over the whole tube $(\beta - \beta' - \frac{1}{4})\beta \in [0.0151,\ 0.0188]$: it contains $1/60$ and contains neither $1/48$ nor the value at the heat-kernel point	Only the 50 intervals at the five places $\beta \approx 16$, 32, 64, 96, 128. That the curve enters U_σ is not shown	Computer-assisted (conditional)
The curvature outside the layer and the weight of the layer	On the curve $\kappa/\beta \ge A(h) - C/\beta > 0$, the weight of the layer $\le Ce^{-c\beta}$	Leading term for large β	Assumption (no interval arithmetic done)

$\sigma \mapsto \sigma/16 + 1/3840$ in (6) is the case $k = 2$ of the flow of the shape coefficient in Table 1 ($\ell_2 = 1/180$, $(16\sigma + 12/180)/256 = \sigma/16 + 1/3840$).

What ultraviolet means in this model. Even if the starting β is made large within a finite range and the flow is run for K steps, the weight at the last scale lies within $\Lambda^K \times$ (the starting distance) of a point on the curve, and the error is uniform in K Computer-assisted (a statement on the range $\beta \in [15.26, 64]$; β falls by about $1/4$ per step, so for example from the final $\beta = 16$ up to 64 is $K \approx 190$ steps). That is, the form of (4) of 15-03 § 04 stands, over a finite range in this model, as a computer-assisted inequality.

The range not reached (in the same rows). $\beta > 128$ — the limit of lattice spacing 0 itself — has only the leading saddle-point term, with no estimate carrying the constants of the error Assumption. The attracting curve remains at $[15.26, 64]$ (the input of the graph transform has not been computed at $[64, 128]$). Interval arithmetic for the curvature outside the layer and the weight of the layer is not reached. Separating $c_* = 1/60$ from $1/48$ is impossible in principle over the whole of a tube centred on the heat kernel (a tube centred on the heat kernel, and also a recentred tube whose radius is $10^{-3}/3$ or more, contains the heat kernel itself, so the envelope of β' over the whole tube always contains $1/48$ Proved here); it was reached only on intervals at five places, on a tube with radius narrowed to 10^{-4} (Table 1). That the curve enters that tube is not reached. $\beta < 2.46$ is not reached with the same tube, but if the tube is narrowed as a function of β, its lower end comes down to 0.95 and is joined to the band side (next paragraph). Above all, it closes here because this is **a model in which the weight of one plaquette returns to the weight of one plaquette in one step**. Structurally it has no multi-link couplings, no re-fixing of the gauge and no large-field remainder, and the "two rows absent from the sources" of 15-02 (the one-step remainder including large fields, and the source of

observables) are empty in this model. The hierarchical model confines even at $U(1)$, so it is no test of non-commutativity either. $\frac{1}{4}$ is close to the four-dimensional one-loop value ($8b_0 \ln 2 \approx 0.258$ for factor 2), but there is no basis for comparing $c_* = 1/60$ with the four-dimensional two-loop coefficient Reading .

One chain from the ultraviolet to strong coupling (hierarchical model). The earlier record for the same model contains the tube D_T on the band side and a computer-assisted theorem that carries it in finitely many steps to the strong-coupling criterion region ($\kappa > 0$) ($a_1 \in [0.0144, 0.21)$, that is, $\beta \in [0.354, 0.961)$; the criterion region is $\beta \le 0.366$). With its coordinates aligned to those of the tube of Table 1 ($\beta \in [2.46, 64]$), it is $a_1 = e^{-3/(2\beta)}$, and as it stands a gap of $\beta \in [0.961, 2.46)$ remains Proved here (matching the coordinates). If we lay down the tube $\hat{U}$ with radius and weights narrowed piecewise as functions of β — the same $|e_n| \le 10^{-3}(q_n^2 s^4 + 10 q_n s^3)$ as in Table 1 for $\beta \ge 2.16$, $4 \times 10^{-4} q_n^2 s^4$ for $\beta \le 1.16$, narrowed linearly on pieces of width 0.05 in between, with the tail parameters taken per cell of width 0.1 — the following hold Computer-assisted (interval arithmetic, interval width 0.002).

- On all of the 755 intervals of $\beta \in [0.95, 2.46]$ and the 27 intervals of the seam ($[2.46, 3.0]$), $T(\hat{U}) \subset \hat{U} \cup D_T$ (maximum ratio of the image 0.934). In one step β drops by 0.2196–0.2990.
- The images leaving the lower end (the 139 intervals of $\beta \in [0.95, 1.228]$) all enter D_T (maximum ratio 0.9888).
- Hence the points of $\hat{U}$ with $\beta \le 64$ enter D_T within 289 steps, and then enter the criterion region within 557 steps: within 846 steps in all Proved here (assembling the step counts; Lean: `mk_full_chain_64`, `tube_exit`, `cell_descent`, `chain_compose`; the assumptions are the inputs from interval arithmetic and the one-step drop $\delta = 0.2196$). The orbit leaving the heat kernel at $\beta_0 = 64$ enters D_T in 211–289 steps.
- Combined with the result of extending the tube above to $[64, 128]$ (Table 1; the one-step drop is at least 0.2296 on all intervals), the points of $\beta \le 128$ enter D_T within 580 steps and then the criterion region within 557 steps: within 1137 steps in all Proved here (assembling the step counts; Lean: `mk_full_chain_128`, `tube_exit_top`, `chain_top`; the assumptions are the inputs from interval arithmetic, $b_{\mathrm{lo}} = 0.95$, $\delta = 0.2196$, and the cell index ≤ 556 on the band side).

What worked at the seam was not including the tube itself in D_T, but **comparing the image of the lower end of the tube with the boundary of D_T**. The upper boundary of D_T lies only 1–7% above the image of the heat kernel in the coefficient of $n = 2$, so inclusion with the shape of the tube kept fails at every radius Reading . Conditions (in the same rows): (a) the interval arithmetic is our own, and the derivations of the Taylor remainder and of the upper bounds on the cumulants are on paper (not yet in Lean); (b) the band side uses the computer-assisted certificate of the earlier record as it is, and is not recomputed here; (c) the reference curve of D_T was rebuilt following the construction in the record, and has been matched against the numbers of the record at one point; (d) the 557 steps are a coarse upper bound counted by the number of cells; (e) by convention, points on the boundaries of the pieces belong to the smaller tube. It passes positive controls (string-for-string agreement with the 27 intervals of Table 1;

agreement of a recomputation of the 200 intervals of $[60, 64]$) and negative controls (thinning the tube to 1/5, breaking the coefficients of the image, narrowing D_T by 3%). $\beta > 128$ **(the limit of lattice spacing 0) is not closed.** The chain closes here because this is a model in which the weight of one plaquette returns to the weight of one plaquette in one step, and it implies nothing about how things close on the four-dimensional lattice. Even at $U(1)$ it flows to strong coupling in the same way, so it is no test of non-commutativity either.

This chain does not carry over to four dimensions — its relation to the sandwich. The line that starts from the four-dimensional Wilson action and, through a sequence of representatives in which the Migdal–Kadanoff image is shrunk uniformly (Tomboulis's sandwich), puts it onto the chain above, breaks at two places Reading . (i) What the sandwich proves is an upper and a lower inequality for one number, the partition function (the upper bound by Migdal–Kadanoff decimation, from translation invariance and convexity, using no property of the group; the lower bound from reflection positivity) Cited . The step that represents the twisted and the untwisted partition functions by the same representative (a common interpolation parameter) is not proved (as K. R. Ito and E. Seiler pointed out) Cited . This step has to be false at $U(1)$, and needs a non-commutative input (the same place as the middle of the renormalisation comparison in 27-02). (ii) The representative is the image shrunk uniformly, and shrinking pushes the transverse coordinate up by at least $(5/3)(1-\alpha)$ Proved here (Lean: `shrink_shift_lower`; assumptions $0 < \alpha$, $q \geq 3$); to stay in the tube above, $1-\alpha$ must be at most of the order of 10^{-2} in the tube on the band side and of 10^{-7} in the ultraviolet tube Reading . This is incompatible with the device of "keeping α away from 1" that the sandwich argument needs. When the one-step sandwich of the same observable is measured in small boxes ($4^4 \to 2^4$, $8^4 \to 4^4$, $\beta \in \{2, 4, 8\}$, Metropolis), it holds in all 12 rows for both $SU(2)$ and $U(1)$, and does not distinguish the groups Reading . Hence this chain is a theorem inside the hierarchical model, and not a seam to the four-dimensional lattice.

A note on the proposal of a one-step inequality with the comparison fixed. One of the candidates for the middle of the renormalisation comparison of the band is $\sigma[T_{\mathrm{MK}} w] \leq 4\sigma[w]$: the string tension of the one-step Migdal–Kadanoff image does not exceed the string tension of the true coarse theory Assumption (not shown for any β). Used at each step of the orbit $w_k = T_{\mathrm{MK}}^k w$, it gives $\sigma[w] \geq 4^{-k}\sigma[w_k]$ without monotonicity of the map Proved here (a finite induction; Lean: `orbit_lower`, `lower_bound_four`; the one-step inequality is assumed). Testing it in a finite box needs two notes. (i) The twist free energy $\Delta F = -\ln(Z^-/Z^+)$ is dual to the string tension, and goes to 0 under confinement. What corresponds to $\sigma \cdot A$ is $-\ln\tanh(\Delta F/2)$, and the true step keeps ΔF on the box (the parity of the twist is preserved by blocking), so the ratio 4 of the tensions and the ratio $1/4$ of the areas cancel, and the threshold of the one-step inequality is 1, not 4, that is, $R := \Delta F(2^4; T_{\mathrm{MK}} w)/\Delta F(4^4; w) \geq 1$ Proved here (an identity from the definitions), Cited (the dual reading). (ii) In periodic boxes of 4^4 and 2^4, for the Wilson-start step at $SU(2)$ ($\beta = 1.5$, 2.0, 2.3) and also for one step of an orbit lying on the Migdal–Kadanoff form (effective β from 2.47 to 2.19), $R \geq 1$, with $R = 1.34(7)$ in the closest row (conservative lower bound 1.19, two independent seeds) Reading (Metropolis, multi-stage BAR, twist $n_{01} = 1$). It is, however, a weak test: the do-nothing map also passes it in the same box ($R = 2.46$ at $\beta = 2.3$). Raising the same orbit step to 6^4 and 3^4 gives $R = 1.17(20)$, and the margin converted to Wilson's β shrinks

from 0.15 at $4^4/2^4$ to 0.02(2) ($R = 0.97(10)$ at the point of effective $\beta = 2.8$): **within the resolution of the measurement, it does not decide** Reading (by statistical error; the ΔF measurement has a systematic floor of 0.1–0.25 without a consistent sign, cause not identified). There is neither evidence that the inequality is broken nor numerical support for the side on which it holds inside the band. The 7% discretisation of the 2^4 box may have made most of the first margin. At $U(1)$ too, one step goes the same way ($1.265(19)$ at $\beta = 1.3$), and a violation that distinguishes the groups does not show up in one step in a small box Reading (one Migdal–Kadanoff step from Wilson raises the effective β, so there is no step crossing the transition in the first place; by Bessel bookkeeping). This note does not change "does not carry over to four dimensions" above.

Questions. We set down three questions that come out of this chapter (numbered consecutively through Part 15; the index is in 15-09).

Table 2 Questions from the hierarchical model

Q.	Question	Size	What moves if settled	What does not move even if settled
4	Extend the invariance of the tube and the attracting curve of the hierarchical model to $\beta > 64$ (for $\beta \le 64$ and below the chain has been joined down to the band side; the only gap left is the upper end). **Answer (partial):** the tube up to $\beta \le 128$ (6277 intervals), the chain from $\beta \le 128$ in 1137 steps. The gaps left are $\beta > 128$ and the curve for $[64, 128]$: join the asymptotic estimate for large β (with the constants of the error) seamlessly to the interval-arithmetic range, making a statement uniform up to $\beta \to \infty$ (there is a proposal to recentre the tube on $\mathrm{HK} \cdot e^{\theta^4/3600}$ instead of the heat kernel)	Paper and computer-assisted	The ultraviolet control of the hierarchical model closes down to lattice spacing 0	Ultraviolet control for the four-dimensional lattice (because the model has no multi-link couplings, no re-fixing and no large fields)
5	On the curve of the hierarchical model, confirm by interval arithmetic the lower bound $\kappa/\beta \ge A(h) - C/\beta$ on the curvature outside the layer and the weight of the layer $\le Ce^{-c\beta}$, and at the same time separate $c_* = 1/60$ from $1/48$. **Answer (partial):** $(\beta - \beta' - \frac{1}{4})\beta \in [0.0151, 0.0188]$ on the tube U_σ with radius narrowed to 10^{-4} (50 intervals at five places). That the curve enters U_σ, and the curvature and the weight of the layer, are still open	Computer-assisted	The "Assumption" row of Table 1, and the coefficients of the two terms of the flow of the coupling	Same as above

Q.	Question	Size	What moves if settled	What does not move even if settled
31	Run the recentred tube U_σ ($R = 10^{-4}$, $A = 0$) through the whole of $[15.26, 128]$ with the interval width kept at 0.02 (adding the cubic term of δs to the test so that the remainder is only the transverse part), take the input of the graph transform at $[64, 128]$, and turn the curve entering U_σ and the envelope of c_* on the curve into an unconditional computer-assisted statement. At the same time, cover $\beta \geq 128$ by one asymptotic interval	Computer-assisted and paper	The conditions of the c_* row of Table 1, and the gap at $\beta > 128$	Same as above

06

What can be said in this class

Table 3 lines up how each component of the decomposition looks in three smallest systems.

Table 3 The three smallest systems and the components of the decomposition (all are maps applied to the weight of one plaquette, not one step on the four-dimensional hypercubic lattice)

System	Class	Flow of the coupling	Transverse contraction λ	Source	Handling of the remainder
Convolution of the one-link heat kernel	Heat kernel $\{p_\tau\}$ (one parameter)	$\tau \mapsto \tau + t$ Proved here	— (no direction leading off)	0 Proved here	Remainder 0, the telescoping sum
Two-dimensional open strip (combining two plaquettes)	The heat-kernel class (the fixed form is the heat kernel)	$\beta \mapsto \beta/2$	$1/4$ Proved here (leading saddle-point term)	0 Proved here	The deviation starting from Wilson only contracts
Migdal–Kadanoff ($d = 4$, $b = 2$)	The invariant curve $\mathrm{HK} \cdot \theta^4/3600 + \theta^6/963900 + \cdots$	$\beta \mapsto \beta - \frac{1}{4} - c/\beta$, on the curve $c_* = \frac{1}{60}$ Proved here (leading term)	$1/16$ (θ^4), $1/256$ (θ^6) Proved here (leading term); on the tube < 1 ($\beta \in [5.88, 64]$) Computer-assisted	$1/3840$ (θ^4) Proved here (leading term)	Approaches the curve with $\Lambda \leq 0.735$ ($\beta \in [15.26, 64]$) Computer-assisted. The tube is invariant for $\beta \in [2.46, 128]$ Computer-assisted

What to read from Table 3 Reading. The class of the class-closure condition of 15-03 is either the heat-kernel class itself (remainder 0) or the invariant curve that deviates from the heat kernel only by a shape of $O(1)$ (the remainder is bounded by a supremum through the transverse contraction); in both the only parameter is the coupling. The source comes from the operation of bundling in parallel, and the contraction from the operation of combining in series. The two-dimensional open strip has no bundling in parallel and so no source; Migdal–Kadanoff has

one. One step on the four-dimensional lattice contains both operations, and moreover multi-link couplings are born. The smallest system for that is the bundle network of 15-05.

07

What cannot be said

(a) All results of this chapter are statements about maps applied to the weight of one plaquette (the convolution of the one-link heat kernel, the two-dimensional open strip, the map on the hierarchical lattice), and imply nothing about one four-dimensional step containing many links, couplings between plaquettes, re-fixing of the gauge and large fields.

(b) The form (2) of the curvature for the one-link heat-kernel class is a reading of numerics over the registered range of β and t.

(c) (5)–(7) and b^{d-4m} are estimates from the leading saddle-point term, dropping the higher orders of the fluctuations and the other terms of the sum over images. The constants of the error are not given.

(d) The tube of the hierarchical model and the chain to the band are computer-assisted statements over a finite range of $\beta \le 128$, and the attracting curve over a finite range of $\beta \le 64$ (the band side of the chain uses the certificate of the earlier record as it is); the limit of lattice spacing 0 ($\beta \to \infty$) is not closed. The curvature outside the layer and the weight of the layer have not been confirmed by interval arithmetic. The envelope of c_* is a statement only on intervals at five places on the recentred tube, and it is not shown that the curve enters that tube.

(e) The hierarchical model confines even at $U(1)$, so what closed here is no test of non-commutativity. $c_* = 1/60$ has not been compared with the four-dimensional two-loop coefficient.

(f) That the chain of § 05 does not carry over to four dimensions is a reading from checking the sources of the sandwich (where the unproved step lies) and from numbers for observables in small boxes. The size of $1-\alpha$ for the representatives of the sandwich has not been measured.

(g) The note in § 05 on the proposal of a one-step inequality with the comparison fixed consists of Monte Carlo numbers in periodic boxes of $4^4/2^4$ and $6^4/3^4$ (in the latter the margin is within the resolution and does not decide; the cause of the systematic floor of the ΔF method is not identified) and the correction of the threshold (an identity from the definitions and the dual reading). The one-step inequality is not shown for any β. Since the do-nothing map also passes in the same box, the numbers say no more than "no violation was seen". The discretisation of 2^4 alone carries a systematic error of about 7%.

08

The table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope

Row	Statement (one line)	Grade	Basis	Scope
[01]	The heat-kernel class closes as a semigroup; quantities are functions of the parameter at that moment, and the change over the steps is a telescoping sum	Proved here	§ 01 (the semigroup property)	One link, within the heat-kernel family
[02]	The curvature outside the layer in divided coordinates is positive, with $\kappa/\beta = A + B/\beta$. In one four-dimensional step, $O(1/\beta^2)$	Reading	§ 01 (two independent implementations)	One link, $SU(2)/\mathbb{Z}_2$, outside the layer, $\beta \in \{2, \dots, 8\}$, $t \le 0.5$
[03]	In one MK step the image approaches the invariant curve at $1/16$ per step, $\beta \mapsto \beta - \frac{1}{4} - 0.0215/\beta$	Reading	§ 02 (numerics for the same system, with a negative control)	One class function, the hierarchical lattice, $\beta \le 32$
[04]	Saddle-point estimate (5)–(7): source $1/3840$, contraction $1/16$, fixed point $1/3600$, contraction b_{d-4m} of θ^{2m}	Proved here	§ 03 (leading saddle-point term)	Large β; higher orders of the fluctuations dropped
[05]	One step on the two-dimensional open strip is a one-link convolution, $\lambda = 1/4$, no source	Proved here, Reading	§ 04	Axial gauge, open strip
[06]	Hierarchical model: invariance of the tube, pointwise contraction, flow of the coupling at heat-kernel points	Computer-assisted	§ 05 Table 1 (interval arithmetic, with controls)	$\beta \in [2.46, 128]$, $[5.88, 128]$, $[2.5, 128]$, $R = 10^{-3}$
[07]	Hierarchical model: the attracting curve (ratio $\Lambda \le 0.735$)	Computer-assisted, Proved here	§ 05 Table 1 (input by interval arithmetic; an elementary lemma on graph transforms)	$\beta \in [15.26, 64]$
[08]	Flow of the shape coefficient, $c(\sigma) = \frac{1}{48} - 15\sigma$, $c_* = \frac{1}{60}$	Proved here	§ 05 Table 1 (leading term)	Large β, no constants for the error
[09]	Lower bound on the curvature outside the layer on the curve, and the weight of the layer. Extension to $\beta > 128$, and the curve for $[64, 128]$	Assumption	§ 05 (the questions "extending the tube of the hierarchical model" and "interval arithmetic for curvature outside the layer")	—
[10]	The chain of the hierarchical model: the narrowed tube $\hat{U}$ closes under $\beta \in [0.95, 2.46]$, the image of the lower end enters the band tube D_T, and points with $\beta \le 64$ reach the criterion region within 846 steps, points with $\beta \le 128$ within 1137 steps	Computer-assisted, Proved here (assembling the step counts)	§ 05 (interval arithmetic, with controls)	The MK map, one plaquette of $SU(2)$; the band side from the earlier record; $\beta > 128$ not included

Row	Statement (one line)	Grade	Basis	Scope
[11]	This chain does not carry over to four dimensions: the unproved step of the sandwich (the common interpolation parameter), and the representatives not staying in the tube	Reading (the push-up by shrinking is Proved here)	§ 05	Checking the sources of the sandwich; observables for $4^4 \to 2^4$, $8^4 \to 4^4$; $\beta \in \{2, 4, 8\}$
[12]	The envelope $[0.0151, 0.0188]$ of c_* on the recentred tube U_σ (contains $1/60$, does not contain $1/48$)	Computer-assisted	§ 05 Table 1 (interval arithmetic; matching by independent quadrature in position space)	$R = 10^{-4}$, $A = 0$, width 0.002, 50 intervals at five places; that the curve enters U_σ is open
[13]	Note on the proposal of a one-step inequality with the comparison fixed: the one-step threshold in a finite box is 1 ($R \geq 1$); one step of the orbit gives $R = 1.34(7)$ ($4^4/2^4$) and $1.17(20)$ ($6^4/3^4$, within the resolution); a weak test, and no distinction between the groups appears in a one-step box	Proved here (the identity for the threshold, the finite induction), Reading (the numbers)	§ 05	$4^4/2^4$ and $6^4/3^4$ periodic; Metropolis; BAR; $SU(2)$ and $U(1)$; at $6^4/3^4$ it does not decide within the resolution; the one-step inequality open for every β

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV SEMI   telescoping sum in the heat-kernel class          (one link)
[02] RDG HK1    curvature outside the layer A + B/β ⟸ [01]          (one link, regist
[03] RDG MK     invariant curve, 1/16, flow of β ⟸ [01]             (hierarchical lat
[04] PRV SADDLE source 1/3840, contraction 1/16, 1/3600 ⟸ [01]     (leading saddle-p
[05] PRV STRIP  open strip is a one-link convolution ⟸ [01][04]
[06] CMP TUBE   tube invariance, contraction, flow of β ⟸ [03][04]  (β ∈ [2.46, 128]
[07] CMP CURVE  attracting curve ⟸ [06]                             (β ∈ [15.26, 64]
[08] PRV FLOW   flow of the shape coefficient, c* = 1/60 ⟸ [04]     (leading term)
[09] ASM MKUV   β > 128, curvature, weight of the layer ⟸ [02][07][08]  (question)
[10] CMP CHAIN  narrowed tube, seam to the band, 846 steps, 1137 steps ⟸ [06]  (β ∈ [
[11] RDG NO4D   does not carry over to four dimensions ⟸ [10]      (checking the sand
[12] CMP CSTAR  envelope of c* on the recentred tube ⟸ [06][08]    (five places, curv
[13] RDG SD     note on the one-step inequality with the comparison fixed (threshold 1
```

To check it, see by eye that in every "⟸" the numbers on the right are smaller than the number on the left. There are no cycles. [09] is an assumption row, in the form of a question. Among the nodes of this chapter there is no node saying that four-dimensional ultraviolet control has been shown.

What this chapter can say

Can say	Cannot say
Proved here The semigroup and the telescoping sum of the heat-kernel class; the saddle-point estimate for one MK step (source, contraction, fixed point, b^{d-4m}, c_*); that one step on the open strip is a convolution	The constants of the error of the saddle-point estimate
Computer-assisted For the hierarchical model: invariance of the tube ($\beta \in [2.46, 128]$), pointwise contraction ($[5.88, 128]$), the attracting curve ($[15.26, 64]$), the flow of the coupling at heat-kernel points, the chain from the narrowed tube via the band side to the strong-coupling criterion region ($\beta \in [0.95, 128]$; the band side from the earlier record), the envelope of c_* on the recentred tube (five places)	$\beta > 128$ (the limit of lattice spacing 0). The curve for $[64, 128]$. Interval arithmetic for the curvature outside the layer and the weight of the layer. The separation of c_* on the curve
Reading The form of the one-link curvature $A + B/\beta$; the MK numerics (ratios 16.06, 17.00, the flow of β); the order of the path dependence on the open strip	One step containing many links and couplings between plaquettes. The source and the contraction for one step on the four-dimensional lattice (15-05)
Reading That this chain does not carry over to four dimensions (the unproved step of the sandwich, the margin of the tube)	A non-commutative proof of the common interpolation parameter. The size of $1 - \alpha$ for the representatives
Proved here That the threshold of the proposal of a one-step inequality with the comparison fixed is 1 in a finite box; the finite induction from the one-step inequality. Reading The numbers in small boxes	The one-step inequality for any β. A step that distinguishes the groups

Sources and reproduction

Item	Type	Source, tools
Conventions of one Migdal–Kadanoff step, and $(1 - b^{-2})/3$	Cited	18-04
The heat-kernel semigroup, the telescoping sum, the saddle-point estimate (5)–(7)	Proved here	The text of this chapter
Curvature for the one-link heat-kernel class	Reading	Record of numerics from this book's study (convolving a one-link class function with the heat kernel for one step; two implementations, in the character series and in position space; as negative controls, the undivided $SU(2)$ and $U(1)$)
The invariant curve and the contraction of one Migdal–Kadanoff step; the open strip	Reading	Record of numerics from this book's study (applying one MK step to the class function of one plaquette; as a negative control, replacing the coefficients by Wilson's at each step; the open strip with a check against independent samples)
The tube, the attracting curve and the flow of the coupling for the hierarchical model	Computer-assisted	Record of computer-assisted work from this book's study (the interval-arithmetic tube, 3077 intervals, with positive and negative controls; the curve by an elementary lemma on graph transforms)

Item	Type	Source, tools
The narrowed tube, the seam to the band, the number of steps of the chain	Computer-assisted, Proved here	Record of computer-assisted work from this book's study (interval arithmetic; overlapping checks at interval widths 0.002 and 0.001; decisions on the small exit intervals; matching by independent quadrature in position space), and the earlier record for the band side. The assembly of the step counts is in the Lean bundle (15-09 Table 3, re-verification: done)
Tomboulis's sandwich and the unproved step	Cited	E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930, arXiv:0803.3019. T. Kanazawa, Phys. Lett. B 670 (2009) 421, arXiv:0805.2742
The push-up of the transverse coordinate by shrinking	Proved here	The text of this chapter. The algebra is in the Lean bundle (15-09 Table 3)
Extension of the tube to $[64, 128]$; the recentred tube and the envelope of c_*; the 1137 steps of the chain	Computer-assisted, Proved here	Record of computer-assisted work from this book's study (the same test run by the main implementation and by our own interval arithmetic; agreement on the 3200 intervals of $[64, 128]$ and the 200 intervals of $[60, 64]$, cross-checked on two machines; negative controls; the recentred tube only on intervals at five places; independent cross-check by the trapezoidal rule in position space). The assembly of the step counts is in the Lean bundle (15-09 Table 3, re-verification: done)
The threshold of the proposal of a one-step inequality with the comparison fixed, and the numbers in small boxes	Proved here, Reading	Record of this book's study (our own Metropolis; multi-stage BAR and thermodynamic integration along the twist interpolation; reruns with independent seeds; positive controls (the classical flux of $\beta = 4$ at $U(1)$; 0 for a double twist); negative control (a map that is too fast)). The induction is in the Lean bundle (15-09 Table 3)
Numbers for the one-step sandwich of observables	Reading	Record of numerics from this book's study (Metropolis; periodic; $4^4 \to 2^4$ and $8^4 \to 4^4$; $\beta \in \{2, 4, 8\}$; $SU(2)$ and $U(1)$; errors from 20 bins)

Terms used in this chapter — the heat-kernel class, semigroup, the telescoping sum, one Migdal–Kadanoff step (bond moving, decimation), the invariant curve, the transverse contraction λ, the source, balance, the tube, the attracting curve. Their meanings are in the A-02 Glossary.

Next chapter: 15-05 The flux-network theorem — how the two numbers, source and contraction, are fixed in the smallest system with many links.

The flux network theorem — transverse contraction and the flow of the parameter from one projection

This chapter is this book's own investigation. § 08 is a map of open questions.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — If we fix the angle θ of the loop W of the single coarse plaquette, restrict to commuting configurations and minimise the tree-level action $\sum_p \phi_p^2/2$, the minimiser is unique, $\phi = \theta f$, $f = Ps/\|Ps\|^2$ (s is a surface with boundary W, and P is the orthogonal projection onto the space of plaquette angles that come from links) (Unit Bundle Theorem). From this, the ratio of the coarse coupling $\beta'/\beta = \sum f^2$, the factor $\sum f^4$ by which the fine quartic term maps to the coarse quartic term, and the one-step factor of the deviation from the heat-kernel class (the transverse contraction) $\lambda = \sum f^4 / \sum f^2$ are all determined, and $\lambda \le \max f^2$ follows in one line. In a three-dimensional box with free boundary, $\sum f^2$ is the effective resistance of a resistor network. We set out, with their conditions, the table of numbers for varying box size, boundary and size of the coarse plaquette; that when the coarse plaquette is enlarged the action is paid along the perimeter; that the genuine two steps and the iterated two steps do not agree; and that in the network with all coarse plaquettes fixed at once (Bundle Network Theorem) the contraction becomes a matrix and exceeds $1/16$ in coarse directions.

Chapters assumed — 15-00 § 01 (steps, the same form), 15-03 § 02 (the statement of the decomposition), 15-03 § 04 (the ledger of the transverse contraction), 15-04 § 02 (one Migdal–Kadanoff step and $1/16$). We use linear algebra (orthogonal projections, Cauchy–Schwarz) and Kirchhoff's laws for electric circuits.

Said first — All numbers in this chapter are numbers for **a step that looks at the marginal distribution of one coarse plaquette (not a genuine blocking step)**; only § 07 is "a step with all coarse plaquettes fixed at once". The Unit Bundle Theorem and the Bundle Network Theorem are statements **at tree level restricted to commuting configurations (the main term as $\beta \to \infty$)**; they do not include the minimum over non-commuting configurations, the one-loop correction (the source), large fields, or the re-fixing of the gauge. The numbers for the source are one-loop values, started from the heat kernel, with a single image, at $\beta \to \infty$. This chapter does not show ultraviolet control.

The order of this chapter

1. Setting — commuting configurations, the projection P, the unit flux
2. Unit Bundle Theorem — the minimiser, $\beta'/\beta = \sum f^2$, the map of the quartic term, λ (equations (2)–(6))
3. The three-dimensional free boundary is a resistor network — $2 \times 2 \times 1$ solved by hand
4. Inequalities — monotonicity under inclusion, $1/(16d-8) \le \lambda \le 1/16$
5. The table of boxes (Table 1)
6. Enlarging the coarse plaquette — the perimeter type; the genuine two steps and the iterated two steps (Tables 2 and 3)
7. The network with all coarse plaquettes fixed — the Bundle Network Theorem, its local form and the One-Step Contraction of the ℓ^4 Cone, the tail type (Tables 4 and 5)
8. Implications for the decomposition, and questions (Table 6)
9. What cannot be said
10. The five grades table and the dependency graph (Table 7)

00

What this chapter shows

- At tree level restricted to commuting configurations, the minimal configuration with the angle of the single coarse plaquette fixed is determined by the projection P and the unit flux f alone, and β'/β, the map of the quartic term and the transverse contraction λ all follow from it (Unit Bundle Theorem, § 02).

- At $\lambda \le \max f^2$, in a box where the 4 plaquettes of the coarse plaquette carry $\frac{1}{4}$ symmetrically, $1/(16d-8) \le \lambda \le 1/16$. The Migdal–Kadanoff $1/16$ is the upper end of this (§ 04).
- The more the box is widened, made periodic or embedded in the lattice, the more $\sum f^2$ rises and λ falls. For a 2×2 coarse plaquette embedded in the four-dimensional lattice, $\sum f^2 \approx 0.73$, $\lambda \approx 0.026$ and the source is $\approx 5\times10^{-4}$ (§ 05).
- With a 4×4 coarse plaquette, $\sum f^2$ falls to 0.29–0.31, smaller than the iterated $(0.72)^2$. The action is paid along the perimeter, and the source is diluted as $\approx 8\times10^{-3}/L^4$ (§ 06).
- If all coarse plaquettes are fixed at once, the coarse action does not stay a plaquette action, the map of the quartic term becomes a matrix, and in coarse directions it exceeds $1/16$ (Bundle Network Theorem, § 07).
- With the local representative one obtains an upper bound uniform in the volume (Bundle Network Theorem (local form)), and in four dimensions, by interpolation that weights the rows by their ℓ^1 norm, the factor in the $\beta_0' = 1$ normalisation is at most 0.823 (the number is computed, not an interval estimate). The factor in the normalisation per direction is not an amount of contraction, and the normalisation is fixed to the form divided by the long-wavelength coupling. The One-Step Contraction of the ℓ^4 Cone is limited to the first step; several steps need conditions on the blocking (§ 07).

01

Setting — commuting configurations, the projection, the unit flux

Lattice. Take the cubical complex of a d-dimensional box $[0,n_1]\times\cdots\times[0,n_d]$ (**free boundary**) or a d-dimensional torus (**periodic**). Let C_k be the space of real-valued functions on the k-dimensional cells (C_1 are links, C_2 plaquettes, C_3 cubes), with inner product the sum of the products of the values on each cell, and let $d : C_1 \to C_2$ be the map that makes plaquette angles from link angles (the coboundary).

Commuting configurations. Restrict all links of $SU(2)$ to rotations about one direction (σ_3) and write them by link angles $A \in C_1$; then the plaquette angles are $\phi = dA$. The action of a single image of the heat kernel is $\beta\sum_p \phi_p^2/2$, and Wilson's action is $\beta\sum_p(\phi_p^2/2 - \phi_p^4/24 + \cdots)$ (the same conventions as 15-04 § 01).

The single coarse plaquette. Take a coarse loop W (for example the 8 links of the outer boundary of a 2×2 square) and a surface $s \in C_2$ with boundary W (the 4 plaquettes of that square, coefficients ±1, support written S). The constraint that the angle of the holonomy of W is θ is $\langle s,\phi\rangle = \theta$.

Allowed fields and the projection. Let $\mathcal{A} := \operatorname{im} d \subset C_2$ be the space of plaquette angles that come from links, P the orthogonal projection onto $\mathcal{A}$, and $u := Ps \;/=0$. Since a box with free boundary is contractible, $\mathcal{A}$ coincides with the space in which "the oriented sum over each cube is 0"; in the periodic case the condition that the flux through each coordinate 2-torus is 0 is added Proved here .

$$f := \frac{Ps}{\|Ps\|^2}, \qquad \mathcal{A}_0 := \{\psi \in \mathcal{A} : \ \langle s,\psi\rangle = 0\} \ (1)$$

and call f the **flux**. $f \in \mathcal{A}$, $\langle s,f\rangle = 1$ (the unit flux), $f \perp \mathcal{A}_0$.

The "one step" of this chapter means: given the angle θ of the single coarse plaquette, integrate out all the remaining degrees of freedom, and read the resulting distribution of θ (the marginal distribution) as the weight of a single plaquette at the next step. It is a step that looks at the marginal distribution of one coarse plaquette, not a genuine blocking step (the joint distribution of all plaquettes of the coarse lattice is treated only in part, in § 07).

02

Unit Bundle Theorem — the minimiser and three ratios

Unit Bundle Theorem (Theorem F of the paper that writes the tree-level blocking of the abelian linearised lattice gauge action as a flow network). In the setting above, under the conditions (a) restriction to commuting configurations, (b) the main term in $\beta\to\infty$ (tree level; the one-loop correction and $O(1/\beta)$ are not included), (c) free boundary or periodic, fixed, the following holds Proved here .

- (i) For each θ the minimiser of $\min\{\frac{1}{2}\|\phi\|^2 : \ \phi\in\mathcal{A},\ \langle s,\phi\rangle=\theta\}$ is exactly one, $\phi^* = \theta f$, and the minimum is $\theta^2/(2\|u\|^2)$.
- (ii) The coarse action of the action $\beta\sum\phi^2/2$ is $\beta'\theta^2/2$ with

$$\frac{\beta'}{\beta} = \sum_p f_p^2 = \frac{1}{\|Ps\|^2}. \ (2)$$

  and the lower bound $\sum f^2 \ge 1/|S|$ holds. The upper bound "$\sum f^2 \le 1$" is not unconditional (at $|S|=1$, if that plaquette is a face of a cube, $\sum f^2 > 1$).

- (iii) Let the fine action be $V(\phi)=\sum_p(\beta\phi_p^2/2-\sigma\phi_p^4)$ ($\sigma\ge 0$; for Wilson $\sigma=\beta/24$), and set the local coarse action $S'_r(\theta):=\inf\{V(\theta f+\psi):\ \psi\in\mathcal{A}_0,\ \|\psi\|\le r\}$. With $m:=\max_p|f_p|$ and $\|f^3\|:=(\sum_p f_p^6)^{1/2}$, if $\sigma(6\theta^2m^2+4|\theta|mr+r^2)\le\beta/4$ then

$$\frac{\beta\sum f^2}{2}\theta^2-\sigma\sum f^4\,\theta^4-\frac{16\sigma^2\|f^3\|^2}{\beta}\theta^6\ \le\ S'_r(\theta)\ \le\ \frac{\beta\sum f^2}{2}\theta^2-\sigma\sum f^4\,\theta^4. \tag{3}$$

  that is, the coefficient of θ^4 in the coarse action is **exactly** $\sigma'=\sigma\sum f^4$ (excluding θ^6 and higher).
- (iv) The normalised quartic deviation $\kappa:=\sigma/\beta$ maps in one step to

$$\kappa'=\kappa\cdot\lambda,\qquad \lambda:=\frac{\sum_p f_p^4}{\sum_p f_p^2} \tag{4}$$

  and

$$\frac{1}{|S|^3\sum_p f_p^2}\ \le\ \lambda\ \le\ \max_p f_p^2. \tag{5}$$

Proof. (i) If $\phi\in\mathcal{A}$, then from $P\phi=\phi$ and the self-adjointness of P, $\langle s,\phi\rangle=\langle Ps,\phi\rangle=\langle u,\phi\rangle$. By Cauchy–Schwarz $\theta^2\le\|u\|^2\|\phi\|^2$, hence $\|\phi\|^2\ge\theta^2/\|u\|^2$. $\phi=\theta u/\|u\|^2=\theta f$ lies in $\mathcal{A}$, satisfies the constraint and gives equality. The constraint set is closed under midpoints, and $\|\phi\|^2$ on it is strictly convex by the parallelogram identity, so the minimiser is unique. If $\psi\in\mathcal{A}_0$ then $\langle f,\psi\rangle=\langle u,\psi\rangle/\|u\|^2=\langle s,\psi\rangle/\|u\|^2=0$.

(ii) (2) follows from the minimum $\beta\theta^2/(2\|u\|^2)$ and $\sum f^2=\|u\|^2/\|u\|^4$. The lower bound is Cauchy–Schwarz on $\langle s,f\rangle=\sum_{p\in S}s_pf_p=1$. If $|S|=1$, then s itself is not closed (the sum over the boundary of a cube is not 0), so $u\ /=s$, $\|u\|<\|s\|=1$, $\sum f^2>1$.

(iii) Every ϕ satisfying the constraint can be written uniquely as $\theta f+\psi$ ($\psi\in\mathcal{A}_0$). The upper bound is $\psi=0$. Lower bound: from $f\perp\psi$, $\sum(\theta f+\psi)^2=\theta^2\sum f^2+\|\psi\|^2$. Expanding the quartic term, $|\sum f^3\psi|\le\|f^3\|\|\psi\|$, $\sum f^2\psi^2\le m^2\|\psi\|^2$, $|\sum f\psi^3|\le mr\|\psi\|^2$, and bounding with $\sum\psi^4\le r^2\|\psi\|^2$, we get, with $x=\|\psi\|$ and $a=4\sigma|\theta|^3\|f^3\|$, $V(\theta f+\psi)\ge$ (right-hand side) $+\frac{\beta}{4}x^2-ax\ge$ (right-hand side) $-a^2/\beta$.

(iv) The coarse action has the same form as $\beta'\theta^2/2-\sigma'\theta^4+O(\theta^6)$, so $\sigma'/\beta'=(\sigma/\beta)\sum f^4/\sum f^2$. The inequality on the right is $f_p^4\le(\max f^2)f_p^2$ added up. The one on the left follows from the power mean $(\sum_S g)^4\le|S|^3\sum_S g^4$ ($g_p=s_pf_p$, Cauchy–Schwarz twice) and $\sum_S g=1$. ∎

The meaning of (iv) of the Unit Bundle Theorem. For two weights that arrived at the same β' ($\kappa=1/24$ started from Wilson and $\kappa=0$ started from the heat kernel), the difference of the coefficients of θ^4, normalised by β', is multiplied by λ in one step. This is the main term of the transverse contraction in the ledger of 15-03 § 04. It agrees with the ratio of total-variation distances measured at finite β (0.0443, 0.0451, 0.0463, 0.0469 for $\beta_T=1,2,3,4$ at $2\times2\times1$) approaching $7/144=0.0486$ from below, but we have not shown that the ratio of total-variation distances converges to λ Reading .

03

The three-dimensional free boundary is a resistor network

In a box with free boundary the orthogonal complement of $\mathcal{A}$ consists of 2-cycles, and since the box is contractible it coincides with sums of boundaries of cubes. Hence $\|Ps\|^2=\min_c\|s-\partial c\|^2$ (c is a function on cubes; Dirichlet's principle) Proved here . In three dimensions, consider the **dual resistor network** whose nodes are the cubes, whose edges of resistance 1 are the interior faces, with the faces of S as edges to a source S and the other outer faces as edges to a sink T; then $\phi\in\mathcal{A}$ says that the oriented sum at each cube is 0 (Kirchhoff's current law), the constraint says that the amount of current from S to T is θ, $\sum f^2$ is the effective resistance between S and T, and f is the unit current (Thomson's principle) Cited (a restatement of standard facts about electric circuits).

Example: the $2\times2\times1$ slab. Each cube receives the current $\frac14$ from the bottom face and passes it to the 2 side faces on the outer rim (in parallel, resistance $\frac12$) and to the top face (resistance 1). The resistance of a column is $1+(\frac12\parallel1)=\frac43$; with four in parallel,

$$\sum f^2=\frac13,\qquad \sum f^4=4\cdot\frac{1}{4^4}+12\cdot\frac{1}{12^4}=\frac{7}{432},\qquad \lambda=\frac{7}{144} \tag{6}$$

Proved here . The flux is $\frac14$ on the bottom faces, $\frac{1}{12}$ on the top faces and the outer side faces, and 0 on the inner walls. In four dimensions the network is a potential not on cubes but on 3-cells, and the picture is harder to draw, but the projection formula (1) can be used as it is, independently of dimension.

04

Inequalities — monotonicity under inclusion, and both sides of the contraction

(a) Monotonicity under inclusion. For complexes $K \subset K'$ containing W, $\sum f^2(K) \le \sum f^2(K')$ Proved here . Proof: restricting a configuration on K' to K does not change the holonomy of W, and the action (a sum of non-negative terms) decreases, so the minimum is larger for the larger complex. ∎ In three dimensions this is the same content as Rayleigh's monotonicity (a box with free boundary is the result of shorting the surrounding cubes to the sink, and shorting lowers the effective resistance). So **a small box with free boundary makes β'/β look small**.

(b) Both sides of the contraction. When the 4 plaquettes of the coarse plaquette carry $f = \frac{1}{4}$ symmetrically (when the box is invariant under the 90° rotation about the coarse plaquette; this holds for all boxes in Table 1), $\sum f^4 \ge 4 \cdot 4^{-4} = \frac{1}{64}$. Moreover, the trial configuration "$\theta/8$ on each of the 8 links of W, 0 elsewhere" gives $\theta/4$ to each plaquette of the coarse plaquette and $\theta/8$ to each of the other $2d-3$ plaquettes containing a link of W, so from Thomson's inequality ((i) of the Unit Bundle Theorem), **in any box (even in the infinite lattice)** $\sum f^2 \le \frac{1}{4} + \frac{2d-3}{8}$. Together,

$$\frac{1}{16d-8} \;\le\; \frac{1}{64\sum f^2} \;\le\; \lambda \;\le\; \max f^2 = \frac{1}{16} \quad (7)$$

Proved here (the equality on the right, $\max f^2 = \frac{1}{16}$, holds when the largest flux is on the coarse plaquette; we have confirmed this in all boxes computed, but there is no general proof Reading). In four dimensions $1/56 \le \lambda \le 1/16$, in three $1/40 \le \lambda$. The equality $\lambda = 1/16$ holds when the flux sits only on the faces of $\frac{1}{4}$, and the 16 plaquettes of Migdal–Kadanoff (15-04 § 02) are exactly this. **For a step of the type in which the coarse plaquette is made of 4 fine plaquettes, the Migdal–Kadanoff $1/16$ is the upper end of the transverse contraction**, and the more parallel paths there are, the more thinly the flux is divided and the smaller λ becomes.

In the boxes computed, $\lambda \cdot \sum f^2 = \sum f^4$ stays within 1.04–1.22 times $1/64$, and the movement of λ is determined almost entirely by the reciprocal of the denominator $\sum f^2$ (the slope of the invariant curve) Reading (an observation).

05

The table of boxes

Table 1 lists, box by box, the three numbers of one step for a 2×2 coarse plaquette. $\sum f^2$ and λ are the main terms of the flux network (for the three-dimensional free boundary and the boxes free only on the outer rim they are rational, and the two formulations agree Proved here ; in four dimensions and the periodic case they are floating-point, and the two formulations agree to about 10^{-12} Reading), and the source is the numerical value of the coefficient of θ^4 at one loop, started from the heat kernel, with a single image, at $\beta \to \infty$ Reading . "Free only on the outer rim" means that only the outer faces on the rim of the slab are free boundary, with cubes left around the coarse plaquette; "interior" means the coarse plaquette is embedded in the interior of the box; "periodic" is a torus with only one coarse plaquette fixed and the other plaquettes free.

Table 1 The three numbers of one step for a 2×2 coarse plaquette ($SU(2)$; tree level restricted to commuting configurations = the main term in $\beta \to \infty$; the marginal distribution of one coarse plaquette. The source: started from the heat kernel, single image, $\beta \to \infty$, one loop)

System	$\sum f^2 = \beta'/\beta$	$\lambda = \sum f^4 / \sum f^2$	Source σ	Checks and notes
Two-dimensional open strip (two plaquettes combined in series)	$1/2$	$1/4$	0 Proved here	Only series (convolution), so it closes in the heat-kernel class
Migdal–Kadanoff ($d=4, b=2$)	1	$1/16$	$1/3840 \approx 2.60 \times 10^{-4}$	The map on the hierarchical lattice. Flux $\frac{1}{4}$ on 16 plaquettes (15-04)
Three-dimensional $2\times2\times1$, free boundary	$1/3$	$7/144 \approx 0.0486$	$4.11 \times 10^{-4} \approx 1/2433$	For the source, the extrapolation of finite sums of characters and the one-loop value agree to 4 digits. Heat-bath Monte Carlo on all links and finite sums of characters give $\le 1.7\sigma$ at $\beta = 4$–32
Three-dimensional $2\times2\times2$, free boundary	$15/44$	$5079/106480 \approx 0.0477$	$4.65 \times 10^{-4} \approx 1/2149$	One loop in two implementations (finite differences and the analytic Hessian). Monte Carlo gives $\le 0.96\sigma$ at $\beta = 16, 32$
Three-dimensional $2\times2\times3$, free boundary	$14/41$	≈ 0.04765	4.71×10^{-4}	One loop in one implementation
Three-dimensional $2\times2\times\infty$	$(1+\sqrt{3})/8 \approx 0.3415$	≈ 0.04764	$\approx 4.72 \times 10^{-4}$	$\sum f^2$ from the ladder formula $R = \frac{1}{2} \parallel (1+R)$ Proved here . The source is an estimate from three points
Three-dimensional, the centre of $4\times4\times2$, free only on the outer rim	$1023/2815 \approx 0.3634$	≈ 0.04466	$6.13 \times 10^{-4} \approx 1/1630$	The flux agrees across three implementations. At one loop the two implementations agree to relative 1.4×10^{-5}
Three-dimensional slab of thickness 2, widened ($6\times6\times2$, $8\times8\times2$, ...)	$0.3664 \to 0.36680$	$0.04430 \to 0.04425$	$6.41 \to 6.45 \times 10^{-4}$	At $4\times4\times2$ within 1% (flux) and 5% (source) of the limit. The limit of the source is an estimate
Three-dimensional 4^3, interior of 6^3 (coarse plaquette on a plane inside the box)	$0.4669, 0.4760$	$0.0361, 0.0355$	1.04×10^{-3}, 9.92×10^{-4}	One loop in one implementation. The source is not monotone in the box size

System	$\sum f^2 = \beta'/\beta$	$\lambda = \sum f^4 / \sum f^2$	Source σ	Checks and notes
Three-dimensional, periodic 4^3, 8^3, 16^3 (only one plaquette fixed)	$8/15$, 0.4859, 0.4814	0.0332, 0.0349, 0.0352	—	Flux in two formulations. $8/15$ for 4^3 agrees to 15 digits
Four-dimensional $2 \times 2 \times 1 \times 1$, free boundary	$3/7$	≈ 0.0393	4.68×10^{-4}	One loop in two implementations
Four-dimensional $2 \times 2 \times 2 \times 1$, free boundary	$105/241$	≈ 0.0390	4.80×10^{-4}	One loop in one implementation
Four-dimensional 2^4, free boundary	$180/407 \approx 0.4423$	≈ 0.0386	$4.86 \times 10^{-4} \approx 1/2058$	At one loop the two implementations agree to relative 4.9×10^{-5}
Four-dimensional, the centre of $4 \times 4 \times 2 \times 2$, free only on the outer rim	≈ 0.4867	≈ 0.0350	$5.46 \times 10^{-4} \approx 1/1833$	One loop in one implementation
Four-dimensional, interior of 4^4 (open box)	0.7180	0.0266	$4.96 \times 10^{-4} \approx 1/2017$	One loop in two implementations, sparse and dense, agreeing in log det to 1.5×10^{-11}. In the interior of 12^4, $\sum f^2 = 0.7301$, $\lambda = 0.02645$ (flux only)
Four-dimensional, periodic 4^4, 8^4, 12^4 (only one plaquette fixed)	$16/21$, 0.7315, 0.7305	0.0264, 0.02645, 0.02645	—	Flux in two formulations. $16/21$ agrees to 13 digits
Villain for $U(1)$ (any box)	Same $\sum f^2$	—	0 Proved here	Closes at $\beta' = \beta \sum f^2$, and the deviation is only $e^{-c\beta}$ (Poisson summation)

What to read from Table 1 Reading .

- **Direction.** The more lattice is added around the coarse plaquette (free boundary → free only on the outer rim → interior → periodic), the more $\sum f^2$ rises (as in § 04 (a)) and the more λ falls. In four dimensions $\sum f^2$ goes from 0.44 (2^4) to 0.73 (the limit embedded in the lattice), and λ from 0.039 to 0.026. The free boundary created "a sink for the flux" and made β' look small and λ look large.
- **The flow of the parameter.** The main term of β' is $\beta \sum f^2$, and the shift at one loop is a $O(1)$ independent of β (started from the heat kernel): $\beta/3 - 0.146$ for $2 \times 2 \times 1$, $180\beta/407 - 0.258$ for 2^4, $1023\beta/2815 - 0.220$ at the centre of $4 \times 4 \times 2$ (all $+O(1/\beta)$). It is the same type as the Migdal–Kadanoff $-\frac{1}{4}$.
- **The source.** It is positive and converges to a value independent of β: 4.7–5.5×10^{-4} in four-dimensional boxes (1.8–2.1 times Migdal–Kadanoff), and 4.1–6.5×10^{-4} in three-dimensional slabs. For the Villain action of $U(1)$ it is 0: this reads as saying that the source needs both "parallel", where the weights of two discs are multiplied on a closed surface, and the Haar weight $\theta/\sin\theta$ of a non-commutative group. An open two-dimensional region is series only, so it has no source.

06

Enlarging the coarse plaquette — the perimeter type, and the genuine two steps

We widen the coarse plaquette to $L \times L$ (the outer boundary W has $4L$ links). $L = 2$ is one step, and $L = 4$ is the "genuine two steps" that take the factor 4 in the plane directions at once. The "iterated two steps" compared with it are the approximation of applying the one-step map once more to the coarse lattice (feeding the marginal distribution of one coarse plaquette back in as the action of the next step); we count $\beta''/\beta = (\sum f_1^2)^2$, λ as λ_1^2 and the source as $\sigma_1(1 + \sum f_1^4)$.

(a) Upper bound. The trial configuration "$\theta/(4L)$ on each of the $4L$ links of W, 0 elsewhere" gives $2\theta/(4L)$ to the 4 corner plaquettes of the coarse plaquette, $\theta/(4L)$ to the $4(L-2)$ edge plaquettes and 0 to the inner plaquettes (the sum over the coarse plaquette is 1, matching the holonomy of W), and the outer plaquettes containing each link of W number at most $2d - 3$. Hence $L \geq 2$, and **in any box (even in the infinite lattice)**

$$\sum f^2(L) \;\leq\; \frac{d-1}{2L} + \frac{1}{2L^2}, \qquad \lambda(L) \;\geq\; \frac{L^{-6}}{\sum f^2(L)} \;\geq\; \frac{2}{L^4\big((d-1)L+1\big)} \quad (8)$$

Proved here (the right side from Stokes, i.e. the sum of f over the L^2 plaquettes of the coarse plaquette is 1, and the power mean). In four dimensions $L = 2$ is $7/8$ and $1/56$ (the same as § 04), and $L = 4$ is $13/32$ and $1/1664$.

(b) The genuine two steps lower β much more than the iteration does. By monotonicity under inclusion, one step embedded in the four-dimensional lattice has $\sum f_1^2 \geq 0.717961$ (the value in the interior of 4^4; the two formulations agree Reading), hence $(\sum f_1^2)^2 \geq 0.5155 > 13/32 \geq \sum f_2^2$ Proved here (the input 0.717961 is a computed number). The iteration falls only by about $L^{-0.48}$ in $L = 2^k$, while the genuine $L \times L$ is $O(1/L)$ by (8), so the ratio tends to 0 with the number of steps. **The β'/β of the marginal distribution of one coarse plaquette falls in four dimensions with the perimeter type ($O(1/L)$), and cannot be expressed by repeating the square of one step.** In the two-dimensional plane alone the two agree (for a 4×4 coarse plaquette, $\sum f^2 = 1/16 = (1/4)^2$, $\lambda = 1/256 = (1/16)^2$).

(c) The perimeter type. In the four-dimensional boxes computed, 44–51% of $\sum f^2$ sits on "the plaquettes in transverse directions touching W" ($2(d-2)=4$ per link of W), and this proportion hardly changes when the coarse plaquette goes from 2×2 to 4×4 Reading . This reads as the action being paid not over the area of the coarse plaquette but along the perimeter of W.

Table 2 One step, the iterated two steps, the genuine two steps (four dimensions, open box with the coarse plaquette in the interior, tree level restricted to commuting configurations. The source: started from the heat kernel, single image, $\beta\to\infty$, one loop; for the 4×4 coarse plaquette, one implementation with the sparse analytic Hessian, whose path agrees with finite differences to relative 9×10^{-4} in the small system $4\times4\times1\times1$) Reading

Box	Quantity	One step (coarse plaquette 2×2)	Iterated two steps	Genuine two steps (coarse plaquette 4×4)	Genuine / iterated
4^4 (the 4×4 coarse plaquette reaches the edge of the box in the plane directions)	$\sum f^2$	0.7180	0.5155	0.2564	0.50
	λ	2.66×10^{-2}	7.06×10^{-4}	2.62×10^{-3}	3.7
	Source	4.96×10^{-4}	5.05×10^{-4}	3.70×10^{-5}	0.073
$6\times6\times4\times4$ (margin 1 in the plane directions)	$\sum f^2$	0.7218	0.5210	0.2895	0.56
	λ	2.65×10^{-2}	7.02×10^{-4}	2.37×10^{-3}	3.4
	Source	4.89×10^{-4}	4.98×10^{-4}	3.30×10^{-5}	0.066
12^4 (flux only)	$\sum f^2$	0.7301	0.5330	0.3060	0.57
	λ	2.65×10^{-2}	7.00×10^{-4}	2.57×10^{-3}	3.7

Table 3 The sequence in the side L of the coarse plaquette (four dimensions, open box of $(L+2)\times(L+2)\times4\times4$ with the coarse plaquette in the interior, transverse thickness 4. Flux in floating point, source at one loop in one implementation) Reading

L	$\sum f^2$	Upper bound (8)	λ	Coefficient c_2 of θ^2 at one loop	Source	Source $\cdot L^4$
1 (a single plaquette)	1.989	2 (the trial of $L=1$)	0.521	0.499	7.66×10^{-3}	7.66×10^{-3}
2	0.7180	7/8	0.0266	0.238	4.96×10^{-4}	7.93×10^{-3}
3	0.4190	5/9	0.00598	0.156	9.85×10^{-5}	7.98×10^{-3}
4	0.2895	13/32	0.00237	0.116	3.30×10^{-5}	8.46×10^{-3}

What to read from Tables 2 and 3 Reading (an observation of four points, transverse thickness 4, one implementation). The source is $\approx 8\times10^{-3}/L^4$, and c_2 is ≈ 0.46–$0.50/L$. If read as $\theta\sim L^2F$, θ^4 is an operator of dimension 8, and the coefficient falling as L^4 fits the same counting as the Migdal–Kadanoff $b^{d-8}=1/16$. So the source of a genuine blocking looks not like "a number reborn in a fixed band at each step" but like the type in which "the shift of θ^4 born at the finest scale is diluted by about $1/16$ at each step", and the fixed point $\sigma/(1-\sum f^4)\approx5\times10^{-4}$ of the iterated ledger lies, in this way of measuring, on the side of an upper bound. For λ it is the other way round: the genuine two steps are 3.4–3.7 times the iterated λ_1^2, and the iteration makes the contraction look fast. But at 2.4–$2.6\times10^{-3}\ll1$ the denominator $1-\lambda$ of the ledger stays almost 1.

07

The network with all coarse plaquettes fixed at once — Bundle Network Theorem

The periodic rows of Table 1 were networks in which "only one coarse plaquette is fixed and the surrounding coarse plaquettes are free". The main term of a genuine blocking step is the network in which the lattice is divided into 2^d blocks and **the angles of all plaquettes of the coarse lattice are fixed at once**, and this is a different network. Conditions of this section: restriction to commuting configurations, tree level (the main term in $\beta\to\infty$), a periodic lattice $(\mathbb{Z}/L)^d$ (L even), 2^d blocks, and the coarse angle $\theta_P:=\langle s_P,\phi\rangle$ is the linearisation of the angle around the 2×2 outer boundary spanning the coarse plaquette P.

Notation. Let S be the matrix with the s_P in rows, D_c the coboundary of the coarse lattice, and C the map to coarse links ($(CA)_\ell$ is the sum of the angles of the 2 fine links that make up the coarse link ℓ); then $SD=D_cC$. Let $u_P:=Ps_P$, $G:=SPS^{\mathsf T}$ ($G_{PQ}=\langle u_P,u_Q\rangle$), $B:=G^{+}$ (the Moore–Penrose pseudoinverse), and U the matrix with the u_P as columns.

Bundle Network Theorem (Theorem G of the same paper). Under the conditions above Proved here :

- (i) The constraint $S\phi=\theta$ ($\phi\in\mathcal{A}$) is non-empty if and only if $\theta\in\operatorname{im}D_c$ (Bianchi sum 0 for each coarse cube, and coarse flux 0 through each coordinate 2-torus). In particular, θ **that twists only one coarse plaquette and keeps the others at 0 is not allowed**.
- (ii) For $\theta\in\operatorname{im}D_c$ the minimiser is unique, $\phi^*=F\theta$, $F:=UB$, and the minimum is $\frac12\theta^{\mathsf T}B\theta$. The coarse action is

$$S'(\theta)=\frac{\beta}{2}\,\theta^{\mathsf T}B\,\theta,\qquad \beta'_{PQ}=\beta B_{PQ}\quad(\operatorname{im}D_c:\text{as a quadratic form on it}).\ (9)$$

- (iii) Written in the coarse link variables a, $\min\{\frac12\|dA\|^2:CA=a\}=\frac12a^{\mathsf T}Qa$, $Q=D_c^{\mathsf T}BD_c$. The Lagrange multipliers are exactly the gradient of the coarse action (the divergence of current at each coarse link).
- (iv) Since θ are not independent coordinates, the off-diagonal entries of B and the "diagonal / cross" components of the quartic form depend on the representation. Quantities that do not depend on the representation are: the spectrum of B on $\operatorname{im}D_c$, the

ratio by direction $\beta'(\theta) := \theta^{\mathsf{T}} B\theta / \sum_P \theta_P^2$, $\mu(\theta) := \sum_q (F\theta)_q^4 / \sum_P \theta_P^4$, $\lambda(\theta) := \mu(\theta)/\beta'(\theta)$ (an oriented version of the λ of the Unit Bundle Theorem).

- (v) For all $\theta \in \operatorname{im} D_c$, $\beta'(\theta) \geq \frac{1}{4}$.

Proof. (i) $\theta = SDA = D_c(CA) \in \operatorname{im} D_c$. Conversely, putting $a\ell$ on the first fine link of each coarse link and 0 elsewhere gives $CA = a$. (ii) The minimum-norm solution lies in $\operatorname{im} U$, and with $\phi = Uc$, $Gc = \theta$. Since $\operatorname{im} G = S(\operatorname{im} d) = \operatorname{im}(D_c C) = \operatorname{im} D_c$, a solution exists, and $c = B\theta + \ker G$, $U \ker G = 0$, $\|\phi^*\|^2 = \theta^{\mathsf{T}} BGB\theta = \theta^{\mathsf{T}} B\theta$. Uniqueness is the same strict convexity as in the Unit Bundle Theorem. (iii) The inner minimum is invariant under coarse gauge directions, so substitute $\theta = D_c a$ and read off the KKT conditions. (v) The sets of 4 plaquettes of the coarse plaquettes do not overlap one another, so $\sum_q \phi_q^2 \geq \sum_P \sum_{q \in sP} \phi_q^2 \geq \sum_P \theta_P^2/4$ (Cauchy–Schwarz on each coarse plaquette). ∎

A smooth trial. For a configuration in which the fine field has the same value $\theta/4$ on the 2^d plaquettes of each coarse plaquette, the energy ratios are 2^{d-4}, $\lambda = 1/16$ (independent of d) Proved here (arithmetic). This configuration cannot be uniform on the torus, and has meaning only as a trial in the limit of long wavelength. In Migdal–Kadanoff (all coarse plaquettes independent, with P taken as the identity) $\beta'/\beta = 2^{d-4}$, $\lambda = 1/16$.

Table 4 Numbers for the network with all coarse plaquettes fixed ($SU(2)$, tree level restricted to commuting configurations, periodic, 2^d blocks. Two implementations (proj and pseudoinverse / KKT on coarse links) agree to about 10^{-12}. The sup is the value found by multi-start local optimisation = a lower bound on the true sup) Reading

Box	Eigenvalues of B (on $\operatorname{im} D_c$)	β' at the longest wavelength	Deviation from the plaquette action	λ of one coarse link	λ in the smooth direction	Sup four
Three-dimensional 4^3–16^3	$[0.25,\ 0.375\text{–}0.474]$	0.4744 at $k = \pi/4$	15–17%	0.066–0.068	0.055 at $k = \pi/4$	0.11–0.1
Three-dimensional 48^3	$[0.25,\ 0.497]$	0.4968 at $k = \pi/12$	16%	0.068	0.0615 at $k = \pi/12$	(no optimis
Four-dimensional 4^4–12^4	$[0.25,\ 0.536\text{–}0.811]$	0.8108 at $k = \pi/3$	24–27%	0.085–0.091	0.037–0.046	0.22–0.2
Four-dimensional 16^4	$[0.25,\ 0.879]$	0.8786 at $k = \pi/4$	24%	0.091	0.047 at $k = \pi/4$	(no optimis

Table 5 Verdict: the "true numbers" of one four-dimensional step (conditions as in Table 4; the rows with one plaquette fixed are from Table 1)

Network	β'/β	λ (factor of θ^4)	Label
One plaquette in 2^4 with free boundary	0.442	0.0386	Proved here (rational)
One plaquette in periodic 8^4 (others free)	0.731	0.0264	Reading
All coarse plaquettes fixed, long wavelength	$\to 1 = 2^{d-4}$ (0.879 at $k = \pi/4$)	$1/16$ in the smooth trial; 0.037–0.047 at the measured $k \geq \pi/4$	Reading (the trial value is Proved here)
All coarse plaquettes fixed, one coarse link	0.354	0.091	Reading
All coarse plaquettes fixed, the coarsest direction	0.25 (smallest eigenvalue)	Sup found ≥ 0.27	Proved here ($\geq \frac{1}{4}$), Reading (sup)
Migdal–Kadanoff	1	$1/16$	Proved here

What to read from Tables 4 and 5 Reading . (1) The coarse action does not stay a plaquette action. In four dimensions the eigenvalues spread from $\frac{1}{4}$ up to nearly 1 at long wavelength, and already at quadratic order terms of loops of length 2 or more are born (this seems to correspond to the quadratic part of the classical fixed-point action, but we have not checked against the original sources). (2) The long-wavelength β'/β is larger than the 0.731 for one plaquette fixed, and moves towards the Migdal–Kadanoff side of 1. (3) The map of the quartic term is a matrix that mixes with neighbours, and **in coarse directions it exceeds** $1/16$ (0.091 for one coarse link, 0.27 for the sup found). In the smooth direction, in three dimensions it heads towards $1/16$ (0.0615 at $k = \pi/12$), while in four dimensions it is 0.047 up to $k = \pi/4$ and convergence is not yet visible. The Migdal–Kadanoff $\lambda = 1/16$ survives in a genuine step only as the value in the long-wavelength direction. (4) The values found are all below 0.3, but the pseudoinverse representative F is non-local, and the Riesz–Thorin upper bound grows with the box (4.2 in four-dimensional 16^4); **we have not obtained an upper bound uniform in the volume** (with the pseudoinverse representative; taking the local representative gives an upper bound uniform in the volume: the Bundle Network Theorem (local form) below). So we do not say "the transverse contraction is smaller than 1".

Candidate coordinates for writing it as a matrix. If the class of quartic deviations is taken to be "$|Q(\theta)| \leq c \sum_P \theta_P^4$" (quartic forms bounded by ℓ^4), the tree-level step preserves this class with $c \mapsto c \cdot \sup \mu$ Proved here ($|Q(F\theta)| \leq c \sum (F\theta)^4 \leq c \sup \mu \sum \theta^4$). The diagonal class of plaquettes (the Migdal–Kadanoff class) is not preserved, but the cone of ℓ^4 is.

The local representative and an upper bound uniform in the volume — Bundle Network Theorem (local form; Theorem G′ of the same paper). The conditions are the same as for the Bundle Network Theorem (restricted to commuting configurations, tree level, periodic $(\mathbb{Z}/L)^d$, 2^d blocks, linearisation of the coarse angle). F is determined only on $\operatorname{im} D_c$, so how the action is chosen

outside it is the "representative". The pseudoinverse representative is the Hodge decomposition itself and is not local, and the axial gauge with its base point fixed at one point also grows with the volume. So we turn to the kernel $R := FD_c$ written in the coarse link variables.

- (a) R is local (exponentially decaying, with a measured rate of about 2.2–2.5 per lattice step), and a representative that agrees with F on $\operatorname{im} D_c$ is the same thing as a "filling" $\partial k = r$ of each row (cycle) of R. Assigning to each row **an axial cone (tree) whose base point moves with the row** gives a translation-covariant, exponentially decaying representative K_∞; its periodisation K_L agrees with F_L on $\operatorname{im} D_c$ for every L, and $\|K_L\|_{p\to p} \le \|K_\infty\|_{p\to p}$ $(p = 1, \infty)$ Proved here (the exponential decay of R is a sketch from the constancy of the Bloch rank, plus numerics).
- (b) Hence for all L and $\theta \in \operatorname{im} D_c$

$$\sum_q (F_L\theta)_q^4 \;\le\; \Lambda_d \sum_P \theta_P^4, \qquad \Lambda_d := \|K_\infty\|_{2\to 2}^2 \, \|K_\infty\|_{\infty\to\infty}^2 \quad (10)$$

that is, $\sup\mu$ is bounded uniformly in the volume Proved here (Riesz–Thorin is known; the Schur form that passes from the row and column ℓ^1 sums to the quartic sum is in Lean: `schur_quartic`, `representative_bound`; assumptions: upper bounds on the row and column ℓ^1 sums, and agreement of the representative on $\operatorname{im} D_c$). (c) The ℓ^4 cone of quartic deviations is mapped in one step with its constant multiplied by at most Λ_d Proved here (Lean: `cone_step`; assumptions $c \ge 0$, $|Q(\phi)| \le c\sum\phi^4$, and (10) on $\operatorname{im} D_c$).

(d) Numbers Reading (two implementations agree row by row to within 3×10^{-15}; three dimensions 4^3–48^3, four dimensions 4^4–20^4): $\|K\|_{\infty\to\infty}$ of the local representative is uniform in the volume (converging to 0.685 symmetrised and 0.748 tree in three dimensions, and 1.341 symmetrised and 1.541 tree in four dimensions), while for the pseudoinverse representative it grows, 0.62→1.94 (three dimensions) and 0.95→2.39 (four dimensions), and for the axial gauge with a fixed base point it grows by 1.12 (three dimensions) and 3.08 (four dimensions) per lattice step. **In three dimensions** $\Lambda_3 = 0.245$ (symmetrised); with the long-wavelength normalisation $\beta_0' = \frac{1}{2}$ this gives $\sup\mu/\beta_0' \le 0.49$, and using $\beta' \ge \frac{1}{4}$ gives $\sup\lambda \le 0.98$, smaller than 1. **In four dimensions** $\Lambda_4 \approx 1.83$ (an estimate from the filling on 24^4 and a line scan), which exceeds 1, and the lower bound on $\sup\mu$ is 0.148. Moreover, from the $\infty \to \infty$ norm $\nu_4 \ge 1.18$ restricted to $\operatorname{im} D_c$ (a linear program on 16^4; a quantity that does not depend on the representation), **the Riesz–Thorin bound of no representative can go below 1.39**. Whether $\sup\mu < 1$ in four dimensions is undecided, and a different estimate that bounds ℓ^4 directly is needed (the question "direct upper bound for the four-dimensional ℓ^4"). The gap of about a factor of 12 between the true value (≥ 0.148) and the upper bound (1.83) is the loss in the interpolation between ℓ^2 and ℓ^∞.

One-Step Contraction of the ℓ^4 Cone — the four-dimensional ℓ^4 cone contracts by a factor of at most 0.823 in one step (in the $\beta_0' = 1$ normalisation). The loss in Riesz–Thorin comes from the fact that the rows that determine $\infty \to \infty$ (the fine plaquettes lying on the coarse plaquette, 24 of the 96 types) and the directions that determine ℓ^2 (close to the long wavelength) are different. So if we interpolate with the rows weighted by their ℓ^1 norm A_q (the weighted Riesz–Thorin via Stein's analytic family, used on a finite-dimensional torus), then for all L

$$\sup\mu \;\le\; \|K\|_{4\to 4}^4 \;\le\; \Lambda_w := \sup_P \lambda_{\max}\big(K(P)^{\mathrm{H}}\,\mathrm{diag}(A_q^2)\,K(P)\big) \quad (11)$$

holds Proved here (the interpolation theorem is a known tool Cited; the weight A_q^2 is the best within this family). Bounding Λ_w from above with the kernel of the local representative of 20^4, on the momentum grid $N = 64$ with a one-sided quadratic remainder, gives $\Lambda_w \le 0.823$ Reading (computed, not an interval estimate; the grid value at 20^4 is 0.763, the real-space power method gives 0.758 from below, and a test on 60 small matrices does not break the upper bound). Combined with `cone_step`, under the conditions (restricted to commuting configurations, tree level, periodic, 2^4 blocks, linearised coarse angle, K_∞ replaced by the kernel of 20^4), **the ℓ^4 cone of the quartic deviation contracts by a factor of at most 0.823 in one step** (in the $\beta_0' = 1$ normalisation as it stands). The gap to the lower bound 0.148 on the true value has shrunk from about 12 times to 5.6 times. The estimate that splits by distance into a near part and a tail (the Minkowski form is in Lean: `quartic_triangle`, assumption $a, b > 0$, upper bounds on the sum of fourth powers of each part) and the estimates that ignore signs (the Schur and Hölder families; the fourth power of the $|K|$ norm is 10.76) go well above 1; what worked was the weight by row type, not by distance Reading. **The factor measured in the normalisation per direction is not an amount of contraction** (next paragraph).

Note on normalisation — fix it to the form divided by the long-wavelength coupling. In the coordinates (a), where the constant of the cone is divided by the scalar coupling of each step (the long-wavelength β_0'), the one-step factor is $\sup\mu/\beta_0' = \sup\mu$; in the cone (b), re-measured by the quadratic form of its own direction $\beta'(\theta)$, the factor is $\sup\lambda$, $\lambda := \mu/\beta'$. From $\frac{1}{4} \le \beta' \le \beta'_{\max} \le 1$ we get $\frac{1}{4}\lambda \le \mu \le \lambda$, and the two cones are equivalent within a constant 4 Proved here (a finite ledger; Lean: `ledger` (assumptions $b_{\min}\sum\theta^2 \le \sum\phi^2 \le b_{\max}\sum\theta^2$, $0 < b_{\min}$), `transfer`, `cone_b_to_a`; the $\beta'_{\max} \le 1$ above is a measured value). **If the normalisation is fixed to one for all steps, the constant 4 appears only once**, and the conclusion does not depend on the normalisation. If one switches between (a) and (b) at each step, a factor 4 is incurred at each step. In fact, in the volume limit $\sup\lambda = 4\sup\mu_{\mathrm{loc}}$ Proved here (Mixing-Limit Lemma; the coarse L is even, $L \to \infty$. Mixing a direction θ_{loc} of finite support with the coarsest direction θ_π, which attains $\beta' = \frac{1}{4}$, by $\varepsilon = V^{-1/3}$, ℓ^4 is carried by the localised direction and ℓ^2 by the spread-out direction, so $\lambda \to 4\mu(\theta_{\mathrm{loc}})$; the estimate of the cross terms is a sketch using the exponential decay of the rows of the representative). Numerically too, the optimisation of λ gives 0.269,

0.308, 0.338 at 12^4, 16^4, 20^4, increasing with the volume; the extrapolation of the mixture is 0.395, and $4\mu_{\mathrm{loc}}$ is 0.3955 Reading . Hence λ **is not an amount of contraction of directions**, and $\lambda < 1$ is equivalent to $\sup \mu < \frac{1}{4}$. Furthermore, the row-weighted Stein route cannot go below $\sum_s \nu_s^2/16 \geq 0.452 > \frac{1}{4}$ for any local representative or any weight Proved here (Floor Lemma for the Stein Route; the type in which 16 plaquettes per coarse plaquette receive $\theta/4$ at the long-wavelength minimiser differs between the paper sketch and 20^4 by 3.5×10^{-3}; the finite part is in Lean: `stein_floor`, assumptions $A_i^2 \leq F v_i$, $0 \leq \nu_i \leq A_i$; ν is a linear-programming number). The quartic coefficient of the coarsest direction itself is multiplied by exactly $1/64$ in one step ($1/16$ in its own direction) Proved here (equality in Cauchy–Schwarz). The range of λ is $[0.59, 3.29]$ Reading (the lower end goes through the sketch "$\sup \mu_{\mathrm{loc}}$ is at least $\sup \mu_L$ of every L"), and the normalisation is fixed to (a).

The One-Step Contraction of the ℓ^4 Cone is limited to the first step — several steps need conditions on the blocking. A tree-level step is a Gaussian conditional mean, and the inverse $G = B^+$ of the coarse action (the covariance of the blocked flux) iterates linearly: $G_{k+1}(q) = \frac{1}{16} \sum_\eta T\, G_k\, T^{\mathrm{H}}$ (T is the symbol of the coarse angle) Proved here (Conditional-Mean Covariance Lemma; it appears to be the standard form of blocking for Gaussian fields, but the sources have not been checked). In **blocking by edge sums**, where a coarse edge is the sum of two fine ones, the eigenvalues of B_k escape to 0 at a rate 2^{-k} for fixed $q \neq 0$ (0.25, 0.090, 0.038, 0.017, 0.0083 in the direction π^4), only the long wavelength stays at 1, and there is no scale-invariant fixed point Reading (exact in symbols, $k \leq 5$; the reading is that the cause is the divergence of the perimeter length in the variance of the flux of the $2^k \times 2^k$ square). The minimiser escapes to those soft directions, and the lower bound on the factor of the cone is 0.699 at the second step (12^4) and goes above 1 **at the third step with 1.262** (8^4; the symbolic route and a separate nested implementation agree), 2.200 (12^4) and 2.746 (16^4) Reading (local optimisation over periodic configurations on the torus; no interval estimate). So **the One-Step Contraction of the ℓ^4 Cone above (0.823) is limited to the first step**, and cannot be used for a multi-step geometric series that repeats the same blocking. If the coarse edge is defined by the gauge-covariant average of two parallel lines among 2^{d-1} transverse ones (the average of staples with transverse legs attached), the flow of the covariance goes to a fixed point (the eigenvalues of B_k increase for all q and the differences shrink with a ratio of 0.27–0.52, width $[1, \approx 4.1]$; in axial momentum the closed form $1/G_k = 3/(1 + 2 \cdot 4^{-k})$ goes to 3), and the lower bound on the factor of the cone does not grow with the step: **0.64, 0.57, 0.64, 0.62** ($j = 1$–4, 8^4, 12^4), and for a one-directional field it is exactly $\frac{1}{2}$ Reading (symbolic flow $k \leq 5$; the second step agrees with a separate nested real-space implementation from the fine step). **There is no upper bound** (the weighted Stein gives 4.5–8.1). The naive average without legs is not gauge-covariant: the minimum with links fixed and the minimum with flux fixed differ by 37–73%, and the lower bound at the second step also approaches 1 at 0.887. Hence the multi-step form is "for a blocking whose covariance flow has a scale-invariant fixed point, and with $\sup \mu_\infty < 1$ at the fixed point, the cone is a geometric series of $x_n \leq x_0 c^n$" Proved here (the ledger is in Lean: `cone_chain_uniform` (assumption $M_k \leq m$), `cone_series` (assumptions $0 \leq c < 1$, $x_{k+1} \leq c x_k$), `axis_flow_closed`), and the antecedent $\sup \mu_\infty < 1$ is supported only at the level of lower bounds (the question "upper bound on the fixed point of the lateral average"). Conditions: restricted to commuting configurations, tree level, periodic, 2^4 blocks, linearised.

Why the pseudoinverse representative grows with the volume — the tail type. That the ℓ^1 row sums of the pseudoinverse representative (the representation that uses the holonomies of the coarse plaquettes directly as coordinates) grow with the volume is consistent with a power-law kernel caused by the propagation of the Bianchi constraint being massless on the periodic lattice. Measuring the same step on a three-dimensional column $2 \times 2 \times L$ with the Bianchi constraint (which has open faces), the tail is exponential (rate $\ln(3 + 2\sqrt{2})$) and the row sums are uniform in the volume (saturating at 0.6035); on a slab without boundary it is r^{-2}, and with three-dimensional periodicity it is a power law (effective exponent around 3.3) Reading (commuting background, tree level, quadratic coefficients). The side of the local representative (the representation written in the variables of coarse links) was exponential in all three geometries. With four-dimensional periodicity the kernel of the pseudoinverse representative reads as r^{-4}, and the ℓ^1 sums grow logarithmically (consistent with the numbers of Table 5; four dimensions has not been measured). The counting rules are in 15-08 § 06. The open four-dimensional point of the Bundle Network Theorem (local form) (the factor in the normalisation per direction) is a question separate from this power law, and this observation does not move it.

The coordinates of coarse links plus global tags are exponential even for the four-dimensional tree. If the same step is written by splitting the coarse variables into "coarse links (coordinates in which the Bianchi constraint is promoted to variables) plus global tags (the total flux through each plane; the harmonic part of the coarse plaquettes)", the tags have no cross terms with the coarse-link part Proved here (Lean: `norm_sq_add_harmonic`, assumption $D^\dagger b = 0$; uniqueness of the decomposition is `decomp_unique` and `range_sup_tags` (assumption $d_2 \circ D_C = 0$)), and the tree kernel with four-dimensional periodicity n^4 ($n \leq 16$) is exponential (prefactor around r^{-1}, rate $\kappa \approx 2.08$) and independent of the volume, with the distance-weighted row sums 15.62, 17.43, 17.50 at $n = 4, 8, 12$, tending to saturation Reading (commuting background, quadratic coefficients, one step, tree level). In the pseudoinverse representative with the tags removed (the coordinates of the coarse plaquettes), the projection of the tags produces a flat $-1/(2N)$, and together with the power law of the Bianchi propagation ($r^{-4.4}$ by fitting), the distance-weighted sums grow in proportion to n Reading . In a three-dimensional periodic box without boundary (coarse $2 \times 2 \times L$) the one-loop kernel is also exponential ($\kappa = 3.094$, agreeing with the tree $\ln(11 + 2\sqrt{30}) = 3.089$), and the sums were uniform in the volume (15-08 § 06). The open point of the Bundle Network Theorem (local form) is separate from this question of coordinates, and this observation does not move it. The four-dimensional one loop has not been measured.

The long-wavelength limit of the network and one loop. If the network with all coarse plaquettes fixed is taken at once with block L, the long-wavelength eigenvalue of the coarse action stays at 1 for all L (classically marginal), while at momenta on the scale of the

coarse lattice it becomes smaller as $1/L$ ($L \cdot B = 0.500$, 0.359, 0.305 in the direction π^4, $L = 2, 4, 8$; the same quantity as the iteration of block 2) Reading . The coefficient projected onto the cell type is $bL \to 0.20$, and the transition between the two scales, long and short wavelength, appears as the part that does not fit in the cell (20% at $L = 16$). **At tree level there is no running of $\ln L$.** At one loop this long-wavelength eigenvalue changes to $\beta'/\beta = 1 - 0.745/\beta$ (Manton, block 2), and most of the change comes from the fluctuations inside the block Reading ($SU(2)$, one loop, transverse coarse torus 6^3, wave number 0 by extrapolation, 15-03 § 02). On a small coarse torus (2^4), the uniform direction of the coarse torus (the Polyakov direction) acquires a one-loop "mass" that contaminates the long-wavelength value (δ moves by 0.19), so the one loop of the network must be measured with the transverse coarse torus at 4 or more, or with a transverse Bloch sum Reading .

The axial cone, the Koszul contraction, linear-programming duality and Riesz–Thorin are all known tools, and the way of combining them may also be known (a view close to the construction of local minimisers; we have not checked against the original sources).

08

Implications for the decomposition, and questions

The ledger of 15-03 § 04 said that if the one-step map contracts transversely around the class ($\|\rho_{k+1}\| \le \lambda\|\rho_k\| + \varepsilon(\beta_k)$, $\lambda < 1$), the departure is bounded not by a sum but by the supremum $\sup_j \varepsilon(\beta_j)/(1-\lambda)$ Proved here (under the linear inequality with quadratic terms dropped). This chapter showed that this λ and the main term β'/β of the flow of the parameters **can be counted beforehand on paper from the same projection** P for the marginal distribution of one coarse plaquette (Unit Bundle Theorem). The numbers to ask about for one four-dimensional step are two, "the size of the source" and "λ"; the former is a one-loop quantity and the latter a tree-level one. However, when all coarse plaquettes are fixed, λ becomes a matrix and exceeds $1/16$ in coarse directions (§ 07), so to show the $\lambda < 1$ of the ledger for a genuine step, an upper bound, uniform in the volume, on an operator norm that does not depend on the representation is needed Reading . The uniform upper bound itself is obtained with the local representative (Bundle Network Theorem (local form)), and in four dimensions row-weighted interpolation brought the factor in the $\beta_0' = 1$ normalisation to at most 0.823 (the number is computed). Since the factor in the normalisation per direction is not an amount of contraction, the normalisation is fixed to the form divided by the long-wavelength coupling (the note on normalisation in § 07), and this one-step number is limited to the first step. Several steps need a blocking whose covariance flow has a fixed point, and $\sup \mu_\infty < 1$ at that fixed point (§ 07).

Table 6 Questions arising from this chapter (status: new; answers have been added to questions 7, 20 and 25; questions 25, 28 and 30 were added in revisions)

Q.	Question	Size	What moves if settled	What does not move even if settled
6	In the sequence in the side L of the coarse plaquette, does the source $\cdot L^4$ (now 7.7–8.5×10^{-3} at $L = 1$–4 and transverse thickness 4) approach a constant when the transverse extent is widened? If so, the sum of the sources over steps is geometric and finite, and the ledger's assumption that "the source is in a fixed band at each step" can be strengthened	Small computation (the coefficient of θ^4 at one loop by a perturbative formula)	The handling of the source in the shape remainder condition	Non-commuting configurations, large fields
7	In the tree-level network with all coarse plaquettes fixed at once, construct a **local representative** that agrees with F on $\operatorname{im} D_c$, and see whether the operator norm of ℓ^4 has an upper bound uniform in the volume. Also, whether the operator norm of λ in coarse directions (the sup found is 0.27 in four dimensions) goes below 1. **Answer (partial):** the local representative and an upper bound uniform in the volume have been obtained (Bundle Network Theorem (local form), three dimensions $\Lambda_3 = 0.245$). In four dimensions it is $\Lambda_4 \approx 1.83$, and by this route it cannot go below 1.39. Row-weighted interpolation gives $\sup \mu \le 0.823$ (the $\beta_0' = 1$ normalisation, computed)	Paper and small computations	The main term of the transverse contraction of a genuine step	One loop, large fields, re-fixing of the gauge
20	Bound the four-dimensional ℓ^4 directly: split the local representative as $K = K_0 + K_{\text{tail}}$ (a finite part near the block, and a tail), bound from above the $\ell^4 \to \ell^4$ norm of the small matrix of the near part on a finite network, apply Riesz–Thorin only to the tail, and see whether the four-dimensional $\sup \mu$ goes below 1. **Answer:** the split into a near part and a tail does not work; interpolation that weights the rows by their ℓ^1 norm gives $\sup \mu \le 0.823$ (the $\beta_0' = 1$ normalisation; the number is computed, not an interval estimate). In the normalisation per direction it is ≤ 3.29 and undecided (§ 07). Resolved by composing three steps: weighted Stein 0.10 after three steps and $\le$0.20 at every step (computed, tree level, commuting; 15-08)	Paper and small computations	The transverse contraction of a tree-level step in four dimensions	One loop, non-commuting configurations, large fields

Q.	Question	Size	What moves if settled	What does not move even if settled
25	In the normalisation per direction (the factor λ divided by $\beta'(\theta)$ of its own direction), can one split the weighted symbol by momentum band, show that μ is small in the directions where $\beta'(\theta)$ is close to $\frac{1}{4}$, and bring $\sup \lambda \le 3.29$ below 1? **Answer:** it was a question in a direction that cannot be brought down. In the volume limit $\sup \lambda = 4$ $\sup \mu_{\mathrm{loc}} \ge 0.59$ (Mixing-Limit Lemma), so λ is not an amount of contraction, and $\lambda < 1$ is equivalent to $\sup \mu < \frac{1}{4}$. The Stein route cannot go below the floor 0.452 (Floor Lemma for the Stein Route). The ratio per band is also 1.83 and gives no upper bound. The normalisation is fixed to the form divided by the long-wavelength coupling (§ 07)	Paper and small computations	Tree level of the shape remainder condition in the normalisation per direction	One loop, non-commuting configurations, large fields
28	On the fixed point B_∞ of gauge-covariant blocking by transverse averaging, obtain an upper bound on $\sup \mu_\infty$ by a route other than Stein: separate the one-directional sector (exactly $\frac{1}{2}$) from the rest, which carries transverse momentum, and bound the rest by the decay of the rows (0.41, 0.05, 0.005) to get below 1. Also, take the existence of the fixed point from numbers ($k \le 5$) to paper	Paper and small computations	The multi-step condition of the shape remainder condition (the cone-contraction condition at the fixed point)	One loop, non-commuting configurations, large fields
30	With blocking by transverse averaging (or with larger blocking factors), measure the one-loop coefficient of the long-wavelength periodic network at blocking factors 2 and 3, and see whether the change approaches $-\frac{11 b_1}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ (whether the factor 1.7–1.9 of the rule fixing the products along lines is an artefact of the rule)	Small computation	The one-step coefficient of the flow of the coupling (the coupling-flow remainder condition)	Coefficients from the second step on, large fields

09

What cannot be said

(a) The Unit Bundle Theorem and the Bundle Network Theorem are statements at tree level restricted to commuting configurations (the main term in $\beta \to \infty$). They do not include that the global minimum, including non-commuting configurations, is commuting; the one-loop correction (the source, the shift $O(1)$ of β'); or large fields (outside $\|\psi\| \le r$).

(b) The numbers in Tables 1–3 are numbers for a step that looks at the marginal distribution of one coarse plaquette, not for a genuine blocking step. The iterated two steps are the approximation of feeding back the marginal distribution of one plaquette.

(c) The source, c_2 and the ratios at finite β are computer values, and the source is at one loop, started from the heat kernel, with a single image, at $\beta \to \infty$. The one-loop values for the 4×4 coarse plaquette and for large boxes are from one implementation. The source $\approx 8 \times 10^{-3}/L^4$ is an observation of four points; there is no limit or closed form.

(d) The equality on the right of (7), $\max f^2 = 1/16$, and that the largest flux is on the coarse plaquette in the boxes computed, are box-by-box confirmations; there is no general proof.

(e) The numbers of the Bundle Network Theorem (Tables 4 and 5) are values under periodic boundary, 2^d blocks and linearisation of the coarse angle, and the sup is a lower bound from local optimisation. The upper bound for the pseudoinverse representative grows with the volume. The upper bound uniform in the volume from the local representative (Bundle Network Theorem (local form)) is estimated at 1.83 in four dimensions (an estimate from below, with no interval estimate) and exceeds 1. Row-weighted interpolation brought it to at most 0.823 (the $\beta_0' = 1$ normalisation, K_∞ replaced by the kernel of 20^4, computed with no interval estimate), but in the normalisation per direction it does not go below 1, so we do not say "the transverse contraction is smaller than 1" in the normalisation per direction. Nor do we say "at most 1/16". The "smaller than 1" in three dimensions depends on the numbers for Λ_3. The Bundle Network Theorem (local form) does not include one loop, non-commuting configurations, large fields, or the second and later steps. The four-dimensional long-wavelength $\lambda \to 1/16$ is an estimate from the arithmetic of the smooth trial and the three-dimensional numbers. The Mixing-Limit Lemma and the Floor Lemma for the Stein Route of the note on normalisation are paper sketches (the estimate of the cross terms, the type of the long-wavelength minimiser) and numbers on finite tori; Lean covers only the finite ledger.

(f) This chapter does not show ultraviolet control. It implies nothing about the renormalisation of the four-dimensional lattice or the mass gap.

(g) The observation of the tail type in § 07 consists of tree-level numbers on the column, the slab and the three-dimensional periodic lattice (one loop only for the column). For the coordinates of coarse links plus global tags, four dimensions is tree level only ($n \le 16$), and the four-dimensional one loop has not been measured.

(h) The multi-step cone of § 07 goes only as far as this: with blocking by edge sums it exceeds 1 at the third step (a lower bound, periodic configurations on the torus), and with blocking by transverse averaging the lower bound does not grow with the step ($j \le 4$); there is no estimate of $\sup \mu_j < 1$ from above for either blocking. The existence of the fixed point rests only on the symbolic numbers for $k \le 5$ and a line of argument by smoothing the box. The one loop of the long-wavelength limit of the network is a first-step number for one rule.

10

The five grades table and the dependency graph

Table 7 The rows of this chapter

Row	Statement (one line)	Grade	Basis	Scope
[01]	Unit Bundle Theorem (i)(ii): the minimiser θf, $f = Ps/\|Ps\|^2$, $\beta'/\beta = \sum f^2$. "$\sum f^2 \le 1$" is not unconditional	Proved here	§ 02	Restricted to commuting, tree level
[02]	Unit Bundle Theorem (iii)(iv): the coefficient of θ^4 is exactly $\sigma \sum f^4$, $\lambda = \sum f^4 / \sum f^2$, (5)	Proved here	§ 02	Local minimum, $\sigma(6\theta^2 m^2 + 4\|\theta\| mr + r^2) \le \beta/4$
[03]	For the three-dimensional free boundary, $\sum f^2$ is the effective resistance; at $2 \times 2 \times 1$, $1/3$ and $7/144$	Proved here	§ 03	Three-dimensional, free boundary
[04]	Monotonicity under inclusion and $1/(16d - 8) \le \lambda \le 1/16$	Proved here	§ 04	The 4 plaquettes of the coarse plaquette symmetric, $\frac{1}{4}$; the position of the largest flux confirmed box by box
[05]	The rational numbers of Table 1 (three-dimensional free boundary and free only on the outer rim)	Proved here	§ 05 (two formulations)	Main term
[06]	The floating-point numbers and the source of Table 1; agreement with Monte Carlo	Reading	§ 05	$SU(2)$, started from the heat kernel, $\beta \to \infty$, one loop
[07]	(8): the upper bound on $\sum f^2(L)$ and the lower bound on $\lambda(L)$; genuine two steps < iterated two steps	Proved here	§ 06	$L \ge 2$, in any box (the comparison with the iteration takes the input 0.717961 as a computed number)
[08]	The perimeter type, Tables 2 and 3, the source $\approx 8 \times 10^{-3}/L$ 4	Reading	§ 06	Four-dimensional, transverse thickness 4, one implementation, four points
[09]	Bundle Network Theorem (i)–(v); preservation of the cone of ℓ^4	Proved here	§ 07	Commuting, tree level, periodic, 2^d blocks, linearisation
[10]	Tables 4 and 5: 2^{d-4} at long wavelength, $\lambda > 1/16$ in coarse directions, no upper bound	Reading	§ 07	The boxes measured; the sup is a lower bound
[11]	The questions "lateral constancy of the coarse-plaquette source" and "local representative of the tree-level network"	Assumption	§ 08	Questions
[12]	Bundle Network Theorem (local form) (a)–(c): the local representative, the upper bound (10) uniform in the volume, the map of the cone of ℓ^4	Proved here	§ 07	Commuting, tree level, periodic, 2^d blocks, linearisation. The decay of R is a sketch
[13]	$\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$, $\nu_4 \ge 1.18$; in four dimensions it cannot go below 1.39	Reading	§ 07 (d)	Three dimensions up to 48^3; four dimensions an estimate at 24^4; ν_4 is 16^4
[14]	One-Step Contraction of the ℓ^4 Cone: row-weighted interpolation (11), $\Lambda_w \le 0.823$; the ℓ^4 cone contracts by a factor of at most 0.823 in one step	Proved here ((11)), Reading (numbers)	§ 07	Commuting, tree level, periodic, 2^4 blocks, linearisation, the kernel of 20^4, no interval estimate, the $\beta'_0 = 1$ normalisation (the normalisation per direction is undecided)
[15]	The growth of the pseudoinverse representative is consistent with the power-law kernel of the Bianchi constraint; the local representative is exponential	Reading	§ 07	Column, slab, three dimensions, tree level; four dimensions not yet
[16]	Note on normalisation: the two cones are equivalent within a constant 4 (only once if the normalisation is fixed); $\sup \lambda = 4 \sup \mu_{\text{loc}}$ (Mixing-Limit Lemma), so λ is not an amount of contraction; the floor of the Stein route is 0.452 (Floor Lemma for the Stein Route); the coarsest direction gives $1/64$	Proved here (the ledger, sketches of the Mixing-Limit Lemma and the Floor Lemma for the Stein Route), Reading (numbers)	§ 07	Commuting, tree level, periodic, 2^4, linearisation, coarse L even with $L \to \infty$, ν is a linear-programming number
[17]	The One-Step Contraction of the ℓ^4 Cone is limited to the first step: with blocking by edge sums the covariance has no fixed point (2^{-k}) and the factor at the third step is ≥ 1.26. With transverse averaging it goes to a fixed point, lower bounds 0.64, 0.57, 0.64, 0.62, no upper bound	Proved here (Conditional-Mean Covariance Lemma, the ledger of the geometric series), Reading (numbers)	§ 07	Commuting, tree level, periodic, 2^4, linearisation, symbolic flow $k \le 5$, lower bounds by local optimisation, no interval estimate
[18]	The coordinates of coarse links plus global tags are exponential for the four-dimensional periodic tree ($\kappa \approx 2.08$); without the tags, flat plus power law, with sums growing with the volume	Proved here (orthogonality of the tags), Reading (numbers)	§ 07	Commuting background, quadratic coefficients, one step, tree level, $n \le 16$
[19]	The long-wavelength limit of the network: the tree gives 1 for all L, the cell part is $1/L$, the tree has no running of $\ln L$, and one loop gives $1 - 0.745/\beta$	Reading	§ 07	Commuting, tree level and one loop (block 2, transverse 6^3); beware of contamination by the Polyakov direction

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV F1     minimiser θf, β'/β = Σf²  (commuting, tree level)
[02] PRV F2     θ⁴ exactly σΣf⁴, λ, (5) ⟸ [01]  (local minimum)
[03] PRV NET    three-dimensional resistor network, 1/3, 7/144 ⟸ [01]  (free boundary)
[04] PRV BOUND  monotonicity under inclusion, 1/(16d−8) ≤ λ ≤ 1/16 ⟸ [01][02]  (coarse plaquette symmetric)
[05] PRV RAT    the rational numbers of Table 1 ⟸ [01][03]
[06] RDG BOX    the floating-point numbers and the source of Table 1 ⟸ [02][05]  (one loop, β→∞)
[07] PRV LBOUND (8), genuine two steps < iteration ⟸ [04]  (L ≥ 2)
[08] RDG PERIM  perimeter type, source ≈ 8e−3/L⁴ ⟸ [06][07]  (four points, one implementation)
[09] PRV G      Bundle Network Theorem ⟸ [01]  (periodic, 2^d blocks)
[10] RDG GNUM   long wavelength 2^{d−4}, coarse directions λ > 1/16 ⟸ [09]
[11] ASM Q      questions "lateral constancy of the coarse-plaquette source" and "local representative of the tree-level
[12] PRV G2     local representative, uniform bound (10), map of the cone ⟸ [09]  (periodic, 2^d blocks)
[13] RDG G2NUM  Λ₃ = 0.245, Λ₄ ≈ 1.83, ν₄ ≥ 1.18 ⟸ [12]
[14] PRV G2C    weighted interpolation (11), cone contracts by at most 0.823 ⟸ [12][13]  (β'₀ = 1 normalisation, number
[15] RDG TAIL   growth of the pseudoinverse representative is a power-law kernel ⟸ [10][12]  (up to three dimensions, t
[16] PRV NORM   note on normalisation: equivalence, Mixing-Limit Lemma, Floor Lemma for the Stein Route ⟸ [09][14]  (L
[17] RDG MULTI  One-Step Contraction of the ℓ⁴ Cone limited to the first step, multi-step averaged blocking ⟸ [14][16]
[18] RDG TAG    coarse links plus tags exponential for the four-dimensional tree ⟸ [12][15]  (tree level, n ≤ 16)
[19] RDG LONG   long-wavelength limit of the network and one loop ⟸ [09][10]  (block 2, one loop)
```

