Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 16 Where things stand, and the questions that remain

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2026-10-01 · chapter Yang–Mills and the mass gap Part 16 Where things stand, and the questions that remain timeline and where things stand

How far the world has come — reading where things stand from a timeline and tables by dimension

> In this chapter, § 01– § 06 are established knowledge, § 07 is a map of open questions and § 08 is guide material.

What this chapter shows — A timeline of the main results from 1954 to 2026 (37 rows, including the four-dimensional construction published in 2026). A board showing, for each dimension, how far through the six steps there are theorems, and how it got filled in. A comparison with abelian groups and scalar fields. Where on the number line of β the lattice theorems lie.

Prerequisite chapters — 7-10 Where four dimensions stops. The six stages are in 7-01, and the content for each dimension in 7-02, 7-06 and 7-07.

Said first — This chapter rearranges known literature and shows no theorem. The existence of four-dimensional Yang–Mills (a theory satisfying the Wightman axioms, as a limit of Wilson's lattice gauge theory) was shown by the construction published in 2026, as a formalisation in Lean, and the committee assessed it as “proving” this Assessment (Part 6). The uniqueness of the infinite-volume limit and the mass gap have not been shown (16-02). The timeline covers what was checked in the main text or the abstract (some rows from the reference list of a review), and is not exhaustive. The articles on this site have not moved any cell of the board (the label Assessment means "according to the committee's assessment report"; for details, see “Labels” in 6-00).

The order of this chapter

1. Reading one result along six axes
2. Timeline I (1954–1999)
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6. Comparison with neighbouring theories: the scope of "it has been done in two and three dimensions"

7. Where the lattice theorems lie on the number line of β — moving figure, computed (the β knob)
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01

Reading one result along six axes

Every theorem about Yang–Mills comes with conditions on "where it holds". Drop the conditions, and a two-dimensional theorem looks like a statement about four dimensions, a lattice theorem like a statement about the continuum. This chapter gathers the conditions into six axes and calls them the **address**.

$$\text{address} = \big(d,\ \text{lattice or continuum},\ \text{volume},\ \text{coupling side},\ G,\ \text{matter fields}\big)\ \text{(1)}$$

d is the dimension of Euclideanised spacetime, and the coupling side is whether the lattice inverse coupling $\beta = 2N/g^2$ (table of notation) is small (strong coupling) or large (weak coupling). The address of the problem statement is "4, continuum, $\mathbb{R}^4$, any compact simple group, none", and there it is required to pass through all six stages. Known The continuum limit of the four-dimensional lattice is believed to lie on the $\beta \to \infty$ side (Chatterjee 2018 § 5). Physics

Example: Driver 1989 reaches the ultraviolet-limit step and the infinite-volume step at (2, continuum, $\mathbb{R}^2$, —, connected compact, none), and Shen–Zhu–Zhu 2022 gives exponential decay on the lattice at (any, lattice with fixed spacing, infinite, strong, $SU(N)$, none). The "Scope" in the timeline abbreviates this address; the year is the year of first public appearance (the arXiv year if there is one), and the year in parentheses is the year of publication. Rows that combine several papers give a range of years.

02

Timeline I (1954–1999): words, tools, and the first constructions

The first half-century had three strands. In physics, the theory was written down (1954), and asymptotic freedom was found in perturbation theory (1973). In the framework, the axioms (1973, 1975) and the lattice (1974) were put in place. In construction came the strongly coupled lattice (1978), ultraviolet estimates in three and four dimensions (1985, 1987–89), the two-dimensional continuum measure (1989), and in four dimensions an outline of a construction that removes the ultraviolet cutoff while keeping the infrared one (1993). Known The formulas of 1974 and 1973 are the following two (5-01, 4-06).

$$S_W = \beta \sum_P \Big(1 - \tfrac{1}{N} \operatorname{Re} \operatorname{tr} U_P\Big), \qquad \mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11N}{3} \cdot \frac{1}{16\pi^2} \tag{2}$$

Example: for $SU(2)$, $\beta = 2.5$ is $g^2 = 2N/\beta = 1.6$. The sign $b_0 > 0$ is asymptotic freedom. The "Comparison" rows are neighbouring theories, abelian groups and scalar fields, and do not move cells. The physics rows continue in Part 17 (17-04, 17-11, 17-09; lattice numbers in 5-08).

Known Table 1 Timeline I (except rows with the Physics label). Pressing a row sets Figure 1 to that year.

Year	Event	Scope
1954	Yang–Mills: classical theory of non-abelian gauge fields	Classical
1973	Gross–Wilczek, Politzer: asymptotic freedom	Coefficient in perturbation theory
1973, 1975	Osterwalder–Schrader: Euclidean axioms and reconstruction of a Wightman theory (conditions revised in 1975)	Axioms
1974	Wilson: lattice gauge theory (a gauge-invariant lattice action)	All dimensions, lattice
1974	't Hooft: the large-N expansion	Physics
1976	Feldman–Osterwalder: the Wightman axioms and a mass gap for weakly coupled ϕ^4_3	Comparison, 3D
1978	Osterwalder–Seiler: reflection positivity and the transfer matrix on the lattice; infinite-volume limit and area law at strong coupling (convergence of the cluster expansion in outline)	Lattice, strong coupling
1978	Singer: a global continuous gauge fixing cannot in general be taken	Geometry
1979–81	Brydges–Fröhlich–Seiler: the two-dimensional abelian Higgs model up to the OS axioms	Comparison, 2D
1980, 1982	Guth, Fröhlich–Spencer: a non-confining phase at large β for four-dimensional lattice U(1)	Comparison, lattice
1981–82	Aizenman, Fröhlich: ϕ^4 is trivial for $d \geq 5$	Comparison
1983–87	Gross, King, Driver: continuum limit of U(1) (3D, 4D, including with Higgs)	Comparison
1985	Bałaban: ultraviolet stability on the three-dimensional lattice (torus; uniform upper and lower bounds on the partition function)	3D, lattice, finite volume
1987–89	Bałaban: ultraviolet stability on the four-dimensional lattice; Theorem 1 of the final paper assumes the effective coupling is small	4D, lattice, finite volume
1989	Driver, Gross–King–Sengupta: the continuum measure on the plane $\mathbb{R}^2$ (Driver also convergence of the lattice approximation)	2D, continuum, infinite volume

Year	Event	Scope
1992–93	Sengupta: the measure on closed surfaces	2D, continuum, finite volume
1993	Magnen–Rivasseau–Sénéor: the "foundations" for constructing the Schwinger functions of four-dimensional SU(2) without ultraviolet cutoff, with the infrared cutoff held fixed (in a regularised axial gauge) (no detailed proof)	4D, continuum
1994	Seiberg–Witten: gap and confinement in a supersymmetric theory	Physics
1997–98	Maldacena, Witten: holography, and models that see the gap as geometry	Physics
1999	Morningstar–Peardon: lattice numbers for SU(3) glueballs	Physics Numerical

03

Timeline II (2000–2026): after the problem statement

On 24 May 2000 the Clay Mathematics Institute announced the Millennium Prize Problems in Paris. The problem statement (Jaffe–Witten) describes the state of four dimensions as follows. Known

> 'at present we do not know any non-trivial relativistic field theory that satisfies the Wightman (or any other reasonable) axioms in four dimensions.' (JW § 6)

Since then, new constructions in two dimensions, the state space and dynamics in three dimensions, and new theorems on the lattice have been added, but a 2004 report writes 'The state of the art remains the works of Balaban and of Magnen, Rivasseau and Sénéor'. Known A 2018 review put the question on the lattice into a formula (the lattice correlation-length problem (Chatterjee's Problem 5.1); 5-09). $f_\beta(x)$ is the correlation of the Wilson loops of the plaquettes nearest to the origin and to the point x.

$$\lim_{|x|\to\infty} \frac{\log f_\beta(x)}{|x|} = -\frac{1}{\xi(\beta)}, \quad \xi(\beta) \in (0,\infty) \ \ (\text{for all } \beta > 0), \qquad \lim_{\beta\to\infty} \xi(\beta) = \infty \ (3)$$

Example: there are 17 rows from 2000 on (11 of them in the 2020s), and the rows that changed a colour on the board are the 2 rows of 2022 (Shen–Zhu–Zhu and Adhikari–Cao) and the 1 row of the construction published in 2026 (three cells). Computed The lattice correlation-length problem is a question on the lattice, and is a different thing from the requirements of the problem statement (continuum, $\mathbb{R}^4$, axioms, non-triviality) (0-03).

Known Table 2 Timeline II.

Year	Event	Scope
2000	The Clay prize problem (the problem statement of Jaffe–Witten)	Problem
2001 (2003)–10	Lévy: random holonomy fields on closed surfaces (non-trivial bundles in 2006)	2D, continuum
2004	Douglas's report: the state of the art is still Bałaban and MRS	Assessment
2013 (2014)	Hairer: regularity structures (renormalisation of stochastic PDEs)	Tool
2018	Chatterjee's review: the lattice correlation-length problem (Problem 5.1) and the problem of the continuum limit of Wilson loops (5.2)	Lattice formulation
2019 (2021)	Aizenman–Duminil-Copin: the limit of four-dimensional ϕ^4 is Gaussian	Comparison
2020 (2022)	Chandra–Chevyrev–Hairer–Shen (CCHS): the two-dimensional stochastic Yang–Mills heat flow and the space of gauge orbits	2D, T^2
2021 (2024)	Cao–Chatterjee: the three-dimensional state space (under the assumption that the theory behaves like a Gaussian free field [GFF; 1-09])	3D, conditional
2022 (2024)	CCHS: stochastic quantisation of three-dimensional Yang–Mills–Higgs (local solutions and a Markov process; the measure not yet reached)	3D, T^3
2022	Chevyrev's review: the three-dimensional measure is not constructed even on T^3	Assessment
2022 (2023)	Shen–Zhu–Zhu: uniqueness and a positive mass gap on the strongly coupled lattice, in an explicit range independent of N	Lattice, strong coupling
2022 (2025)	Adhikari–Cao: exponential decay of correlations at weak coupling on the four-dimensional lattice for finite groups (including non-abelian)	Lattice, weak coupling
2023 (2026)	Chevyrev–Shen: the measure on T^2 is the unique invariant measure of the Langevin dynamics, independent of the lattice action	2D, T^2
2024 (2026)	Chatterjee: a special scaling limit of SU(2) Yang–Mills–Higgs is a Gaussian field	Comparison, Higgs
2025	Cao–Nissim–Sheffield: the area law in the 't Hooft regime (in 't Hooft's convention, multiplying the action by βN, the side with β small independently of N; 7-10 § 05) (not peer-reviewed)	Lattice, strong coupling
2026	Chevyrev–Klose–Mohamed, Chhaibi et al.: representations of the two-dimensional measure, and a construction not via the lattice (not peer-reviewed)	2D

Year	Event	Scope
2026	The published construction (a formalisation in Lean; submitted 2026-10-28, published 2026-11-10 together with the committee's assessment report; the committee's statement is dated 2026-11-09): for any compact simple group, builds a theory satisfying the Wightman axioms as a limit of four-dimensional Wilson lattice gauge theory. In a periodic box it converges along the whole sequence of cutoffs, and it passes to infinite volume along a subsequence. The uniqueness of the infinite-volume limit and the mass gap are not shown Assessment (6-00, 16-02)	4D, continuum (the committee's assessment)

Words in Table 2: **stochastic quantisation** = a method that builds a measure as the invariant measure (a measure whose distribution does not change over time) of some stochastic time evolution. **Langevin dynamics** = a gradient flow with noise that leaves the measure invariant. **Stochastic Yang–Mills heat flow** = the flow in the direction that decreases the Yang–Mills action (heat flow), with noise added. **Regularity structures** = a framework that gives meaning to singular stochastic partial differential equations and renormalises them. All of these are in 7-09. **Markov process** = a stochastic process whose future distribution is determined by the present state alone (7-02).

04

The board: moving through the years

We transfer the timeline onto cells. The rows are continuum non-abelian Yang–Mills (no matter fields) in $d = 2, 3, 4$ and the four-dimensional lattice, and the columns are the six steps (7-01). The regularisation step and the finite-volume step build the measure $\mu_{a,L}$ on the lattice, and close on the lattice (7-10 § 03). In three and four dimensions the finite-volume measure has not yet been built while staying in the continuum, so the finite-volume step there is written "Closed (lattice)". The construction published in 2026 does not build a continuum probability measure either (16-02 § 03), so the four-dimensional finite-volume cell stays "Closed (lattice)".

The names of the states are the same as in Table 2 of 7-01: **Closed** (there is a theorem), **Partial** (with assumptions, in outline, up to estimates of the partition function, or only part of the requirements; "with a hypothesis" in Table 2 of 7-10 also goes here), **Not yet**, and — (the question is hard to pose in that form; 7-02). The state in year y is fixed by the record of the last change (y_k, c_k, v_k) at or before y.

$$s_C(y) = v_k, \qquad k = \max\{\, k : \; y_k \le y, \; c_k = c \,\} \quad (4)$$

Figure 1 Moving the year with the knob fills the cells with the theorems up to that year (press a row of Table 1 or Table 2, or select a row and press Enter, to go to that year)

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If JavaScript is disabled, see Table 3 and Tables 1 and 2.
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Computed The numbers in this figure were computed on this machine (formula (4)). How the cells are divided and how the states are decided is this chapter's choice.

Computed Table 3 The board in 2026 (state, and the year it reached that state). The three four-dimensional cells of 2026 are at the level of the committee's assessment. The three cells of the lattice row do not correspond to the step columns.