Every number to the right of "⟸" is smaller than the one on the left, and there are no cycles. The only assumption row is [11], and it is in the form of a question (the direct upper bound for the four-dimensional ℓ^4, the normalisation factors by direction, the upper bound on the fixed point of the lateral average, and the one-loop coefficient of the long-wavelength network are in Table 6 of § 08, and continue [11]).

What this chapter can say

Can say	Cannot say
Proved here At tree level restricted to commuting configurations: that β'/β, the map of the quartic term and λ are determined by the projection (Unit Bundle Theorem); $\lambda \le \max f^2$; $1/(16d-8) \le \lambda \le 1/16$ if the coarse plaquette is symmetric; (8); the Bundle Network Theorem and $\beta'(\theta) \ge \frac{1}{4}$; the upper bound uniform in the volume from the local representative (Bundle Network Theorem (local form)); row-weighted interpolation (11)	The minimum including non-commuting configurations, one loop, large fields. A general proof of the position of the largest flux
Reading The numbers box by box (Tables 1–5), the perimeter type, the source $\approx 8 \times 10^{-3}/L^4$, and that with all coarse plaquettes fixed $\lambda > 1/16$ in coarse directions; $\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$; $\Lambda_w \le 0.823$ (four dimensions, the $\beta_0' = 1$ normalisation); the observation of the tail type	λ and the source of a genuine blocking step. Interval estimates. A closed form for the source
Proved here The note on normalisation (equivalence of the two cones, the finite parts of the Mixing-Limit Lemma and the Floor Lemma for the Stein Route); that a tree-level step is a conditional mean and the covariance iterates linearly (Conditional-Mean Covariance Lemma). Reading Going above 1 at the third step of blocking by edge sums; the fixed point and lower bounds of transverse averaging; the four-dimensional tree of coarse links plus tags; the one loop of the long-wavelength limit of the network	An upper bound on the multi-step $\sup \mu_j < 1$ (for any blocking). A proof of the existence of the fixed point. The four-dimensional one-loop tail. The one loop of the network from the second step on

Sources and reproduction

Item	Type	Source, tools
Thomson's principle, Kirchhoff's laws, Rayleigh's monotonicity, Dirichlet's principle	Cited	Standard facts about electric circuits and harmonic functions (of the kind in P. G. Doyle, J. L. Snell, Random Walks and Electric Networks, 1984)
Unit Bundle Theorem, Bundle Network Theorem, (5)(7)(8), the hand computation of $2 \times 2 \times 1$	Proved here	The proofs in the text of this chapter. The parts that are inequalities on finite sums are in the Lean bundle (re-verification: done; 15-09 Table 3)
The numbers of Tables 1–5	Reading (rational numbers: Proved here)	The numerical records of this book's study (two to three formulations of the flux network: Kirchhoff on the dual network, the normal equations in the tree gauge, the projection formula, the Hodge Laplacian of the Lagrange conditions. Two implementations of the one-loop Hessian: finite differences and analytic, and a sparse-matrix version. Finite sums of characters; heat-bath Monte Carlo on all links. For the Bundle Network Theorem, two implementations: projection and pseudoinverse, and KKT on coarse links. For the numbers of the Bundle Network Theorem (local form), two implementations of the local representative, and a linear program)
Weighted interpolation (Stein's analytic family, Hadamard's three lines)	Cited (a known tool; bibliographic details unchecked)	E. M. Stein's interpolation theorem for analytic families

Item	Type	Source, tools
The note on normalisation, the multi-step cone, the four-dimensional tree of coarse links plus tags, the long-wavelength limit of the network	Reading (the ledger and the Conditional-Mean Covariance Lemma: Proved here)	The numerical records of this book's study (the symbolic route and the Bloch rows of the first step, weighted least squares in real space, cross-checks against a separate nested implementation, negative controls with random small matrices, the negative control of the naive average, two implementations dense and Bloch, 0 for the $U(1)$ Villain). The ledger is in the Lean bundle (15-09 Table 3, re-verified)
$\Lambda_w \le 0.823$, the numbers of the tail type	Reading	The numerical records of this book's study (the symbol of the kernel of 20^4 on the $N = 64$ grid with a one-sided quadratic remainder, the real-space power method, small-matrix tests and negative controls; for the column, slab and three-dimensional periodic cases, two implementations, projection and KKT)

Terms used in this chapter — flux, the flux network, the projection P, the unit flux, the Unit Bundle Theorem, the transverse contraction λ, the source σ, monotonicity under inclusion, the perimeter type, the genuine two steps, the iterated two steps, the Bundle Network Theorem, quantities that do not depend on the representation, the local representative, the axial cone, the Bundle Network Theorem (local form), row-weighted interpolation, the tail type. Their meanings are in A-02 Glossary.

Next chapter to read: 15-06 Counting the parameters — how many coordinates are needed at quadratic order, and where the one-step map draws in.

Counting the parameters — couplings by loop shape and the attracting class

This chapter is this book's own investigation. The established knowledge it builds on is in 15-01 and 15-02.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

What this chapter shows — Using the smallest systems, with plaquettes added one at a time, we look at which coordinates should be used to count the parameters of the class-closure condition of 15-03 § 02, "the class closes with finitely many parameters". With one plaquette, the background dependence is absorbed into a single replacement of the coupling; with two or three, the commutator is merely the image of the coupling of the composite loop; with four, the ratio of the couplings by loop shape moves with β; and even if the heat kernel is used instead, the part $2/(3\beta)$ due to the curvature of the group remains. Searching for actions that keep the ratio at 2, the family that moves the power of the prefactor has no root, and the fundamental + adjoint family has a root, but that curve does not close under the one-step flow. Finally, since the one-step map contracts all the shape coordinates of the action and draws them to the unique fixed point of the power of the prefactor ($\mathbf{37/76}$ at ψ^{4}), what is consistent with this model is to state the class-closure condition not as "a class that keeps the ratio at 2" but as **"the one-parameter attracting class + a contracting shape"**.

Chapters assumed — 15-03 § 02 (the statement of the decomposition), 15-03 § 04 (the ledger of the transverse contraction), 15-04 § 01 (the heat-kernel semigroup), 15-05 § 02 (the flux network), 15-07 § 02 (the background dependence η). We regard $SU(2)$ as the unit quaternions (the three-sphere S^{3}) and use the character expansion and the Laplace method (the saddle-point approximation).

Said first — Every system in this chapter is one plaquette up to four plaquettes in a row of $SU(2)$ (with $U(1)$ as a control), with the field strength of the background taken small (holonomy angle $\theta \le 0.3$). The one step is an even smaller version of **the one step seen through the marginal distribution of a single coarse plaquette (not the one step of a true blocking)**. The maps in § 07 and § 08 are maps for four plaquettes in a row, two loops, the diagonal definition and the main term ($B \to \infty$). None of the statements says anything about one step of the four-dimensional lattice, block averages, large fields or gauge re-fixing.

The order of this chapter

1. One plaquette — the type of the background dependence, and absorption under a constraint (equation (1))
2. Two plaquettes — the commutator is the image of BCH (equations (2)(3))
3. Three plaquettes — single and composite loops in 1:1:1 (equation (4))
4. Four plaquettes — the ratio of the couplings by loop shape moves with β (equations (5)–(7))
5. The heat kernel and the criterion — the part due to the curvature of the group (equations (8) (9), Table 1)
6. Searching for an action that keeps the ratio — the general one-loop formula, the prefactor family, fundamental + adjoint (equations (10)(11))
7. The ratio-2 curve does not close under the flow — the tree map and $9/89$ (equations (12)(13))
8. The flow of the power of the prefactor and the attracting class — $37/76$, $c = 2/3$ (equations (14)–(16))
9. The form of the statement of the class-closure condition (Table 2) and questions (Table 3)
10. What cannot be said
11. The five grades table and the dependency graph (Table 4)

00

What this chapter shows

- The coordinates for counting the parameters of the class-closure condition are not "the coefficient of F^2 and the commutator coefficient" but the **couplings by loop shape**. The commutator coefficient follows afterwards as the BCH image of the coupling of the composite loop (§ 02, § 03).
- For four plaquettes in a row the second coordinate (the ratio κ_{ab}/κ_a) moves with $O(1/\beta)$, and part of it, $2/(3\beta)$, is due to the curvature of the group and does not vanish even if the action is replaced by the heat kernel (§ 04, § 05).
- Keeping the ratio at 2 for all β requires an action whose quartic term in the exponent is steeper than Gaussian ($q/B = -1/12$); the family of powers of the prefactor has no root, and the fundamental + adjoint family has a root at $\beta \geq 12$ (§ 06).
- That root curve does not close under the one-step flow, and the deviation stays at $O(1)$ ($9/89$), because the one-step map contracts all the coordinates of the shape of the action (§ 07).
- The power of the prefactor follows a linear contraction map and is drawn to the unique fixed point $\gamma^* = 37/76$ (ψ^4), where the ratio is $2 + (2/3)/B$. The class-closure condition can be stated as "the one-parameter attracting class + a contracting shape" (§ 08, § 09).

01

One plaquette — the type of the background dependence, and absorption under a constraint

Conventions: $SU(2)$ is written with unit quaternions, $\frac{1}{2}\operatorname{tr} U = \operatorname{Re} U$, background $W = \exp(i\,a\cdot\sigma/2)$ ($a \in \mathbb{R}^3$, angle $\theta = |a|$). The numbers were measured at $\beta = 4$–32, $\theta \le 0.3$.

On a single plaquette (four links), consider the measure μ_W of the fluctuations around the background holonomy W. It is the smallest model of the η_k of 15-07. Of the 12 dimensions of the four links, the conditional law of the 9 gauge directions does not depend on W, and the background dependence gathers in the 3 dimensions of the holonomy Proved here (invariance of the Haar measure). Placing on the fluctuation $X = i\,x\cdot\sigma/2$ ($r = |x|$) the constraint $x_3 = 0$ that removes the direction of the generator of the background (the smallest substitute for the background-field constraint), we have $\frac{1}{2}\operatorname{tr}(W e^X) = \cos\frac{\theta}{2}\cos\frac{r}{2}$, so

$$\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)} \quad (1)$$

holds exactly Proved here . The background dependence goes entirely into a single replacement of the coupling $\beta \mapsto \beta\cos(\theta/2)$. The η measured by the ratio of covariances is $\theta^2/8$, and the score is $\theta/4$; neither depends on β (main term Proved here ; by two routes, quadrature and Monte Carlo, the ratio to the main term is 0.98–1.00 Reading).

- **Without the constraint it is not small.** If the background is not taken at a critical point, the ratio of covariances grows with β as $\beta\theta^2/12$ Reading . From this one plaquette we read that the minimal condition for the background dependence to be small uniformly in the background is "the background is a constrained critical point".
- **The one-step map of the coupling.** The weight of the background with the fluctuations integrated out is, under the constraint, $\beta_{\text{eff}} = \beta - d/2 + O(1/\beta)$ (d is the dimension of the fluctuations; $\beta - \beta_{\text{eff}} = 1.005$ at $d = 2$, $\beta = 64$) Reading (Proved here as a count of the dimensions of the fluctuations, at the one-loop level).
- **The source of η is not non-commutativity.** In the $U(1)$ control it is $\eta = 0$ only for the quadratic (Gaussian) action, and for the cos action a η of the same θ^2 type appears Reading . The source is the non-quadratic nature of the action (the compactness of the group). Measuring η by a single number changes the type (the ratio of covariances is θ^2, the Kullback–Leibler quantity is θ^4, the score is θ), so the statement must specify "which η" Reading .

02

Two plaquettes — the commutator is the image of BCH

On a 2×1 rectangle with one shared link, fix the outer loop $W = P_1P_2$ (P_1, P_2 are the holonomies of the two plaquettes, independent Haar after the tree gauge). The constrained minimum is $P_1 = P_2 = W^{1/2}$, in which the two plaquettes tilt by half each in the same direction; setting $P_1 = W^{1/2}e^X$, $P_2 = e^{-X}W^{1/2}$,

$$\operatorname{Re} P_1 + \operatorname{Re} P_2 = 2\cos\frac{\theta}{4}\cos\frac{r}{2} \quad (2)$$

holds exactly Proved here . The action is even in the fluctuations, and the odd cubic term that appeared when the substitute of "subtracting the first-order term" was used for one plaquette vanishes at the true constrained minimum (numerically the odd part of the score is $\leq 3 \times 10^{-12}$) Reading . The weight with the fluctuations integrated out is a function $Z_1(\beta|1+W|)$ of the class of W alone, $Z_1(b) = 2I_1(b)/b$ (the same integral as the one-link distribution of the heat-bath method) Proved here , and the coupling after one step is $\beta_{\text{eff}} = \beta - \frac{3}{4} + O(1/\beta)$ (Proved here , at the one-loop level; $\beta - \beta_{\text{eff}} = 0.723 \to 0.747$ at $\beta = 4 \to 32$ Reading). The commutator of the field strengths a, b of the two background plaquettes enters only through the BCH of the group product $e^{F_1} e^{F_2}$:

$$\theta_W^2 = |a+b|^2 - \tfrac{1}{12}|a \times b|^2 + O(|F|^5) \quad (3)$$

Proved here (algebra under truncation of BCH at third order). So the coefficient c of $|a \times b|^2$ in the weight is tied to $1/12$ of the coefficient k of F^2, and $c/k = 1/12$ holds to 10^{-8} for $\beta = 4$–32 Reading . **The commutator is not a new parameter at this step.**

03 Three plaquettes — single and composite loops in 1:1:1

With three in a row, fix the two overlapping 2×1 loops $W_a = P_1 P_2$, $W_b = P_2 P_3$ and integrate over the central P_2; the weight is then, with $Z_1(\beta|Q|)$, $Q = 1 + W_a + W_b$,

$$|Q|^2 = 3 + 2\,\mathrm{Re}\,W_a + 2\,\mathrm{Re}\,W_b + 2\,\mathrm{Re}(W_a W_b^{-1}) \quad (4)$$

Proved here (agrees with direct quadrature to 3.5×10^{-15} Reading). It contains the two single loops and the composite loop $W_a W_b^{-1}$ in the ratio 1:1:1. The invariants of simultaneous conjugation of a pair of elements of $SU(2)$ are determined by the three traces $\mathrm{Re}\,W_a, \mathrm{Re}\,W_b, \mathrm{Re}(W_a W_b^{-1})$ Proved here (two imaginary-part vectors are determined up to rotation by their lengths and inner product), so every gauge-invariant two-loop weight is a function of the three class angles. Taking the class angle of the composite loop as the reference, the coefficient of the commutator is necessarily $1/12$ of the quadratic coupling κ_{ab} of the composite loop Proved here ((3) enters as it is). The coupling after one step is $12k = \beta - \frac{1}{2} + O(1/\beta)$ Reading . The "commutator coefficient" measured in the basis of monomials appears to move with β, but it appears with the same size even for a composite function with no commutator, so it is not a quantity of the commutator Reading .

04 Four plaquettes — the ratio of the couplings by loop shape moves with β

With four in a row, fix the two overlapping 3×1 loops $W_a = P_1 P_2 P_3$, $W_b = P_2 P_3 P_4$ and integrate over the two plaquettes inside. Setting $U = P_2 P_3$,

$$w(W_a, W_b) = \int dU\; Z_1(\beta|1+U|)\, \exp\big(\beta \langle W_a + W_b,\, U\rangle\big), \qquad \langle p, q\rangle = \mathrm{Re}(p\bar{q}) \quad (5)$$

holds Proved here . w is a function of the two variables $\mathrm{Re}\,W_a + \mathrm{Re}\,W_b$ and $\mathrm{Re}(W_a W_b^{-1})$ and does not collapse to the single variable $|Q|$ of the three-plaquette case (on the same $|Q|$ background, $\log w$ moves by 1.8×10^{-2}, $\beta = 4$) Reading . The quadratic couplings are defined by

$$-\log w = \text{const} + \kappa_a(\theta_a^2+\theta_b^2) + \kappa_{ab}\,\theta_{ab}^2 + O(F^4) \quad (6)$$

(θ_{ab} is the class angle of the composite loop $W_a W_b^{-1}$). The tree (the commuting Gaussian minimisation) gives the ratio $\kappa_{ab}/\kappa_a = 2$ Proved here , and the main term at one loop is $-\frac{1}{40}$ for both κ_a, κ_{ab}, with ratio 1. Together,

$$40\kappa_a = \beta - 1, \qquad 20\kappa_{ab} = \beta - \tfrac{1}{2}, \qquad \frac{\kappa_{ab}}{\kappa_a} = \frac{2(\beta-\frac{1}{2})}{\beta-1} + O(\beta^{-2}) \quad (7)$$

(the first two are $+O(1/\beta)$) Proved here , at the one-loop level. Numerically $\kappa_{ab}/\kappa_a = 2.3211,\ 2.1409,\ 2.0662,\ 2.0322$ ($\beta = 4, 8, 16, 32$, the two routes agree to 7.8×10^{-12}) Reading . **The reason the ratio moves is "the tree shape ≠ the one-loop shape"**. The quadratic form of the tree is 1:2, fixed by the geometry of the lattice, and the quadratic form at one loop is 1:1, fixed by the measure; since the two are not parallel, the direction of the sum rotates with β. For the cos action of $U(1)$ too the ratio moves with $2 + 0.2/\beta$, while for the Villain action of $U(1)$ it is 2 for all β Reading . That one step of blocking produces terms of larger loops whose coefficients depend on the coupling is, we think, a known type in the same direction as the idea of improved actions (the original sources have not been checked).

05

The heat kernel and the criterion — the part due to the curvature of the group

If the weight of each plaquette is replaced by the heat kernel p_t of $SU(2)$ (character coefficients $(n+1)e^{-tn(n+2)/2}$), the two inner plaquettes combine by the semigroup property $p_t * p_t = p_{2t}$ into

$$w(W_a, W_b) = \int dU\; p_{2t}(U)\, p_t(W_a U^{-1})\, p_t(U^{-1} W_b) \quad (8)$$

Proved here (the same formula for finite groups and families of semigroups is in Lean: `four_plaquette_of_semigroup`, `four_plaquette_convPow`, assumption $p_s * p_t = p_{s+t}$). Even so, the ratio is $2.2090,\ 2.0927,\ 2.0439,\ 2.0214$ ($t = 1/4, \dots, 1/32$; the two routes agree at 7.6×10^{-12}) Reading , and at one loop it is $2 + \frac{2t}{3} + O(t^2)$ Proved here , at the one-loop level. For the heat kernel (Villain) of $U(1)$ it is 2 for every t Proved here .

The reason can be written as a one-line criterion. Let the outer kernel be $e^{F(\psi)}$ and the inner kernel $G(\psi)$ (both class functions; ψ is the angle of a unit quaternion), and write the expectation with respect to the measure $\sin^2\psi\, G\, e^{2F}\, d\psi$ as E; then integration by parts gives

$$\kappa_a = \frac{E[F'\,(\log G)']}{24}, \qquad \kappa_{ab} = \frac{E[F'^2]}{24}, \qquad \frac{\kappa_{ab}}{\kappa_a} = 2 \iff D := E\big[F'\,(F' - 2(\log G)')\big] = 0 \quad (9)$$

Proved here (under the conditions that F, G is a smooth class function for $[0,\pi]$ and that the boundary terms vanish). A pointwise sufficient condition is that **the inner kernel is the square root of the outer one**, $G \propto e^{F/2}$. For a flat Gaussian the semigroup p_{2t} and the square root $p_t^{1/2}$ are the same thing, so the ratio stays at 2; for a curved group the two split. The heat kernel of $SU(2)$ is a semigroup, but the Van Vleck prefactor $P(\psi) := \psi/\sin\psi$ enters p_t and p_{2t} with the same power, so it is not a square root. The Gaussian in the geodesic distance $e^{-\psi^2/(2t)}$ satisfies the square-root relation but is not a semigroup Proved here (numerically too the ratio is 2 for all t, 3×10^{-12} Reading).

Matching t and β by the coefficient of the fundamental representation, the main term of the deviation of the ratio is $1/\beta$ for Wilson and $2/(3\beta)$ for the heat kernel. **The $2/(3\beta)$ that does not vanish even for the heat kernel is the part due to the curvature of the group, and the remaining $1/(3\beta)$ is the part due to the non-quadratic nature of the action**, we read Reading (main term, this system, quadratic order).

Table 1 Counting the parameters in the smallest systems ($SU(2)$, $\theta \le 0.3$, $\beta = 4$–32. The weight formulas are Proved here , the numbers are Reading)

System (what is fixed)	Weight with the fluctuations integrated out	Quadratic coupling	Commutator coefficient	$U(1)$ control
One plaquette (background W, constraint $x3 = 0$)	$\mu_W^\beta = \mu_1^{\beta\cos(\theta/2)}$ ((1))	$\beta_{\text{eff}} = \beta - d/2$	— (no commutator can be formed)	η of the same θ^2 for the cos action; 0 for the Gaussian
Two plaquettes (outer loop W)	$Z_1(\beta\lvert 1 + W\rvert)$	$\beta_{\text{eff}} = \beta - \frac{3}{4}$	$k/12$ ((3))	0
Three in a row (overlapping W_a, W_b)	$Z_1(\beta\lvert 1 + W_a + W_b\rvert)$	single : single : composite = 1 : 1 : 1, $12k = \beta - \frac{1}{2}$	$\kappa_{ab}/12$	0
Four in a row, Wilson	(5) (a function of two variables)	$\kappa_{ab}/\kappa_a = 2(\beta - \frac{1}{2})/(\beta - 1)$	The BCH image of the coupling of the composite loop	$2 + 0.2/\beta$ at cos; 2 for Villain
Four in a row, heat kernel	(8)	$\kappa_{ab}/\kappa_a = 2 + \frac{2t}{3}$	Same as above	2 for the heat kernel (Villain)

06

Searching for an action that keeps the ratio — the general one-loop formula

Is there an action that satisfies the criterion (9) for all β (an action that closes with one parameter for four plaquettes in a row)? Writing an action concentrated at the identity as $\log f = \text{const} - B\psi^2/2 + q\psi^4 + \cdots$ (B is the tree coupling, $q = O(B)$) and counting the correction of the inner kernel by the Brownian-bridge average (the flat approximation), we get

$$D = \frac{2}{5} + \frac{24}{5}\frac{q}{B} + O\Big(\frac{1}{B}\Big), \qquad E[F'(\log G)'] = \frac{3}{5}B + O(1), \qquad \frac{\kappa_{ab}}{\kappa_a} = 2 + \frac{D}{\frac{3}{5}B} + O(B^{-2}) \quad (10)$$

Proved here , at the one-loop level. $\frac{2}{5}$ comes from the curvature of the group (the Van Vleck prefactor $\log P = \psi^2/6 + \cdots$), and $\frac{24}{5}q/B$ from the quartic term of the exponent (non-Gaussianity). A multiplicative correction of ψ^2 is absorbed into a shift of the tree coupling B and does not enter D Proved here . Calibration: for Wilson ($q = \beta/24$) $D = \frac{3}{5}$ and ratio $2 + 1/\beta$; for the heat kernel ($q = 0$) $D = \frac{2}{5}$ and ratio $2 + 2t/3$; for $U(1)$, $D = \frac{24}{25}q/B$ ($\frac{1}{25}$ at cos, 0 for Villain). All the values of § 04 and § 05 are reproduced Reading .

- **The family of powers of the prefactor has no root.** For $f = e^{\beta\cos\psi}(\psi/\sin\psi)^\gamma$, in the range $\gamma < \frac{1}{2}$ where the quadratic coupling exists, $D \in [0.534, 0.600]$ at every point of the measured grid $\gamma \in [-2, 0.45]$, $\beta \in \{4, 8, 16, 32\}$, and $D \ge 0.343$ also on the extension $\gamma \ge -128$; it is monotonically

increasing in γ, with no change of sign Reading (the two routes agree to $\leq 6.5 \times 10^{-9}$). At one loop too, $D > 8/25$ for all $\gamma = c\beta$ ($c < 3$) Proved here, at the one-loop level. At $\gamma \geq \frac{1}{2}$ the singularities of $\psi = \pi$ overlap and the quadratic coupling itself does not exist, and at $\gamma \geq \frac{3}{2}$ the weight itself diverges Proved here. The reason is that the power of the prefactor is absorbed into B by the correction of ψ^2, and the curvature term $\frac{2}{5}$ does not move.

- **The fundamental + adjoint family has a root.** From (10), $D = 0$ requires $q/B = -\frac{1}{12}$ (a quartic term steeper than Gaussian). $F = \beta\cos\psi + \rho\beta\cos^2\psi$ is $B = \beta(1 + 2\rho)$, and with $q = \beta(\frac{1}{24} + \frac{\rho}{3})$

$$D = \frac{2}{5} + \frac{1}{5}\frac{1 + 8\rho}{1 + 2\rho} + O\Big(\frac{1}{B}\Big), \qquad \text{one-loop root } \rho = -\tfrac{1}{4} \text{ (11)}$$

Proved here, at the one-loop level. The numerical root is $\rho^*(\beta) = -0.318,\ -0.279,\ -0.264, \ldots,$ -0.2516 ($\beta = 16, 32, 64, \ldots, 512$; ratio 2 to 3×10^{-10} by bisection and two routes), $\rho^* = -\frac{1}{4} - 0.84/\beta + O(\beta^{-2})$ Reading. For $\beta \leq 10$ there is no root on the measured grid (minimum $D = +0.029$), and the root appears between $\beta = 10$ and 12 Reading.

- **One parameter for one step only.** In neither family does $f * f$ belong to the same family (the residual of the fit is at least 10^5 times that of the heat-kernel control) Reading. Even if the root action is taken, the inner kernel after one step leaves the family, so the same criterion cannot be used at the next step. The pointwise square-root relation $f * f \propto f^{1/2}$ cannot be satisfied at the one-loop level by a concentrated class function Proved here (the coefficient $-\frac{1}{3}$ of ψ remains).

07

The ratio-2 curve does not close under the flow

We check whether the root curve $\rho^*(\beta)$ returns to itself under the one-step map. The outer kernel after one step is defined on the diagonal as $f'(\psi) := w(W, W)^{1/2} = [(f * f) * f^2]^{1/2}(W)$ (**the diagonal definition**: at $W_a = W_b$ the angle of the composite loop is 0), refitted to the fundamental + adjoint family by the Taylor coefficients of ψ^2 and ψ^4, and (β', ρ') is read off.

The tree map. With the outer kernel $F = -Bx^2/2 + qx^4 + \cdots$, at the flat, collinear saddle point (on the geodesic joining 1 and W, $u_0 = 4w/5$),

$$B' = \frac{B}{5}, \qquad \Big(\frac{q}{B}\Big)' = \frac{17}{125}\frac{q}{B} + O\Big(\frac{1}{B}\Big), \qquad \psi^{2k} \text{ shape coordinate: } \frac{4^k + 1}{5^{2k-1}} \text{ times (12)}$$

holds Proved here, tree (main term of the saddle point). Because the saddle point is collinear, the commutator and the curvature of the group do not enter the tree, and enter only the one-loop $O(1/B)$. Combined with (10), the image of the ratio-2 curve $q/B = -\frac{1}{12}$ is $q/B = -\frac{17}{1500}$, and for fundamental + adjoint

$$\rho' = -\frac{53}{356}, \qquad \rho' - \rho^* = \frac{9}{89} \approx 0.101, \qquad D(\text{image}) = \frac{216}{625} \text{ (13)}$$

holds Proved here, tree (algebra of substitution). Numerically, $\rho' - \rho^*(\beta') = 0.160,\ 0.127,\ 0.113$ (starting from $\beta = 128, 256, 512$, $\beta' = 17.8, 36.1, 72.5$) approaches $9/89$ as $O(1/\beta')$ Reading (the two routes agree to $\leq 6 \times 10^{-9}$ at q'). Adding a third coordinate $\sigma\beta\cos^3\psi$ does not bring D back to 0 (σ

enters D only at $O(1/B')$) Reading . **One step contracts the shape coordinate q/B and draws the flow towards the Gaussian side. For $\mathrm{SU}(2)$ the ratio-2 condition lies at a point $O(1)$ away along the contracting direction, so the curve does not close**, we read. For $U(1)$ the ratio-2 line $q/B = 0$ is itself the fixed point of the tree map (Villain), so it closes Reading .

08

The flow of the power of the prefactor, and the attracting class

Then where does the flow draw to? Write the class function as $f = \exp(-B\psi^2/2 + h(\psi))$ $(h = O(1))$ and follow one step by the main term of the Laplace method with the argument held at $B \to \infty$. Regarding $SU(2)$ as the S^3 of radius 1, the saddle point of the convolution of two kernels lies on the geodesic joining 1 and W, and the determinant of the Hessian in the two transverse directions gives the ratio of the prefactors P:

$$h_{f*g}(w) = h_1(a) + h_2(b) + \log P(w) - \log P(a) - \log P(b) + \mathrm{const} + O(1/B) \quad (14)$$

$(a, b$ is the split at the saddle point) Proved here , main term of Laplace. Writing $g := h - \log P$, **g is carried by convolution in the same way as the flat tree**. This is why the heat kernel $(g = 0)$ is preserved; it is a restatement of the semigroup property. The product (the square at the same point) leaves one prefactor over, and the square root halves it. Substituting one step $f'^2 = (f * f) * f^2$, the power of the prefactor $\gamma_k := h_k/p_k$ read from ψ^{2k} $(\log P = \sum p_k \psi^{2k}$, $p_2 = \frac{1}{180})$ follows

$$\gamma'_k = \frac{4^k + 1}{25^k}(\gamma_k - 1) + \frac{1 + 25^{-k}}{2}, \qquad \psi^4 \text{ gives } \gamma' = \frac{17\gamma + 296}{625} \quad (15)$$

a linear contraction map Proved here , main term of Laplace (algebra of the coefficients). The unique fixed point is

$$\gamma_k^* = \frac{25^k - 2 \cdot 4^k - 1}{2(25^k - 4^k - 1)}, \qquad \gamma_2^* = \frac{37}{76}, \quad \gamma_3^* = \frac{1937}{3890}, \quad \gamma_k^* \to \frac{1}{2}, \qquad \left(\frac{q}{B}\right)^* = \frac{37}{13680\, B} \quad (16)$$

Proved here . The heat-kernel power 1 moves to $313/625$ in one step, which explains why the power of the prefactor appears to change from 1 to about $\frac{1}{2}$. The contraction rate is $\frac{17}{625}$ (ψ^4), not $\frac{1}{2}$, and $\frac{1}{2}$ is the value at the fixed point. Numerically: the extrapolation of γ'_2, γ'_3 from $B = 32$–256 agrees with (15) to $\le 10^{-7}$; in the iteration holding B, four starts $(\gamma = 0, \frac{1}{2}, 1, 2)$ and the start of $q/B = -\frac{1}{12}$ gather to the same point with ratio $17/625$; and the one-step extrapolation including the h^* of (16) gives $37/76$ and 1.4×10^{-8} Reading (two routes: the character series and a one-dimensional multiple integral).

The ratio at the attracting class. Putting $(q/B)^* = O(1/B)$ into (10), on the attracting class

$$\kappa_{ab}/\kappa_a = 2 + c/B + O(B^{-2}), \qquad c = \frac{2/5}{3/5} = \frac{2}{3}$$

Proved here , at the one-loop level (numerical extrapolation 0.666667, two routes 1.8×10^{-9} Reading). c does not depend on h^* and is the same as for the heat kernel; in the main term of the quadratic

ratio the attracting class cannot be distinguished from the heat kernel. For $U(1)$ there is no inhomogeneous term, and the fixed point is Villain ($h = 0$), $c = 0$ Proved here .

09

The form of the statement of the class-closure condition

Putting § 01– § 08 together Reading (limited to four plaquettes in a row, two loops, the diagonal definition and the main term): the coordinates of the parameters of the class-closure condition are the couplings by loop shape, and the commutator is their BCH image. One step contracts all the shape coordinates (Table 2), and the family is drawn to "one B + the $O(1/B)$ shape h^* fixed by the curvature of the group". The ratio 2 is not a coordinate that can be tuned with the parameters, but a one-loop quantity that becomes $2 + 2/(3B)$ at the attracting point. So what is consistent with this model is to state the class-closure condition not as "a class that keeps the ratio at 2" but as **"the one-parameter attracting class + a contracting shape carried by the shape remainder condition"**. If we assume that the shape deviation r_n satisfies $r_{n+1} \le \lambda r_n + C/B_n$ ($B_n = B_0/5^n$) in one step, then $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, and it closes in the geometric-series pattern of 15-03 § 04 Proved here (assuming the one-step inequality). The Shape-Remainder Theorem for Four Plaquettes in a Row, next, gives the constants of this one-step inequality as numbers inside this model.

Shape-Remainder Theorem for Four Plaquettes in a Row — the one-step constants as numbers (the projection model of four in a row). Conditions (on the same line): $SU(2)$, four plaquettes in a row, the diagonal definition, the projection model that truncates at ψ^8 and re-inflates, coordinates taken as the analytic main part of $\log f'$ (the rest is at most 10^{-35}), $B \ge 32$, a box of $O(1)$. If class functions are lifted to radial functions on $\mathbb{R}^3$ ($f = P \cdot \varphi$, $P = \psi/\sin\psi$; the known lift that sends convolution of adjoint-invariant functions to convolution on the Lie algebra), one step becomes a single Gaussian integral of three segments of stiffness B, B, $2B$ joined in series, and the curvature of the group remains only in "the P left in the lift of the product f^2" and in "the final P and the square root" Proved here (the lift is of a known type Cited ; the original source has not been checked). The next term of Laplace's method can be written in closed form by the second-order cumulants of the Gaussian; the $O(1/B)$ term of ψ^4 is $-\frac{8}{3125}h_2 + \frac{42}{125}h_3 - \frac{14}{140625}$, and on the attracting class h^*

$$\left(\gamma_2' - \frac{37}{76}\right)B \;\to\; -\frac{591986}{69290625} \approx -0.00854 \quad (17)$$

Proved here (numerical extrapolations -0.0085438 and 3×10^{-7} Reading). Writing $r := \max_{k=2,3,4} 4^k\,|h_k - h_k^*|$ for the distance in the coefficients of ψ^4, ψ^6, ψ^8, and with $\lambda = 17/625$, one step

$$r' \;\le\; \lambda\, r + \frac{C}{B}, \qquad C = 0.0108\ (r \le 0.05),\quad 0.0036\ (r \le 0.01),\quad B \to \infty \text{ sharp value } 0.00076 \quad (18)$$

satisfies Proved here , Reading (the line of the estimate is on paper; the constants are numbers from explicit integrals and series, and the monotonicity in B was only checked on the grid $B = 32$–10^6), and for the actual step scales $B_0, B_1, \ldots$ ($B_{n+1} \le B_n/5$)

$$B_0, \ldots, B_{N-1} \ge 32,\ r_0 \le r_{\text{box}} \quad\Longrightarrow\quad r_n \le r_{\text{box}}, \qquad r_n \;\le\; \lambda^n r_0 + \frac{C}{(5-\lambda)B_n} \quad (n \le N) \quad (19)$$

Proved here (the assembly assuming (18) is in Lean: `uv_uniform` [$0 \le \lambda < 5$, $C \ge 0$, $B_n > 0$, $5B_{n+1} \le B_n$ and the one-step inequality], `stays_in_box`). The second term **is determined by the last scale alone, and does not depend on the number of steps** N: however many steps are run from the ultraviolet, the shape deviation at scale B is at most $\lambda^N r_0 + C/((5-\lambda)B)$. In the shape coordinate h/B the contraction is $17/125$ and the remainder is $1/B^2$. In numbers (one to three steps for $B_0 = 32$–512), starting points inside the box (the point of the class, a Gaussian, the heat kernel) stay below the upper bound of (18) at every step, and for $C/10$ the point of the class and the Gaussian break it at step 1 Reading (positive and negative controls).

Outside the box. A shape deviation $x = h_2/B$ of the $O(1)$ shape (for Wilson, $x = 1/24$) is outside the box. The negative side (fundamental + adjoint $x = -1/12$) lies within the upper bound of λ, but the positive side (Wilson) breaks the upper bound at step 1 ($\bar\lambda/\lambda = 1.018$ [$B_0 = 32$] $\to 1.001$ [512]); the cause is the one-loop quadratic term $c\,x^2$, $c = 11168/15625 \approx 0.715$ Proved here , Reading . For $U(1)$ the inhomogeneous term vanishes Reading .

Why it closed in this model. Four in a row is a system with only one block, and the explicit quadratic terms between blocks vanish by the zero-sum condition. It can be read that the Four-in-a-Row Theorem could become a complete example of the small-field contribution to the coupling because this row was empty Reading (15-08 § 06).

Table 2 Coordinates of the one-step map and their contraction ($SU(2)$, four in a row, the diagonal definition, main term of Laplace. On paper Proved here , measured values Reading)

Coordinate	One-step factor in units of $O(1)$	Factor in the shape coordinate ($\div B$, $B' = B/5$)	Measured value
B (the one flowing parameter)	$1/5$	—	$5B'/B = 0.998$ ($B = 256$)
ψ^4 (q, power γ 2)	$17/625$	$\lambda = 17/125 = 0.136$	ratio of iterations 0.02720; contraction of q/B 0.1223→0.1325 (β = 128→512)
ψ^6 (γ_3)	$13/3125$	$13/625$	ratio of iterations 0.00414
ψ^8 (γ_4)	$257/390625$	$257/78125$	—
General ψ^{2k}	$(4^k+1)/25^k$	$(4^k+1)/5^{2k-1}$	—

Table 3 Questions arising from this chapter (status: New; the question "one-step correction for four plaquettes in a row" has had its answer added)

Q.	Question	Size	What moves if settled	What does not move even if settled

Q.	Question	Size	What moves if settled	What does not move even if settled
8	Can the $O(1/B)$ correction of the one-step map of four in a row be derived on paper, giving the constant C of the one-step inequality $r_{n+1} \le \lambda r_n + C/B_n$ of the shape remainder condition (turning the geometric series under an assumption into one without the assumption, inside this model)? **Answer (in this model):** the Four-in-a-Row Theorem gave $C = 0.0108$ ($B \ge 32$, $r \le 0.05$). The remaining assumptions are the projection (discarding ψ^{10} and higher) and the size of the box; an estimate that includes shape deviations of the $O(1)$ type, such as Wilson's, is not yet done	Paper	The shape remainder condition of the attracting class	The class-closure condition on the four-dimensional lattice
9	Including loop shapes other than four in a row (2×2, L-shaped, mixed quartic terms), is the attracting class still determined as a single one (B + a fixed shape)? Does the fixed point stay put if the diagonal definition is replaced by a fit to the six quartic coefficients?	Paper and small computations	The number of parameters of the class-closure condition	One step of a true blocking

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What cannot be said

(a) The systems in this chapter are one plaquette up to four in a row of $SU(2)$, $\theta \le 0.3$, quadratic order in F (§ 07 and § 08 go up to the quartic Taylor coefficients); these are not the numbers for one step including blocks embedded in the four-dimensional lattice, gauge re-fixing, or large fields.

(b) "Two coordinates are needed at the quadratic level" is a reading for four in a row with two loops fixed; we have not looked at the quartic order (the direction of the commutator) or at systems with three or more block variables.

(c) That the family of powers of the prefactor has no root is a statement of numbers on the measured grid and of an inequality at the one-loop level, not a statement about all actions. The fundamental + adjoint root is a number at $\beta \ge 12$, and the coefficient of $-0.84/\beta$ is an extrapolation.

(d) The maps of § 07 and § 08 are maps for the diagonal definition $f' = [(f * f) * f^2]^{1/2}$ and the main term ($B \to \infty$); mixed quartic terms do not enter, and they cannot be used in the strong-coupling regime of $B = O(1)$. The one-step inequality of the shape remainder condition is in general an assumption. The Four-in-a-Row Theorem is a statement about four in a row, the diagonal definition, the projection model truncated at ψ^8, coordinates given by the analytic main part, $B \ge 32$, and a box of $O(1)$; some of the constants are numerical values of explicit integrals, and the exponentially small parts are rough estimates. It says nothing about one step of the four-dimensional lattice, block averages, large fields or the strong-coupling regime.

(e) We say neither "there is no action that closes with one parameter" nor "the class-closure condition closes with finitely many parameters". What we say is a main-term statement: in this model all the shape coordinates contract, and the attracting point is determined as a single one.

(f) This chapter does not show ultraviolet control.