	1 Regularisation	2 Finite volume	3 Ultraviolet limit	4 Infinite volume	5 Axioms	6
2D continuum	Closed 1974	Closed 1974	Closed 1989	Closed 1989	Partial 1989	—
3D continuum	Closed 1974	Closed (lattice) 1974	Partial 1985	Not yet	Not yet	N
4D continuum	Closed 1974	Closed (lattice) 1974	Closed 2026 (Partial from 1989)	Partial 2026	Closed 2026	N
Lattice cells	**Strong coupling**		**Weak coupling, continuous group**		**Weak coupling, finit**	
4D lattice (fixed spacing)	Closed 2022 (Partial from 1978)		Not yet		Closed 2022	

Example: the board moved in six years, 1974, 1978, 1985, 1989, 2022 and 2026, and the continuum cells did not move from 1989 to 2025. Of the 21 cells, 12 are Closed, 3 Partial, 5 Not yet and 1 —. Computed The three cells of 2026 (the four-dimensional ultraviolet limit, infinite volume and axioms) are filled in with the published construction, at the level at which the committee assessed it as “proved in Lean” Assessment . The infinite-volume cell is made "Partial" because the limit is taken along a subsequence and uniqueness is not shown (16-02 § 02). Table 2 of 7-10 makes the infinite-volume step of the lattice "Closed" at strong coupling by OS 1978. The lattice cells of this chapter count the infinite-volume step and the uniform-gap step (gap) as one, so 1978 is "Partial". The axiom step in two dimensions goes as far as Euclidean invariance of gauge-invariant expectations (Corollary 6.7 of Driver 1989), with no claim in the form of the OS axioms, so it is "Partial". The uniform-gap step in two dimensions has no propagating degrees of freedom and the question is hard to pose, so it is "—" (both as in Table 2 of 7-01). Known

The strong-coupling lattice cell becomes "Partial" in 1978 and "Closed" in 2022. Osterwalder–Seiler 1978 gave the convergence of the cluster expansion in outline, and the gap (exponential clustering) follows from that convergence. The proof of uniqueness and a positive mass gap in an explicit range is Shen–Zhu–Zhu 2022 (both according to the introduction of SZZ; the abstract of OS 1978 states the existence and analyticity of the infinite-volume limit at strong coupling and Wilson's confinement estimate; 5-02 § 08). Known That the continuum cells did not move from 1989 to 2025 is not because there was no progress. Progress *inside* the cells, such as the characterisation of the two-dimensional measure by Langevin dynamics and the dynamics in three dimensions, did not show up on the coarse board (this chapter's reading; for the words of the literature step by step, see 7-10).

05

Tables by dimension

We rewrite the rows of the board in the language of addresses. The columns are the lattice (fixed spacing), the inside and the outside of the double limit (7-01), and the axioms and the gap. $\mu_{a,L}$ is the measure on a box with lattice spacing a and side L.

$$\mu = \lim_{L\to\infty} \underbrace{\lim_{a\to 0} \mu_{a,L}}_{\text{continuum, finite volume}} \tag{5}$$

Known Table 4 Where things stand, by dimension (non-abelian, no matter fields).

d	Lattice (a fixed)	Continuum, finite volume	Continuum, infinite volume	Axioms, gap
2	Exactly solvable (7-02)	Closed surfaces, T^2: Sengupta, Lévy, CCHS, Chevyrev–Shen	$\mathbb{R}^2$: Driver, Gross–King–Sengupta	The literature makes no claim (the observables are loops)
3	Infinite volume and exponential decay at strong coupling	Ultraviolet stability (estimates of the partition function). The measure not constructed; up to the state space and dynamics	Not yet	Not yet

d	Lattice (a fixed)	Continuum, finite volume	Continuum, infinite volume	Axioms, gap
4	Infinite volume and exponential decay at strong coupling. At weak coupling, finite groups only	The continuum limit in a periodic box (the published construction, 2026; correlation functions converge along the whole sequence of cutoffs; no continuum probability measure is built) Assessment . Before that, ultraviolet stability with a hypothesis (Bałaban) and an outline with an infrared cutoff (MRS)	Limits along subsequences and phase decomposition (the published construction). Convergence and uniqueness of the limit are not shown Assessment	The Wightman axioms phase by phase (the published construction) Assessment . The gap: not yet

That **two dimensions** is closed up to continuum and infinite volume is due to its exactly solvable structure (independence of plaquettes and the heat kernel of the group; 7-02). **Three dimensions** is said to be easier, but the continuum measure is 'open, even in finite volume' (7-06). In **four dimensions**, up to 2025 the ultraviolet-limit step was with a hypothesis and in outline in finite volume, and there were no theorems from the infinite-volume step to the uniform-gap step (7-07) Known . The construction published in 2026 filled the ultraviolet-limit step, the infinite-volume step (along a subsequence) and the axiom step Assessment (6-01). The uniform-gap step remains open (16-02 § 04).

Example: the four-dimensional "Continuum, finite volume" of Table 4 corresponds to the ultraviolet-limit step of Table 3 (Closed from 2026), and the "Lattice" column to the lattice row of Table 3. The outline of MRS removes the ultraviolet cutoff but keeps the infrared cutoff. The published construction takes the ultraviolet limit in a periodic box and then passes to infinite volume along a subsequence (8-05).

06

Comparison with neighbouring theories: the scope of "it has been done in two and three dimensions"

Saying "it has been done in two and three dimensions" is correct for non-abelian Yang–Mills only in two dimensions. What it fits up to three dimensions is abelian groups and scalar fields (16-08). We line up the "Comparison" rows of the timeline. None of them moves a cell of the board. Known

Known Table 5 How far neighbouring theories have got (the "Comparison" rows of the timeline).

Theory	d	How far it has got
U(1) Higgs (an abelian gauge field and a charged scalar field)	2, 3	In two dimensions up to the OS axioms (Brydges–Fröhlich–Seiler). The problem statement calls it "the only complete example of an interacting gauge theory satisfying the axioms". In three dimensions, the continuum infinite-volume limit (King)

Theory	d	How far it has got
ϕ^4_3 (weak coupling)	3	The Wightman axioms (JW § 6.2) and a mass gap (Feldman–Osterwalder)
Pure U(1) gauge theory	4	The limit from lattice to continuum (Driver 1987). The limit is expected to be the free electromagnetic field (the explanation in 7-10). The lattice has a non-confining phase at large β (Guth, Fröhlich–Spencer)
ϕ^4_4	4	The limit of lattice-regularised $\lambda\phi^4$ at infinite volume and zero lattice spacing is Gaussian (Aizenman–Duminil-Copin). Triviality for $d \geq 5$ is Aizenman, Fröhlich
Two-dimensional O(3) and O(4) sigma models (asymptotically free, with non-abelian global symmetry)	2	There is a manuscript with claims in `github.com/openai/math` (result group 215), made public on 2026-10-06. Exponential decay of correlations on the lattice at all temperatures (n ≥ 3, uniform over finite subgraphs) comes **with a Lean formalisation**. The continuum limit of O(3) (OS axioms, a unique vacuum, a positive mass gap, non-Gaussian) and the asymptotics of the lattice mass for O(4), m ∼ 32e^(π/4−1/2) √ β e^(−πβ) (the same constant as predicted by Hasenfratz–Maggiore–Niedermayer 1990), are **in a paper manuscript only, unverified**

Two-dimensional sigma models, unlike two-dimensional YM (7-02), have asymptotic freedom and dimensional transmutation, so they are the closer example of "a problem of the same shape" as four-dimensional YM. They have no gauge symmetry and no confinement. This row does not move the four-dimensional cells of the board (Table 3).

The two four-dimensional rows are both examples where the limit is a free (Gaussian) field. For asymptotically free (4-06) Yang–Mills a non-trivial limit is expected, but there is no proof (7-08). Physics Even in the same four dimensions, ϕ^4 becomes strongly coupled at short distances, while Yang–Mills becomes weakly coupled. The directions are opposite, so the triviality theorems for ϕ^4 do not apply to Yang–Mills. Example: comparing in three dimensions, ϕ^4_3 has passed all six steps (Table 2 of 7-01). Non-abelian Yang–Mills is Partial at the ultraviolet-limit step, and Not yet from the infinite-volume step on (Table 3).

07

Where the lattice theorems lie on the number line of β

At strong coupling the high-temperature expansion (expansion in powers of small β, also called the strong-coupling expansion; 5-02) works routinely, and gives exponential decay even for continuous Lie groups (according to the introductions of Adhikari–Cao and Shen–Zhu–Zhu; Osterwalder–Seiler 1978 abstract only). At weak coupling, uniqueness of the infinite-volume limit of lattice gauge theory is in general not known for $d > 2$ with large β (Chatterjee 2018 § 3). That is the side the continuum limit needs (5-05). Known

Converting the conditions of two theorems to one β

Converting the conditions of Shen–Zhu–Zhu, with weight $\exp(N\beta\,\mathrm{Re}\sum_P \mathrm{Tr}\,U_P)$, and of Adhikari–Cao, with weight $\exp(-\beta\sum_P \mathrm{Re}(\chi(1)-\chi(U_P)))$ (χ is the character of a representation), to the β_W of formula (2) gives

$$\beta_W = N^2\beta_{\mathrm{SZZ}} < \frac{N^2}{16(d-1)}, \qquad \beta_W = 2\beta_{\mathrm{AC}} \geq \frac{2\,(114+4\log|G|)}{\Delta_G}, \qquad \Delta_G = \min_{g\neq 1} \mathrm{Re}(\chi$$

(Δ_G is the notation of Adhikari–Cao, not the mass gap Δ_G of 16-05 and 16-08). It is related to the β of 7-10 § 05 (Chatterjee's convention $e^{-\beta\sum_p \mathrm{Re}\,\mathrm{tr}(I-U_p)}$) by $\beta_W = N\beta_{\mathrm{Ch}}$, and the $|\beta| < N/48$ of 7-10 is the same condition as $\beta_W < N^2/48$ ($d=4$) in this chapter. Adhikari–Cao allow any unitary representation ρ of G, with $\chi = \mathrm{tr}\,\rho$. Here we take ρ to be the inclusion $G \subset SU(2)$, which gives Wilson's model with link variables restricted to G ($\chi(1)=2$, so $\beta_W = 2\beta_{\mathrm{AC}}$). log is the natural logarithm.

The former is $\beta_W < 0.25,\ 0.125,\ 0.083$ for $d = 2, 3, 4$ with $SU(2)$, and 9/4 times that for $SU(3)$. Computed Example: $\beta_W < 1/12$ at $d=4$ is 1/27 of the $\beta_W = 2.25$–2.60 ($SU(2)$, Lucini–Teper 2001) Physics that lattice numbers use to extrapolate to the continuum, and the condition $\beta_W \geq 697$ for the binary icosahedral group is 268 times the upper end of that range. Computed

Figure 2 Choosing a finite group $G \subset SU(2)$ with the knob moves the position where the Adhikari–Cao theorem starts to apply (the left end of the green on the right). The green on the left is the strong-coupling theorem, and the dashed lines are lattice numbers

Computed The numbers in this figure were computed on this machine (the groups built by closure from generators with their orders checked, and formula (6) computed). The range of lattice numbers is from the literature.

Computed Table 6 Finite subgroups of $SU(2)$ (defining representation) and the condition of formula (6). The defining representation is the matrices themselves, and the character is the trace (1-02). The smallest angle is the θ of the eigenvalues $e^{\pm i\theta}$ of the element nearest the identity.

Group	Order	Smallest angle	Δ_G	$\beta_W \geq$
$\{\pm 1\}$	2	180°	4	58.4

Group	Order	Smallest angle	Δ_G	$\beta_W \geq$
Quaternion group Q_8	8	90°	2	122.3
2T (binary tetrahedral group)	24	60°	1	253.4
2O (binary octahedral group)	48	45°	0.5858	442.1
2I (binary icosahedral group)	120	36°	0.3820	697.2
$2D_{50}$ (binary dihedral group)	200	3.6°	0.00395	68512

How to read Table 6

(a) The thresholds are constants that appear in the proofs, not the positions of phase transitions. (b) The finite subgroups of $SU(2)$ are only the cyclic groups, the binary dihedral groups, and three exceptions of order 24, 48 and 120. Known The groups whose names carry "binary" are rotation groups in $SO(3)$ pulled back by the double cover $SU(2) \to SO(3)$ (1-02). The binary tetrahedral, octahedral and icosahedral groups are the inverse images of the rotation groups of the regular polyhedra, and the binary dihedral group $2D_n$ is the inverse image of the rotation group of the regular n-gon (the dihedral group of order $2n$); each has twice the order of the original. For $2D_n$ (order $4n$), $\Delta_G = 2 - 2\cos(\pi/n) \approx \pi^2/n^2$, so the condition recedes roughly in proportion to n^2. Computed (c) With the one-loop $\xi \propto e^{3\pi^2 \beta_W/11}$ ($SU(2)$), each reduction of the lattice spacing to 1/10 increases β_W by about 0.86 (a one-loop rough guide; for two-loop corrections and the fact that asymptotic scaling does not yet hold in this range of β, see 5-05). Physics Why results for finite groups do not carry over to continuous groups is in 5-10, and the strong-coupling expansion in 5-02.

The β knob — two shores and a band on one axis

On the number line of Figure 2 we overlay the correlation length ξ in lattice units. The left shore is the strong-coupling theorems (the condition (9) of SZZ, and the paper of this site Paper ; 23-01 The line of construction and the diffusive-relaxation-time assumption) and the leading strong-coupling term $\xi = -1/(4 \ln u)$; the right shore is the two-loop formula (16-05 (10), $C = 185$) and Bałaban's hypothesis. The hypothesis is that the effective coupling is small enough, and no value is written (7-07 § 04), so a rough guide reading $g^2 \leq \gamma$ as $\beta \geq 4/\gamma$ is moved with the γ knob. In the middle are the numerical window 2.25–2.60 (Lucini–Teper 2001) and the band 1.75–2.6 where neither leading term is within 5% for the plaquette (16-05 Figure 1). Physics

The lower panel is the range of β over which the statements of 23-01 The line of construction and the diffusive-relaxation-time assumption are needed (this book's arrangement). The role of crossing the band falls to the infinite-volume uniqueness statement, the physical-unit gap statement and the relaxation-time band at the

bookkeeping length, which are needed at each β. The reflection-positivity preservation statement holds at every β; the Wilson-loop limit statement, the rotation-recovery statement and the non-triviality statement are statements about the $\beta \to \infty$ limit; and the OS reconstruction statement is a theorem independent of β. Known Theorems exist only at the left end, and the smaller γ is made, the further the rough guide for the right shore recedes.

Figure 3 Moving the vertical line with the β knob shows below the ξ at that β (leading strong-coupling term, two loops, numerical) and what exists there. The γ knob moves the side of Bałaban's hypothesis (the rough guide $\beta \geq 4/\gamma$). Both axes logarithmic



Computed The curves are (9)(10) of 16-05 computed on the spot (Bessel functions by series; Table 7 gives the same values with scipy). The points and window values are Lucini–Teper 2001 Physics .

Computed Table 7 Markers of β in Figure 3 and ξ (in lattice units; the column "ξ numerical" is the $1/(am)$ of 16-05 Table 3).

β	ξ leading strong-coupling term	ξ two loops (C=185)	ξ numerical	What is at this β
0.08333	0.065	—	—	Edge of condition (9) of SZZ (β < 1/12)
0.2722	0.093	—	—	Edge of the condition of this site's paper (β < 0.2722)
1.5	0.234	0.127	—	Right end where the leading strong-coupling term is within 5% for the plaquette
1.75	0.266	0.233	—	Left end of the band (neither leading term within 5%)

β	ξ leading strong-coupling term	ξ two loops (C=185)	ξ numerical	What is at this β
2.25	0.333	0.806	0.72	Left end of the numerical window
2.3	0.340	0.914	0.92	
2.4	0.354	1.17	1.05	
2.475	0.365	1.42	1.33	
2.55	0.375	1.72	1.71	
2.6	0.383	1.95	1.95	Right end of the numerical window and of the band
2.75	0.405	2.85	—	Left end where the leading weak-coupling term is within 5% for the plaquette
3.243	—	10	—	ξ = 10 at two loops (rough guide for extrapolation)
4.136	—	100	—	ξ = 100 at two loops (rough guide for extrapolation)
5.022	—	1000	—	ξ = 1000 at two loops (rough guide for extrapolation)
8	—	2.49×10^6	—	Side of Bałaban's hypothesis if γ = 0.5 (rough guide β ≥ 4/γ)
20	—	1.8×10^20	—	Side of Bałaban's hypothesis if γ = 0.2 (rough guide β ≥ 4/γ)

08

Detour: what the timeline leaves out

A section you may skip.

One comes across postings claiming to have proved the Yang–Mills problem. This chapter puts a row in the timeline under two conditions.

$$\text{include} \iff (\text{text or abstract checked}) \wedge (\text{address (1) can be written}) \quad (7)$$

Preprints that have not been peer-reviewed are included with a mark. Postings claiming to have proved the prize problem itself are not included unless there is a record of verification by experts (there are also cases where arXiv moderators withdrew one on grounds of the standard of its content). Physical numbers and holographic models measure after putting that "the theory exists" (17-13), and do not move cells. Example: the 1999 row of lattice numbers does not change the colour of any cell. The study in Part 23 of this book (23-01) is also outside the board.

The construction published in 2026 is included because there are the committee's assessment report and statement.

What this chapter can say

	Content
What is said	The content of the 37 rows of the timeline (the kind of checking is in the table of sources) Known . The row of the construction published in 2026, and the three 2026 cells of the board, are at the level of the committee's assessment Assessment
What is said	The board and the counts of rows under this chapter's way of deciding the cells. The conversion of formula (6), and the orders, Δ_G and condition values of the finite subgroups Computed . The ξ of Figure 3 and Table 7 restate the formulas and numbers in the text, and the assignment of β ranges to the eight statements of construction is this book's arrangement
What is not said	That the timeline is exhaustive. Other ways of dividing the cells. The grounds for the 1978 judgement of the strong-coupling cell are statements in the introductions of Shen–Zhu–Zhu and Adhikari–Cao; the main text of Osterwalder–Seiler 1978 has not been read. That the limit of Driver 1987 is the free electromagnetic field (main text: bibliographic data only). Drawing conclusions about the relation between finite-group models and weakly coupled $SU(2)$ from the thresholds of Table 6. This site shows nothing about any of the steps in four dimensions from the ultraviolet-limit step to the uniform-gap step (what moved cells in 2026 is the published construction). Whether the proof of the published construction is correct (cited as the committee assessed it)

Sources and reproduction

Item	Kind	Source / tools
The quotation of the problem statement in § 03, ϕ^4_3 and "the only complete example" in Table 5	Text	A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Math. Inst. (2000) § 6, § 6.2, § 6.4. Date and place of the announcement from claymath.org
The 2004 sentence	Text	M. R. Douglas, report for the Clay Math. Inst. (2004)
The continuum-limit side in § 01, the lattice correlation-length problem in § 03, the uniqueness sentence in § 07, the abelian-group rows of Table 5	Text § 3, § 4, § 5, § 6	S. Chatterjee, arXiv:1803.01950
The state of three dimensions	Text § 1	I. Chevyrev, arXiv:2202.13359 / Chevyrev–Shen, arXiv:2302.12160

Item	Kind	Source / tools
The weights and conditions of formula (6). The judgement of the strong-coupling cell (convergence of the cluster expansion in OS 1978 is in outline, and the gap follows from that convergence)	Text § 1	Shen–Zhu–Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 (equations (1.2)(1.3), introduction) / Adhikari–Cao, arXiv:2202.10375v3, Ann. Probab. 53 (2025) 140–174 (equations (1.2)(1.3), Theorem 1.1, introduction)
Driver, Bałaban, MRS	Opening of the text (material check)	Driver, CMP 123 (1989) 575–616 (Corollary 6.7; the text makes no claim about the axioms) / Bałaban, CMP 102 (1985) 255–275, 122 (1989) 355–392 / Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325–383 (abstract and § I)
The range of years 1987–89 for Bałaban in four dimensions	Bibliographic data only (the 1987 and 1988 papers)	Counted from the four-dimensional paper CMP 109 (1987) 249–301 to CMP 122 (1989) 355–392 (the same counting as 7-07 and 7-01; the foundations of 1983–85 shared with three dimensions are not counted)
Other rows from 2000 on	Abstract (material check)	arXiv:2006.04987, 2201.03487, 2111.12813, math/0101239 (Lévy; submitted January 2001, Mem. AMS 166 no.790 (2003); non-trivial bundles in PTRF 136, 2006), 1303.5113, 1912.07973, 2401.10507, 2509.04688, 2607.22236, 2607.24640
The row of the construction published in 2026, and the 2026 cells of the board	Assessment	Code of the published construction `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, submitted 2026-10-28). The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening and § 8. The committee's statement (the third statement on the page of the Kavli Institute for Theoretical Physics, 2026-11-09; signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov, Nikita Nekrasov). For how to read them, see 6-00
Range of lattice numbers	Text, Table 1 (material check)	Lucini–Teper, JHEP 0106 (2001) 050

Item	Kind	Source / tools
Original papers before 2000: Yang–Mills 1954, Gross–Wilczek / Politzer 1973, Osterwalder–Schrader 1973 / 1975, Wilson 1974, 't Hooft 1974, Feldman–Osterwalder 1976, Osterwalder–Seiler 1978, Singer 1978, Brydges–Fröhlich–Seiler 1979–81, Guth 1980, Fröhlich–Spencer 1982, Aizenman 1981–82, Fröhlich 1982, Gross 1983, King 1986, Driver 1987, Gross–King–Sengupta 1989, Sengupta 1992–93, Seiberg–Witten 1994, Maldacena 1997, Witten 1998, Morningstar–Peardon 1999	Bibliographic data only	The content is from the text and reference list of Jaffe–Witten, Chatterjee § 4 and § 6, and the statements in the introductions of Adhikari–Cao and Shen–Zhu–Zhu. Bibliographic examples: Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 / Aizenman, PRL 47 (1981) 1, CMP 86 (1982) / Fröhlich, NPB 200 (1982) / Driver, CMP 110 (1987) 479–501 / Maldacena, hep-th/9711200 / Witten, ATMP 2, 505 / Sengupta, JFA 108, Ann. Phys. 221 / Singer, CMP 60 / Fröhlich–Spencer, CMP 83 / Morningstar–Peardon, PRD 60
Table 7, Figure 3	Computed	Python and SciPy (beta_axis.py; 16-05 (9)(10), $C = 185$). The numerical window is Lucini–Teper Table 1 (16-05 Table 3, 23-01 Table 2)
Classification of the finite subgroups of $SU(2)$	Known (bibliographic data not checked)	Standard classification. Orders checked by computing closures
Table 3, Figure 1, counts of rows (recounted with the three 2026 cells added) / Table 6, Figure 2, the numbers in § 07	Computed	Python on this machine (counted year by year with formula (4)) / Python and NumPy (groups built by closure from quaternion generators, with orders, determinants and unitarity checked; $\ln 10 \cdot 11/(3\pi^2) = 0.855$)

Next chapter to read: 16-02 Five open questions. The questions left by the published construction are read following the five in the assessment report. The places where the ways physics and mathematics dig come apart are in 16-05.

Five open questions

This chapter is a map of what is unsolved. § 01 is existing knowledge (quotations from the assessment report).

What this chapter shows — For the construction of the existence of four-dimensional Yang–Mills published in 2026, we read one by one the five questions that the committee's assessment report lists as "the main points this work does not address". For each question we set out (1) what is needed, as mathematics, to answer it, (2) what is present on the side of the construction's Lean code as preparation for it, and (3) what is absent. Present: polynomial estimates uniform in the phase, uniqueness of the vacuum in each component, and parallel transport along lattice paths. Absent: exponential decay, continuum Wilson loops, and boundary conditions other than periodic ones.

Chapters this one relies on — 6-04 What is not shown, 12-05 Arzelà–Ascoli and the properties that pass to the limit, 12-07 Phase decomposition, 4-11 The definition of the mass gap.

Said first — The results of the published construction are cited at the level of "the committee's assessment is that they are proved in Lean". What this book's study has checked goes only as far as reading the definitions and the forms of the statements in the source. "Absent" means "not found within the range of a search of names and documentation strings"; the possibility that the same content exists under another name cannot be excluded. This chapter is a map of questions, and answers none of them.

Order of this chapter

1. The five questions of the assessment report — a summary of § 8 of the report
2. Convergence and uniqueness — the ultraviolet along the whole sequence, infinite volume along a subsequence
3. A continuum probability measure — the table of correlation functions and the reconstruction space
4. A uniform mass gap — polynomial estimates, and uniqueness of the vacuum in each component

5. Wilson loops and the area law — as far as parallel transport along lattice paths
6. One-particle states — tools for single-insertion states
7. Summary (Table 1) — what is needed, what is present, what is absent
8. Notes on reading — what "absent" means, and how the questions relate

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Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see “Labels” in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book.

Names of the sources: the code of the published construction is `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, submitted 2026-10-28), and the assessment report is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). Locations in the code are given as relative paths within that repository, with declaration names and line numbers.

01

The five questions of the assessment report

At the opening, the assessment report judges that the submission proves that, for any simple compact group, a relativistic quantum field theory satisfying the Wightman axioms can be constructed as a scaling limit of four-dimensional Wilson lattice gauge theory, and writes that it does not prove that the infinite-volume limit is unique, nor the mass gap Assessment (the summary at the opening of the report). The way of construction — correlation functions first take the limit in a finite physical volume, and pass to infinite volume along a subsequence obtained by compactness — is also explained in the same place Assessment . § 8 lists five main points the construction does not address Assessment . In summary:

1. **Convergence and uniqueness.** To show convergence and uniqueness of the infinite-volume limit, rather than finding a convergent subsequence.
2. **A continuum probability measure.** To define a probability measure of the continuum theory on a suitable space, and have the Schwinger functions of the construction arise from it.
3. **A uniform mass gap.** To show a uniform mass gap: exponential decay of the corresponding correlations, and a gap in the spectrum of the Osterwalder–Schrader

Hamiltonian.

4. **Wilson loops and the area law.** To build Wilson-loop observables and connect them with the local observables of the construction. To show the area law for Wilson loops.
5. **One-particle states.** To build one-particle states and analyse the energy–momentum spectrum. The report adds that this analysis has been carried out in many constructive models in two and three dimensions.

What the Jaffe–Witten problem statement (0-03) asks for is existence and a mass gap. Of the five, 3 corresponds to the latter half; 1, 2, 4 and 5 are questions for bringing the construction closer to "a complete description of Yang–Mills theory". Below we read each question in the order: what is needed as mathematics, what is present on the side of the code, and what is absent.

02

Convergence and uniqueness

What is needed. That the correlation functions converge along the whole sequence of increasing volumes (without taking a subsequence), and that the limit does not depend on how the sequence is chosen. In other words, that the infinite-volume limit is determined uniquely Proved here .

What is present.

- The ultraviolet limit (the limit sending the lattice spacing to 0) is taken along the whole sequence of cutoffs once the volume is fixed. The conclusion of the construction theorem `constructed` (`Construction.lean:15`) contains the position statement `FullVolume.PositionsUV` along the whole sequence (`Extraction.Sequence.full`) Source . In the public theorem `JointExistence` (`ChallengeSpec.lean:186`) as well, the convergence field `Converges` (`PublicWilson.lean:85-90`) requires, in each volume, convergence along the whole sequence of cutoffs; only the volume side is a subsequence Source . The strong form with kernels and differences (`PositionsUV … Sequence.full`) is present only in the construction theorem `constructed` Source .
- The infinite-volume limit is taken as a subsequence, by Arzelà–Ascoli from boundedness and equicontinuity (`Extraction.first_surface`, `YM/Extraction/Lipschitz.lean:166`; `crops`, `YM/Extraction/Held.lean:136`) Source . For how to read this, see 12-05.
- The uniqueness lemma for limits `Extraction.Order.unique` (`YM/Extraction/Orders.lean:70`) is the general fact about Hausdorff spaces that two limits along the same subsequence are equal Source . It does not say that the limit is

independent of the choice of the sequence of volumes. In the record of how declarations are used, it is not reached from any headline theorem (outside) Source .

What is absent. Estimates of the difference when the volume is changed, decay of correlations, and material for uniqueness of the limit do not enter where the infinite-volume limit is taken Source . The only inputs are upper bounds for boundedness and equicontinuity.

Uniqueness in the volume and the "uniqueness of the vacuum in each component" of the phase decomposition (§ 04) are different questions. The former is uniqueness of the limiting measure; the latter is uniqueness of the vacuum in each component within one limit.

03

A continuum probability measure

What is needed. A probability measure on a space of distributions (or a suitable quotient of the space of connections) whose moments are the Schwinger functions of the construction Proved here . It would also be the foundation for handling non-local observables such as Wilson loops.

What is present. The objects of the construction are a table of correlation functions indexed by labels of local fields (`Table`) and the Osterwalder–Schrader space built from it Inferred (for how to read this, see 12-08). On the lattice side, the Wilson measure at each lattice spacing and each volume is defined as a probability measure (`law` in `YM/Wilson/Law.lean:87`) Source .