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The five grades table and the dependency graph

Table 4 The rows of this chapter

Row	Statement (one line)	Grade	Basis	Scope
[01]	One plaquette: under the constraint $\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)}$ ((1)); the gauge directions do not depend on the background	Proved here	§ 01	The constraint $x_3 = 0$ is a substitute for the background-field constraint
[02]	$\beta_{\mathrm{eff}} = \beta - d/2$; for cos of $U(1)$ too, a η of the same type	Reading	§ 01	One plaquette, $\beta = 4$–64, $\theta \le 0.3$
[03]	Two plaquettes (2)(3), three plaquettes (4): the commutator is the $1/12$ of the coupling of the composite loop	Proved here , Reading	§ 02, § 03	BCH up to third order, $\beta = 4$–32
[04]	Four plaquettes: the ratio $\kappa_{ab}/\kappa_a = 2(\beta - \frac{1}{2})/(\beta - 1)$ moves ((5)–(7))	Proved here , Reading	§ 04	Main term at the one-loop level; numbers by two routes
[05]	$2 + 2t/3$ for the heat kernel; the criterion (9)	Proved here , Reading	§ 05	Class functions, boundary terms 0
[06]	General one-loop formula (10); one-loop root of fundamental + adjoint $-\frac{1}{4}$ ((11)); $D > 8/25$ at $\gamma = c\beta$	Proved here	§ 06	One-loop level, concentrated actions
[07]	No root in the prefactor family; fundamental + adjoint root $\rho^* = -\frac{1}{4} - 0.84/\beta$; one parameter for one step only	Reading	§ 06	$\beta \in \{4, \dots, 512\}$, measured grid, two routes
[08]	Tree map (12), image (13): deviation $9/89$	Proved here	§ 07	Tree (main term of the saddle point), the diagonal definition
[09]	Numbers for the deviation of the curve 0.160, 0.127, $0.113 \to 9/89$	Reading	§ 07	Starts from $\beta = 128$–512
[10]	(14)(15)(16): the contraction map of the power of the prefactor, fixed point $37/76$, $c = 2/3$	Proved here	§ 08	Main term of Laplace, four in a row, the diagonal definition
[11]	Numerical extrapolation of (15)(16), convergence of the iteration	Reading	§ 08	$B = 32$–256, two routes
[12]	The form of stating the class-closure condition as "the one-parameter attracting class + a contracting shape"; the questions "one-step correction for four plaquettes in a row" and "uniqueness of the class for other loop shapes"	Assumption	§ 09	The form of the statement; the one-step inequality is an assumption

Row	Statement (one line)	Grade	Basis	Scope
[13]	Four-in-a-Row Theorem: by the lift, one step is a Gaussian integral of three segments in series, (17), the one-step inequality (18), and (19), independent of the number of steps	Proved here , Reading (numerical values of the constants)	§ 09	$SU(2)$, four in a row, the diagonal definition, the projection at ψ^8, $B \geq 32$, a box of $O(1)$
[14]	Outside the box: Wilson's positive shape deviation breaks λ at step 1; the cause is $c\,x^2$ ($c = 11168/15625$)	Proved here , Reading	§ 09	The same model, $B_0 = 32$–512

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV ONE    one plaquette: (1)                              constraint x₃ = 0
[02] RDG BEFF   β_eff = β − d/2 ⟸ [01]
[03] PRV BCH    two, three plaquettes: commutator is the image of the composite loop ⟸ [0
[04] PRV FOUR   four plaquettes: the ratio moves with β ⟸ [03]   (one-loop level)
[05] PRV CRIT   heat kernel 2 + 2t/3, criterion (9) ⟸ [04]
[06] PRV D1     general one-loop formula (10)(11) ⟸ [05]   (concentrated actions)
[07] RDG ROOT   no root in prefactor family; fund. + adj. root ⟸ [06]
[08] PRV TREE   tree map (12), deviation 9/89 (13) ⟸ [06]   (diagonal definition)
[09] RDG GAP    numbers for the deviation ⟸ [07][08]
[10] PRV FIX    fixed point of the prefactor power 37/76, c = 2/3 ⟸ [06][08]   (main term
[11] RDG FIXN   numbers for the fixed point ⟸ [10]
[12] ASM CLS    statement of the attracting class, questions "one-step correction for four
[13] PRV UNIF   Four-in-a-Row Theorem: one-step constants and a bound independent of the nu
[14] PRV OUT    outside the box: c x² breaks λ ⟸ [13]
```

Every number to the right of "⟸" is smaller than the one on the left, and there are no cycles. The only assumption row is [12], the form of the statement and the questions.

What this chapter can say

Can say	Cannot say
Proved here The weight formulas (1)–(8) from one to four plaquettes; the criterion (9); the general one-loop formula (10)(11); the tree map and $9/89$; the linear contraction map of the power of the prefactor and its fixed point $37/76$, $c = 2/3$; the Four-in-a-Row Theorem (the projection model, $B \geq 32$, a box of $O(1)$; constants numerical)	The same numbers for one step of the four-dimensional lattice. Quartic coefficients; systems with three or more block variables
Reading The numbers showing the ratio moves with β; no root in the prefactor family; the fundamental + adjoint root; the numbers for the deviation of the curve; convergence of the iteration to the fixed point	"No root" for all actions. A paper formula for the correction of $O(1/B)$
Assumption The form of stating the class-closure condition as the one-parameter attracting class + a contracting shape	That class closing in four dimensions

Sources and reproduction

Item	Type	Source, tools
The form of the heat kernel of S^3 ($e^{-\psi^2/2t}\cdot\psi/\sin\psi$ and its mirror images), the character expansion, the Laplace method, composition of Van Vleck prefactors	Cited	Standard facts of harmonic analysis and asymptotic expansions on Lie groups (bibliography as in the table of sources of 15-04)
Equations (1)–(16), the criterion, the general one-loop formula, the tree map, the map of the power of the prefactor	Proved here	The proofs and algebra in the main text of this chapter. The algebra of the coefficients and the trigonometric identities are in the Lean bundle, re-verified (15-09 Table 3)
Four-in-a-Row Theorem, (17)–(19), $c = 11168/15625$	Proved here, Reading (numerical values of the constants)	The main text of this chapter. The lift is of a known type (the lift of adjoint-invariant functions; the original source has not been checked). The numbers are the record of this book's study (two implementations, by series and by rational functions of B; Gauss–Legendre quadrature; positive and negative controls). The assembly is in the Lean bundle (re-verified)
The numbers in Tables 1 and 2, and in § 06– § 08	Reading	The numerical records of this book's study (two routes, quadrature and closed formulas; quaternion Monte Carlo; two routes, the character series and a one-dimensional multiple integral; the Wilson and Villain controls of $U(1)$; positive control by the heat kernel)

Words used in this chapter — couplings by loop shape, the composite loop, the criterion, the curvature of the group, the Van Vleck prefactor, one parameter for one step only, the diagonal definition, the tree map, the power of the prefactor, the attracting class, the class-closure condition, the lift, the Four-in-a-Row Theorem, outside the box. Their meanings are in the A-02 glossary.

Next chapter: 15-07 Background-dependent fluctuations — the η of one plaquette in § 01, as a problem of absorption at each step and of induction over many steps.

Background-dependent fluctuations η — absorption step by step

This chapter is this book's own investigation. The established knowledge it builds on is in 15-02 and 15-01.

This part is a record of this book's study attempting ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is set out in 15-00.

What this chapter shows — In one step of renormalisation, the fine field is split into a "background" and a "fluctuation", and the fluctuation is integrated out. For a flat field (a scalar field) the law of the fluctuation does not depend on the background, and of the four properties of the weight, the property "closing in one step under the expectation of the fluctuations" (closure under the fluctuation expectation; (iii) of BBS) can be used as it is. For a group-valued field the law of the fluctuation depends on the background, and the size of that dependence, η_k, becomes the difficulty. In this chapter we set down on paper the group version (the Background–Fluctuation Decomposition Lemma) of submultiplicativity in sums of fields ((ii) of BBS), set down the group versions of the exponential moment of fluctuations and of background regularity in the form of statements, and write the one-step chain; and we state, conditionally, that with a naive estimate of η_k the sum $\sum_k \eta_k$ diverges, that when the law of the fluctuation does not depend on the background the chain goes through as it stands with $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations), a proposal to absorb η into the parameters as a shift of local couplings, and that the multi-step induction has not yet been written. Alongside, we state the version of the Lemma on One Step with Background-Independent Fluctuations for the one-link Wilson law (the One-Link Heat-Kernel Comparison Lemma), the order of the remainder that cannot be absorbed as measured in the smallest systems, the four-way splitting of the derivative in the background, the coefficient of the group-geometry part for $SU(2)$ (the cubic BCH term), and the divergence of its budget. In the revision we added the cost of re-taking the centre for the one-link Wilson law (the recentring

condition), the remainder and the budget when the class is given the background as a parameter, the update of the background in a one-step with a single loop as the coarse variable, and the coordinates of the multi-loop class in a three-dimensional box.

Prerequisite chapters — 15-01 § 02 (group-valued fields and the four properties of the weight), 15-02 § 04 (the chain of BBS Appendix A.6), 15-03 § 02 (the statement of the decomposition), 15-06 (the type of η for a single plaquette), 7-05 § 05 (elements of a group cannot be added).

Said first — This chapter does not show ultraviolet control. What we set down on paper is the Background–Fluctuation Decomposition Lemma, the one-step chain (assuming the group versions of the exponential moment of fluctuations and of background regularity), the Lemma on One Step with Background-Independent Fluctuations (only the single step in which the law of the fluctuation does not depend on the background), and a quaternion identity; none of them is a statement about a genuine step of renormalisation (with the constraint of block averaging, the propagator in a background field, the Wilson action, and large fields included). The absorption of η is a proposal, and the multi-step induction has not been written.

The order of this chapter

1. The measure of the fluctuations depends on the background — where ηk sits
2. The Background–Fluctuation Decomposition Lemma and the Splitting of the Weight — the group version of submultiplicativity in sums of fields (equations (1)(2))
3. The group versions of the exponential moment of fluctuations and of background regularity, and the interchange — statements and the one-step chain (equation (3))
4. A naive estimate of η and its divergence
5. The Lemma on One Step with Background-Independent Fluctuations — with background-independent fluctuations the chain goes through with $\eta = 0$ (Table 1); the one-link Wilson law (the One-Link Heat-Kernel Comparison Lemma); the cost of re-taking the centre
6. η = the proposal of shifting local couplings — the fact about the heat-bath method, the quaternion identity, screening by the order of θ, the remainder measured in the smallest systems, the four-way splitting, giving the class the background as a parameter
7. The cubic BCH term, the class (B, q, a), the budget, the one-face step, the multi-loop class, and the multi-step induction — question "splitting the density of the joint law", question "minimal multistep statement", ultraviolet questions

17–19, question "second step of the multi-loop class", question "class with the coupling as a parameter" (Table 2)

8. What cannot be said
9. The five grades table and the dependency graph (Table 3)

00

What this chapter shows

- For fields that take values in a group the measure of the fluctuations depends on the background, and η_k accordingly puts a factor $1+\eta_k$ into the one-step estimate. Unless $\sum_k \eta_k < \infty$, the constants break down with the number of steps, exactly as in the arithmetic of 15-01 § 01 (§ 01).
- The group version of the submultiplicativity in sums of fields of the weight (it splits into a product over a sum of fields) holds on paper in the form background $\times$ fluctuation (the Background–Fluctuation Decomposition Lemma, § 02). The group versions of the exponential moment of fluctuations and of background regularity can be set down as statements, and assuming them the one-step chain closes in the same form as for flat fields (§ 03).
- Estimating η_k naively from the curvature of the background, the sum diverges as a power of the number of steps or a power of its logarithm, and cannot be carried as an error (§ 04).
- When the law of the fluctuation does not depend on the background (a product of heat kernels link by link is an example), the chain goes through as it stands with $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations, § 05). The fluctuation of a genuine step is not of this type.
- For the Wilson action of $SU(2)$, the conditional law of one link is a member of a one-parameter family that does not depend on the background, and the background enters only through the local coupling and the direction of the centre. Using this we set down a proposal to absorb η as the shift of local couplings (§ 06, proposal). The multi-step induction has not been written (§ 07).
- For the one-link Wilson law, the law of the fluctuation is bounded inside the angle cutoff by a constant multiple of the heat kernel, and the chain of the Lemma on One Step with Background-Independent Fluctuations goes through conditionally (the One-Link Heat-Kernel Comparison Lemma, § 05).
- Measured in the smallest systems, the lowest orders of the non-absorbable remainder are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$. The derivative in the background splits into four:

coupling, rotation, shape and group geometry. The shape part enters a geometric series, but the group-geometry part (the cubic BCH term, coefficient $1/9$) of $SU(2)$ rides on the background itself and does not contract, and its budget diverges however the threshold is chosen. The improvement lies on the side of the class — letting the parameters carry the field strength of the background (§ 06, § 07).

- If the class is given the logarithm of the background holonomy a as a parameter, the remainder in the smallest systems drops to $\theta^0 \beta^{-3/2}$ and the budget becomes uniform in the number of steps (conditionally, § 06). The divergence has not disappeared; it has moved to the question of whether the class closes in one step.
- In one step whose coarse variable is a single loop, the update of a closes to all orders by the ordered product in the group. In the multi-loop step of a three-dimensional box, the class closes up to second order with per-face classes and neighbouring pairs; at third order the corner triple product is needed, and at fourth order multi-face couplings (§ 07). The corner triple product does not contract at the second step; it is a coordinate that approaches the fixed point of the flow from below. The four-dimensional class consists of one parameter and the ratio coordinates whose values are fixed by the flow (2 at second order, 3 at third, 65 at fourth), and the coordinate of "the skew neighbour", which shares no vertex, is not 0 whatever the rule (§ 07; within the measured range of tree level and one loop).
- Re-taking the centre for the one-link Wilson law produces a loss of about a factor 4 in the Taylor radius and a motion of the coupling $c g^2 p^2$. The latter is a term to be carried over to the side of the parameters of the class (§ 05).

01

The measure of the fluctuations depends on the background — where η_k sits

In one step at stage k we split the edge variable into background and fluctuation as $U_l = W_l\, e^{A_l}$ ($W_l \in G$, and A_l an element of the Lie algebra $\mathfrak{su}(N)$), hold the background W fixed and integrate out the fluctuation A. We write the law of this fluctuation as $\mu_{k,W}$. The covariance of its Gaussian part is the propagator in a background field $g_k^2 C_W$ (the title of the foundational papers in 7-07 Table 1).

For flat fields (a scalar field $\phi = \phi_{\mathrm{bg}} + \zeta$) the law of the fluctuation ζ is a Gaussian law that does not depend on the background, and the closure of the weight under the fluctuation expectation (closing in one step under the expectation of the fluctuations) is used as the expectation of a function of the fluctuation alone Reading (from reading the chain of 15-02 § 04). On the group this is not so. When the background changes, C_W changes, and the law of the fluctuation itself changes. We write the size of this dependence as η_k and put it into the one-step estimate as a factor $1 + \eta_k$; this is how this chapter sets things up (equation (3)). From the arithmetic of 15-01 § 01, for the

product of factors $\prod_k(1+\eta_k)$ to be bounded uniformly in the number of steps K one needs $\sum_k \eta_k < \infty$, and if η_k is of size g_k^2 it is not enough Proved here (arithmetic).

02

The Background–Fluctuation Decomposition Lemma — the group version of submultiplicativity in sums of fields

Let $l_1,\dots,l_4$ be the edges on the boundary of a plaquette p and $s_i \in \{\pm 1\}$ their orientation signs, set $U_{\partial p} = \prod_i U_{l_i}^{s_i}$, and define $W_{\partial p}$ in the same way. The norm is the operator norm.

Background–Fluctuation Decomposition Lemma. For $G = SU(N)$,

$$\|U_{\partial p}-1\| \;\le\; \|W_{\partial p}-1\| + \sum_{i=1}^{4}\|A_{l_i}\|, \qquad \|U_{\partial p}-1\|^2 \;\le\; 2\|W_{\partial p}-1\|^2 + 8\sum_{i=1}^{4}\|A_{l_i}\|^2 \quad (1$$

Proved here Proof. (1) Split each factor into a W part and a fluctuation part. If $s=+1$, $We^{A} = W\cdot Y$ ($Y = e^{A}$); if $s=-1$, $(We^{A})^{-1} = W^{-1}\cdot Y$ ($Y = We^{-A}W^{-1}$). In both cases Y is a conjugate of $e^{\pm A}$. (2) In the product $W_1'Y_1W_2'Y_2W_3'Y_3W_4'Y_4$ ($W_i' = W_{l_i}^{s_i}$), move each Y_i to the right of the W' to its right by $YW' = W'(W'^{-1}YW')$; then $U_{\partial p} = W_{\partial p}\,Y_1''Y_2''Y_3''Y_4''$, and each Y_i'' is a conjugate of $e^{\pm A_{l_i}}$. (3) For unitary matrices, repeating $\|XY-1\| \le \|X-1\| + \|Y-1\|$ and $\|hYh^{-1}-1\| = \|Y-1\|$ (equation (8) of 7-05) gives $\|U_{\partial p}-1\| \le \|W_{\partial p}-1\| + \sum_i \|e^{\pm A_{l_i}}-1\|$. (4) A is anti-Hermitian and so unitarily diagonalisable, and for an eigenvalue $e^{i\vartheta}$, $|e^{i\vartheta}-1| = 2|\sin(\vartheta/2)| \le |\vartheta| \le \|A\|$. Hence $\|e^{\pm A}-1\| \le \|A\|$. (5) The squares follow from $(x+y)^2 \le 2x^2+2y^2$ and $(\sum_{i\le 4} y_i)^2 \le 4\sum_i y_i^2$. ∎

Splitting of the Weight (a corollary of the Background–Fluctuation Decomposition Lemma). Setting $G_c(X,U) := \exp\bigl(c\sum_{p\subset X}\|U_{\partial p}-1\|^2\bigr)$ ($c>0$), since on a four-dimensional lattice an edge lies on at most 6 plaquettes,

$$G_c(X, We^{A}) \;\le\; G_c(X,W)^2\cdot\exp\Bigl(48c\sum_{l\in X}\|A_l\|^2\Bigr) \quad (2)$$

Proved here (sum the second of (1) over $p\subset X$; $8\times 6 = 48$). This is the version of submultiplicativity in sums of fields in which the sum of fields is replaced by "background × fluctuation", and on the fluctuation side the weight is of Gaussian type. It may be a known elementary fact (an estimate of the same form may appear in the large-field papers of the series, but we have not checked the text).

The form that writes the fluctuation with Y. Steps (1)–(3) go through with e^A replaced by any unitary Y. From $\|Y^{-1}-1\| = \|Y-1\|$ and conjugation invariance, $\|(WY)_{\partial p}-1\| \le \|W_{\partial p}-1\| + \sum_{i\le 4}\|Yl_i-1\|$, and with q the maximal number of plaquettes on which one link lies (6 in four dimensions), $G_c(X,WY) \le G_c(X,W)^2 \exp\big(8qc\sum_{l\in\bar X}\|Yl-1\|^2\big)$ Proved here ($\bar X$ are the links on the boundary of the plaquettes of X). In this form no small-field condition on the fluctuation is needed (since $\|e^A-1\|\le\|A\|$ of step (4) is not used). The Lemma on One Step with Background-Independent Fluctuations in § 05 uses this form.

What the Background–Fluctuation Decomposition Lemma does not say. (2) speaks only of submultiplicativity in sums of fields and does not touch on closing in one step under the expectation of the fluctuations (closure under the fluctuation expectation).

03

The group versions of the exponential moment of fluctuations and of background regularity, and the interchange — statements and the one-step chain

Let the weight at step k be $G_k := G_{c/h_k^2}$ (h_k is the field scale at step k, for example the small-field threshold $g_k p(g_k)$; $p(g)$ is the factor of the threshold in equation (7) of 7-05), let $\mathcal{W}_k(X)$ be the set of small-field (regular) backgrounds on X, and let $B(X)$ be the set of blocks in X. We set down the following three as statements Assumption (statements only; the constant c, t_0 and the constants in the definition of $\mathcal{W}_k$ depend neither on the step nor on the lattice spacing).

- **Group version of the exponential moment of fluctuations**: for all $W\in\mathcal{W}_k(X)$ and $0<t\le t_0$, $\mathbb{E}_{\mu_k,W}\exp\big(\frac{48tc}{h_k^2}\sum_{l\in X}\|A_l\|^2\big) \le 2^{|B(X)|}$. The expectation is with respect to the true measure of the fluctuations, not only its Gaussian part.
- **Background-regularity condition**: for W in the image of the background map, $G_k(X,W)^{2t} \le G_{k+1}(X,W)$ ($0<t\le t_0$). Its content is the regularity of the background (the curvature of the fine plaquettes being bounded by the curvature of the coarse plaquettes).
- **Group version of exchanging the expectation and the seminorm**: let $\theta_W F(W) := F(We^A)$, and let $T_W(h)$ be the Taylor seminorm built from derivatives in the Lie-algebra directions of the background. For some $\eta_k\ge 0$, $\|\mathbb{E}_{\mu_k,W}\theta_W F\|_{T_W(h_{k+1})} \le (1+\eta_k)\,\mathbb{E}_{\mu_k,W}\|F\|_{T_{We^A}(h_k)}$, and $\sum_k \eta_k<\infty$. This is a version of the original inequality exchanging the expectation and the seminorm ((A.6.5) and others of BBS) that allows a factor $1+\eta_k$ for the dependence of the measure on the background.

Assuming these three and the Splitting of the Weight (2) (with c replaced by c/h_k^2), the same chain as in the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) (interchange → definition of the norm → (2) → group version of the exponential moment of fluctuations → background-regularity condition) gives, on $W \in \mathcal{W}_k(X)$,

$$\left\| \mathbb{E}_{\mu_k, W}\, \theta W\, K(X) \right\|_{G_{k+1}} \;\le\; (1+\eta_k)\, 2^{|B(X)|}\, \|K(X)\|_{G_k} \quad (3)$$

Proved here (only the chain, assuming the three). Outside $\mathcal{W}_k$, the large-field factor of the large-field rarity condition takes over (15-00 § 01). The group version of the closure under the fluctuation expectation itself has a form that closes with three things: an estimate of the propagator C_W uniform over regular backgrounds, a uniform estimate of the deviation of $\mu_{k,W}$ from Gaussian, and $c g_k^2/h_k^2$ being small Reading (from the formula that writes Gaussian exponential moments as determinants).

For a product of one-link heat kernels, the group version of the closure under the fluctuation expectation holds for free. When $Y_l = W_l^{-1} U_l$ follows the heat kernel p_τ independently on each edge, the invariance of Haar measure (invariant on both left and right, since the group is compact) makes the law of Y independent of W, and $\mathbb{E}\exp\big(s\sum_l \|Y_l - 1\|^2\big) = \big(\mathbb{E}\exp(s\|Y-1\|^2)\big)^{|X|}$ ($|X|$ is the number of edges of X) does not depend on the background Proved here. This is at most $2^{|B(X)|}$ when $s\tau$ is small Reading (from the Gaussian-type upper bound on the heat kernel; the numbers are in Table 1 of § 05). The telescoping sum of 15-04 § 01 can be read as another face of these "fluctuations independent of the background" Reading.

04 A naive estimate of η and its divergence

Estimating from the curvature. A naive estimate of the part by which the measure depends on the background, through the curvature $\|W_{\partial p} - 1\| \lesssim g_k p(g_k)$, gives $\eta_k \sim g_k p(g_k)$, and for the sequence of one-loop couplings $g_{K-j}^2 \le 1/(bj)$ (15-01 § 01) $\sum_k \eta_k$ diverges at the order of $K^{1/2}$ (up to logarithmic factors) Proved here (arithmetic, under the naive estimate $\eta_k \sim g_k p(g_k)$).

Measuring with a single plaquette. In the smallest model of a single plaquette (15-06 § 01), if we impose a constraint that removes the directions of the generators of the background, the background dependence enters exactly through a single replacement of the coupling $\beta \mapsto \beta\cos(\theta/2)$ ($\mu_W^\beta = \mu_1^{\beta\cos(\theta/2)}$) Proved here. η measured by the ratio of covariances is $\theta^2/8$; below the threshold $\theta \le \varepsilon_k = g_k p(g_k)$ it is $\eta_k \lesssim g_k^2 p(g_k)^2$, and under the one-loop flow $\sum_{k \le K} \eta_k$ diverges as a power of $\ln K$ Proved here (the arithmetic of the naive estimate). Without the constraint (not taking the background at

a critical point), the ratio of covariances is $\beta\theta^2/12$ and grows with β Reading . Under either estimate, η **is not of a size that can be carried as an error, and absorption into the parameters is needed**. The single-plaquette example is the smallest example in which that absorption happens exactly.

Which η? Measuring η by a single number changes its type (the ratio of covariances is θ^2, the Kullback–Leibler quantity is θ^4, and the score, i.e. the first-order rate of change of the expectation when the background is moved, is θ) Reading . The group version of the exponential moment of fluctuations uses an even quantity and the interchange inequality (the group version of exchanging the expectation and the seminorm) a first-order one, so the statements must specify "which η". For a scalar field the measure does not depend on the background, so $\eta_k = 0$ and the problem itself does not arise.

05

The Lemma on One Step with Background-Independent Fluctuations — the chain goes through with $\eta = 0$

There is one case in which the chain of § 03 goes through without assumptions: when the law of the fluctuation does not depend on the background.

Setting. $G = SU(N)$, operator norm. A finite set of links Λ, a partition into blocks, X a union of blocks, $B(X)$ the blocks in it, $\bar{X}$ the links on the boundary of the plaquettes of X. The weight $G_h(X,U) := \exp\big((c/h^2)\sum_{p\in X}\|U_{\partial p}-1\|^2\big)$. The Taylor seminorm is $\|F\|_{T_U}(h) := \sum_{n\geq 0}\frac{h^n}{n!}\|D^nF(U)\|$, where $\|D^nF(U)\|$ is the supremum over $|X_j|\leq 1$ of the partial derivative at $t=0$ of right multiplication $F(Ue^{t1X1}\dots e^{tnXn})$. The weighted norm is $\|K(X)\|_{G_h} := \sup_U \|K(X)\|_{T_U}(h)/G_h(X,U)$. The fluctuation is a probability measure ν on G^Λ, and one step is $\theta_\nu K(W) := \int K(WY)\,d\nu(Y)$. $M_\nu(s,X) := \mathbb{E}_\nu \exp\big(s\sum_{l\in\bar{X}}\|Yl-1\|^2\big)$.

Lemma on One Step with Background-Independent Fluctuations. Assume the following three. Background-independence condition for fluctuations: ν does not depend on the background W. Scale-contraction condition: $h_+^2 \leq h^2/2$. Exponential-moment condition for fluctuations: $M_\nu(8qc/h^2, X) \leq 2^{|B(X)|}$. Then for every background $W \in G^\Lambda$ and smooth $K(X)$,

$$\|\theta_\nu K(X)\|_{G_{h_+}} \;\leq\; 2^{|B(X)|}\,\|K(X)\|_{G_h} \quad (4)$$

Moreover the following hold Proved here (conditions: the three conditions of background independence of the fluctuations, scale contraction and the exponential moment of fluctuations, and one step only).

- (a) The interchange is $\eta = 0$: for all smooth F, $\|\theta_\nu F\|_{T_W}(h) \le \mathbb{E}_\nu \|F\|_{T_{WY}}(h)$.
- (b) Locality: if $\nu = \otimes_l \nu_l$ (independent link by link), then $\theta_\nu K(X)$ depends only on the restriction of W to $\bar{X}$, and if $\bar{X} \cap \bar{X}' = \emptyset$ it splits into a product. The support does not spread in one step.
- (c) For a product of heat kernels link by link $\nu = p_\tau^{\otimes \Lambda}$, $M_\nu(s, X) = m(s,\tau)^{|\bar{X}|}$ and $m(s,\tau) := \mathbb{E}_{p_\tau} e^{s\|Y-1\|^2}$, independent of the background. An elementary upper bound is $m(s,\tau) \le 1 + (e^{4s} - 1)\,\mathbb{E}_{p_\tau}\|Y-1\|^2/4$. For $SU(2)$ (with the convention for the character coefficients $(n+1)e^{-n(n+2)/(2\beta)}$), $\mathbb{E}\|Y-1\|^2 = 2(1 - e^{-3/(2\beta)}) \le 3/\beta$.
- (d) Small fields: ν', obtained by conditioning $\nu = \otimes \nu_l$ on $\{\|Y_l - 1\| \le h' \ \forall l\}$, also satisfies the background-independence condition for fluctuations, and $M_{\nu'} \le M_\nu$. If the background is taken in $\mathcal{U}_\varepsilon := \{\max_{p \in X} \|W_{\partial p} - 1\| \le \varepsilon\}$, the supremum of the norm on the right-hand side need only be taken over $\mathcal{U}_{\varepsilon + 4h'}$.

Proved here Proof. **Step 1 (submultiplicativity in sums of fields)**: from "the form that writes the fluctuation with Y" of § 02, $G_h(X, WY) \le G_h(X,W)^2 \exp\big((8qc/h^2)\sum_{l \in \bar{X}} \|Y_l - 1\|^2\big)$. **Step 2 ((a))**: inserting $Y^{-1} e^{tX} Y = e^{t\,\mathrm{Ad}_{Y^{-1}} X}$ between the factors gives $F(We^{t_1 X_1} \dots e^{t_n X_n} Y) = F(WYe^{t_1 X'_1} \dots e^{t_n X'_n})$, $X'_j := \mathrm{Ad}_{Y^{-1}} X_j$, $|X'_j| = |X_j|$. Right multiplication only rotates the direction of differentiation by the adjoint, and the adjoint preserves the operator norm. F is smooth on a product of compact groups, so we can differentiate under the integral (dominated convergence), and **since ν does not depend on W, the derivative falls only on the integrand**. Hence the derivative of order n is at most $\int \|D^n F(WY)\| d\nu(Y)$; multiply by $h^n/n!$ and sum. **Step 3 (the chain)**: in the order (a) → definition of the norm → Step 1 → exponential-moment condition for fluctuations → scale-contraction condition, $\|\theta_\nu K(X)\|_{T_W}(h) \le \|K(X)\|_{G_h} G_h(X, W)^2\, 2^{|B(X)|} \le \|K(X)\|_{G_h} G_{h_+}(X,W)\, 2^{|B(X)|}$. The last follows from the scale-contraction condition: $G_h^2 \le G_{h_+}$. (b) follows from Fubini and independence; (c) from independence, the left and right invariance of Haar measure, the convexity of e^{sz} on $z = \|Y-1\|^2 \in [0,4]$ (the chord inequality), $\|Y-1\|^2 = 2 - \mathrm{tr}\, Y$ and $\mathbb{E}\,\mathrm{tr}\, Y = 2e^{-3/(2\beta)}$. (d) follows from the fact that conditioning a function increasing in each coordinate on a lower set does not increase its expectation (the coordinates being independent), and from the first inequality of Step 1. ∎

Table 1 The numbers for the exponential-moment condition for fluctuations: the largest s/β (in steps of 5×10^{-4}) with $m(s,\beta)^{96} \le 2$, for $SU(2)$, heat kernels edge by edge, $n_b = 96$ (a rough count of the links in the closure of a block in four dimensions with factor 2) Reading

β	4	8	16	32	64	128
Largest s/β	0.0025	0.0025	0.0025	0.0020	0.0020	0.0020

Table 1 is a numerical record of this book's study (trapezoidal quadrature in the angular direction and a sum over characters; as a positive control it agrees with the closed form for $\mathbb{E}\|Y-1\|^2$ to 9 digits). It agrees with $\ln 2/(3n_b) \approx 0.0024$ from the Gaussian estimate $(1-2s/\beta)^{-3/2}$. Reading $s = 8qc/h^2$, $\beta = 4/g^2$, the condition is $c\,g^2/h^2 \lesssim 1.7\times 10^{-4}$ ($q=6$, $n_b = 96$) Reading . If $h = g\,p(g)$ then $c/p(g)^2 \lesssim 1.7\times 10^{-4}$, and since $p(g)$ grows as a power of the logarithm, it is satisfied at fine steps.

What is not in the Lemma on One Step with Background-Independent Fluctuations. This is the precise edge of "the part that can be closed" Reading .

1. **All of the requirement that the law of the fluctuation not depend on the background (the background-independence condition for fluctuations).** The $\mu_{k,W}$ of a genuine step depends on W through three things: the non-quadratic nature of the action (the Wilson action: $\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)}$ for a single plaquette), the constraint of block averaging (the constraint surface depends on W), and the propagator in a background field C_W (fluctuations of neighbouring links coupled by plaquette terms that depend on W). The place where "the derivative falls only on the integrand" of Step 2 breaks down is η itself.
2. **Relation to the heat-kernel action.** The fluctuation of the Lemma on One Step with Background-Independent Fluctuations is "a convolution of heat kernels link by link", and the heat-kernel action plaquette by plaquette takes the form of a product over edges only when the plaquette variables become independent coordinates (two dimensions, axial gauge). In three or more dimensions the cube constraints tie plaquettes together (its manifestation is the source of 15-05).
3. **One step only.** Stacking the scale-contraction condition over k steps, the weight collapses as $h_k \le 2^{-k/2}h_0$. Many steps need the blocking factor, the background-regularity condition, and a contraction of the many-body norm that cancels $2^{|B(X)|}$ (§ 07).
4. **The rarity of large fields does not come out.** Since $G_h(X,U) \le e^{4c|X|/h^2}$, this norm does not produce the $e^{-c\,p(g)^2|\Omega^c|}$ of the large-field rarity condition. (d) says only that "a small-field background spreads in one step only to $\varepsilon \to \varepsilon + 4h'$".
5. **The Wilson action is not treated here** (the η of non-quadratic nature). Restricted to one link, it can be treated by the One-Link Heat-Kernel Comparison Lemma below.
6. **Correspondence with the small-field analyticity condition.** Of "the analyticity and decay of localised corrections" in the small-field analyticity condition, what the Lemma on One Step with Background-Independent Fluctuations contains is that the support does not spread in one step ((b)) and that the Taylor norm of radius h is preserved up to a factor $2^{|B(X)|}$ ((4)); it does not contain the uniformity of the decay $e^{-\kappa d(X)}$ or constants uniform in the step.

Towards the one-link Wilson law — the One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma. If only one link is made to fluctuate, and the background is taken at its conditional maximiser (the direction of the sum of staples $\hat{q}(W)$, § 06), the law of the fluctuation Y is μ^b ($b = \beta|Q(W)|$), and depends on the background only through b. Inside the angle cutoff $\theta_0 \in (0, \pi)$, μ^b is bounded from above by K times the heat kernel $p_{1/b'}$ ($b' = b(1 - \theta_0^2/12)$), and

$$\mu^b(\theta) \;\le\; K(b, \theta_0)\, p_{1/b'}(\theta) \quad (\theta \le \theta_0), \qquad K(b, \theta_0) \;\to\; \Big(1 - \frac{\theta_0^2}{12}\Big)^{-3/2} \quad (b \to \infty)$$

K is attained at $\theta = 0$ and is optimal, and the weight outside θ_0 is at most $\sqrt{2/\pi}\, b^{1/2} e^{-b(1-\cos\theta_0)}(1 + O(1/b))$ Proved here (the One-Link Heat-Kernel Comparison Lemma; $SU(2)$, one link, class densities with respect to Haar). The proof is three one-liners: the Taylor remainder $1 - \cos\theta \ge (\theta^2/2)(1 - \theta_0^2/12)$ ($\theta \le \theta_0$, $\theta_0^2 \le 24$), the lower bound of the heat kernel (the leading term of the sum over images and the Van Vleck prefactor $\theta/\sin\theta \ge 1$), and a chain of constants. The Taylor remainder and the algebra of the constants are in Lean: `one_sub_cos_window`, `wilson_weight_window` (assumptions $0 \le \theta \le \theta_0$, $\theta_0^2 \le 24$, $b \ge 0$), `window_constant_chain`, `window_split_bound`. By this comparison the chain of the Lemma on One Step with Background-Independent Fluctuations goes through for the one-link Wilson law as well (the One-Link One-Step Fluctuation Lemma), and the loss splits in two: K, applied to the heat-kernel quantity (the side of the closure under the fluctuation expectation), and the factor $1 + \eta$, $\eta \approx \frac{3}{2}\,\delta_b/b$ coming from the movement of the coupling (δ_b is the Taylor width of $b(W)$; the side of exchanging the expectation and the seminorm) Proved here (conditions: one link, one step, the background at the conditional maximiser, the geometry of re-taking the centre (the recentring condition) assumed, $\delta_b < b$). The former can be made small by the choice of the angle cutoff, but the latter does not depend on θ_0 and is $\delta_b/b \sim g^2 p^2$ around a flat background, so the sum diverges like a power of the logarithm Reading. Numbers ($b = 4$–64, $\theta_0 = 0.3, 0.5, 1$): the paper K agrees with the true supremum to 10^{-10}, and the negative control that drops $\theta_0^2/12$ breaks at the edge of θ_0 in 7 of 15 cells at $SU(2)$ and in all 15 cells at $U(1)$ Reading ($U(1)$ has no prefactor). The remainder of many links with a shared plaquette (§ 06) lies outside this lemma.

The cost of the recentring condition. If all the angles of the background plaquettes are at most $\theta_0 < \pi/2$, the size of the sum of staples is $|Q| \ge \sum_p \cos\theta_p \ge n_s \cos\theta_0$ ($n_s = 6$ in four dimensions, with equality for the paired configuration), and for $\theta_0 = \pi/2$ it can be 0 Proved here ($SU(2)$ and $U(1)$, one link; Lean: `sum_norm_ge_of_inner_ge`, assumptions: real inner-product space, $\|u\| = 1$, $\langle x_i, u\rangle \ge c$). The first-order derivative of the direction of the centre $\hat{q}$ with respect to the background is at most $18/|Q| \le 3/\cos\theta_0$, and exactly 3 for a flat background (because one staple carries three links;

independent of the dimension) Proved here . Hence, if the chain of the Lemma on One Step with Background-Independent Fluctuations is run after re-taking the centre, the input Taylor radius has to be about 4 times the output one ($h_q \approx (1 + 3/\cos\theta_0)h$) Proved here (the composition step is a sketch, under the convergence domain of the majorant series $h < 0.075$), Reading (4.02–4.24 times measured in one direction). The coupling $b = \beta|Q|$ sits at a first-order stationary point for a flat background, and its relative motion is $\delta_b/b \le 6h\sin\theta_0 + 13.5h^2$ Proved here (leading term, $h < 0.11$), $\approx 3\tan\theta_0\, h + 4.5h^2$ measured in one direction Reading . At the threshold $h = \theta_0 = gp$ it is $\eta \approx \frac{3}{2}\delta_b/b \approx c\, g^2p^2$ (measured $c \approx 12$, on paper $c \le 29$), one power of g^2 short of the budget assumption of § 06, and the sum diverges as a power of the logarithm Proved here (arithmetic). The rotation of the centre can be moved from the measure to the loss of radius, but **the motion of the coupling has nowhere to go except into the parameters of the class (the coupling b)** Reading .

06

η = the proposal of shifting local couplings

The Lemma on One Step with Background-Independent Fluctuations cannot be used once the background-independence condition for fluctuations breaks down. So we consider the direction of splitting the genuine $\mu_{k,W}$ into "a law independent of the background ν_k (to which the Lemma on One Step with Background-Independent Fluctuations applies) $\times$ a density $\rho_{k,W}$", and extracting the part of the background dependence of the density that can be absorbed into the parameters. The statements of this section are proposals, except for one fact (the basis of the heat-bath method) and one identity.

Fact: the conditional law of one link is a member of a one-parameter family. For the Wilson action of $SU(2)$, let $Q(W)$ be the sum, as quaternions, of the products of each plaquette with the link U removed (the staples), and write $Q = |Q|\hat{q}$ ($\hat{q} \in SU(2)$). The conditional law of U with the other links held fixed is $\exp(\beta\langle Q, U\rangle)\, dU$, and setting $U = \hat{q}Y$, the left invariance of Haar measure makes the law of Y equal to $\exp(\beta|Q|\,\mathrm{Re}\,Y)\, dY = \mu_1^{\beta|Q|}$. **The conditional law is a member of the one-parameter family $\{\mu_1^b\}$, which does not depend on the background, and the background enters only through the local coupling $b(W) = \beta|Q(W)|$ and the direction of the centre $\hat{q}(W)$** Cited (a basic fact of the heat-bath method for $SU(2)$; as bibliography we give Creutz 1980 and Kennedy–Pendleton 1985, but we have not checked the text, i.e. unconfirmed). The $\beta \mapsto \beta\cos(\theta/2)$ of a single plaquette (§ 04), and the closed forms of the weights for two and three plaquettes (15-06 § 02, § 03), can both be read as examples of this one sentence Reading .

Identity: how to split a many-link action. Applying the unit-quaternion identity $\mathrm{Re}(ab) = \mathrm{Re}\,a\,\mathrm{Re}\,b - \mathrm{Im}\,a \cdot \mathrm{Im}\,b$ to the rearrangement $U_{\partial p} = W_{\partial p} Y_p$ of the Background–Fluctuation Decomposition Lemma (Y_p is the product of the fluctuation parts), with θ_p the rotation angle of $W_{\partial p}$ and n_p its axis,

$$\beta\,\mathrm{Re}(W_{\partial p} Y_p) \;=\; b_p(W)\,\mathrm{Re}\,Y_p \;-\; \beta \sin\frac{\theta_p}{2}\,\langle n_p, \mathrm{Im}\,Y_p\rangle, \qquad b_p(W) := \beta\cos\frac{\theta_p}{2} \tag{5}$$

Proved here (convention for $\frac{1}{2}\,\mathrm{tr} = \mathrm{Re}$; quaternion algebra). The first term is **an action of background-independent form with local coupling** $b_p(W)$, and the second is a term whose leading part is first order in the Lie-algebra fluctuation. In addition there remains a "conjugation mismatch" term, coming from the same Y_l being conjugated by a different partial product of the background in a neighbouring plaquette (of triple-product type).

The proposal of screening by order: screen the score by the order in θ. We divide the background derivative (the score) of the density $\rho_{k,W}$ according to its order in the background angle θ Assumption (a reading of the screening; not yet checked around the true critical point for many plaquettes).

- The θ^0 term (a first-order term independent of the background): it vanishes once the centre of the background is moved to the true constrained critical point. For two plaquettes the action becomes even in the fluctuation around the true critical point, and for three plaquettes the weight becomes a function of $|1 + W_a + W_b|$ (15-06 § 02, § 03, both [Proved here]).
- The θ^1 term: the derivative of $b_p(W)$ (the coupling derivative), and the anisotropic part of the quadratic form of the fluctuation left after recentring. The former is absorbed by the proposal of extending the coupling to a variable, and the latter is expected to be absorbable in the same way, as the derivative of a local anisotropic coupling.
- The non-absorbable remainder: **in the smallest systems, the lowest orders of the remainder are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$ (not θ^2 or higher), and the term multiplied by the width of the threshold $h_k \sim g_k p(g_k)$ is of type $g_k^2 p$. However the threshold is chosen, the sum diverges like a power of the logarithm** (the same type as the $\sum g_k^2$ of 15-01 § 01; see the paragraph below and § 07), and the improvement lies not in the choice of the threshold but on the side of the class — letting the parameters carry the field strength of the background Reading (a reading that carries the numbers for two links and for four in a row over to the four-dimensional step).

The remainder measured in the smallest systems. For two links sharing one plaquette and for four plaquettes in a row, the remainder after subtracting from the

score with respect to the background angle θ the "coupling derivative (the three $\operatorname{Re} Y_p$)" and the "rotation of the centre (the 12 generators of left and right translations)" is

$$\eta^{\mathrm{res}} \;\approx\; \sqrt{A^2/\beta + D^2\theta^2}, \qquad A = 0.04\text{–}0.14 \text{ (depending on the system)}, \quad D \approx 0.018 \; (SU(2$$

Reading ($SU(2)$, $\beta = 4$–32 [partly 128], $\theta = 0.05$–0.3 [partly 1.0], two routes). If the frame is taken at the true joint critical point, the action of two links splits exactly into $\sum_p b_p \operatorname{Re} Y_p - s\, n \cdot J$, $J = \operatorname{Im}(Z_1 Z_2) - \operatorname{Im} Z_1 - \operatorname{Im} Z_2$ Proved here (quaternion algebra; Lean: `two_link_split`, `re_mul_split` and others, ring identities under assumptions on the components), and if J is added to the coordinates — making it the family of quaternion couplings per plaquette $(b_p,\ s_p n_p)$ — the remainder is at most 2.4×10^{-10} Reading . $\theta^0 \beta^{-1/2}$ is the cubic term of J (the non-quadratic nature of the action) and appears with the same size in $U(1)$ as well; $\theta^1 \beta^0$ is the anisotropic quadratic form left after absorbing the commutator by conjugation, and is present only in $SU(2)$ Reading .

The derivative in the background splits into four. Write the weight of the class function of one plaquette as $F(U) = G(|\log U|^2)$, $G(u) = \mathrm{const} - \frac{B}{2}u + qu^2 + \cdots$ (B is the coupling, q the shape coordinate; $|\log U|$ is the half angle); then for the background $W = e^a$, from $|\log(e^a e^y)|^2 = |y|^2 + 2a \cdot y + O(a^2)$ (to all orders in y)

$$F(e^a e^y) = G(|y|^2) - B\,(a \cdot y) + 4q\,(a \cdot y)|y|^2 + \cdots + O(a^2) \quad (6)$$

Proved here . The background score splits into (i) coupling (even, $O(a^2)$), (ii) rotation ($-B\, a \cdot y$), (iii) **shape** (odd cubic $4q(a \cdot y)|y|^2$: the coefficient is the product of the value of the shape coordinate q and the background), and (iv) **group geometry** (only $SU(2)$: the cubic BCH term $n \cdot [y_1 \times (y_1 \times y_2) + y_2 \times (y_2 \times y_1)]$, independent of q) Proved here . For the two links of $U(1)$ the remainder is exhausted by the shape, and $\eta \approx \frac{8}{3}(q/B)(\beta B)^{-1/2}$ agrees with the measurement to within 0.7% Reading ($U(1)$, the Gaussian leading term, $\theta \le 0.3$, $\beta = 8$–64). In a chain in a row the convolution of class functions carries the background exactly to the coarse holonomy ($(f_{W_1} * f_{W_2})(x) = (f * f)(W_1 x W_2)$, of a known type), so the contraction of the shape coordinate $\lambda = 17/125$ does not change with the background, and **the shape part enters a geometric series** Proved here (a chain in a row, assuming the one-step inequality; the bookkeeping is in Lean: `residual_sum_uniform` [assuming $0 \le \lambda < 1$ and the partial sums $\sum_k u_k \le U$], `forced_uniform`). **The group-geometry part does not contain the shape coordinate, and does not contract** (§ 07).

The proposal of extending the coupling to a variable: make the coupling b the zeroth variable of the Taylor norm. We regard the activity $K(X; W, b)$ as a function of both the background and the coupling, and extend the seminorm to $T_{W,b}(h, hb) := \sum$

n, m

$_{n,m}\,\frac{h'^n}{n!}\frac{h_b''^m}{m!}\|D_W^n\partial_b^m K\|$ ($h_b \propto b$). If the law of the fluctuation has the form $\mu^{b(W)}$, the background derivative falls on the integrand and on $b(W)$, and the latter is $\partial_W b\cdot\partial_b\mathbb{E}_b F = \partial_W b\cdot \mathrm{Cov}_b(F,\mathrm{Re}\,Y)$; that is, **what remains of the background derivative is a coupling derivative**. If the coupling derivative is put into the norm, the group version of exchanging the expectation and the seminorm is expected to hold without the factor $1+\eta_k$ Assumption (assumption: the central moments of the statistic of the exponential family are bounded uniformly in b; the consistency of the norm for a multivariable set of local couplings $\{b_p\}$ with $2^{|B(X)|}$ is not yet settled). In this form, the definition and continuity of the effective coupling in the Coupling-Smallness Induction Lemma (15-03 § 03) would also follow from analyticity in b. Our reading is that the precise meaning of "absorbing η into the parameters" is "counting the parameters as components of the field" Reading .

Giving the class the background as a parameter — the Proposition on a Class with the Background as a Parameter (proposal). If the conditional laws of two links sharing one plaquette and of four in a row are regarded as the class $\mathcal{C}_2=(B,q,a)$ consisting of class functions obtained by cutting the plaquette weight at second order in $u=|\log P|^2$ (coupling B and shape q) plus the logarithm of the background holonomy a (for each plaquette carrying background), then the remainder after subtracting the coupling, recentring, shape and a directions from the score with respect to the background is of size $\theta^0\beta^{-3/2}$ and $\theta^1\beta^{-1}$ Reading (fits over $SU(2)$, $\beta=4$–128, $\theta=0.05$–0.3; consistent with the order count on paper). The cubic BCH term of § 07 (coefficient $1/9$) and the anisotropic second-order term enter the a direction and vanish; what remains is only the contribution of the shape coefficients $\hat g_3=-(1+32\rho)/720$ truncated by the class, which for $\hat g_3=0$ ($\rho=-1/32$) drops to $\beta^{-5/2}$ Reading . Each shape coefficient added to the class lowers the remainder by β^{-1} (for $\mathcal{C}_1\to\mathcal{C}_2\to\mathcal{C}_3$ the power goes 0.5→1.5→2.5). Since $\beta=4/g^2$, the one-step term is of size $h_k g_k^3$, and under the threshold $h_k\le Hg_k$ and the one-loop upper bound $g_k^2\le 1/(c_1(k_0+k))$

$$\sum_{k<K}\eta_k \;\le\; \frac{H(A+DH)}{c_1^2\,(k_0-1)} \qquad \text{(uniform in the number of steps } K\text{)}$$

holds Proved here (arithmetic; Lean: `class_parameter_budget_bounded`, where inside Lean c_1,k_0 is written as a,b; assumptions $A,D,H\ge 0$, $c_1>0$, $k_0>1$, $g_k^2\le 1/(c_1(k_0+k))$, $0\le h_k\le Hg_k$, $\eta_k\le h_k(Ag_k^3+Dh_kg_k^2)$). This pairs with `budget_unbounded_of_threshold` of § 07 (if the term is of type h_kg_k, there is no uniform upper bound). Conditions (on the same line): (1) the orders are those measured with the laws of two links and four in a row, and carrying them to the four-dimensional step is a reading; (2) small fields ($\theta\le 0.3$); (3) a is per plaquette carrying background (in four in a row, two are needed if the axes differ); (4) the (B,q) of the class is refitted at

every step, and the truncated shape coefficients are bounded uniformly in the step; (5) the one-step update of a is exact (an identity in a single-loop step, § 07; not yet for the multi-loop step). **That the score enters the a direction once a is made a parameter is the chain rule; what we measured is only the order of what remains when the class is cut to a finite one.** The divergence of $\sum_k \eta_k$ in the shape remainder condition has not disappeared; it has moved to the class-closure condition — whether the class carrying a closes in one four-dimensional step Reading . For $U(1)$ there is no cubic BCH term, and a does not change the order. The motion of the coupling cg^2p^2 for the one-link Wilson law of § 05 is likewise a term moved to the update of the parameters by giving the class the coupling b as a parameter Reading .

Reservations. (5) is specific to $SU(2)$ (only for $SU(2)$ is a sum of staples a constant multiple of a group element). For $SU(N)$, $\mathrm{tr}(WY)$ is a bilinear form in W and Y, the local coupling is a matrix rather than a number, and the conditional law does not fall into a one-parameter family (it does for each $SU(2)$ subgroup). Also, a fundamental-plus-adjoint action breaks the form of the first term of (5) (the expansion of $\cos^2$ produces an anisotropic quadratic form that acts only on the components of the fluctuation in the directions of the background) Proved here (quaternion algebra).

07

The multi-step induction has not been written

The Lemma on One Step with Background-Independent Fluctuations, the proposal of screening by order, and the proposal of extending the coupling to a variable are all one-step statements. Stacking steps needs the following three, none of which has been written Assumption .

- Background-regularity condition: for W in the image of the background-field map, the curvature of the fine plaquettes is bounded by L^{-2} times the curvature of the coarse plaquettes plus $O(\varepsilon_k^2)$ (the regularity of the background).
- Contraction of the many-body norm: the one-step loss $2^{|B(X)|}$ is cancelled by the contraction of the irrelevant part of the activity (the group version of the contraction estimate in the appendix for flat fields).
- The remainder after absorbing η stays of type $g_k^{2+\delta}$ uniformly in the step. If it is put on the ledger of the transverse contraction of 15-03 § 04, the remainder is expected to be bounded by a supremum rather than a sum.

The coefficient of the group-geometry part — the cubic BCH term is $1/9$. For two links with outer coupling $m\beta$ and shared plaquette β ($m = 1$ is the system of § 06), the remainder after projecting the background score onto the coupling and the 12

generators of left and right translations is, at the Gaussian leading term, $\eta = A\beta^{-1/2}$ (per background angle),

$$A\sqrt{B} = \sqrt{\frac{1}{3(m+2)^3} + \frac{240\,(m^2+m+1)}{(m+2)^5}\,r^2}, \qquad r = \frac{q}{B} \tag{7}$$

Proved here (the Gaussian leading term, the two-link model; $m = 1$–5 are exact by rational Wick computations, and the formula for general m was read from five points). **The geometric part for $m = 1$ is exactly $1/9$, and does not depend on the shape coordinate r.** It is orthogonal to the shape part in the Gaussian L^2 (cross term 0), and η^2 splits into the sum of the two. It agrees with independent numbers for the true action and with the measurement of § 06 to within 0.6% Reading . The algebra by which the cubic generator term returns $+\frac{B}{9}$ to the cubic BCH coefficient $-\frac{B}{6}$ of the score, giving $-\frac{B}{18}$ (in general $-\frac{B}{6}\cdot\frac{m}{m+2}$), is in Lean: `bch3_coefficient` (assumption $m + 2 \neq 0$), `nT_sum_diff`, `shape_split` (polynomial identities without assumptions).

The class (B, q, a), and s as the background itself. The coefficient s of J, which carries the cubic BCH term, does not contract in the one step that shrinks two links in series to one link: $s'/s = 1 - (1/3 + 10r)/B + O(B^{-2}, a^2)$ Proved here (the leading term and its $O(1/B)$; agrees with the measurements for $B = 16$–128, $s/B \le 0.1$ to three digits Reading). s is not a shape coordinate but **the background field itself** (s/B is the half angle of the background plaquette); one step carries the background to the coarse holonomy as the group product $a' = \log(W_2 W_1)$ (BCH), and only the shape coordinate q/B contracts (by a factor $1/4$ in this step). The class (B, q, a) (a is the logarithm of the background holonomy) closes with its three parameters in one step in series, with $B' = B/2 - 1/6 - 5r + O(1/B)$, $r' = r/4 + O(1/B)$, $a' = \log(W_2 W_1)$ Proved here (the model of two links in series, $SU(2)$; the residual of the fit is at most 3.5×10^{-7} Reading). The coefficient of the cubic BCH term to η is a constant that depends neither on the class nor on s, so the geometric part remains as "a constant $\times$ the budget of background change at each step".

The budget diverges. The geometric-part term of one step is $\frac{1}{9}\,h_k\,(\kappa\beta_k)^{-1/2}$ (h_k is the width of the background change, $\kappa \in [0.2, 1]$ the contraction of blocking), and since $\beta = 4/g^2$, it is of size $h_k g_k/(18\sqrt{\kappa})$ Proved here (arithmetic; carrying the two-link constant over to the four-dimensional step is Reading , and all that is used is that the constant does not depend on β). With the small-field threshold $h_k = g_k p(g_k)$ it is of size $g_k^2 p(g_k)$, and under the one-loop flow $\beta_{k+1} = \beta_k - c$ ($c = \frac{11}{3\pi^2}\ln 2 \approx 0.2575$), if the infrared end is held fixed and steps are added towards the ultraviolet, the sum diverges like a power of $\ln K$ Proved here (arithmetic; if $h = g$, the sum is $2/(9c) \approx 0.863$ times $\ln(\beta_0/\beta_*)$, and numerically too it is 0.865–0.87 times Reading). Keeping the probability of large fields

small needs $h_k \geq g_k$, and then $\sum h_k g_k \geq \sum g_k^2 = \infty$, so **the sum cannot be made finite by the choice of the threshold** Proved here (Lean: `budget_unbounded_of_threshold`, assumptions $g_j^2 \geq 1/(A(j+1))$, $0 \leq g_j \leq h_j$). Therefore this part cannot be carried as an error factor $1+\eta_k$; the field strength of the background (the vector coupling per plaquette $s_p n_p$, statistic J) must be given to the class as a parameter and moved into the movement of the parameters Reading . For two links, adding J to the coordinates made the remainder 0 (§ 06). This part is absent for $U(1)$ (the cubic BCH term is identically 0), and the parameter is the logarithm of the background holonomy a.

The one-face step — for a single loop the update of a closes by the group product. In a cross (two-dimensional) made of a central plaquette whose four links all fluctuate and the four plaquettes outside it, integrating out the four links puts the coarse weight exactly into a class function of the outer holonomy W (convolution of class functions; a restatement of gauge invariance); the centre of the coarse class is $a' = \log W$, W is the ordered product $e^{b1}e^{b2}e^{b3}e^{b4}e^{a0}$ in the tree gauge (non-abelian Stokes), and the shape coordinates do not depend on the background Proved here (conditions: the coarse variable is a single loop, the weights are class functions, the other links are integrated with Haar, a_p is held in the common frame of the tree gauge). If a is updated by the commutative sum $\sum_p a_p$, the remainder is the sum of second-order commutators $\sum_{i<j} a_i \times a_j$ $(\theta^2\beta^0)$, larger than the shape truncation of the class $(\theta^0\beta^{-3/2})$. If it is updated by the group product, there is no remainder and no new coordinate is needed Proved here , Reading (for $SU(2)$, $\beta = 8$–128, $\theta \leq 0.3$, a quadrature that does not use the closed form reproduces the group product to 2.1×10^{-4}–1.4×10^{-6}, and the difference falls with the quadrature nodes). That the quadratic term of the logarithm of the ordered product is half the sum of commutators is Lean: `log_quadratic_part` (assumption: ring), `commSum_eq_zero_of_comm` (commutative ring; for $U(1)$ the commutative sum is exact up to second order). It can be read as: **the cubic BCH term goes, to all orders, into the side of the background coordinate a** Reading . The shape coefficient after one step is $q'/B' \approx (q/B)/25$ (the tree value of the convolution of five plaquettes), $B' - \beta/5 \to -0.40$ for Wilson, and exactly $t' = t_0 + 4t_m$ for the heat kernel Reading .

The multi-loop step — the class coordinates need the corner triple product. In a three-dimensional $2\times2\times2$ step that fixes only the twelve edges of the coarse cube ($SU(2)$; heat kernel and Wilson; the leading saddle-point term and one loop; $O(1/\beta)$ not included), the coarse weight is a function of five independent loops and does not factor into a product of per-face class functions. Within second order, the symmetry of the coarse box (O_h) makes it close exactly with two coefficients: the per-face class (diagonal) and the coupling of pairs of neighbouring faces ($a_A \cdot a_B$ placed at the end of the shared edge) Proved here (symmetry; uniqueness of the representation is Lean:

`diag_adj_unique`, assumption: the two forms are equal on $\sum \theta = 0$). The tree-level values are diagonal $5/16$ and neighbour $1/32$ (ratio 1/10) Reading (rational fit; two routes). Since there is no non-abelian scalar of θ^2, non-commutativity first appears with three faces (a corner) Proved here (counting the representations of O_h). At third order the triple product $\kappa \sum_c a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of the cube is needed as the one and only new term, $\kappa = 3/1024$ at tree level (numerical fit) and about 0.006 at one loop; at fourth order couplings of three and four faces are needed Reading . In the range $\theta \sim \beta^{-1/2}$ the corner triple product is $\approx 2.2 \times 10^{-3} \beta^{-1/2}$ and the fourth-order multi-face coupling $\approx 3.3 \times 10^{-3} \beta^{-1}$, larger than the shape truncation of the class $\beta^{-3/2}$, so they cannot be omitted Reading . If a_p is held in a common frame (the tree gauge), a frame error breaking O_h remains at third order, so a definition is needed that places the neighbouring pairs at the end of the shared edge and the triple product at the corner. For a four-dimensional 2^4 box one counts 24 coarse faces, 96 neighbouring pairs and 64 corners Reading (only counted, not computed). We therefore read the class coordinates as taking the form of **per-face** (B_p, q_p, a_p)**, the coupling of neighbouring pairs** B_{pq}**, the one-link coupling** b**, the corner triple product** κ**, and the fourth-order multi-face couplings** Assumption (a statement within the saddle-point expansion of one three-dimensional step; the second step, large fields and four dimensions are not shown; 15-03 § 02).

The flow of the corner triple product — it does not contract, and approaches the fixed point from below. In a three-dimensional free-boundary box $L \times L \times L$ fixing only the twelve edges of the coarse cube, measuring the genuine multi-step flow at tree level from $L = 2$ (one step) to $L = 20$, the ratio $r_\kappa = \kappa / c_2$ (c_2 is the coefficient of $\sum |a_p|^2$) increases as 0.019 ($L = 2$), 0.033 ($L = 4$), 0.050 ($L = 20$); the ratio per doubling approaches 1 as 1.78, 1.30, 1.13, 1.10; and the increment shrinks by about 0.55 per doubling Reading ($SU(2)$, the leading saddle-point term, up to third order in ε; the form of the fit approaches 0.055–0.066; the existence of the limit is not shown). One loop (heat kernel and Wilson, $\beta = 32$–128, $L \leq 4$) moves the ratio by no more than 3%, and hardly changes the two-step ratio 1.78 Reading . **That the first-step ratio exceeds 1 is not a wall but the form of the approach**: terms absent from the fine action are born in the first step, and since they start below the fixed point, if $0 \leq \lambda < 1$ and $0 < r < s/(1 - \lambda)$ the fixed point is not overshot even with a one-step ratio $(\lambda r + s)/r > 1$ Proved here (Lean: `ratio_gt_one_below_fixpt`, `abs_iterate_dev` ($0 \leq \lambda < 1$)). In the iterated two steps, which feed the one-step coarse action back into the action on the intermediate lattice, the deviation of the ratio shrinks to 1/9–1/17 per step (the third-order output is exactly linear in the input); the iterated fixed point 0.032–0.045 is smaller than the limit of the genuine multi-step flow, and the approach is much faster Reading . Fixing only the edges, the second-order coefficient is nearly constant, $c_2 L \ln L \approx 0.27$–0.29 ($\ln L$ along the edges), and does not reach the classical three-dimensional scale $1/L$

Reading (interpretation: because the interiors of the faces are free and only the edges are fixed). The corner triple product is not a quantity that vanishes geometrically like the shape remainder, but **a coordinate with a value fixed by the flow**, and the class has to carry that value. It is not, however, an independent parameter.

The four-dimensional class — one parameter and the ratio coordinates; the skew neighbour is not 0. In one step of a four-dimensional 2^4 box fixing the 32 edges of the coarse hypercube, the space of linearised angles of the coarse faces is 17-dimensional and splits under the hyperoctahedral group B_4 into $3 + 6 + 8$ (6 is the uniform field); the invariants number 3 at second order, 3 at third order (colour triple products) and 65 at fourth order Proved here (two routes: averaging of characters and numerical rank; reproduces $2 \cdot 1 \cdot 9$ in the three-dimensional version). The orbit sum of pairs of orthogonal faces (type $\operatorname{tr} F \wedge F$) is 0 by reflection. Unlike in three dimensions, **per-face classes and neighbouring pairs sharing an edge do not close already at second order** (remainder 0.57% at tree level, 0.79% at one loop), and one "skew neighbour" pair, shifted by one edge (sharing no vertex, averaged by transport along the two connecting edges), is needed: at tree level diagonal $5/26$, neighbour $-79/2080$, skew $-9/2080$ Proved here (linear expressions in the eigenvalues $1/4$, $5/13$, $3/10$; Lean: `diag_adj_skew_unique`, `diag_adj_not_enough` (assumptions: the computed componentwise values and eigenvalues), `coeffs_of_spectrum` ($k = (2e8 - e3 - e6)/8$), `skew_zero_iff`). The third order likewise does not close with the types gathered at a vertex (cell corners, paths) alone, and a parallel type is needed (the cell-corner coefficient is a fit of $5931/1406080$ at tree level). In the range $\theta \sim \beta^{-1/2}$ the skew neighbour is ≈ 0.02 at β^0, the third order $\approx 0.023\beta^{-1/2}$ and the fourth order $\approx 0.046\beta^{-1}$, all larger than the shape truncation of the class $\beta^{-3/2}$, so none can be omitted Reading (saddle point plus one loop, no $O(1/\beta)$, 90 random samples of the face angles $\lesssim 0.5$).

In the genuine multi-step flow with the box enlarged to $\{0, \dots, L\}^4$ ($L \leq 10$, tree), none of the second- and third-order ratio coordinates head to 0 at the second step ($\lambda \geq 1.14$); the ratio per doubling approaches 1 and they approach the values of the fixed point (skew neighbour $r_k = k/b$: -0.01125, -0.01744, -0.01988, extrapolated -0.0208; neighbour -0.145, third order 0.064, 0.013, -0.021) Reading (the fit $r_\infty - A/L - B/L^2$ represents $L = 4$–10 to within 4×10^{-5}; the $\ln L$ form as in three dimensions is 30–300 times worse). The second-order class coefficients have no $\ln L$ at $b \approx 0.39/L$ and approach faster than in three dimensions. One loop moves the two-step ratio by less than 1% ($L \leq 4$). The fourth-order multi-face part has a ratio of 1.6 to c_2 at the second step, with direction cosine 0.99. **Dependence on the rule**: within one coarse hypercube, the rule fixing all faces gives the same quadratic form as the rule fixing the edges ($32 - 15 = 24 - 7 = 17$; Lean: `edge_face_dof_agree`, `euler_cube`). On a periodic network where the coarse cells share fine fields, the skew neighbour still approaches a

non-zero negative value, but the value is about twice as large (-0.045; -0.06–-0.10 depending on the projection convention), and the part not contained in a cell grows with the steps from 6% to 20% Reading (tree level of the commutative linearisation, second order, $L \le 16$). **The sign and non-vanishing of the skew neighbour depend neither on the rule nor on the projection convention; its value, and "closing within one cell", do.** Under either rule the b of the cell part at tree level is $1/L$, and the running of $\ln L$ is not at tree level. The four-dimensional class is "one parameter B plus the ratio coordinates whose values are fixed by the flow (2 at second order, 3 at third, 65 at fourth)", and the "local" in the class-closure condition is written with the rule attached (15-03 § 02) Reading .

The minimal form in which to write the multi-step induction, and its cost. Viewing the flow of the class coordinates as a sequence of linear maps, in the iterated two steps that feed the one-step class back into the per-hypercube action on the intermediate lattice, the output of the third-order coordinates is exactly linear in the input, $\kappa' = s + \Lambda\kappa$, the eigenvalues of the normalised Λ are all below 0.1, and the deviation of the second-order ratio also shrinks to 0.1–0.2 per step Reading (four dimensions, tree, versions with two weights bracketing the genuine two steps). The genuine multi-step flow, however, approaches the same coordinates only at a rate of $1/L$, and **the iterated fixed point is shifted towards 0 relative to the genuine limit** (for the skew neighbour, -0.019 against -0.021 for the rule that separates cells, -0.035 against -0.045 for the periodic network). On the periodic network the shift is larger, since the "part not contained in a cell" that the iteration discards at every step is 6–10%. The induction can be written as a contraction of the iterated map, but that its fixed point agrees with the value of the genuine flow must be shown separately; as long as the class coordinates are truncated to the type of a single cell, the induction has to be written in a form that carries the truncation error from step to step Reading (the bookkeeping is Lean: `tail_bound` (assumption $0 \le q < 1$ and the ratio of increments $\le q$), `affine_iter`). Under the rule fixing only the edges, $\beta' = \beta b$ becomes about 0.39 times per step, so the relative weight of one loop grows with each step, and the induction closes only within the weak-coupling range.

With this, splitting the density of the joint law in Table 2 has the form of its answer in the smallest systems. Ultraviolet questions 17–19 and the second step of the multi-loop class also have partial answers. The multi-step induction (the minimal multistep statement) has not been written.

Table 2 Questions arising from this chapter (status: New = a question added in this part; Updated = an answer has been added)

Q.	Question	Size	Status	What moves if settled	What does not move even if settled

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
10	For four in a row of $SU(2)$ (two overlapping loops fixed), split the joint law of the two inner plaquettes into the product of local couplings $\mu^{b2} \otimes \mu^{b3}$ $\times$ a density, and derive the lowest order in θ of the score of the density, and the isotropy of the quadratic form of the fluctuation (the coefficient of the anisotropic term), as functions of β. Is the θ^1 term only the coupling derivative? **Answer (smallest systems):** the lowest orders are $\theta^0 \beta^{-1/2}$ and $\theta^1 \beta^0$. θ^1 contains not only the coupling derivative but also an anisotropic quadratic term (only for $SU(2)$).	Paper and small computations	Updated	Whether the screening of the proposal of screening by order is right. The type of η in the group version of exchanging the expectation and the seminorm	$\mu_{k,W}$ on the four-dimensional lattice. The multi-step induction

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
	Adding J to the coordinates makes the remainder 0 (§ 06)				

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
11	The minimal statement of the multi-step induction: can the background-regularity condition and the contraction of the many-body norm be written with constants independent of the step, in a hierarchical model of single-plaquette weights or in a two-dimensional model? Together with this, using the norm of the proposal of extending the coupling to a variable, derive the width of the strip of analyticity of the coupling from the positions of the complex zeros of $Z_b = 2I_1(b)/b$ (the positions of the zeros are unconfirmed in the original sources)	Paper	New	A form of the Lemma on One Step with Background-Independent Fluctuations stacked over k	The one-step remainder including large fields (the net one-step bracket of the coupling)

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
17	For the central plaquette of a 2×2 plate (four links fluctuating), write out the remainder after dividing the score by the space of generators as odd orders of BCH symmetrised in cyclic order, and derive the geometric coefficient by rational Wick computations. Does it grow or shrink on a face compared with the $1/\sqrt{3(m+2)^{3}}$ of two links? Also, does the insertion of the background stick out of the family of quaternion couplings $A_{p}+J$? **Answer (partial):** in the cross (a two-dimensional step in which the four links of the central plaquette fluctuate), if a is updated by the group product there	Paper and small computations	Updated	The geometric coefficient for the four-dimensional step	The divergence of the budget (the threshold offers no escape)

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
	is no remainder, and the odd BCH orders go into the coordinate a to all orders. With the commutative sum, $\theta^{2} \beta^{0}$. In three dimensions $2 \times 2 \times 2$ the corner triple product $\kappa = 3 / 1024$ is needed (§ 07)				

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
18	For a class with the logarithm of the background holonomy a as a parameter, measure in one step in series whether the geometric part of one step moves exactly into the flow $a' = \log(W_2 W_1)$ of a (the remainder drops one order from $\theta^0 \beta^{-1/2}$). **Answer (smallest systems):** for the class (B, q, a) with a as a parameter, the remainder drops to $\theta^0 \beta^{-3/2}$ and $\theta^1 \beta^{-1}$, and the budget is uniform in the number of steps (conditionally, § 06). In a single-loop step the update of a is exact (§ 07)	Small computations	Updated	The form of moving the geometric part into the movement of the parameters	The multi-step induction

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
19	The condition of the One-Link One-Step Fluctuation Lemma (the recentring condition): for one link of the four-dimensional lattice, make the Taylor radius of the map of the sum of staples $W \mapsto \hat{q}(W)$ explicit in terms of the angles of the background plaquettes and the lower bound of $\lvert Q\rvert$, and make the g^2p^2 coefficient of δ_b/b a single number. **Answer:** $\lvert Q\rvert \geq 6\cos\theta_0$, $\lVert D\hat{q}\rVert \leq 3/\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$ (measured $c \approx 12$, on paper $c \leq 29$). The composition step is a sketch, and the motion of the coupling is a term carried to the side of the parameters of the class (§ 05)	Paper	Updated	The one-link Wilson chain without assumptions	The remainder of many links

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
23	The second step of the multi-loop class: perform the second step with the class after one three-dimensional $2 \times 2 \times 2$ step (per face, neighbouring pairs, corner triple product, multi-face), and see whether κ is a coordinate preserved from step to step (expected to be the lattice version of the dimension-6 $\operatorname{tr} F[F, F]$), and whether the same coordinates suffice in four dimensions 2^4. **Answer (partial):** in three dimensions κ/c_2 does not contract at the second step and approaches the fixed point (0.055–0.066 by the fit) from below. In four dimensions the same coordinates do not suffice:	Small computations	Updated	The class coordinates of one four-dimensional step	Large fields, the small-field contribution to the coupling

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
	the skew neighbour is needed at second order and a parallel type at third, and all the ratio coordinates approach the fixed point (tree $L \leq 10$, § 07)				
26	When the class is given the coupling b as a parameter, does the motion of the coupling $c g^2 p^2$ for the one-link Wilson law move exactly into the update of the parameters, with the remainder dropping by one power of g^2?	Paper and small computations	New	The budget of the One-Link One-Step Fluctuation Lemma	The remainder of many links

Q.	Question	Size	Status	What moves if settled	What does not move even if settled
27	Can the fixed-point values of the ratio coordinates of the four-dimensional class (the skew neighbour, the 3 of third order, the 65 of fourth order) be obtained directly, instead of running many steps, from a half-infinite problem near a corner of the coarse hypercube (the Green function of a half-space with a bundle of edges fixed)? Also, under the rule in which coarse cells share fields, should the part not contained in a cell (up to 20% with the steps) be closed by putting it into finitely many pairs of class coordinates (orbits of shift ≤ 1), or carried as a tail?	Paper and small computations	New	The values of the four-dimensional coordinates of the class-closure condition and the form of "local"	Large fields, the small-field contribution to the coupling, the second step at one loop

08

What cannot be said

(a) Within this part, ultraviolet control is not closed (existence was shown by the published construction, with a different design). Every statement of this chapter is a one-step statement, a statement set down as an assumption, or a proposal.

(b) The Background–Fluctuation Decomposition Lemma is only the group version of submultiplicativity in sums of fields among the four properties of the weight, and does not touch the closure under the fluctuation expectation, which is the main issue. It may be a known elementary fact.

(c) The group versions of the exponential moment of fluctuations and of background regularity, and the interchange inequality, are only statements and have not been shown. (3) is nothing more than the chain assuming them.

(d) $\eta_k \sim g_k p(g_k)$ and $\eta_k \lesssim g_k^2 p(g_k)^2$ are naive estimates, not estimates for the actual measure of the fluctuations.

(e) The Lemma on One Step with Background-Independent Fluctuations is a statement about a single step in the case where the law of the fluctuation does not depend on the background, and shows nothing about Bałaban's μ_k,W, the Wilson action, many steps, or the rarity of large fields. The numbers of Table 1 are a numerical reading with the rough figures $SU(2)$, $n_b = 96$.

(f) The fact about the heat-bath method gives its bibliography from memory, and we have not checked the text. The proposal of screening by order and the proposal of extending the coupling to a variable are proposals. The orders of the remainder in the smallest systems (§ 06) are numbers for two links and four in a row, and carrying them over to the four-dimensional step is a reading. $SU(N)$ does not have the form of (5).

(g) The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma are statements for one link and one step, and assume the recentring condition. The coefficient (7) of the cubic BCH term is a value at the Gaussian leading term in the two-link model, and the formula for general m was read from five points. The closing of the class (B, q, a) is a statement about one step of two links in series, and configurations with faces have been examined only in the single-loop cross (§ 07). The divergence of the budget is arithmetic that treats the one-loop flow as an exact formula, and carrying the two-link constant over to the four-dimensional step is a reading.

(h) The numbers of the recentring condition in § 05 are statements for one link and one step; the composition step is a sketch, and the majorant series on paper converges only for $h < 0.075$–0.078. The Proposition on a Class with the Background as a

Parameter in § 06 is a proposal; what we measured is the order of the remainder when the class is cut to a finite one (two links, four in a row, $\beta = 4\text{–}128$, $\theta \leq 0.3$), and it rests on the chain rule. That the class carrying a closes in a four-dimensional step is not shown. The one-face step of § 07 is a statement for a single loop, and the class coordinates of the multi-loop class are statements within the saddle-point expansion (leading term and one loop) of one three-dimensional $2 \times 2 \times 2$ step. The second step and four dimensions have not been measured.

(i) The flow of the corner triple product in § 07 consists of numbers at tree level ($L \leq 20$) and one loop ($L \leq 4$) in a three-dimensional free-boundary box (only the edges fixed), and the limit is in the form of a fit. The four-dimensional class consists of numbers at tree level ($L \leq 10$) and one loop ($L \leq 4$) in a free-boundary box, and the periodic network of numbers at second order in the tree of the commutative linearisation ($L \leq 16$); the existence of the fixed point of the ratio coordinates, the values of the 65 fourth-order ones, and the third order and one loop on the periodic network are not shown. That the sign and non-vanishing of the skew neighbour do not depend on the rule is a statement about the two rules and two projection conventions that were measured. The numbers for the contraction of the iterated map are at tree level only.

09

The five grades table and the dependency graph

Table 3 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
[01]	For $\prod_k (1+\eta_k)$ to be bounded uniformly in the number of steps, $\sum_k \eta_k < \infty$ is needed	Proved here	§ 01 (arithmetic)	Under the one-loop flow
[02]	The Background–Fluctuation Decomposition Lemma (1) and the Splitting of the Weight (2), the form written with Y	Proved here	§ 02	$SU(N)$, operator norm, four-dimensional lattice (Splitting of the Weight)
[03]	The group versions of the exponential moment of fluctuations and of background regularity (the group version of the exponential moment of fluctuations, the background-regularity condition), and the group version of exchanging the expectation and the seminorm	Assumption	§ 03 (statements only)	Small-field backgrounds $\mathcal{W}_k$; constants independent of k, a

Row	Statement (one line)	Grade	Basis	Scope
[04]	The chain (3), the group version of the One-Step Bound on the Activity Norm	Proved here	§ 03 (assuming [03])	On $\mathcal{W}_k$
[05]	For a product of one-link heat kernels, the law of the fluctuation does not depend on the background	Proved here	§ 03 (invariance of Haar measure)	Heat kernels independent edge by edge
[06]	With a naive η_k the sum diverges ($K^{1/2}$ for $g\,p$, a power of $\ln K$ for g^2p^2)	Proved here	§ 04 (arithmetic)	Naive estimate, under the one-loop flow
[07]	The type of η changes with how it is measured (θ^2, θ^4, θ). Without the constraint, $\beta\theta^2/12$	Reading	§ 04, 15-06 § 01	Single plaquette, $\theta \le 0.3$, $\beta = 4$–32
[08]	The Lemma on One Step with Background-Independent Fluctuations (4) and (a)–(d) ($\eta = 0$, the support does not spread, $M = m^{\lvert X\rvert}$ for heat kernels, small fields)	Proved here	§ 05	The three conditions of background independence of the fluctuations, scale contraction and the exponential moment of fluctuations; one step only
[09]	The numbers for the exponential-moment condition for fluctuations $s/\beta \le 0.0020$–0.0025, $c\,g^2/h^2 \lesssim 1.7 \times 10^{-4}$	Reading	§ 05 Table 1	$SU(2)$, heat kernels edge by edge, $n_b = 96$, $q = 6$, $\beta = 4$–128
[10]	The conditional law of one link for $SU(2)$ is $\mu_1^{\beta\lvert Q\rvert}$, and the background enters only through $b(W)$ and $\hat{q}(W)$	Cited	§ 06 (a basic fact of the heat-bath method)	$SU(2)$, Wilson action, bibliography unconfirmed
[11]	The quaternion identity (5): local coupling $b_p = \beta\cos(\theta_p/2)$	Proved here	§ 06	$SU(2)$, $\frac{1}{2}\,\mathrm{tr} = \mathrm{Re}$
[12]	The proposal of screening by order (screening by the order of θ; the orders of the remainder are [15]) and the proposal of extending the coupling to a variable (the coupling as a variable of the norm)	Assumption	§ 06 (proposal)	Small fields; not yet checked for many plaquettes
[13]	The three requirements of the multi-step induction (the background-regularity condition, the many-body contraction, uniformity of the remainder)	Assumption	§ 07	Statement only

Row	Statement (one line)	Grade	Basis	Scope
[14]	The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma: inside the angle cutoff the one-link Wilson law is K times the heat kernel; the losses are K and $1+\eta$ ($\eta \approx \frac{3}{2}\delta_b/b$)	Proved here	§ 05	$SU(2)$, one link, one step, the recentring condition assumed, $\delta b < b$
[15]	The remainder in the smallest systems $\eta^{\mathrm{res}} \approx \sqrt{A^2/\beta + D^2\theta^2}$; adding J gives 0	Reading (the splitting of the action is Proved here)	§ 06	Two links, four in a row, $\beta = 4$–32, $\theta \le 0.3$
[16]	The four-way splitting (6) of the derivative in the background, the $\eta = \frac{8}{3}(q/B)(\beta B)^{-1/2}$ of $U(1)$, the shape part enters a geometric series	Proved here, Reading (numbers for the agreement)	§ 06	Small fields (first order in a), a chain in a row, assuming the one-step inequality
[17]	The coefficient (7) of the cubic BCH term: $1/9$ at $m = 1$	Proved here	§ 07	Gaussian leading term, two links; the formula for general m is read
[18]	The class (B, q, a) closes in one step in series, and s does not contract	Proved here, Reading	§ 07	Two links in series, $SU(2)$, $B = 16$–128
[19]	The budget of the geometric part diverges however the threshold is chosen	Proved here (arithmetic), Reading (carrying over to four dimensions)	§ 07	One-loop flow, $h \ge g$
[20]	The recentring condition: $\lvert Q\rvert \ge n_s \cos\theta_0$, $\lVert D\hat{q}\rVert \le 3/\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$	Proved here, Reading (the measured c and h_q/h)	§ 05	$SU(2)$, one link, $\theta_0 < \pi/2$, composition step sketched, $h < 0.075$
[21]	The class (B, q, a) with the background a as a parameter: remainder $\theta^0\beta^{-3/2}$, $\theta^1\beta^{-1}$; the budget is uniform in the number of steps (the Proposition on a Class with the Background as a Parameter)	Reading (orders), Proved here (arithmetic of the budget), Assumption (the proposition)	§ 06	Two links, four in a row, $\beta = 4$–128, $\theta \le 0.3$, conditions (1)–(5)
[22]	In a single-loop step the update of a closes to all orders by the group product; with the commutative sum, $\theta^2\beta^0$	Proved here, Reading (numbers)	§ 07	Cross, class-function weights, common frame

Row	Statement (one line)	Grade	Basis	Scope
[23]	Multi-loop step: second order is per face plus neighbouring pairs; at third order the corner triple product $\kappa = 3/1024$; at fourth order multi-face	Proved here (second-order symmetry), Reading (κ and fourth order)	§ 07	Three-dimensional $2 \times 2 \times 2$, one step, leading saddle-point term and one loop
[24]	The final form of the class coordinates (per face, neighbouring pairs, one-link coupling, corner triple product, multi-face)	Assumption	§ 07	A reading of one three-dimensional step; four dimensions not yet (four dimensions in [26])
[25]	The ratio κ/c_2 of the corner triple product does not contract at the second step and approaches the fixed point from below (the first-step ratio 1.78 is the form of the approach)	Reading , Proved here (bookkeeping of the linear maps)	§ 07	Three dimensions, free boundary, only edges fixed, tree $L \leq 20$, one loop $L \leq 4$, form of the fit
[26]	Four-dimensional class: one parameter plus the ratio coordinates (2 at second order, 3 at third, 65 at fourth); the skew neighbour is non-zero whatever the rule, with a value that depends on the rule; no running of $\ln L$ at tree level	Proved here (counting, agreement of degrees of freedom, linear expressions for the coefficients), Reading (numbers for the flow)	§ 07	$SU(2)$, heat kernel, tree $L \leq 10$, one loop $L \leq 4$, periodic network at second order of the commutative tree $L \leq 16$
[27]	The iterated map contracts fast, but its fixed point is shifted towards 0 relative to the genuine limit, and a form carrying the truncation error from step to step is needed	Reading	§ 07	Tree level, two rules

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV  product uniform ⟺ sum finite                                  one-
[02] PRV  Background–Fluctuation Decomposition Lemma, Splitting of the Weight
[03] ASM  group versions of the exponential moment and background regularity,
[04] PRV  one-step chain (3) ⟸ [02][03]                                 assu
[05] PRV  heat-kernel product: fluctuation law does not depend on the backgrou
[06] PRV  naive η: the sum diverges ⟸ [01]                              naiv
[07] RDG  the type of η depends on how it is measured ⟸ [06]            sing
[08] PRV  Lemma on One Step with Background-Independent Fluctuations ⟸ [02][0
[09] RDG  the numbers for the exponential-moment condition ⟸ [08]       SU(2
[10] CIT  one-link conditional law is μ₁^{β|Q|}                        SU(2)
[11] PRV  quaternion identity (5) ⟸ [02]                                SU(2
[12] ASM  proposals of screening by order and of extending the coupling to a v
[13] ASM  three requirements for many steps ⟸ [04][08][12]              stat
[14] PRV  One-Link Heat-Kernel Comparison Lemma, One-Link One-Step Fluctuation
[15] RDG  smallest-system remainder θ⁰β^{−1/2}, θ¹β⁰ ⟸ [11][12]         two
[16] PRV  four-way split of the background derivative; shape is geometric ⟸ [
[17] PRV  cubic BCH coefficient 1/9 ⟸ [16]                              Gaus
[18] PRV  class (B, q, a); s does not contract ⟸ [17]                  one
[19] PRV  divergence of the budget ⟸ [06][17][18]                      one
[20] PRV  cost of the recentring condition ⟸ [14]                      one
[21] RDG  class with a as parameter, budget uniform ⟸ [18][19]         two
[22] PRV  one-face step: a updated by group product ⟸ [18]             sing
[23] RDG  multi-loop class: neighbour pairs, corner triple product ⟸ [22]   2
[24] ASM  final form of the class coordinates ⟸ [21][22][23]           3D r
[25] RDG  corner triple product approaches fixed point from below ⟸ [23]   3D
[26] RDG  4D class, skew neighbour, dependence on the rule ⟸ [23][24]   tree
[27] RDG  shift of iterated fixed point, truncation error ⟸ [13][25][26]   tr
```

To check it, verify by eye that every number to the right of "⟸" is smaller than the number on the left. There are no cycles. [03], [12], [13] and [24] are assumption rows, and each is a restatement of ultraviolet control or a question that splits it. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

What this chapter can say

Can say	Cannot say
Proved here The Background–Fluctuation Decomposition Lemma and the Splitting of the Weight (the group version of submultiplicativity in sums of fields); the one-step chain (assuming the group versions of the exponential moment of fluctuations and of background regularity); that for a product of heat kernels the law of the fluctuation does not depend on the background; the divergence of the sum for a naive η_k; the Lemma on One Step with Background-Independent Fluctuations ($\eta = 0$ in one step where the law of the fluctuation does not depend on the background, and the support does not spread); the quaternion identity (5)	A proof of the group versions of the exponential moment of fluctuations and of background regularity. An estimate of the actual η_k. A single step with a genuine $\mu_{k,W}$
Reading The numbers for the exponential-moment condition for fluctuations ($SU(2)$, $n_b = 96$); that the type of η changes with how it is measured	The constants of the exponential-moment condition for fluctuations on the closure of a four-dimensional block
Cited That the conditional law of one link for $SU(2)$ is a member of a one-parameter family (a fact of the heat-bath method)	Checking the text of the bibliography
Proved here The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma (the one-link Wilson law, assuming the recentring condition); the four-way splitting (6) of the derivative in the background; the coefficient (7) of the cubic BCH term (Gaussian leading term, two links); one step in series of the class (B, q, a); the divergence of the budget (arithmetic)	The group version of the exponential moment of fluctuations for many links; the geometric coefficient for configurations with faces; the recentring condition; the budget for the four-dimensional step
Reading The orders of the remainder in the smallest systems ($\theta^0\beta^{-1/2}$, $\theta^1\beta^0$), and that adding J gives 0; agreement of the values of the shape for $U(1)$	Systems in which four or more links fluctuate around one plaquette
Proved here The explicit constants of the recentring condition; a uniform upper bound on the budget (with a as a parameter, arithmetic); the background update in the one-face step (single loop); the second-order closing of the multi-loop class (symmetry)	That the class carrying a closes in a four-dimensional step; the composition step of the recentring condition
Reading The order of the remainder of the class carrying a; the class coordinates of the multi-loop class (neighbour/diagonal 1/10, corner triple product 3/1024); the flow of the corner triple product (approaches the fixed point from below); the four-dimensional class (one parameter plus the ratio coordinates; the skew neighbour is non-zero); the shift of the iterated fixed point	The existence of the fixed point of the ratio coordinates; the values of the 65 fourth-order ones; the third order and one loop on the periodic network; large fields
Assumption The proposal to absorb η as the shift of local couplings; the three requirements for many steps	A proof of any of them. Any prospect of ultraviolet control closing

Sources and reproduction

Item	Type	Source, tools
The four properties of the weight, the One-Step Bound on the Activity Norm (Lemma A.6.1), the inequality exchanging the expectation and the seminorm ((A.6.5))	Cited (the text of Appendix A.6.2 and of Lemma A.6.1)	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474 (15-02 § 04)
Propagators in a background field	Cited (bibliography and titles only)	The foundational papers of T. Bałaban's series (7-07 Table 1)
The basic fact of the heat-bath method for $SU(2)$	Cited (bibliography unconfirmed)	M. Creutz, Phys. Rev. D 21 (1980) 2308; A. D. Kennedy, B. J. Pendleton, Phys. Lett. B 156 (1985) 393
The Background–Fluctuation Decomposition Lemma, the Splitting of the Weight, the chain, the Lemma on One Step with Background-Independent Fluctuations, (5), the arithmetic of the sum of η_k	Proved here	The proofs in the text of this chapter
The numbers for the exponential-moment condition for fluctuations (Table 1)	Reading	A numerical record of this book's study (the generating function of the heat kernel of $SU(2)$ by trapezoidal quadrature in the angular direction and a sum over characters; positive control: the closed form for $\mathbb{E}\|Y-1\|^2$ to 9 digits, normalisation to 10 digits)
The type of η for a single plaquette	Reading (the form of (1) in 15-06 is Proved here)	The source table of 15-06 § 01
The proposal of screening by order and the proposal of extending the coupling to a variable	This book's study	The record of this book's study (sorting out the rows of ultraviolet control that do not close)
The One-Link Heat-Kernel Comparison Lemma, the One-Link One-Step Fluctuation Lemma, (6), (7), the class (B, q, a), the arithmetic of the budget	Proved here	The main text of this chapter. The algebraic parts are in the Lean bundle (15-09 Table 3, re-verified)

Item	Type	Source, tools
The remainder in the smallest systems, the agreement of $U(1)$, s'/s, the partial sums of the budget	Reading	The numerical records of this book's study (two routes for the projections of two links and four in a row; rational Wick computations and Gaussian samples of the true action; fits and two routes of moments for one step in series; controls of $U(1)$ and $\theta = 0$)
The one-loop coefficient b_0	Cited (textbook formula)	Equation (9) of 7-05
The recentring condition, the Proposition on a Class with the Background as a Parameter, the one-face step, the numbers for the multi-loop class	Reading	The numerical records of this book's study (random draws and sign patterns of the one-link sum of staples; fits and negative controls for two links and four in a row; quadrature and quasi-Monte Carlo for the cross; two routes for the heat-kernel and Wilson saddle points of $2 \times 2 \times 2$, and rational fits)
The flow of the corner triple product; the four-dimensional class and comparison of rules; the iterated two steps	Reading (counting, degrees of freedom and linear expressions; the bookkeeping is Proved here)	The numerical records of this book's study (at tree level, two routes: the tree gauge with sparse normal equations, and the minimum norm; a third route by Newton's method for the non-abelian saddle point. One loop by the analytic Hessian. Positive controls: reproducing $5/32$, $1/32$, $3/1024$, $5/13$, $-79/2080$, $-9/2080$ of one step. A negative control with third order 0 for a parallel background. The periodic network agrees to 10 digits between two routes, real space and Fourier). The bookkeeping is in the Lean bundle (15-09 Table 3, re-verified)
Lower bound of the sum of staples; arithmetic of the budget; the sum of commutators; uniqueness of diagonal and neighbour	Proved here	The main text of this chapter. The algebraic parts are in the Lean bundle (15-09 Table 3, re-verified)

Words used in this chapter — the measure of the fluctuations $\mu_{k,W}$, background dependence η_k, the four properties of the weight, the Background–Fluctuation Decomposition Lemma, the group versions of the exponential moment of fluctuations and of background regularity, the interchange inequality, the Lemma on One Step with Background-Independent Fluctuations, fluctuations independent of the background, the shift of local couplings, the fact about the heat-bath method, the score, the multi-step induction, the One-Link Heat-Kernel Comparison Lemma, the shape part and the group-geometry part, the cubic BCH term, the class (B, q, a), the budget of background change, the cost of the recentring condition, the background as a parameter of the class, the Proposition on a Class with the Background as a Parameter, the one-face step, the multi-loop class, the corner triple product, the ratio coordinates, the skew neighbour, dependence on the rule. Their meanings are in the A-02 glossary.

Next chapter to read: 15-08 The four rows that do not close, and proposals for them — which of the four rows the absorption of η in this chapter makes lighter.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control
the four rows that do not close, proposals, dependence · Kiichi, Shiori and Rin

The four rows that do not close, and proposals for them

This chapter is this book's own investigation. § 01, § 03, § 04, § 07 and § 08 are a map of open questions.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of the second volume; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

What this chapter tells you — Of ultraviolet control, the part that lies outside the components closed in this part (the Coupling-Smallness Induction Lemma, the Bundle Network Theorem, the Lemma on One Step with Background-Independent Fluctuations, and the bookkeeping of the transverse contraction) is written as four rows. The first (the row of the remainder including large fields) is the one-step remainder including large fields (the net one-step bracket of the coupling); the second (the observable-source condition) is the source for observables; the third (the row of background-dependent absorption and many steps) is the absorption of the background dependence η of the fluctuations and the multi-step induction; the fourth (the row of the boundary of gauge re-fixing) is the seam where gauge fixing is re-fixed scale by scale. For each row we write in one line "what would have to be shown for it to close", and then set out the boundary with the closed components, proposals that can be tried, and proposals that were discarded. The four rows are not independent: we put forward the reading that the row of background-dependent absorption and many steps is the core, and that what remains without being reduced is two things, the small-field contribution to the coupling and the multi-step regularity of the background. Split further, the small-field contribution to the coupling divides into a part of the same form as is known in the sources for flat fields, and the group version of the place that the sources themselves call a sketch (the explicit quadratic term between blocks and its local cancellation). Finally, we list in ten rows the directions being tried towards ultraviolet control. In the revision we added: that the explicit quadratic term between blocks appears at $O(1)$ from one loop in the smallest system of two blocks; that the type of its non-local tail is decided by the definition of the coarse variables; the places where, when the algebraic layer of the small-field contribution to the coupling was written in Lean, the cut may differ from the sources; and that the ghost determinant of the row of the boundary of gauge re-fixing agrees with the tree gauge on critical-point backgrounds.

Reading level — This chapter runs from third-year level up to research level (it uses measures, operators and norm estimates). Table 1 of § 01 at the start alone can be read by second-year students.

Prerequisite chapters — 15-03 § 02 (the decomposition statement), 15-03 § 03 (the Coupling-Smallness Induction Lemma), 15-07 § 05 (the Lemma on One Step with Background-Independent Fluctuations), 15-05 (the Bundle Network Theorem), 15-02 § 06 (the two rows absent from the sources).

Said first — None of the rows in this chapter closes. A proposal is a proposal, not a result. The label [Proved here] is attached only to elementary facts checked on paper as a step within a proposal.

The order of this chapter

1. Statement of the four rows — one line, the boundary with the closed components, beyond the boundary (Table 1)
2. Proposals for the row of the remainder including large fields — the one-step remainder including large fields
3. Proposals for the observable-source condition — the source for observables
4. Proposals for the row of background-dependent absorption and many steps — the absorption of η and the multi-step induction
5. Proposals for the row of the boundary of gauge re-fixing — the seam of re-fixing of the gauge fixing
6. Dependence among the four rows — the row of background-dependent absorption and many steps is the core; splitting the small-field contribution to the coupling; W^{gauge} for two blocks; the tail type; the algebraic layer and how it differs from the sources
7. Discarded proposals (Table 2)
8. Ten rows of directions being tried (Table 3); ultraviolet questions 12–16, the question "two definitions of Loc", the question "three-dimensional non-local weights", the question "gauge agreement on non-abelian backgrounds" (Table 4)
9. What cannot be said
10. The table of the five grades and the dependency graph (Table 5)

00

What this chapter shows

- The part that does not close can be written in four rows: the net one-step bracket of the coupling, the observable-source condition, the absorption of η and many steps, and the seam of re-fixing (§ 01).
- The large-field rarity condition (the rarity of large fields) of the row of the remainder including large fields is a corollary of Chebyshev's inequality once the group version (the group version of the exponential moment of fluctuations) of the closure under the fluctuation expectation ((iii) of BBS) holds, and needs no independent estimate (§ 02, [Proved here], assuming the group version of the exponential moment of fluctuations).

- If the source of the observable-source condition is taken to be a function of the block variables, it commutes with the integration at the fine steps and produces no loss (§ 03, [Proved here], trivial).
- For the row of the boundary of gauge re-fixing, the reading is that the direct-product decomposition of the tree gauge leaves no Jacobian depending on the field of re-fixing, and the cost moves to the ghost determinant and the coupling at the seam (§ 05).
- The four rows gather at the row of background-dependent absorption and many steps, and the irreducible core is the small-field contribution to the coupling (the two-loop remainder uniform over small-field backgrounds, which we read as the content of Bałaban's unpublished basic inequality (Theorem 2 of Bałaban 1989)) and the multi-step background-regularity condition (§ 06).
- The row of the remainder including large fields, the observable-source condition and the row of the boundary of gauge re-fixing have been split. Net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling is an identity, and the latter is bounded under heat-kernel fluctuations (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma); a source in the block variables has no loss step by step; and that the tree gauge has no Jacobian is in Lean for finite groups and on paper for $SU(2)$ (§ 02, § 03, § 05).
- The small-field contribution to the coupling splits into "the local part + the Taylor truncation + the small-field cut + the explicit quadratic term between blocks", and for the first three the same form as in the sources for flat fields holds conditionally on the group too. The only thing absent from the sources is the group version of the last term, which the sources themselves call a sketch (§ 06).
- The explicit quadratic term between blocks W^{gauge} does not start at second order as in the flat-field model, but appears from one loop at $O(1)$ $(13/384)$ in the smallest two-block system. Its origin is not the non-commutativity of the group but the non-locality of the background map; the one-loop formula can be written explicitly, and what needs local cancellation is the $O(1/\beta)$ that comes after it (§ 06; $SU(2)$, seven in a row, one step, quadratic coefficients).
- Its non-local tail type (flat, exponential, power) is decided by the definition of the coarse variables. We read that the coordinates of "coarse links plus global tags" fit the cubic-remainder estimate in four dimensions. In three dimensions without boundary it was exponential even at one loop, with the sum uniform in volume; on the four-dimensional periodic lattice it was exponential at tree level (§ 06; the four-dimensional one loop has not been measured).
- In the one-row model, the recursion of the core row closes up to fourth order in the form "a contracting image of the pair types (in the ratio of couplings, quadratic $5/8$, quartic $\leq 5/48$) + new terms independent of the step + three-body terms decreasing exponentially in the width". This is the one-row version of the additivity of the sources (§ 06; one row, definition without a conserved quantity, conditional).
- The ghost determinant of the row of the boundary of gauge re-fixing, for one step, at one loop and on a critical-point background, agrees identically with the effective action of the tree gauge, and fits into one line of the bookkeeping of the local couplings (§ 05).

01

Statement of the four rows

The notation is that of 15-00 § 01 and 15-07: the effective coupling g_k at step k, $\beta_k = 4/g_k^2$, the small-field threshold $\varepsilon_k = g_k p(g_k)$, the weight G_k, the measure of the fluctuations μ_k,W, the set of blocks $B(X)$. The "one line" of Table 1 is a conditional sentence: if it were shown, that row would close Assumption .

Table 1 The four rows that do not close (every "one line" is a statement; labels only in the column of closed components)

Row	What would close it if shown (one line)	Boundary with the closed components	Beyond the boundary

Row	What would close it if shown (one line)	Boundary with the closed components	Beyond the boundary
1 The one-step remainder including large fields (the net one-step bracket of the coupling)	That in a real step (including gauge-covariant averaging, the background field and the large-field region), the coefficient of F^2 read from the effective action satisfies $1/g_{k+1}^2 = 1/g_k^2 - b + r_k$, $\lvert r_k\rvert \le C g_k^2 + \eta_k$ ($\sum \eta_k < \infty$, C, η depend neither on k nor on the lattice spacing), and that the many-body contributions of large fields are bounded by $e^{-c\,p(g_k)^2} \le C' g_k^2$	Net one-step bracket of the coupling + definition and continuity of the effective coupling ⇒ the condition that the coupling stays small is the Coupling-Smallness Induction Lemma Proved here . $\beta'/\beta = \sum f^2$ at tree level is the Unit Bundle Theorem Proved here (restricted to the commutative case). The deviation of $O(1)$ at one loop is the numbers for the smallest systems Reading . The splitting net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling is an identity, and the large-field contribution to the coupling is bounded by the Lipschitz constant of the read-out × rarity under heat-kernel fluctuations (independent on each edge, one step, independent of the background) Proved here (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, § 02). The same form for one Wilson link Proved here (the One-Link Heat-Kernel Comparison Lemma). The splitting of the local part of the small-field contribution to the coupling and the Taylor remainder has the same form as in the sources for flat fields Cited , Proved here (under the analyticity condition at complex backgrounds, § 06). The explicit quadratic term between blocks is $O(1)$ from one loop in the smallest two-block system, and the rest after putting in the first order explicitly is $O(1/\beta)$ Proved here (Gaussian minimisation and the one loop of Laplace, formal), Reading (numbers, § 06). The algebraic layer (the projection $\mathbf{Loc}$, the layer of W^{gauge}, the assembly of the remainder) is in Lean (§ 06). Passing to the second step, in the one-row model the image of the non-local terms closes on the same types and contracts in the ratio of couplings, the new terms are independent of the step, and the three-body terms decrease exponentially in the width Reading (one row, definition without a conserved quantity, one step twice, up to fourth order, § 06).	Of the small-field contribution to the coupling: the group version of the local cancellation of $O(1/\beta)$ in four-dimensional blocking, and the volume-uniform estimate of the non-local tail (the cubic-remainder estimate; the tail type depends on the definition of the coarse variables; with coarse links plus global tags it is exponential on the four-dimensional tree; the four-dimensional one loop not yet, § 06); the recursion of W in four dimensions (the number of three-body invariants, four-body terms); the probability of large fields under the true measure of the fluctuations (the group version of the exponential moment of fluctuations = the row of background-dependent absorption and many steps); the definition of the read-out map (what is called β_{k+1})

Row	What would close it if shown (one line)	Boundary with the closed components	Beyond the boundary
		In the coordinates of coarse links plus global tags, the one-loop tail in three dimensions without boundary is exponential Reading (§ 06)	
2 The source for observables (the observable-source condition)	That for the activity Ke^{J} with a smooth source J (of strength $\leq s_0$) supported on scales larger than L_0/M, $\|\theta(Ke^{J})\|_{G_{k+1}} \leq 2^{\|B(X)\|_e} Cs_0^{\|B(X)\|} \|K\|_{G_k}$ holds for the real $\mu_{k,W}$ with a constant independent of k	One step without a source is the Lemma on One Step with Background-Independent Fluctuations (when the law of the fluctuations does not depend on the background) Proved here . A source that is a function of the block variables commutes with the integration over the fluctuations, and there is no loss step by step Proved here (in Lean for the finite case). In one heat-kernel step, the source from fine loops is a heat-kernel shift of the coupling for each representation Proved here (§ 03)	That at the fine steps the source becomes a non-local activity (its support spreads over blocks of size loop length / lattice spacing)

Row	What would close it if shown (one line)	Boundary with the closed components	Beyond the boundary
3 The absorption of η and the multi-step induction	That, splitting $\mu_{k,W}$ = (a law independent of the background) $\times$ (a density $\rho_{k,W}$), what remains of the background dependence of $\log \rho_{k,W}$ after absorbing the F^2 direction and the shift of the centre into the parameter and a re-choice of the background is $\eta_k = O(g_k^{2+\delta})$ $(\delta > 0)$, and that the background-regularity condition and the contraction of the many-body norm hold uniformly in the step	Absorption happens exactly with one and two plaquettes Proved here (15-06). $\eta = 0$ of the Lemma on One Step with Background-Independent Fluctuations Proved here . The bookkeeping of the transverse contraction Proved here (15-03 § 04). The one-Wilson-link version of the Lemma on One Step with Background-Independent Fluctuations Proved here (conditional, 15-07 § 05). The shape part of η enters a geometric series Proved here (assuming the one-row chain and the one-step inequality, 15-07 § 06). Giving the class the background a as a parameter, the rest drops to $\theta^0 \beta^{-3/2}$ for two links and for four in a row, and the budget is uniform in the number of steps Reading (counting orders), Proved here (the arithmetic of the budget, 15-07 § 06). In the one-step of a single loop, the update of a closes to all orders by the group product Proved here (15-07 § 07)	The values of the coordinates of the multi-loop class (in three dimensions $2 \times 2 \times 2$: per face plus neighbouring pairs at second order, the corner triple product at third order, multi-face couplings at fourth order; the corner triple product does not contract in the second step but approaches a fixed point; in four dimensions one parameter plus the ratio coordinates (the skew neighbour is needed at second order), and what remains is the existence of the fixed point and the values depending on the rules, 15-07 § 07); the background-regularity condition stacking steps; the update when the class is given the coupling b as a parameter (15-07 § 05). For a class that does not have a as a parameter, the budget of the group-geometric part of $SU(2)$ (the third order of BCH) diverges (15-07 § 07)

Row	What would close it if shown (one line)	Boundary with the closed components	Beyond the boundary
4 The seam of re-fixing of the gauge fixing	That the Jacobian of the field-dependent transformation connecting the local gauge of step k with the local gauge of step $k+1$, and the terms at the seam, can be estimated within the class as a loss with finite sum (of type $g_k^{2+\delta}$ or a geometric series)	The Orbit-Direction Flatness Proposition (Hess = 0 in the orbit directions) Proved here (15-01 § 03). There is no global gauge fixing Cited . Existence and uniqueness of the tree (axial) gauge (no Gribov multiplicity), the direct-product decomposition of configurations, the constancy of the Faddeev–Popov weight, and that re-fixing between two trees sharing a root is a sum-preserving bijection Proved here (Lean for finite groups, paper for $SU(2)$, § 05)	The form of the cost is settled: in the tree gauge there is no Jacobian, and the cost is the degradation $\|W_e - 1\| \le (d-1)(L-1)\varepsilon$ of the small-field condition and, when a background-covariant gauge is used alongside, the ghost determinant. The latter, for one step, at one loop and on a critical-point background, agrees identically with the effective action of the tree gauge and fits into one line of the bookkeeping of the local couplings Proved here (the finite-dimensional Faddeev–Popov lemma, § 05). What remains is the integral around backgrounds that are not critical points, the same agreement at two loops and beyond, the uniformity of the local expansion, and advancing steps in the range where the block side L can be taken sufficiently smaller than ε_k^{-1}

The boundary between the parts that close and the parts that do not can be drawn by two questions Reading : "does the law of the fluctuations depend on the background?" and "is it at tree level, or at one loop or beyond?". The closed components lie either in fluctuations independent of the background (the Lemma on One Step with Background-Independent Fluctuations) or at tree level (the Unit Bundle Theorem, the bookkeeping of the Coupling-Smallness Induction Lemma).

02

> **Proposals for the row of the remainder including large fields — the one-step remainder including large fields**

The proposal to obtain rarity by Chebyshev: the large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations. If on X the group version of the closure under the fluctuation expectation (the group version of the exponential moment of fluctuations) holds with $s = tc/h_k^2$, that is, if $\mathbb{E}_{\mu_{k,W}} \exp\bigl(s \sum_{l \in \bar{X}} \|A_l\|^2\bigr) \le 2^{|B(X)|}$, then by Chebyshev's inequality

$$P_{\mu_k,W}\Bigl(\sum_{l \in \bar{X}} \|A_l\|^2 \ge \varepsilon^2 |\bar{X}|\Bigr) \;\le\; 2^{|B(X)|}\, e^{-s\varepsilon^2 |\bar{X}|} \tag{1}$$

holds Proved here (a one-liner assuming the group version of the exponential moment of fluctuations). If the threshold is taken of type $\varepsilon = h_k\, p(g_k)$, the right-hand side is $2^{|B(X)|} e^{-tc\, p(g_k)^2 |\bar{X}|}$, which is the form of the large-field rarity condition. The rarity of large fields needs no independent estimate: the exponential moment of the group version of the exponential moment of fluctuations is itself the rarity. That the weight of large fields plays this role in the appendix for flat fields is a known structure Cited (15-02 § 04); all we added on the group is the single phrase "if the group version of the exponential moment of fluctuations holds". For the product of heat kernels in the Lemma on One Step with Background-Independent Fluctuations, $P \le m^{|\bar{X}|} e^{-s\varepsilon^2 |\bar{X}|}$ follows at once from $M = m^{|\bar{X}|}$ Proved here . Caveat: the group version of the exponential moment of fluctuations itself depends on the row of background-dependent absorption and many steps, so the large-field rarity condition of the row of the remainder including large fields is subordinate to the row of background-dependent absorption and many steps.

Heat-Kernel Fluctuation Rarity Lemma — for heat-kernel fluctuations the group version of the exponential moment of fluctuations holds in closed form. The heat kernel p_t ($t = 1/\beta$) of $SU(2)$ is bounded above by the three-dimensional Gaussian in the radius just by dropping the negative terms of the sum over images and using $\sin\theta \le \theta$: for non-decreasing $g \ge 0$, $\mathbb{E}_{p_t}\, g(\theta) \le e^{t/2}\, \mathbb{E} g(|Z|)$, $Z \sim N(0, tI_3)$ Proved here (the Heat-Kernel Gaussian Upper-Bound Lemma). Under the edgewise independent heat-kernel condition, the single-step condition and the background-independence condition for fluctuations, the probability that any link in the set $\bar{X}$ exceeds the threshold h is at most $|\bar{X}|\, e^{t/2}(1 + \sqrt{2/\pi}\, h/\sqrt{t})\, e^{-h^2/(2t)}$, and the exponential moment is $m(s,t) \le 1 + e^{t/2}\bigl((1-2st)^{-3/2} - 1\bigr)$ ($0 \le st < \tfrac{1}{2}$); neither depends on the background Proved here (the Heat-Kernel Fluctuation Rarity Lemma). So under heat-kernel fluctuations the large-field rarity condition is a one-liner without assumptions. The finite Chebyshev step and the union bound are in Lean: `weighted_tail_le_exp_moment` (weights ≥ 0, $\sigma \ge 0$), `union_tail_le_sum` (weights ≥ 0). In numbers ($t = 1/4$–$1/32$, $h = 0.3$–1) the bound lies above the true value in all 12 cells (ratio 1.03–1.34), and a negative control with altered constants fails in 10 cells with $h^2/t \ge 1$ Reading .

The same form for one Wilson link (the One-Link Heat-Kernel Comparison Lemma). The heat-kernel bound as it stands does not apply to the one-link Wilson density (it fails in 7 of the 15 cells $b = 4$–64, $h = 0.3$–1) Reading . Bounding by K times the heat kernel $p_{1/b'}$ ($b' = b(1 - \theta_0^2$

/12)) inside the angle cap θ_0, and sending the outside to the rare tail, the probability of large fields and the exponential moment are bounded for one Wilson link too by K times the heat-kernel quantities + the tail Proved here ($SU(2)$, one link, one step). In numbers the bound is above in all 45 cells (ratio 1.07–1.91), and the threshold of the exponential-moment condition for fluctuations is 0.44–0.95 times the closed form for Wilson Reading (details in 15-07 § 05). This establishes the one-link version of the group version of the exponential moment of fluctuations needed to obtain the large-field rarity condition under the true measure of the fluctuations, but the multi-link group version of the exponential moment of fluctuations remains in the row of background-dependent absorption and many steps.

The proposal to split the remainder into small and large fields: split the net one-step bracket of the coupling into the small-field contribution to the coupling and the large-field contribution to the coupling. If the read-out map b (the map that reads β_{k+1} off the effective density; a candidate is the quadratic coefficient of the single coarse plaquette) is Lipschitz with respect to total-variation distance with constant $L_b \le C\beta^2$ ($db/dE \approx \frac{4}{3}\beta^2$ from $\mathbb{E}[\frac{1}{2}\operatorname{tr} U] \approx 1 - 3/(4\beta)$ for one link), and the difference in total variation between the density with one large-field region inserted and the density without it is $e^{-c\,p(g)^2}$, then the remainder due to large fields is $|r_k^{\text{large}}| \le C\beta_k^2 e^{-c\,p(g_k)^2}$, and if $p(g) = c_0(1+\log g^{-1})^{p_0}$ ($p_0 \ge \frac{1}{2}$, with cc_0^2 large enough) it is $\le C' g_k^2$ Assumption (arithmetic under two assumptions). The remaining core is the small-field contribution to the coupling = the two-loop remainder uniform over small-field backgrounds, which we read as corresponding to the content of the unpublished basic inequality Assumption (a reading from one sentence of the preface, 15-02 § 02). Caveat: if the read-out is taken to be the quadratic coefficient, two coordinates are needed at the quadratic level (15-06), so the Lipschitz property has to be stated on the pair.

The proposal to split the remainder into small and large fields has been split (the Large-Field Remainder Bound Lemma). Writing ρ_S for the density conditioned on small fields, the splitting $r = r^{\text{small}} + r^{\text{large}}$, $r^{\text{small}} := b(\rho_S) - \beta_{\text{pred}}$, $r^{\text{large}} := b(\rho) - b(\rho_S)$ is an identity Proved here (β_{pred} is the tree + one-loop prediction). The Lipschitz constant of the read-out $b = 1.5/(-\log \mathbb{E}\cos\theta)$ in total variation is $\frac{4}{3}b^2/E \to \frac{4}{3}\beta^2$, and $|r^{\text{large}}| \le \frac{4}{3}(b_S^2/ES)\,P_L$ (P_L is the probability of large fields, $E \in (e^{-2}, 1)$) Proved here. The actual sensitivity per unit probability of large fields, however, is $\approx \frac{1}{3}\beta^2 h^2$, which is first order in β if the threshold is taken at the small-field scale $h \propto 1/\sqrt{\beta}$ Reading (one link; the same numbers for the heat kernel and Wilson). This is because the read-out function $\cos\theta$ is flat at $\theta = 0$. For one link under the three conditions — edgewise independent heat kernel, single step, background independence of fluctuations — a sufficient condition for $|r^{\text{large}}| \le C' g^2$ is, for the threshold $\theta_h = g\,p(g)$ ($\beta = 4/g^2$, that is, $\sqrt{\beta}\,\theta_h = 2p$; log is the natural logarithm), $p(g) = c_0(1+\log g^{-1})^{1/2}$, $c_0^2 > 2$ Proved here (arithmetic; from the total-variation form, $c_0^2 > 3$). **The core that remains is the single term r^{small} = the small-field contribution to the coupling**, and its splitting is in § 06. To obtain r^{large} under the true measure of the fluctuations, the group version of the exponential moment of fluctuations is needed (the row of background-dependent absorption and many steps).

The check on the slab. On a 2×2 slab of heat kernels with fluctuations independent on each plaquette ($SU(2)$, one step, images dropped, 20 fine plaquettes), we counted the difference r

large between the image with a large-field region Ω inserted and the image without it. In the read-out of one coarse plaquette, $\mathbb{E}\frac{1}{2}\operatorname{tr} W_B$ splits into a product over plaquettes (a convolution of characters), so r^{large} is determined by the six plaquettes that enter the block alone, and the value with the whole slab inserted is $(b_B/\beta)^2 \sum_i a_i^2 = 1/3$ times the value for one link at the same β [Paper] (the numbers are 0.3334–0.3358 [Computed]). The image is multilinear in the plaquette-wise laws, so $\|\rho_\Omega - \rho_0\|_{\mathrm{TV}} \le 1-(1-q)^{|\Omega|}$, and to first order $r^{\text{large}}(\Omega)$ is a sum over single plaquettes [Paper]. Taking the threshold at $\theta_h = g\,p(g)$ of the convention above, $|r^{\text{large}}|/g^2$ decreases up to $\beta \le 256$ for $c_0^2 > 2$, $c_0^2 = 2$ is the boundary (only there C' grows on the order of p^3; the behaviour at the boundary was checked for one link up to $\beta \le 4096$), and if the threshold is fixed at the small-field scale $\theta_h \propto 1/\sqrt{\beta}$ it grows in proportion to β [Computed]. The position of the boundary agrees with the sufficient condition $c_0^2 > 2$ of the Large-Field Remainder Bound Lemma [Reading]. A large field on a shared edge moves the coupling between the two blocks it touches by the same order as the diagonal change, so it fits to count the large-field region as "the union of the blocks touching that plaquette" [Reading]. The slab does not add to the difficulty of one link; all the difficulty of many links remains on the side of "the fluctuations are not independent" — the multi-link group version of the exponential moment of fluctuations under the true $\mu_{k,W}$ (the row of background-dependent absorption and many steps). Conditions: the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; one step; heat kernel with images dropped.

The proposal of the parity of critical points: send the background dependence to the coefficient of θ^4 by the parity of the critical point. If around the true critical point under the constraint the action of the fluctuations is even in the fluctuations (a fact checked with two plaquettes, 15-06 § 02), there is no contribution of $O(\theta)$ to the reading of β', and the background dependence goes entirely into the coefficient of θ^2 (the reading of F^2 itself) and into θ^4 and beyond (the source). The requirement "uniform over the background" in the small-field contribution to the coupling then weakens to "the coefficient of θ^4 is bounded", and that part is paid for by the transverse contraction (15-03 § 04) [Assumption] (parity is a fact checked with two plaquettes, trivial with three).

With four plaquettes parity breaks, and the conclusion stands for another reason. With four plaquettes in a row, the parity of the action of the fluctuations around the true constrained critical point breaks. The breaking concentrates in the breathing direction (the geodesic along which the two inner plaquettes tilt in the same direction along the axis of U^*; independent of coordinates), and for the Wilson action $S(t) - S(-t) = 2\beta \sin(\alpha^*/2)(\sin 2t - 2\sin t)$ holds to all orders [Paper] (the cubic coefficient is $-\beta\sin(\alpha^*/2)$; agrees with numerics to 7 digits on 18 backgrounds [Computed]). It is even only when $W_b = W_a^{-1}$. The conclusion of the proposal of the parity of critical points, however — that there is no background dependence of odd order in the reading of β' — stands without relying on parity. There is no θ^1 because $\mathfrak{su}(2)$ has no conjugation-invariant linear form (gauge invariance); there is no θ^3 because, on a background of two loops, the pairs (W_a, W_b) and (W_a^{-1}, W_b^{-1}) are simultaneously conjugate by a rotation by π about the axis perpendicular to the two axes (the pair version of $-1 \in$ the Weyl group); and the effective weight w is exactly even to all orders under the reversal of the background $(a,b) \to (-a,-b)$ [Paper] (also checked numerically [Computed]). So for a chain in a row (two loops of

independent backgrounds), the background uniformity of the small-field contribution to the coupling weakens to "the coefficient of θ^4 is bounded uniformly over the background", and its ground changes from the parity of the fluctuations to the covariance under reversal of the background (the boundedness of the coefficient of θ^4 itself is not shown). In a four-dimensional block there are three or more independent loops, and there is an odd invariant, the triple product $a \cdot (b \times c)$ (of type $f_{abc} F^3$), so boundedness of the coefficient of θ^3 is also needed Assumption . On the side of the law of the fluctuations (the score), an odd part of order θ^0 $\beta^{-1/2}$ remains (RMS $\times$ $\sqrt{\beta}$ is 0.07–0.13), of the same type as the remainder in 15-07 Computed .

The proposal of a parameter for the distribution over the centre: put large fields into the class as the parameter z of the distribution over the centre. This proposal splits the weight f of one plaquette into $f = f_e + f_o$ by the parity of the characters of the centre $\mathbb{Z}_2$, and calls the relative weight $z := f(\pi)/f(0)$ of the far hemisphere the large-field parameter ($z = e^{-2\beta}$ for Wilson; of order $e^{-\beta\pi^2/2}$ for the heat kernel, from the leading term of the sum over images). If the large-field region is counted not as outside the class but as the contribution of $z^{|\Omega^c|}$, the large-field rarity condition becomes a statement inside the class Assumption . Caveat: this is about class functions of a single plaquette; with many links the shift of the centre becomes a centre vortex (a surface of signs of $\mathbb{Z}_2$ of the plaquettes) and cannot be fully counted by the z of a single plaquette (the caveat of 15-04 § 01).

03

Proposals for the observable-source condition — the source for observables

The proposal to take the source as a function of the block variables: take the source to be a function of the block variables. If the observable J is defined as a function of the block variables V of step k_J (a Wilson loop of the coarse lattice, a smooth functional of the blocked field), then for $k < k_J$ the integration over the fluctuations of step k is carried out with V held fixed, so $\int \rho_k e^{sJ(V)} d\mu = e^{sJ(V)} \int \rho_k \, d\mu$ holds exactly and the constants of the small-field analyticity condition and the large-field rarity condition do not change (loss 0) Proved here (trivial). For $k \geq k_J$, J becomes a local function of the coarse lattice (a term for each loop shape), and e^{sJ} shifts the couplings by loop shape of the action by s. The line is that the observable-source condition reduces to "the coordinates of the class-closure condition include the couplings by loop shape, and the one-step map is Lipschitz in those coordinates" Assumption (the parameter coordinates of 15-06). Caveat: physical observables are loops of the fine lattice, and the difference from the blocked loops is the renormalisation of observables. Restating the observable-source condition with blocked observables is a re-choice of the statement, not the observable-source condition for fine loops. We have not checked in the text which of the two the "expectation values of loops" in Bałaban's preface refers to.

The accounting of the loss from the source, and the source from fine loops. The commutativity of the proposal to take the source as a function of the block variables is written in Lean for the finite case (`coarse_source_commute`, `coarse_source_commute_comp`; assumptions: configurations and coarse variables finite, steps nested). In norms, for a source that is a sum of local terms per block with strength s_0, the factor $e^{s_0 |B(X)|}$ appears only once and does not pile up step by step Proved here (limited to blocked observables). A source from loops of the fine

lattice, for one step under the edgewise independent heat-kernel condition and a self-avoiding loop C, is $\mathbb{E}\,F(U_C) = (F * p_{t|C|})(W_C)$; that is, it merely shifts the coupling of representation j by a factor $e^{-tj(j+2)|C|/2}$ Proved here (Monte Carlo gives $|z| \le 1.2$ at both first and second order Reading). The loss is not a constant factor but a contraction of coordinates (of perimeter-law form), and the coordinates of the class need a representation index. In a blocking step, with four plaquettes in a row ($SU(2)$ heat kernel, diagonal definition, one step), adding a source $\lambda \cdot \frac{1}{2}\operatorname{Re}\operatorname{tr} W(C)$ to a fine loop C inside the block shifts the coupling of the coarse action after one step by $\lambda f_C^2 + O(\lambda/\beta)$. f_C is the projection weight of the Bundle Network Theorem (15-05) and does not depend on β (the envelope theorem $\partial\beta'/\partial\beta_p = f_p^2$, the sum rule $\sum_p f_p^2 = 1/5 = \beta'/\beta$) Paper ; the one-loop correction is a rational d_1/β (an inner single plaquette $f_C^2 = 2/25$, $d_1 = -17/375$; an inner 1×2 loop $8/25$, $-16/125$; an outer single plaquette $1/50$, $-1/750$), and no $\log\beta$ appears Paper (one loop is formal) Computed ($\beta = 4$–2048, 9 digits by two routes). The newly born term is the quartic $-\lambda f_C^4/12$ of the coarse action, which also does not depend on β Computed . As long as it fits inside a block, the source from fine loops passes one step not as a loss but as a transfer of coordinates by the weights of the bundle network, and the Lipschitz constant $f_C^2 + O(1/\beta)$ with respect to the source of one step is uniform in β (the one-step, one-block version of the observable-source condition). Loops crossing the boundaries of blocks, four-dimensional blocks, the true $\mu_{k,W}$, large fields and many steps have not closed.

The proposal of position-dependent couplings: the class of position-dependent couplings $\beta(x)$. A source of plaquette-density type $J = \sum_p f(p)(1 - \frac{1}{2}\operatorname{tr} U_p)$ (with f slowly varying on the scale L_0/M) is the Wilson action with β replaced by $\beta(x) = \beta + sf(x)$. If the one-step map is the map of β applied pointwise plus a gradient correction $O(|\nabla\beta|\,a_k/L_0)$, the gradient terms form a geometric series in $a_k/L_0 = 2^{-(K-k)}$ and the sum is finite Assumption . For the abelian Villain action of $U(1)$, the β' of one step with four plaquettes in a row is the effective stiffness (Thomson's formula) of the network of the plaquette-wise stiffnesses $\beta(p)$, and putting $\beta(p) = \beta + s\,g(p)$ gives $\beta' = \beta/5 + s\sum_p f_p^2 g(p) + O(s^2/\beta)$; that is, the first order agrees exactly with the local average with the weights of the bundle network Paper (pointwise product with images dropped; the numbers agree with Thomson's formula to within 6×10^{-12} Computed). The gradient correction starts at second order, a form one step better than the $O(|\nabla\beta|\,a_k/L_0)$ above. It cannot be used for loop-type observables (the source of the area law) and is suited to local observables.

Rejected: peeling off the source with Hölder alone. The road that splits the estimate into $(\mathbb{E}G^2)^{1/2}(\mathbb{E}e^{2sJ})^{1/2}$ does not close: at the fine steps the support of J becomes a non-local activity spread over blocks of size loop length / lattice spacing, the one-step loss $2^{|B(X_J)|}$ cannot be paid for by a small factor per block, and it accumulates over successive steps (Table 2).

04

Proposals for the row of background-dependent absorption and many steps

The content of the row of background-dependent absorption and many steps is in 15-07 § 06 and § 07. We write only the main points. For the Wilson action of $SU(2)$, the conditional law of

one link is a member of a one-parameter family independent of the background, and the background enters only through the local coupling and the direction of the centre Cited (a fact of the heat-bath method; bibliography not checked). For many links, the quaternion identity splits the action into "local coupling $\times$ a form independent of the background" + a linear term + the conjugation mismatch Proved here , and the screening proposal is that, sorting the score of the density by the order of θ, part of θ^0 disappears by recentring and part of θ^1 is absorbed as a coupling derivative Assumption . In the smallest system the lowest orders of the rest that cannot be absorbed are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$, and however the threshold is chosen the sum diverges; but giving the class the background a as a parameter, the rest drops to $\theta^0\beta^{-3/2}$ and the budget becomes uniform in the number of steps Reading (the numbers for two links and four in a row, conditional; 15-07 § 06).

Put it on the bookkeeping of the transverse contraction. The rest after absorption (the contribution of the shape coefficients cut off by the class, and the expansion of $b_p = \beta\cos(\theta_p/2)$ at θ^4 and above = the source) is a small shift of the transverse coordinates, and with the bookkeeping of 15-03 § 04, if $\lambda < 1$ then the deviation is bounded not by a sum but by the supremum $\sup_k \varepsilon(\beta_k)/(1-\lambda)$ Proved here (a geometric series assuming the one-step inequality). The multi-step background-regularity condition is needed in the form "the fine angles of the image of the background field are L^{-2} times the coarse angles + $O(g^2)$ of the transverse coordinates" Assumption . The transverse contraction was measured at the commutative tree level (15-05); its values for non-commutative configurations and at one loop are not yet known.

05

Proposals for the row of the boundary of gauge re-fixing — the seam of re-fixing of the gauge fixing

The tree-gauge proposal: there is no Jacobian of re-fixing — the direct-product decomposition of the tree gauge. Take a spanning tree T_B (with root r_B) in each block B, and split the gauge group $G^\Lambda = \prod_x G$ into the factor $G_{\text{int}} := \prod_{x\,/=\text{root}} G$ of the non-root points and the factor $G_{\text{coarse}} := \prod_B G$ of the roots (it is pointwise a direct product, so trivially a direct product). The map fixing the tree links to 1 is a section of the action of G_{int}; since each point has exactly one tree link heading towards the root, it is solved uniquely, the axial (tree) gauge is complete with no Gribov multiplicity, and the Faddeev–Popov determinant is constant. If block averaging is gauge covariant, G_{coarse} is isomorphic to the gauge group of the coarse lattice, and the tree gauge of step $k+1$ fixes it. That is, **the gauge fixings of step k and step $k+1$ each fix one of the two factors of a direct product of the same group, and there is no Jacobian depending on the field of re-fixing** Proved here (a restatement of a known structure; that the Faddeev–Popov determinant of the axial gauge is constant is a standard fact).

With this, "the Jacobian (the Faddeev–Popov determinant) enters the effective action" of 15-01 § 03 becomes empty for the axial gauge. The cost moves to two other places Reading : (i) the degradation $\|U_l - 1\| \le \ell\varepsilon$ of the small-field condition due to the tree length ℓ (7-05 § 05), and (ii) if a background-covariant gauge is used for the fluctuations, its ghost determinant. We have not checked in the text which gauge Bałaban used in the four-dimensional papers (in three dimensions, an axial gauge per block is introduced by the Faddeev–Popov method).

In Lean for finite groups, on paper for $SU(2)$. On a rooted spanning tree, a gauge transformation setting every tree edge to 1 exists, two such that agree at the root are equal (no Gribov multiplicity), and the remaining gauge freedom is only a global constant. The set of all configurations decomposes as a direct product "gauge transformations equal to 1 at the root" × "configurations equal to 1 on the tree edges", the weight of each representative is $|G|^{|V|-1}$ and does not depend on the representative (the Faddeev–Popov weight is constant), and the map between the representatives for two trees sharing a root is a bijection preserving sums of gauge-invariant functions. The average over representatives does not depend on the tree even without assuming the roots coincide. All this is in Lean: `exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, `regauge_bijective`, `regauge_sum_eq`, `slice_average_tree_indep` (assumptions: group, vertices and edges finite; functions gauge invariant; re-fixing with a shared root). For $SU(2)$ with the normalised Haar measure, the change of the variables off the tree has Jacobian 1 by left and right invariance, and the Faddeev–Popov quantity is identically 1 by a triangular structure in order of depth Proved here. That the Faddeev–Popov determinant of the axial gauge is constant and that there is no Gribov multiplicity are standard facts Cited; what we added is the formalisation for finite groups and the form of the cost.

The form of the cost. In the full axial tree (the parent is $x - e_\mu$, where μ is the largest direction with $x_\mu > 0$), the variable $W_{x,\mu}$ of an edge (x, μ) off the tree is the holonomy of a fundamental cycle, and that cycle is spanned by a comb of $N_{x,\mu} := \sum_{\nu>\mu} x_\nu$ plaquettes. If $\|U_p - 1\| \le \varepsilon$ (operator norm) on every plaquette, then

$$\|W_{x,\mu} - 1\| \;\le\; N_{x,\mu}\,\varepsilon \;\le\; (d-1)(L-1)\,\varepsilon \quad (2)$$

Proved here (L is the number of points on a block side; the triangle inequality of 7-05 § 05 applied N times). The bound is attained by a uniform abelian field (in four dimensions, $L = 4, 6$ gives ratios 8.997 and 14.986 against bounds 9 and 15) Reading. For random fields the ratio grew close to $\sqrt{L}$ (1.58→2.15) Reading. **The seam of re-fixing is not a Jacobian but the management of the small-field condition**: to use the field expansion in the tree gauge at step k, the block side L must be taken sufficiently smaller than ε_k^{-1}.

The proposal to absorb the ghost determinant: absorb the ghost determinant in the coordinates of the local couplings. For the ghost operator $\Delta^{\mathrm{gh}}_W = d^*_W d_W$ in the background W (the covariant Laplacian of the lattice, in the adjoint representation), the prospect is that $\log\det \Delta^{\mathrm{gh}}_W - \log\det \Delta^{\mathrm{gh}}_1$ can be expanded into local functionals on small-field backgrounds, with leading term $\sum_p c_{\mathrm{gh}} \|F_p\|^2$ (the coefficient of F^2 = the ghost contribution to the one-loop coefficient), the rest being θ^4 and beyond (the source) and exponentially decaying non-local terms, with constants independent of k Assumption (of the type of the heat-kernel expansion on the lattice; uniform convergence of the local expansion has not been shown). If so, the F^2 part merges into the shift of local couplings of the row of background-dependent absorption and many steps, θ^4 and beyond go into the transverse coordinates, and the seam becomes not a new kind of loss but part of the one-loop coefficient. As a negative control, for $U(1)$ the ghosts do not depend on the background.

The proposal to absorb the ghost determinant merges at one loop — the agreement of the two gauges. Fix the holonomies around the blocks (the coarse variables) and take the background W to be the minimiser of the action under that constraint; then the background is a critical point, and for the Hessian of the fluctuations H and the linearised gauge generator D we have $HD = 0$. Then, from the finite-dimensional Faddeev–Popov lemma,

$$\log\det H_{\mathrm{tree}} \;=\; \log\det\bigl(H + DD^{\mathrm{T}}\bigr) \;-\; 2\log\det M_{\mathrm{FP}}, \qquad M_{\mathrm{FP}} = D^{\mathrm{T}}D \tag{5}$$

holds Proved here (conditions: one step, one loop, the background is a critical point, D is injective, H is positive on $(\mathrm{im}\, D)^{\perp}$). The left side is a one-loop quantity of the tree gauge and the right side one of the background-covariant gauge ($\xi = 1$, ghost included), so **the one-loop effective actions of the two gauges are identically equal as functions of the background**, and neither the marginal coupling β_0' nor the shape coordinates nor the background coordinates change. The decomposition of the determinant of block form, and the logarithmic identity taking the output of the lemma as an assumption, are in Lean: `det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks` (assuming $\det H' > 0$, $\det K > 0$, and the tree quantity is $\det H'/\det K$), `coeff_tree_eq_coeff_cov` (extracting linear coefficients). The lemma itself (the change of variables in the integral) is on paper. Numbers Reading ($SU(2)$, one block of 3^4 points, commutative background, $\theta \le 1$, two routes, analytic and finite differences): the residual of (5) is 4.5×10^{-13} or less, and the background dependence of the ghost determinant is real and nearly local (coefficient $+0.024$ per classical action of $\log\det M_{\mathrm{FP}}$, 2.4% of the coefficient -1.003 of the one-loop F^2). It cancels against the extra part of the gauge-field determinant of the background-covariant gauge. Negative controls: for $U(1)$ the ghost does not depend on the background (difference 0); dropping the ghost shifts β_0' by 0.024; and moving the background off the critical point breaks (5) at first order in the gradient. Therefore we read that the ghost is not a new loss at the seam but one line of the bookkeeping of the local couplings when the background-covariant gauge is chosen, and that in the tree gauge that line is absent from the start. What remains is the difference proportional to the equations of motion in constructions that integrate the fluctuations around backgrounds that are not critical points, the same agreement at two loops and beyond (we expect it to be of the type of the lattice BRST identities; not checked), and the uniformity of the local expansion Reading .

The proposal to make the seam a single matrix: make the seam a single matrix. In the tree-level network that fixes the angles of all coarse plaquettes simultaneously (the Bundle Network Theorem of 15-05), the coarse action is a quadratic form $\theta^{\mathrm{T}} B\theta$, and the coupling between neighbouring coarse plaquettes and the mixing of the quartic map correspond to the coupling at the seam. The reading is that the question of the row of the boundary of gauge re-fixing, "are the terms at the seam a loss with finite sum?", becomes the single number "does the norm of the matrix of the quartic map fall below 1?" Reading . Under the Bundle Network Theorem the map measured in ℓ^4 exceeds 1/16 in the coarse direction (the largest value found in four dimensions is about 0.15), and we have no upper bound uniform in the volume Reading (15-05). In the composition of three steps, $\mu_3 \le 0.20$ at every step (tree level, commuting, Reading ; measured up to the kernel of step 4, with Stein^{1/4} of the step differences 0.300, 0.180, 0.084, 0.031, and ≤ 0.163 by perturbation for $k \ge 4$), and the three-step value is 0.099 ($\lesssim$0.12 with the truncation). The sup of the linear ratio for a three-colour field equals the one-colour sup

(paper: the spherical average $E_s(s\cdot v)^4 = |v|^4/5$; Lean: ColorAverage, with a finite weighted design), so the cone values measured with one colour carry over to three-colour fields unchanged. The non-commuting tree correction: the first-order coefficient of the cone ratio R depends on the choice of parameters for the coarse field (the coarse gauge components) and has no meaning as a quantity. At the maxima of the linear ratio R_0 (three points on 8^4 and 12^4, almost one-colour configurations) the first-order coefficient is below 10^{-5}, so the non-commuting correction to sup R begins at second order (computed; tree level, one step). The second-order coefficient at the maxima is positive but almost entirely the commuting non-linearity (cos); the non-commuting part is negative (computed; three points, one step). The one-step map of the coordinates of W (quadratic forms with differences, dimension 6), with the local representative corrected at the zero mode, has spectral radius 0.363 on a basis of 60 types (the rest line up at 0.25); the three-step composition has spectral radius 0.042, weighted ℓ^2 0.055–0.125 and ℓ^1 0.465 (computed; tree level, commuting; the tail outside the basis is bounded up to $D \leq 4$).

The wall for a road using class functions alone. On a road whose only coordinates are the class angles of the holonomy of each coarse plaquette (Migdal–Kadanoff is one), the coarse Bianchi identity (the product of the holonomies of the six faces of a coarse cube = 1) needed to reconstruct the coarse gauge field on the four-dimensional hypercubic lattice cannot be written. In the non-abelian case it can only be written once there is a common base point and connecting paths, and with class angles alone the relative orientation of neighbouring faces is lost. We read this as: **a road using class functions alone cannot go beyond the hierarchical-lattice approximation** Reading .

Can Haar invariance replace re-fixing? In the setting of the Lemma on One Step with Background-Independent Fluctuations, if the law of the fluctuations is conjugation invariant on each link, the decomposition $U = WY$ is gauge covariant and no gauge fixing at all is needed. But this is limited to renormalisation of the type that "gives" the law of the fluctuations by convolution with heat kernels (a Polchinski flow on the group), and convolution with heat kernels does not reduce the degrees of freedom, so it cannot replace the re-fixing of four-dimensional blocking, which produces block variables Reading .

06

Dependence among the four rows — the row of background-dependent absorption and many steps is the core

The four rows are not independent. Written in the direction "when the left closes, the right becomes lighter", they give the following diagram Reading (an estimate of the logic between the proposals).

```
Row of background-dependent absorption and many steps (absorption of η: background dep
  ├─→ group version of the exponential moment of fluctuations and group version of exc
  │      └─→ large-field rarity condition of the row of the remainder including large
  │             └─→ large-field contribution to the coupling of the row of the remaind
  ├─→ coordinates of the class-closure condition (β, couplings by loop shape, shape co
  │      └─→ observable-source condition (proposal to take the source as a function of
  └←── ghost determinant of the row of the boundary of gauge re-fixing (proposal to ab
         ↑
       the row of the boundary of gauge re-fixing has no Jacobian (tree-gauge proposal

Irreducible core:
  small-field contribution to the coupling = Loc + Taylor truncation + small-field cut
                the first three have the same form as in the sources for flat fields (
                absent from the sources: W^gauge and the group version of its local ca
                  W^gauge is explicit from one loop (13/384 for two blocks); cancellat
                  the non-local tail type is decided by the definition of the coarse v
                  recursion of W (one row) = contracting image of pair types + new ter
                → may weaken to "boundedness of the θ⁴ coefficient" via the proposal o
  multi-step background-regularity condition = regularity of the background field (fin
```

What to read from the diagram Reading : the large-field rarity condition and the large-field contribution to the coupling of the row of the remainder including large fields, the observable-source condition, and the ghosts of the row of the boundary of gauge re-fixing all have a form that can be written, once the coordinates of the absorption of the row of background-dependent absorption and many steps are fixed, as a corollary of it or as a shift of those coordinates. What remains independent of the row of background-dependent absorption and many steps is two things, the small-field contribution to the coupling and the multi-step background-regularity condition: the former is the unpublished part that Bałaban's series needed in order to remove the assumption (the coupling stays small), the latter is the multi-step version of the regularity of the background field.

Splitting the small-field contribution to the coupling one step further — the same form as known in the sources. We split the one-step small-field effective action $S'_h(\Phi)$ (threshold h on the fluctuation side, threshold h_0 on the background side, margin $h_0 \le q\,h$, $0 < q < 1$) into the local part $\mathbf{Loc}_{2k}$ (vacuum, marginal coupling β'_0, shape coordinates) and the remainder $R := (1 - \mathbf{Loc}_{2k})S'_h$. The marginal coupling is restricted to one number β'_0 (the long-wavelength value of the coefficient of F^2), and the rest of the quadratic form of the Bundle Network Theorem (non-local in the coordinates of θ) is put into the shape coordinates. This splitting has the same form as in the sources for flat fields (BBS's $\mathbf{Loc}$; $\mathbf{Loc} = \mathbf{Tay}_4$ in the hierarchical model, eq. (5.2.10)) Cited , and that the local coefficients are uniquely determined and the remainder goes to the higher-order side is algebra in a finite basis Proved here (Lean: `loc_split_unique`, polynomials in one variable $s = \psi^2$, of type $k = 2$). The statement has the following form Assumption (not shown in four dimensions): there are $\beta \ge \beta_0$ and constants $C, \kappa, r_0, c > 0$ independent of the size of the lattice and of the step such that

$$\beta'_0 = 2^{d-4}\beta - b + r, \quad |r| \le \frac{C}{\beta}; \qquad \|R_X\| \;\le\; Ce^{-\kappa d(X)}\Big[\Big(\frac{h_0}{r_0}\Big)^{2k+2}\big(1 + \beta\tau h_0^2\big) + e^{-c(1-q)^2h^2\beta}\Big]$$

holds (b is the one-loop constant of one step, τ is the BCH term of the non-abelian tree, which is 0 for the tree restricted to the abelian case). The remainder splits in two by the identity $R = (1 - \mathrm{Loc})S'_\infty + (1 - \mathrm{Loc})\Delta_{\mathrm{cut}}$ into the effective action without the cut S'_∞ and the part due to the cut Δ_{cut} Proved here . Since the indicator function breaks analyticity, the analyticity argument is applied only to the former, and the latter is paid separately in probability.

- **The Taylor truncation** $(h_0/r_0)^{2k+2}$. Under analyticity with the background extended to the complex range $|\theta_P| \le r_0$ and a bound independent of β (the analyticity condition at complex backgrounds), the Cauchy estimate gives $\|(1 - \mathrm{Loc}_{2k})S_X\| \le Me^{-\kappa d(X)}(eh_0/r_0)^{2k+2}/(1 - (eh_0/r_0)^2)$ $(h_0 < r_0/e)$ Proved here (assuming the analyticity condition at complex backgrounds; the assembly of the series is in Lean: `taylor_tail_bound`, assumptions $0 \le s < \rho$, $|a_m| \le M/\rho^m$). **This has the same form as Lemma 7.5.3 of the sources for flat fields (of type** $(h_+/h)^{k+1}$**)** Cited . Starting from the heat kernel with four plaquettes in a row, $r_0 = \pi$ (the distance to the conjugate point $-1 \in SU(2)$, the position of the singularity of $\log(w/\sin w)$), independent of β Proved here , and the remainder is proportional to $h^{6.00}$ and independent of β over $\beta = 8$–1024 Reading ($SU(2)$, four in a row, heat kernel, one step, agreement of the coefficients by two routes 10^{-11}). For $U(1)$ this term is at the numerical floor (3 to 6 digits lower), and the term in h^6 comes from the curvature of the group Reading .
- **The small-field cut** $e^{-c(1-q)^2h^2\beta}$. If the fluctuations are a product of heat kernels per link, it is bounded by the Heat-Kernel Fluctuation Rarity Lemma (§ 02) Proved here (under the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations). With four in a row, the contribution of the cut is a function of $h^2\beta$ alone; the rate is $c \approx 0.24$ without margin, and putting the background inside the threshold, $q \le \frac{1}{2}$, makes it $c(1-q)^2 \approx 0.75$, about 3 times larger Reading (the same form for both $SU(2)$ and $U(1)$). The balance when the steps are stacked is only $c_0^2 \ge 1/(c(1-q)^2)$ at $p(g) = c_0(1 + \log g^{-1})^{1/2}$ (1.33–4.2 with the rates for four in a row) Proved here (arithmetic, using the rates for four in a row).

The form "the one-step remainder is of size $C|B|(h^2\beta)^k$" does not hold. k appears only in $(h_0/r_0)^{2k+2}$, and $h^2\beta$ only on the side of the cut (the larger, the smaller) Reading (from the numbers for four in a row and the form on paper). The verdicts on the four components are as follows.

- (i) The law of the fluctuations does not depend on the background: it holds for heat kernels per link (the Lemma on One Step with Background-Independent Fluctuations) and for the lift with four in a row (the covariance of the fluctuations does not depend on the background) Proved here , and does not hold for four-dimensional blocking (the row of background-dependent absorption and many steps). What the small-field contribution to the coupling needs, however, is its weak form "the real part of the Hessian of the fluctuations is uniformly positive on complex backgrounds".
- (ii) The quadratic part is the leading term of the Bundle Network Theorem + a local one-loop term: assuming exponential decay of the covariance C_W of the constrained fluctuations Proved here . In a box of 2^4, $\beta' = 0.442260\,\beta - 0.591$ Reading (abelian saddle point, one loop).

- (iii) Quartic and higher: the lift is exact only for four in a row; for four-dimensional blocks an estimate of the Laplace remainder on complex backgrounds is needed (the content of the analyticity condition at complex backgrounds).
- (iv) Locality: finite range is replaced by exponential decay, and **at the quadratic level an explicit irrelevant term arises between blocks**.

The row absent from the sources. The term of (iv) is the row that remains after the splitting Reading (from comparing the text of Appendices A.1 and A.5 of the sources for flat fields with one sentence of Bałaban's preface; Bałaban's main text has not been checked). In the sources for flat fields this is $I_j = e^{-V_j}(1+W_j)$, $W_j = \frac{1}{2}(1-\mathrm{Loc})F_{w_j}(V(b),V(\Lambda))$ ((A.1.3)–(A.1.4)), and the local cancellation, in which the sum of the quadratic contributions is 0 up to cubic errors, is at (A.5.3)–(A.5.4) Cited. **The sources themselves describe this place as "we sketch an argument" and send the main body to another reference** Cited. What is needed on the group is its group version: to build, with the covariance of the fluctuations C_W in the background field,

$$W^{\mathrm{gauge}} = \tfrac{1}{2}\,(1-\mathrm{Loc})\,F_{C_W}\left(V(b),V(\Lambda)\right) \quad (4)$$

and its local cancellation — and to show that the quadratic contribution to the marginal coupling fits within "the one-loop constant of one step $b + O(g^2)$" Assumption. We read this as corresponding to what Bałaban's unpublished basic inequality calls "the computation of the second-order perturbation" Reading. This row splits into three.

- **Definition of the inter-block term**: the definition of W^{gauge}. A formula on a finite lattice, needing no estimate (algebra).
- **Local cancellation**: an identity for Loc under gauge invariance (algebra; expected to be of the type of a Ward identity).
- **Cubic-remainder estimate**: the rest is $O(g^3)$, uniformly in complex backgrounds and volume (analysis: positivity of the Hessian on complex backgrounds, decay of C_W, contraction of the shape coordinates uniform in volume).

The four-dimensional $O(1/\beta)$ layer of the local cancellation. In four dimensions we assembled one step with block ratio 2 in every direction. The fine box 4^4 is split into 16 blocks 2^4, a soft heat-kernel constraint is put on the coarse links of length 2 joining the block centres, and the $O(1/\beta)$ quadratic term is taken on shell (conditioned on the coarse variables held fixed) Computed. The zero directions are exactly as many as the gauge directions (1827); the on-shell quadratic term agrees in two tree gauges to 2×10^{-9} (the precision of the finite differences), and agrees with the value in the background-field gauge (with the Faddeev–Popov determinant) to relative 2×10^{-7}. That the on-shell quadratic term does not depend on how the gauge is taken has been checked directly in four dimensions Computed. Splitting the quadratic term across blocks by the positions of the variables and taking Loc up to second order at the midpoint of each pair, what remains is, in wave numbers with the block spacing as unit, 0.24–0.86% of the whole pair at twice the wavelength and 3–16% at the shortest wavelength of the next step. This is of the same order as for the neighbouring pairs of the two-dimensional plate (0.6%, 11%) Computed. What remains shrinks as the fourth power of the wave number

(dimension 8) for pairs point-symmetric about their midpoint and for sums across blocks at point-symmetric positions (exponent 3.87 for pairs touching at a vertex, 3.93 for pairs touching across a face placed at the centre of the box, 3.89 for the sum of all pairs across blocks; approaching 3.95–3.98 each time the wave number is halved). For individual pairs that are not point-symmetric, the exponent drops to 3 at small wave numbers, and terms of dimension 7 remain with small coefficients Computed . On a translation-symmetric lattice, the value per volume of the sum of all pairs across blocks does not depend on the phase of the plane wave and is an even function of the wave number. Therefore, after taking **Loc** up to second order, no term of dimension 7 remains, and what remains starts at dimension 8 Paper . The singular values of the pairs across blocks are degenerate according to the multiplicities of the representations of the stabiliser of the pair. For pairs touching at a vertex, the three standard components of the permutation representation of the four directions are equal, and they do not form a product of a few graded factors Computed . In an open box, the ratio of what remains when the whole is measured by **Loc** at one point grows with the size of the box (from 7% to 28% at the shortest wavelength of the next step); this is not due to far pairs, but to the tails of the links far from that point growing fat in a Taylor expansion around one point Computed . Splitting the quadratic term across blocks into pairs of fluxes of coarse faces (a gauge-invariant split) and taking **Loc** of each pair at its midpoint, what remains after taking **Loc** up to second order is about 2% of the whole at the shortest wavelength of the next step and about 0.1% at twice the wavelength, independent of the size of the box (exponent 3.98–4.00, dimension 8) Computed . If the centre of the block is taken as the single point of **Loc**, the coarse faces lie away from that point and what remains is about 30%. The books of BBS require the point of **Loc** of each term to be taken inside the support of that term Reading . On the side of one row and the two-dimensional plate, the algebra of the decomposition of pairs across blocks — the rational arithmetic of the model of one row (Lean: LocalCancellationOneRow), the identity splitting one product into block-wise squares and the square of the difference (Lean: RankOneRemainder), and the block deflation that makes a remainder vanishing from both sides on uniform-field and gradient test fields (Lean: BlockDeflation) — has been checked in Lean (all are identities with real coefficients over finite indices; the sizes are on the side of the numbers). Conditions: a small coarse lattice (at most 24 coarse faces), one step, the on-shell $O(1/\beta)$ quadratic term, finite-difference precision of order 10^{-9}, exponents judged from four wave numbers, exponents not measured in a periodic box.

What the row absent from the sources contains (revised). Written in the block-spin coarse variables, the quadratic term between blocks W^{gauge} starts at O(1) from one loop (13/384 for the row of seven), but this is an artefact of the coordinates. Taking Loc (monomials plaquette by plaquette) in the coordinates of local polymers of the smeared (fine) background field $\phi^* = F(\Phi)$, the O(1) quadratic term at one loop goes into the plaquette-wise sum (for SU(2): 5/12 on the inner faces, 1/4 on the end faces) and into pairs inside the same block, and 13/384 is its pull-back (computed, rational). The genuine remainder across blocks vanishes at O(1) and starts at O(1/β) when the blocks form a tree, and when the coarse variables carry no conserved quantity it decays in distance as e^{−κd} ($\kappa = \ln(5+2\sqrt{6}) \approx 2.29$, the rate of the fluctuation covariance) (U(1) and SU(2), computed). In systems with cycles an O(1) quadratic term across blocks does appear, but for the smallest cycle all of it is a constant times the square Θ^2 of the holonomy around the vertex (SU(2), rational; 0 for U(1)), a small term of $(\partial F)^2$ type that vanishes for uniform fields (10^{-3} of the plaquette-wise term) and is localised on small sets. In

form, the core row therefore returns to that of the sources — "local polynomial in the fine background field + a remainder localised on small sets" — but in four dimensions the remainder is not small (see the caveat below). What remains is the cubic-remainder estimate (the volume-uniform estimate of the remainder and the cancellation on small sets) and the treatment of the remainder at fourth order and above (the O(1) one-loop bubbles) Reading . Conditions: one row and a two-dimensional plate; one step; formal Laplace; the O(1/β) part for SU(2) and the quartic remainder have not been done on the plate. On the torus plate (4×4 to 10×10, rational), pulling the one-loop O(1) quadratic term in the coordinates of the smeared background field back to the coarse field a splits it into $\Sigma_B\,\rho\,\Sigma a_B^{\,2}$ plus a remainder of $(\partial a)^2$ type; the density ρ takes the same value on every block (ρ_L = 0.080253, 0.080704, 0.080710, 0.080710 for L = 4, 6, 8, 10) and agrees with the symbolic $\hat{k}(0)$. The term $u_B u_{B'}$ of neighbouring pairs of blocks (coefficient 1.6×10^{-6}) always splits, by an identity, into a uniform part and a gradient part, and the uniform part, 10^{-5} of ρ, goes into ρ (a renormalisation of the plaquette-wise coefficient). On an open plate ρ shifts in the boundary layer (+40% at the corners, +15% at the edges), but this is the boundary effect of the background $\bar{\phi} = F\cdot1$ and has the form absorbed by a block-wise counterterm P(B). That is, the marginal part is a renormalisation of a uniform density and the remainder across blocks is of $(\partial a)^2$ type (computed, rational; conditions: two dimensions, one step, formal one loop; the closed form of ρ not yet obtained). The above are results for one row and for the two-dimensional plate. In four dimensions, even with the same fine definition of Loc, the remainder of pairs of plaquettes belonging to different blocks does not become small. Even with the choice that minimises the remainder (constrained least squares), from 4^4 to 12^4 (block average along the covariant path) the remainder across blocks is 0.063–0.071 of the plaquette-wise term (largest ratio) and 0.104–0.109 (Frobenius), independent of the volume (computed). The 10^{-3} of the plate is a smallness that came from plaquettes not sharing fluctuations in two dimensions. In four dimensions, therefore, the statement of the row absent from the sources — "W^{gauge} is O(1) from one loop" — stands as it is; what carries over in the coordinates of the smeared background field is the form (plaquette-wise polynomial + pairs inside the same block + a remainder between blocks) and the bookkeeping method (fine Loc, counterterms on small sets), not the smallness of the remainder. What sets this size is how the average is taken: about 7% both for the Bałaban-type average over parallel lines and for the covariant path, and down to about 1% for a single straight path, but straight paths do not compose into an average and cannot be used for the books (computed, cited). What sets the rate of decay is the pair of average and gauge fixing: with a Feynman-type fixing (odd points) the pole of the kernel of the multiplier gives 0.95–1.09 per block, and with an axial-type fixing on a tree inside the block 2.2–3.5 (expected from the roots; reading). The one-loop quadratic term pulled back to the coarse field agrees for every pair of average and fixing (relative 2×10^{-15}), and the difference between the pairs appears only in the off-shell part (computed). For every pair, the books with the volume weight of BBS (needing $(\eta-1)\kappa > L\hat{}d \log 2$) fall short by more than 8 orders (Lean: LargeSetWeight (bbs_weight_vs_decay, summable_shell_iff, card_shell); the form of the inequality, and that the series of the weight A converges $\Leftrightarrow A < e\hat{}\kappa$ when the number of shells is polynomial); with the tree-length weight of Bałaban–Dimock ($e\hat{}\{\kappa\, d_M\}$ and M-cubes) it is of a form that goes on the books for $M \geq M_0$, but M_0 remains an estimate of about 30–55 blocks (about 2000 with the upper limit of the combinatorial constant) (reading). Hence the form of the plate (the return to the form of the sources) does not hold in four dimensions for any of the pairs examined. The value for a uniform field (the $q \to 0$ limit of

the coefficient of F^2) is not determined by the numbers up to 10^4 (computed). The distribution of the small-set terms, the row sums and the decomposition of Loc (the (1−Loc) side vanishing for a uniform field) are in Lean: SmallSetRowSum; the affine recursion for the contraction $\mu_k \leq \bar{\mu}$ per step is in Lean: ThetaSquareOneStepAffine (on the plate the one-step contraction is ½, sup $\leq 1/10080$).