What is absent. No definition of a probability measure of the continuum theory has been found Inferred (within the range of a name search). There is neither a statement taking the limit of the sequence of lattice measures as a measure, nor a definition giving a name to the limiting measure. For the effective density at a fixed coarse step, there is as far as convergence of that sequence (16-03).

04

A uniform mass gap

What is needed. That there is some $m > 0$ such that the spectrum of the reconstructed Hamiltonian H lies in $[m, \infty)$ away from the vacuum. In the language of correlations, the same thing is that two-point correlations decay exponentially in the distance at a rate at least m (4-11). If the limit splits into components (phases), that m must also be uniform over the components Proved here .

What is present.

- **Polynomial estimates uniform in the phase.** For a finite word w, for all phases i and $u \in (0,1]$, the spectral weight at energy $(0,u]$ is $\leq M\,u^2$ (`EveryPhase.light_bound`, `YM/EveryPhase/Light.lean:221`). For the heat semigroup, for all phases and $s \geq 1$, $0 \leq \langle A, (e^{-sH} - P_0)A\rangle \leq M\,s^{-2}$ (`EveryPhase.inverse_square`, `YM/EveryPhase/Decay.lean:173`; P_0 is the projection onto the zero-energy space) Source . The constant M does not depend on the phase.
- **Uniqueness of the vacuum in each component.** That, in each component of the phase decomposition, the zero-energy space consists only of multiples of the vacuum (`UniqueHeat`, `YM/Phase/Mixture.lean:35`) is stated for all components (`EveryPhase.every`, `YM/EveryPhase/Vacuum.lean:146`) Source . No speed (rate) is attached.

The polynomial estimates come from a far-distance estimate that comes from local commutativity (`EveryPhase.late`, `YM/EveryPhase/Light.lean:208`) Source . The estimate of the weight by u^2 is of a form compatible even with a theory whose spectrum continues down to 0, and does not contain a mass gap Inferred . In other words, the mass-gap question is to "make this u^2 weight 0 below some m" Inferred .

What is absent. No statement of exponential decay (clustering with a rate, a spectral gap) has been found Source (within the range of a name search).

05

Wilson loops and the area law

What is needed. (a) To define Wilson-loop observables (the trace of parallel transport along a closed curve) in the continuum theory. On the lattice they can be written as products of links along a path, but in the continuum the connection itself is a distribution, so arguments of regularisation and limits are needed. (b) To connect them with local fields (the basic observables of the construction). (c) That for large loops the expectation becomes exponentially small in the enclosed area (the area law) Proved here .

What is present. There are parallel transport along lattice paths, `Lattice.travel` (`YM/Lattice/Path.lean:105`), and lemmas for its concatenation, reversal and gauge transformation Source . These are the material for writing Wilson loops on the lattice. The predictor of the probability kernel of one step of renormalisation is also built from framed parallel transport (9-04) Source .

What is absent. Continuum loop observables, the area law, and the relation between loops and local fields have not been found Inferred (within the range of a name search). The lattice Wilson measure is defined only on the periodic torus Source , and other

boundary conditions such as 't Hooft twists, or a quantity corresponding to the vortex free energy, have not been found Inferred (within the range of a name search).

This book's study has the area law in the strong-coupling window on the lattice (20-06) and estimates of the vortex ratio with twisted boundary conditions (Part 21). Both are statements with the lattice spacing fixed, and do not answer (a)–(c) of this question. Their place is described in 16-04.

06

One-particle states

What is needed. A subspace with an isolated spectral shell of mass m ($p^2 = m^2$ in energy–momentum), and tools to build scattering states from it (of Haag–Ruelle type) Proved here . Usually the mass gap (§ 04) is presupposed.

What is present. There is a module dealing with single-insertion states and the spectral representation (`YM/TraceExtension/Damping.lean` , titled "Single insertion point states and spectral representation") Source . Its documentation says it "does not assume mass-gap or cone-support results" Source . Whether this amounts to preparation for one-particle analysis has not been checked Inferred .

What is absent. A definition of one-particle states, isolated shells and scattering states have not been found Inferred (within the range of a name search).

07

Summary (Table 1)

Question	What is needed	What is present on the side of the code	What is absent
Convergence and uniqueness	Convergence along the whole sequence of volumes, uniqueness of the limit	The ultraviolet (cutoff) along the whole sequence, also in the public theorem; infinite volume along an Arzelà–Ascoli subsequence Source . The strong form with kernels and differences only in `constructed` Source . Uniqueness on the same subsequence, `Order.unique` Source	Estimates of the difference when the volume changes, decay, material for uniqueness Source
A continuum probability measure	A measure whose moments are the Schwinger functions	The table of correlation functions and the reconstruction space Inferred . The lattice Wilson measure Source	A definition of the continuum measure Inferred (within the range of a name search)

Question	What is needed	What is present on the side of the code	What is absent
A uniform mass gap	Exponential decay, a spectral gap, a rate uniform over components	Polynomial estimates u^2, s^{-2} uniform in the phase Source . Uniqueness of the vacuum in each component (no rate) Source	A statement of exponential decay Source (within the range of a name search)
Wilson loops and the area law	Continuum loop observables, their relation to local fields, the area law	Parallel transport along lattice paths, `travel` Source	Continuum loops, the area law Inferred (within the range of a name search). Boundary conditions other than periodic Inferred (within the range of a name search)
One-particle states	Isolated shells, scattering states	Tools for single-insertion states and the spectral representation Source	A definition of one-particle states Inferred

Table 1 The five open questions, and the preparation on the side of the code. The wording of the questions follows § 8 of the assessment report Assessment .

08

Notes on reading

- **What "absent" means.** It means not found by searching names, documentation strings (moddoc) and the record of how declarations are used. The possibility that the same content exists under another name cannot be excluded.
- **The level of "present".** What is present is part of what "the committee assessed as proved in Lean", and this book's study has only read the forms of the statements. Which declarations enter the dependency chain of the public theorem is stated only for those checked by following the chain (6-02). For many of the declarations listed in this chapter, the chain has not been checked.
- **How the questions relate.** 3 (the mass gap) is tied to 1 (uniqueness): clustering with a rate is usually a tool for deriving uniqueness of the limit Inferred . 5 (one particle) usually presupposes 3. 4 (the area law) and 3 are different statements, and no general argument deriving one from the other is known Inferred (16-09).
- **This chapter does not answer.** For none of the questions does this book show an answer. The holdings of this book's study and their distance from the questions are described in 16-03 and 16-04. A map of the ideas this book's study has tried for the band is in 16-11.

What this chapter says

What is said	What is not said
Assessment The assessment report accepts the construction of existence, and writes that uniqueness in infinite volume and the mass gap are not proved. The five questions of § 8	Whether the judgements of the assessment report are right
Source Also in the public theorem, the ultraviolet (cutoff) is along the whole sequence and infinite volume along a subsequence (`Converges`; `first_surface`, `crops`). The strong form with kernels and differences is present only in the construction theorem `constructed`	A route to uniqueness in the volume
Source There are polynomial estimates uniform in the phase (`light_bound`, `inverse_square`), uniqueness of the vacuum in each component (`UniqueHeat`, `every`), lattice parallel transport (`travel`), and tools for single-insertion states (`TraceExtension/Damping.lean`)	Whether these enter the dependency chain of the public theorem (not checked)
Inferred A continuum probability measure, continuum Wilson loops, the area law, boundary conditions other than periodic, and one-particle states are not found. The u^2 estimate does not contain a mass gap	Excluding the possibility that they exist under other names

Sources and reproduction

Item	Kind	Source, tools
The five open questions, the judgement at the opening	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), opening and § 8
Public theorem, construction theorem	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `ChallengeSpec.lean:186` `JointExistence`, `Construction.lean:15` `constructed`
The infinite-volume subsequence and the uniqueness lemma	Source	`YM/Extraction/Lipschitz.lean:166` `first_surface`, `YM/Extraction/Held.lean:136` `crops`, `YM/Extraction/Orders.lean:70` `unique`
Polynomial estimates, uniqueness of the vacuum	Source	`YM/EveryPhase/Light.lean:208` `late`, `:221` `light_bound`, `YM/EveryPhase/Decay.lean:173` `inverse_square`, `YM/Phase/Mixture.lean:35` `UniqueHeat`, `YM/EveryPhase/Vacuum.lean:146` `every`
Lattice parallel transport, the Wilson measure, single insertion	Source	`YM/Lattice/Path.lean:105` `travel`, `YM/Wilson/Law.lean:87` `law`, `YM/TraceExtension/Damping.lean` (documentation)
The problem statement	Cited	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute). 0-03

Words used in this chapter — open question, infinite-volume limit, uniqueness, subsequence, probability measure, Schwinger function, mass gap, spectral gap, phase

decomposition, uniqueness of the vacuum, Wilson loop, area law, one-particle state. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-03 The infrared edge of the published construction — reads what is and is not said where the construction stops (the step with lattice spacing 1).

The infrared edge of the published construction

This chapter is existing knowledge (a reading of the published construction). § 05 and § 06 are a map of what is unsolved.

What this chapter shows — We read where the sequence of renormalisation steps of the published construction ends, and what is said there. Every step is carried within the weak-coupling band; the coarsest step has lattice spacing 1, and the coupling there is matched to a large reference value $\bar{\beta}$. The effective density at that step converges uniformly along the whole sequence of cutoffs once the volume is fixed. On the other hand, its limiting measure is not defined as an object, and reflection positivity, the relation to coarse Wilson loops, estimates uniform in the volume, and renormalisation to coarser scales are not stated. We set out precisely that restating the remaining questions as "long-distance questions of the effective theory at the step with lattice spacing 1" is likely right in direction, but that some statements needed to assert this are missing.

Chapters this one relies on — 16-02 Five open questions, 8-02 Sequences of spacings and of volumes, 9-01 The block kernel, 11-10 Convergence at a fixed step, and the step that removes the step index.

Said first — The results of the published construction are cited at the level of "the committee's assessment is that they are proved in Lean"; what this book's study has checked goes only as far as reading the definitions and the forms of the statements in the source. "Not stated" means not found within the range searched. This chapter describes the form of the remaining questions; it does not show a route to answering them.

Order of this chapter

1. A map of the steps — every step at weak coupling, step 0 at spacing 1 — the band of couplings and the reference value
2. The effective density at step 0, and its convergence — uniform in the cutoff, along the whole sequence
3. Correlations are read from fine steps — step 0 is where units are fixed
4. What is not stated about the objects at step 0 — five things

5. Restating the remaining questions — what is right and what is missing — the direction, the points that are not precise, the missing statements
6. From weak coupling to strong coupling — correspondence with this book's "crossover band"
7. What this chapter does not say

Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see “Labels” in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book. Sources: relative paths, declaration names and line numbers in `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0); the assessment report is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10).

01

A map of the steps — every step at weak coupling, step 0 at spacing 1

The construction starts from the Wilson measure at cutoff N (the finest step), applies one step of renormalisation by a probability kernel (9-01) N times, and coarsens through steps $N, N-1, \dots, 0$.

- **The lattice.** The lattice at step m is a periodic lattice of side $L_p \cdot D_m$, with lattice spacing $a_m = 1/D_m$ (`Schedule.side`, `YM/Lattice/Scale.lean:67`) Source . Step 0 has side L_p and lattice spacing 1, and is the coarsest step. The volume L_p is measured in the units of step 0 Source .
- **The coupling.** At every step $m \le N$, the coupling $\beta(N, p, m)$ of step m lies in the band $[\hat\beta_m/2,\ 2\hat\beta_m]$ (the field `Conv.band` of the induction hypothesis, `YM/RG/History.lean:218`) Source . $\hat\beta_m$ is the reference value $\bar\beta$ plus the one-loop running $s \log(1/a_m)$ (`Calendar.hatBeta`, `YM/RG/History.lean:79`; `scale`, `YM/Lattice/Scale.lean:115`), and at step 0, since $a_0 = 1$, $\hat\beta_0 = \bar\beta$ Source . s is the one-loop slope, with value $11C_g/(24\pi^2)$ (`slopeConstant`, `YM/Gauge/Colors.lean:184`) Source .
- **Matching at step 0.** For all N, p, $\beta(N, p, 0) = \bar\beta$ (`Selected.reaches`, `YM/RG/History.lean:239`) Source . The bare coupling at each cutoff is chosen by the

shooting method so as to satisfy this condition (10-04).

The size of the reference value. In the record of the construction (`Paid.bar`, `YM/Price/Intent.lean:148`), the reference value $\bar{\beta}$ is set equal to the barrier value `Bar.value`, and the barrier is subject to the condition "$x^{-1/2} e^{(\log\log x)^2} \le 1$ for all $x \ge \bar{\beta}$" (`Bar.threshold`, `YM/Price/Intent.lean:127`) Source. From this condition alone, $\bar{\beta}$ comes out at roughly 9×10^5 or more (the threshold being 8.97×10^5) Reading (a floating-point computation of the threshold $x \approx 8.97 \times 10^5$ beyond which the inequality holds for all x). The other condition is that the entry value `betaEntry` (same file, `:116`) is at most $\bar{\beta}$ (same file, `:126`), but the source gives no formula or number for the entry value, only its existence Source. Since the coupling at every step is at least $\hat{\beta}_m/2 \ge \bar{\beta}/2$ Source, it reads that **every step is within the weak-coupling band, and the strong-coupling region is never passed through** Inferred. This is a design choice of the construction, in a form that avoids the difficulty on the strong-coupling side (that there is no estimate that works with the same constants in every band) Inferred.

02

The effective density at step 0, and its convergence

The density. The effective density $\mathrm{eval}_{N,p,m}$ at step m, for cutoff N and volume p, is a function on configurations of the coarse lattice at step m; it is non-negative, and its integral against the reference measure (the product of Haar probability measures over links) equals the partition function Z_N (`History.partition_eq`, `YM/RG/History.lean:242`) Source. Successive steps are linked by the push-forward identity with the probability kernel that uses the coupling $\beta(N, p, m+1)$ of step $m+1$ (`Transfer`, `YM/RG/History.lean:223`) Source. In the language of measures, the normalised $\mathrm{eval}_{N,p,m}/Z_N$ is the density of the probability measure obtained by pushing the bare Wilson measure forward $N - m$ times by the kernel Inferred (combining the three identities; there is no statement in the form of an identity of measures).

Since the kernel depends on $\beta(N, p, m+1)$, this push-forward is not "the Wilson measure sent by a fixed map"; the map itself moves with N Source.

Uniform convergence in the cutoff. Fixing the step m and the volume p, and viewing the density coals_N, with the vacuum constant $e^{-v|\Lambda|}$ removed, as a sequence in N:

- It is uniformly bounded and uniformly Cauchy in the sup norm on the configuration space (`FullVolume.Retreat.Flux.fanned`, `YM/FullVolume/Fields/Cement.lean:50`) Source.

- Its integral (the normalisation constant) is also Cauchy (`Flux.lacks`, same file, `:56`) Source .
- Within the construction, this is obtained for every step $m \geq 0$ (including step 0) and every p, along the whole sequence of cutoffs (without subsequences) (`sees`, `YM/FullVolume/Fields/Filled.lean:30`) Source .

Uniform convergence of the normalised density eval/Z_N is not a named statement, but it follows in one line from the uniform Cauchy property of the numerator, the Cauchy property of the denominator, and a positive lower bound on the denominator independent of N (`lingers`, `YM/FullVolume/Fields/Till.lean:20`) Inferred . Since the configuration space is compact and the reference measure is a probability measure, convergence in the sup norm includes convergence in total variation, and is stronger than weak convergence Inferred .

03

Correlations are read from fine steps

Continuum correlations are read not from step 0 but from a fine step j. The difference between the value computed on the representation at step j using the step-j image of the observable and the true lattice correlation is bounded by a quantity independent of N (`Sever.echoes`, `YM/FullVolume/Fields/Peat.lean:42`), and becomes small as $j \to \infty$ under a condition on the separation of points Source . Combining this with the Cauchy property at a fixed step j (`latter`, `YM/FullVolume/Fields/Timber.lean:58`), the true lattice correlation becomes Cauchy in N (`arrives`, same file, `:74`) Source .

In this line of argument, step 0 is "where units are fixed by $\bar{\beta}$", and is not used in estimating correlations Inferred . For distances longer than step 0, the volume L_p is simply placed in the units of step 0, and the infinite-volume limit is taken by compactness along a subsequence (12-05) Source .

04

What is not stated about the objects at step 0

1. **The limiting measure.** The $N \to \infty$ limit of the measure at step 0 (or at any step) is not defined as an object. No definition giving a name to a limiting density or limiting representation has been found Source (within the range of a name search). What there is goes only as far as the materials of the representation (couplings, activities, path data) being Cauchy in N, or converging to some limit Source . That the limit has a representation of the same form, or that it is consistent across steps, is also not stated.

2. **Reflection positivity.** The statement of reflection positivity (`Slice.reflected_positive`, `YM/Slice/Reflection.lean:98`) is only about the bare Wilson measure Source. Reflection positivity of the pushed-forward measure at step 0 is not stated Source (within the range of a name search). Since the support of the kernel can be large enough to straddle the mirror plane (`fitsPatch`, `YM/RG/History.lean:110`), we infer that the push-forward does not automatically inherit reflection positivity Inferred.
3. **The relation to coarse Wilson loops.** The relation between Wilson loops of configurations at step 0 and Wilson loops of the fine field (or the continuum) is not stated Source (within the range of a name search). Since a coarse bond is a prediction from framed parallel transport multiplied by a fluctuation (`connector`, `YM/RG/Block.lean:39`), we infer that a coarse loop is close to "parallel transport along a thick path, with fluctuations" Inferred.
4. **Estimates uniform in the volume.** Estimates for the objects at step 0, such as the lower bound on the normalisation constant (`till`, `YM/FullVolume/Fields/Till.lean:9`) and the constant in the difference estimate (`refreeze`, `YM/FullVolume/Fields/Flight.lean:28`), contain exponentials of the number of lattice points $|\Lambda|$, and are not uniform in the volume p Source. What could be read as estimates uniform in the volume are things like the size of observables measured by the chessboard root on the side of the Wilson measure (`Extraction.Wcmain`, `YM/Extraction/Center.lean:38`; for how to read this, see 12-02), and these are not estimates of the measure at step 0 Source. The bookkeeping of one step itself (the norms of activities and the contraction constants) consists of pointwise sums and does not depend on the volume Source.
5. **Renormalisation to coarser scales.** Step numbers are natural numbers, and step 0 is the coarsest step. No theorem continuing renormalisation to scales coarser than step 0 has been found Source (within the range of a name search). The group of modules with ultraviolet extension in their names (`YM/Extension/UV/`) is a separate route on the ultraviolet side, not a continuation to the coarse side Source (documentation).

05

Restating the remaining questions — what is right and what is missing

Among the open questions of the assessment report, uniqueness in infinite volume, the mass gap and the area law (1, 3 and 4 of 16-02) are all questions about long-distance behaviour. We consider restating them as "long-distance questions, as $L_p \to \infty$, of the effective lattice theory at step 0 (with large coupling $\bar{\beta}$ and lattice spacing 1)".

Facts that can be read from the source Source.

- Every step is carried within the weak-coupling band, and at step 0, $\beta = \bar{\beta}$.
- The volume L_p is measured in the units of step 0.
- Behaviour at distances longer than lattice spacing 1 (exponential decay, the area law, uniqueness of the volume limit) is not dealt with in any theorem Source (within the range of a name search).

On the basis of these facts, restating the remaining questions as "long-distance questions of the theory at step 0 ($\bar{\beta}$ large, $L_p \to \infty$)" is likely right in direction Inferred. Since the coupling at that step is on the weak-coupling side, it is this book's inference that they can be further restated as "the question of following the effective lattice theory at spacing 1 from weak coupling to strong coupling" Inferred.

Neither can be asserted. The theory at step 0 is not a theory with the Wilson action but a theory with an effective action given by a representation made of a truncated action, activities and path data, and $\bar{\beta}$ is a coefficient in that representation. The representation is not unique Source. Reflection positivity of that measure is not stated either (§ 04). What can be written is only "the main part of the remaining questions is likely to be reducible to exponential decay at long distances in the effective theory at step 0" Inferred.

Points where this is not precise.

- (a) Continuum correlations are not functions of the measure at step 0 alone. The true correlation can be sent to step 0 as the push-forward of a density with the observable inserted, but to write it as an expectation under the measure at step 0, the image of the observable at step 0 is needed Source.
- (b) Convergence of the image of an observable at step m (`Rising.hail`, `YM/FullVolume/Fields/Capital.lean:57`) is stated only for m at or above a reference step for each observable. For observables whose reference step is above 0, the image at step 0 is not dealt with Source (a field of the hypotheses).
- (c) The estimate of the difference between a fine step and the true correlation (§ 03) is used only in the direction $j \to \infty$, and the error of reading out at step 0 is not small Source.

Statements missing for the restatement (the same root as § 04).

1. The existence and naming of the $N \to \infty$ limit of the measure at step 0. It is likely to follow from the convergence of § 02 Inferred, but there is no statement.
2. Reflection positivity of the pushed-forward measure at step 0 (or at any step). Because of the support of the kernel, it is not obvious Inferred.

3. The relation between Wilson loops of coarse configurations and Wilson loops of the fine field (or the continuum).
4. Estimates for the objects at step 0 that are uniform in the volume p.
5. The limit of the images of observables at step 0 (at every order), and an identity connecting expectations at step 0 with continuum correlations.

In particular, without the three of 1 (the limiting object), 4 (local estimates uniform in the volume) and 5 (images of observables at step 0), one cannot state that "the remaining questions are long-distance questions of the effective theory at step 0". With all five, the remaining questions could be stated as "questions about one named family of lattice measures". Until they are all in place, the restatement is a guide to direction.

06

From weak coupling to strong coupling

The coupling $\bar{\beta}$ at step 0 is large (weak coupling), and the lengths at which confinement and the mass gap become visible are expected to lie much further on, even in the units of that step Inferred . If the one-loop running is formally extended to the side coarser than step 0, the length at which the coupling becomes of order 1 is of order $e^{\bar{\beta}/s}$ in the units of step 0 Inferred (an extrapolation, not a statement of the construction). Getting there would need new steps flowing from step 0 further to the coarse side until the strong-coupling region is entered Inferred . What is needed at the entrance to them is 1–3 of § 05 (the limiting measure, reflection positivity, the relation to coarse Wilson loops) Inferred . This is a different set from the three needed to turn the restatement into a statement (1, 4 and 5 of § 05).

This book has called the intermediate region connecting weak and strong coupling "the crossover band" (26-00). If the remaining questions may be restated as above, they correspond to questions of crossing this band Inferred . The holdings of this book's study lie at the two ends of the band, and the middle remains open (16-04).

One caution. The renormalisation of this book's study (the push-forward by a deterministic block map, 19-04) and the push-forward of the published construction (a coupling-dependent probability kernel) are not the same, even if their objects are similar. Convergence of one does not give convergence of the other Inferred .

07

What this chapter does not say

- It shows nothing about the long-distance behaviour of the theory at step 0.

- It says neither that the five statements of § 05 hold nor that they do not. About 2, it only infers that it is not obvious.
- "Not stated" is the result within the range searched. The possibility that the same content exists under another name cannot be excluded.
- Whether the proof of the published construction is correct stays at the level of the committee's assessment.

What this chapter says

What is said	What is not said
Source The coupling at every step is in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$; step 0 has lattice spacing 1 and $\beta = \bar{\beta}$ (`Conv.band`, `Selected.reaches`, `Schedule.side`)	An explicit value of $\bar{\beta}$ (from the condition, roughly 9×10^5 or more Reading ; the entry value is given only as existing)
Source The effective density at each step, including step 0, is uniformly Cauchy in the sup norm along the whole sequence of cutoffs with the volume fixed, and the normalisation constant is also Cauchy (`Flux.fanned`, `lacks`, `sees`)	Convergence uniform in the volume
Inferred Convergence of the normalised density in total variation follows in one line	Its form as a named statement
Source The limiting measure, its reflection positivity, the relation to coarse Wilson loops, estimates uniform in the volume for the objects at step 0, and renormalisation coarser than step 0 are not stated within the range searched	Excluding the possibility that they exist under other names
Source Every step is in the weak-coupling band, step 0 is at the reference value, the volume is in the units of step 0, and distances longer than spacing 1 are dealt with in no theorem (within the range of a name search). Inferred On that basis, restating the remaining questions as "long-distance questions of the theory at step 0" is likely right in direction, and they can be further restated as "the question of following it from weak coupling to strong coupling". Five statements are missing to make this a statement	The answer to that question

Sources and reproduction

Item	Kind	Source, tools
Lattices of the steps, the band of couplings, matching at step 0, the mass of the density, push-forward	Source	`yang-mills-group/OpenAI-YM-20261028`: `YM/Lattice/Scale.lean:67` `side`, `YM/RG/History.lean:218` `Conv.band`, `:223` `Transfer`, `:239` `Selected.reaches`, `:242` `partition_eq`, `:110` `fitsPatch`

Item	Kind	Source, tools
Uniform convergence of the density	Source	`YM/FullVolume/Fields/Cement.lean:50` `Flux.fanned`, `:56` `lacks`, `Filled.lean:30` `sees`, `Till.lean:9` `till`, `:20` `lingers`
Reading out from fine steps	Source	`YM/FullVolume/Fields/Peat.lean:42` `Sever.echoes`, `Timber.lean:58` `latter`, `:74` `arrives`, `Flight.lean:28` `refreeze`, `Capital.lean:57` `Rising.hail`
Reflection positivity, the chessboard root, prediction of coarse bonds	Source	`YM/Slice/Reflection.lean:98` `reflected_positive`, `YM/Extraction/Center.lean:38` `Wcmain`, `YM/RG/Block.lean:39` `connector`
Open questions	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8

Words used in this chapter — renormalisation step, lattice spacing, band of couplings, reference value, probability kernel, push-forward, effective density, uniformly Cauchy, total variation, reflection positivity, chessboard root, crossover band. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-04 Where this book stands — sets out at which end, weak coupling or strong coupling, the holdings of this book's study lie, and guides the reading of Volume 3.

Where this book stands

> This chapter is a guide. § 02 is a map of what is unsolved.

What this chapter shows — We line up the theorems that this book's study has machine-checked in Lean along the axis of the coupling. The holdings lie at both ends. At the strong-coupling end there are the infinite-volume limit, the mass gap (in lattice units), the area law and an upper bound on the vortex free energy; as things that hold at any coupling there are the vortex floors and properties of the vortex ratio; and on the weak-coupling side there is the rarity of large plaquettes. The middle (the crossover band) remains open. Finally, we guide the reader through what each part of Volume 3 deals with.

Chapters this one relies on — 16-02 Five open questions, 16-03 The infrared edge of the published construction.

Said first — Every theorem listed here is a statement with the lattice spacing fixed, and a formalisation of a result of a known type. Neither the continuum limit nor the mass gap in the sense of the problem statement is shown. Statements of kinds that the published construction does not deal with are included, but this does not mean that any open question has been answered.

Order of this chapter

1. Holdings on the axis of the coupling — Table 1
2. The middle remains open — what is missing in the crossover band
3. Overlaps and differences with the published construction — the same direction, different kinds
4. How to read Volume 3 — one line per part, and three reading orders
5. What this chapter does not say

Labels — Source = the source of the published construction was read / Inferred = this book's inference / Assessment = according to the committee's assessment report (for details, see "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are labels used throughout the book; Reading is this book's reading of numbers, and Lean is a Lean theorem of this book.

01

Holdings on the axis of the coupling

We view the coupling β of lattice gauge theory as a single axis. The side of small β is strong coupling, the region where confinement and the mass gap can be shown by expansions of known type (5-02). $\beta \to \infty$ is weak coupling, and the continuum limit is taken at this end (5-05). Placing the theorems of this book's study on this axis gives Table 1.