The remainder pulled back to the coarse field (the core). The size of the remainder also changes depending on whether the one-loop quadratic term is viewed in the fine field or pulled back to the coarse field. In the quadratic term pulled back to the coarse field (an on-shell quantity, independent of the choice of gauge fixing), the pole of the kernel of the multiplier cancels algebraically by the identity $\Lambda_1 K = -Y$ Paper , Computed , and the remaining rate of decay becomes 1.76 per block, the mass of the physical mode of the fluctuations (a double pole without images) Computed . If the local part is represented by single plaquettes only, the remainder is about 10% (Frobenius 25–40%), but if the local part also takes in a plaquette together with the two plaquettes touching it (rectangles, chairs and plaquettes touching at a corner = improvement terms of dimension 6, with coefficients 2–3% of the plaquette-wise one), what remains is about 1% (Frobenius about 6%), independent of the volume (8^4–12^4) Computed . That is, nine tenths of the remainder seen in the fine field are the coefficients of finitely many improvement terms, and on the books they can be carried as couplings that shrink by L^{-2} per step Paper (counting dimensions). The remaining about 1% are pairs of plaquettes that do not touch, about 10 times away from the 0.1% of the plate. Moving the non-touching pairs into improvement terms type by type as well, the remainder drops by a factor of about 0.6–0.9 per type, and falls below 0.1% with about ten types Computed . Further, if the block average is made a 4 : 6 mix of the covariant path and the straight path, the root of the physical mode rises to 2.54 per block, and the core remainder after counting single plaquettes and about ten improvement terms in the local part is about 0.05% of the plaquette-wise term (Frobenius about 0.5%, 10^4–12^4), on a par with the plate (0.1%, 2.29) in both size and rate Computed (conditions: the multi-step composition of the mixed average tends to a self-similar average (information dimension 1.92) different from Bałaban's average (the uniform average over a cube), and estimates of regularity on it have not been checked. The multi-step flow of the coefficients of the improvement terms has been checked up to two steps, and the one-loop shift shrinks by a factor of 0.3–0.5 per step).

In the zero-sum system, this row is empty. A system with only one block (four in a row, a single box) has $W = 0$ by the zero-sum condition, as in the hierarchical model (the type of the sentence in Appendix A.1 of the sources) Reading . It can be read that the Shape-Remainder Theorem for Four Plaquettes in a Row of 15-06 could be a complete example in that model not because the model is easy but because this row was empty. Conversely, **this row cannot be tested with the numbers from the smallest systems**. Testing it needs a system in which two blocks are connected by the covariance of the fluctuations (the question "smallest system that is not zero-sum"). The answer for that system is the next paragraph.

The smallest two-block system — W^{gauge} starts explicitly from one loop. In seven in a row, made by placing two copies of four in a row side by side so that they share the boundary plaquette ($SU(2)$, heat kernel, one step; the coarse variable is defined by "the two diagonal loops are equal"), we measured the quadratic coefficients of the coarse action: the coefficient

between the blocks has, besides $-\beta/32$ at tree level, a one-loop $O(1)$, whose value approaches $13/384$ (difference 7.9×10^{-10}) Reading ($\beta = 32$–512; the coefficients agree to 10^{-10} by two routes). The W of the sources for flat fields starts at second order, but with this coarse variable it appears from first order (one loop). Its origin is not the non-commutativity of the group but **the non-locality of the background map** built from the block constraints: the commutative Wilson action of $U(1)$ (the action of cos) also gives $51/4096$, and for the Gaussian action (Villain) everything is 0 Reading . The one-loop term of $SU(2)$ can be written explicitly as the pull-back by the background map of an expression local in the fine background (the determinant coming from the prefactor of the heat kernel and the Laplace–Beltrami of S^3) Proved here (one loop of Gaussian minimisation and the Laplace method; the asymptotic expansion is formal, and no estimate of the remainder is attached), and the coefficients for the two blocks are $K_{AA} = \frac{7\beta}{32} - \frac{67}{384}$, $K_{AB} = -\frac{\beta}{32} + \frac{13}{384}$. Its local part (the row sums) is absorbed into the marginal coupling β_0', and the rest (the square of the difference of the coarse curvatures, of dimension 6 in four dimensions) goes into the shape coordinates. The rest after putting in the first order explicitly is $O(1/\beta)$ (coefficient $\approx -1/384$, from a fit), and it is this part that needs local cancellation Reading . The non-commutative tree term is the quartic $-\frac{\beta}{768}|a \times b|^2$ and vanishes on uniform fields ($a = b$), so it does not touch the local couplings and enters only on the side of W Reading . The decomposition of quadratic forms into local and non-local parts and its uniqueness, and the absorption by linearity of the row sums, are in Lean: `loc_split` (K is symmetric), `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_unique`, `one_loop_numbers` (only the arithmetic of substitution). With this, the question "smallest system that is not zero-sum" has the form of its answer in the smallest system: the content of the row absent from the sources divides into "W^{gauge} starts with an explicit one-loop formula from first order" and "the local cancellation of the $O(1/\beta)$ that follows", and for the former the formula and the numbers agree in this system. Conditions: seven in a row, the diagonal definition, one step, quadratic coefficients, $\beta \geq 32$. Nothing is said about one step on the four-dimensional lattice or about volume-uniform estimates.

Passing to the second step — the group version of the recursion of the coordinates of W (one row). In the same one-row model, we added the non-local terms born in the first step (the type of the difference of two neighbouring plaquettes $|x - y|^2$, and three quartic types containing two or more differences) to the fine action, ran the second step, and looked at how they map onto the coarse field, by numerical integration of the integrand without approximation ($\beta = 32$–512) and by rational numbers with the background substituted. The non-local terms of the second step split into "the image of the first-step terms (the non-local part of the conditional expectation of the fine terms given the coarse field)" and "the terms newly born in the second step" Proved here (a first-order Taylor formula in the non-local terms). The image closes on the same four types and local terms (automatic for the two-block output by counting invariants Proved here), and the quadratic type maps to $15/128$ times, the quartic types to at most $5/256$ times Proved here (tree rational numbers; the characteristic polynomial of the quartic block is in Lean: `quartic_block_charpoly`), Reading (the correction of $1/\beta$). Measured in the ratio of coarse couplings ($\beta' = 3\beta/16$), quadratic is $5/8$ and quartic $5/48$ or less, and all contract step by step. On the other hand, the terms newly born at each step are

numbers independent of the step ($1/64$ at tree level, $9/384$ when the one-loop coefficient of ψ^2 is included in the coupling), so the non-local terms are not driven to 0 but **are drawn geometrically towards a fixed value** Proved here (the one-loop formula is the formal leading term of Laplace; the bookkeeping is in Lean: `affine_iter` ($\lambda \neq 1$), `affine_bound_uniform` ($0 \leq \lambda < 1, 0 \leq C$), `new_term_gamma`). In the continuum field of the sources this is the part written by additivity, "the non-local terms of the previous step remain without changing the kernel, and the part of the new step is added" ($w_j = C_1 + \cdots + C_j$ of (A.1.4)–(A.1.5) and the identity E_C $\theta F_w(A,B) + F_C(A_+,B_+) = F_{w+C}(A_+,B_+)$ from Exercise 2.2.4) Cited ((A.1.4)–(A.1.5) checked in the text; the identity from Exercise 2.2.4 has not been confirmed in the text Reading); on the group it becomes "a contracting image + new terms independent of the step" Reading . The image of the quadratic type produces $|a \times b|^2$ at $-13/768$ from the BCH of the group (a numerical reading, no derivation on paper). Conditions: $SU(2)$ heat kernel (with $U(1)$ Wilson as a control), one row, the diagonal definition, one step of the two-block seven twice, the four types of neighbouring pairs, first order in the non-local terms, up to fourth order in the coarse field. The ℓ^4 cone $|Q(\phi)| \leq c \| \phi \|_4^4$ contains every quartic many-body term whose coefficients have finite ℓ^1 at each point (Lean: QuarticCone), so the three-step contraction gives the four-dimensional version of the "contracting image" without counting invariants (tree level, commuting). Flowing the coordinates of W (a quadratic form in dimension 6) step by step gives 0.38 after three steps (the value for a particular input, not an operator norm) Reading .

The many-body tail — the form in which the one-row recursion closes. When three or more super-blocks are lined up, the image of the non-local terms contains quartic terms involving three coarse fields at once (six of them, such as $|a|^2(b \cdot c)$ and $(a \cdot b)(a \cdot c)$), and the very form "pair types" is not preserved from step to step. In a row with coarse variables that give the boundary plaquette half to each side (no conserved quantity), we measured these coefficients for each of three sources (the image of the previous step's quartic pairs, the image of the quadratic pairs, and the tree and one-loop terms newly born at that step): all decrease as $e^{-\kappa d}$ in the end-to-end width d of the triple, and $\kappa = \ln(5 + 2\sqrt{6}) \approx 2.29$ was the same as the rate of the background map Reading (two routes, rational and floating point, 5×10^{-17}; Laplace on the group and full quadrature with finite β, 4×10^{-13} at tree level and 4×10^{-10} at one loop; for the tree and the image of quartic pairs the prefactor is constant, for one loop and the image of quadratic pairs the prefactor is linear in d; $L \leq 8$, $d \leq 7$). The three-body sum containing one block is independent of the volume, and the three-body image contracts further to 5×10^{-3} or less Reading (tree step, linear background). The bookkeeping is in Lean: `three_body_sum_le` (assuming $0 \leq \rho < 1$, $|q(d)| \leq C\rho^d$), `pair_tail_uniform` (assuming $0 \leq \lambda, \mu < 1$ and two recursive inequalities). Going back to a definition with a conserved quantity, the same three-body coefficients become flat in distance, and the origin is neither the group nor the one loop but the definition of the coarse variables. Therefore, **in the one-row model, the recursion of the core row closes up to fourth order in the form "a contracting image of the pair types + new terms independent of the step + three-body terms decreasing exponentially in distance"** — this is the one-row version of the row absent from the sources Reading . However, the three-body coefficients at $d = 2$ are the same size as the coefficients of neighbouring pairs (the head is not small), so short three-body terms must be included in the set of types from the start. Conditions: one row, definition without a conserved quantity, heat kernel, one step, up to

fourth order, tree and one loop, first order in the previous step's non-local terms. Four dimensions (three-body invariants are many, as pairs of orientation and position), four-body terms, and the self-image of three-body terms at the one-loop level have not been measured. The quartic terms newly generated at each step at one loop are +0.038 of $\| \theta \|_4^4$ on the 6^4 lattice (with the quadratic part subtracted in Landau gauge; +0.046 raw, +0.014 ultra-local; subtracting the quadratic term requires a definition of the gauge) Reading . One loop also generates cubic terms (of the type of the three-colour triple product $\varepsilon_{abc}\theta_a\theta_b\theta_c$, $f_3/\| \theta \|_3^3 \approx 0.02$, zero when restricted to two colours); since both $\pm\theta$ lie in the cone they do not cancel in the sup, and a separate step bounding the cubic cone $|C(\theta)| \le c_3 \| \theta \|_3^3$ is needed (non-commuting). The cubic AM–GM inequality, the three-colour triple product $|u\cdot(v\times w)| \le (|u|^3+|v|^3+|w|^3)/3$ and the pull-back of the cone are in Lean: CubicCone. The three-step ℓ^3 bound is 0.57 / 0.30 for two kernels, below 1 at every step (computed, tree level).

The non-local tail type is decided by the definition of the coarse variables. What the volume-uniform estimate of the cubic-remainder estimate needs is that the weights of $(1-\mathrm{Loc})W^{\mathrm{gauge}}$ decrease fast enough with distance. Measuring on a chain of L copies of four in a row, a two-dimensional slab, three-dimensional columns $2\times 2\times L$ with the Bianchi constraint, and periodic lattices, the tail type was decided not by the group nor by the one loop but by how the fine variables straddle between coarse variables Reading ($SU(2)$, heat kernel, one step, quadratic coefficients of the coarse field, commutative background).

- **Flat**: defining the coarse variable by "the two diagonal loops are equal", the conjugacy class of the shared plaquette is conserved along the chain, and the one-loop coefficient between blocks is $(4L+5)/(6(3L+2)^2)$ independent of distance ($13/384$ at $L=2$). For $L=2$–8 and $\beta=32$–512 the difference over distances 1–4 is 4×10^{-12} relative Reading . Adding the conserved quantity to the coarse variables removes the non-locality Proved here . The single global constraint of total flux 0 on a periodic lattice likewise produces a flat $-1/(2N)$.
- **Exponential**: giving the boundary plaquette half to each of the two blocks (sharing it by weights), the conserved quantity disappears, the coefficient is $C(r+\alpha)e^{-\kappa r}$, and κ is the tree rate $\ln(5+2\sqrt{6}) \approx 2.29$ (2.43 on the two-dimensional slab). The representation that writes the coarse faces in the variables of coarse links (promoting Bianchi to the coarse variables) is also exponential on columns, slabs and in three dimensions throughout; on columns it is $\ln(9+4\sqrt{5}) \approx 2.887$, and one loop has the same rate with prefactor r Reading (chain $L \le 21$, $r \le 8$; columns tree $L \le 20$, one loop $L \le 16$).
- **The constraint-propagation type**: in the representation that takes the holonomies of the coarse faces directly as coordinates (the pseudoinverse representative), the Bianchi constraint is held implicitly, and the tail is decided by the propagation of the constraint. On a column with open faces it is exponential $e^{-1.763r}$ (rate $\ln(3+2\sqrt{2})$), on a periodic slab without boundary r^{-2}, and in three-dimensional periodic it is a power (effective exponent around 3.3) Reading (tree level).

The distance-weighted sum $\sum_r r|Q(r)|$ grows with the volume if flat, saturates independently of the volume if exponential (0.0447 on the chain, 9 digits on the column), and if a power r^{-p} it is uniform only when $p-d>1$, taken together with the shells of dimension d Proved here

(arithmetic; Lean: `flat_not_uniform_geometric` (for the flat formula), `weighted_sum_uniform_linear` (assuming $|q(r)| \le C(r+1)\sigma^{2r}$, $0 \le \sigma < 1$), `power_sum_uniform` ($s > 1$), `power_sum_unbounded` ($s \le 1$)). The counting rule is as follows Reading : conserved quantities made by sharing through δ and global constraints are flat; sharing by weights and constraints promoted to coarse variables are exponential (the rate is the transfer rate of the fine field); coordinates holding local constraints implicitly are of the constraint-propagation type (exponential if there is a boundary; a power if there is no boundary and it spreads in two or more directions). For four-dimensional blocking, we read that if the coarse variables are taken as "coarse links plus global tags (the flux through the torus, the magnetic flux of the centre $\mathbb{Z}_2$, the Polyakov direction)", the local part is exponential, and the global tags go onto a separate bookkeeping as flat terms of small rank Reading . In four dimensions only the tree level was measured (next paragraph).

The coordinates of the cubic-remainder estimate — coarse links plus global tags are exponential at one loop in three dimensions without boundary and on the four-dimensional tree. Writing one step with all coarse faces fixed at once (commutative background, quadratic coefficients of the coarse field) in a periodic box in three dimensions without boundary, splitting the coarse variables into "coarse links" and "global tags (the 3 total fluxes through each plane = the harmonic part of the coarse faces)", the one-loop non-local weights of the $SU(2)$ heat kernel are $(r+\alpha)e^{-\kappa r}$, $\kappa = 3.094$ (agreeing with the tree rate $\ln(11+2\sqrt{30}) = 3.089$), and the distance-weighted sum saturates independently of the volume (2.7×10^{-6} relative at $L = 12 \to 16$; for all partner rows the increments are 0.172, 0.024, 0.0019) Reading ($O(1)$ of $\beta \to \infty$, coarse $2 \times 2 \times L$, $L \le 16$, two routes, analytic Hessian and difference Hessian, differing by $\le 10^{-10}$). The tags have no cross terms with the coarse-link part, and the space of faces satisfying Bianchi splits uniquely into the direct sum of the curvature of the coarse links and the tags Proved here (Lean: `norm_sq_add_harmonic` (assuming $D^\dagger b = 0$), `range_sup_tags` (assuming $d_2 \circ D_C = 0$), `decomp_unique`, `finrank_ker_eq`). The global direction newly appearing at one loop (the Polyakov direction) became exponentially smaller, by about $1/75$ each time the coarse length increases by 2. Writing the same step in coordinates with the tags removed (the holonomies of the coarse faces), the projection of the tags produces a flat $-1/(2N)$ to relative 10^{-12}. On the tree of the four-dimensional periodic n^4 ($n \le 16$), the kernel in the coarse-link coordinates is exponential (prefactor around r^{-1}, $\kappa \approx 2.08$) and the sum tends to saturate, while the coordinates with the tags removed are a power ($r^{-4.4}$ from a fit) plus flat, and the sum grows in proportion to n Reading . Therefore the coordinates of the cubic-remainder estimate are coarse links plus global tags, supported by [Reading] numbers up to the four-dimensional tree level. The four-dimensional one loop has not been measured.

Writing the algebraic layer in Lean — the same form as in the sources, and where the cut may differ. Of the statement (3) of the small-field contribution to the coupling, we wrote the algebraic layer and the assembly of the real inequalities in Lean (two bundles, 40 theorems, all on the three standard axioms). (A) Defining $\mathbf{Loc}$ as the projection that keeps or discards matrix entries, it is linear and idempotent, the decomposition into a local part and a purely non-local part is unique, and the image of $\mathbf{Loc}$ and the image of $1 - \mathbf{Loc}$ are orthogonal in the Frobenius inner product (`locM_idem`, `split_unique`, `frob_loc_nonloc`, no assumptions; the trace is preserved when the set of kept pairs is reflexive). (B) Taking $\mathbf{Loc}$ to be the block-diagonal

projection, $W^{\text{gauge}} = \frac{1}{2}(1 - \mathrm{Loc})F_C$ can be written only with the covariance entries straddling blocks, and for two blocks it is $W(V(b), V(\Lambda)) = \frac{1}{2} V_A^{\mathrm{T}} C_{AB} V_B$ (`Wg_explicit`, `two_block_W`). (C) Receiving the Cauchy-type bound on the coefficients and the bound on the cut as assumptions, the one-step remainder is bounded in the bracketed form of (3) (`one_step_remainder`, margin h $0 \le qh$), and under the schedule $g_k^2 = 1/(A(k+1))$, $h = c_0 g p$, $\beta = 4/g^2$, if $c_0^2 \ge 1/(c(1-q)^2)$ then the sum over stacked steps is uniform in the number of steps K (`total_remainder_uniform`, assuming $A \ge 1$, $r_0^2 \ge 1$, $m \ge 3$ and others). Analyticity (the analyticity condition at complex backgrounds) and the probability of the cut come in from outside as assumptions.

The same form as in the sources Cited , Reading (the correspondence is a reading of the equation numbers of the sources; we have not reopened the text of the sources): that Loc is a projection and the decomposition is unique (BBS § 5.2, Appendix A); that W can be written only with the covariance entries straddling blocks ((A.1.3)–(A.1.4)); the projection layer of the local cancellation, "the part entering W does not contribute to the marginal coupling" (up to just before (A.5.3)–(A.5.4)); and the form of the Taylor remainder (Lemma 7.5.3). **Where the cut may differ from the sources** Reading :

1. Taking Loc as a projection of matrix entries. The Loc of the sources is a projection onto a space of local polynomials, and for quadratic forms it is the type that keeps the zero-momentum symbol (the row sums). If the shift of the marginal coupling is defined by a trace, the off-diagonal entries within the same block do not enter, and the meaning of the numbers may differ from the zero-momentum type. The properties of the projection themselves (idempotent, unique, orthogonal) do not depend on the choice.
2. The reason for zero sum. In Lean, $W = 0$ follows from the definition of Loc, whereas the zero sum of the hierarchical model in the sources follows from the structure of the covariance. Even for the same "$W = 0$", the reason differs.
3. Paying for the cut separately by probability. Applying analyticity only to the effective action without the cut, and paying for the part due to the cut separately by a probability estimate, is a different road from the sources (the regulator method for flat fields).
4. How the sum over steps is taken. Bounding the bound of each step by $1/(k+1)^2$ and adding (a p series) differs from the combination of the contraction of the shape coordinates (a geometric series) and the flow of the couplings in the sources, and for a coarse estimate assumes Loc_4 ($m \ge 3$).

None of these points changes the form of the statement of the small-field contribution to the coupling; they are notes to separate how far it has the same form as in the sources and from where the cut is different (the question "two definitions of Loc"). The first point became numbers in the two-block system: the quadratic non-local rest of $O(1/\beta)$ is, with Loc of row sums, only $\frac{1}{768}|a-b|^2/\beta$ containing differences and makes no new coordinate, while with the projection of entries $-\frac{1}{384} a \cdot b/\beta$ without differences remains Proved here (the expansion of the closed formula is formal), Reading (10^{-11} at $\beta = 32$–2048; $SU(2)$ heat kernel, seven in a row, the diagonal definition, one step). The type of the sources is the row sum, and the two definitions are connected by the identity $s_{\text{row}} = s_{\text{comp}} + \frac{1}{2} N_{\text{comp}}(x, x)$ (Lean: `sRow_eq_sComp_add`, `row_sum_cancels` (assuming F is symmetric), `row_split_unique`). The content of the group

version of the local cancellation — that the sum of the quadratic contributions over polymers in (A.5.3)–(A.5.4) is 0 up to a cubic error — has not been put into Lean.

In summary, what could be said about the small-field contribution to the coupling is that **the same form as the structure known in the sources for flat fields** (Loc + **the Taylor remainder, the tail of the cut, the quadratic terms between blocks) holds on the group too, under the analyticity condition at complex backgrounds and margin** q. It does not claim a new structure. What is absent from the sources is only the group version of the place the sources themselves call a sketch — the explicit quadratic term between blocks W^{gauge} and its local cancellation — of which two are algebraic and one is an analytic problem. In the smallest two-block system, W^{gauge} starts with an explicit formula from one loop, what needs local cancellation is the part $O(1/\beta)$, and the non-local tail type is decided by the definition of the coarse variables. The algebraic layer can be written in Lean, and at four places in it the cut may differ from the sources.

07

Discarded proposals

Table 2 Discarded proposals and why (the labels for the reasons are per cell)

Proposal	Aim	Why discarded
A class dominated by heat kernels	Take as class $A^{-1}p\beta \le f \le Ap\beta$ the weights f squeezed above and below by heat kernels, and make the large-field rarity condition a statement about the tail of the character expansion	The Wilson action is not in this class: on the far hemisphere it is $fW(\pi)/fW(0) = e^{-2\beta}$, the heat kernel is of order $e^{-\beta\pi^2/2}$, and the ratio is $e^{(\pi^2/2-2)\beta} \approx e^{2.9\beta}$ Proved here (arithmetic up to polynomial factors). This excess is the z of the proposal of a parameter for the distribution over the centre. It is meaningful if the theory is defined with the heat-kernel action (on hold). Restricted to inside the angle cap θ_0, Wilson is bounded by K times the heat kernel (the One-Link Heat-Kernel Comparison Lemma, 15-07 § 05)
Peeling off the source with Hölder alone	Separate the observable-source condition with bounded powers of the closure under the fluctuation expectation	At the fine steps the source becomes a non-local activity, and the one-step loss $2^{\lvert B(X_J)\rvert}$ cannot be paid Reading
Moving η to the side of the action with a fundamental + adjoint action	Build a one-parameter class with an action that keeps the quadratic ratio 2	The expansion of $\cos^2$ produces an anisotropic quadratic form acting only on the component of the fluctuations in the direction of the background, and the conditional law of one link is no longer a member of a one-parameter family (15-07 § 06) Proved here . In addition, the curve of ratio 2 does not close under the one-step flow (15-06) Reading
Four dimensions with class functions alone	Never fix the gauge at all	The coarse Bianchi identity needs connecting paths, so it cannot go beyond the hierarchical-lattice approximation Reading (§ 05)
Removing re-fixing by Haar invariance	Make gauge fixing unnecessary through the conjugation invariance of the law of the fluctuations	Limited to renormalisation of the heat-kernel-convolution type; it does not reach four-dimensional blocking Reading (§ 05)

Proposals of the type "with enough computing resources the constants become explicit" (making the constants explicit with interval arithmetic) are not the core of the four rows, so they are not in this table.

08

Ten rows of directions being tried

Table 3 sets out in ten rows the directions being tried towards ultraviolet control (moved from Table 3 of 26-11, with the components added in this part written into "What can be said now"). "Smallest setting" is the smallest object on which the direction can first be tried, "What can be said now" is the current state with labels, and the last column is the risk that the direction ends up as a restatement of something already known. We do not give numbers for the prospects. No row closes ultraviolet control.

Table 3 Directions being tried ("Conditions attacked" are the three conditions of ultraviolet control in 15-00 § 01; "Sources of difficulty" are those of 24-07 Table 3)

Direction	Conditions attacked / sources of difficulty	Smallest setting	What can be said now	Risk of restatement
1 Rewriting the series on the small twisted T^4	Analyticity, rarity, the coupling stays small / dimensionless coupling, lack of gauge fixing	$T^4_{L_0}$, $n_{12} = 1$	A proposal for the ultraviolet statement for a small twisted four-dimensional torus Assumption . Correspondence with the sources in 15-02 § 06 Reading	The fine steps are the same as for the periodic T^4, so it becomes a rewriting of the series itself
2 The group-valued sigma model first	Analyticity, rarity, the coupling stays small / dimensionless coupling	The twisted cylinder for two-dimensional $O(4)$	The transplantation goes as far as the background-field stage Cited (15-02 § 03)	Two-dimensional results imply nothing for four dimensions. Only the form of the statements carries over

Direction	Conditions attacked / sources of difficulty	Smallest setting	What can be said now	Risk of restatement
3 The weight for group-valued fields	Rarity, analyticity / lack of gauge fixing	A single block	The group version (the Background–Fluctuation Decomposition Lemma) of submultiplicativity in sums of fields ((ii) of BBS) Proved here . In one step with background-independent fluctuations, the group versions of submultiplicativity in sums of fields, closure under the fluctuation expectation, monotonicity to the next step and the exchange all hold and the chain closes (the Lemma on One Step with Background-Independent Fluctuations) Proved here . (The group versions of) closure under the fluctuation expectation and monotonicity to the next step for the real $\mu_{k,W}$ only as statements Assumption (15-07)	The Background–Fluctuation Decomposition Lemma and the Lemma on One Step with Background-Independent Fluctuations do not touch the background dependence η

Direction	Conditions attacked / sources of difficulty	Smallest setting	What can be said now	Risk of restatement
4 Polchinski flow on the group (multi-scale flowing with the heat kernel)	Rarity, analyticity / lack of gauge fixing	A one-link class function	In the one-link heat-kernel class the losses form a telescoping sum Reading (15-04). The estimate of the weighted norm in one step closes with $\eta = 0$ if the fluctuations are a product of heat kernels link by link (the Lemma on One Step with Background-Independent Fluctuations) Proved here . Measures that include couplings between many links not yet done	Convolution with heat kernels does not reduce the degrees of freedom and produces no block variables
5 Paying for large fields all at once with the chessboard	Rarity / lack of gauge fixing, dimensionless coupling	Periodisation of T^4	Reflection positivity of the block variables is needed Assumption (26-05)	The large-deviation induction at each step ("worst-case geometry" of 15-00 § 03) cannot be avoided
6 Self-containment of the flow of the coupling	The coupling stays small / dimensionless coupling	Anywhere	Coupling-Smallness Induction Lemma: net one-step bracket of the coupling + continuity of the coupling ⇒ the coupling stays small (15-03 § 03) Proved here . The bookkeeping part is elementary	Unless the net one-step bracket of the coupling is shown, it remains a tidying of the bookkeeping part of "the coupling stays small"
7 Making the constants explicit with computer assistance	Analyticity, the coupling stays small / dimensionless coupling, lack of gauge fixing	The linearised operator of one block	There is a record of floating-point estimates, but no interval arithmetic yet Reading	It only makes the constants explicit and does not supply the induction itself
8 Regularity structures at criticality (stochastic quantisation)	Analyticity / dimensionless coupling	Cannot be written	The key condition of the solution theory is $d < 4$, and four dimensions is the borderline Cited (24-06 § 08)	The tools need rebuilding, and no minimal statement can be set down

Direction	Conditions attacked / sources of difficulty	Smallest setting	What can be said now	Risk of restatement
9 The determinant in self-dual coordinates	Analyticity, the coupling stays small / dimensionless coupling, lack of gauge fixing	Near the twisted isolated flat connections	Only a proposal for a one-loop check Assumption	Reproducing the one-loop coefficient is a restatement of one loop and can only serve as a negative control. There is no self-dual decomposition on the lattice
10 The twisted one-point model and the Nahm transform	Analyticity / dimensionless coupling	The one-point model	The Nahm transform is a correspondence of classical self-dual configurations; no correspondence at the level of the measure is known Assumption (sources not checked)	The problem of the order of large N and $\beta \to \infty$ remains, and it merely swaps the ultraviolet and the infrared

What to read from Table 3 Reading (a physical estimate). The directions with components closed on paper are directions 3 and 4 (the Background–Fluctuation Decomposition Lemma, the Lemma on One Step with Background-Independent Fluctuations) and direction 6 (the Coupling-Smallness Induction Lemma), and all lie outside the real issue (submultiplicativity in sums of fields, background-independent fluctuations, the bookkeeping part). The real issue — the group version of closure under the fluctuation expectation for the real $\mu_{k,W}$, and the one-step remainder including large fields (the net one-step bracket of the coupling) — is the single point towards which directions 3 and 4 and direction 6 each head, and they meet where the dependence of the measure of the fluctuations on the background takes effect. This is the same place as "the row of background-dependent absorption and many steps is the core" in § 06.

Questions arising from this chapter. Table 4 places, one for each of the four rows, the smallest question that can be tried next. The question "smallest system that is not zero-sum" arises from the core row in § 06. The question "two definitions of Loc" and the question "three-dimensional non-local weights" arise from the revision of § 06, the question "gauge agreement on non-abelian backgrounds" from § 05, and the question "quartic width of the three-body tree" from the third revision of § 06.

Table 4 Questions arising from this chapter (status: New = a question added in this part; Updated = part of the answer has been written in)

Q.	Row	Question (one line)	Size	Status	What does not move even if settled

Q.	Row	Question (one line)	Size	Status	What does not move even if settled
12	1	For one link, derive on paper the Lipschitz constant of the read-out map from the derivative of the Bessel ratio; on the slab, put a number on the difference in total variation between the density with one large-field region inserted and the density without it; and see whether the form large-field contribution to the coupling $\le C' g^2$ holds in the smallest system. **Answer (partial):** done for one link (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, § 02; Lipschitz constant $\frac{4}{3}\beta^2$, actual sensitivity $\approx \frac{1}{3}\beta^2 h^2$). On the 2×2 slab with heat-kernel fluctuations, at the threshold $\theta_h = g\,p(g)$ ($\beta = 4/g^2$), the form holds for $c_0^2 > 2$, $= 2$ is the boundary, and with the threshold fixed it does not hold. The value on the slab is 1/3 of that for one link and is local (§ 02). Many links under the true measure not yet done (the row of background-dependent absorption and many steps)	Paper and small computations	Updated	The small-field contribution to the coupling (the content of the basic inequality (unpublished))
13	2	With four plaquettes in a row, add a source $s\,\mathrm{Re}\,W_a$ to one loop, and see whether the shift of the couplings by loop shape after one step is linear in s with a coefficient independent of β. Also, on paper, whether one step of the position-dependent coupling $\beta(x)$ in the Villain action of $U(1)$ becomes a local average. **Answer (partial):** in one heat-kernel step (without blocking), the source from fine loops is a shift of the coupling for each representation, exact at both first and second order (§ 03). In one blocking step with four in a row, the shift by a source on a loop inside the block is the main term f_C^2 (independent of β) plus a rational $O(1/\beta)$, with no $\log\beta$. The $\beta(x)$ of the $U(1)$ Villain action is, to first order, the local average with the weights of the bundle network (§ 03). Loops crossing boundaries and many steps not yet done	Paper and small computations	Updated	The observable-source condition for loops of the fine lattice

Q.	Row	Question (one line)	Size	Status	What does not move even if settled
14	3	Whether the parity of the critical point holds with four plaquettes (the odd part of the action of the fluctuations around the true constrained critical point), and if so whether the background uniformity of the small-field contribution to the coupling weakens to boundedness of the coefficient of θ^4 (the proposal of the parity of critical points). Paired with the question "splitting the density of the joint law" (15-07). Answer (partial): parity breaks with four plaquettes in the breathing direction (closed formula). The conclusion of the proposal of the parity of critical points stands for a chain in a row by the covariance under reversal of the background (§ 02). The row for θ^3 in four dimensions not yet done	Small computation	Updated	The multi-step background-regularity condition
15	4	Split the one-loop Hessian of a block with factor 2 into a gauge-field part and a ghost part, and put the coefficient of F^2 of the latter into numbers (negative control: 0 for $U(1)$). At the same time, formalise the direct-product decomposition of the tree gauge for finite groups. **Answer:** the second half is done (Lean for finite groups, § 05). The coefficient of F^2 of the ghost is $+0.024$ per classical action for one block of 3^4 points on a commutative background (2.4% of one loop), and on a critical-point background the one loops of the two gauges agree identically by (5) (§ 05)	Small computation, Lean	Updated	Uniform convergence of the local expansion (of the type of the small-field analyticity condition)

Q.	Row	Question (one line)	Size	Status	What does not move even if settled
16	1	In the smallest system that is not zero-sum — two copies of four in a row placed side by side sharing the fine links between them, so that the two blocks are connected by the covariance of the fluctuations — write W^{gauge} ((4)), and put into numbers, for $\beta = 32$–512, the amount by which the quadratic contribution to the marginal coupling β_0' deviates from the sum of the two blocks (the breaking of the local cancellation). If the cancellation holds, $O(1/\beta)$; if not, $O(1)$. **Answer (smallest system):** in seven in a row W^{gauge} is $O(1)$ from one loop ($13/384$), and the rest after putting in the first order explicitly is $O(1/\beta)$. On chains, slabs and columns of L blocks the tail type is decided by the definition of the coarse variables (§ 06)	Small computations and paper	Updated	The cubic-remainder estimate in four dimensions (the cubic estimate)
21	1	Define $\mathbf{Loc}$ in two ways — by row sums (zero momentum) and by matrix entries — and give the difference of W^{gauge} in rational numbers for the covariance of a one-row two-block system. If the difference is 0 the cut is the same as in the sources; if not 0, the place where it differs is identified by numbers. **Answer:** the difference is not 0. The quadratic non-local rest of $O(1/\beta)$ is $\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums (only terms containing differences), and $-\frac{1}{384}a\cdot b/\beta$ with the projection of entries. The type of the sources is the row sum ($SU(2)$ heat kernel, seven in a row, one step, § 06)	Small computations and Lean	Updated	The local cancellation in four dimensions
22	1	In three dimensions without boundary (columns with periodic cross-section, or 8^3), do the one-loop non-local weights $Q^{(1)}(r)$ of $SU(2)$ in the coordinates of coarse links plus global tags remain exponential, as at tree level? **Answer (partial):** exponential on a periodic coarse $2\times 2\times L$ ($L\le 16$) ($\kappa = 3.094$, agreeing with the tree rate), and the sum is uniform in volume. The four-dimensional periodic tree is also exponential ($\kappa\approx 2.08$). Boxes wide in the transverse direction and the four-dimensional one loop are not yet done (§ 06)	Small computation	Updated	The cubic-remainder estimate in four dimensions

Q.	Row	Question (one line)	Size	Status	What does not move even if settled
24	4	Measure (5) and the ghost coefficients on a non-commutative background (a critical point with commutator F), and see whether the leading term of the local expansion is a single number for $\sum_p \lvert F_p\rvert^2$ (whether it is the same for commutative F and for the commutator's F)	Small computation	New	Agreement at two loops and beyond
29	1	In a four-dimensional box of coarse links plus global tags, measure the dependence on width of the three-body quartic (commutator type) at tree level, and see whether the rate agrees with the two-body tree rate (the first test of carrying the one-row recursion of W over to four dimensions). At the same time, measure in one row the four-body terms and the contraction μ of the self-image of three-body terms at the one-loop level	Small computation	New	The local cancellation and the cubic-remainder estimate in four dimensions

09

What cannot be said

(a) Within this part, ultraviolet control does not close (existence was shown by the published construction, with a different design). None of the four rows closes. Every proposal in this chapter is a proposal, not a result.

(b) The "one line" of Table 1 is a conditional statement — if it were shown, the row would close — and none of them has been shown. That the small-field contribution to the coupling corresponds to the content of the basic inequality (unpublished) is a reading from one sentence of the preface.

(c) The proposal to obtain rarity by Chebyshev is a one-liner assuming the group version of the exponential moment of fluctuations, and the group version of the exponential moment of fluctuations has not been shown. The Lipschitz constant and the difference in total variation of the proposal to split the remainder into small and large fields are assumptions and have not been estimated.

(d) The commutativity of the proposal to take the source as a function of the block variables is a trivial fact limited to functions of the block variables, and says nothing about observables given by loops of the fine lattice.

(e) "There is no Jacobian" of the tree-gauge proposal is a restatement of a structure limited to the axial (tree) gauge, and we have not checked the choice of gauge in Bałaban's four-dimensional papers. The local expansion of the ghost determinant is a prospect.

(f) "The row of background-dependent absorption and many steps is the core" is an estimate of the logic between the proposals, and none of its implications has been shown. The directions of Table 3 are candidates, and neither their consistency nor their prospects have been assessed.

(g) The splitting (3) of the small-field contribution to the coupling in § 06 is a statement with the analyticity condition at complex backgrounds and margin q put in, and an estimate under them. The analyticity condition at complex backgrounds has not been shown for four-dimensional blocking. The numbers in § 06 are for four in a row ($SU(2)$, heat kernel, one step) and one-loop numbers for a box of 2^4, not numbers for one step of the four-dimensional lattice. The row absent from the sources (the group version of W^{gauge} and its local cancellation) has not been shown. "The same form as in the sources" speaks of agreement of form and is not a claim of a new structure.

(h) The Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma and the accounting of the source are statements under the edgewise independent heat-kernel condition, the single-step condition and the background-independence condition for fluctuations (the One-Link Heat-Kernel Comparison Lemma is for one Wilson link), and say nothing about the true measure of the fluctuations. The Lean for the tree gauge is a statement about finite groups; the Haar measure of continuous groups is on paper.

(i) The numbers for the two-block system, chains, slabs and columns of § 06 are statements for one step, quadratic coefficients of the coarse field and commutative backgrounds (the tail type), and the one loop is a formal asymptotic expansion by the Laplace method. The coefficient of $O(1/\beta)$ is from a fit, with no formula. The counting rule for the tail type is a reading from the numbers of one-, two- and three-dimensional models, and in four dimensions only the tree level in the coordinates of coarse links plus global tags ($n \le 16$) was measured. The 40 Lean theorems are finite-dimensional projections and identities and the assembly of real inequalities, with analyticity (the analyticity condition at complex backgrounds) and the probability of the cut entering as assumptions. The four points where the cut may differ from the sources are readings, not rechecked against the text of the sources.

(j) The recursion of W and the many-body tail of § 06 are statements for one row, two blocks (three-body for $L \le 8$, $d \le 7$), the definition without a conserved quantity, first order in the non-local terms and up to fourth order in the coarse field, and the one loop is the formal leading term of Laplace. $-13/768$ is a reading from numbers. The one loop of the coordinates of the cubic-remainder estimate is only for three-dimensional quasi-one-dimensional boxes (coarse $2 \times 2 \times L$), and four dimensions only at tree level. The difference between row sums and the projection of entries is a statement for one step of seven in a row.

(k) The agreement of the two gauges (5) of § 05 is a statement for one step, one loop and a critical-point background. The numbers for the ghost coefficients are values for $SU(2)$, blocks of 3^4 points and $4 \cdot 4 \cdot 3 \cdot 3$ points, and commutative backgrounds; non-commutative backgrounds, $\xi \neq 1$, two loops and multiple steps have not been measured.

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The table of the five grades and the dependency graph

Table 5 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
[01]	The "one line" of the four rows that do not close (the net one-step bracket of the coupling, the observable-source condition, the absorption of η and many steps, the seam of re-fixing)	Assumption	§ 01 Table 1 (statement)	Constants independent of k, a
[02]	The closed components lie in background-independent fluctuations or at tree level	Reading	§ 01	The range of the components of this part
[03]	The large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations (1)	Proved here	§ 02 the proposal to obtain rarity by Chebyshev	Assuming the group version of the exponential moment of fluctuations
[04]	Net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling, $\lvert r^{\mathrm{large}}\rvert \le C\beta^2 e^{-cp^2}$	Assumption	§ 02 the proposal to split the remainder into small and large fields	Assuming the Lipschitz property and the difference in total variation, $p_0 \ge \frac{1}{2}$
[05]	Sending the background dependence to the coefficient of θ^4 by the covariance under reversal of the background; the large-field parameter z	Assumption	§ 02 the proposal of the parity of critical points and the proposal of a parameter for the distribution over the centre	Parity holds with two plaquettes and breaks in the breathing direction with four. The conclusion stands for a chain in a row by the covariance under reversal of the background. In four dimensions boundedness of the coefficient of θ^3 is also needed
[06]	If the source is taken as a function of the block variables, it commutes with the integration at the fine steps	Proved here	§ 03 the proposal to take the source as a function of the block variables	Blocked observables
[07]	The source is a shift of the coordinates of the class; the class of position-dependent couplings	Assumption	§ 03	Proposal
[08]	By the direct-product decomposition of the tree gauge there is no Jacobian of re-fixing	Proved here	§ 05 the tree-gauge proposal	Axial (tree) gauge; a restatement of a known structure
[09]	The local expansion of the ghost determinant and its merging into the local couplings	Assumption	§ 05 the proposal to absorb the ghost determinant	Small fields; uniform convergence not yet shown

Row	Statement (one line)	Grade	Basis	Scope
[10]	A road with class functions alone cannot go beyond the hierarchical lattice; removing re-fixing by Haar is limited to the heat-kernel type	Reading	§ 05	—
[11]	Dependence among the four rows: the row of background-dependent absorption and many steps is the core; the remaining core is the small-field contribution to the coupling and the multi-step background-regularity condition	Reading	§ 06 (estimate)	—
[12]	Wilson is not in the class dominated by heat kernels (ratio $e^{(\pi^2/2-2)\beta}$)	Proved here	§ 07 Table 2	$SU(2)$, one link, up to polynomial factors
[13]	The ten rows of directions being tried, and the single point where the real issue converges	Reading	§ 08 Table 3 (physical estimate)	Candidates
[14]	The Heat-Kernel Gaussian Upper-Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma: for heat-kernel fluctuations the group version of the exponential moment of fluctuations and the large-field rarity condition hold in closed form. For one Wilson link, K times + the tail by the One-Link Heat-Kernel Comparison Lemma	Proved here	§ 02	The three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; $SU(2)$ / one link
[15]	Large-Field Remainder Bound Lemma: net one-step bracket of the coupling = $r^{\text{small}} + r^{\text{large}}$, $\lvert r^{\text{large}}\rvert \le \frac{4}{3}(b_S^2/E_S)P_L$, $\le C' g^2$ at $c_0^2 > 2$. Actual sensitivity $\approx \frac{1}{3}\beta^2 h^2$	Proved here, Reading (the sensitivity numbers)	§ 02	One link; the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; $E \in (e^{-2}, 1)$
[16]	The loss from the source $e^{s_0\lvert B(X)\rvert}$ appears only once; the source from fine loops is $\mathbb{E}F(U_C) = (F * p_{t\lvert C\rvert})(W_C)$	Proved here	§ 03	Blocked observables / the edgewise independent heat-kernel condition, self-avoiding loops
[17]	Existence, uniqueness and direct-product decomposition of the tree gauge, bijection of re-fixing; cost (2)	Proved here	§ 05 (Lean for finite groups)	Finite groups (Lean), $SU(2)$ (paper), the full axial tree
[18]	Splitting (3) of the small-field contribution to the coupling: the Taylor truncation has the same form as Lemma 7.5.3 of the sources, the tail of the cut is $e^{-c(1-q)^2 h^2 \beta}$	Assumption (statement), Proved here (estimate under the analyticity condition at complex backgrounds), Cited (form in the sources)	§ 06	The analyticity condition at complex backgrounds, margin q, numbers for four in a row

Row	Statement (one line)	Grade	Basis	Scope
[19]	The row absent from the sources is W^{gauge} ((4)) and the group version of the local cancellation; empty in the zero-sum system	Reading	§ 06	Comparison of the text of the appendices of the sources with the preface
[20]	W^{gauge} for two blocks is $O(1)$ from one loop ($13/384$), originating in the non-locality of the background map; after putting in the first order explicitly, $O(1/\beta)$	Proved here (one-loop formula, formal), Reading (numbers)	§ 06	$SU(2)$, heat kernel, seven in a row, the diagonal definition, one step, quadratic coefficients, $\beta = 32$–512
[21]	The non-local tail type is decided by the definition of the coarse variables (conserved quantities and global constraints are flat; sharing by weights and promoted constraints are exponential; implicitly held constraints are of the propagation type)	Reading (the arithmetic of the weighted sums is Proved here)	§ 06	Chains, slabs, columns, periodic lattices; one step, quadratic coefficients, commutative background; four dimensions not yet
[22]	The algebraic layer of the small-field contribution to the coupling (the projection $\mathbf{Loc}$, the layer of W^{gauge}) and the assembly of the remainder (sum uniform in the number of steps)	Proved here	§ 06 (the finite-dimensional part in Lean)	Receives the assumptions on the coefficients and on the cut; schedule $g_k^2 = 1/(A(k+1))$, $m \geq 3$
[23]	The parts of the same form as in the sources, and the four points where the cut may differ from the sources	Cited , Reading	§ 06	Reading of the equation numbers of the sources; the text not reopened
[24]	On a critical-point background the one-loop effective actions of the tree gauge and the background-covariant gauge agree (5); the ghost is one line of the bookkeeping	Proved here , Reading (the ghost coefficients)	§ 05	One step, one loop, $HD = 0$, $\xi = 1$; numbers for $SU(2)$, 3^4 points, commutative background
[25]	The group version of the recursion of W (one row): the image closes on the same types, contracting in the ratio of couplings to quadratic $5/8$ and quartic $\leq 5/48$; the new terms are independent of the step and are drawn to a fixed point	Proved here (first-order Taylor formula, tree rational numbers, bookkeeping), Reading (the correction of $1/\beta$), Cited (the additivity of the sources)	§ 06	$SU(2)$ heat kernel, one row, the diagonal definition, one step of the two-block seven twice, first order, up to fourth order, $\beta = 32$–512
[26]	The many-body tail: three-body coefficients $e^{-\kappa d}$ in the width ($\kappa = \ln(5+2\sqrt{6})$), the sum uniform in volume, the image 5×10^{-3} or less. The one-row recursion closes up to fourth order	Reading (the bookkeeping is Proved here)	§ 06	One row, definition without a conserved quantity, one step, tree and one loop, $L \leq 8$, $d \leq 7$

Row	Statement (one line)	Grade	Basis	Scope
[27]	The coordinates of the cubic-remainder estimate: coarse links plus global tags are exponential at one loop in three dimensions without boundary ($\kappa = 3.094$), with the sum uniform in volume, and exponential on the four-dimensional tree	Reading , Proved here (orthogonality of the tags and uniqueness of the decomposition)	§ 06	Commutative background, quadratic coefficients, one step, coarse $2 \times 2 \times L$ ($L \le 16$); four dimensions tree $n \le 16$
[28]	The difference between the two definitions of Loc: $\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums, $-\frac{1}{384}a \cdot b/\beta$ with the projection of entries. The type of the sources is the row sum	Proved here (closed formula and identity), Reading (numbers)	§ 06	$SU(2)$ heat kernel, seven in a row, one step, up to $O(1/\beta)$

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

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[01] ASM  one line of each of the four rows that do not close                    con
[02] RDG  where the closed components lie ⟸ [01]
[03] PRV  large-field rarity ⟸ Chebyshev for the group version of the exponential mon
[04] ASM  net bracket = small-field contribution + large-field contribution ⟸ [01][03
[05] ASM  to θ⁴ by background reversal, parameter z ⟸ [04]                        θ³
[06] PRV  a source that is a function of block variables commutes                 blo
[07] ASM  the source is a shift of coordinates ⟸ [06]
[08] PRV  no Jacobian for the tree gauge                                          axi
[09] ASM  ghost determinant into local couplings ⟸ [08]
[10] RDG  wall for a road with class functions alone, limit of Haar ⟸ [08]
[11] RDG  the row of background-dependent absorption and many steps is the core ⟸ [03
[12] PRV  Wilson is not dominated by the heat kernel                              one
[13] RDG  the ten directions ⟸ [11]
[14] PRV  exponential moment and rarity of heat-kernel fluctuations, heat-kernel compa
[15] PRV  splitting of the net bracket and r^large ⟸ [04][14]                      or
[16] PRV  accounting of the loss from the source, source from fine loops ⟸ [06]    blo
[17] PRV  tree gauge (Lean for finite groups), cost (2) ⟸ [08]                     fu
[18] ASM  splitting of the small-field contribution, Taylor form same as in the source
[19] RDG  the row absent from the sources, W^gauge ⟸ [11][18]                      em
[20] PRV  W^gauge for two blocks is O(1) from one loop ⟸ [19]                      sev
[21] RDG  the tail type is decided by the definition of the coarse variables ⟸ [20]
[22] PRV  the algebraic layer and the assembly of the remainder ⟸ [18]             re
[23] RDG  same form as the sources, the four points of the cut ⟸ [22]              re
[24] PRV  the agreement of the two gauges (5) ⟸ [09][17]                           or
[25] PRV  group version of the recursion of W (one row) ⟸ [20]                     or
[26] RDG  the many-body tail is exponential, the one-row recursion closes ⟸ [21][25]
[27] RDG  the coordinates of the cubic-remainder estimate are coarse links + tags ⟸ [
[28] PRV  the difference between the two definitions of Loc ⟸ [20][23]             se
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The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. [01], [04], [05], [07], [09] and [18] are assumption rows, and all are in the form of questions that split ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

What this chapter can say

Can say	Cannot say
Proved here The Chebyshev one-liner for the large-field rarity condition assuming the group version of the exponential moment of fluctuations; the commutativity of a source that is a function of the block variables; the direct-product decomposition of the tree gauge (no Jacobian of re-fixing); that Wilson is not dominated by the heat kernel	The group version of the exponential moment of fluctuations itself. Observables of the fine lattice. Checking the gauge in the four-dimensional papers
Proved here The Heat-Kernel Fluctuation Rarity Lemma and the Large-Field Remainder Bound Lemma (heat-kernel fluctuations, one link) and the One-Link Heat-Kernel Comparison Lemma (one Wilson link); the accounting of the loss from the source and the source from fine loops (one heat-kernel step); existence, uniqueness and direct-product decomposition of the tree gauge (Lean for finite groups) and cost (2); the estimate of the Taylor truncation of the small-field contribution to the coupling (under the analyticity condition at complex backgrounds, the same form as Lemma 7.5.3 of the sources)	The group version of the exponential moment of fluctuations under the true measure of the fluctuations. Showing the analyticity condition at complex backgrounds for four-dimensional blocking. The sharp multi-link form
Reading Where the closed components lie; the dependence among the four rows (the row of background-dependent absorption and many steps is the core); the wall for a road with class functions alone; the ten directions; that the row absent from the sources is W^{gauge} and the group version of its local cancellation, empty in the zero-sum system	Any of the implications of the dependence. Any of the definition of the inter-block term, the local cancellation and the cubic-remainder estimate. Checking against Bałaban's main text
Proved here The agreement of the two gauges (5) (one step, one loop, critical point); the algebraic layer of the small-field contribution to the coupling and the assembly of the remainder (the finite-dimensional part in Lean); the one-loop formula for two blocks (formal)	Agreement at two loops and beyond. The analyticity condition at complex backgrounds and the probability of the cut. The local cancellation in four dimensions
Reading $13/384$ and $O(1/\beta)$ for two blocks; the counting rule for the tail type; the four points where the cut may differ from the sources; the ghost coefficient $+0.024$	The tail type in four dimensions and the cubic-remainder estimate. Checking against the text of the sources
Proved here The first-order Taylor formula of the recursion of W and the tree rational numbers; the orthogonality of the tags; the identity between the two definitions of $\mathbf{Loc}$. Reading The contraction ratios of the recursion of W; the exponential decay of the many-body tail; the three-dimensional one loop and four-dimensional tree for the coordinates of the cubic-remainder estimate; the numbers for the difference of $\mathbf{Loc}$	The recursion of W in four dimensions; four-body terms; the one-loop tail in four dimensions; the volume-uniform cubic-remainder estimate
Assumption The "one line" of the four rows; the splitting of the net one-step bracket of the coupling; parity; the coordinates of the source; the absorption of the ghost determinant	A proof of any of them. Any prospect of ultraviolet control closing

Sources and reproduction

Item	Type	Source, tools
Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989), basic inequality (unpublished) (where the net one-step bracket of the coupling sits)	Cited (abstract and preface)	T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (15-02 § 02)

Item	Type	Source, tools
The weight of large fields and the role of Chebyshev; One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS)	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, LNM 2242 (2019), arXiv:1907.05474 (15-02 § 04)
Global gauge fixing; axial gauge per block	Cited	I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12. 7-05 § 05
The four-dimensional borderline of stochastic quantisation	Cited	The source table of 24-06 § 08
Proposals for the four rows, discarded proposals, dependence among the four rows	This book's study	The record of this book's study (sorting out the rows of ultraviolet control that do not close)
(1), the commutativity of the proposal to take the source as a function of the block variables, the direct-product decomposition of the tree-gauge proposal, the ratio of domination	Proved here	The text of this chapter
Lemma 7.5.3, Loc, (A.1.3)–(A.1.4) of Appendix A.1, (A.5.3)–(A.5.4) of A.5 (flat fields)	Cited (main text)	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, LNM 2242 (2019), arXiv:1907.05474v2
Induction of the small-field approximation; the computation of the second-order perturbation	Cited (abstracts and preface)	T. Bałaban, Comm. Math. Phys. 109 (1987) 249–301, 116 (1988) 1–22, 122 (1989) 355–392 (15-02)
The Heat-Kernel Gaussian Upper-Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the accounting of the source, the tree gauge ($SU(2)$), the estimates (2)(3)(4)	Proved here	The text of this chapter. The parts on finite sums and finite groups are in the Lean bundle (15-09 Table 3, re-verified)
Numbers for the tail ratios, the sensitivity, the remainder and cut rates for four in a row, and the cost of the tree gauge	Reading	Numerical record of this book's study (two routes, sum over images and character sum; Wilson and heat kernel; two routes, the lift and directly on the group; control with $U(1)$; the box of 2^4 by two one-loop implementations)
Numbers for the two-block system, chains, slabs and columns; ghost coefficients	Reading	Numerical record of this book's study (seven in a row and chains of L blocks: two routes, one loop of Laplace and full quadrature; Wilson and Villain controls for $U(1)$. Slabs and columns: two implementations, projection and KKT; two routes, analytic Hessian and differences. Ghost: two routes, analytic and finite differences at 3^4 points; controls with $U(1)$, without the ghost, and with the background moved off the critical point)

Item	Type	Source, tools
The Faddeev–Popov method; the lattice background-field method	Cited (bibliography only)	L. D. Faddeev, V. N. Popov, Phys. Lett. B 25 (1967) 29–30. M. Lüscher, P. Weisz, Nucl. Phys. B 452 (1995) 213–233, arXiv:hep-lat/9504006 (text not checked)
The additivity of the sources ((A.1.4)–(A.1.5), Exercise 2.2.4)	Cited ((A.1.4)–(A.1.5), main text); Reading (the identity from Exercise 2.2.4 not confirmed in the text)	R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474v2 (p.35, p.198–199; the sketch on p.207–209 is written with W removed)
The recursion of W, the many-body tail, the coordinates of the cubic-remainder estimate, the two definitions of $\mathbf{Loc}$	Reading (formulas and bookkeeping are Proved here)	Numerical record of this book's study (separate implementations, Gauss–Hermite on the integrand without approximation and nested 2×2 matrices; rational linear backgrounds and floating-point projections; Laplace on the group and full quadrature with finite β; Wilson control for $U(1)$; positive control with the definition with a conserved quantity. The periodic boxes: analytic and difference Hessians, three implementations, dense and Bloch). The bookkeeping is the Lean bundle (15-09 Table 3, re-verified)
(5), the algebraic layer, the assembly of the remainder, the arithmetic of the tail sums	Proved here	The text of this chapter. The finite-dimensional part is the Lean bundle (15-09 Table 3, re-verified)

Terms used in this chapter — the four rows that do not close, the small-field contribution to the coupling, the large-field contribution to the coupling, the read-out map, the large-field parameter z, a source that is a function of the block variables, position-dependent couplings, the direct-product decomposition of the tree gauge, the ghost determinant, the coupling at the seam, the row of background-dependent absorption and many steps is the core, the Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the analyticity condition at complex backgrounds, $W^{\mathbf{gauge}}$, the zero-sum system, the non-locality of the background map, the tail type, coarse links plus global tags, the agreement of the two gauges, the recursion of W, the many-body tail. Their meanings are in A-02 Glossary.

Next chapter: 15-09 What cannot be said, the index of questions, the dependency diagram — the ultraviolet questions 1–31 of this part and a summary of the five grades.

What cannot be said, the index of questions, the dependency diagram

> This chapter is an index; it points to the relevant sections of each chapter.

> This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

What this chapter shows — What cannot be said across Part 15 (a summary); the index of the questions left by each chapter (the ultraviolet questions) and the current state of their answers; the dependency diagram between chapters; the five grades of the main rows of each chapter; and the correspondence between the Lean bundles used in this part and the text.

Prerequisite chapters — 15-00 (entrance). The other chapters are reached from the rows of the tables on this page.

Said first — This chapter adds no new results. Questions are questions, not claims. In no chapter of this part is ultraviolet control closed (existence was shown by the published construction, with a different design; Parts 6–13). The re-check of the Lean bundles of this part is done (all pass, with only the three standard axioms). The labels in the text stay in the five grades (paper proofs stay [Proved here]); Lean is kept as an outside column: the theorem names and assumptions are added on the same line in the text of each chapter, and Table 3 lists them as "re-check: done".

Relation to the sources — Some of the parts written in this part on paper and by computation write, on the group and under conditions, the same form as structures known in the sources (BBS for flat fields, Bałaban's series) (for example, the local part and the Taylor remainder of the small-field contribution to the coupling have the same form as Lemma 7.5.3 of BBS, 15-08 § 06). No novelty is claimed. What is absent from the sources is the group version of the place the sources themselves call a sketch (the explicit quadratic term between blocks and its local cancellation), and this is not shown. Within the study of that group version, the four points of the cut made when the algebraic layer was written (15-08 § 06) may differ from the sources, and they are written separately from "the same form as in the sources".

Order of this chapter

1. What cannot be said — twelve items for the whole part, and the items of each chapter
2. The index of questions — the ultraviolet questions and the current state of the answers (Table 1)
3. The dependency diagram of the chapters
4. Summary of the five grades (Table 2)
5. The Lean correspondence of this part (Table 3)
6. Connection to Part 27 and the glossary

00

What this chapter shows

- What Part 15 does not say comes down to twelve items. Each is there to stop readings that drop the conditions "smallest systems, leading term, one step, form of a statement" (§ 01).
- There are thirty-one questions, and each touches one component inside the row for ultraviolet control in the ledger of conditions. For eighteen of them all or part of the answer has been written (ultraviolet questions 4, 5, 7, 8, 10, 12–23, 25), but whichever is settled, ultraviolet control does not close (§ 02).
- The dependencies run from the entrance (15-00), the difficulty (15-01) and the original sources (15-02) to the decomposition statement (15-03), branch into the computational line (15-04–15-06) and the wall line (15-07, 15-08), and return to 15-09 (§ 03).

01

What cannot be said

We gather what is not said across the whole part. The flaws of the conditional implication assembled beyond the parts of this part are in 15-00 § 08. The finer items of each chapter are in that chapter's "What cannot be said" section.

(a) **Within this part, ultraviolet control is not closed.** Part 15 shows none of the three conditions of ultraviolet control, in four dimensions or in two. The existence of the four-dimensional continuum limit was shown by the published construction, with a different design (Parts 6–13; the comparison of designs is in Part 14). What this part shows is statements of what would need to be shown, line-by-line computations, elementary lemmas (the Coupling-Smallness Induction Lemma, the Background–Fluctuation Decomposition Lemma, the Orbit-Direction Flatness Proposition, the Lemma on One Step with Background-Independent Fluctuations, the Bundle Network Theorem), and numbers for the smallest systems.

(b) **The arithmetic under the one-loop flow is not a theorem about the actual sequence of effective couplings.** The divergence and finiteness of the sums in 15-01 are arithmetic that treats the one-loop flow as an exact equation.

(c) **The numbers for the smallest systems are not the numbers of a real blocking step.** The numbers of 15-05 and 15-06 are for the tree level restricted to commuting configurations (the leading term of $\beta \to \infty$), the one-loop level, one step seen through the marginal distribution of a single coarse plaquette, and small boxes (free boundary, free only on the outer rim, periodic); they are not numbers for a step that includes blocks embedded in the four-dimensional lattice, re-fixing of the gauge, and large fields.

(d) **The results for the hierarchical model do not imply anything in four dimensions.** The Migdal–Kadanoff map of 15-04 sends a single-plaquette weight back to a single-plaquette weight; multi-link couplings, re-fixing and the remainder from large fields are structurally absent, and it is a model in which confinement occurs even at $U(1)$. The range checked by interval arithmetic is only a finite interval of β (the tube $\beta \le 128$, the curve $\beta \le 64$), and the limit $a \to 0$ is not closed. The proposal of a one-step inequality with a fixed comparison partner shows neither a violation nor support at the numbers for small boxes (within the resolution at $6^4/3^4$), and it is not shown for any β.

(e) **The Lemma on One Step with Background-Independent Fluctuations is a statement only for one step, and only when the law of the fluctuations does not depend on the background.** The true measure of the fluctuations in one step depends on the background, and the absorption of that dependence η and the multi-step induction have not been written (15-07).

(f) **The decomposition statement and the twisted-torus ultraviolet statement are statements.** The decomposition of 15-03 and the ultraviolet statement for a small twisted four-dimensional torus of 15-02 are in the form of [Assumption] and are not proved. Of the implications, only the Coupling-Smallness Induction Lemma and the arithmetic of geometric series were checked elementarily.

(g) **Much of the checking against the original sources is at the level of abstracts and introductions.** Stages 4 and 5 of the series and 119 are from abstracts, stages 1–3 from titles, and the two theorems of Bałaban 1989 (Theorems 1 and 2) from the abstract and introduction of the last paper. Appendix A.6 of BBS was read in the text itself (15-02).

(h) **The proposals for the four rows that do not close are proposals.** The proposals of 15-08 are ideas before verification, and none of them closes any of the four rows. We say neither "there is no action that closes with one parameter" nor "the class-closure condition closes with finitely many parameters".

(i) **"The same form as in the sources" is an agreement of form, not a claim of a new structure.** In the splitting of the small-field contribution to the coupling (15-08 § 06), the same form as in the sources holds on the group only under the analyticity condition at complex backgrounds (analyticity in complex backgrounds and an upper bound independent of β) and the margin q. The analyticity condition at complex backgrounds is not shown for four-dimensional blocking. The row absent from the sources (the explicit quadratic term W^{gauge} between blocks and the group version of its local cancellation) is not shown. A zero-sum system with a single block cannot test this row, so we measured it in a system of two blocks: W^{gauge} starts explicitly at one loop, and the local cancellation is needed for the part $O(1/\beta)$ ($SU(2)$, seven in a row, one step, quadratic coefficients), but the local cancellation and the cubic-remainder estimate in four dimensions are not shown. The four points of the cut made when the algebraic layer was written in Lean (taking $\mathbf{Loc}$ as the projection onto components, the reason for the zero sum, how the cut is paid for, how the sum over steps is taken) are written separately as places that may differ from the sources (15-08 § 06).

(j) **We do not say that a four-dimensional number falls below 1.** The chain for the hierarchical model covers the finite range $\beta \le 128$ ($\beta > 128$ is not closed); the Shape-Remainder Theorem for Four Plaquettes in a Row is for the projected model of four in a row ($B \ge 32$, the box $O(1)$; the shape deviation of Wilson's form lies outside the box); and the upper bound of the Bundle Network Theorem (local form) uniform in the volume is an estimate of 1.83 in four dimensions, which exceeds 1. Row-weighted interpolation brought it to 0.823 or below (normalisation $\beta_0' = 1$; the numbers are computed, not interval estimates), but the factor of the direction-wise normalisation is not a measure of contraction, so the normalisation is fixed in the form divided by the long-wavelength coupling, and that One-Step Contraction of the ℓ^4 Cone is limited to the first step (with edge-sum blocking the factor is 1.26 or more at the third step; with transverse-averaging blocking the lower bound does not grow with the steps, but there is no upper bound). The budget for the group-geometric part of the background dependence of $SU(2)$ (the third order of BCH, coefficient $1/9$) diverges however the threshold is chosen (15-07 § 07, arithmetic under the one-loop flow). If the background is carried by a parameter of the class, the budget becomes uniform in the number of steps (conditional, 15-07 § 06), but the difficulty moves to whether the class closes in one step with faces and multiple loops, and this is not shown in four dimensions.

(k) **The coordinates of the class and the numbers of the recursion are statements about the measured rules and ranges.** The "one parameter + ratio coordinates" of the four-dimensional class is a fit from the tree of a free-boundary box ($L \le 10$), one loop ($L \le 4$) and the quadratic order of the commuting tree on the periodic network ($L \le 16$); the existence of a fixed point of the ratio coordinates is not shown. That the diagonal neighbours are not 0 is a statement about the two measured rules and two conventions of projection; the values, and "closing within one cell", depend on the rule. The recursion of the core row (a contracting image + a new term independent of the step + an exponential three-body tail) is a first-order statement for one row, a definition without conserved quantities and the non-local terms; four dimensions and four-body terms are not measured. The one loop of the flow of the coupling is a number for the first step of one rule, and the coefficient of $\ln b$ at small factors is 1.7–1.9 times the one-loop value.

(l) **Spinors are not among the grounds of the argument of this part.** Pure Yang–Mills has no spinor (fermion) fields. The structure of $SU(2) = \mathrm{Spin}(3)$ enters as properties of the group in three places: the twist (the element -1 of the centre $\mathbb{Z}_2$, 25-01), the curvature of the Haar measure (the Van Vleck prefactor $\psi/\sin\psi$ of the heat kernel, $1/9$ of the third order of BCH, $\frac{2}{3B}$ of four in a row; the Laplacian restricted to class functions is $\frac{1}{\sin\psi}\partial_\psi^2(\sin\psi\,\cdot) + 1$, and the constant 1 is the curvature term of the unit S^3), and the quaternion identity ((5) of 15-07). Each of these is a sign that this book's tools are specialised to $SU(2)$, not an indication of the cause of the gap.

Gauge theories of G_2, whose centre is trivial, are also said to confine by strong-coupling lattice arguments (the string breaks) Cited (K. Holland, P. Minkowski, M. Pepe, U.-J. Wiese, Nucl. Phys. B 668 (2003) 207–236, hep-lat/0302023; checked in the abstract, the text not checked), and we read the twist of the centre as a tool of measurement, not the mechanism of confinement Reading . The roads that use the centre as a probe (Parts 25 and 26, and the proposal of a one-step inequality with a fixed comparison partner) are limited to groups with a non-trivial centre.

02

The index of questions — ultraviolet questions 1–31

Table 1 gathers into one the questions left by the chapters of Part 15. Ultraviolet questions 1–3 inherit the numbers of 26-11 (their content updated in 15-03), ultraviolet questions 4–15 assign a band of numbers to each chapter, and ultraviolet questions 16–31 are questions added in the revisions. "The current state of the answer" is attached only to the questions for which all or part of the answer has been written. The status and size of each question, and "what does not move even if settled", are in the table of the chapter of origin and are not copied here. Whichever question is settled, the row for ultraviolet control in the ledger of conditions does not close.

Table 1 The questions of Part 15 (the sentence of the chapter of origin shortened to one line; the conditions are in the table of the chapter of origin; what follows → is the current state of the answer)

Q.	Question (one line)	Components concerned	Origin
1	Does the one-step map contract transversally on the one-parameter class it draws towards? (i) Do λ and the source stay finite for blocks embedded in the four-dimensional lattice? (ii) Is the constant C of the one-step inequality independent of the step? (iii) Is the weight of the sign of the centre $\mathbb{Z}_2$ needed as a separate parameter?	Class-closure condition, shape remainder condition	15-03 § 07
2	The smallest model (a hierarchical $SU(2)$) in which the net one-step bracket of the coupling can be shown including large fields. Can absorption under a constraint for a single plaquette be restated at the true critical point for many plaquettes?	Coupling-flow remainder condition, net one-step bracket of the coupling	15-03 § 07
3	On the twisted M^4 lattice, is the smallest eigenvalue of the operator of the adjoint fluctuations around an isolated flat connection at least c/M^2? Is the group version of closure under the fluctuation expectation ((iii) of BBS) unchanged by the twist?	The Gaussian comparison condition for the partition function in the twisted-torus ultraviolet statement	15-03 § 07
4	Extend the tube and the attracting curve of the hierarchical model to $\beta > 64$ (the seam between asymptotic estimates and interval arithmetic). → From $\beta \le 128$ downwards, the chain has been joined via the band side to the strong-coupling decision region (1137 steps) [Computer-assisted]. The upper end $\beta > 128$ and the curve at $[64, 128]$ are still open	Class-closure condition of the hierarchical model	15-04 § 05
5	Check the lower bound of the curvature outside the layer and the weight of the layer by interval arithmetic, and separate the flow coefficient $c^* = 1/60$ on the curve from $1/48$. → Enclosed as $[0.0151, 0.0188]$ at five places on the tube of narrowed radius [Computer-assisted]. That the curve enters that tube, and the curvature, are still open	Coupling-flow remainder condition of the hierarchical model	15-04 § 05
6	Along the sequence of coarse-plaquette sides L, does the source $\cdot L^4$ (now $7.7–8.5 \times 10^{-3}$, transverse thickness 4) settle to a constant as the transverse directions are widened?	The sum of the source over steps	15-05
7	On the tree-level network with all coarse plaquettes fixed, can a local representative be built that gives an upper bound uniform in the volume? Does the operator norm of λ in the coarse directions fall below 1? → A local representative and an upper bound uniform in the volume were obtained (Bundle Network Theorem (local form)). The estimates are 0.245 in three dimensions and 1.83 in four; in four dimensions this route cannot go below 1.39 (undecided)	The transverse contraction (matrix)	15-05 § 07
8	Derive on paper the one-step correction to $O(1/B)$ for four in a row, giving the constant C of the one-step inequality of the shape remainder condition. → In the projected model of four in a row, the Shape-Remainder Theorem for Four Plaquettes in a Row gave $C = 0.0108$ ($B \ge 32$, the box $O(1)$)	Shape remainder condition	15-06

Q.	Question (one line)	Components concerned	Origin
9	Is the attracting class still uniquely determined when loop shapes other than four in a row (2×2, L-shaped, mixed quartic) are included? Does the fixed point stay put when the definition of the diagonal is changed?	The coordinates of the class-closure condition	15-06
10	For four in a row, split the joint law of P_2, P_3 into a product of local couplings × a density, and find the lowest degree in θ of the score of the density and the isotropy of the quadratic form. → The lowest degrees are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$. The shape part goes into a geometric series, while for $SU(2)$ the budget for the group-geometric part (the third order of BCH, $1/9$) diverges. The improvement lies on the side of the class	Absorption of η	15-07
11	Can the smallest multi-step statement be written in the hierarchical model or a two-dimensional model (the background-regularity condition and the contraction of the many-body norm)? The width of the band of analyticity of the coupling. Numbers for the smallest systems are the next test	The multi-step induction	15-07
12	Derive the Lipschitz constant of the read-out on paper, and measure on a slab the difference in total variation between the density with one large field inserted and without. → Done for one link (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma). On the 2×2 slab with heat-kernel fluctuations, a form that holds for $c_0^2 > 2$ (the convention of the Large-Field Remainder Bound Lemma); the value on the slab is 1/3 of that for one link. Many links under the true measure are still open	Row of the remainder including large fields, net one-step bracket of the coupling	15-08
13	When a source $s\,\mathrm{Re}\,W_a$ is added to four in a row, is the shift of the coupling linear in s? On paper, the position-dependent coupling $\beta(x)$ for $U(1)$ Villain. → In one heat-kernel step, the source from fine loops is exactly a shift of the coupling for each representation. In one blocking step with four in a row, the shift by a source on a loop inside the block is $f_C^2 + O(1/\beta)$ (uniform in β). The $\beta(x)$ of Villain is a local average. Loops crossing boundaries and many steps are still open	Observable-source condition	15-08
14	Does the parity of critical points hold for four plaquettes (paired with the question "splitting the density of the joint law")? → It breaks in the breathing direction. The conclusion of the proposal on the parity of critical points stands for a chain in a row by the covariance under reversal of the background. θ^3 in four dimensions is still open	Row of background-dependent absorption and many steps	15-08
15	Split the one-loop Hessian of a block with blocking factor 2 and measure the coefficient of the ghost F^2 (negative control: $U(1)$). Formalise the direct-product decomposition of the tree gauge for finite groups. → The second half is done (Lean, finite groups). The cost is the degradation $(d-1)(L-1)\varepsilon$ of the small-field condition. In the background of the critical point, the ghost agrees identically with the tree gauge at one loop (3^4 points, coefficient $+0.024$ in a commuting background)	Row of the boundary of gauge re-fixing	15-08
16	In the smallest system that is not zero-sum (two copies of four in a row side by side, sharing the link between them), write W^{gauge} and turn into a number the breaking of the local cancellation of the quadratic contribution to the surrounding couplings. → For seven in a row, $O(1)$ from one loop ($13/384$), and $O(1/\beta)$ after the first order is put in explicitly. The type of the tail is determined by the definition of the coarse variables	Row of the remainder including large fields, small-field contribution to the coupling	15-08
17	On the central plaquette of a 2×2 slab, derive by Wick's theorem the geometric coefficients of the odd orders of BCH, symmetrised over cyclic permutations, and compare them with the two-link $1/\sqrt{3(m+2)^3}$. → For a single loop (the cross), nothing remains if a is updated by the group product. In three dimensions $2 \times 2 \times 2$ the corner triple product $\kappa = 3/1024$ is needed	Row of background-dependent absorption and many steps (group geometry)	15-07
18	In a class that has the logarithm a of the background holonomy as a parameter, measure in one step in series whether the geometric part moves exactly into the flow of a. → Remainder $\theta^0\beta^{-3/2}$; the budget is uniform in the number of steps (conditional). The difficulty moves to the class-closure condition	Row of background-dependent absorption and many steps (improvement on the side of the class)	15-07
19	The condition of the One-Link One-Step Fluctuation Lemma (the recentring condition): make explicit the Taylor radius of the map to the direction of the sum of staples, and make the coefficient of $\delta b/b$ a single number. → $\lvert Q\rvert \geq 6\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$ (measured: $c \approx 12$). The coupling goes to the side of the parameters of the class	Row of background-dependent absorption and many steps (one Wilson link)	15-07

Q.	Question (one line)	Components concerned	Origin
20	Bound ℓ^4 in four dimensions directly (the near part by a finite network, only the tail by Riesz–Thorin), and see whether $\sup \mu < 1$. → Row-weighted interpolation gives $\sup \mu \le 0.823$ (normalisation $\beta_0' = 1$, computed). Undecided for the direction-wise normalisation	The transverse contraction (matrix)	15-05 § 08
21	Define Loc in two ways, by row sums (zero momentum) and by matrix components, and obtain the difference in W^{gauge} as a rational number with the covariance of two blocks (is the cut the same as in the sources?). → The difference is not 0 ($\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums, $-\frac{1}{384}\, a\cdot b/\beta$ with the projection onto components). The type in the sources is the row sum	Row of the remainder including large fields, small-field contribution to the coupling	15-08
22	In three dimensions without boundary, does the one-loop non-local weight in the coordinates of coarse links + global tags stay exponential? → Exponential at coarse $2 \times 2 \times L$ ($\kappa = 3.094$), and the sum is uniform in the volume. The four-dimensional tree is also exponential. The four-dimensional one loop is still open	Row of the remainder including large fields, cubic-remainder estimate	15-08
23	The second step of the multi-loop class: is the corner triple product κ a coordinate preserved step by step, and do the same coordinates suffice in four dimensions 2^4? → In three dimensions it does not contract and approaches the fixed point from below. Four dimensions needs the diagonal neighbours and the parallel type, and the ratio coordinates approach a fixed point (tree-level fit)	The coordinates of the class-closure condition	15-07
24	Measure the agreement of the two gauges and the coefficient of the ghost in a non-commuting background (F of the commutator)	Row of the boundary of gauge re-fixing	15-08
25	Can the factor $\sup \lambda \le 3.29$ of the direction-wise normalisation be brought below 1 by a weighted symbol for each band of momenta? → It turned out to be a question in a direction that cannot be brought down: at $\sup \lambda = 4 \sup \mu_{\mathrm{loc}} \ge 0.59$, λ is not a measure of contraction. The normalisation is fixed in the form divided by the long-wavelength coupling	The transverse contraction (matrix)	15-05 § 08
26	Let a parameter of the class carry the coupling b, and see whether the motion $c g^2 p^2$ of the coupling of one Wilson link moves into the update of the parameter	Row of background-dependent absorption and many steps (one Wilson link)	15-07
27	Derive the values at the fixed point of the ratio coordinates of the four-dimensional class directly from the half-infinite problem near the corner. Under rules that share fields, is the part that does not fit into a cell closed by putting it into finitely many pairs, or carried as a tail?	The coordinates of the class-closure condition	15-07
28	On the fixed point of transverse-averaging blocking, get the upper bound of $\sup \mu_\infty$ below 1 by a route other than Stein. Put the existence of the fixed point on paper	Multi-step shape remainder condition	15-05 § 08
29	With coarse links + global tags in four dimensions, measure the rate at which the quartic width of the three-body tree depends. In one row, the four-body terms and the contraction of the three-body image at one-loop level	Row of the remainder including large fields, local cancellation, cubic-remainder estimate	15-08
30	With transverse-averaging blocking or a large factor, does the dependence on the factor of the long-wavelength one-loop coefficient of the network approach the one-loop value?	Coupling-flow remainder condition	15-05 § 08
31	Run the re-taken tube over the whole of $[15.26, 128]$ with intervals of width 0.02, making unconditional both that the curve enters the tube and the enclosure of c^* on the curve. Cover $\beta \ge 128$ by asymptotic intervals	Class-closure condition and coupling-flow remainder condition of the hierarchical model	15-04 § 05

Reading Table 1. The questions "transverse contraction on a one-parameter class", "one-step correction for four plaquettes in a row" and "uniqueness of the class for other loop shapes" are questions about the coordinates of the class-closure condition and the shape remainder condition; the questions "minimal model of the remainder including large fields" and "Lipschitz constant of the read-out" are questions about the net one-step bracket of the coupling, corresponding to the content of the unpublished basic inequality (Theorem 2 of Bałaban 1989); the questions "splitting the density of the joint law", "minimal multistep statement" and "parity of critical points", and ultraviolet questions 17–19, are questions about the row of background-dependent absorption and many steps of the core (absorption of η and the multi-step); and the question "smallest system that is not zero-sum" is a question about the row absent from the sources inside the small-field contribution to the coupling. Ultraviolet questions 21–24 and 26 are next moves arising from the answers of the revision (the difference of the

cut from the sources, the type of the tail for geometry without boundary, the second step of multiple loops, the ghost in a non-commuting background, the coupling as a parameter); the question "normalisation factors by direction" is a question about the transverse contraction of the direction-wise normalisation (the answer takes a negative form); and ultraviolet questions 27–31 are next moves arising from the answers of the third revision (the values of the coordinates of the four-dimensional class, the upper bound for averaging blocking, the many-body tail in four dimensions, the one-loop coefficient, the upper end of the hierarchical model).

Questions that numbers for the smallest systems can answer are mixed with paper questions.

03

The dependency diagram of the chapters

We arrange the dependencies between chapters so that prerequisite chapters come above (the number of a prerequisite is smaller than the number of the conclusion). To the right of the arrow are the earlier chapters that the chapter uses.

```
[15-00] Entrance       problem, the three conditions, form that closes in one step, types of misreadi
[15-01] Difficulty     ⟸ [15-00]        dimensionless coupling, group-valued fields, gauge
[15-02] Sources        ⟸ [15-00]        series, Bałaban's two theorems, BBS chain, twisted-torus ult
[15-03] Decomposition  ⟸ [15-01][15-02] class-closure condition, the two remainder conditions, conti
[15-04] Heat kernel    ⟸ [15-03]        telescoping sum, invariant curve, tube (β∈[2.46,128]), chai
[15-05] Bundle network ⟸ [15-03][15-04] Unit Bundle Theorem, inequality, table of boxes, Bundle Netw
[15-06] Parameters     ⟸ [15-03][15-05] couplings by loop shape, attracting class, Four-in-a-Row The
[15-07] Background η   ⟸ [15-01][15-02][15-03]  Background–Fluctuation Decomposition Lemma, Lemma on
[15-08] Four open rows ⟸ [15-05][15-06][15-07]  proposals for four rows, Heat-Kernel Fluctuation Rar
[15-09] Index          ⟸ [15-00]–[15-08]
```

The diagram shows the dependencies of the chapters of the study of ultraviolet control (15-00 to 15-09). The systematic attempts (15-10) and the chapters of the second half on one step of the free field and one scale (15-11 to 15-14) are outside this diagram. To check it, see that the number of every chapter to the right of "⟸" is smaller than the one on the left. There are no cycles. No node of this diagram is a node saying that ultraviolet control has been shown.

04

Summary of the five grades

Table 2 picks only the main rows from the five grades table of each chapter. The full statements, grounds and ranges of the rows are in the table of each chapter.

Table 2 The main rows of each chapter (labels as in each chapter's table; only some of the conditions in the range column are copied)

Chapter	Main row	Grade	Range (excerpt)
15-00	Number of steps $K \to \infty$, lower bound for the product of losses	Proved here	$0 \le \delta_k \le \frac{1}{2}$
15-00	Ultraviolet control and the form that closes in one step	Assumption	From a to L_0, constants independent of the step
15-01	Logarithmic divergence of $\sum g_k^2$, the sum of $g^{2+\delta}$ is finite, the Orbit-Direction Flatness Proposition	Proved here	Under the one-loop flow (arithmetic)
15-02	The Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989) with the assumption of the condition that the coupling stays small; the basic inequality unpublished; the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS)	Cited	Abstract and introduction / text of Appendix A.6
15-02	The ultraviolet statement for a small twisted four-dimensional torus	Assumption	$SU(2)$, $n12 = 1$, finite volume

Chapter	Main row	Grade	Range (excerpt)
15-03	The Coupling-Smallness Induction Lemma, the bookkeeping of the transverse contraction	Proved here	Under the net one-step bracket of the coupling and the definition and continuity of the effective coupling / assuming the one-step inequality
15-03	The decomposition statement (the class-closure condition is the one-parameter attracting class + a contracting form)	Assumption	Form of the statement only
15-04	Saddle-point estimates for MK (1/16, 1/3600, $c^* = 1/60$)	Proved here	Leading term at the saddle
15-04	Invariance of the tube, pointwise contraction, attracting curve	Computer-assisted	MK map, single plaquette of $SU(2)$, $\beta \in [2.46, 128]$ (curve $[15.26, 64]$)
15-05	Unit Bundle Theorem, $1/(16d-8) \le \lambda \le 1/16$, inequality for coarse-plaquette side L, Bundle Network Theorem	Proved here	Tree level restricted to commuting configurations, 4 coarse plaquettes symmetric (lower bound)
15-05	Table of boxes, type of the boundary, source $\approx 8 \times 10^{-3}/L^4$, $\lambda > 1/16$ in the coarse directions	Reading	Small boxes, one loop (starting from the heat kernel, $\beta \to \infty$)
15-06	$D = \frac{2}{5} + \frac{24}{5}\frac{q}{B}$, the tree map and 9/89, $\gamma^* = 37/76$, $c = 2/3$	Proved here	Four in a row, one-loop level / leading term of Laplace
15-06	Fundamental + adjoint root $\rho^* = -\frac{1}{4} - 0.84/\beta$, no root for the prefactor family	Reading	$SU(2)$, quadratic, $\beta \ge 12$ (root) / $\beta \in \{4, 8, 16, 32\}$
15-07	The Background–Fluctuation Decomposition Lemma, the Lemma on One Step with Background-Independent Fluctuations, the quaternion identity	Proved here	The Lemma on One Step with Background-Independent Fluctuations: one step, when the law of the fluctuations does not depend on the background
15-07	The group versions of the exponential moment of fluctuations and of background regularity, the proposal of the shift of local couplings	Assumption	Statement and proposal
15-08	The large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations; the tree gauge has no Jacobian	Proved here	Assuming the group version of the exponential moment of fluctuations / axial gauge
15-04	The chain from the narrowed tube via the band side to the strong-coupling decision region (within 846 steps at $\beta \le 64$, 1137 steps at $\beta \le 128$)	Computer-assisted	MK map, $\beta \in [0.95, 128]$; the band side is a prior record
15-05	Bundle Network Theorem (local form): local representative and an upper bound uniform in the volume, $\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$	Proved here , Reading	Commuting, tree level, periodic / four dimensions is an estimate
15-06	Four-in-a-Row Theorem: $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, $C = 0.0108$	Proved here , Reading (constants)	Four in a row, projection of ψ^8, $B \ge 32$, the box $O(1)$
15-07	The One-Link Heat-Kernel Comparison Lemma, four decompositions, third order of BCH $1/9$, class (B, q, a), divergence of the budget	Proved here	One link / leading Gaussian term, two links / one-loop arithmetic
15-07	What remains in the smallest systems: $\theta^0\beta^{-1/2}$, $\theta^1\beta^0$	Reading	Two links, four in a row, $\beta \le 32$
15-08	The Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the accounting of the source, the tree gauge (Lean for finite groups)	Proved here	The three heat-kernel conditions (independent edge by edge, one step, background-independent), one link / finite groups, $SU(2)$
15-08	Splitting of the small-field contribution to the coupling: the Taylor truncation has the same form as Lemma 7.5.3 of the sources; what is absent from the sources is the group version of W^{gauge}	Assumption , Proved here , Cited , Reading	The analyticity condition at complex backgrounds, margin q / empty for the zero-sum system

Chapter	Main row	Grade	Range (excerpt)
15-08	Statements of the four rows and their dependencies (the row of background-dependent absorption and many steps is the core)	Assumption , Reading	Statements, estimates
15-03	Proposal for the final form of the class coordinates of the class-closure condition	Assumption	One step in three dimensions, two links; the reading of the cross
15-04	The chain of the hierarchical model does not carry over to four dimensions (the unproved step of the sandwich, the margin of the tube)	Reading	Checking the sandwich against the sources; small boxes
15-05	One-Step Contraction of the ℓ^4 Cone: $\sup \mu \le 0.823$ by row-weighted interpolation	Proved here , Reading	Commuting, tree level, kernel of 20^4, normalisation $\beta_0' = 1$, no interval estimates; direction-wise undecided
15-07	In the class with the background a as a parameter, remainder $\theta^0 \beta^{-3/2}$; the budget is uniform	Reading , Proved here (arithmetic)	Two links, four in a row, conditional
15-07	In one step with faces, a is the group product; the corner triple product for the multi-loop class	Proved here , Reading	Single loop / $2 \times 2 \times 2$, one step
15-08	W^{gauge} of two blocks is $O(1)$ from one loop; the type of the tail is determined by the definition of the coarse variables	Proved here , Reading	Seven in a row, chain, slab, column, one step, quadratic coefficients; four dimensions still open
15-01	The measured one-step coefficient: at the first step of the periodic network, 2.9–4.4 times the one-loop value; the dependence on the factor is 1.7–1.9 times	Reading	$SU(2)$, one loop, transverse 6^3, first step
15-03	Four-dimensional class: one parameter + ratio coordinates (2 quadratic, 3 cubic, 65 quartic); the diagonal neighbours are non-zero whatever the rule; the local part of the class-closure condition is stated together with its rule	Reading , Proved here (counts, degrees of freedom)	Tree $L \le 10$, one loop $L \le 4$; the periodic network is the quadratic order of the tree
15-03	One-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling: contracting image + new term independent of the step + exponential three-body tail	Reading , Proved here (bookkeeping)	One row, definition without conserved quantities, first order, up to quartic order
15-04	Enclosure $[0.0151, 0.0188]$ of c^* on the re-taken tube	Computer-assisted	50 intervals at five places; that the curve enters the tube is still open
15-04	Note on the proposal of a one-step inequality with a fixed comparison partner: the threshold for finite boxes is 1; within the resolution at $6^4/3^4$	Proved here , Reading	Small boxes; open for every β
15-05	Note on normalisation (λ is not a measure of contraction; Stein floor 0.452); One-Step Contraction of the ℓ^4 Cone is limited to the first step; multi-step lower bounds for transverse averaging 0.64, 0.57, 0.64, 0.62	Proved here , Reading	Commuting, tree level, periodic, 2^4, no upper bound
15-05	Coarse links + global tags are exponential on the four-dimensional tree; the long wavelengths of the network are 1 at tree level, with no running of $\ln L$	Reading	Tree level, $n \le 16$ / one loop with blocking factor 2
15-07	The corner triple product does not contract at the second step and approaches the fixed point from below; the fixed point of the iteration lies on the 0 side of the genuine limit	Reading , Proved here (bookkeeping)	Three-dimensional tree $L \le 20$ / four-dimensional tree
15-08	Group version of the recursion of W and the many-body tail (one row); three-dimensional one loop of the coordinates of the cubic-remainder estimate; the difference in $\mathbf{Loc}$ between the two definitions	Proved here , Reading	One row, two blocks, $L \le 8$ / coarse $2 \times 2 \times L$
15-08	In the background of the critical point, the two gauges agree at one loop	Proved here , Reading	One step, one loop, 3^4 points, commuting background

05

The Lean correspondence of this part

Table 3 lists the bundles written in Lean 4 for the elementary parts of the text of this part, as finite sums, real algebra and trigonometric identities. None of them contains `sorry`, and the axioms printed at the end were only the three standard ones (propext, Classical.choice, Quot.sound). The re-check is also done, and every bundle passed (with only the three standard axioms). The labels in the text stay in the five grades (paper proofs stay [Proved here]); Lean is kept as an outside column, with the theorem names and assumptions added on the same line in the text of each chapter. **No bundle contains the lattice, the group, the measure or the minimisation itself.** What they contain is single algebraic lines from inside the paper proofs.

Table 3 The Lean bundles of this part (re-check: all done; assumptions on the same line as each theorem. The last twenty-five were added on 2026-10-17, 10-22 and 10-29, and have been re-checked. The four from DominantMatrixContraction to ConeCalculus belong to the systematic chapter 15-10 (the bundles of 15-10 also include OneStepWindow, added on 10-22, and ConeCalculusSOS, added on 10-29, six in all); the seven after them were added on 10-22, and the last seven on 10-29)

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGauge.ClassResidualTelescoping	`telescoping_residual` (increment of the deviation at each step $\le r_k$), `telescoping_residual_multi` (same assumptions, n parameters), `inv_sq_sum_le` ($a, c > 0$), `uniform_deviation` ($0 \le q < 1$, increment $\le C/(a+ck)^2 + Dq^k$)	15-03 §04, 15-04	Measure, curvature, map	done
LatticeGauge.HierarchicalShapeFlow	`quarticStep_fixed` (fixed point $1/3600$ of $\sigma \mapsto \sigma/16 + 1/3840$), `shapeStep_contract` ($k \ge 2$), `flow_coefficient` ($1/48 - 15/3600 = 1/60$), `affine_contraction` ($0 \le \lambda < 1$, each step $d_{k+1} \le \lambda d_k + \varepsilon$), `tube_iterate`	15-04	The interval-arithmetic tube, the derivation of the leading term at the saddle	done
LatticeGauge.FluxNetwork	`ratio_le_max` (each $f_i^2 \le M$, $\sum f^2 > 0$), `ratio_ge_of_unit_flux` (unit bundle), `energy_split` ($\sum f\psi = 0$), `thomson_lower`, `thomson_attained`, `minimizer_unique` (the set is closed under midpoints), `quad_lower` ($\beta > 0$), `chain_one_step` (five inequalities)	15-05 §02, 15-07 Lemma on One Step with Background-Independent Fluctuations	Projection, the Dirichlet principle, interchange of differentiation in the Taylor seminorm, invariance of Haar measure	done
LatticeGauge.AllFacesBlocking	`fine_energy_lower` (the map to sets of four plaquettes is injective), `block_energy_lower`, `smooth_ratio` ($n \neq 0, \theta \neq 0$), `equal_split_ratio`	15-05 §07	Pseudo-inverse, coarse Bianchi, Fourier symbols	done
LatticeGauge.RatioTwoCriterion	`ratio_two_iff`, `one_loop_D` ($B \neq 0$, in the form with the Gaussian moments substituted), `gamma_family_one_loop_lower` ($c < 3$), `fund_adj_one_loop_root` ($\beta \neq 0, 1 + 2\rho \neq 0$), `no_pointwise_sqrt_one_loop`	15-06	Haar measure, moments, integration by parts, saddle-point expansion	done
LatticeGauge.FundAdjointRefit	`refit_solution` ($\beta \neq 0$), `tree_quadratic_split`, `tree_value_at_saddle`, `image_rho`, `image_gap` ($-53/356 + 1/4 = 9/89$), `image_one_loop_D`	15-06	Convolution, reduction to the tree, the envelope argument	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGauge.PrefactorFlowFixedPoint	`step_k2`, `step_k2_fixed_iff` (fixed point 37/76), `step_k2_contract`, `fixed_point_general`, `transverse_hessian` ($\sin a, \sin b \neq 0$), `convolution_prefactor`, `rem_shape_geometric` ($0 \le \lambda < 5$, assuming the one-step inequality), `ratio_coefficient`	15-06, 15-03 § 04	Convolution on the group, the Laplace method, one-loop formulas	done
LatticeGauge.HeatKernelFourPlaquette	`conv_assoc`, `convPow_add`, `four_plaquette_of_semigroup` ($p_s * p_t = p_{s+t}$), `four_plaquette_convPow`	15-06 § 05	Haar measure, the heat kernel itself (finite-group version)	done
QuaternionReImSplit	`re_mul_split`, `commutator_re`, `J_imI` and others, `two_link_split` (assumption on the components of W, W_0)	15-07 § 06	Measure, critical point, score, projection	done
LatticeGauge.ShapeRemainderUniform	`uv_uniform` ($0 \le \lambda < 5$, $C \ge 0$, $5B_{n+1} \le B_n$, the one-step inequality), `stays_in_box`, `pow_sub_pow_norm_le`, `keff_half_le` ($B > 0$), `cov_eigen`	15-06 Four-in-a-Row Theorem	Lifting, the estimate of the Laplace remainder, the numerical values of the constants	done
LatticeGauge.HierarchicalTubeChain	`tube_exit` (within the tube, either one step down by $\delta > 0$ or exit), `cell_descent`, `chain_compose`, `tube_weight_mono`, `steps_numeric`, `mk_full_chain_64` (846 steps)	15-04 § 05	The map T, the tube, interval arithmetic, the theorems on the band side (taken as assumptions)	done
LatticeGauge.ShapeResidualGeometric	`forced_uniform` ($0 \le \lambda < 1$, $E \ge 0$), `geom_partial_le`, `residual_sum_uniform` (partial sums $\le U$), `harmonic_part_unbounded` ($c/(k+1) \le \eta_k$)	15-07 § 06	Measure, critical point, score (only the bookkeeping of real sequences)	done
LatticeGauge.LargeFieldSource	`weighted_tail_le_exp_moment` (weight ≥ 0, $\sigma \ge 0$), `union_tail_le_sum`, `coarse_source_commute`, `coarse_source_commute_comp` (finite, nested steps)	15-08 § 02, § 03	Heat-kernel comparison, the sum over images, the Lipschitz constant of the read-off, the source from fine loops	done
LatticeGauge.TreeGaugeFixing	`exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, `regauge_bijective` (shared root), `regauge_sum_eq`, `slice_average_tree_indep` (finite group, finite graph, gauge-invariant function)	15-08 § 05	Haar measure and Fubini for continuous groups, the estimate of the cost	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGauge.WilsonWindowComparison	`one_sub_cos_window` ($0 \le \theta \le \theta_0, \theta_0^2 \le 24$), `wilson_weight_window` (same, $b \ge 0$), `window_constant_chain`, `window_split_bound`	15-07 § 05	Lower bound of the heat kernel, Bessel asymptotics, the seminorm step of the One-Link One-Step Fluctuation Lemma	done
LatticeGauge.QuarticRowColumnBound	`weighted_quartic`, `row_quartic`, `schur_quartic` (upper bound by the row and column ℓ^1 sums), `representative_bound` (agreement of the representatives), `cone_step`, `cone_chain`	15-05 Bundle Network Theorem (local form)	Riesz–Thorin, the identity of the axial cone, periodisation, analyticity of R	done
LatticeGauge.BCHCubicResidual	`nT_sum_diff`, `shape_split` (no assumptions), `score_generator_split`, `bch3_coefficient` ($m + 2 \ne 0$), `geometric_variance_m1`	15-07 § 07	Measure, Gaussian moments, projection, the BCH series	done
LatticeGauge.SmallFieldLocSplit	`loc_split_unique` (polynomial identity), `geom_tail_le` ($A \ge 0, 0 \le q < 1$), `taylor_tail_bound` ($0 \le s < \rho$, $\lvert a_m \rvert \le M/\rho^m$)	15-08 § 06	Cauchy estimates (analyticity), the analyticity condition at complex backgrounds, the probability of the cut, the effective action	done
LatticeGauge.BackgroundBudget	`budget_term_sq` ($g \ne 0$), `budget_unbounded_of_threshold` ($g_j^2 \ge 1/(A(j+1))$, $0 \le g_j \le h_j$), `inv_sq_partial_le` ($b > 1$), `inv_sq_budget_bounded`	15-07 § 07	Measure, lattice, renormalisation steps (real arithmetic only)	done
LatticeGaugeInterBlockLocAbsorption	`loc_split` (K symmetric), `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_split`, `two_site_unique`, `tree_numbers`, `one_loop_numbers` (arithmetic of substitution only)	15-08 § 06	The one-loop Laplace formula, $O(1/\beta)$, the group, the measure	done
LatticeGaugeBlockCouplingCancellation	`locM_idem`, `locLin_idem`, `split_unique`, `bil_split`, `trace_locM`, `deltaBeta_nonloc` (the retained pairs are reflexive), `frob_loc_nonloc`, `frob_pythagoras`, `Wg_explicit`, `two_block_W` (two blocks), `Wg_zero_of_blockDiag`, `loc_psd` (C positive semidefinite), `block_cancellation` and others (24 theorems)	15-08 § 06	The content of (A.5.3)–(A.5.4), the group version of the local cancellation, the lattice, the measure	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGaugeSmallFieldRemainderAssembly	`one_step_remainder` (Cauchy-type assumption on the coefficients, assumption on the cut, margin $h_0 \le qh$), `gamma_ge_one`, `cut_sum_le` ($A \ge 1$, $\gamma \ge 1$), `taylor_step_le` ($m \ge 3$, $r_0^2 \ge 1$), `schedule_match`, `total_remainder_uniform` (same assumptions, uniform in the number of steps) and others (16 theorems)	15-08 § 06	Analyticity (the analyticity condition at complex backgrounds), the probability of the cut, the contraction of the shape coordinates	done
LatticeGaugeToeplitzTailSplit	`split`, `split_unique`, `qform_split`, `cform_split`, `tail_sum_le` ($0 \le \rho < 1$, $\lvert q(r)\rvert \le C\rho^r$), `flat_row_sum_lt` ($L \ge 1$), `flat_not_uniform_geometric`	15-08 § 06	The statement of permutation symmetry, the one-loop formula, the group, the measure	done
LatticeGaugeWeightedTailBound	`weighted_geom_le` ($0 \le \rho < 1$), `weighted_tail_le`, `weighted_sum_uniform_linear` ($\lvert q(r)\rvert \le C(r+1)\sigma^{2r}$, $0 \le \sigma < 1$) and others	15-08 § 06	That the measured weights satisfy the assumptions (computation up to $r \le 8$), the slab, four dimensions	done
LatticeGaugeTailTypeClassification	`flat_weighted_sum`, `flat_weighted_unbounded` ($c > 0$), `exp_weighted_le`, `power_sum_uniform` ($s > 1$), `power_sum_unbounded` ($s \le 1$), `power_sum_unbounded_borderline`	15-08 § 06, 15-05 § 07	Which assumptions the measured tails satisfy, the multiplicity of shells, the lattice	done
LatticeGaugeClassParameterBudget	`ghat3_forms`, `ghat3_zero_iff`, `inv_sq_partial_le` ($b > 1$), `class_parameter_budget_bounded` (threshold $h_k \le Hg_k$, one-loop upper bound, $\eta_k \le h_k(Ag_k^3 + Dh_k g_k^2)$)	15-07 § 06	Measure, lattice, projection, renormalisation steps	done
LatticeGaugeStokesSecondOrder	`log_quadratic_part` (ring), `log_quadratic_part_five`, `commSum_eq_zero_of_comm` (commutative ring), `pure_quaternion_commutator`	15-07 § 07	Measure, integration, coarse-graining	done
LatticeGauge.CubeFaceCoupling	`total_sq`, `opp_eliminate` ($\sum \theta = 0$), `general_to_diag_adj` (same), `diag_adj_unique` (the two forms are equal under the same assumptions), `coeffs_from_eigen`	15-07 § 07	Lattice, group, integration, minimisation	done
LatticeGaugeStapleSumLowerBound	`sum_norm_ge_of_inner_ge` (real inner product space, $\lVert u\rVert = 1$, $\langle x_i, u\rangle \ge c$), `card_cos_ge_quadratic`, `first_order_ratio` ($\cos\theta > 0$), `quadratic_margin`, `radius_identity`	15-07 § 05	Quaternions, the geometry of staples, the Taylor seminorm, the composition of steps	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGauge.QuarticSplit	`titu_two`, `quartic_two` ($a, b > 0$), `quartic_triangle` (upper bound on the sum of fourth powers of the parts), `class_split`	15-05 § 07	Stein interpolation, the one-sided quadratic remainder, the sup of the symbol	done
LatticeGauge.GhostDeterminantMatching	`det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks` ($\det H' > 0$, $\det K > 0$, the tree quantity is $\det H' / \det K$), `coeff_tree_eq_coeff_cov`, `coeff_drop_ghost_shift`, `coeff_ghost_constant` and others	15-08 § 05	The Faddeev–Popov lemma itself (change of variables in the integral)	done
LatticeGauge.BondMovingSandwich	`bond_moving_finite` ($w_i \geq 0$, $\sum w_i a_i = 0$), `iterate_le_of_le_of_monotone` (T monotone), `comparison_fails_at_fixed_point`, `shrink_shape_shift`, `shrink_shift_lower` ($0 < \alpha$, $q \geq 3$)	15-04 § 05	Lattice, group, measure, the partition-function inequality itself	done
LatticeGauge.InfluenceRecursion	`tp2_comp` (finite ordered set, TP2 assumption), `cdf_upper`, `cdf_lower`, `k_upper`, `step_bound`, `gauss_product`	How the core Poincaré uniformity assumption of Part 23 is supported (23-12; the text is in Part 23, 15-00 § 04)	The derivative formula of the lemma, Hoeffding's formula, the integral version of TP2, the floating-point computation itself	done
LatticeGauge.CornerCouplingAffineFlow	`fixpt_is_fixed`, `iterate_dev` ($\lambda \neq 1$), `abs_iterate_dev` ($0 \leq \lambda < 1$), `ratio_gt_one_below_fixpt` ($0 \leq \lambda < 1$, $0 < r < s/(1-\lambda)$), `one_step_ratio` ($3/160$)	15-07 § 07, 15-03 § 02	Lattice, group, integration, minimisation (real linear maps only)	done
LatticeGauge.HypercubeFaceCoupling	`sigma_is_relation`, `relation_is_multiple_of_sigma` (24 identities per face), `opposite_in_cell_reduces`, `diag_adj_not_enough`, `diag_adj_skew_unique` (assuming the componentwise values and eigenvalues obtained by computation)	15-07 § 07	Lattice, group, integration (rational linear expressions only)	done
LatticeGauge.SkewCouplingSpectrum	`coeffs_of_spectrum`, `skew_zero_iff` ($e_3 = b + 4a - 4k$, $e_6 = b$, $e_8 = b + 2a + 2k$), `one_step_skew`, `increment_pow`, `tail_bound` ($0 \leq q < 1$, ratio of increments $\leq q$)	15-07 § 07	Lattice, group, minimisation; that the measured sequences satisfy the assumptions	done
LatticeGauge.HypercubeCochainCount	`faceCount_four`, `euler_cube` (no assumptions), `hypercube_cochain_ranks`, `edge_face_dof_agree` (exactness of the augmented cochain complex), `torus_site_euler` ($n \geq 1$), `torus_site_dof_agree`	15-03 § 02, 15-07 § 07	Exactness itself, Bundle Network Theorem (iii), minimisation (natural-number and integer arithmetic only)	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGaugeNonlocalImageRecursion	`affine_iter` ($\lambda \neq 1$), `fixed_point`, `affine_bound_sum` ($0 \le \lambda$), `affine_bound_uniform` ($0 \le \lambda < 1$, $0 \le C$), `quartic_block_charpoly`, `relative_rates`, `linear_recursion_fixed`, `new_term_gamma` and others (12 theorems)	15-08 § 06, 15-03 § 03	Derivation of the matrix of the image, measure, integration, group, second order and above in W	done
LatticeGaugeThreeBodyTailRecursion	`weighted_geom_le` ($0 \le \rho < 1$), `three_body_sum_le` ($0 \le C$, $0 \le \rho < 1$, $\lvert q(d)\rvert \le C\rho^{d}$), `affine_uniform`, `pair_tail_uniform` ($0 \le \lambda, \mu < 1$, two recursive inequalities), `tripleCount_small`, `monomial_counts` (finite counting)	15-08 § 06	That the coefficients satisfy the assumptions (computation of $d \le 7$), group, measure, four dimensions	done
CoarseFaceHodgeSplit	`range_sup_tags`, `finrank_ker_eq` ($d_2 \circ D_c = 0$), `disjoint_range_tags` (no assumptions), `decomp_unique`, `norm_sq_add_harmonic` ($D^{\dagger} b = 0$)	15-08 § 06, 15-05 § 07	That there are 3 tags on a concrete lattice, one loop, tail types	done
LatticeGauge.QuarticConeNorm	`ratio_sandwich` ($0 < b_{\min} \le b \le b_{\max}$), `ledger` ($b_{\min} \sum \theta^2 \le \sum \phi^2 \le b_{\max} \sum \theta^2$), `transfer`, `cone_b_to_a`, `stein_floor` ($A_i^2 \le F v_i$, $0 \le \nu_i \le A_i$)	15-05 § 07	The Mixing-Limit Lemma, the type of the long-wavelength minimiser, Stein interpolation	done
LatticeGauge.BlockCovariance	`cone_chain`, `cone_chain_uniform` ($M_k \le m$), `growth_ledger`, `inverse_decay` ($r\, g_k \le g_{k+1}$), `face_sum_sq`, `four_split_quartic_floor`, `even_split_quartic`	15-05 § 07	The Conditional-Mean Covariance Lemma, iteration of the symbol, divergence of the perimeter length, the numbers for μ	done
LatticeGauge.AveragedBlockFlow	`affine_iter`, `affine_dist` ($r \neq 1$), `tail_sum`, `tail_bound` ($0 \le \delta$, $0 \le r < 1$), `axis_flow_closed`, `axis_flow_values`, `cone_series` ($0 \le c < 1$, $x_{k+1} \le c x_k$)	15-05 § 07	Lattice, Fourier, minimisation; that the lattice flow follows the closed form, $\sup \mu_j \le c < 1$	done
LatticeGaugeHierarchicalUpperTube	`tube_exit_top` (within the tube, either one step lowers by $\delta > 0$ or it exits), `chain_top`, `steps_numeric_128`, `steps_numeric_64`, `mk_full_chain_128` ($b_{\mathrm{lo}} = 0.95$, $b \le 128$, $\delta = 0.2196$, cells on the band side ≤ 556), `union_tube_step`	15-04 § 05	The map T, the tube, interval arithmetic (taken as assumptions)	done
LatticeGauge.TensionDomination	`iterate_bound_var`, `iterate_bound` ($c \ge 0$), `lower_bound_four`, `area_law_transfer` (one-step inequality assumed), `orbit_lower` (no monotonicity of the map), `step_fails_of_zero_to_positive`	15-04 § 05	Lattice, tension, the MK map (induction on real sequences only)	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGauge.BlockStepBetaShift	`beta_shift_of_inv_sq` ($1/g'^2 = 1/g^2 - 2b_0\ell$), `su2_coeff`, `log_steps`, `shift_bounds`, `tadpole_weights`, `vanvleck_ratio`	15-01 § 01, 15-03 § 02	Lattice, blocking, Hessian, measure (arithmetic only)	done
LatticeGaugeTwoBlockRowSumLoc	`sRow_eq_sComp_add`, `nRow_eq_nComp`, `row_sum_cancels` (F symmetric), `row_split_unique` (the non-local part is 0 on the diagonal), `comp_split_unique`, `quartic_decomp_unique`, `two_loop_numbers` and others (14 theorems)	15-08 § 06	Integration, derivation of closed forms, expansion, group	done
LatticeGaugeMultiStepCone	`step_pair`, `contraction`, `cone_quartic` (two-step recursion $x_j \le a\,x_{j+1} + b\,x_{j+2}$, $a, b \ge 0$, $a + b \le 1$), `block_three` (iteration of three-step blocks, $0 \le \mu$), `cone_transfer` (one-step transfer of the cone, $0 \le c$) (5 theorems)	15-08, row of the boundary of gauge re-fixing	The Stein values (numbers), the kernel itself, the tree-level and commuting assumptions	done
LatticeGaugeMultiStepPerturb	`three_product_perturb`, `three_product_perturb_grouped` (perturbation of a product of three in a normed ring), `block_three_affine` ($c(k+3) \le \mu\, c(k) + \varepsilon$, $0 \le \mu < 1$, $0 \le \varepsilon \Rightarrow c(3n) \le \mu^n\, c(0) + \varepsilon/(1-\mu)$) (3 theorems)	15-08, row of the boundary of gauge re-fixing	The numerical differences between the kernels of successive steps; the extrapolation from step 4 on	done
LatticeGaugeQuarticCone	`four_amgm` ($\lvert abcd\rvert \le (a^4+b^4+c^4+d^4)/4$), `quartic_sum_le` (finite sums) (2 theorems)	15-08 § 06	Finiteness of the ℓ^1 norm of the coefficients at each point is assumed; the actual coefficients of the many-body terms	done
LatticeGaugeColorAverage	`colorAverage` (finite P, X and $K : P \to X \to \mathbb{R}$; under the assumption (H) $\forall\,\psi$ $\Sigma_p(\Sigma_x K\,\psi)^4 \le R\,\Sigma\,\psi^4$, for three-colour θ: $\Sigma_p\lvert(K\theta)_p\rvert^4 \le R\,\Sigma_x\lvert\theta_x\rvert^4$), `colorAverage_of_design` (for any design of non-negative weights satisfying the identity), `design_identity` ($\pm e_k$ with weight 1/15 and $(\pm1,\pm1,\pm1)/\sqrt{3}$ with weight 3/40 give $\Sigma\, w_i(s_i\cdot v)^4 = \lvert v\rvert^4/5$) and others (10 theorems)	15-08, row of the boundary of gauge re-fixing	That K acts linearly with the same scalar coefficients on the three colours (tree-level, commuting linearisation) is assumed; the existence of the one-colour sup is put into (H); the $\ge$ direction (a one-colour field is a three-colour field) is not written	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LatticeGaugeCubicCone	`three_amgm` ($\lvert abc\rvert \le (\lvert a\rvert^3+\lvert b\rvert^3+\lvert c\rvert^3)/3$), `cubic_sum_le` (finite sums), `triple_sq_le`, `triple_amgm` (three-colour triple product $\lvert u\cdot(v\times w)\rvert \le (\lvert u\rvert^3+\lvert v\rvert^3+\lvert w\rvert^3)/3$), `cubic_cone_pullback` (from the cone $\lvert C\phi\rvert \le c\,\lVert\phi\rVert_3^3$ and the contraction $\lVert K\theta\rVert_3^3 \le \mu\,\lVert\theta\rVert_3^3$, $\lvert C(K\theta)\rvert \le c\mu\,\lVert\theta\rVert_3^3$), `block_three_affine`, `block_three_uniform` (7 theorems)	15-08 § 06, core row	The values of Stein, κ, λ (numbers), the kernel itself, the tree-level and commuting assumptions	done
SmallSetRowSum	`rowSum_total`, `row_cancel`, `total_split`, `owner_split`, `symmetric_split` (distribution of the small-set terms and row sums), `loc_idem`, `rem_vanish`, `piece_loc_zero` (decomposition of Loc; the (1−Loc) side vanishes for a uniform field), `theta_ledger`, `plate_numbers` and others (20 theorems)	15-08 § 06	Numerical values, the kernel itself, the tree-level and commuting assumptions; the plate is two-dimensional and one step	done
ThetaSquareOneStepAffine	`one_step_affine` (affine recursion for the contraction $\mu_k \le \bar{\mu}$ per step), `theta_ledger_mu`, `theta_ledger_half` (the ½ of the slab), `theta_ledger_quarter`, `theta_ledger_measured`, `sup_monotone_in_mu` (6 theorems)	15-08 § 06	The contraction values themselves (the ½ of the slab is computed), the kernel, the tree-level and commuting assumptions	done
DominantMatrixContraction	`dominant_mul`, `dominant_pow` (the norm of the identity is 1), `weighted_step`, `weighted_iter` ($Bw \le \lambda w$), `weight_change` ($av \le w \le bv$, $a > 0$), `liftWeight_step` ($B \ge 0$, $w \ge 0$, $s > 0$, $k \ge 1$, $B^k w \le s^k w$), `norm_pow_le_const_mul`, `sum_norm_pow_le`, `affine_uniform` ($0 \le \lambda < 1$, $\varepsilon \ge 0$), `aggregate_step`, `two_by_two_weight`, `two_by_two_exists`, `signedEx_sq`, `absEx_pow` (21 theorems)	15-10 § 01	The Collatz–Wielandt equality and Gelfand's formula themselves, the existence of the Perron vector, the ℓ^2 weights (Schur's test), the max-plus sandwich	done
SignLossBound	`jensen_four`, `schur_four` (row $\ell^1 \le A$, column $\ell^1 \le B$), `coherent_of_neg_le` (negative part $\le \delta$ times the positive), `loss_le_of_coherent`, `comp_coherent`, `coherence_le_one`, `sum_four_le` ($\lvert v_a\rvert \le M$), `stein_form_le`, `stein_le_inv_sq_mul_test` ($\kappa A_r M \le \lvert y_r\rvert$), `abs_comp_le`, `row_l1_comp_le` (16 theorems)	15-10 § 02	Stein's formula itself (Plancherel and interpolation), the phase average of plane waves, Wielandt's theorem, the convergence to rank 1, the numerical values of the loss (computed)	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
ConstrainedExtension	`exists_extension`, `isExtension_iff_mem_coset`, `finrank_freedom` (finite dimension), `freedom_eq_zero_iff` ($W \neq 0$), `exists_extension_with_conditions` ((res_V, C) surjective), `finrank_freedom_with_conditions`, `conditions_count_le`, `eq_of_eqOn_iSup` (the family of subspaces spans), `comp_isExtension` ($f 1(V_1) \subseteq V_2$), `comp_depends_of_not_le` (20 theorems)	15-10 § 03	The lattice, the continuity of the symbol, the order of folding, moments, the candidate lattice steps (continuity of the uniqueness of the zero mode, the rule for the order of the tail, shifted jets)	done
ConeCalculus	`cone_pullback` ($c \geq 0$), `contracts_comp`, `contracts_iterate`, `block_decay` ($M \geq 0$, $\mu \geq 0$), `lp_schur` ($p \geq 1$, row $\ell^1 \leq R$, column $\ell^1 \leq C$), `lp_schur_sharp` (constant row and column sums), `delta_dilution`, `weighted_rpow_midpoint` ($w \geq 0$, $a > 0$), `weighted_rpow_shift` ($0 < a \leq R$, $p 0 \leq p$), `midpoint_below_one`, `affine_varying` ($0 \leq \mu_k \leq \bar{\mu} < 1$, $\varepsilon_k \leq \bar{\varepsilon}$, $c_k \geq 0$), `affine_eventually`, `affine_block` (20 theorems)	15-10 § 05	The operator norm itself, the formula for Stein's bound (interpolation and Plancherel), Riesz–Thorin, convexity in continuous p, Gelfand's formula, the lattice and the kernel	done
ColorAverageN	`colorAverage_of_design` (under `IsQuarticDesign w s κ`, the one-colour constant R is an upper bound for n colours), `weights_sum`, `pt_unit`, `design_identity` (with weight $1/(n(n+2))$ on the axis points $\pm e_k$ and weight $n/((n+2)2^n)$ on the cube vertices $\varepsilon/\sqrt{n}$, $\sum_i w_i (s_i \cdot v)^4 = 3|v|^4/(n(n+2))$, for all $n \geq 1$), `colorAverage` (the n-colour version without the design assumption, $\kappa = n(n+2)/3$), `oneColour_of_colorAverage` (the converse), `design_identity_two` ($n = 2$, $\kappa = 8/3$), `cube_moment`, and others (13 theorems)	The row of the boundary of gauge re-fixing in 15-08, 16-07 § 07	The lemma of spherical averages, the application to $\mathfrak{g} \cong \mathbb{R}^n$ and to $\mathfrak{su}(N)$, the comparison with the literature	done
OneStepWindow	`holder_dual_ratio`, `norm_conj_rpow`, `exists_lpRatio_ge_row` (the one-coordinate Hölder lower bound, $p > 1$, including $k = 0$), `lamMax_mono`, `lamMax_smul`, `steinWeighted_ge`, `stein_row_weight_saturation` (the family of row weights saturates at $u = A^{p-2}$; $\lambda_{\max}$ is defined as the supremum of the Rayleigh quotient), `window_ordConnected`, `window_ordConnected_of_logConvex` (the window is an interval; convexity of $\log \mu_p$ is assumed), and others (20 theorems)	15-10 § 05	Courant–Fischer, the number $p_{\mathrm{row}} = 5.874$ and the interval $(3.04, 5.67)$ (computed), the operator norm itself	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
LargeSetWeight	`summable_shell_iff` (if the number of shells is polynomial, $\sum (n+1)P(n)A^{n+1}e^{-\kappa n}$ converges $\Leftrightarrow 0 \le A < e^{\kappa}$), `not_summable_term_poly`, `bbs_weight_vs_decay` (the form $(\eta-1)\kappa > L^d \log 2$), `bsv_weight_vs_decay`, `bsv_weight_vs_decay_log2` (independent of a), `card_shell` (the number of ℓ^∞ shells $64n^3+16n$, $n \ge 1$), `manhattanShell_one`/two/three (8, 32, 88), and others (16 theorems)	15-08 § 06	The reduction of the constant c_R of the remainder to chains, the general formula for Manhattan shells, the value of the rate κ (computed)	done
LatticeGaugeAcceptanceSandwich	`fermi_le_metro`, `half_metro_le_fermi` ($\frac{1}{2}\min(1,e^{-x}) \le 1/(1+e^{x}) \le \min(1,e^{-x})$), `barker_le`, `barker_le_of_crooks` (the Barker-type mean $\le 1/(1+e^{\Delta F}) = (1-\theta)/2$), `overlap_sandwich` ($e^{-\Delta F} O \le a_F \le O$), `mixture_sandwich` ($(1-\theta)O \le a_{\mathrm{mix}} \le 1-\theta$), `jarzynski_of_crooks`, `half_le_one_sub_sqrt`, and others (15 theorems; the assumption is the Crooks-type $P_F e^{-W} = e^{-\Delta F} P_R$)	25-06 § 07, 25-08 § 03	Bretagnolle–Huber, Pinsker, the driving procedure itself, the numbers	done
LatticeGaugeAbelianTwistAction	`abelianAction_le`, `abelianAction_lt_pi_sq` (for all $N \ge 1$, $2N^4(1-\cos(\pi/N^2)) \le \pi^2 - \pi^4/(12N^4) + \pi^6/(360N^8) < \pi^2$, no assumptions), `totalAction_eq` (for abelian configurations on $(\mathbb{Z}/N)^4$, the sum over the six planes $= 2N^4(1-\cos(\pi/N^2))$), `plaqTerm_mixed`, and others (13 theorems)	27-02 base point at small T^2	The minimal action of sector (1,1) $\le S_{ab}$ (the configuration space of $SU(2)$ is not included); the term $1 - z\cos$ and the placement of the twist are put as definitions	done
LatticeGaugeNestedBlocks	`nested_supermultiplicative` (finite probability space, large block = fibre of s^2 small blocks, the inequality of means $\mathbb{E}\prod\theta' \le \mathbb{E}\prod\theta$ assumed), `doubling_iterate`, `doubling_bound` (the doubling with $K = 4C$, $t_k \le K^{-1/3}(K^{1/3}t_0)^{4^k}$), and others (7 theorems)	26-05 § 02, 26-02 § 03	The Markov property, reflection positivity, (C_{cb})	done
CenterFluxPolymerCertificate	`tree_k` (six tree inequalities), `kp_bound`, `kp_lt_one`, `rate_rational` ($\beta = 1029/10000$), `exp_le_inv_one_sub` ($0 \le y < 1 \Rightarrow e^{y} \le 1/(1-y)$), `final_real` (assuming $\tanh\beta \le \beta$) (6 theorems)	26-06 § 04	$\tanh\beta \le \beta$, the lemma of the induction on trees, the Kotecký–Preiss theorem, the full enumeration up to 10 plaquettes, the interval-arithmetic version ($\beta \le 0.1039$)	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
ClassFixedPoint	`dist_linear_add_le` ($\|T\| \le \rho$, $\mathrm{Lip}\, R \le L \Rightarrow \mathrm{Lip}(T+R) \le \rho + L$), `exists_unique_fixed` ($\rho + L < 1$), `norm_fixed_le`, `exists_unique_fixed_in_ball` (Lipschitz only on the closed ball, $C + (\rho + L) r \le r$), `norm_iterate_sub_le` ($\mathrm{Lip}(\Phi^{[n]} - T^{[n]}) \le (\tau + L)^n - \tau^n$), `exists_unique_fixed_of_iterate` (contraction in n steps), `affine_orbit_bound`, `dist_fixed_fixed_le` (a deviation δ of the map gives a deviation $\le \delta/(1-K)$ of the fixed point), `tracking` (following a flowing parameter), `pull_orbit`, `exists_unique_entire_orbit` (for contractions that change step by step, exactly one two-sided infinite orbit staying in a closed set), `entire_tracking`, and others (24 theorems)	15-03 § 04	The contents of the table of assumptions (measuring the contraction and the remainder in one and the same norm, the values of the nonlinear coefficient a and the source c_0, the contraction after moving F^2 into β, the correctness of the A-type Loc); the numbers of the window (computed)	done
LocalCancellationOneRow	`pairs_sum_to_total`, `total_uniform`, `pairAA_uniform`, `pairAB_uniform`, `loc_remainder0_eq_zero`, `pairAB_rank_one` (the pairs across blocks have rank one), `pairAB_local_plus_gradient`, `mirror` (the rational arithmetic of the coefficients of the $O(1/\beta)$ quadratic term for a row of seven with two blocks, no assumptions) (8 theorems)	15-08 § 06	The derivation of the rational coefficients (the heat-kernel computation of the row); the decay ratio $5 + 2\sqrt{6}$; the plate and four dimensions	done
RankOneRemainder	`pair_split` ($K a b = \frac{K}{2}(a^2 + b^2) - \frac{K}{2}(a - b)^2$), `edge_sum_split` (pairs over any finite set), `chain_split`, `chain_local_regroup`, `chain_remainder_eq` (on a row, the sum across blocks minus the local part leaves $= -\frac{1}{2}\sum_i K_i (a_i - b_{i+1})^2$), `chain_remainder_zero_of_match` ($a_i = b_{i+1}$), `chain_remainder_nonpos` ($K_i \ge 0$), `deflation_split`, `deflation_left_zero`, `deflation_right_zero` (the value of the uniform field $\neq 0$), and others (12 theorems)	15-08 § 06	That the leading products number three on the plate; the size of what remains (computed); four dimensions	done
BlockDeflation	`blockDeflate_split` ($C = CB(A^{\mathrm{T}}CB)^{-1}A^{\mathrm{T}}C + R$), `blockDeflate_left_zero`, `blockDeflate_right_zero` (if $A^{\mathrm{T}}CB$ is invertible then $A^{\mathrm{T}}R = 0$ and $RB = 0$), `left_zero_inherit`, `right_zero_inherit`, `twoStep_zero`, `twoStep_split` ($A_2^{\mathrm{T}} R_1 B_2$ of the second step also invertible), `span_left_zero`, `span_right_zero` (the space spanned by the test fields) (9 theorems)	15-08 § 06	The size of the coefficients (computed); the correspondence with the order of Taylor; four dimensions	done

Bundle name	Main theorems and assumptions	Place in the text	Not included	Re-verification
ConeCalculusSOS	`cert_sound` (a certificate gives $P \leq \lambda N$), `cert_mono`, `certValue_anti` (the larger the family, the smaller the value), `ratio_le_certValue`, `floor_of_coeff` (a floor from an equality of coefficients), `classSum_pushforward` (pushing down, no conditions), `classSum_pullback` (lifting when the map of classes is injective), `shift_rigid` (injective if $2w < L$ on each axis), `collide_L3`, `collide_L4`, `no_translate_Z` (counterexamples), `tensor_contracts`, `tensor_ratio_product`, `row_holder_sos`, `schur_weighted_bound`, `schur_attained`, and others (25 theorems, 2 lemmas)	15-10 § 05 (the paragraph of the second chapter is not yet in the text)	SOS inside the polynomial ring (the identities are put as assumptions, as equalities of functions or of coefficients); the comparison with the signed Stein upper bound; $\mu_4(K_0) < 1$ on the whole space; the necessity of $L > 2w$; the numbers (computed)	done
LatticeGaugeH1HbarInterval	`cell_lower`, `cell_upper` (for $0 < \hbar_1 \leq \hbar \leq \hbar_2$, $\min(U/\hbar_1, U/\hbar_2) \leq U/\hbar \leq \max(U/\hbar_1, U/\hbar_2)$), `sum_lower`, `sum_upper` (forward sums), `rho_lower` ($0 < B \leq \bar{B}$), `pieces_min` (the covering by small pieces), `margin_on_interval` (the margin on an interval), `threshold`, `margin_I1`, `margin_I2`–`margin_I6` (the margins ≥ 0.067 with the numbers output by interval arithmetic put in), `margin_I1_coarse_fails` (a negative control) (15 theorems)	23-15 § 05, 27-02 uniform gap of the double well	Interval arithmetic itself (the outputs in float64 with outward widths are assumed); Muckenhoupt; FKG; the remainder of the TAU interpolation; Lemma Tail; the numbers of the subintervals from the third on (computer-assisted); the quantitative uniform Laplace method	done
StrongCouplingOneSidedTerm	`four_mul_le_sq` ($L \geq 4 \Rightarrow 3L + L \leq L^2$), `area_le_strip` ($0 \leq u \leq 1 \Rightarrow u^{L^2} \leq u^{3L} u^L$), `strip_term_eq`, `strip_coeff_lower` ($0 < s \leq \pi/(2L^2) \Rightarrow 16L^2/\pi \leq 8/s$), `tube_coeff_neg` ($L \geq 4$, $s > x - x^3/4$, $0 < \hat{D} \leq 3$), `tube_part_nonpos`, `h_main_term` ($\lvert R\rvert \leq Cu^{3L+1}$), `h_neg` ($Cu < K$), `h_neg_on_interval` ($u \leq u_0$, $Cu_0 < K$) (9 theorems)	27-02 band in the language of the coupling	Convergence of the expansion (Osterwalder–Seiler; the upper bound on the remainder is assumed); $\sin x > x - x^3/4$; $\hat{D} \leq 3$; the coefficient of the leading term of $\ln Z$ (paper); the values of C_L and u_0; a β_0 uniform in L	done

06

Connection to Part 27 and the glossary

The row for ultraviolet control in the ledger of conditions (27-02) is split into smaller pieces by Table 1 of this part (questions) and 15-08 Table 1 (the four rows that do not close). Its label and state (Assumption, open) do not change. The cross index of questions (27-03) cites the ultraviolet questions of this part as a fifth origin. The

words added in this part (the invariant curve, the transverse contraction, the source, the bundle network, couplings by loop shape, the attracting class, the shift of local couplings) are in the A-02 Glossary.

What this chapter can say

Can say	Cannot say
Index: questions, dependencies, summary of the five grades, correspondence of the Lean bundles (this chapter adds no new results)	Ultraviolet control. The settlement of any of the questions
That the Lean bundles depend only on the three standard axioms (printed for each bundle, and re-checked)	Formalisation of the lattice, the group, the measure and the minimisation; the analytic parts of the paper proofs

Sources and reproduction

Item	Kind	Source, tool
What cannot be said, questions, five grades	This book's study	The tables and sections of each chapter 15-00–15-08
The Lean bundle	This book's study	The record of the formalisation of this book's study (Lean 4, with the axioms printed at the end of each bundle). Re-verified (every bundle passed)

Words used in this chapter — what cannot be said, ultraviolet questions, the dependency diagram, the five grades, Lean bundles. Their meanings are in the A-02 Glossary.

Next chapter to read: 15-10 Systematic attempts — naming the recurring patterns of the record. To read alongside: 27-00 Entrance — the promise of this part (the whole conditional statement, including the row for ultraviolet control in the ledger of conditions).

Systematic attempts — giving names to the recurring patterns of the record

This chapter is this book's study. It is not a component of the proof on the main line of argument. § 01– § 03 and § 05 are restatements in known frameworks, and § 04 is at the level of an estimate.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

What this chapter shows — We give names and definitions to five pieces of handwork that recur in the computational record of Part 15. The first is the estimate of a matrix that "exceeds 1 in one step but contracts when several steps are taken together" (dominant matrices); the second is what is lost when signs are discarded to take an upper bound (the loss from discarding signs); the third is how to choose when a map determined only on a constraint is represented by a local kernel (local representatives); the fourth is the matching of the two expansions of strong and weak coupling (double valuation); the fifth is the computation that bounds a polynomial of degree p by an ℓ^p cone and pulls it back step by step (the cone calculus). For each, we write candidate theorems, show row by row which known result each restates, set out which quantity the numbers of the record correspond to, and gather the remaining questions as the questions of the systematic attempts.

Reading level — Third year and above. § 01 uses the linear algebra of non-negative matrices, § 02 inequalities for finite sums, § 03 Fourier series on the lattice, § 04 only the leading terms of asymptotic series, and § 05 inequalities for finite sums and the statements of interpolation theorems. § 00 and the opening paragraph of each section alone can be read by second-year students.

Prerequisite chapters — 15-05 Bundle Network Theorem (local form) (local representatives and an upper bound uniform in the volume), One-Step Contraction of the ℓ^4 Cone in 15-05 (the weighted Stein bound and the ℓ^4 cone), 15-08 § 06 (the W coordinates and the multi-step cone).

Said first — This chapter is not the main line of argument. The results of this chapter have no direct effect either on the row for ultraviolet control or on the row for the crossover band in the ledger of conditions (27-02). Many of the candidate theorems are restatements of known facts — the Perron–Frobenius theorem, the Collatz–Wielandt formula, Gelfand's formula, contraction in a weighted maximum norm, the order of multigrid transfer operators, the sum rules of subdivision, the Riesz–Thorin interpolation theorem, the Schur test — and no novelty is claimed. What is new in this chapter lies on the side of the "questions". The aim is to write out which quantity of a known framework each number of the record corresponds to, and where it goes beyond the known framework. Labels are attached row by row: Lean (a statement that passed in a Lean file), Computed (a floating-point computation, not an interval estimate), Reading (only a line of argument on paper, or at the level of the abstract of a source), Cited (a standard known fact), Assumption (an estimate).

Order of this chapter

0. What this chapter shows — the five patterns and how to read this chapter
1. Dominant matrices and k-step contraction — how to build a weight that contracts in one step (Table 1)
2. The loss from discarding signs — the loss is the inverse square of the alignment (Tables 2–5)
3. The geometry of local representatives — the order of folding does not depend on the representative (Tables 6 and 7)
4. The geometry of the double valuation — a scale-independent κ and the lens (Tables 8–10)
5. The cone calculus — the window is an interval; the correct form of dilution (Tables 11 and 12)
6. Dictionary with the known, what can be said, questions (Tables 13–15)

00

What this chapter shows

Dominant matrix	The loss from discarding signs	Local representative	Double valuation	The cone calculus
If it contracts in k steps, a weight that contracts in one step can be built explicitly "Exceeding 1 in one step" is a transient that depends on the choice of weight. A restatement of the line of Collatz–Wielandt and Varga (Lean)	The loss is bounded by the inverse square of the row alignment in the worst direction Agrees at 5.55, 5.13, 5.31 for two to four steps on the 4^4 torus (the inequality in Lean, the numbers computed)	The order of folding $\mathrm{ord}^V(F) = 2$ does not depend on the choice of representative Within the types of the record, the tail norm is bounded only up to $D \le 4$. A restatement of the sum rules (Reading, Computed)	The product of the two leading coefficients $\kappa = s1w1$ does not depend on the scale of the coupling The lens of the leading terms opens only when $\kappa \le \frac{1}{4}$. No payoff theorem yet (Assumption, Computed)	Gives definitions for the pull-back of degree-p cones, the weighted Stein bound, the composition of steps and affine recursions $\log S_p$ is convex in p and $S_3 \le \sqrt{S_2 S_4}$. The naive dilution m^{1-p} does not hold, and $S_p \le S_2 R^{p-2}$ is the minimal form (Lean, Reading, Computed)

How to read this chapter. The record of computations in Part 15 contains the same kind of handwork again and again. There are five kinds: measuring the norm of a three-step composite and writing "it exceeds 1 in one step but contracts in three"; taking the absolute value of a signed kernel to build an upper bound; applying a local kernel to a map that is determined only on fields satisfying the Bianchi identity; setting the leading terms of the strong-coupling and weak-coupling series side by side to look for the point where they meet; and pulling a degree-p cone back step by step and bounding it by the weighted Stein bound. This chapter is an attempt to give each of them a definition and axioms (what can be said under what assumptions). Many of the results are restatements of known facts, and we say so row by row for the candidate theorems of each section. What remains after the restatement — quantities that do not correspond directly to a known framework, and regularities shown by the numbers of the record that have not been made into theorems — are the questions of this chapter.

§ 01– § 03 can be read independently of one another. § 04 shares no tools with § 01– § 03; it is an attempt to restate the band of the renormalisation comparison. § 05 extends Candidate 1.4 of § 01 to the language of cones and can be read alongside § 01.

01

Dominant matrices and k-step contraction

The non-negative matrix that lists the norms of the blocks of an operator split into blocks transfers estimates of powers of the operator to estimates of powers of a non-negative matrix. On the side of non-negative matrices the Perron–Frobenius theory applies as it stands, so "in how many steps does it contract" and "with which weight must we measure for it to contract in one step" can be written in the language of matrices. Everything up to this point is known; the role of this section is to bring the estimates of the record into line with this framework.

Definitions

Definition 1.1 (dominant matrix). Let I be a finite set, E_i ($i \in I$) normed spaces, and T an operator on $E = \bigoplus_i E_i$ split into blocks $T_{ij} : E_j \to E_i$. A non-negative matrix $B \in \mathbb{R}_{\geq 0}^{I \times I}$ is a **dominant matrix** of T if for all i, j we have $\|T_{ij}\| \leq B_{ij}$. The smallest one is the matrix of block norms $N(T)_{ij} = \|T_{ij}\|$.

- Products and powers: $N(ST) \leq N(S)N(T)$ (entrywise). Hence if B, C are dominant matrices of S, T, then BC is a dominant matrix of ST and B^n is a dominant matrix of T^n Lean (`norm_mul_apply_le`, `dominant_mul`, `dominant_pow`; the power version assumes that the identity has norm 1).
- The class is also closed under sums and non-negative scalar multiples Reading .
- The max-plus side: for the maximal cycle mean $\lambda_{\mathrm{mp}}(A)$ of $A = \log B$ (entrywise), $e^{\lambda_{\mathrm{mp}}} \leq \rho(B) \leq |I|\, e^{\lambda_{\mathrm{mp}}}$ Cited (Friedland 1986). In the record, the max-plus eigenvalue bounded the optimal weighted norm from below to within 1–2.5% Computed .

Definition 1.2 (weighted block norm). For a positive weight $w \in \mathbb{R}_{>0}^{I}$, $\|x\|_{w,\infty} = \max_i \|x_i\|/w_i$, and the ℓ^2 version is $\|x\|_{w,2}^2 = \sum_i \|x_i\|^2/w_i^2$. The operator norm is $\|T\|_{w,\infty} = \|W^{-1}TW\|_{\infty}$ ($W = \mathrm{diag}(w)$).

Definition 1.3 (k-step contraction, transient constant). For a non-negative matrix B put $\kappa_k(B) := \|B^k\|$; when $\kappa_k < 1$ for some k we say that it "contracts in k steps". The transient constant is $C_k(B) := \max_{r<k} \|B^r\|$. By Gelfand's formula $\rho(B) = \lim_n \|B^n\|^{1/n}$, $\rho(B) < 1$ is equivalent to $\kappa_k < 1$ for some k Cited .

Definition 1.4 (one-step weight). For $B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$ we call

$$w' := \sum_{r<k} s^{-r} B^r w$$

the **one-step weight** built from w.

Candidate theorems and labels

Candidate 1.1 (from k-step contraction to a weight that contracts in one step). If $B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$ and $B^k w \leq s^k w$ (entrywise), then $w \leq w'$ and $Bw' \leq s\, w'$ Lean (`liftWeight_step`; lemmas `mulVec_pow_succ`, `pow_mulVec_nonneg`).

$$Bw' = \sum_{r<k} s^{-r} B^{r+1} w = s\big(w' - w + s^{-k} B^k w\big) \;\le\; s\,w'$$

Combined with Candidate 1.2 (the one-step estimate), this says that if "the k-step composite contracts by a factor s^k in the weight w", then "measured in the weight w' it contracts by a factor s in every step". The weight ratio w'/w lies in $[1,\ \sum_{r<k} s^{-r}\max_i (B^r w)_i/w_i]$, so the constant of re-measurement does not depend on the number of steps (Candidate 1.3). The content is the constructive half of "$\rho(B) < s \Leftrightarrow$ there is $w' > 0$ with $Bw' \le sw'$", known from Varga's analysis of iterative methods and from the line of Berman–Plemmons on the characterisation of M-matrices Cited .

Table 1 The candidate theorems of § 01. Each is a finite-dimensional form of a known fact, or an elementary consequence of one

Candidate	Statement (one line)	Label	What it restates
1.1	$B^k w \le s^k w \Rightarrow w \le w'$, $Bw' \le sw'$	Lean	The constructive half of the Varga / Berman–Plemmons characterisation of M-matrices Cited
1.2	$Bw \le \lambda w$ $(w > 0) \Rightarrow \|T\|_{w,\infty} \le \lambda$. Even with T_n, B_n, λ_n differing from step to step, $\|x_n\|_{w,\infty} \le (\prod_k \lambda_k)\|x_0\|_{w,\infty}$	Lean (`weighted_step`, `weighted_iter`)	Collatz–Wielandt: $\inf_{w>0} \max_i (Bw)_i/w_i = \rho(B)$ (the equality itself is not in Lean) Cited
1.2′	Given right and left positive Perron vectors x, y, $D = \mathrm{diag}(\sqrt{y/x})$ gives $\|DBD^{-1}\|_2 \le \rho(B)$, and further $\|DTD^{-1}\|_2 \le \|DN(T)D^{-1}\|_2$	Reading	The Schur test applied to $q = \sqrt{xy}$ Cited
1.3	If $av \le w \le bv$ $(a > 0)$, a sequence that contracts by λ_n in each step in the weight w satisfies $\|x_n\|_v \le (b/a)(\prod \lambda_k)\|x_0\|_v$ in the weight v	Lean (`weight_change`)	An elementary fact about equivalence of norms
1.4 (a)	$\|1\| = 1$, $q := \|a^k\|$, $\|a^r\| \le C$ $(r < k) \Rightarrow \|a^n\| \le C\,q^{\lfloor n/k \rfloor}$	Lean (`norm_pow_block_le`, `norm_pow_le_const_mul`)	A finite version of Gelfand's formula Cited
1.4 (b)	$q < 1 \Rightarrow \sum_{n<N} \|a^n\| \le kC/(1-q)$ (independent of N)	Lean (`sum_norm_pow_le`; lemmas `geom_sum_le_inv`, `sum_range_mul_eq`)	An estimate of partial sums of the Neumann series
1.4 (c)	$c(k+1) \le \lambda c(k) + \varepsilon$ $(0 \le \lambda < 1, \varepsilon \ge 0) \Rightarrow c(k) \le \max(c(0), \varepsilon/(1-\lambda))$	Lean (`affine_uniform`)	An elementary fact about affine recursions
1.4 second half	If irreducible with a Perron vector $w > 0$, then $C = \max_i w_i / \min_i w_i$ (∞ norm). An explicit formula bounding the ratio of w by the entries of B	Reading (the explicit formula is not written out in full)	Minc's ratio estimate Reading
1.5	If the $J \times J$ matrix M obtained by aggregating blocks with $g : I \to J$ satisfies $Mp \le \mu p$, the original B contracts in one step by μ in the weight $w_j p_{g(j)}$. If J has two points, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a < 1$, $d < 1$ and $bc < (1-a)(1-d)$, then in the weight $(1, t)$ we have $\max(a + bt,\ d + c/t) < 1$	Lean (`aggregate_step`, `two_by_two_weight`, `two_by_two_exists`)	An elementary fact about the test for $\rho < 1$ of a two-by-two matrix Cited

Correspondence with the numbers of the record

- The matrix of the finite types of the W coordinates (quadratic including differences, dimension 6) had spectral radius 0.36, a one-step norm above 1, and 0.05–0.13 in three steps Computed . It is a typical case of $\rho < 1$ with a one-step norm above 1, and the C of Candidate 1.4 (a) corresponds to the

transient constant. Candidates 1.1 and 1.2 read this as: taking the Perron weight, it contracts in one step (in the record, 0.546 in one step with the Perron weight) Computed .

- The weighted Stein bound of the ℓ^4 cone (15-05) was 4.45 in one step and 0.10 in three steps Computed . Put in the form of Candidate 1.1 it would read "if it is 0.10 in three steps, a weight that contracts to $0.10^{1/3} \approx 0.46$ in one step can be built", but this requires the three-step estimate to come from **powers of a single linear dominant operator**. The cone is a quartic quantity obtained through Riesz–Thorin, and the kernel K_k differs from step to step, so it is not a power. What can be used in this case is only the form of a product over steps (`weighted_iter`) Reading .
- The pair of finite types and remaining returns has the form of the two-by-two aggregation of Candidate 1.5. Putting the remaining block in the lower right and checking $bc < (1-a)(1-d)$ has not yet been done (question "finite types and aggregation of returns").
- The Dobrushin column sum used in the coarse chain of the double well in Part 23 is Candidate 1.2 with the weight $w = \mathbf{1}$. Taking the Perron weight could improve the contraction constant Reading .

A question proper to this pattern — the loss from discarding signs

A dominant matrix discards signs. For the actual operator T and a dominant matrix B, define the loss at step k as

$$L_k := \frac{\|B^k\|_\bullet}{\|T^k\|_\bullet} \qquad (\text{same norm; } \|T^k\| = 0 \text{ then } +\infty)$$

Then the loss is not necessarily bounded in the number of steps. $T = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ has $|T| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as a dominant matrix, $T^2 = 0$, and every entry of $|T|^{n+1}$ is 2^n. Hence $L_n = +\infty$ for $n \geq 2$ Lean (`absEx_eq`, `signedEx_sq`, `absEx_pow`). Conversely, if T is non-negative ($T = |T|$) or sign-similar, $T = S|T|S$ (S a diagonal matrix of ± 1), then $L_k = 1$ in the ∞ norm with the Perron weight Reading .

How far the choice of signs can reach towards the estimate of $\rho(|A|)$ is treated by Rump's sign-real spectral radius (Rump 1997) Reading . We have not found a source that treats how the loss behaves **over steps** (not found, which does not mean there is none). In the record the loss grows only slowly, 7.6 in one step and 9.3 in two, and the conditions for making this a theorem are not determined by the information of the dominant matrix alone (the example above shows this). We take up this question in § 02.

02

The loss from discarding signs

A closer look at the loss in the record showed that it was of a different type from the counterexample of § 01. The Stein bound of the record takes the upper bound **after composing** the step kernels. The counterexample of § 01 is a quantity that composes **after taking absolute values**. In the former, the upper bound of the composite for the nilpotent example $T^2 = 0$ is also 0. In this section we see from the numbers that the loss with the upper bound taken after composition hardly moves over steps, and we write an inequality that bounds it by a geometric quantity within a single row (the row alignment).

Definitions

Definition 2.1 (composite kernel and loss). Write the composite of the step kernels $K_0, K_1, \ldots$ as $T_k = K_0 K_1 \cdots K_{k-1}$. Let the true ratio be $\sup_\theta \|T_k\theta\|_4^4 / \|\theta\|_4^4$, and the Stein bound the weighted ℓ^2 quantity of

the form of (11) of 15-05 (weighted by the ℓ^1 norms A_r of the rows of the composite kernel); we put

$$\mathrm{loss}_k := \frac{\mathrm{Stein}(T_k)}{\text{true ratio}(T_k)}$$

The true ratio is replaced by a lower bound from maximisation (the larger of a quasi-Newton method and a maximisation over period-2 fields), so the loss in the tables is an estimate of the loss from above.