Place	Theorem	Range	Chapter
Strong-coupling end	Exponential decay of correlations (constants free of the volume)	$18 \tanh \lvert\beta\rvert < 1$, etc.	20-02
	The infinite-volume limit (no subsequence), translation invariance, clustering	$\beta \neq 0$, $18 \tanh \lvert\beta\rvert < 1$	20-03
	The mass gap of the vessel of reconstruction (lattice units, a lower bound)	The same window	20-04
	The area law for Wilson loops (finite volume, infinite volume)	In a finite box $\beta < 1.135 \cdot 10^{-3}$ (the version with constant 441); in infinite volume $\beta < 1.22 \cdot 10^{-4}$ (the version with constant 4096; also $\beta > 0$ and $18 \tanh \beta < 1$)	20-06
	An upper bound on the vortex free energy (the rate per unit area does not depend on the transverse size; the prefactor N^2 (the transverse area) remains)	$\beta < 1.22 \cdot 10^{-4}$ in the basic version (constant 4096), $\beta < 1.135 \cdot 10^{-3}$ in the sharp version (constant 441)	20-06

Place	Theorem	Range	Chapter
At any coupling	Reflection positivity (three forms)	All real β (depending on the form)	20-01
	The thin-vortex floor (the rate depends on the transverse size N)	All β	21-02
	The thick-vortex floor, the minimal action of a vortex	For each sufficiently large β (the threshold only as existing)	21-03, 21-05
	The sign of θ, pairs of Polyakov loops	Even vertical size M	21-06
	Continuity of the vortex ratio, the $\beta \to \infty$ limit, the largest solution of the matching, a lower bound on the deficit of the twisted box	With the lattice fixed	21-07
Weak-coupling side	Rarity of large plaquettes (uniform in the volume)	$\beta \geq 81$ (explicit threshold)	22-01

Table 1 Where the firm theorems of this book's study are placed. Each is a formalisation in this book's Lean, with the lattice spacing fixed, of a result of a known type Lean . The values of the windows are floating-point readings Reading .

Two notes on reading it.

- **The strong-coupling end is "the side where things are easier to show".** The mass gap and area law here are statements different from the mass gap of the continuum theory that the problem statement asks for, and they lie on the opposite side from the weak-coupling end where the continuum limit is needed.
- **What holds at any coupling is not uniform in the transverse size.** The rate of the vortex floor tends to 0 as the transverse size N is made large (the thin floor), or the threshold depends on N (the thick floor). Restricted to the strong-coupling window, there is a form whose rate does not depend on N. The prefactor N^2 (the transverse area) remains, and outside the window there is no uniform form (20-06).

02

The middle remains open

The strong-coupling window for the infinite-volume limit and the mass gap is $|\beta| < 1/19$ (≈ 0.053; an explicit sufficient condition for 18 $\tanh|\beta| < 1$, the window itself being $\operatorname{artanh}(1/18) \approx 0.0556$; 20-03), and the threshold for rarity is $\beta \geq 81$; the band of interest in physics (β around 2 to 3) falls in neither. We have called this middle "the crossover band" (26-00). What is missing to cross the band is set out in 19-07 as statements written with defined objects only (matching, convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band). Each of them asks for estimates **uniform** in the lattice spacing and the size of the box, and they do

not follow from the firm theorems of this book (each a statement at one lattice spacing). A map of the ideas this book's study has tried for the band is in 16-11, and a list of the models on which arguments about the band are tested is in 26-12.

Seen from the side of the published construction too, the remaining questions are in the same place. The construction carries every step at weak coupling and stops at the step with lattice spacing 1 (16-03). The main part of the remaining questions (the mass gap, the area law, uniqueness) is likely to be reducible to exponential decay at long distances in the effective theory at that step Inferred . However, the theory at step 0 is not one with the Wilson action but a theory with an effective action given by a representation (the reference coupling $\bar{\beta}$ is a coefficient in it, and the representation is not unique), and its reflection positivity is not stated either Source . To state this, three things are needed: the limiting object, local estimates uniform in the volume, and images of observables at step 0. Under this expectation, the questions gather around the band from both sides Inferred .

About the band, what this book's study has is a conditional implication and a list of premises (27-01, 28-04). It has not shown that the premises are satisfied.

03

Overlaps and differences with the published construction

Things in the same direction. Reflection positivity of the lattice Wilson measure and the Chessboard Estimate are present in both Source . The published construction has, for the original Wilson measure, reflection positivity in planes through lattice points and in planes through midpoints of edges, along all four axes ($\beta \geq 0$), and also the triangle inequality for the chessboard root Source (12-01, 12-02). The line of argument that takes subsequential limits and transfers properties to them is also of the same type.

Kinds of things not found in the published construction (within the range searched).

- **Rarity of large plaquettes.** The published construction uses the Chessboard Estimate to measure the size of observables, not to estimate the probability of sets of large plaquettes Inferred (22-00).
- **Twisted boundary conditions and the vortex free energy.** The Wilson measure of the published construction is defined only on the periodic torus, and no statement about twists has been found Inferred (within the range of a name search).
- **The mass gap and area law inside the strong-coupling window.** The published construction does not pass through the strong-coupling region (16-03 § 01), so statements of this kind appear to be absent Inferred (within the range of a name search).

This difference is not a comparison of which has got further. The published construction gives the existence of the four-dimensional continuum theory, as assessed by the committee Assessment . This book's study does not have that. Conversely, the three items above are statements with the lattice fixed, and answer none of the open questions of the assessment report (16-02). This book's strong-coupling results (the infinite-volume limit, the mass gap, the area law, the vortex floors) are statements at the **opposite end** (strong coupling, with the lattice spacing fixed) from the remaining questions (a uniform mass gap of the continuum theory, and so on). They deal with the same kinds of quantities in a different region, and we do not say that they give a route to the open questions.

04

How to read Volume 3

Part	Title	What it deals with
Part 16	Where things stand, and the questions that remain	The timeline, the open questions, the infrared edge of the published construction, where this book stands, the seams, tasks, misconceptions, the table of seven stages, the map of tools
Part 17	From the side of physics	Mechanisms of confinement, the string picture, large N, the mass gap as geometry. An arrangement of views that are not mathematical theorems
Part 18	A dictionary of two languages	Correspondence between the words of physics and the words of mathematics; this book's assumptions in the language of physics
Part 19	Objects, and the statements to be shown	Definitions of the lattice, the Wilson measure, twists, reflection and the block map (with Lean definitions), and the statements to be shown
Part 20	Firm theorems — the vessel and strong coupling	Reflection positivity, decay at strong coupling, the infinite-volume limit, reconstruction and the mass gap, the vessel with fermions, the area law
Part 21	Firm theorems — the vortex floor and Polyakov loops	Sign systems, the thin- and thick-vortex floors, the minimal action of a vortex, the sign of θ, the vortex ratio — continuity, limit, matching
Part 22	Firm theorems — the rarity of large plaquettes	The Chessboard Estimate, and rarity uniform in the volume at weak coupling
Parts 23 and 24	The first road — from the direction of time	The record of the study heading for the band from relaxation times and correlation lengths
Part 25	The second road — from the direction of space	The record of the study heading for the band from small twisted tori and confinement
Part 26	The crossover band	The correct form of the band, reduction, what cannot be crossed by structure alone, the comparison with renormalisation

Part	Title	What it deals with
Part 27	Closing — conditional statements	The conditional statement and the ledger of conditions, the index of questions and of what is not said
Part 28	The tree of premises	Three implications, the list of premises, what depends on design choices, the places that are open

Table 2 The parts of Volume 3.

Suggested reading orders.

- **If you want only what is firm.** Part 19 (objects) → Parts 20–22 (firm theorems). The last chapter of each part has a table of "what it says, and what it does not".
- **If you want the shape of the remaining questions.** This part (16-02 to 16-04) → 19-07 (the statements to be shown) → Part 26 (the band) → Part 28 (the tree of premises).
- **If you want to start from motivation.** Parts 17 and 18 (the physical view and the dictionary) → this part → Parts 20–22.

Parts 23–25 are records of the study along the two roads towards the band, and have not been rewritten. Most of their conclusions are gathered in Parts 26–28.

05

What this chapter does not say

- It does not claim to have shown anything about the mass gap proper (a mass gap of the continuum theory, uniform in the lattice spacing).
- Every theorem of Table 1 is a formalisation of a result of a known type, and no novelty is claimed.
- What is "not found" in the published construction is the result within the range searched; the possibility that it exists under another name cannot be excluded.
- It says neither that the band statement is true nor by which route it could be shown.

What this chapter says

What is said	What is not said
`Lean` The theorems of Table 1 (the strong-coupling end, any coupling, the weak-coupling side) are machine-checked in Lean. Details are in each chapter	Estimates uniform in the lattice spacing, the continuum limit, the band

What is said	What is not said
Source Reflection positivity and the chessboard are present in both. Inferred The published construction does not pass through the strong-coupling region (depending on the size of $\bar{\beta}$)	—
Inferred Rarity of large plaquettes and twisted boundary conditions are not found in the published construction (within the range of a name search). This book's strong-coupling results are statements at the opposite end from the remaining questions	Answers or routes to the remaining questions

Sources and reproduction

Item	Kind	Source, tools
The theorems of Table 1	Lean	The tables of sources in each chapter (of the types of Osterwalder–Seiler, Dobrushin, 't Hooft, Fröhlich–Israel–Lieb–Simon) and the Lean declarations
The infrared of the published construction and reflection positivity	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0). Locations are in 16-03 and 12-01
The assessment of the construction of existence, the open questions	Assessment	The assessment report (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), opening and § 8

Words used in this chapter — strong coupling, weak coupling, crossover band, area law, vortex floor, vortex free energy, rarity of large plaquettes, statements to be shown, tree of premises. Their meanings are in the A-02 Glossary.

Next chapter to read: 16-05 Where the two ways of digging part — maps where the two roads of this book's study are cut off. To read from the firm theorems, go to 19-00.

Where the two ways of digging part — seeing the mismatch of two tunnels in seven seams

This chapter is a map of open questions. § 05 is this book's own investigation.

What this chapter shows — For each of the seven seams, the sentence of physics, the mathematical statement and the partial bridges, and three types. Comparing, for a local quantity and for a mass, how far the two shores are from the numbers (calculations on this machine).

Prerequisite chapters — 7-11 Obstacles beyond divergence and 17-13 Where the way physics digs parts from mathematics. The tools are 4-10 (reflection positivity), 5-01 (the lattice) and 4-07 (dimensional transmutation).

Said first — This chapter names the seams; it fills none of them. The count of seven and the types are this book's arrangement. The record of this book's study has worked only around the seam of "there is one scale", and has not reached a theorem. This chapter shows neither the existence of the mass gap nor the construction of the continuum limit. The seam of existence has been bridged by the construction published in 2026 (the committee's assessment Assessment ; Part 6). The label Assessment means "according to the committee's assessment report" (“Labels” in 6-00).

The order of this chapter

1. Two ways of digging — from existence, or from values. A parallel table of walls (six obstacles and seven moves)
2. A map of the seams — seven seams and three types (Table 1)
3. The type in which a premise becomes a goal — existence, a single scale
4. The type of limits and uniformity — shores, finite and infinite, extra dimensions
5. Measuring between the shores — the plaquette and the mass (Figure 1, computed)
6. The type in which the same word points to different things — spacetime and space, particles
7. Overlaying the two lists — partial bridges

01

Two ways of digging

There are two directions in which to dig at this problem (0-02). Mathematics starts from "there is no theory yet" and passes through the six stages (7-01) in order; physics starts from "the theory exists" and computes quantities inside it. As a metaphor, they are two tunnels dug into the same mountain from both sides; more precisely, there are places where a sentence on one side cannot be handed over as it is to the other. The goal of mathematics has the following form (0-03) Known .

$$\forall G \;\; \exists \mathcal{T}_G : \quad \text{axioms} \;\wedge\; \text{non-trivial} \;\wedge\; \text{agrees with asymptotic freedom at short distances} \;\wedge\; \text{s}$$

G is a compact simple group, $\mathcal{T}_G$ the quantum Yang–Mills theory on $\mathbb{R}^4$, and H its Hamiltonian. The physical sentence "computing the quantity Q in $\mathcal{T}_G$ gives q" includes $\exists \mathcal{T}_G$ among its premises. We translate a sentence of physics P into a mathematical statement M, and call a place where the translation does not become a theorem a **seam**.

$$\text{seam} = (P, M) : \quad P \text{ is used in physics}, \quad M \text{ is not a theorem, or points at something other t}$$

Example: the $m_{0^{++}}/\sqrt{\sigma} = 3.844(61)$ of $SU(2)$ (Lucini–Teper 2001) Physics is a prediction in physics, and in mathematics a target that the constructed theory must hit (17-13 § 01).

A parallel table of walls — the six obstacles of mathematics and the seven moves of physics

The seams of Table 1 were made by pairing rows of the lists on the two sides. Before that, we give a parallel table of the rows by name. On the left are the six obstacles of 7-11 Table 4, and on the right the seven moves of 17-13 Table 5. The stock phrases are the expressions each side uses when pointing at that wall. Some rows exist on one side only.

Parallel table Names of the walls (the correspondence is this book's arrangement)

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
Obstacle 1 No small parameter in the infrared	Moving to a neighbouring theory and coming back with a knob, Looking from the side of a small parameter	"No uniform estimate across the scale where the coupling is $O(1)$" / "Below Λ it is non-perturbative"	Both sides

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
Obstacle 2 Invisible to perturbation theory	Looking from the side of a small parameter, Regarding as one	"All Taylor coefficients of $e^{-c/\alpha}$ are 0" / "$\Delta = c\Lambda$"	Both sides
Obstacle 3 Not collapsing to a free field	Measuring after putting that it exists	"Show that the limit is not a free field" / "The theory exists"	Both sides
Obstacle 4 The state space is curved	Looking after choosing a gauge	"There is no global gauge fixing" / "Compute with the gauge fixed"	Both sides ("Outside the table: gauge" in Table 4)
Obstacle 5 Few regularisations preserve reflection positivity	—	"A regularisation that preserves positivity" / —	**Mathematics only**. Numerical calculations in physics also use improved actions that lose positivity (the entry "improved action" in 18-04)
Obstacle 6 The structure of infinitely many degrees of freedom	—	"The representations are inequivalent (Haag's theorem)" / —	**Mathematics only**. Perturbative calculations in physics proceed using the interaction picture
—	Taking σ from the string picture (§ 03, § 04)	— / "A string is stretched, the area law"	**Physics only**. Confinement (the area law) is outside what the problem statement requires, an extension problem (17-02)

Name in mathematics (7-11)	Name in physics (17-13)	Stock phrase (mathematics / physics)	One side only?
— (outside the six: the ultraviolet side)	Extrapolating from finite β	"Construction of the continuum limit" / "Extrapolate in a^2"	Not among the six of 7-11. In mathematics it is a problem on the ultraviolet side (7-07, 7-10)

The four rows on both sides call the same wall by different words. The rows on one side only are places that, in that side's language, are not seen as walls. Positivity and equivalence of representations pass unasked in the procedures of physics, and the area law passes unasked in the mathematical problem statement. Rearranging this parallel table into seams gives Table 1, and lining it up with the partial bridges gives Table 4 of § 07.