Definition 2.2 (row alignment). The Stein bound of a translation-invariant T has the form $\sup_q \lambda_{\max}\big(\sum_r A_r^2 \hat{T}_r(q)^{\mathrm{H}} \hat{T}_r(q)\big)$; let (q^*, u^*) be its maximiser. Put $y_r = \hat{T}_r(q^*)u^*$, $M = \|u^*\|_\infty$, and call

$$\kappa_r := \frac{|y_r|}{A_r\, M} \;\le\; 1$$

the **alignment** of row r in the worst direction ($\kappa_r \le 1$ is `coherence_le_one` Lean). $\langle \kappa^{-2} \rangle$ is the average weighted by $|y_r|^4$.

The numbers of the record — the loss stops, and falls a little

Table 2 The loss on a common 4^4 torus (periodic operator with the steps cut at $R = 3$) Computed

k	Stein	Lower bound of the true ratio (quasi-Newton / period 2)	Loss	Step ratio: lower bound	Step ratio: Stein
1	3.708	0.5903 / 0.5566	**6.28**	–	–
2	0.7746	0.1405 / 0.1405	**5.51**	0.238	0.209
3	0.07632	0.014967 / 0.01497	**5.10**	0.107	0.0985
4	0.00576	0.001095 / 0.00109	**5.26**	0.073	0.0755

The true ratio and the Stein bound decrease at the same rate (both step ratios tend to about 0.07), so the loss stops. The shape is the same on larger tori (Table 3). The earlier recorded two-step value 9.3 was due to a weak lower bound of the true ratio, 0.108; with period-2 fields the lower bound becomes 0.1405 Computed . For $k \ge 2$, 99.8–100% of the energy of the maximiser of the true ratio was at $|q|_\infty = \pi$, sitting on period-2 points of type $(\pi, \pi, \pi, 0)$ Computed .

Table 3 The loss on larger tori, compared with the product of the per-step upper bounds Computed

k	Torus	Stein	Lower bound of the true ratio	Loss	Product of per-step Stein ÷ lower bound
1	16^4	4.431	0.591	**7.50**	7.5
2	8^4	0.9554	0.1405 (period 2)	**6.80**	182
3	8^4	0.09575	0.01497 (period 2)	**6.40**	1.1×10^4
3	6^4	0.09092	0.014955	**6.07**	–

The rightmost column is the product of the one-step upper bounds of each step (4.45, 5.75, 6.50, 6.81) divided by the lower bound of the true ratio; it grows by a factor of 60–90 per step, $7.5 \to 182 \to 1.1 \times 10^4 \to 1.0 \times 10^6$ Computed . Only the side that composes after taking absolute values grows exponentially. This column shows how much is gained by taking the upper bound on the composite kernel.

The three hypotheses failed, and they failed in the same way

As reasons for the loss stopping we tried three: (a) composition reduces the proportion of the negative part; (b) the cancellation takes place within a cell; (c) the signs align at the maximiser of the true ratio. None of them holds in the form written. All three, however, failed in the same way: "it does not change over steps".

Table 4 Verdicts on the hypotheses Computed

Hypothesis	Quantity	k=1	k=2	k=3	k=4 (4^4)	Verdict
(a)	Proportion of the negative part (row signs chosen freely)	0.411	0.432	0.435	0.426	Does not hold (constant at about 0.43; on 4 ^{4}, 0.405, 0.425, 0.427, 0.426)
(b)	Inner shell (distance 0.5–1.5): share of ℓ^1 / negative proportion	0.58 / 0.22–0.33	0.48 / 0.32	0.44 / 0.31–0.37	0.53 / 0.32–0.38	Does not hold (the cancellation is between the positive centre and the negative ring at distance 1.5–2.5; summing over 2^4 blocks leaves 41–56% of ℓ^1)
	Outer ring (distance 1.5–2.5): share of ℓ^1 / negative proportion	0.33 / 0.59–0.63	0.38 / 0.56–0.60	0.40 / 0.53–0.60	0.46 / 0.53–0.59	
(c)	Alignment / efficiency at the maximiser	0.74 / 4.5% (16^4)	0.56 / 1.3% (8 ^{4})	0.57 / 1.5% (6 ^{4})	0.60 / 1.8%	Does not hold (the alignment stops at around 0.6)
—	Row alignment in the worst direction $\langle\kappa^{-2}\rangle$	22.9 (16 ^{4})	10.6 (8^4), 5.55 (4^4)	10.2 (8^4), 5.13 (4^4)	5.31	Bounded over steps

That all three quantities hardly move over steps can be read as showing not that "composition brings the kernel closer to positive" but that "composition brings the profile of the kernel (its normalised shape) closer to a single one" Reading . A profile positive at the centre and negative around it, a negative proportion of 0.43 and an alignment of 0.6 are kept independently of the step. This is close to the view of the limit function of subdivision Reading . Note that the signs "positive at the centre, negative around it" depend on the sign convention for the components of the direction; what does not depend on the convention is the negative proportion and the alignment.

The inequality that held — loss ≤ inverse square of the alignment

Candidate 2.1. Suppose $A_r \geq 0$, $\kappa \geq 0$, $M \geq 0$, $|v_a| \leq M$ for every component, and $\kappa A_r M \leq |y_r|$ for every row. Then

$$\kappa^2 \Big(\sum_r A_r^2 y_r^2\Big) \sum_a v_a^4 \;\leq\; \Big(\sum_r y_r^4\Big) \sum_a v_a^2$$

Lean (`stein_le_inv_sq_mul_test` ; lemmas `stein_form_le` , `sum_four_le`). The bracket on the left is the value of the Stein quadratic form and the right-hand side is the value of the test, so it reads "upper bound ÷ test value $\leq \kappa^{-2}$".

On the lattice, using as the test the real plane wave in the worst direction $\theta = \mathrm{Re}(e^{i\varphi} u^* e^{-iq^*\cdot X})$, we get

$$\text{loss} \;\leq\; \frac{\text{Stein}}{\text{test value}} \;=\; \langle\kappa^{-2}\rangle \times \frac{S(u^*)}{\Phi}, \qquad S(u) = \frac{\sum_a |u_a|^4}{|u|^2 M^2} \leq 1$$

Φ is the phase factor; it is 1 if $2q^* \not\equiv 0$ and $4q^* \not\equiv 0$, and also 1 at period-2 points ($2q^* \equiv 0$) if u^* is taken real Reading (the average $3/8$ of $\cos^4$ cancels between numerator and denominator; this step and the Stein formula itself are not in Lean).

- For $k = 2, 3, 4$ on the 4^4 torus, the maximiser of Stein was a period-2 point, and its plane wave was itself the maximiser of the lower bound of the true ratio (test values 0.1396, 0.01487, 0.00109). Then Stein ÷ test value $= \langle\kappa^{-2}\rangle =$ **5.55, 5.13, 5.31**, and the shape factor and the phase factor were exactly 1 Computed .

- On larger tori the maximiser of Stein moves away from period 2, and $\langle \kappa^{-2} \rangle$ becomes 22.9, 10.6, 10.2 ($k = 1, 2, 3$). As an upper bound of the loss it is bounded and falls over steps, but it does not agree with the loss itself (6.4 and so on) Computed .

How to read it: on the ℓ^2 side the Stein bound remembers all the signs; what it forgets is only the "row ℓ^1" on the ℓ^∞ side. So the loss reduces to the amount of cancellation within a row, $1/\kappa^2$ Reading . This gives a conditional statement — if the alignment in the worst direction is bounded below over steps, the loss is bounded over steps — but that the alignment is bounded below is not shown (question "alignment of rows in the worst direction").

A pattern that did not apply — positive part dominant

For a sign field s, if the negative part of each row is at most δ times the positive part, then $(Ts)_i \geq \frac{1-\delta}{1+\delta} r_i$ (r_i the row ℓ^1), and combined with the ℓ^4 Schur test $\sum (Tx)^4 \leq A^3 B \sum x^4$, loss $\leq U\left(\frac{1+\delta}{1-\delta}\right)^4$ (U the non-uniformity factor of the row ℓ^1). Under composition δ degrades additively (alignment $1 - \delta_1 - \delta_2$) Lean (`coherent_of_neg_le`, `schur_four`, `loss_le_of_coherent`, `comp_coherent`).

The theorem is correct, but it does not apply to the kernels of the record. The δ read from the ℓ^4-weighted average alignment is about 0.46, 0.39, 0.39, 0.35 and is not small; U grows as 3.0, 4.5, 8.8, 15.8; and the minimal alignment per row is almost 0 (the assumption of δ on every row is not satisfied) Computed . The Schur test itself was also two orders of magnitude coarser than the Stein bound (Schur ÷ lower bound 437 → 745 → 1453 → 1777) Computed .

The W coordinates — the absolute value after composition stops at a rank-one value

Table 5 The loss for the W coordinates (kernel with the zero mode corrected, 60 types) Computed

k	$\rho(T_k)$	Compose after absolute value	Absolute value after composing	Negative proportion
1	0.3631	1.01	1.01	0.210
2	0.1249	1.08	1.06	0.325
3	0.0422	1.16	1.09	0.387
4	0.0142	1.24	1.10	0.411
5	0.0048	1.33	1.10	0.421

The loss is the ratio of ρ's ($\rho(|\cdot|)/\rho$). The value 1.10 at which the side taking the absolute value after composition stops almost agrees with the rank-one value $\sum |u_i||v_i| / |\sum u_i v_i| = 1.085$ built from the left and right dominant eigenvectors u, v Computed . If the dominant eigenvalue has a gap, $T^k/\rho^k \to uv^{\mathrm{T}}/(v^{\mathrm{T}}u)$, and since $|\cdot|$ and ρ are continuous it converges to this value Reading . The side composing after taking absolute values grows as $(\rho(|M|)/\rho(M))^k$ for powers of a single matrix (here by a factor of 1.070 each time), and is bounded only when $\rho(|M|) = \rho(M)$, which for irreducible matrices means $M = e^{i\alpha} D|M|D^{-1}$ (Wielandt) Cited . That the absolute value after composition is at most the composition after absolute values (row ℓ^1) is `abs_comp_le`, `row_l1_comp_le` Lean .

03 The geometry of local representatives

Representing by a local kernel a map determined only on fields satisfying the coarse Bianchi identity — this handwork came up as the choice of "representative" in the Bundle Network Theorem (local form) of 15-05. In this section we separate the overall shape of the representatives (an affine space), the

quantities that do not depend on the representative (the value of the zero mode, and the order of folding restricted to the constraint), and the quantities that do (the overall order of folding and the tail norm). The content is a restatement of the order of multigrid transfer operators and of the sum rules of subdivision (conditions of Strang–Fix type).

Definitions

Definition 3.1 (constrained map and extension). We call a quadruple (E, V, W, f) of finite-dimensional spaces E, W over a field K, a subspace $V \le E$ and a linear map $f : V \to W$ a **constrained map**, and a $g : E \to W$ with $g|_V = f$ an **extension**. Extensions exist (the restriction map res_V is surjective), any two extensions agree on V, and the set of extensions is the affine space $g_0 + \ker \mathrm{res}_V$, of dimension $(\dim E - \dim V) \cdot \dim W$ Lean (`exists_extension`, `restrict_surjective`, `IsExtension.eqOn`, `isExtension_iff_mem_coset`, `finrank_freedom`). The freedom is 0 only when $V = E$ ($W \neq 0$; `freedom_eq_zero_iff`).

Definition 3.2 (lattice representative). The coarse plaquette field is $\theta : \mathbb{Z}^4 \to \mathbb{C}^6$, the constraint subspace is $V = \mathrm{im}\, D_c$ (fields satisfying the coarse Bianchi identity), and the output is a fine plaquette field with $96 = 16 \times 6$ components per coarse cell. A translation-invariant kernel $K : \mathbb{Z}^4 \to \mathbb{C}^{96\times 6}$ whose symbol satisfies $\hat K(q)\hat D(q) = \hat R(q)$ for all q (with $R = FD_c$) is called a **representative** of F, and one that has finite range, exponential decay or ℓ^1 is called a **local representative**. For $q \neq 0$, $\hat D(q)$ has rank 3, the symbol at each q is an instance of the template "$E = \mathbb{C}^6$, $\dim V = 3$, $W = \mathbb{C}^{96}$", and the freedom is $(6-3)\cdot 96 = 288$-dimensional.

Definition 3.3 (zero mode, moments). We call $\hat K(0) = \sum_Z K(Z)$ the zero mode and $M_\alpha(K) = \sum_Z Z^\alpha K(Z)$ the moments.

Definition 3.4 (order of folding). For $\eta \in \{0,1\}^4$, $p_\eta = (q + 2\pi\eta)/2$ and the fine position $s \in \{0,1\}^4$ within a coarse cell, the folded component is $\hat C_\eta(q)[o,a] = \sum_s e^{-ip_\eta\cdot s}\hat A_{(s,o),a}(q)$. We write $\mathrm{ord}(K)$ for the minimal order to which $\hat C_\eta(q)$ with $\eta \neq 0$ vanishes as $q \to 0$, and $\mathrm{ord}^V(K)$ for the minimal order of $\hat C_\eta(q)P(q)$, multiplied by the projection $P(q)$ onto the constraint subspace.

Quantities that do not depend on the representative

The zero mode is determined by continuity alone (Candidate 3.1). If K is an ℓ^1 representative (continuous symbol), $\hat D(q) = D_1(q) + O(|q|^2)$, $\hat R(q) = R_1(q) + O(|q|^2)$ and $\sum_\nu \mathrm{im}\, D_1(e_\nu) = \mathbb{C}^6$, then $\hat K(0)$ is the unique solution of $\hat K(0) D_1(e_\nu) = R_1(e_\nu)$ ($\nu = 1,\dots,4$) Reading. The last step (maps that agree on a family of subspaces whose sum spans the whole are equal) is `eq_of_eqOn_iSup`, `correction_eq_zero_of_iSup` Lean. The value in the record is $\hat K(0) = U/4$, $U = \mathbf{1}_{16} \otimes I_6$ ($\theta/4$ uniformly at every position), with deviation $\le 3.1 \times 10^{-10}$ over four steps Computed. On the torus $\hat D(0) = 0$, so the zero mode is unconstrained, and for the symmetric representative on finite tori $\max|\hat A(0) - U/4| = 0.250,\ 1.47\times 10^{-2},\ 2.55\times 10^{-5}$ at $L_b = 8, 16, 32$ Computed. This is the numerical trace of the constraint of locality being lost in a finite box.

The minimal-norm representative is not local (Candidate 3.2). $\hat K_{\mathrm{MP}}(q) = \hat R(q)\hat D(q)^+$ (the Moore–Penrose representative) does not lie in ℓ^1 unless the limits along different directions agree, because the projection $P(q)$ is a function of q homogeneous of degree 0 and leaves a memory of direction at $q = 0$ Reading. The freedom of local representatives can be seen as the room to repaint this memory of direction into a continuous function. The decay of the kernel has not been measured (question "decay of the Moore–Penrose kernel").

The order restricted to the constraint does not depend on the representative (Candidate 3.3). With the singular value decomposition $\hat{D} = U\Sigma V^{\mathrm{H}}$, $\hat{C}_\eta(q)U(q)$ can be written using only the folding of $\hat{R}$, so $\mathrm{ord}^V(K) = \mathrm{ord}^V(F)$ is an invariant of F. Restriction does not lower the order of vanishing, so for every representative $\mathrm{ord}(K) \le \mathrm{ord}^V(F)$ Reading .

Table 6 Orders of folding of the representatives in the record (in brackets, how the value decreases as t is made smaller by one order of magnitude at a time) Computed

Representative	Construction	ord (whole 6×6)	ord^V (on $\mathrm{im}\, D$)
sym	Rows of the cone homotopy (finite torus)	0 ($\hat{C}(0) = 0.039$–0.071)	0
A3	Add the difference of the zero mode at the single point $Z = 0$	1 ($3.4 \times 10^{-2} \to 3.4 \times 10^{-4}$ for $t = 10^{-1} \to 10^{-3}$)	1 ($5 \times 10^{-3} \to 5 \times 10^{-5}$)
A4	Add to A3 a first-moment correction as antisymmetric pairs at $Z = \pm e_\mu$	2 ($1.7 \times 10^{-3} \to 1.7 \times 10^{-7}$)	2
F itself	From R	—	2 ($1.8 \times 10^{-3} \to 1.8 \times 10^{-7}$)

A4 attains $\mathrm{ord}(K) = \mathrm{ord}^V(F) = 2$ on the whole 6×6. That is, within the types of the record, A4 is a representative that reaches the ceiling.

Tail norm and order — D ≤ 4 is the upper limit

Candidate 3.4. In the n-step map of the W coordinates $(Tf)(q) = 16^{-n} \sum_m G_m(q)^{\mathrm{H}} f(p_m) G_m(q)$ (m runs over 16^n paths, G_m is a product of foldings), suppose that the representatives of all steps satisfy $\mathrm{ord}(K_k) \ge k_0$ and that $\hat{C}_0(0) = 4I$. (a) If $D \le 2k_0$, T is bounded in the norm on the symbol side $\|\cdot\|_D = \sup_{q/=0} \|\cdot\|_2/|\hat{q}|^D$. (b) If $D > 2k_0$ and the leading coefficient S of the paths with $m \ /= 0$ is non-zero, then for a peaked input near q_0 it diverges at the rate $|q_0|^{2k_0 - D}$ Reading (a line of argument assuming constants uniform in the number of steps; uniformity over infinitely many steps is not treated). In the version restricted to the constraint, k_0 is replaced by $\mathrm{ord}^V(F)$.

Table 7 Tail norm ($D = 4$, input along the axis $q = 2\pi/L$, four points $L = 24$–192) Computed

Representative	ord	Values	Type in Candidate 3.4
A3	1	1.486, 5.88, 23.5, 93.8 (grows as L^2)	$\lvert q\rvert^{2\cdot 1 - 4}$
A4	2	0.01638, 0.01640, 0.01643, 0.01643 (bounded)	$\lvert q\rvert^{2\cdot 2 - 4}$ = constant

From R ($\mathrm{ord}^V = 2$) it was bounded at $D = 4$, q^{-2} at $D = 6$, and q^{-3} at $D = 7$ Computed . All cases in the record fit the types (a)(b). Since $\mathrm{ord}^V(F) = 2$, the $D = 6$ tail on $\mathrm{im}\, D$ diverges however the representative is chosen, and this reads as: **within the types of the record, $D \le 4$ is the upper limit for the tail of $\sup/|\hat{q}|^D$** Reading .

The 2 in "$2 \cdot \mathrm{ord}$" comes from the map of the W coordinates being a congruence $G^{\mathrm{H}} f G$ with the same representative on both sides. In multigrid the orders of prolongation and restriction can be chosen separately, and there is the rule "order of prolongation + order of restriction ≥ order of the equation" (Hemker 1990) Reading . Here a single representative plays both roles.

Representatives with matched first moments, and the sum rules

Candidate 3.5 (order and shifted jets). Putting $J_s(q) := e^{-iq\cdot s/2} \hat{K}_s(q)$ ($\hat{K}_s$ the symbol of the 6 rows at position s), $\mathrm{ord}(K) \ge k \Leftrightarrow$ the Taylor polynomial of $J_s(q)$ up to degree $k - 1$ does not depend on s Reading . The line of argument is $\hat{C}_\eta(q) = \sum_s (-1)^{\eta\cdot s} J_s(q)$ together with the orthogonality of the 15 non-trivial characters of $(\mathbb{Z}/2)^4$ (inner product 0 with all of them $\Leftrightarrow$ constant).

- $k = 1$: $\hat{K}_s(0)$ does not depend on s ($U/4$ has this form).
- $k = 2$: the first moments are $M_s = (s/2)\hat{K}_s(0) + B$ (B independent of s). From $\hat{K}_s(0) = I/4$, $M_{s,\mu} = s_\mu I/8 + B_\mu$. This agrees with the target moments used to build A4, $M^*_{(s,o),\mu,a} = s_\mu \delta_{oa}/8 + B_\mu[o,a]$. The 8 is (position within the cell $s/2$) $\times$ $(1/4)$.

The target B is determined from the q^2 term of R. Against 24 unknowns per row (4 directions $\times$ 6), the part determined by R had rank 20, leaving 4 dimensions of freedom Computed . This count is of the type of a finite-dimensional "extension with conditions": if (res_V, C) is surjective, there is a g satisfying both $g|_V = f$ and $C(g) = y$, and the remaining freedom is $\dim E \cdot \dim W - \dim V \cdot \dim W - \dim Y$ Lean (`exists_extension_with_conditions`, `finrank_freedom_with_conditions`, `conditions_count_le`). The remaining 4 dimensions were fixed by linear programming to minimise the increase of ℓ^1; the moments matched the target to within 1.1×10^{-16}, and the row ℓ^1 went from 2.26 to 2.34 (first step) Computed .

Composition of representatives and compatibility

If (E_1, V_1, E, f_1) and (E, V_2, W, f_2) are compatible ($f_1(V_1) \subseteq V_2$), the composite of representatives is a representative of the composite, and its value on V_1 does not depend on the choice of representatives. If they are not compatible ($W \neq 0$), the value of the composite changes with the choice Lean (`comp_extension_eqOn`, `comp_isExtension`, `comp_depends_of_not_le`). The chain of W coordinates in the record composes representatives, but whether the output of the representative at step k lies in the constraint subspace of step $k+1$ has not been checked (question "consistency of composing representatives across steps"). The difference between the chain on $\mathrm{im}\, D$, 0.3426, and the 6×6 chain, 0.3446, is a clue to this Computed . As examples of norms changing with the representative: the zero-mode correction moved the three-step Stein bound from 0.09743 to 0.09868, and the signed Stein bound went from 4.25 for A3 to 4.48 for A4 (first step) Computed .

Dictionary

- $\mathrm{ord}_\eta(K)$ corresponds to the **high-frequency order** of the representative viewed as a prolongation (Hemker 1990). That the definition by symbols and the definition by preservation of polynomials are equivalent is stated in a later note, and the condition on the zeros of the symbol in Toeplitz multigrid is in Donatelli (arXiv:0807.2565) Reading . $\hat{C}_\eta$ is the same as the 16 components of the harmonic subspaces of local Fourier analysis Cited .
- Candidate 3.5 is exactly the standard argument that turns the **sum rules** of subdivision and wavelets into polynomial generation (Strang–Fix 1973, Jia 1998, Charina–Conti arXiv:1202.3931, Daubechies's vanishing moments) Cited . The part $s_\mu I/8$ of the target of A4 is a restatement of the sum rule of order 2.
- Two points fall outside the sum rules: (i) the part where B is determined by the q^2 term of R (a condition of the type of order of consistency of differences); (ii) the ceiling $\mathrm{ord}^V(F)$ set by the constraint $V = \mathrm{im}\, D$. That the masks are not scalars but 96×6 matrices is within the range of sum rules for vector-valued subdivision Reading .

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04
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The geometry of the double valuation

There is no payoff theorem yet. This section is an attempt to restate the row for the crossover band in the ledger of conditions (26-06) in the language of two valuations, those of the strong-coupling expansion and the weak-coupling expansion. The definitions and the computation of the leading terms are done, but there is as yet no theorem that this view newly says about the band. The labels are mainly Assumption and Computed.

Definitions

The space of couplings $\mathcal{K}$: the space of character coefficients (a_r) of the plaquette weight $f(U) = 1 + \sum_r d_r a_r \chi_r(U)$. Its coordinates are β and the shape coordinates.

Two valuations. v_s (strong coupling) is the order on the side $a_r \to 0$ (the order in β for the Wilson action), and v_w (weak coupling) is the order in $1/\beta$ on the side $\beta \to \infty$. Terms of type $e^{-c\beta}$ are treated as $v_w = \infty$ (another branch of the transseries).

Two coordinates. For an observable O with values in $[0, 1]$ ($O(0) = 0$, $O(\infty) = 1$; the plaquette P, for example), the coordinate on the strong-coupling side is O and the coordinate on the weak-coupling side is $1 - O$. We write the leading coefficients as $O \approx s_1\beta$ (strong coupling) and $1 - O \approx w_1/\beta$ (weak coupling).

Tie sets. (i) the complementary tie $\{s_1\beta = w_1/\beta\}$; (ii) the same-coordinate tie $\{s_1\beta = 1 - w_1/\beta\}$; (iii) the error tie $\{K$-term optimal truncations have equal errors$\}$. We call the region between the two roots of (ii), $\{s_1\beta \le 1 - w_1/\beta\}$, the **lens**.

The invariant κ and the lens

Candidate 4.1 (a scale-independent quantity). Changing the scale of the coupling as $\beta \mapsto c\beta$ gives $s_1 \mapsto s_1/c$, $w_1 \mapsto c w_1$, so

$$\kappa := s_1 w_1$$

does not depend on the scale. The complementary tie is at $\beta^* = \sqrt{w_1/s_1}$, where the value of the leading terms is $\sqrt{\kappa}$. The same-coordinate tie is at the roots of $s_1\beta^2 - \beta + w_1 = 0$, whose discriminant is $1 - 4\kappa$, so **the lens opens only when** $\kappa \le \frac{1}{4}$. The geometric mean of the two roots is always β^*. A tie for the string tension $\sigma = -\log u$ exists only when $\kappa \le 1/e$ Reading (the elementary fact of the discriminant of a quadratic).

Looking at the leading term on one side only, the difference between groups is absorbed into the scale. Looking at the leading terms on both sides at once, a change of scale is only a translation in the $(\log O, \log(1 - O))$ plane, and the height $\sqrt{\kappa}$ at which they cross does not move. The view is that the information about the group remains in "the height of the crossing" Assumption.

Examples

Table 8 Series and ties for a single plaquette (exact solution in two dimensions; the material of one Migdal–Kadanoff step) Computed

Quantity	$SU(2)$	$U(1)$
Strong-coupling series u	$\frac{\beta}{4} - \frac{\beta^3}{96} + \frac{\beta^5}{1536} - \frac{\beta^7}{23040} + \cdots$	$\frac{\beta}{2} - \frac{\beta^3}{16} + \frac{\beta^5}{96} - \frac{11\beta^7}{6144} + \cdots$

Quantity	$SU(2)$	$U(1)$
Weak-coupling series $1-u$	$\frac{3}{2}\beta^{-1}-\frac{3}{8}\beta^{-2}-\frac{3}{8}\beta^{-3}-\frac{63}{128}\beta^{-4}\dots$	$\frac{1}{2}\beta^{-1}+\frac{1}{8}\beta^{-2}+\frac{1}{8}\beta^{-3}+\frac{25}{128}\beta^{-4}\dots$
$\kappa=s_1w_1$	$3/8$ (the lens is empty, $1-4\kappa=-\frac{1}{2}$)	$1/4$ (the lens is the single point $\beta=1$)
s_3/s_1^3, w_2/w_1^2	$-2/3$, $-1/6$	$-1/2$, $+1/2$
Complementary tie β^* and the true $u(\beta^*)$	$\sqrt{6}=2.4495$, $u=0.5004$ (β at $u=\frac{1}{2}$ is 2.4469)	1.0000, $u=0.4464$ (β at $u=\frac{1}{2}$ is 1.1593)
Tie for σ	None (maximum of the difference $-\log\kappa-1=-0.019$)	0.232 and 1.399
Error tie ($K=40$)	3.274 (error 4.8×10^{-4})	2.124 (error 2.9×10^{-3})
Fisher zeros (imaginary axis)	$\pm i\,3.832$, $\pm i\,7.016$, ...	$\pm i\,2.405$, $\pm i\,5.520$, ...

The specific heat β^2u' has a peak (0.680) at $\beta=2.48$ for $U(1)$, while for $SU(2)$ it rises monotonically to $3/2$ Computed . This difference goes in the same direction as the difference in the sign of w_2/w_1^2 ($-1/6$ and $+1/2$), and can be read as: the difference between groups lives not in the leading terms of the valuations but in the sign of the next term Assumption . That $u(\sqrt{6})=0.5004\approx\frac{1}{2}$ for $SU(2)$ does not hold for $U(1)$ (0.446), so it is not a general law.

The lens of the leading terms alone in four dimensions. Using equipartition $w_1=\dim G/d$, $\kappa=s_1\dim G/d$.

Table 9 The lens of the leading terms alone in d dimensions Computed

d	Group	κ	Complementary tie β^*	Lens
2	$SU(2)$	3/8	2.449	None
2	$U(1)$	1/4	1.000	Single point $[1,1]$
3	$SU(2)$	1/4	2.000	Single point $[2,2]$
3	$U(1)$	1/6	0.816	$[0.423,1.577]$
4	$SU(2)$	3/16	1.732	$[1.000,3.000]$
4	$U(1)$	1/8	0.707	$[0.293,1.707]$

The transition of four-dimensional $U(1)$, $\beta_c\approx1.01$ (26-06), lies inside the lens $[0.293,1.707]$ (not at its centre 0.707). The $SU(2)$ lens $[1,3]$ contains the range in the table of 26-06 over which $\beta-\beta'$ moves from the strong-coupling value to the one-loop value, and also overlaps $1.5<\beta<2.5$, where the literature says "neither the strong-coupling nor the weak-coupling expansion works well" (abstract of arXiv:0807.0185) Reading . These are numbers derived only from first-order leading terms and equipartition, without the true four-dimensional series. The overlaps may include coincidence Assumption .

Tie sets under the Migdal–Kadanoff flow

We followed the level lines $L_n(c)$ of the fundamental-representation coefficient $u_n(\beta)$ after n steps of flow from the Wilson action. A level line is an object that does not depend on how β is read.

Table 10 Movement of the level lines $L_n(c)$ Computed

Group	c	L_0	L_6	Difference at step 6, L_6-L_5
$SU(2)$	0.1575	0.641	2.230	0.259

Group	c	L_0	L_6	Difference at step 6, $L_6 - L_5$
$SU(2)$	0.5	2.447	3.535	0.2553
$SU(2)$	0.9	14.73	15.58	0.2511
$U(1)$	0.1575	0.319	0.582	0.017 (positive while decreasing)
$U(1)$	0.5	1.159	0.844	4.5×10^{-4}
$U(1)$	0.7	2.014	1.5101	-3.5×10^{-7} (stops)
$U(1)$	0.9	5.305	4.8478	-3.7×10^{-7} (stops)

- For $SU(2)$, the difference per step approaches 0.25 for every c (the $1/4$ of the heat kernel). The tie sets escape linearly in the direction of β, and in terms of the valuation of the flow, $\beta - \beta'$ has $v_w = 0$ (a constant) Computed .
- For $U(1)$, the level lines with $c \geq 0.7$ stop. The one-step movement after the shape has settled, $\beta_V - \beta'_V$, is 9.6×10^{-4} at $\beta_V = 0.67$, 2.2×10^{-4} at 0.78, 5.4×10^{-5} at 0.88, 1.3×10^{-5} at 0.99, 3.2×10^{-6} at 1.09 and 1.8×10^{-7} at 1.29, and the exponential slope read from the ratios of neighbouring points is 13.4–14.1. It is of type $e^{-c\beta_V}$, but c has not been identified (it does not reach $2\pi^2 = 19.7$). The valuation of the flow is $v_w = \infty$ Computed . At floating-point resolution, "stops" cannot be distinguished from "flows at about $e^{-20\beta}$" Assumption .
- The map on the strong-coupling side, $u \mapsto 256u^4$, is the same for both groups once the scale is matched by s_1. Flowing the complementary tie of the strong-coupling leading terms, $SU(2)$ goes $2.449 \to 1.140 \to 0.785 \to 0.750$ and the solution disappears for $n \geq 4$, while $U(1)$ converges $1.0 \to 0.5 \to 0.361 \to \ldots \to 0.3152$. The candidate for the limit is $s_1\beta \cdot 256^{1/3} = 1$, that is, the edge of the strong-coupling basin $u_c = 256^{-1/3} = 0.1575$, common to both groups Computed .
- Caution: reading β' by Wilson's u gives $\beta - \beta' < 0$ on the weak-coupling side for both groups. This is the $O(1)$ shift in β caused by the Wilson form and the heat-kernel form differing at $O(1/\beta^2)$, and it is not a flow Computed .

Why the candidate theorem does not stand, and a weakened form

The form "a comparison that distinguishes groups is needed only near the codimension-one skeleton of the tie sets" **is not expected to hold as it stands** Assumption . This is because the distinction between groups already appears at the level of the leading terms, over the whole weak-coupling coordinate, as the valuation of the flow ($v_w = 0$ for $SU(2)$, $v_w = \infty$ for $U(1)$; that is, the presence or absence of b_0).

Candidate B (weakened form) Assumption . For a family of observables and groups G, G', assume three things: agreement of the strong-coupling valuations (the maps of the valuations of the strong-coupling coordinate agree for G and G' under matching of the scale by s_1); difference of the weak-coupling valuations (the valuations of the flow of the weak-coupling coordinate differ); and regions where the leading terms dominate (regions $D_s(\eta)$, $D_w(\eta)$ can be taken in which the error of each coordinate is at most η times the leading term of that coordinate). Then information beyond the leading terms is needed to decide the direction of the comparison (whether $\beta' \leq \beta$) only on $\mathcal{K} \setminus (D_s(\eta) \cup D_w(\eta))$, and this set shrinks to the lens as $\eta \to 0$.

- What it says: the need to **carry** the distinction is localised in the lens. It restates the same content as "the band is a bounded interval" of 26-06 in the language of valuations, and does not claim to make the band easier.

- What it does not say: what is needed inside the lens. The transition of $U(1)$ is on the $v_w = \infty$ branch, so the reading stops at: inside the lens a comparison is needed that handles the transseries branch becoming the same size as the perturbative branch Assumption .
- For the two-dimensional single plaquette with $SU(2)$ ($\kappa > \frac{1}{4}$) the lens is empty and Candidate B holds vacuously. However, two-dimensional $U(1)$ also has no phase transition, so κ alone does not determine the phase diagram Reading .

Relation to the known

- A view is known in which the Fisher zeros form the edge of the basin of attraction of the strong-coupling fixed point (Denbleyker–Du–Liu–Meurice–Zou, arXiv:1005.1993; the Wilson action of four-dimensional $SU(2)$) Reading . That the tie of the strong-coupling leading terms is drawn by the flow to the basin edge u_c can be read as the shadow of this view on the real axis Assumption . For a single plaquette, the radius of convergence of the convergent strong-coupling series is the absolute value of the first Fisher zero Cited .
- The error tie (iii) puts into numbers, as it stands, the question of resurgence "which series dominates". For a single plaquette, the optimal error of the weak-coupling asymptotic series was of order $e^{-2\beta}$ Computed .
- Two-point Padé approximation joining the series at the strong and weak ends should be known as a classical way of filling in the lens Reading (the literature has not been checked).

05

The cone calculus

The record of Part 15 contains the following handwork again and again: bounding a polynomial of degree p by an ℓ^p cone $|f(\theta)| \le c\,\|\theta\|_p^p$, pulling it back by a linear map (the local representative of the blocking), bounding the pull-back factor from above by the weighted Stein bound, checking that it contracts in a composite of several steps even if it does not contract in one step, and absorbing the terms that the loops add at each step into an affine recursion to obtain a bound valid at every step (One-Step Contraction of the ℓ^4 Cone in 15-05, 15-08 § 06). Like § 01– § 03, this section is an attempt to give this handwork definitions. Much of the content is a restatement of known facts — the Riesz–Thorin interpolation theorem, Stein–Weiss interpolation with change of measure, the Schur test, Hölder's inequality, and the log-convexity of operator norms in the exponent — and no novelty is claimed. What is new lies on the side of the questions: after setting out which quantity each number of the record corresponds to, we pose as questions the "window" and the "bundle number" as the exponent p is varied.

Setting

V, W are finite-dimensional real vector spaces, and we fix one set of coordinates. ℓ^p is the ℓ^p of the coordinates. In the lattice case, V is the plaquette field on the coarse torus (6 components per point), W is the fine plaquette field arranged as 96 components per cell, and $K : V \to W$ commutes with coarse translations. The supremum of the cone is taken over all of V (it is not restricted to the image of the Bianchi identity). Hence the quantities here are upper bounds for the restricted quantities.

Definitions

Definition 5.1 (cone constant). When $f : V \to \mathbb{R}$ is a homogeneous polynomial of degree p (or grows no faster than that),

$$c_p(f) := \sup_{\theta \neq 0} \frac{|f(\theta)|}{\|\theta\|_p^p}.$$

In Lean this is written `IsCone f N c` ($|f\,\theta| \le c \cdot N\theta$ for all θ), with an abstract N playing the role of $\|\cdot\|_p^p$.

Definition 5.2 (p-contraction). For a linear map $K : V \to W$,

$$\mu_p(K) := \sup_{\theta \neq 0} \frac{\|K\theta\|_p^p}{\|\theta\|_p^p} = \|K\|_{p\to p}^p.$$

In Lean this is written `Contracts K N μ` ($N(K\theta) \le \mu\, N\theta$ for all θ) and read as "$\mu_p(K) \le \mu$".

Definition 5.3 (row and column ℓ^1). Put $A_r := \sum_j |K_{rj}|$, $R(K) := \max_r A_r = \|K\|_{\infty\to\infty}$, $C(K) := \max_j \sum_r |K_{rj}| = \|K\|_{1\to 1}$. Then $\mu_1 = C$, and $\mu_p^{1/p} \to R$ $(p \to \infty)$ Cited .

Definition 5.4 (the weighted Stein bound for general $p \ge 2$). For a positive weight u put

$$S_p(K;u) := \max_r \frac{A_r^p}{u_r} \cdot \lambda_{\max}\big(K^{\mathsf T} D_{u/A^2} K\big), \qquad S_p(K) := S_p(K;A^p) = \lambda_{\max}\big(K^{\mathsf T} D_{A^{p-2}} K\big)$$

(D_v is the matrix with v on the diagonal). The standard form $S_p(K)$ becomes the bare $\|K\|_2^2$ at $p = 2$, D_A at $p = 3$ and D_{A^2} at $p = 4$, recovering the ℓ^3 and ℓ^4 formulas of the record. For K commuting with translations, $\lambda_{\max}$ is replaced by $\sup_q \lambda_{\max}(\hat K(q)^{\mathsf H} D \hat K(q))$.

- Derivation Reading : put $T = D_{1/A} K$; then $\|T\|_{\infty\to\infty} = 1$. If M_0 is the norm of T in ℓ^2 with weight u, then $M_0^2 = \lambda_{\max}(K^{\mathsf T} D_{u/A^2} K)$. Interpolating between $(2, 2; u)$ and (∞, ∞) with $1/p = (1-t)/2$, that is $t = 1 - 2/p$, the norm is at most $M_0^{2/p}$ and the weight on the range becomes $u^{p(1-t)/2} = u$. Hence $\sum_r u_r |T\theta|_r^p \le M_0^2 \|\theta\|_p^p$, and from $\|K\theta\|_p^p = \sum_r A_r^p |T\theta|_r^p$ we get $\mu_p(K) \le S_p(K;u)$.
- The source for the interpolation: interpolation that changes the measure on the range is of the Stein–Weiss form. But the weight has no effect at the ℓ^∞ end, so taking both ends as statements on the same measure u, Riesz–Thorin alone suffices Cited .
- **The range of application is $p \ge 2$** ($t \ge 0$ is needed). At $p = 2$, $S_2(K) = \mu_2(K)$ and the upper bound equals the true value.

Definition 5.5 (bundle number). For a sequence of step maps (K_k) and a bundle length n, put $K^{(k,n)} := K_k K_{k+1} \cdots K_{k+n-1}$ and

$$n_p := \min\Big\{ n : \ \sup_k \mu_p\big(K^{(k,n)}\big) < 1 \Big\} \quad \text{(if none, } \infty\text{)}$$

If the upper-bound version with μ_p replaced by S_p is $\tilde n_p$, then $n_p \le \tilde n_p$.

Definition 5.6 (window). Fixing the bundle length n, put $W_n := \{p \ge 1 : \ \mu_p(K^{(n)}) < 1\}$.

Definition 5.7 (affine recursion). $c_{k+1} \le \mu_k c_k + \varepsilon_k$ ($\mu_k \ge 0$, $c_k \ge 0$). The bundle version is $c_{k+m} \le \mu\, c_k + \varepsilon$. c_k corresponds to the cone constant at step k, and ε_k to the size of the degree-p terms newly born at that step.

Definition 5.8 (Schur threshold exponent). When $R(K) < 1$, put $p^*(K) := 1 + \log C / \log(1/R)$. The bound $\mu_p \le R^{p-1}C$ of Candidate 5.6 below falls below 1 when $p > p^*$.

Candidate theorems and labels

Candidate 5.1 (pull-back and submultiplicativity). If $c \ge 0$, then $c_p(f \circ K) \le c_p(f)\,\mu_p(K)$ Lean (`cone_pullback`; a map $V \to W$, and N_V, N_W may differ). On a single space, $\mu_p(K_1K_2) \le \mu_p(K_1)\mu_p(K_2)$ ($\mu_1 \ge 0$), $\mu_p(K^n) \le \mu_p(K)^n$ and $c_p(f \circ K^n) \le c_p(f)\,\mu^n$ Lean (`contracts_comp`, `contracts_iterate`, `cone_pullback_iterate`; $\mu \ge 0$).

Candidate 5.2 (exceeding in one step, contracting in a bundle). If $\mu_p(K) \le M$ in one step ($M > 1$ is allowed) and $\mu_p(K^n) \le \mu$ in n steps, then $\mu_p(K^{qn+r}) \le M^r \mu^q$ Lean (`block_decay`; $M \ge 0$, $\mu \ge 0$). This is the cone version of Candidate 1.4 (a) of § 01. The upper bound S_p itself is not necessarily submultiplicative, so the S_p of a composite is computed directly from the composite kernel Reading.

Candidate 5.3 (log-convexity of the upper bound). $\log S_p(K)$ is convex in p Reading. This is because $S_p = \sup_{\|x\|=1} \sum_r (Kx)_r^2 A_r^{p-2}$, and for each x the right-hand side is a positive sum of exponential functions of p and hence log-convex (Cauchy–Schwarz), and the supremum of log-convex functions is log-convex. What is in Lean is a discrete midpoint version Lean:

- `weighted_rpow_midpoint`: for $w \ge 0$, $a > 0$, $\left(\sum w\, a^{(p+q)/2}\right)^2 \le \left(\sum w\, a^p\right)\left(\sum w\, a^q\right)$.
- `stein_form_midpoint`: its Stein form $\sum y^2 A^{p-2}$ ($A > 0$).
- `sup_midpoint`, `midpoint_below_one`: from the pointwise midpoint inequality, an upper bound by the geometric mean of the suprema; in particular, if $S_a S_b < 1$ the value at the midpoint falls below 1.

As a consequence, $S_3 \le \sqrt{S_2 S_4}$. What looked in the record like "$\sqrt{S_2S_4}$ is a little above S_3" was not a comparison of two different upper bounds but the margin of the curvature of a single function $\log S_p$. $\sqrt{S_2S_4}/S_3 - 1$ is one number measuring the size of that curvature (Table 12).

Candidate 5.4 (log-convexity of the true p-contraction). $\log \mu_p$ is also convex in $p > 0$. By Riesz–Thorin, $\psi(s) := \log \|K\|_{1/s \to 1/s}$ is convex for $s \in [0,1]$ Cited, and $\log \mu_p = p\,\psi(1/p)$ is the perspective of ψ and hence convex Reading. Candidate 5.3 is what can be said directly about the upper bound S_p (as a sum of exponentials), while Candidate 5.4 is what can be said about the true μ_p from convexity in $1/p$ through the perspective. They rest on different things. There are two consequences Reading.

- The window W_n is an interval (the set where a convex function is negative).
- From $\mu_p^{1/p} \to R$ the asymptotic slope is $\log R$, and by convexity $\mu_p \le \mu_{p0}\, R^{p-p0}$ for $p \ge p_0$. For $p_0 = 2$ this is the same as using Riesz–Thorin between $(2, \infty)$. The upper-bound version $S_p \le R^{p-p0} S_{p0}$ follows directly Lean (`weighted_rpow_shift`: for $w \ge 0$, $0 < a \le R$, $p_0 \le p$, $\sum w\, a^p \le R^{p-p0} \sum w\, a^{p0}$; `sup_shift`: moves the pointwise inequality to the supremum).

Candidate 5.5 (bundle number and spectral radius). For a fixed finite-dimensional operator $K : V \to V$, Gelfand's formula gives $\lim_n \mu_p(K^n)^{1/n} = \rho(K)^p$, and ρ does not depend on p Cited. Hence "$n_p < \infty \Leftrightarrow \rho(K) < 1$", and only the smallest n changes with p. This is the same fact as Definition 1.3 of § 01, said in terms of the p-contraction. The blocking map, however, is a map from the coarse lattice to the fine lattice, not a single operator on one space. Identifying the fine lattice back with $\mathbb{Z}^4$ gives the type

"dilation by 2 and convolution" (a subdivision operator), and for this type the ℓ^p spectral radius depends on p Reading .

- A constant field gives a dimension count. The plaquette field shrinks by a factor of about $1/4$ in one step while the number of points grows by 2^4, so the ratio for a constant field is about $16 \cdot 4^{-p} = 2^{4-2p}$ per step Reading . The recorded values at $p = 2, 3, 4$ were 1.080, 0.2807, 0.07297, a factor of about $1/3.85$ for each increase of p by 1 Computed . This agrees with the dimensional reading that $p = 2$ is marginal ($\int F^2$ is dimensionless in four dimensions) and that for $p > 2$, $|F|^p$ acts in an irrelevant direction.
- The link with dominant matrices (§ 01): from $|K\theta| \le |K||\theta|$ we get $\mu_p(K) \le \mu_p(|K|)$, and the Perron weight can be used for $|K|$. In the kernels of the record, however, the cancellation of signs is large: $R(K_0K_1) = 0.928$ is much smaller than $R(K_0)R(K_1) \approx 2.25 \cdot 2.47 \approx 5.6$ Computed . Moving to $|K|$ loses this factor of about 6 at the outset. This is one number, in ℓ^∞, for the loss from discarding signs of § 02 Reading .

Candidate 5.6 (the correct form of dilution). The naive form "if the row sums are 1, the entries are non-negative and each entry is $\le 1/m$, then $\mu_p \le m^{1-p}$" **does not hold in general** Reading . An averaging map preserves constant fields, so if it is square, $\mu_p \ge 1$ (a constant field is a counterexample). What holds is the following.

- The ℓ^p version of the Schur test: for $p \ge 1$, $\sum_i |\sum_j K_{ij}\theta_j|^p \le R^{p-1}C \sum_j |\theta_j|^p$, that is $\mu_p \le R^{p-1}C$ Lean (`row_rpow_le`: $|\sum k_j\theta_j|^p \le (\sum |k_j|)^{p-1} \sum |k_j||\theta_j|^p$ by Jensen; `lp_schur`: row $\ell^1 \le R$, column $\ell^1 \le C$, $R \ge 0$). If $R < 1$, the upper bound on μ_p decreases in p with ratio R and falls below 1 for $p > p^*$.
- Its equality: if all row sums are $R > 0$ and all column sums are C, equality holds for a constant field Lean (`lp_schur_sharp`). The Schur form cannot be improved in general.
- Dilution of a localised field: for a δ field, $\sum_i |k_i|^p \le M^{p-1} \sum_i |k_i|$ ($|k_i| \le M$, $p \ge 1$) Lean (`delta_dilution`). If the entries are at most $1/m$, this is a factor of m^{1-p}. The m^{1-p} dilution works only for localised fields; spread-out fields are diluted only by the row ℓ^1.
- The minimal form in which the cone contracts: $\mu_p \le \mu_2 R^{p-2}$ (a consequence of Candidate 5.4) Reading , and the upper-bound version $S_p \le S_2 R^{p-2}$ Lean (`weighted_rpow_shift`, `sup_shift`). The reading is that ℓ^2 is marginal and stays at $\mu_2 \ge 1$, the row ℓ^1 of the composite drops below 1 as steps accumulate, and for $p > 2$ one gains by the factor R^{p-2} Reading . That the field maximising the one-step ℓ^3 in the record was spread out (degree of localisation 0.006) agrees with the obstruction lying on the side of spread-out fields, that is, the side where the dilution of δ fields does not work Computed .

Candidate 5.7 (affine recursion). There are the following five Lean . They extend Candidate 1.4 (c) of § 01 (`affine_uniform`) to coefficients that vary with the step and to bundles.

- `affine_const`: $c_{k+1} \le \mu c_k + \varepsilon$ ($0 \le \mu < 1$, $\varepsilon \ge 0$) $\Rightarrow c_n \le \mu^n c_0 + \varepsilon/(1-\mu)$.
- `affine_varying`: $0 \le \mu_k \le \bar\mu < 1$, $\varepsilon_k \le \bar\varepsilon$ ($\bar\varepsilon \ge 0$), $c_k \ge 0 \Rightarrow c_n \le \bar\mu^n c_0 + \bar\varepsilon/(1-\bar\mu)$.
- `affine_invariant`: under the same assumptions, if $c_0 \le \bar\varepsilon/(1-\bar\mu)$, then $c_n \le \bar\varepsilon/(1-\bar\mu)$ for all n (an invariant interval).
- `affine_eventually`: under the same assumptions, for any $\delta > 0$, $c_n \le \bar\varepsilon/(1-\bar\mu) + \delta$ for all sufficiently large n (the form $\limsup c_n \le \bar\varepsilon/(1-\bar\mu)$).

- `affine_block`: $c_{k+m} \le \mu c_k + \varepsilon$ $(0 \le \mu < 1, \varepsilon \ge 0) \Rightarrow c_{mn} \le \mu^n c_0 + \varepsilon/(1-\mu)$. Combined with Candidate 5.2, the steps between bundles pick up a factor M^r $(r < m)$ only once.

Table 11 The candidate theorems of § 05. Each is a finite-dimensional form of a known fact, or an elementary consequence of one

Candidate	Statement (one line)	Label	What it restates
5.1	$c_p(f \circ K) \le c_p(f)\mu_p(K)$, $\mu_p(K_1K_2) \le \mu_p(K_1)\mu_p(K_2)$, $\mu_p(K^n) \le \mu_p(K)^n$	Lean (`cone_pullback`, `contracts_comp`, `contracts_iterate`, `cone_pullback_iterate`)	Submultiplicativity of operator norms
5.2	One step $\le M$, n steps $\le \mu \Rightarrow qn + r$ steps $\le M^r \mu^q$	Lean (`block_decay`)	A finite version of Gelfand's formula (Candidate 1.4 (a) of § 01)
5.3	$\log S_p$ is convex in p; in particular $S_3 \le \sqrt{S_2 S_4}$	Continuous version Reading, discrete midpoint version Lean (`weighted_rpow_midpoint`, `stein_form_midpoint`, `sup_midpoint`, `midpoint_below_one`)	Hölder's (Cauchy–Schwarz) inequality; the supremum of log-convex functions
5.4	$\log \mu_p$ is convex in p, W_n is an interval, $\mu_p \le \mu_{p0} R^{p-p0}$ $(p \ge p_0)$. Upper-bound version $S_p \le R^{p-p0} S_{p0}$	True μ_p: Cited + Reading; upper-bound version: Lean (`weighted_rpow_shift`, `sup_shift`)	Riesz–Thorin (convexity in $1/p$) and the perspective
5.5	For a fixed operator, $n_p < \infty \Leftrightarrow \rho(K) < 1$. The blocking map is of subdivision type, and its ℓ^p spectral radius depends on p. The ratio for a constant field is about 2^{4-2p} per step	Cited, Reading, numbers Computed	Gelfand's formula; the L^p theory of subdivision
5.6	The naive dilution $\mu_p \le m^{1-p}$ does not hold (constant field). What holds is $\mu_p \le R^{p-1}C$ (with equality cases), M^{p-1} for δ fields, and $S_p \le S_2 R^{p-2}$	Counterexample Reading; the three forms Lean (`row_rpow_le`, `lp_schur`, `lp_schur_sharp`, `delta_dilution`, `weighted_rpow_shift`); $\mu_p \le \mu_2 R^{p-2}$ Reading	The Schur test; finite sums for Riesz–Thorin between $(1,\infty)$ and $(2,\infty)$
5.7	Affine recursion: constant coefficients, coefficients varying with the step, invariant interval, upper limit, bundles of m steps	Lean (`affine_const`, `affine_varying`, `affine_invariant`, `affine_eventually`, `affine_block`)	An elementary fact about affine recursions (Candidate 1.4 (c) of § 01)

Correspondence with the numbers of the record

One step is the step-0 kernel K_0 on the 8^4 torus (no truncation); two steps is K_0K_1 (coarse 16^4, kernels truncated at $R = 3$, with the ℓ^1 lost being 0.8% for K_0 and 1.9% for K_1); three steps is $K_0K_1K_2$ (coarse 12^4, $R = 3$). The one-step lower bound is the value of the true ratio from a quasi-Newton method (L-BFGS, two starting points); for $p \ge 12$ it was replaced by the value for the sign field of the largest row. The lower bound is not shown to be the maximum. All are floating-point computations, not interval estimates Computed.

Table 12 The weighted Stein bound S_p and the one-step lower bound Computed

p	One step S_p	One step lower bound	One step constant field	Two steps S_p	Three steps S_p
2	1.9806	1.9806	1.080	3.0189	3.6658
2.5	2.3626	1.4106	0.551	2.2358	–
3	2.8641	1.0233	0.2807	1.6642	0.57864
3.5	3.5442	0.7730	0.1431	1.2501	–
4	4.4457	0.6470	0.0730	**0.9707**	0.09599

p	One step S_p	One step lower bound	One step constant field	Two steps S_p	Three steps S_p
5	7.8474	0.7328	0.0190	0.6144	–
6	14.910	1.2326	0.0049	0.4093	–
8	65.160	5.0155	0.0003	0.2197	–
16	3.91e4	18.2 (sign field)		0.0598	–
32	1.71e10	7.96e6 (sign field)		0.0143	–
∞ ($S_p^{1/p} \to R$)	$R = 2.2524$			$R = 0.9283$	$R = 0.3318$

- **Convexity of** $\log S_p$: the second difference quotients were all positive for one step ($p = 2$–64), two steps (2–32) and three steps $(2, 3, 4)$. The three-point ratios $S_p/(S_a^\lambda S_b^{1-\lambda})$ are 0.858–0.99999, all at most 1 (Candidate 5.3) Computed .
- $\sqrt{S_2 S_4}/S_3$: one step 1.036, two steps 1.029, three steps 1.025 ($K_0K_1K_2$) and 1.016 ($K_4K_4K_4$) Computed .
- **Asymptotic slope**: $S_p^{1/p} \to R$, and R falls with the steps: 2.252 for one step, 0.928 for two, 0.332 for three. The slope of $\log S_p$ at large p agreed with $\log R$ (one step: slope 0.8125 for $p = 48 \to 64$ against $\log R = 0.8120$) Computed .
- **Monotonicity of the slope and extrapolation**: the slope of $\log S_p$ for three steps $K_0K_1K_2$ increases from -1.846 for $2 \to 3$ to -1.796 for $3 \to 4$, heading towards the asymptotic slope $\log R = -1.103$. By the upper-bound version of Candidate 5.4, $S_p \le S_4 R^{p-4} = 0.0960 \cdot 0.332^{p-4}$ for $p \ge 4$, so cones of high p can be bounded from above without recomputation (the inequality in Lean, the numbers Computed).
- **Three values of** p **for three steps**: 3.67, 0.58, 0.092 at $p = 2/3/4$ (for $K_0K_1K_2$, the values 3.6658, 0.57864, 0.09599 of Table 12; from $K_1K_2K_3$ to $K_4K_4K_4$, 3.673–3.682, 0.5715–0.5732, 0.0920–0.0922, almost the same regardless of the step). The one-step upper bound grows with p, but for three steps it shrinks with p: in three steps ℓ^2 stretches while ℓ^3 and ℓ^4 contract Computed .
- **Schur is weak for the kernels of the record**: the column ℓ^1 is large, $C = 23.3$ (one step) and 131.1 (two steps), and $R^{p-1}C$ is 52–6856 for one step and 13–122 for two. The two-step p^* is $1 + \log 131.1/\log(1/0.928) \approx 66.5$, while the Stein bound already falls below 1 at $p = 4$ for two steps. The minimal form $S_2 R^{p-2}$ is also 1.065 at $p = 16$ and 0.587 at $p = 24$ for two steps, a much weaker upper bound than S_p itself Computed .
- **Upper bound** $\tilde{n}_p$ **on the bundle number** (from S_p, bundles starting at K_0): 3 at $p = 3$, 2 for $p = 4$–32 (two-step $S_4 = 0.971$), 2 at $p = \infty$ ($R(K_0K_1) = 0.928 < 1$); at $p = 2$ every one up to three steps exceeds 1 Computed .
- **The true one-step** μ_p: from the lower bound, $\mu_p(K_0) \ge 1$ for $p \le 3$ and $p \ge 6$. For $3.5 \le p \le 5$ the lower bound is 0.65–0.77 and the upper bound 3.5–7.8, and **whether the true one-step** μ_p **falls below 1 is undecided** Computed . The lower bound is U-shaped in p, 1.98 → 0.60 ($p = 4$) → 5.0 ($p = 8$), which does not contradict the convexity of the true $\log \mu_p$ (Candidate 5.4). The possibility that the one-step window W_1 is non-empty remains. Its location, however, can be narrowed. From the one-coordinate Hölder inequality, $\mu_p \ge \max_r \|k_r\|_{p'}^{p}$, which is at least 1 for $p \ge 5.874$, and the ratio of an explicit field (the field alternating on the mesh of the lattice) is at least 1 for $p \le 3.039$ and $p \ge 5.665$ Computed (the inequality is Lean: OneStepWindow (holder_dual_ratio, exists_lpRatio_ge_row)). Since the window is an interval (Candidate 5.4; Lean: window_ordConnected_of_logConvex, convexity of

$\log \mu_p$ assumed), W_1, if non-empty, is a single interval inside $(3.04,\ 5.67)$. On the side of the upper bound, the family of row weights u saturates at $u = A^{p-2}$ (Lean: stein_row_weight_saturation), and for the column weights too $b = 1$ is optimal Computed , so weighted interpolation on $(2, \infty)$ tightens no further. The field giving the lower bound is the pattern alternating on the mesh of the lattice ($|q|_\infty = \pi$), and the obstruction at one step is an ultraviolet pattern, not an infrared one Reading .

In the language of windows, the window given by S_p contains $[\approx 4, \infty)$ for two steps and $[3, 4]$ for three steps. For one step the window given by S_p is empty, and the true window is undetermined. Since $\mu_2 = S_2$ is 1.98, 3.02, 3.67 for $n \le 3$, all at least 1, each W_n ($n \le 3$), being an interval, does not straddle 2 Reading .

A question proper to this pattern — the window, the bundle number, the threshold exponent

- **The extent of the window.** For the sequence of blocking maps, how does the window W_n widen with n? By Candidate 5.4 each W_n is an interval; it is open above if $R_n < 1$ and closed above if $R_n > 1$. Where the lower end moves cannot be told from the numbers of the record. If the true values for one step at $3.5 \le p \le 5$ are settled, whether W_1 is empty is settled (question "motion of the lower edge of the window"). If W_1 is non-empty, it is a single interval inside $(3.04, 5.67)$.
- **Divergence of the bundle number as $p \downarrow 2$.** On the side $p < 2$, the ratio for a constant field is about $2^{4-2p} > 1$ per step, so for large n it is expected not to enter the window Assumption . If the limit of $\mu_p(K^{(n)})^{1/n}$ is determined by the dimension count 2^{4-2p} of the constant field, then n_p diverges like $c/(p-2)$. If it is not, the lower end $p_c > 2$ of W_∞ is a critical exponent determined by the geometry of the lattice. Which of these holds has not been checked Assumption (question "divergence of the number of bundles").
- **The meaning of the threshold exponent p^*.** Does the gap between the Schur p^* (about 66.5 for two steps) and the lower end of the Stein window (about 4 for two steps) come from the size of the column ℓ^1 or from the cancellation of signs? How far can p^* be lowered by a weighted Schur test (positive test functions) (question "Schur threshold and Stein window")?

Dictionary

Handwork of the record	Known framework	Label
Interpolation between ℓ^2 and ℓ^∞; convexity of $\log \|K\|_{p\to p}$	The Riesz–Thorin interpolation theorem (M. Riesz 1927, G. O. Thorin 1939): $\log \|K\|_{p\to p}$ is convex in $1/p$	Cited
Interpolation that changes the weight on the range (Definition 5.4)	Stein–Weiss, Interpolation of operators with change of measures (Trans. AMS 87, 1958). If the measures at both ends are fixed, Riesz–Thorin suffices	Cited , bibliography Reading
Upper bound by the row and column ℓ^1 (Candidate 5.6)	The Schur test (I. Schur 1911). The weighted version (positive test functions) is known in the forms of Aronszajn and Halmos–Sunder	Cited , bibliography Reading
Pointwise inequalities of the cone; the midpoint inequality (Candidate 5.3)	Hölder's inequality, Cauchy–Schwarz, the arithmetic–geometric mean inequality	Cited
Convexity of $\log \mu_p$ in p (Candidate 5.4)	Log-convexity of operator norms (Riesz–Thorin) and the perspective function of convex analysis (Boyd–Vandenberghe, Convex Optimization § 3.2.6)	Cited , Reading

Handwork of the record	Known framework	Label
Computing μ_p itself	For $p \neq 1, 2, \infty$, approximating the $p \to p$ norm of a matrix is NP-hard (Hendrickx–Olshevsky, SIAM J. Matrix Anal. Appl. 2010). For lower bounds, Boyd's power method (D. W. Boyd 1974, Linear Algebra Appl. 9). This is the reason for using the upper bound S_p	Reading
Contracting in a bundle (Candidate 5.5)	Gelfand's spectral radius formula. For operators involving dilation there is the L^p theory of subdivision (R.-Q. Jia, Subdivision schemes in L_p spaces, Adv. Comput. Math. 1995), in which convergence is decided by the p-norm joint spectral radius	Cited , Reading

The bibliography is at the level of abstracts and has not been checked against the texts of the original sources. Every candidate theorem of this section is a finite-dimensional form of one of the frameworks above, or an elementary consequence of one.

06

Dictionary with the known, what can be said, questions

Summary of the five grades

Table 13 The main rows of this chapter (labels as in the rows of the text; conditions on the same row)

Section	Claim	Label	Conditions
§ 01	Dominant matrices are closed under products and powers	Lean	Finitely many blocks with values in a normed ring. For powers, the identity has norm 1
§ 01	Candidate 1.1: if it contracts in k steps, there is a weight w' that contracts in one step	Lean	$B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$, $B^k w \leq s^k w$
§ 01	Candidates 1.2–1.5 (one-step estimate, change of weight, finite Gelfand, affine recursion, aggregation of blocks and the two-by-two test)	Lean	Finite-dimensional. The Collatz–Wielandt equality, Gelfand's formula itself and the existence of Perron vectors are not included
§ 01	Candidate 1.2′ (ℓ^2 weight, the Schur test), explicit formula for the transient constant	Reading	Right and left positive Perron vectors exist
§ 01	The loss from discarding signs is not necessarily bounded over steps (nilpotent example)	Lean	The quantity that composes after taking absolute values
§ 01	Perron–Frobenius, Collatz–Wielandt, Gelfand, the max-plus bounds, the characterisation of M-matrices	Cited	—
§ 02	The loss stops from two steps on and falls a little (6.28, 5.51, 5.10, 5.26)	Computed	4^4 torus, steps cut at $R = 3$, the true ratio is a lower bound, $k \leq 4$
§ 02	Candidate 2.1: upper bound $\div$ test value $\leq \kappa^{-2}$	Lean	$\kappa A_r M \leq \lvert y_r \rvert$ for every row, $\lvert v_a \rvert \leq M$
§ 02	Loss $= \langle \kappa^{-2} \rangle$ (5.55, 5.13, 5.31)	Computed	4^4, $k = 2, 3, 4$, shape and phase factors equal to 1
§ 02	Phase factor $\Phi = 1$, the formula for the plane-wave test	Reading	$2q^* \not\equiv 0$ and $4q^* \not\equiv 0$, or period 2 with u^* real
§ 02	Hypotheses (a)(b)(c) fail, and the way they fail is constant over steps	Computed	$k \leq 3$ (larger tori), $k \leq 4$ (4^4)
§ 02	The profile of the composite kernel approaches a single one over steps	Reading	A reading from the numbers

Section	Claim	Label	Conditions
§ 02	Upper bound of the positive-part-dominant type (Schur and degradation under composition)	Lean	Negative $\leq \delta$ positive on every row. The assumption is not satisfied for the kernels of the record
§ 02	The loss for the W coordinates stops at 1.10, close to the rank-one value 1.085	Computed	Kernel with the zero mode corrected, 60 types, $k \leq 5$
§ 03	Existence of extensions, affine structure, dimension of the freedom, extension with conditions, compatibility of composition	Lean	Finite-dimensional linear algebra
§ 03	Candidates 3.1–3.5 (uniqueness of the zero mode, the Moore–Penrose representative is not local, invariance of ord_V, the rule for the order of the tail, shifted jets)	Reading	The lattice steps are lines of argument on paper. Candidate 3.4 assumes constants uniform in the steps
§ 03	ord: sym 0, A3 1, A4 2; $\operatorname{ord}^V(F) = 2$; the tail is bounded for $D \leq 4$	Computed	Representatives of the record, axial input at $D = 4$, $L \leq 192$
§ 03	Correspondence with sum rules, high-frequency order and Strang–Fix	Cited, Reading	Details of how the original sources count orders (± 1) not checked
§ 04	$\kappa = s_1 w_1$ does not depend on the scale, and the lens opens for $\kappa \leq \frac{1}{4}$	Reading	A count with the leading coefficients only
§ 04	Single-plaquette series, ties, error ties, the lens in d dimensions, MK level lines	Computed	Single plaquette, leading terms with equipartition, the approximate MK map
§ 04	Information about the group remains in the height of the crossing and the sign of the next term; overlap of the four-dimensional lens with the band	Assumption	Leading terms only; may include coincidence
§ 04	Candidate B (the comparison that carries the distinction is localised in the lens)	Assumption	Assumes agreement of the strong-coupling valuations, difference of the weak-coupling valuations, and regions where the leading terms dominate. Statement only
§ 05	Candidates 5.1, 5.2, 5.7 (pull-back of cones, submultiplicativity, exceeding in one step and contracting in a bundle, five forms of affine recursion)	Lean	Statements written with an abstract N (in the role of $\|\cdot\|_p^p$). Theorem-specific assumptions such as $\mu \geq 0$, $0 \leq \mu < 1$, $\varepsilon \geq 0$. The supremum of the operator norm itself is not included
§ 05	Candidates 5.3 and 5.6: midpoint convexity of $\log S_p$, Schur's $\mu_p \leq R^{p-1} C$ and its equality, dilution of δ fields, $S_p \leq S_2 R^{p-2}$	Lean	Finite sums. $w \geq 0$, $a > 0$, $a \leq R$, $p \geq 1$. Convexity in continuous p and the formula for S_p itself (interpolation) are not included
§ 05	Derivation of the weighted Stein bound for general p, convexity of the true $\log \mu_p$, the window W_n is an interval, counterexample to the naive dilution (constant field)	Reading, Cited	$p \geq 2$ ($t = 1 - 2/p \geq 0$). Convexity via Riesz–Thorin and the perspective
§ 05	Second difference quotients of $\log S_p$ all positive; $\sqrt{S_2 S_4}/S_3 = 1.016$–$1.036$; $S_p^{1/p} \to R$ (2.252, 0.928, 0.332); upper bounds on the bundle number (3 at $p = 3$; 2 for $p = 4$–32 and ∞); the true one-step μ_p is undecided for $3.5 \leq p \leq 5$ (lower bound 0.65–0.77, upper bound 3.5–7.8); if W_1 is non-empty, a single interval inside $(3.04, 5.67)$	Computed	One step 8^4 without truncation; two steps coarse 16^4; three steps coarse 12^4; truncated at $R = 3$; the lower bound is not shown to be the maximum

Section	Claim	Label	Conditions
§ 05	n_p diverges like $c/(p-2)$ as $p \downarrow 2$, or there is a lower end $p_c > 2$ of W_∞	Assumption	An estimate from the dimension count 2^{4-2p} of the constant field alone

The Lean of this chapter

None of the five files contains sorry or uses `native_decide`, and the axioms printed at the end were the three standard ones (propext, Classical.choice, Quot.sound) or a subset of them. None of the files contains lattices, groups, measures, minimisation or the Fourier transform itself.

Table 14 The Lean files of this chapter Lean

File (namespace)	Number of theorems	Main theorems and assumptions	Place in the text	Not included
DominantMatrixContraction (`DominantMatrix`)	21	`dominant_mul`, `dominant_pow` (the identity has norm 1), `weighted_step`, `weighted_iter` ($Bw \le \lambda w$), `weight_change` ($av \le w \le bv$, $a > 0$), `liftWeight_step` ($B \ge 0$, $w \ge 0$, $s > 0$, $k \ge 1$, $B^k w \le s^k w$), `norm_pow_le_const_mul`, `sum_norm_pow_le`, `affine_uniform` ($0 \le \lambda < 1$, $\varepsilon \ge 0$), `aggregate_step`, `two_by_two_weight`, `two_by_two_exists`, `signedEx_sq`, `absEx_pow`	§ 01	The Collatz–Wielandt equality, Gelfand's formula itself, the existence of Perron vectors, Candidate 1.2′ (ℓ^2 weight), the max-plus bounds
SignLossBound (`SignLoss`)	16	`jensen_four`, `schur_four` (row $\ell^1 \le A$, column $\ell^1 \le B$), `coherent_of_neg_le` (negative $\le \delta$ positive), `loss_le_of_coherent`, `comp_coherent`, `coherence_le_one`, `sum_four_le` ($\lvert v_a\rvert \le M$), `stein_form_le`, `stein_le_inv_sq_mul_test` ($\kappa A_r M \le \lvert y_r\rvert$), `abs_comp_le`, `row_l1_comp_le`	§ 02	The Stein formula itself (Plancherel and interpolation), the average of the plane-wave phase, Wielandt's theorem, convergence to rank one, the numbers of the loss
ConstrainedExtension (`ConstrainedExtension`)	20	`exists_extension`, `isExtension_iff_mem_coset`, `finrank_freedom` (finite-dimensional), `freedom_eq_zero_iff` ($W \neq 0$), `exists_extension_with_conditions` ($(\mathrm{res} V, C)$ surjective), `finrank_freedom_with_conditions`, `eq_of_eqOn_iSup` (the sum of the family of subspaces is the whole), `comp_isExtension` ($f_1(V_1) \subseteq V_2$), `comp_depends_of_not_le`	§ 03	Lattices, continuity of symbols, the order of folding, moments, the lattice steps of Candidates 3.1–3.5

File (namespace)	Number of theorems	Main theorems and assumptions	Place in the text	Not included
ConeCalculus (`ConeCalculus`)	20	Definitions `IsCone`, `Contracts`. `cone_pullback` ($c \geq 0$), `contracts_comp`, `contracts_iterate`, `cone_pullback_iterate` ($\mu \geq 0$), `block_decay` ($M \geq 0$, $\mu \geq 0$), `row_rpow_le`, `lp_schur` ($p \geq 1$, row $\ell^1 \leq R$, column $\ell^1 \leq C$), `lp_schur_sharp` (row sums constant at $R > 0$, column sums constant at C), `delta_dilution`, `weighted_rpow_midpoint` ($w \geq 0$, $a > 0$), `stein_form_midpoint`, `weighted_rpow_shift` ($0 < a \leq R$, $p 0 \leq p$), `sup_shift`, `sup_midpoint`, `midpoint_below_one`, `affine_const` ($0 \leq \mu < 1$, $\varepsilon \geq 0$), `affine_varying`, `affine_invariant`, `affine_eventually`, `affine_block`	§ 05	The operator norm (the supremum of the true ℓ^p norm) itself, the formula for the Stein bound (interpolation and Plancherel), Riesz–Thorin, convexity in continuous p, Gelfand's formula, lattices and kernels, the numbers of the record
OneStepWindow (`LatticeGauge.OneStepWindow`)	20	`holder_dual_ratio`, `norm_conj_rpow`, `exists_lpRatio_ge_row` (the one-coordinate Hölder lower bound, $p > 1$), `lamMax_mono`, `lamMax_smul` ($\lambda_{\max}$ is defined as the supremum of the Rayleigh quotient), `steinWeighted_ge`, `stein_row_weight_saturation` (the family of row weights saturates at $u = A^{p-2}$), `window_ordConnected`, `window_ordConnected_of_logConvex` (convexity of $\log \mu_p$ is assumed)	§ 05	Courant–Fischer, the number $p_{\mathrm{row}} = 5.874$ and the interval $(3.04, 5.67)$ (computed), the operator norm itself

What is not said

(a) **We do not claim to have set up a new theory or new theorems.** The candidate theorems of § 01–§ 03 and § 05 are all finite-dimensional forms of known facts, elementary consequences of them, or restatements in known frameworks (Perron–Frobenius, multigrid orders, sum rules, Riesz–Thorin interpolation, the Schur test). What this chapter adds is the correspondence that reads the numbers of the record as those quantities, and the questions that come out of it.

(b) **What passed in Lean is only the 97 theorems of Table 14.** Candidate 1.2′, the explicit formula for the transient constant, the phase factor, the lattice steps of Candidates 3.1–3.5, all of § 04, the derivation of the Stein bound for general p in § 05, the convexity of the true $\log \mu_p$, and Candidate 5.5 are lines of argument on paper or estimates.

(c) **The numbers are not estimates in the rigorous sense.** The numbers in § 02– § 04 are floating-point computations, not interval estimates. The true ratio is a lower bound from maximisation and is not shown to be the maximum (for one step on 16^4, two starting points stopped at 0.5767, short of the recorded 0.591).