02

A map of the seams — seven seams and three types

Table 1 lines up the seven seams. The order of mathematics (existence → axioms → gap) and the order of physics (values → mechanisms) run in opposite directions, so stages at the same position do not correspond. As a metaphor this is the mismatch of the tunnels; more precisely, a premise on one side becomes the goal of the last stage on the other.

Table 1 The seven seams (the division and the types are this book's arrangement)

Seam	Sentence of physics	Mathematical statement	Type
1 The premise of existence	Put that the theory exists, and compute quantities in it	Make the limit satisfying the axioms, not as a subsequence	Type in which a premise becomes a goal
2 A single scale	Mass and string tension are both constant multiples of one Λ	Formula (5) holds uniformly in β	Type in which a premise becomes a goal
3 Different shores	Compute on the side of g→0, small β, N→∞	Control every scale at finite N	Type of limits and uniformity
4 Finite and infinite	Measure in finite boxes, and extrapolate in a^2	A gap uniform in the volume (Chatterjee's lattice correlation-length problem, Problem 5.1)	Type of limits and uniformity
5 Spacetime and space	Think in a spacetime where particles fly and there is causality	Come back from a Euclidean measure by reflection positivity	Type in which the same word points to different things

Seam	Sentence of physics	Mathematical statement	Type
6 The word "particle"	The gap is the mass of the lightest glueball	A spectral gap (an isolated one-particle state is an extension problem)	Type in which the same word points to different things
7 Extra dimensions	Get the gap from the geometry of the cigar	The R→0 (λ→0) limit, and exchanging limits	Type of limits and uniformity

The type in which a premise becomes a goal is a seam where a premise of physics becomes a goal of mathematics; **the type of limits and uniformity** is a seam where physics computes at a particular value (a shore) of a parameter p, and mathematics needs the limit that moves p to the value p_* of the problem statement; **the type in which the same word points to different things** is a seam where the same word points at different objects. The type of limits and uniformity has the following form.

$$\text{physics}: \ F(p_{\text{shore}}) \text{ computed}, \qquad \text{mathematics}: \ \lim_{p \to p_*} F(p) \text{ exists, and } \inf_{p} \Delta(p) > 0 \text{ and}$$

Example: in the seam of finite and infinite, $p = L$ (the side of the box) with $p_* = \infty$; in the seam of extra dimensions, $p = R$ (the size of the extra dimensions) with $p_* = 0$.

The following sections are ordered by type (the type in which a premise becomes a goal is 1 and 2, the type of limits and uniformity is 3, 4 and 7, and the type in which the same word points to different things is 5 and 6).

03

The type in which a premise becomes a goal

The seam of the premise of existence

The calculations of physics stand on the existence of the theory (a measure and a Hilbert space), and mathematics has to make that existence. Partway through the making, neither mass nor string tension is defined. The problem statement writes 'one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition' Known . Mathematics shows $\exists\,\mathcal{T}$ as the limit of approximations $\mu_{a,L}$ with lattice spacing a and side L.

$$\mu = \lim_{L \to \infty} \lim_{a \to 0} \mu_{a,L} \qquad \text{(not a subsequence: the limit itself exists and is unique)} \ (4)$$

The problem statement writes, in footnote 2 of § 6.5, that a subsequential limit obtained by compactness (weak existence) alone is not accepted as a solution (unless the properties of the limit are established by another method) Known . Example: in two dimensions the measure has been built up to $\mathbb{R}^2$ (7-02); in three dimensions, 'a construction of the 3D YM measure and a description of its gauge-invariant

observables, even on T^3, remains open' (Chevyrev 2022)Known ; and in four dimensions, up to 2025, the stage reached was ultraviolet stability in finite volume obtained with a hypothesis (7-07).

The construction published in 2026 built, for any compact simple group, a theory satisfying the Wightman axioms as a limit of four-dimensional Wilson lattice gauge theory. The committee's assessment is that this is “proved in Lean”Assessment (6-00). The inner limit $a \to 0$ of formula (4) is taken with the periodic box fixed, along the whole sequence of cutoffs, and the outer limit $L \to \infty$ is along a subsequence obtained by compactness (8-05, 12-05). The assessment report writes that convergence and uniqueness of the infinite-volume limit are not shownAssessment (16-02 § 02). This chapter does not judge what this construction amounts to in the sense of footnote 2. Seen from the side of physics, half of "the theory exists" in this seam has become a mathematical statement. What remains is that the quantities in that theory (the mass gap, the string tension) be determined as mathematics, and that is a question on the side of the seam of "there is one scale" and the seam of finite and infinite.

The seam of "there is one scale"

In physics every quantity with a dimension is taken to be a constant multiple of a single scale Λ (4-07) Physics . Translated into mathematics, for example for the mass $m(\beta)$ and string tension $\sigma(\beta)$ in lattice units ($\beta = 2N/g^2$ is the lattice inverse coupling, not an inverse temperature), it is that the following holds.

$$0 < c \le m(\beta)^2/\sigma(\beta) \le C < \infty \qquad (\beta \ge \beta_0 \text{ uniformly}) \quad (5)$$

This is peculiar to four dimensions because $[g^2] = (\text{mass})^{4-d}$ Known . For $d < 4$, if the theory exists, dimensional analysis gives a single scale; at $d = 4$, $\Lambda = \mu\, e^{-1/(2b_0 g^2(\mu))}$ is born as a quantum effect from the running of the coupling Physics . But dimensional analysis fixes only the form. For example at $d = 3$, g^2 has the dimension of mass, so one can write $m = c_1 g^2$ and $\sigma = c_2 g^4$, but this does not forbid the constants c_1, c_2 from going to 0 or ∞ in the limit. Example: examples where one scale breaks are three-dimensional compact $U(1)$ (said to have $\sigma/m^2 \to \infty$ in the continuum limit with the mass fixed in physical units; the abstract of Göpfert–Mack writes that in the continuum limit with m_D fixed the string tension in physical units is infinite; main text not read) and the supergravity regime ($m^2/\sigma \sim 1/\lambda$) Physics (17-12 § 08). The record of this book's study cut out a sentence close to (5) as the diffusive-relaxation-time assumption and split it into three chains (17-12, 23-01), but it has not become a theorem.

> 04

The type of limits and uniformity

The seam of the different shores

Theorems and calculations work at three shores: perturbation theory at $g \to 0$, the strong-coupling expansion at small β, and holography at $N \to \infty$ together with $\lambda = g^2 N \to \infty$. The theory of the problem statement is on none of these shores. Converting the lattice coupling to the 't Hooft coupling gives the following.

$$\lambda = g^2 N = 2N^2/\beta \quad (6)$$

On the strong-coupling side, the infinite-volume limit and exponential clustering (the gap) at sufficiently small β go back to Osterwalder–Seiler 1978 (range not made explicit; according to the introduction of Shen–Zhu–Zhu) Known . The condition of the theorem with an explicit range (Shen–Zhu–Zhu; arXiv 2022, CMP 2023) is $\beta < N^2/(16(d-1))$ in Wilson's normalisation (16-01) Known . Converted by (6), it is $\lambda > 32(d-1)$, or $\lambda > 96$ in four dimensions, independent of N Computed . The continuum limit is in the direction $\lambda \to 0$, and the lattice numbers are at $\beta = 2.25$–2.60 for $SU(2)$, which is $\lambda = 3.6$–3.1 Computed (a rough guide, since the conventions differ). Example: both strong shores are places where the cutoff and the physical scale are not separated (on the strongly coupled lattice the correlation length is shorter than the lattice spacing, and in supergravity the KK tower stays at the same height as the glueballs; 17-10).

The seam of finite and infinite

In a finite box at finite β the entries of the transfer matrix are positive, so the gap is always positive (17-13) Known . What is asked is that it does not collapse as the box is enlarged.

$$\text{for each } L\,,\ \Delta_L > 0 \qquad \text{and} \qquad \inf_L \Delta_L > 0 \qquad \text{are different claims} \quad (7)$$

On the latter, the problem statement writes 'no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume' Known , and the form on the lattice is Chatterjee's 2018 lattice correlation-length problem (16-08 § 08). Physics measures in finite boxes, checks volume effects with rules of thumb such as $L\sqrt{\sigma} \geq 3$ (L is the physical length of the side of the box), and extrapolates in the lattice spacing with a^2 (17-13 § 05) Physics . Example: for a strip of the two-dimensional Ising model, the gap closes as $1/L$ at the critical point, but it is positive for every L (2-09). $\inf_L$ cannot be read from numbers in finite boxes.

The seam of extra dimensions

In holography the gap comes from the geometry of the extra dimensions (the cigar where a circle shrinks) (17-09) Physics . To return to four dimensions one sends the size of the extra dimensions $R = 1/M_{\mathrm{KK}}$ to 0 with Λ fixed, but from the one-loop estimate

$$\Lambda/M_{\mathrm{KK}} \approx \exp\!\left(-24\pi^2/(11\lambda)\right) \qquad (\lambda \text{ is the value at } M_{\mathrm{KK}}\ ) \quad (8)$$

$R \to 0$ is the same direction as $\lambda \to 0$ (17-10) Physics . Supergravity can be used for $\lambda \gg 1$, and Csáki et al. write 'This is the opposite of the limit that is required for the supergravity description to be valid' Physics . Example: at $\lambda \approx 16.6$ converted from the Sakai–Sugimoto fit (a fitted value from a model with quarks, not a value for pure Yang–Mills; $N = 3$; can be off by a factor of about 2 by convention), (8) is 0.27, and the scales are not separated Computed . Within the range we searched, we found no justification for exchanging the limits.

05

Measuring between the shores — the plaquette and the mass

How far the shores are depends on the quantity. For the four-dimensional $SU(2)$ Wilson action $S = \sum_P \left(1 - \frac{1}{2}\operatorname{tr} U_P\right)$ with weight $e^{-\beta S}$ ($\beta = 4/g^2$; 5-01), we compare the leading terms of the two shores with the numbers, for a local quantity (the plaquette) and a long-distance quantity (the mass). Both leading terms are the standard lowest-order expansions (the strong-coupling expansion is treated in 5-02).

$$\left\langle \tfrac{1}{2}\operatorname{tr} U_P \right\rangle \approx u(\beta) = \frac{I_2(\beta)}{I_1(\beta)} \ \ (\beta \text{ small}), \qquad \left\langle \tfrac{1}{2}\operatorname{tr} U_P \right\rangle \approx 1 - \frac{3}{4\beta} \ \ (\beta \text{ large}) \quad (9)$$

What is inside the two leading terms

I_n is the modified Bessel function, defined by the following integral.

$$I_n(\beta) = \frac{1}{\pi}\int_0^{\pi} e^{\beta\cos\theta}\cos n\theta \, d\theta$$

The leading strong-coupling term u is the average of $\frac{1}{2}\operatorname{tr} U$ with weight $e^{\beta\cdot\frac{1}{2}\operatorname{tr} U}$, treating one plaquette as a $SU(2)$ variable unrelated to its surroundings. Under the Haar measure on the group (5-01 § 06) the density of $\frac{1}{2}\operatorname{tr} U = \cos\theta$ is proportional to $\sin^2\theta$ ($0 \le \theta \le \pi$), and this average is exactly I_2/I_1 Known (agreeing to 10 digits by numerical integration Computed). When β is small the ties with the surrounding plaquettes are weak, and this single-plaquette average is the leading term Known .

The leading weak-coupling term is a count by **equipartition**. Equipartition means that under the weight $e^{-\beta S}$, each degree of freedom entering S quadratically contributes $\frac{1}{2}$ to $\langle \beta S \rangle$ (from $\int x^2 e^{-\beta x^2}\,dx / \int e^{-\beta x^2}\,dx = 1/(2\beta)$). When β is large the plaquettes are pressed close to 1, and S can be approximated by a quadratic form in small fluctuations of the links. Per lattice point there are 4 links, and $SU(2)$ has 3 directions, making 12. Of these, the 3 gauge-transformation directions do not change the action and are not counted, leaving 9. There are 6 plaquettes per lattice point, so per plaquette $\langle 1 - \frac{1}{2}\operatorname{tr} UP \rangle \approx \frac{9/2}{6\beta} = \frac{3}{4\beta}$ Known .

The mass is the rate at which correlations decrease. At strong coupling, joining two distant plaquettes needs the space between them filled with plaquettes in the shape of a tube, and each lattice step the tube grows picks up u^4 for the 4 plaquettes of its side. Correlations decrease with distance t like $u^{4t} = e^{-(-4\ln u)t}$, so $am \approx -4 \ln u$ Known . On the weak-coupling side, expanding am as a power series in g^2 gives coefficient 0 at every order (because of the form e^{-1/g^2}), and only the form comes from asymptotic freedom Physics .

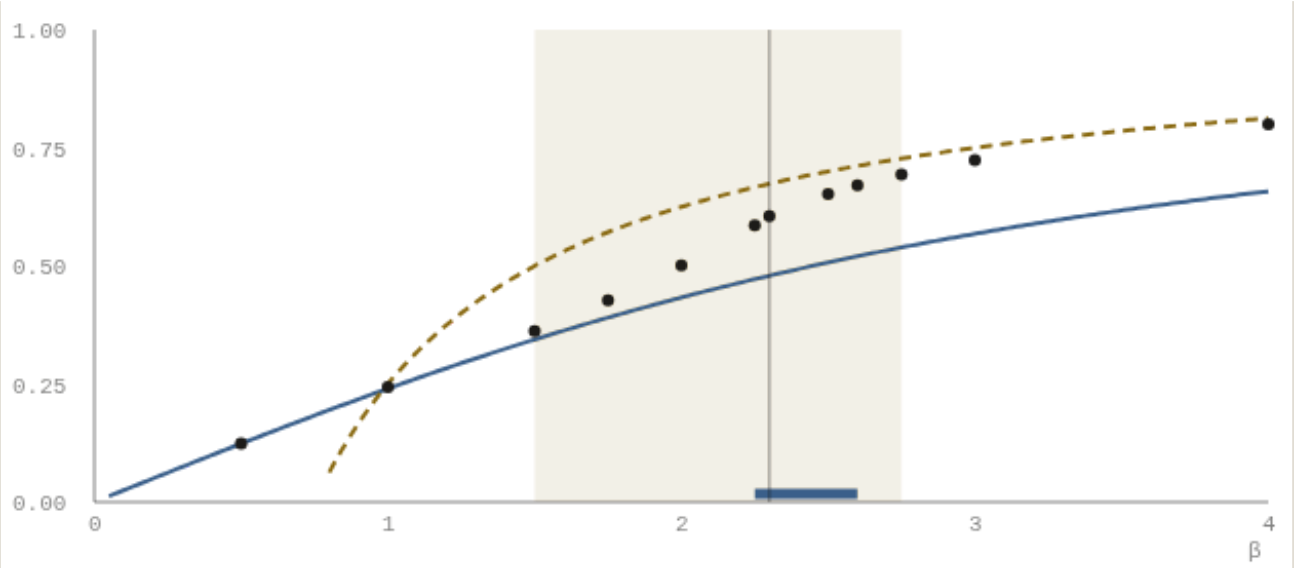
$$am \approx C\,(b_0 g^2)^{-51/121}\, e^{-3\pi^2\beta/11}, \qquad b_0 = \frac{22}{48\pi^2}$$ (10)

This is two-loop asymptotic scaling (4-06). The constant C is not fixed by asymptotic freedom, and it is the very information of the gap.