(d) **That the loss is bounded at every step is not shown.** What is shown is only the numbers for $k \leq 4$ (4^4) and $k \leq 3$ (8^4), and the conditional inequality "if the alignment is bounded below, the loss is

bounded". The 4^4 torus wraps the support of the kernel heavily, so "loss $= \langle \kappa^{-2} \rangle$" there is a fact about the operator of that period.

(e) **The tail rule of Candidate 3.4 is a line of argument assuming constants uniform in the steps.** Uniformity over infinitely many steps is not treated. Whether compatibility of composition holds in the chain of the record has not been checked.

(f) **The four-dimensional numbers of § 04 are derived only from first-order leading terms and equipartition.** The true four-dimensional series is not used. The overlaps with the band and the transition may include coincidence. MK is an approximate map, and whether the stopping level lines of $U(1)$ are true fixed lines has not been distinguished. Candidate B is a restatement and does not claim to make the band easier.

(g) **The results of this chapter have no direct effect on ultraviolet control or on the renormalisation comparison.** The table of correspondence with the record in § 01 is a rereading of the record.

(h) **Much of the literature was read at the level of abstracts and excerpts.** For Hemker, Charina–Conti, Jia, Donatelli, Rump, Varga, Bertsekas–Tsitsiklis, arXiv:1005.1993, arXiv:0807.0185, Stein–Weiss, Hendrickx–Olshevsky and Boyd (power method), the details of the definitions in the original sources have not been checked.

(i) **The windows and bundle numbers of § 05 are windows of upper bounds.** The window given by S_p contains $[\approx 4, \infty)$ for two steps and $[3, 4]$ for three steps, but the values of the ends of the window of the true μ_p have not been obtained. The one-step range $3.5 \le p \le 5$ is undecided, and the dependence of the two- and three-step numbers on truncation and on the size of the coarse torus has not been assessed. The behaviour of n_p as $p \downarrow 2$ is an estimate.

Remaining questions

Table 15 The questions of this chapter (questions of the systematic attempts 1–16)

No.	Question	Section
1	Can the lower bound κ_0 of the row alignment in the worst direction be estimated on the limiting kernel of the steps, and the amount by which κ moves under perturbation of the steps be bounded row by row, so as to say loss $\le \kappa_0^{-2}$ at every step?	§ 02
2	If the part by which the maximiser of Stein moves away from period 2 on larger tori (upper bound 10.2 against a loss of 6.4) is measured by splitting into Stein restricted to period-2 points and the other q, where does the difference come from?	§ 02
3	If one quantity measuring the cancellation on the side of the true operator is fixed (for example, the ratio of $\|T^k\|$ and $\||T|^k\|$ divided by its minimum over the sign-similarity class), what can the regularity of the loss growing only slowly over steps be said to come from?	§ 01, § 02
4	Aggregating the finite types and the remaining returns into a two-by-two matrix, does $bc < (1-a)(1-d)$ hold for the matrix of the record? If so, can it be raised to an interval estimate?	§ 01
5	The record for the W coordinates measures in ℓ^2. Reread with Candidate 1.2′ (the weight of the Schur test), what does the one-step contraction constant become?	§ 01
6	The supremum of constrained sum rules: for any $k \le \operatorname{ord}^V(F)$, can a local representative with $\operatorname{ord}(K) \ge k$ be taken with finite range? Is there a general form for cases where ord^V rises with other steps or other blockings?	§ 03
7	A canonical choice of the degrees of freedom: does a representative minimising a Stein-type norm exist, is it unique, and is it local?	§ 03
8	Consistency of composition: does the output of the representative at step k lie in the constraint subspace of step $k+1$ (the image of Bianchi on the fine side)? If not, how much does the freedom outside $\operatorname{im} D$ affect the chain?	§ 03

No.	Question	Section
9	At what rate does the kernel of the Moore–Penrose representative decay (of type $\lvert Z\rvert^{-4}$, from a discontinuous symbol homogeneous of degree 0)?	§ 03
10	Why does the zero mode of the symmetric representative on finite tori appear to approach $U/4$ faster than exponentially?	§ 03
11	In a triple valuation with a third valuation (the exponent of the transseries) added, does the distinction stand that the band of $U(1)$ is a tie between the weak-coupling and exponential valuations, while that of $SU(2)$ is a tie between the strong-coupling and weak-coupling valuations? What does the MK slope 13–14 for $U(1)$ tend to?	§ 04
12	If the error tie is found with the genuine four-dimensional series (higher strong-coupling terms and higher-loop weak-coupling coefficients), how does the lens overlap the band of 26-06?	§ 04
13	In the space of the two couplings, fundamental and adjoint, (β_F, β_A), does the codimension-one skeleton of the tie sets become a non-trivial polyhedral complex? How does it line up with the end point of the line of first-order transitions?	§ 04
14	For the sequence of blocking maps, how does the lower end of the window $W_n = \{p : \mu_p(K^{(n)}) < 1\}$ move with n? Does the true $\mu_p(K_0)$ fall below 1 for one step at $3.5 \le p \le 5$ (decide whether W_1 is empty, either by tightening the upper bound by optimising the Stein weight u beyond A^p, or by raising the lower bound with starting points that mix localised and spread-out fields or with the power method)? Answer (partial): if W_1 is non-empty, a single interval inside $(3.04, 5.67)$. The family of row weights saturates at $u = A^{p-2}$, and weighted Stein tightens no further (Lean: OneStepWindow). The one-step lower bound at $p = 4$ has risen to 0.6470, but the gap to 1 is undecided	§ 05
15	How does the bundle number n_p diverge as $p \downarrow 2$? Is the limit of $\mu_p(K^{(n)})^{1/n}$ determined by the dimension count 2^{4-2p} of the constant field, giving $n_p \approx c/(p-2)$, or is the lower end $p_c > 2$ of W_∞ determined by the geometry of the lattice?	§ 05
16	Does the gap between the Schur threshold exponent p^* (about 66.5 for two steps) and the lower end of the Stein window (about 4 for two steps) come from the size of the column ℓ^1 or from the cancellation of signs? How far can p^* be lowered by a weighted Schur test (positive test functions)?	§ 05, § 01

Sources

Item	Kind	Source
Perron–Frobenius, Collatz–Wielandt, comparison by domination $\rho(A) \le \rho(\lvert A\rvert) \le \rho(B)$	Cited	The chapter on non-negative matrices in R. A. Horn, C. R. Johnson, Matrix Analysis (Reading)
Contraction in weighted maximum norms and iterative methods, M-matrices	Cited (Reading)	R. S. Varga, Matrix Iterative Analysis. A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences. D. P. Bertsekas, J. N. Tsitsiklis, Parallel and Distributed Computation (1989)
Max-plus eigenvalues and bounds	Cited	S. Friedland (1986). F. Baccelli, G. Cohen, G. J. Olsder, J.-P. Quadrat (1992). P. Butkovič (2010). M. Akian, S. Gaubert, A. Marchesini, arXiv:1309.7319 (Reading)
Sign-real spectral radius	Reading	S. M. Rump (1997)
Order of transfer operators	Reading	P. W. Hemker, On the order of prolongations and restrictions in multigrid procedures, J. Comput. Appl. Math. 32 (1990) 423–429. M. Donatelli, arXiv:0807.2565. U. Trottenberg, C. W. Oosterlee, A. Schüller, Multigrid (local Fourier analysis)
Sum rules, Strang–Fix conditions, order of approximation	Cited (Reading)	G. Strang, G. Fix (1973). R.-Q. Jia, Math. Comp. 67 (1998). M. Charina, C. Conti, arXiv:1202.3931. I. Daubechies, Ten Lectures on Wavelets
Fisher zeros and the strong-coupling basin, crossover	Reading (abstracts)	A. Denbleyker, D. Du, Y. Liu, Y. Meurice, H. Zou, arXiv:1005.1993. arXiv:1112.2734. arXiv:0807.0185

Item	Kind	Source
Riesz–Thorin, Stein–Weiss, the Schur test, the difficulty of computing $p \to p$ norms, the L^p theory of subdivision	Cited (Reading)	M. Riesz (1927), G. O. Thorin (1939). E. M. Stein, G. Weiss, Trans. AMS 87 (1958). I. Schur (1911). J. M. Hendrickx, A. Olshevsky, SIAM J. Matrix Anal. Appl. (2010). D. W. Boyd, Linear Algebra Appl. 9 (1974). R.-Q. Jia, Adv. Comput. Math. (1995). S. Boyd, L. Vandenberghe, Convex Optimization § 3.2.6
The numbers of § 02– § 05	Computed	The record of computations of this book's study (cross-check of the direct action of the rows of the composite kernel against the step-by-step action; two routes, quasi-Newton and period-2 fields; the single-plaquette series built in rational numbers; the S_p of § 05 computed directly from the composite kernels)
The theorems of Table 14	Lean	The Lean files DominantMatrixContraction, SignLossBound, ConstrainedExtension, ConeCalculus, OneStepWindow

Words used in this chapter — dominant matrix, weighted block norm, k-step contraction, transient constant, one-step weight, the loss from discarding signs, row alignment, constrained map, extension, local representative, freedom, zero mode, moments, the order of folding $\mathbf{ord}$ and $\mathbf{ord}^V$, shifted jets, double valuation, complementary tie, same-coordinate tie, error tie, lens, $\kappa = s1w1$, cone constant c_p, p-contraction μ_p, weighted Stein bound S_p, bundle number n_p, window W_n, Schur threshold exponent p^*.

Next chapter to read: 15-11 Entrance — one step and one scale.

Entrance — one step and one scale

This chapter is a guide.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

Scope — The theorems of the second half of this part (15-11 to 15-14) are either statements about the free field, the vacuum background and the linearised blocking, or statements about the Wilson measure at one scale (with the lattice spacing fixed). They say nothing about the renormalised coarse-scale measure, fields with a background, or estimates uniform across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

What this chapter shows — What the chapters of the second half of this part (15-12 to 15-14), and the rarity of large plaquettes that moved to Part 22 (22-01), deal with, and the scope that applies to the whole second half. The second half consists of parts, finished down to Lean, for one step of this book's design (a fixed block map that does not depend on a cutoff); they are placed in this part as part of the attempt at ultraviolet control.

Chapters this one relies on — 19-04 The block map and one step of renormalisation (definitions of the objects of the second half), 19-01 (the Wilson measure).

Said first — The scope is as written directly under the title. Every theorem of the second half is a formalisation, for the objects of this book, of a result of a known type, machine-checked in Lean. The chapters of the second half say nothing about the construction of the continuum limit or the mass gap (the statements to be shown are in 19-07).

Order of this chapter

1. The chapters of the second half — one line per chapter
2. The scope that applies to the whole second half — linear, vacuum, one scale
3. Reading order — going back and forth with the chapters on objects

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01

The chapters of the second half

Chapter	One line
15-12 One step and many steps of the free field	The physical part G_k of the symbol recursion obtained by pushing the free field k times with the path kernel: the upper side, the lower side, the sandwich of the propagator, one-step contraction, contraction uniform in the step, and convergence of quadratic functionals of Γ_k.
15-13 Linearisation of the block map and the covariance of the push-forward	Locality and gauge covariance of the non-linear block map, linearisation around the identity, the covariance $CTC^\dagger$ of the push-forward, and the plaquette covariance of the free field pushed by the linearised blocking.
22-01 Rarity of large plaquettes — the chessboard estimate (Part 22)	For the Wilson measure at one scale, the probability of the event that all plaquettes are large is small at a rate independent of the volume ($\beta \geq 81$).
15-14 What the second half (15-11 to 15-13) says, and what it does not	All the theorems of the second half in one table, with their scope, what they do not say, and their Lean declarations.

02

The scope that applies to the whole second half

- **Linear.** 15-12 and 15-13 § 05 are statements about objects in which the blocking is replaced by its first-order part (the path kernel C). They agree with the non-linear block map B only up to the first-order part (15-13 § 03).
- **Vacuum background.** The linearisation is around the identity $U = 1$, and the field that is pushed is the Gaussian free field. Expansion around a background field is not treated.
- **One scale.** 22-01 is a statement about the Wilson measure on the fine lattice itself, and does not reach the blocked measure at a coarse scale.
- **Design choice.** The constants were computed with the weights $(1/8, 3/4, 1/8)$ and the family of paths with factor 2 (19-04; the discussion is in 28-05).

Estimates uniform across steps are not in the chapters of the second half; they are on the side of the premises (28-04).

03

Reading order

15-12 and 15-13 assume the objects of 19-04 (the family of paths, the block map, the path kernel, the multi-step family). 15-13 § 05 uses the upper and lower sides of 15-12. 22-01 assumes only the Wilson measure of 19-01 and the reflection positivity of 20-01, and can be read independently of 15-12 and 15-13.

What this chapter can say

Can say	Cannot say
The list of the theorems of the second half is in 15-14	The measure at a coarse scale, fields with a background, estimates uniform across steps

Sources and reproduction

Item	Kind	Source / tools
The theorems and sources of each chapter	—	The table of sources and reproduction in each chapter

Terms used in this chapter — path kernel, block map, symbol recursion, push-forward, plaquette covariance, chessboard estimate. Their meanings are in A-02 Glossary.

Next chapter to read: 15-12 One step and many steps of the free field — it begins with estimates, uniform in the step, of the propagator of the free field pushed by the path kernel.

One step and many steps of the free field — weights (1/8, 3/4, 1/8), the vacuum background, linearisation

This chapter is this book's own investigation. The first half of § 01 is established knowledge.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

Scope — The theorems of this chapter are statements about the free field, the vacuum background ($U = 1$) and the linearised blocking. Their object is a family of 4×4 matrices over momentum (the symbol recursion); they say nothing about the renormalised coarse-scale measure, fields with a background, or nonlinear blocking. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

What this chapter shows — We read six facts about the physical part $G_k(q)$ of the symbol recursion obtained by applying the block average of the path kernel C k times to the free field in Feynman gauge: the upper side $G_k \preccurlyeq P/\omega$ (with constant exactly 1), the lower side $\frac{3}{100}P/\omega \preccurlyeq G_k$, a sandwich of the propagator uniform in the step, the one-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$, a contraction uniform in the step with $\tau < 1/170$ over 17 steps, and convergence of quadratic functionals of Γ_k at rate $(3/5)^k$. Every one of them is a statement machine-checked in Lean.

Chapters assumed — 19-04 The path kernel, 19-04 The multi-step family, The weights are a design choice, 7-05 § The free field.

Said first — The scope is as written directly under the title. There are four further points. (1) The momenta are the continuous momenta of infinite volume $q \in \mathbb{R}^4$ ($\omega(q) > 0$); the momenta of a finite torus are a subset of these. (2) That G_m is the physical part of the symbol of the stage covariance in position space (the operator obtained by

pushing the free field m times) is in Lean, but the inequalities of § 02– § 07 are inequalities between symbols (matrices for each momentum), and are not stated as inequalities between operators in position space. (3) The constants are 3 to 186 times looser than the true values Reading , and the constant in the convergence of Γ_k is $\kappa^2 \approx 1.4 \cdot 10^{13}$ Reading . (4) We do not say that “the one-loop coupling converges”. What is shown goes only as far as the convergence of quadratic functionals of Γ_k.

The order of this chapter

1. The object — the symbol recursion and its physical part G_k — the symbol of the path kernel, aliases, $\mathcal{M}$, $G_k = \mathcal{M}^k(P/\omega)$
2. The upper side — $G_k \preccurlyeq P/\omega$ — the constant is exactly 1, from a one-step inequality and preservation of order
3. Detour: the lemma $E_\nu < 4$ inside the upper side — an upper bound for an explicit function, and invariance under reflection
4. The lower side — $\frac{3}{100} P/\omega \preccurlyeq G_k$ — the chain of nearest aliases, and a case split with the sixteen aliases
5. A sandwich of the propagator uniform in the step — the inverse H_k on $\operatorname{ran} P$, and $(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$
6. One-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$ — the true supremum is $125/256$ Reading
7. Contraction uniform in the step — $\tau < 1/170$ over 17 steps — splitting into the nearest-alias term and the rest
8. Convergence of quadratic functionals of Γ_k — the form of the fluctuation on the sum of aliases, rate $(3/5)^k$
9. What is not said — identification with the stage action of the lattice, position space, looseness of the constants

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01

The object — the symbol recursion and its physical part G_k

The introduction is 7-05 § The free field. When the free field is pushed by the linear map of the blocking, the pushed field is again Gaussian, and its covariance is $CTC^\dagger$ Cited . A translation-invariant operator can be written as a matrix for each momentum (its symbol), so stacking steps becomes a recursion of matrices over momenta. In renormalisation group analysis, estimates of the stage propagators of this free field are one of the starting points (Bałaban's series of papers, 7-07) Cited .

The object of this chapter is the one set up in 19-04 § The multi-step family on the free-field side. What 19-04 set up with torus momenta is considered here for all $q \in \mathbb{R}^4$ with $\omega > 0$. We repeat the notation. For a momentum $q \in \mathbb{R}^4$,

$$\omega(q) = \sum_{\nu} 4 \sin^2 \frac{q_\nu}{2}, \qquad s_\nu(q) = \sin \frac{q_\nu}{2}, \qquad P(q) = 1 - \frac{s(q) s(q)^{\mathsf{T}}}{|s(q)|^2}. \tag{1}$$

$P(q)$ is a projection of rank 3, the projection onto the "physical part" with the gauge direction removed. The fine momenta that fall onto a coarse momentum q (the aliases) are the 16 momenta $r_\zeta(q) = q/2 + \pi\zeta$, $\zeta \in \{0,1\}^4$. The one-step map is

$$(\mathcal{M}G)(q) = \frac{1}{16} \sum_{\zeta \in \{0,1\}^4} A_\zeta(q)\, G\big(r_\zeta(q)\big)\, A_\zeta(q)^{\mathsf{T}}, \qquad A_\zeta = 2P(q)\, S_\zeta C(r_\zeta)\, P(r_\zeta)\, N(r_\zeta), \tag{2}$$

where S_ζ is a diagonal of signs, $C(r)$ is a diagonal of cos, and N is an explicit 4×4 matrix coming from the average over the family of paths (the matrix that appears in the symbol of the coefficients of 19-04 (9)). The family at step k is

$$G_0 = P/\omega, \qquad G_k = \mathcal{M}^k(P/\omega) \qquad (\omega(q) > 0). \tag{3}$$

We say precisely what G_k is. Let $Q(q)$ be the symbol of the path kernel, and write $\mathcal{R}$ for the recursion from the symbol X at the fine step to the symbol at the coarse step. Starting from $X_0 = I/\omega$ (the symbol of the free propagator in Feynman gauge), $\mathcal{R}^k X_0$ is determined, and its physical part is G_k.

$$P\,(\mathcal{R}^k X_0)\,P = \mathcal{M}^k(P\,X_0\,P) = G_k. \tag{4}$$

Lean: (4) is `LatticeRec.rec_iterate` (for any X_0) and `LatticeRec.projFam_X0`; the closed form of the symbol of the average over the family of paths is `LatticePath.pathSym_closed`; that the symbol of $CTC^\dagger$ is $\mathcal{R}X$ when the symbol of the one-step torus operator T is X is `LatticeTorus.torus_step` ($(\mathbb{Z}/2L)^4 \to (\mathbb{Z}/L)^4$). That the symbol of the lattice stage covariance obtained by pushing the free-field covariance m times is $\mathcal{R}^m X_0$, and that its physical part is G_m, are `LatticePush.push_phase`, `LatticePush.stageCov_phase` and `LatticePush.projFam_stage`.

In this chapter we call G_k "**the physical part of the symbol recursion**". Every statement is a Loewner inequality for a concrete family of 4×4 matrices, for all $q \in \mathbb{R}^4$ with $\omega(q) > 0$ ($A \preccurlyeq B$ means $\langle x, Ax\rangle \le \langle x, Bx\rangle$ for all x). None is a statement about abstract operators. Lean holds the momentum as $\theta = q/4$; the domain is `OneStepFinal.Qpos`, $\mathcal{M}$ is `OneStepBridge.Mfam`, and the alias map is `OneStepFinal.rqPos`. The weights $(1/8, 3/4, 1/8)$ enter through N and A_ζ, and every

constant that appears here was computed with this choice (19-04 § The weights; the discussion is in 28-05).

02

The upper side — $G_k \preccurlyeq P/\omega$

Upper-side theorem Lean: `OSUpQpos.os_up_Qpos`, `OSUpQpos.Gs_le_G0` (no assumption fields). For all q with $\omega(q) > 0$,

$$\mathcal{M}(P/\omega) \preccurlyeq P/\omega, \qquad \text{and hence for every step } k \quad G_k \preccurlyeq P/\omega. \tag{5}$$

Meaning. However many steps are stacked, the physical part of the pushed propagator does not exceed the starting P/ω. The constant is exactly 1. The true supremum is $1 - 3.4 \cdot 10^{-6}$ Reading (the largest eigenvalue of ωG_1 at $k = 1$, floating point at $2.4 \cdot 10^4$ points), so the constant 1 is sharp.

The route of the proof has three stages.

1. **Reduce the one-step inequality to an upper bound for an explicit function.** Opening $\langle x, \mathcal{M}(P/\omega)x\rangle$ into the sum over the 16 aliases, $\mathcal{M}(P/\omega) \preccurlyeq P/\omega$ follows from the inequality for explicit functions "$E_\nu(\sigma) \le 4$ for all ν". Here E_ν is an explicit rational expression in $\sigma_\mu = \sin^2(q_\mu/4)$ (a variable different from the s of (1)) (`OSUpBridge.os_up`; the definition is in the detour).
2. **$E_\nu \le 4$ at every point.** On $[0, \frac{1}{2}]^4$, $E_\nu < 4$ ($\sigma \neq 0$) is a lemma. A general point is moved into $[0, \frac{1}{2}]^4$ by the coordinate-wise reflection $\sigma_\mu \mapsto 1 - \sigma_\mu$, using that E_ν does not change under this reflection (`OSUpQpos.hEE_all`).
3. **Stack the steps.** $\mathcal{M}$ is a linear map that preserves order. Applying $\mathcal{M}^k$ to $\mathcal{M}G_0 \preccurlyeq G_0$ gives $G_{k+1} \preccurlyeq G_k$, and by transitivity $G_k \preccurlyeq G_0$ Proved here.

The sequence of steps decreases monotonically: $G_0 \succcurlyeq G_1 \succcurlyeq G_2 \succcurlyeq \cdots$. This is the input to § 06 and § 07.

Detour: the lemma $E_\nu < 4$ inside the upper side

You may skip this. For $\sigma \in [0,1]^4$ ($\sigma_\mu = \sin^2(q_\mu/4)$) and $e \in \{0,1\}^4$, set $b_{e,\mu} = 1 - \sigma_\mu$ ($e_\mu = 1$) or σ_μ ($e_\mu = 0$), $H = 4\sum_\mu \sigma_\mu(1-\sigma_\mu)$, $X_e = \sum_\mu b_{e,\mu}$, and

$$E_\nu(\sigma) = \sum_{e \in \{0,1\}^4} \frac{H}{X_e}\, \kappa_e^+ \,(1 - b_{e,\nu})$$

Here κ_e^+ is the square of the larger of two values of explicit rational expressions in X_e, $\sum b^2$ and $\max b$. Lean: `FreeStageOS.E_lt_four_s` ($E_\nu(\sigma) < 4$ on $\sigma \in [0, \frac{1}{2}]^4 \setminus \{0\}$), the momentum form `FreeStageOS.E_lt_four`, and the inequality for rational expressions in two variables `FreeStageOS.eq_lt_four`. The bound 4 is sharp: as $\sigma \to 0$, $E_\nu \to 4$ (for example, 3.999995 at $\sigma = (10^{-6}, 0, 0, 0)$ Reading). The invariance under reflection comes from $e \mapsto e \oplus f$ being a bijection of $\{0,1\}^4$, and from each term $b(1-b)$ of H not changing under $b \mapsto 1-b$ (`OSUpQpos.EE_bbflip`). This lemma is an upper bound for a scalar function, not a theorem about the lattice objects themselves. It is placed here as a part of the upper-side theorem.

03

The lower side — $\frac{3}{100} P/\omega \preccurlyeq G_k$

Lower-side theorem Lean: `HsSplit.lower_side_c`. For any family closed under aliases ($\omega > 0$), and in particular for all q with $\omega(q) > 0$, at every step k,

$$\frac{3}{100}\,\frac{P}{\omega} \preccurlyeq G_k. \quad (6)$$

The assumption fields in Lean are only two properties of the family (that the alias map is $r\zeta$, and that $\omega > 0$); on the whole of momentum space (`Qpos`, `rqPos`) both are discharged by definition. The earlier version `OneStepLower.lower_side` has the same form with constant $(4/\pi^2)(0.04452)^2 \approx 8.03 \cdot 10^{-4}$ Reading .

Meaning. However many steps are stacked, the physical part does not collapse. Together with the upper side, the eigenvalues of ωG_k on $\operatorname{ran} P$ lie in $[3/100,\ 1]$ at every step. The true infimum goes down with the step: for $k = 0, 1, \dots, 5$ it is $1,\ 0.210,\ 0.132,\ 0.110,\ 0.0997,\ 0.0951$ Reading , and the estimate for the limit is about 0.091 Reading . At $k = 4, 5$, the minimum is attained at the corner $q = \pi(1,1,1,1)$ Reading . The constant $3/100$ is about $1/3$ of the true value.

The route of the proof.

1. **The chain of nearest aliases.** Keeping only the alias $\zeta^*(q)$ nearest to the origin among the 16, $\mathcal{M}$ is bounded from below by a single term. `NearAliasB.chain_lower` gives the explicit lower bound obtained by following this for d steps.
2. **The sixteen aliases.** `NearAliasB.near_lower` gives a lower bound in one step keeping all sixteen aliases.
3. **Case split.** At every point with $\omega > 0$, either the chain lower bound (2 steps) or the sixteen-alias lower bound (5 steps) is at least $3/100$ (`HsSplit.hS_split`, no assumptions). This is checked with a certificate for polynomial inequalities that

divides the box of variables into cells Computer-assisted . The certificate is checked by the Lean kernel and supports an analytic proof.

4. **Pass to the steps.** Applying the case-split lower bound step by step gives (6) (`NearAliasB.lower_side_split`).

04

A sandwich of the propagator uniform in the step

$G_k(q)$ is positive definite on $\operatorname{ran} P(q)$, so it has an inverse there. We call that inverse $H_k(q)$ ($P H_k = H_k$, $G_k H_k = P$) the inverse of the propagator, and $\frac{1}{4} H_k$ its quarter. H_k is uniquely determined by G_k Proved here .

Propagator sandwich theorem Lean: `LatticeRec.lattice_bracket` (in the form of the physical part of the symbol recursion $\mathcal{R}^k(I/\omega)$), `OneStepLower.bracket_Qpos_free` (in the form $\mathcal{M}^k(P/\omega)$). No assumption fields. At every step k and for all q with $\omega(q) > 0$,

$$\frac{\omega(q)}{745}\, P(q) \;\preccurlyeq\; \tfrac{1}{4} H_k(q) \;\preccurlyeq\; 312\, \omega(q)\, P(q). \quad (7)$$

In the version with an improved upper constant, `NearAliasB.bracket_Qpos_split`, the right-hand side is $\omega/(4 \cdot \frac{3}{100})\, P = \frac{25}{3}\, \omega P$ (from the lower side of § 03).

Meaning. The inverse of the propagator at step k is sandwiched around ωP by two constants that do not depend on the step. ωP is the symbol of the physical part of the action of the free field at the fine step. In other words, however many steps are stacked, the quadratic form of the pushed free field stays of the same order of size as the form of the original free field (reading $\frac{1}{4} H_k$ as the action of the pushed field is a reading, and that identification is on paper; § 08). The constants are uniform in k and q. The volume does not appear in the statement.

The route of the proof. The right side of (7) is the inverse of the lower side $c P/\omega \preccurlyeq G_k$ (on $\operatorname{ran} P$, taking inverses reverses the Loewner order) Proved here . If $c \geq 1/1248$ the constant is 312; if $c = 3/100$ it is $25/3$ (the lower-side constant of the earlier version is about $8.03 \cdot 10^{-4} \geq 1/1248$). The 745 on the left is the inverse of $G_k \preccurlyeq 186.25\, P/\omega$, a weaker form than the upper side of § 02 (constant 1). Taking the inverse of the upper side would make the left constant $1/4$ Proved here , but there is no Lean declaration in that form.

Numerically, the eigenvalues of ωG_k lie roughly in $[0.091,\ 1]$ Reading , and the constants of (6), (7) and the split version are from 3 times ($3/100$) to 186 times (the 745 on the left)

looser than the true values Reading .

05

One-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$

One-step contraction theorem Lean: `OneStepBridge.Mfam_P_le_row`. For any family closed under aliases (on the whole of momentum space the assumptions are discharged by definition),

$$\mathcal{M}P \preccurlyeq \tfrac{3}{5}\,P. \quad (8)$$

Meaning. Pushing the family of projections P itself (the family not multiplied by $1/\omega$) by one step shrinks it by a factor of at most $3/5$. Whereas P/ω is large at low momenta and is preserved by $\mathcal{M}$ (§ 02), P carries no weight ω, and so it shrinks under the average over aliases. In § 06 this contraction plays the role of making "the contribution of high-momentum aliases" smaller step by step.

The true supremum is $125/256 \approx 0.488$ Reading (as $q \to 0$; $2 \cdot 10^5$ points and local optimisation), and $3/5$ is 1.23 times that. The earlier version `OneStepBridge.Mfam_P_le` had $429/512$. The proof bounds $\langle x, \mathcal{M}Px\rangle$ by splitting it row by row.

Differences between steps also shrink at the same rate. Applying $\mathcal{M}^k$ to $-2P \preccurlyeq \mathcal{M}(P/\omega) - P/\omega \preccurlyeq 2P$ (`CouplingLimit.D_lower`, `CouplingLimit.D_upper`) puts $G_{k+1} - G_k$ between $\pm 2(3/5)^k P$ Proved here . Numerically the values of $\mathcal{M}(P/\omega) - P/\omega$ lie in $[-0.089, -0.024]P$ Reading , so ± 2 is about 22 times looser.

06

Contraction uniform in the step — $\tau < 1/170$ over 17 steps

Theorem of contraction uniform in the step Lean: `TorusLam.stage_tau_17` (no assumption fields). Set

$$\tau = \frac{\pi^2}{4}\left(\frac{1}{4}\right)^{17} + \frac{100}{3}\left(\frac{3}{5}\right)^{17} < \frac{1}{170} \quad (9)$$

Then for every step k, all q with $\omega(q) > 0$, and all x,

$$\big\langle x,\ \mathcal{M}^{17}(\omega\, G_k)(q)\, x\big\rangle \ \le\ \tau\,\omega(q)\,\big\langle x,\ G_{k+17}(q)\, x\big\rangle. \quad (10)$$

Here ωG_k is the family $q \mapsto \omega(q)G_k(q)$. The form (9) is exactly the definition of `StageArith.tauM` in Lean, and the comparison with $1/170$ is also in Lean.

Meaning. Since $G_{k+17} = \mathcal{M}^{17} G_k$, (10) says that "multiplying by ω on the fine side and then pushing 17 steps" is at most $1/170$ times "pushing 17 steps and then multiplying by the coarse ω". At low momenta, the ω of a fine momentum is about 4^{-17} times the ω of the coarse momentum. The contribution from high-momentum aliases is multiplied by $(3/5)^{17}$ through the contraction of § 05. The two terms of (9) correspond to these two contributions.

The route of the proof.

1. **Split.** $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$, where $\mathcal{M}_0$ is the single term of the nearest alias $\zeta^*(q)$ and $\mathcal{M}_1$ is the remaining fifteen terms. Both preserve order. If $0 \preccurlyeq F \preccurlyeq P'$ then $\mathcal{M}^m F \preccurlyeq \mathcal{M}_0^m F + \mathcal{M}^m P'$ (induction, `TauSplit.iterate_split`).
2. **Insert.** $F = \omega G_k$, $P' = P$. $0 \preccurlyeq F$ is $G_k \succcurlyeq 0$, and $F \preccurlyeq P$ is the upper side of § 02.
3. **The nearest-alias term.** $\mathcal{M}_0$ carries the weight along a chain determined by the point, so ω decreases along the chain as $\omega(\text{the } m\text{-th step of the chain}) \le (\pi^2/4) 4^{-m}\, \omega(q)$ (the finite Viète product $\prod_{j<m} \cos(\pi/2^{j+2}) \ge 2/\pi$, `TauSplit.omF_chain_le`).
4. **The remaining terms.** $\mathcal{M}^m P \preccurlyeq (3/5)^m P$ (iterating § 05), which is returned to G_{k+m} by the lower side of § 03, $P \preccurlyeq (100/3)\, \omega G_{k+m}$.

The value of the one-step contraction at step 0 (in the form $\tau \le 31/32$) is not written as a theorem in this chapter. What Lean has goes only as far as the scalar parts, and the stage of assembling the actual alias contributions into that form is on paper. The theorem about concrete objects for contraction uniform in the step is (10).

07

Convergence of quadratic functionals of Γ_k

One step of renormalisation needs the covariance of the fluctuation of the fine field with the coarse field held fixed Cited . Here, as a candidate for its symbol form, we build a matrix Γ_k on the space that views the 16 aliases together. Identifying Γ_k with the covariance of the lattice fluctuation is on paper.

Setting. $\mathcal{B} = \bigoplus_{\zeta \in \{0,1\}^4} \mathbb{R}^4$ (the sum of aliases, 64-dimensional). The covariance on the sum of aliases at step k is $\mathbf{A}_k(q) = \bigoplus_\zeta G_k(r_\zeta(q))$ (block diagonal). The constraint map is $\mathbf{C}(q)u = \frac{1}{4}\sum_\zeta A_\zeta(q) u_\zeta$ ($\mathcal{B} \to \mathbb{R}^4$). A lift E ($\mathbb{R}^4 \to \mathcal{B}$) is required, with $\Pi = E\mathbf{C}$, to satisfy $\mathbf{C}\Pi = \mathbf{C}$ and $\Pi \mathbf{A}_k = \mathbf{A}_k \Pi^\dagger$. Then

$$\Gamma_k(q) = \mathbf{A}_k(q) - \Pi\, \mathbf{A}_k(q)\, \Pi^\dagger. \quad (11)$$

Γ_k does not depend on the choice of a lift satisfying the conditions. For two choices Π, Π' one gets $\Pi^2 = \Pi$, $\Pi\Pi' = \Pi$, $\Pi'\Pi = \Pi'$, and so $\Pi\mathbf{A}\Pi^\dagger = \Pi\mathbf{A} = \Pi'\mathbf{A} = \Pi'\mathbf{A}\Pi'^\dagger$ Proved here .

Convergence theorem for Γ_k Lean: `AliasPoinc.coupling_converges_gamma_closed` (no assumption fields; parts `AliasLiftSize.hlift_holds`, `AliasPoinc.hH_holds`). Let Ξ be a weighted sum of finitely many values of quadratic forms and products of two values of quadratic forms:

$$\Xi(\Gamma) = a + \sum_i w_i \langle x_i, \Gamma(p_i) x_i\rangle + \sum_j w'_j \langle v_j, \Gamma(q_j) u_j\rangle \langle u'_j, \Gamma(q'_j) v'_j\rangle. \quad (12)$$

The weights, vectors and momenta are arbitrary. Then at every step k a lift satisfying the conditions exists, and there is a limit b_∞ with

$$|\Xi(\Gamma_k) - b_\infty| \le \frac{2L'}{2/5}\Big(\frac{3}{5}\Big)^k, \qquad \sum_{k<n} \max\big(b_\infty - \Xi(\Gamma_k), 0\big) \le \frac{2L'}{(2/5)^2}. \quad (13)$$

Here $L' = \kappa^2\big(\sum_i |w_i|\,|x_i|^2 + \gamma \sum_j |w'_j|(\cdots)/2\big)$, $\kappa = 1 + 2\sqrt{4 + 16\cdot 464880^2}$, $\gamma = (1 + 69120/121)/(8/745)$, and none of them depends on the step or the momentum.

Meaning and strength. The rate $(3/5)^k$ comes from the contraction of § 05 (the numerically observed ratio of successive step differences of G_k is 0.40–0.42 Reading , inside it). The constants are uniform, but $\kappa^2 \approx 1.4\cdot 10^{13}$ Reading and $\gamma \approx 5.3\cdot 10^4$ Reading are extremely far from the true values.

A note on reading. The stage coefficient b_k of the one-loop coupling is the second-order coefficient with respect to the background, projected onto the type of the local action. That it can be written as $\Xi(\Gamma_k)$ of the form (12) is on paper, and is not in Lean. So this theorem is not a proof that "the one-loop coupling b_k converges"; it is that "any finite quadratic functional of Γ_k converges geometrically". When Ξ is taken in the form of the one-loop vertex, its limit is expected to be the universal one-loop value (Cited , 7-07), but that identification is on paper, and Lean does not state the value of the limit.

08 What is not said

The theorems of § 02– § 07 are all statements about the physical part G_k of the symbol recursion and about Γ_k built from it. The following are not said.

- **Identification with the stage action of the lattice.** Reading $\frac{1}{4}H_k$ as "the action h_k at step k" (the minimum of the action of the fine free field with the coarse field held

fixed, a Schur complement) is on paper. The Lean objects go only as far as the inverse of G_k on $\operatorname{ran} P$.

- **Operators in position space.** Three things are connected to position space in Lean. One step: if the symbol of a torus operator T is X, the symbol of $CTC^\dagger$ is $\mathcal{R}X$ (`LatticeTorus.torus_step`). The chain of steps and step 0: the operator on $(\mathbb{Z}/L)^4$ obtained by pushing m times the free-field covariance on $(\mathbb{Z}/2^m L)^4$ (with the constant component removed) has, at torus momenta, the symbol with phases $\mathcal{R}^m X_0$ (`LatticePush.push_phase`, `LatticePush.freeCov_phase`, `LatticePush.stageCov_phase`), and its physical part is G_m (`LatticePush.projFam_stage`). The covariance of the plaquette field (`LatticeFace.faceCov_eq_posC`, 15-13). There is no theorem stating the inequalities of § 02– § 07 themselves as inequalities between operators in position space.
- **Finite volume.** The momenta are the continuous momenta of infinite volume $q \in \mathbb{R}^4$ ($\omega > 0$); the momenta of a finite torus $q = 2\pi n/N$ ($n \not\equiv 0 \bmod N$) are a subset of these. The zero momentum is not treated.
- **Complex form.** The statements are in the real form with the phases removed. The identity of the symbols with phases is in Lean (`LatticePush.stageCov_phase`), but there is no theorem stating the inequalities of § 02– § 06 in complex form.
- **Constants.** They are 3 to 186 times looser than the true values Reading , and the constants for Γ_k are of order 10^{13} Reading .
- **Non-linear, background, measure at a coarse scale.** Only the free field pushed by the linearised blocking (the path kernel C) is treated. This chapter says nothing about the non-linear block map B, background fields $U \neq 1$, or the measure at a coarse scale obtained by pushing forward the Wilson measure.
- **Choice of weights.** The constants were computed with the weights $(1/8, 3/4, 1/8)$ and factor 2, and cannot be used as they are for another choice (28-05).

Each theorem is a formalisation, for these weights, of a result of a type known in renormalisation group analysis: estimates, uniform in the step, of the propagator when the free field is blocked Cited . Among the estimates across steps, those needed for non-linear blocking and for fields with a background are on the side of the premises (28-04).

What this chapter can say

Can say	Cannot say
Upper side: for all q with $\omega > 0$ and at every step, $G_k \preccurlyeq P/\omega$ (constant exactly 1). Lean: `OSUpQpos.os_up_Qpos`, `OSUpQpos.Gs_le_G0`	Non-linear blocking, background fields. The form as operators in position space
Lower side: at every step, $\frac{3}{100} P/\omega \preccurlyeq G_k$. Lean: `HsSplit.lower_side_c`	A constant close to the true infimum (about 0.091 Reading)
Sandwich: at every step and for all q, $(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$ (in the split version the upper side is $\frac{25}{3}\omega P$). Lean: `LatticeRec.lattice_bracket`, `OneStepLower.bracket_Qpos_free`, `NearAliasB.bracket_Qpos_split`	Reading $\frac{1}{4} H_k$ as the stage action h_k (on paper). The complex form of the inequalities. Inequalities between operators in position space
One-step contraction $\mathcal{M} P \preccurlyeq \frac{3}{5} P$. Lean: `OneStepBridge.Mfam_P_le_row`	A Lean form of the true supremum $125/256$ Reading
Contraction uniform in the step: $\tau < 1/170$ over 17 steps. Lean: `TorusLam.stage_tau_17`	$\tau \le 31/32$ at step 0 (the assembly is on paper). Identification with the per-step contraction rate used in the argument on paper (not in Lean)
Any finite quadratic functional of Γ_k converges at rate $(3/5)^k$, with constants uniform in the step and the momentum. Lean: `AliasPoinc.coupling_converges_gamma_closed`	Convergence of the one-loop coupling b_k (the identification $b_k = \Xi(\Gamma_k)$ is on paper). The size of the constants ($\kappa^2 \approx 10^{13}$ Reading)
Proved here Γ_k does not depend on the choice of a lift satisfying the conditions (the three lines of algebra in § 07)	—

Sources and reproduction

Item	Kind	Source / tools
That the push-forward of the free field is Gaussian with covariance $CTC^\dagger$, and that estimates of the stage propagators are a starting point of the renormalisation group	Cited	The sources of 7-05 and 7-07 (T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275. J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335)
The universal one-loop value (what § 07 compares with)	Cited	The sources of 7-07 (b_0, Particle Data Group)
The theorems of § 02– § 07	—	Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, checked by a separate re-run). The list is in 15-14

Item	Kind	Source / tools
The case split of the lower side (§ 03)	Computer-assisted	A certificate for polynomial inequalities dividing the box of momenta into cells, checked by the Lean kernel (`HsSplit.hS_split`). Supports an analytic proof
Ranges of eigenvalues ($1 - 3.4 \cdot 10^{-6}$, 0.210–0.0951, about 0.091), $125/256$, $[-0.089, -0.024]$, the ratio of step differences 0.40–0.42, the values of E_ν, the values of κ^2 and γ	Reading	The numerical record of this book's study (the recursion transcribed directly from the Lean definitions, in floating point, at 292 points: random points, points near the origin, edges, and lattice momenta with $N = 2, 4$; $\mathcal{M}(P/\omega)$ at $2.4 \cdot 10^4$ points, $\mathcal{M}P$ at $2 \cdot 10^5$ points)

Terms used in this chapter — symbol recursion, physical part, alias, Loewner order, inverse of the propagator, one-step contraction, contraction uniform in the step, sum of aliases, lift, Γ_k. Their meanings are in A-02 Glossary.

Next chapter to read: 15-13 Linearisation of the block map and the covariance of the push-forward — the position-space side: that the first-order part of the non-linear block map is the path kernel, the covariance of the push-forward, and the covariance of the plaquette field.

2026-11-13 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control Linearisation and push-forward · Kiichi, Shiori and Rin

Linearisation of the block map and the covariance of the push-forward

This chapter is this book's own investigation. The first half of § 01 is established knowledge.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

Scope — The theorems of this chapter are statements about local properties of the nonlinear block map at one scale (locality and gauge covariance), its linearisation around the identity, the covariance identity for the push-forward of an arbitrary measure, and the covariance of the plaquette field obtained by pushing the free field forward through the linearised blocking. They say nothing about the renormalised coarse-scale measure, fields with a background, or nonlinear estimates across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

What this chapter shows — We read four facts about the nonlinear block map B defined in 19-04. (1) The value of a coarse link is determined by the fine links in a cube of radius 2 alone, and is compatible with gauge transformations. (2) When $U = e^{\varepsilon a}$, $B(U)$ lies within $257(\varepsilon M)^2$ of $\exp(\varepsilon C a)$, and its first-order part is the path kernel C. (3) The covariance of a measure pushed forward by a linear map is $CTC^{\dagger}$ (for any measure). (4) The covariance of the field strength of the plaquettes of the free field pushed forward m times by the path kernel has largest eigenvalue ≤ 1, diagonal $\geq \frac{3}{200}(1 - N^{-4})$ and ratio $\Lambda \leq 640/9$, for every step m and every $N \geq 2$.

Chapters this one relies on — 19-04 the family of paths, the block map, the path kernel, the push-forward, the multi-step family; 15-12 (the lower side and the upper side).

Said first — The scope is as written directly under the title. In particular, the plaquette covariance of § 05 is the covariance of the plaquette field strength of **the free field pushed by the linearised blocking (the path kernel C)**. It is not the covariance of the field pushed by the non-linear block map B. The two agree only up to the first-order part of B (§ 03); higher-order terms and large fields are not treated. The linearisation is only around the identity $U = 1$.

The order of this chapter

1. The position-space side — the symbol recursion of 15-12, and the position-space objects of this chapter
2. Locality and gauge covariance — a cube of radius 2, $g(2y)\, B(U)_b\, g(2(y + e_\mu))^*$
3. Linearisation around the identity — an error of $257(\varepsilon M)^2$, and the first-order part being the path kernel
4. The covariance of the push-forward $= CTC^\dagger$ — an identity for any measure
5. The covariance of the plaquette field strength — self-adjoint, $\lambda_{\max} \le 1$, a lower bound on the diagonal, $\Lambda \le 640/9$
6. The route of the proof for the plaquette covariance — from position space to symbols, and from symbols to the two sides of 15-12
7. Detour: the $\frac{1}{2}$ and the $1 - N^{-4}$ in the lower bound on the diagonal — symmetry of directions and the zero momentum
8. What is not said — the non-linear covariance, backgrounds, large fields, the measure at a coarse scale

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01

The position-space side

The introduction is 7-05 § One step. Blocking of a gauge field builds a coarse link by averaging products of fine links along paths Cited . Products along paths are used to keep the gauge structure, and the amount by which the average leaves the group is brought back into the group by some rule. Three properties are commonly asked of a good blocking: that it is local, that it is compatible with gauge transformations, and that near the identity it is close to a linear average Cited .

15-12 dealt with objects on the momentum side (the symbol recursion). This chapter is on the position-space side. The objects are the non-linear block map of 19-04,

$$B(U)_b = \rho_{r_0}\big(q_b(U)\big), \qquad q_b(U) = \sum_{o,A} w(o,A)\, U\big(\gamma_{o,A}(b)\big), \qquad \rho_{r_0}(q) = \frac{q}{\max(|q|, r_0)} \tag{1}$$

its first-order part, the path kernel C, and the push-forwards $B_*\mu$ and $C_*\mu$. The fine lattice is $\Lambda_{m+1} = (\mathbb{Z}/2^{m+1}L)^4$, the coarse lattice is $\Lambda_m = (\mathbb{Z}/2^m L)^4$, and a coarse link $b = (y, \mu)$ starts from the fine point $2y$. Configurations take values in the closed unit ball $\mathbb{B}$ of the quaternions.

§ 02 and § 03 are statements about the non-linear B itself, § 04 is a general identity for push-forward by a linear map, and § 05 is a statement about the free field pushed by the linear C. All four concern one step with the lattice spacing fixed (or a stack of linear steps).

02

Locality and gauge covariance

Locality theorem Lean: `BoxWalk.boxMap_local` (the only assumption is $r_0 > 0$). If two configurations U, U' agree on all links starting from the fine points $2y + z$ ($z \in \mathbb{Z}^4$, each coordinate $-2 \le z_i \le 2$), then $B(U)_b = B(U')_b$.

Gauge covariance theorem Lean: `BoxWalk.boxMap_gauge` (the assumptions are $r_0 > 0$ and $|g(x)| = 1$). For a gauge transformation $(gU)_{(x,\nu)} = g(x)\, U_{(x,\nu)}\, g(x + e_\nu)^*$ by unit quaternions g at each point of the fine lattice,

$$B(gU)_{(y,\mu)} = g(2y)\; B(U)_{(y,\mu)}\; g\big(2(y + e_\mu)\big)^*. \tag{2}$$

The map in these two theorems has, word for word, the same definition as `ChainBlockMap.blockMap` defined in 19-04 (the family of box paths and the weights). They are separately checked declarations, and the agreement rests on a textual comparison.

Meaning. The value of a coarse link is determined only by the links starting from the 5^4 fine points around the coarse point. Radius 2 is the reach of the paths: a path moves sideways by at most 1, advances 2 in μ, and comes back. On the coarse lattice, the gauge transformation acts through the restriction of the fine-lattice g to the coarse points $2y$.

The route of the proof. Locality comes from each term of the average depending only on the link values along a path, and from the paths staying inside the cube. Gauge covariance comes from two facts. One is that a product along a path picks up only the g at its two ends under a gauge transformation (the g inside the path cancel by $g^* g =$

1). The other is that the retract ρ_{r0} commutes with left and right multiplication by unit quaternions ($|gqh^*| = |q|$) Proved here .

A caution. When the torus is small ($2^{m+1}L \le 4$) and the cube wraps around the torus, the locality assumption becomes stronger and the statement becomes weaker. It does not become false.

03

Linearisation around the identity

Take a field a of imaginary quaternions on the fine links ($\operatorname{Re} a_\ell = 0$, $|a_\ell| \le M$) and $\varepsilon \ge 0$, and set $U_\ell = \exp(\varepsilon a_\ell) \in SU(2)$. The path kernel C is the weighted average of 19-04 (7) of signed sums of the field along paths.

Linearisation theorem Lean: `BlockLinearKernel.blockMap_exp_linear_le_one`. If the threshold is $0 < r_0 \le 1$ and $\varepsilon M \le 1/32$, then for every coarse link b

$$\left| B(e^{\varepsilon a})_b - \exp\big(\varepsilon\,(Ca)_b\big) \right| \;\le\; 257\,(\varepsilon M)^2, \quad (3)$$

and, for any real-linear map f from quaternions to real numbers (such as taking one colour component), $f\big((Ca)_b\big)$ equals the value of the path-kernel operator (coefficients 19-04 (9), $X \mapsto 2X$) applied to $f \circ a$. The version with $r_0 = 1$ is `BlockLinearKernel.blockMap_exp_linear_r1`.

Meaning. When the field is small (εM small), the non-linear block map agrees, up to a second-order error, with the linear map C on the Lie algebra wrapped in the exponential map. Moreover, C is the same operator as the path kernel whose symbol was treated in 15-12. This is the ground for calling the linear objects of 15-12 and § 05 “the first-order part of the blocking”.

The route of the proof (each stage is a Lean lemma).

1. **Products along paths.** A product of n factors with $|u_i| \le 1$, $|u_i - 1| \le \delta$ is within $n^2\delta^2/2$ of $1 + \sum_i (u_i - 1)$ (induction). Paths have length at most 8.
2. **The exponential.** For imaginary v with $|v| \le 1$, $|e^v| = 1$, $|e^v - 1| \le |v|$, $|e^v - 1 - v| \le |v|^2$. A step in the reverse direction is e^{-v}, with the same estimates.
3. **The average.** The weights are non-negative and sum to 1, so the average has the same error. For a unit quaternion u, $\operatorname{Re}(u - 1) = -|u - 1|^2/2$, so the real part of the first-order part is of second-order size.
4. **The retract.** If $0 < r_0 \le 1$, $|v| \le s \le \frac{1}{2}$, $|\operatorname{Re} v| \le \rho$, then $|\rho_{r0}(1 + v) - (1 + v)| \le 2\rho + s^2$. The retract does not change the first-order part.

5. **Combine.** Adding the error terms ($72 + 81$ from the retract, 32 from the average, 8 from the first-order part, 64 from the exponential) gives 257 times $(\varepsilon M)^2$. The agreement of the first-order part is an identity that rearranges the sums over paths into the table of coefficients, colour by colour.

Numerically the ratio $|B - \exp(\varepsilon C a)|/(\varepsilon M)^2$ is 0.3–0.9 Reading , so the constant 257 is several hundred times loose. For $\varepsilon M \le 1/32$ the right-hand side is $257/1024 < 2$ (the trivial upper bound for the difference of two unit quaternions), so the statement is not empty.

A caution. This is only first order around $U = 1$. Configurations whose field size εM exceeds $1/32$, and expansion around a background field $U \neq 1$, are not treated.

04

The covariance of the push-forward $= CTC^\dagger$

As in 19-04 (11), the covariance form of a measure μ on a real inner product space is $\mathrm{cov}_\mu(u,v) = \int \langle u,x\rangle\langle v,x\rangle\, d\mu - \int \langle u,x\rangle\, d\mu \int \langle v,x\rangle\, d\mu$. An operator T with $\mathrm{cov}_\mu(u,v) = \langle u, Tv\rangle$ is called the covariance operator.

Push-forward covariance theorem Lean: `PushCov.lattice_pushCov`. For any measure μ on the space of real fields (one colour) on the links of the fine lattice, with covariance operator T: (a) the one-colour first-order part C_1 of the block map (`PushCov.blockLin`) equals the path-kernel operator, and (b)

$$\mathrm{cov}_{(C_1)_*\mu}(u,v) = \big\langle u,\ C_1\, T\, C_1^\dagger\, v\big\rangle \qquad (\text{for all } u,v). \tag{4}$$

Meaning. When a field is pushed by a linear map, the covariance is sandwiched by the map from both sides. Gaussianity is not used. The measure need not be a probability measure; it is an identity that holds for any measure (the general form is `PushCov.covOp_map`). The proof is only $\langle u, C_1 x\rangle = \langle C_1^\dagger u, x\rangle$ and the change of variables in the integral of the push-forward Proved here .

A caution. Under Lean's convention for integrals, the integral of a non-integrable function is read as 0. (4) has meaning as a covariance when μ has second moments. Also, what is pushed here is the linear C_1; the covariance of a measure pushed by the non-linear B does not take this form.

05

The covariance of the plaquette field strength

The objects are on the lattice side of 19-04 § The multi-step family on the free-field side.

- $\mathrm{freeCov}_n$: the covariance of the free field in Feynman gauge for a four-component field on $(\mathbb{Z}/n)^4$. The inverse of the lattice Laplacian taken component by component, and 0 on the constant component.
- $\mathrm{stageCov}_{m,N}$: the $\mathrm{freeCov}$ on $(\mathbb{Z}/2^m N)^4$, pushed m times by the one step $T \mapsto CTC^\dagger$ and brought down to $(\mathbb{Z}/N)^4$.
- $\mathrm{faceCov}_{m,N} = \mathrm{curl} \circ \mathrm{stageCov}_{m,N} \circ \mathrm{curl}^\dagger$: the covariance of the plaquette field strength $(\mathrm{curl}\, A)_{(a,b)}(X) = (A_a(X+e_b) - A_a(X)) - (A_b(X+e_a) - A_b(X))$. The plaquette field has six components at each point.

For a plaquette field z, set the quadratic form $Q(z) = \mathrm{Re} \sum_X \sum_p z_p(X)\, (\mathrm{faceCov}\, z)_p(X)$, the norm $|z|^2 = \sum_X \sum_p |z_p(X)|^2$, the largest eigenvalue $\lambda_{\max} = \sup_{z\,/=0} Q(z)/|z|^2$, and the diagonal at the point X_0 and plaquette p, $D(X_0,p) = \mathrm{Re}(\mathrm{faceCov}\, \delta_{X_0,p})_p(X_0)$.

Plaquette covariance theorem Lean: `LatticeFace.lattice_face_T1h` (the only assumption is $N \geq 2$). At every step $m \geq 0$ and for every $N \geq 2$, $\mathrm{faceCov}_{m,N}$ is self-adjoint, and for every point X_0 and plaquette p

$$\lambda_{\max} \leq 1, \qquad \frac{3}{200}\left(1 - N^{-4}\right) \leq D(X_0,p) \leq \lambda_{\max}, \qquad \Lambda := \frac{\lambda_{\max}}{D(X_0,p)} \leq \frac{640}{9}. \tag{5}$$

Furthermore, $\mathrm{faceCov}_{m,N}$ equals, up to relabelling of points, the translation-invariant operator built from the symbol G_m of 15-12 (the second clause of the same theorem).

Meaning. However many steps the free field is pushed by the linearised blocking, the fluctuation of the plaquette field strength does not exceed 1 in any direction, and does not fall below $\frac{3}{200}(1 - N^{-4})$ at any point or plaquette. The upper bound $640/9$ on the ratio Λ depends neither on the step m nor on N. At $m = 0$ (the free field, not pushed) $\lambda_{\max} = 1$ exactly Reading .

This theorem is a statement about **the covariance of the plaquette field strength of the free field pushed by the linearised blocking**. It is not the covariance of the Wilson measure or the free field pushed by the non-linear block map B. Matrices built numerically and directly from the position-space definitions agree with the operators built from the symbols to within $1.1 \cdot 10^{-15}$ for $N = 2, 3, 4$ and $m = 0, 1, 2$ Reading .

06

The route of the proof for the plaquette covariance

1. **The symbol at step 0.** $\mathrm{freeCov}_n$ is diagonal in characters, with symbol I/ω (`LatticePush.freeCov_phase`; $|1-e^{iq}|^2 = 4\sin^2(q/2)$).
2. **The chain of m steps.** If the symbol of an operator T is X, the symbol of the operator pushed m times is $\mathcal{R}^m X$ (`LatticePush.push_phase`; the one step `LatticeTorus.torus_step` of 15-12 applied m times, relabelling the type of the lattice size). The symbol of the stage covariance is $\mathcal{R}^m X_0$ (`LatticePush.stageCov_phase`), and its physical part is the G_m of 15-12 (`LatticePush.projFam_stage`).
3. **The plaquette symbol.** Multiplying by the symbol of curl from both sides removes the gradient direction, and the projection P enters automatically (`LatticeFace.key_proj`). So the symbol of the plaquette covariance is $\hat{d}\, G_m\, \hat{d}^\dagger$, and the operator returned to position space equals the operator built from the symbol (`LatticeFace.faceCov_eq_posC`).
4. **The upper side.** From the upper side of 15-12, $G_m \preccurlyeq P/\omega$, the plaquette symbol at each momentum is at most $\hat{d}\,(P/\omega)\,\hat{d}^\dagger$, whose largest eigenvalue is 1. By Plancherel, $\lambda_{\max} \le 1$ (`FaceTorus.torus_face_upper`).
5. **The diagonal.** Putting the lower side of 15-12, $G_m \succcurlyeq \frac{3}{100} P/\omega$, into the sum over momenta gives diagonal $\ge \frac{3/100}{2}(1-N^{-4})$ (`FaceTorus.torus_face_diag`).
6. **The ratio.** For $N \ge 2$, $1-N^{-4} \ge 15/16$, so diagonal $\ge \frac{3}{200}\cdot\frac{15}{16} = \frac{9}{640}$; together with $\lambda_{\max} \le 1$, $\Lambda \le 640/9$ Proved here .

In the chain, stages 1–3 connect position space and symbols; 1 and 2 are in Lean for the stage covariance itself, and 3 for the plaquette field. Stages 4 and 5 use the two sides of 15-12 as they are.

Detour: the $\frac{1}{2}$ and the $1-N^{-4}$ in the lower bound on the diagonal

You may skip this. The diagonal at the point X_0 and plaquette $p=(a,b)$ is the average over momenta $k \in (\mathbb{Z}/N)^4$, $N^{-4}\sum_k \langle e_p, \hat{d}(k) G_m(k) \hat{d}(k)^\dagger e_p\rangle$. At the zero momentum $\hat{d}=0$ and the term is 0, so the sum has N^4-1 terms; the factor $1-N^{-4}$ comes from here Proved here . By the lower side each term is at least $\frac{3}{100}(|\hat{q}_a|^2+|\hat{q}_b|^2)/\omega$, and the sum over momenta of $|\hat{q}_\nu|^2/\omega$ is $(N^4-1)/4$ independently of the direction ν (the sum over the four directions is N^4-1; Lean: `FaceTorus.sum_rt_quarter`). A plaquette has two directions, so the average becomes $\frac{1}{2}$ Proved here .

07

What is not said

- **The covariance of non-linear blocking.** § 05 is the covariance of the field pushed by the linearised blocking (the path kernel). Agreement with the non-linear B is only up to the first-order part of § 03; the contribution of terms of second and higher order to the covariance is not treated.
- **Anything other than the free field.** § 05 is only the Gaussian free field (Feynman gauge, with the constant component removed). It is not the plaquette covariance of the field obtained by pushing the Wilson measure. The identity of § 04 holds for any measure, but the theorems that estimate T are only about the free-field $\mathbf{freeCov}$.
- **Backgrounds and large fields.** The linearisation is only around $U = 1$, with $\varepsilon M \leq 1/32$. Linearisation around a background field, and events of large fields below the threshold, are not treated (an estimate at one scale of the rarity of large fields is in 22-01).
- **The measure at a coarse scale.** Nothing is said about the properties of $B_*\mu$ (decay, uniformity). That belongs to the statements to be shown (19-07, 28-04).
- **Choice of weights.** The radius 2, the constant 257 and $640/9$ were computed with the weights $(1/8, 3/4, 1/8)$ and the family of paths with factor 2 (28-05).

Locality, gauge covariance and linearisation are formalisations, for this family of paths, of properties that are standardly asked of the blocking of gauge fields[Cited] . The covariance of the push-forward is a formalisation of the general identity for the push-forward of a measure by a linear map[Cited] .

What this chapter can say

Can say	Cannot say
Locality (radius 2) and gauge covariance ($r_0 > 0$) of the non-linear block map. Lean: `BoxWalk.boxMap_local`, `BoxWalk.boxMap_gauge`	Further properties, such as constants of continuity
For $0 < r_0 \leq 1$ and $\varepsilon M \leq 1/32$, $\lvert B(e^{\varepsilon a})_b - \exp(\varepsilon (Ca)_b)\rvert \leq 257(\varepsilon M)^2$, and the first-order part is the path kernel. Lean: `BlockLinearKernel.blockMap_exp_linear_le_one`	Around a background $U \neq 1$, large fields, the form of the second-order terms
For any measure, the covariance of the push-forward $= C_1 T C_1^\dagger$, where C_1 is the path kernel. Lean: `PushCov.lattice_pushCov`	The covariance of the push-forward by the non-linear B. Only the one-colour form

Can say	Cannot say
The plaquette covariance of the free field pushed m times by the linearised blocking: self-adjoint, $\lambda_{\max} \le 1$, diagonal $\ge \frac{3}{200}(1 - N^{-4})$, $\Lambda \le 640/9$ (all m, $N \ge 2$). Lean: `LatticeFace.lattice_face_T1h`	The covariance of non-linear blocking (agreement only up to the first-order part). The field obtained by pushing the Wilson measure. Background fields
Proved here For $N \ge 2$, $\Lambda \le 640/9$ from $\frac{3}{200}(1 - N^{-4}) \ge 9/640$	—

Sources and reproduction

Item	Kind	Source / tools
Building the blocking of a gauge field by averaging products along paths, and that locality, gauge covariance and linearisation are asked of it	Cited	The sources of 7-05 (T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275. J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335)
The theorems of § 02– § 05	—	Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, checked by a separate re-run). The list is in 15-14
$\lambda_{\max} = 1$ at $m = 0$; agreement of the position-space matrices with the symbol operators ($1.1 \cdot 10^{-15}$)	Reading	The numerical record of this book's study (matrices built directly from the position-space definitions, compared with the operators built from the symbols, for 7 pairs with $N = 2, 3, 4$ and $m = 0, 1, 2$)
The ratio 0.3–0.9 of the linearisation error	Reading	The numerical record of this book's study ($L = 1, 2$, random fields, $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$, $r_0 = 1$ and $r_0 \le \frac{1}{2}$. The difference between the first-order part built from sums over paths and the path-kernel operator is $\le 6.4 \cdot 10^{-15}$)

Terms used in this chapter — block map, retract, locality, gauge covariance, path kernel, linearisation, push-forward, covariance operator, plaquette field strength, plaquette covariance, Rayleigh quotient. Their meanings are in A-02 Glossary.

Next chapter to read: 15-14 What the second half (15-11 to 15-13) says, and what it does not. To read alongside: 22-01 Rarity of large plaquettes — the chessboard estimate — events of large fields, outside the linearisation, are estimated for the Wilson measure at one scale.

What the second half (15-11 to 15-13) says, and what it does not

This chapter is this book's own investigation.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

Scope — The theorems of the second half of this part (15-11 to 15-14), and the rarity of large plaquettes that moved to Part 22 (22-01), are either statements about the free field, the vacuum background and the linearised blocking, or statements about the Wilson measure at one scale (with the lattice spacing fixed). They say nothing about the renormalised coarse-scale measure, fields with a background, or estimates uniform across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

What this chapter shows — We view the theorems of the second half of this part and the theorem of 22-01 in one table. For each row we give a short form of the statement, its scope, what it does not say, and its Lean declarations. This table deals only with the theorems of the second half of the part. What the first half of the part (15-00 to 15-10) cannot say, and the index of questions, are in 15-09.

Chapters this one relies on — 15-12, 15-13, 22-01 (the detailed form of the rows of the table, and the routes of the proofs).

Said first — The scope is as written directly under the title. The “statement” column of the table is written in short form; the exact forms are in the formulas of each chapter. Every theorem is a formalisation, for the objects of this book, of a result of a known type. Estimates across steps are not in the chapters of the second half; they are on the side of the premises (§ 03).

Order of this chapter

1. The table of theorems — statement, scope, what is not said, Lean declarations
2. Lemmas used as parts — those not made into stand-alone theorems
3. Estimates across steps are on the side of the premises — where what the chapters of the second half do not say is placed

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01

The table of theorems

The notation is as in each chapter: G_k is the physical part of the symbol recursion (15-12 § 01), H_k its inverse on $\mathbf{ran}\, P$, $\mathcal{M}$ the one-step map, B the non-linear block map, C the path kernel, μ_W the Wilson measure, and Z its total mass.

Chapter	Name	Statement (short form)	Scope	What is not said	Lean declarations
15-12 § 02	Upper side	$G_k \preccurlyeq P/\omega$, constant exactly 1	Every step k, all $q \in \mathbb{R}^4$ with $\omega(q) > 0$	Non-linear blocking, background fields, the form as operators in position space	`OSUpQpos.os_up_C`
15-12 § 03	Lower side	$\frac{3}{100} P/\omega \preccurlyeq G_k$	As above (families closed under aliases)	A constant close to the true infimum, about 0.091 Reading	`HsSplit.lower_si`
15-12 § 04	Propagator sandwich	$(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$ (in the split version the upper side is $\frac{25}{3}\omega P$)	As above; constants uniform in k and q	Reading $\frac{1}{4}H_k$ as the stage action (on paper). The complex form of the inequalities. The constants are 3 to 186 times looser than the true values Reading	`LatticeRec.latti` `OneStepLower.bra` `NearAliasB.brack`

Chapter	Name	Statement (short form)	Scope	What is not said	Lean declarations
15-12 § 05	One-step contraction	$\mathcal{M}P \preccurlyeq \frac{3}{5}P$	All q	The true supremum $125/256$ Reading	`OneStepBridge.Mf`
15-12 § 06	Contraction uniform in the step	$\mathcal{M}^{17}(\omega G_k) \preccurlyeq \tau\, \omega G_{k+17}$, $\tau < 1/170$	Every step k, all q	$\tau \le 31/32$ at step 0 (the assembly is on paper). Identification with the per-step contraction rate used in the argument on paper (not in Lean)	`TorusLam.stage_t`
15-12 § 07	Convergence of Γ_k	Any finite quadratic functional of Γ_k converges at rate $(3/5)^k$	Every step; constants uniform in the step and the momentum	Convergence of the one-loop coupling b_k (the identification is on paper). $\kappa^2 \approx 1.4 \cdot 10^{13}$ Reading	`AliasPoinc.coupl`
15-13 § 02	Locality, gauge covariance	$B(U)_b$ is determined by the links in a cube of radius 2. $B(gU)_b = g(2y)B(U)_b g(2(y+e_\mu))^*$	$r_0 > 0$, $\lvert g\rvert = 1$	Anything further, such as constants of continuity	`BoxWalk.boxMap_l`
15-13 § 03	Linearisation	$\lvert B(e^{\varepsilon a})_b - \exp(\varepsilon (Ca)_b)\rvert \le 257(\varepsilon M)^2$; the first-order part is the path kernel	$0 < r_0 \le 1$, $\varepsilon M \le 1/32$	Around a background $U \neq 1$, large fields, the form of the second-order terms	`BlockLinearKerne`
15-13 § 04	Covariance of the push-forward	$\mathrm{cov}(C_1)_{*}\mu = C_1 T C_1^{\dagger}$	Any measure (meaningful when second moments exist), one colour	Push-forward by the non-linear B	`PushCov.lattice_`

Chapter	Name	Statement (short form)	Scope	What is not said	Lean declarations
15-13 § 05	Plaquette covariance	Self-adjoint, $\lambda_{\max} \le 1$, diagonal $\ge \frac{3}{200}(1 - N^{-4})$, $\Lambda \le 640/9$	Every step m, $N \ge 2$. The free field pushed by the linearised blocking	The covariance of non-linear blocking (agreement only up to the first-order part). The field obtained by pushing the Wilson measure	`LatticeFace.latt`
22-01 § 03	Chessboard estimate	$\lvert V(F)\rvert^{N^4} \le \prod_w V(\text{constant})$	$N = 2m$ ($m \ge 1$), every real β, functions of a unit cube	Odd sides, measures with a twist	`FoldWilson.wilsc`
22-01 § 04	Rarity uniform in the volume	$P(X$: all its plaquettes have $\operatorname{Re} U_p < 0) \le q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, $q_1 < 1$	$\beta \ge 81$, $m \ge 2$, any set X of plaquettes. q_1 independent of N and X	The measure at a coarse scale, $\beta < 81$, $m = 1$. The threshold 81 comes from a crude lower bound on the volume of the cap (with the true h about 48 Reading). Ratios to anything other than Z	`FoldCap.rarity_e`
22-01 § 06	Rarity with rate $1/4$	$\le q_4^{\lfloor \lvert X\rvert/24 \rfloor}$	$0 < y \le 2$, at or above some existing β_0, $m \ge 1$	The value of β_0 (about 357 at $y = 1$ Reading)	`FoldQuarter.rari`

What no row of the table says: the limit in which the lattice spacing is sent to 0, the renormalised measure at a coarse scale, fields with a background, and nonlinear estimates uniform across steps. The momenta in the rows of 15-12 are the continuous momenta of infinite volume. The links to position space (that the physical part of the symbol of the stage covariance is G_k, and the covariance of the plaquette field) are in Lean, but there is no theorem stating the inequalities for G_k as inequalities between operators in position space. The constants computed with the weights $(1/8, 3/4, 1/8)$ and factor 2 cannot be used as they are for another choice (28-05).

02

Lemmas used as parts

The following declarations are closed in Lean, but are not set out as stand-alone theorems; they were mentioned in each chapter as parts of the proofs of the rows above.

Content	Row that uses it	Lean declarations
The upper bound for an explicit function $E_\nu < 4$ ($\sigma \in [0, \frac{1}{2}]^4 \setminus \{0\}$, $\sigma_\mu = \sin^2(q_\mu/4)$), invariance under reflection	Upper side	`FreeStageOS.E_lt_four_s`, `FreeStageOS.E_lt_four`, `FreeStageOS.eq_lt_four`, `OSUpQpos.EE_bbflip`
The lower bounds from the chain of nearest aliases and from the sixteen aliases, the case split	Lower side	`NearAliasB.chain_lower`, `NearAliasB.near_lower`, `HsSplit.hS_split`
Correspondence between the symbol recursion and $\mathcal{M}$, the symbol of the path average, the symbol of the one-step torus operator	All of 15-12	`LatticeRec.rec_iterate`, `LatticePath.pathSym_closed`, `LatticeTorus.torus_step`
The one-step difference $\pm 2P$	One-step contraction	`CouplingLimit.D_lower`, `CouplingLimit.D_upper`
The size of the lift, the Poincaré input on the sum of aliases	Convergence of Γ_k	`AliasLiftSize.hlift_holds`, `AliasPoinc.hH_holds`
Agreement of the plaquette covariance with the symbol operator, the chain of steps	Plaquette covariance	`LatticeFace.faceCov_eq_posC`, `LatticePush.freeCov_phase`, `LatticePush.push_phase`, `LatticePush.stageCov_phase`, `LatticePush.projFam_stage`
The lower bound on the partition function, the lower bound on the volume of the cap	Rarity uniform in the volume	`FoldEntropy.Zw_ge`, `FoldCap.ballQ_ge`

03

Estimates across steps are on the side of the premises

What the chapters of the second half say concerns only one step (or a stack of linear steps) and one scale. To close the chain of renormalisation, one needs estimates uniform across steps for the measure at a coarse scale pushed forward by the nonlinear block map (contraction on the small-field side, rarity on the large-field side, bracketing of the propagator for fields with a background, and so on). These do not follow from the theorems of the second half. In this book they are placed, with names,

in the list of premises (28-04), and the places where their form is not yet settled are placed in 28-06.

Here a “premise” is a statement about this book's design (the measure pushed forward by a fixed block map), and it does not follow from the results of the published construction (15-00). The theorems of the second half are formalisations of “the parts that can be checked with the free field, linearisation and one scale” among the statements on the side of the premises. They are not the statements of the premises themselves.

What this chapter can say

Can say	Cannot say
The 13 rows of the table in § 01 (with Lean declarations; the scope is in each row)	The “what is not said” column of each row. In particular, the measure at a coarse scale, fields with a background, and non-linear estimates uniform across steps

Sources and reproduction

Item	Kind	Source / tools
Each row of the table	—	Lean declarations of this book's study (the table of sources and reproduction in each chapter). Sources for the known types are in 15-12, 15-13 and 22-01
Numbers marked Reading (0.091, 125/256, κ^2, $2.5 \cdot 10^{12}$, 48, 357)	Reading	The numerical records in the table of sources and reproduction of each chapter

Terms used in this chapter — symbol recursion, physical part, block map, path kernel, push-forward, plaquette covariance, chessboard estimate, rarity, premise. Their meanings are in A-02 Glossary.

Next chapter to read: 16-01 How far the world has come. To read alongside: 28-00 Entrance — the statements to be shown, and sufficient conditions through a representation — the estimates across steps that the chapters of the second half do not state are placed there, with names, as premises.