Setting them beside the numbers

The numbers are, for the plaquette, the heat-bath method on a 6^4 lattice on this machine Computed , and for the mass, Table 1 of Lucini–Teper 2001 Physics . The heat-bath method is a Monte Carlo method that repeats updates redrawing one link at a time directly from its conditional distribution with the surrounding links fixed (5-07).

Figure 1 Choosing the quantity with the buttons and moving β with the knob changes the readings at the vertical line. The solid line is the leading strong-coupling term, the dashed line the leading weak-coupling term (for the mass, (10) fitted at β = 2.6), and the points are the numbers. Only on the plaquette plot are drawn the shaded band (the interval containing the measured points β = 1.75–2.6 where neither leading term is within 5%; at its ends, 1.5 and 2.75, one of them is within 5%) and the thick line at the bottom (β = 2.25–2.60, where there are numbers for the mass)



β = 2.30, leading strong-coupling term 0.4793, leading weak-coupling term 0.6739, numerical 0.6051 (deviation −20.8%, +11.4%)

Computed The plaquette points (Table 2) and the curves ((9) and (10) computed on the spot) were computed on this machine. Physics The mass points are Table 1 of Lucini–Teper 2001.

Computed Table 2 $\langle \frac{1}{2} \operatorname{tr} UP \rangle$: heat-bath method on 6^4 (in parentheses a rough guide to the statistical error; extracted from the 15 points measured) and the two leading terms. Deviations are percentages of the numerical value.

β	Numerical	Strong coupling u	Deviation	Weak coupling	Deviation
0.50	0.1234(2)	0.1237	+0.3%	—	—
1.00	0.2438(3)	0.2402	−1.5%	0.2500	+2.5%
1.50	0.3616(4)	0.3441	−4.8%	0.5000	+38.3%
1.75	0.4271(4)	0.3906	−8.6%	0.5714	+33.8%
2.00	0.5010(7)	0.4331	−13.6%	0.6250	+24.7%
2.25	0.5856(7)	0.4720	−19.4%	0.6667	+13.8%
2.30	0.6051(9)	0.4793	−20.8%	0.6739	+11.4%
2.50	0.6526(5)	0.5072	−22.3%	0.7000	+7.3%
2.60	0.6705(4)	0.5203	−22.4%	0.7115	+6.1%
2.75	0.6935(4)	0.5391	−22.3%	0.7273	+4.9%
3.00	0.7235(2)	0.5679	−21.5%	0.7500	+3.7%
4.00	0.7997(1)	0.6580	−17.7%	0.8125	+1.6%

Table 3 Mass am (lattice units). Physics The numerical am column is Table 1 of Lucini–Teper (the master copy when this book cites the six SU(2) points; $a\sqrt{\sigma}$ and lattice sizes

are in 5-09 Table 1). Computed The columns $\xi = 1/(am)$, strong coupling and (10) are calculations on this machine, with (10) fitted to the point $\beta = 2.6$ ($C = 185$).

β	Numerical am	ξ	Strong coupling $-4\ln u$	Strong coupling / numerical	(10)
2.25	1.390(30)	0.72	3.004	2.16	1.241
2.30	1.090(33)	0.92	2.942	2.70	1.095
2.40	0.953(19)	1.05	2.825	2.96	0.851
2.475	0.754(10)	1.33	2.742	3.64	0.705
2.55	0.586(10)	1.71	2.663	4.55	0.583
2.60	0.514(8)	1.95	2.613	5.08	0.514

Example: for the plaquette, the leading strong-coupling term is within 5% of the numbers for $\beta \le 1.5$, and the leading weak-coupling term for $\beta \ge 2.75$ (the agreement at $\beta = 1$ is only where the curves cross); at the points with $1.75 \le \beta \le 2.6$ neither is within 5% Computed . The $\beta = 2.25$–2.60 used to extrapolate the continuum value of the mass is inside this band. For the mass, the leading strong-coupling term is 2.2–5.1 times the numbers, and (10) reproduces the other five points to within 11%, but C is a fitted number Computed . For a local quantity the numbers fill the space between the two leading terms; for a long-distance quantity the weak-coupling shore has no value (7-11 § 03), and the strong-coupling shore does not reach.

The physical continuum value 3.844(61) is the ratio $m/\sqrt{\sigma}$ of the six points in this band ($\xi \le 2$) extended to $a^2\sigma \to 0$ (Lucini–Teper Table 5; for how it is extended, see 17-13 § 05) Physics . This is an extrapolation to the continuum limit ($a \to 0$), and volume effects are checked separately with rules of thumb for finite boxes (the seam of finite and infinite).

06

The type in which the same word points to different things

The seam of spacetime and space

The intuition of physics lives on spacetime (Minkowski), where particles fly and there is causality. The construction of mathematics is a probability measure on Euclidean $\mathbb{R}^4$, with neither light cones nor causality (0-04). The way back is reflection positivity (4-10). With Θ the reflection of time, F, G functions on the positive-time side, and τ_t translation in time,

$$\langle F, G\rangle_{\mathcal{H}} := \langle \Theta F\, G\rangle, \qquad \langle F, F\rangle_{\mathcal{H}} \ge 0, \qquad \langle F, e^{-tH} G\rangle_{\mathcal{H}} = \langle \Theta F\, \tau_t G\rangle \quad (11)$$

the inequality in the middle is reflection positivity, and only with it can $\mathcal{H}$ and $H \geq 0$ be built (the Osterwalder–Schrader reconstruction)Known . "Time", "energy" and "particle" are words that acquire meaning through (11). That the Wilson action preserves reflection positivity is due to Osterwalder–Seiler 1978 Known . On the other hand, the lattice itself breaks rotational symmetry. The problem statement writes that when reflection positivity is lost through regularisation, there are few ways to recover it (quotation in 0-03 § 03) Known . Example: for the Pauli–Villars two-point function an F with $\langle F, F\rangle_{\mathcal{H}} < 0$ can be built (7-11 § 06).

The seam of the word "particle"

Physics counts glueball masses Physics . What the problem statement asks for is a spectral gap, and the existence of an isolated one-particle state at the bottom is set aside among the 'natural extensions' (0-03 § 09) Known .

$$\text{problem statement}: \ \operatorname{spec}(H) \cap (0, \Delta) = \varnothing, \qquad \text{physics}: \ m_{0^{++}} \text{ is an isolated eigenvalue of}$$

($\mathbf{P}$ is the momentum.) Example: the way correlations measured on the lattice decrease tells the position of the bottom, but not whether it is an isolated point or the edge of a continuous spectrum (16-08 § 05). The unit GeV does not come out of pure Yang–Mills either, and Athenodorou–Teper note 'Strictly speaking this is of course not possible: the pure gauge theory and QCD … are two different theories' Physics .

07

Overlaying the two lists — obstacles, moves, partial bridges

7-11 listed the obstacles on the side of mathematics, and 17-13 the moves of physics. The seams pair their rows. The partial bridges that already exist all have the following form.

$$A \implies (M \text{ , in part}), \qquad A \text{ in four dimensions, at weak coupling, in the continuum, in infin}$$

Example: in Shen–Zhu–Zhu, A is $\lambda > 96$ (an explicit range; below $N^2/48$ in Wilson's β, with the conversion in Table 4 of 5-01), the conclusion is a gap in infinite volume, and A does not hold in the direction of the continuum limit $\lambda \to 0$.

Known Table 4 The seams with the obstacles of 7-11, the moves of 17-13 and partial bridges (the correspondence is this book's arrangement)

Seam	Obstacle	Move	Partial bridge
1 Existence	3, 6	Putting that it exists	The measure on $\mathbb{R}^2$ (7-02). In four dimensions, the published construction (the continuum limit in a periodic box, infinite volume along a subsequence, the Wightman axioms; the committee's assessment Assessment , Part 6). There is no uniqueness of the infinite-volume limit
2 One scale	2	Regarding as one (scale)	Not found within the range we searched (17-12 is at the stage of cutting it out as an assumption)
3 Shores	1	Expansion, neighbouring theories	The infinite-volume limit, area law and exponential clustering at strong coupling (Osterwalder–Seiler 1978; range not made explicit). A gap and a log-Sobolev inequality in the explicit range $\lambda > 96$ (Shen–Zhu–Zhu)
4 Finite and infinite	1, 6	Extrapolation, the string picture	Exponential decay at weak coupling for finite groups (Adhikari–Cao). The lattice correlation-length problem for continuous non-abelian groups remains open. The published construction passes to infinite volume along a subsequence, but there is no gap uniform in the volume (16-02)
5 Spacetime and space	5	All the premises	On finite lattices, reflection positivity of the Wilson action and a positive transfer matrix (preserved in weak limits too). The recovery of rotational symmetry remains
6 Particles	—	Regarding as one (seeing the gap as the mass of a particle; 17-13 § 09)	Within the range we searched, not found in four dimensions
7 Extra dimensions	1	Neighbouring theories	The eigenvalue problem after fixing the background (17-09). No exchange of limits found
Outside the table: gauge	4	Choosing a gauge	A wall common to both sides. A conditional theorem stating confinement in terms of gauge-invariant quantities (Wilson loops) without choosing a gauge (Chatterjee 2021: if centre symmetry is unbroken there is confinement; 17-13 § 08, 16-08 § 06)

Apart from the bridge over the seam of existence, the partial bridges all lie on the side of the shores (strong coupling, low dimensions, finite lattices), and within the range we searched none extends towards the problem statement. The bridge over the seam of existence (the published construction), too, carries every step of renormalisation at weak coupling and stops at the step with lattice spacing 1. It does not enter the behaviour at long distances (a uniform gap, the area law) (16-03). Gauge is not put among the seven because local gauge fixing is a wall common to both mathematics (7-05) and physics. Chatterjee's 2021 theorem is one road that goes on in front of this wall without choosing a gauge. Topological sectors and θ (7-11) could also be counted as a

seam, in that the problem statement does not touch θ. What would need to be known to move which rows is lined up in 16-07, and the list of questions remaining after the published construction in 16-02.

What this chapter can say

	What is said	What is not said
Known	For each seam, the sentence of physics, the mathematical statement, and the current state and partial bridges as stated by the primary sources	That seven is exhaustive (the counting, the types and the correspondences of Table 4 are this book's arrangement)
Computed	The plaquette on 6^4, and the β where the two leading terms are within 5%. The ratio of the leading strong-coupling term to the masses of the primary source. That u equals the single-plaquette average I_2/I_1 (numerical integration). The conversions (6) and (8)	Infinite-volume values (finite-volume effects not estimated). The 5% line is this chapter's choice
Physics	The estimates (8) and (10), and the reading of the examples where one scale breaks	That asymptotic scaling and the exchange of limits are correct
Overall	—	This site fills none of the seams (the bridge over the seam of existence belongs to the published construction). It does not show the existence of the mass gap

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement, the lattice correlation-length problem, three dimensions	Text	Jaffe–Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000) / Chatterjee, arXiv:1803.01950 (2018) / Chevyrev, arXiv:2202.13359 (2022)
The published construction in § 03	Assessment	The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening, § 5 and § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28)
Lattice numbers, the quotation in § 06	Text	Lucini–Teper, JHEP 0106 (2001) 050, hep-lat/0103027 (Tables 1 and 5) / Athenodorou–Teper, arXiv:2007.06422 (2020)

Item	Kind	Source / tools
Theorems on the strong-coupling side	Abstract, part of the text	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 (abstract; that the gap (exponential clustering) follows from convergence of the cluster expansion is according to the introduction of Shen–Zhu–Zhu: 'as explained in [OS78], the existence of a mass gap (exponential clustering) follows from convergence of the cluster expansion') / Adhikari–Cao, arXiv:2202.10375 (abstract) / Shen–Zhu–Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 (abstract, introduction and equation (1.2) (normalisation of the action) checked; the conversion $\beta = N^2\beta_{\mathrm{SZZ}}$ to Wilson's β is arithmetic on this machine)
The quotation in § 04, non-separation of KK, $m^2/\sigma \sim 1/\lambda$, $\lambda \approx 16.6$	Text	Csáki–Ooguri–Oz–Terning, hep-th/9806021 / Klebanov–Strassler, hep-th/0007191 / KMMW, hep-th/0311270 (equation (2.7)) / Sakai–Sugimoto, hep-th/0412141, hep-th/0507073 (the conversion to λ and substitution into (8) are arithmetic on this machine)
Centre symmetry, three-dimensional compact $U(1)$	Text, abstract	Chatterjee, arXiv:2006.16229 (text) / Göpfert–Mack, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit with m_D fixed is a free scalar field; main text not read; the year is Crossref's year of publication, INSPIRE gives 1981)
(9), $-4\ln u$, the coefficients of (10), (11)	Known, bibliographic data only	(9) and $-4\ln u$ are the standard lowest-order strong-coupling expansion and equipartition (the strong-coupling expansion is treated in 5-02). Review: Drouffe–Zuber, Phys. Rep. 102 (1983) 1–119 (bibliographic data only) / the two-loop coefficients (4-06) / the reconstruction theorem (4-10)
Tables 2 and 3, Figure 1, (6), (8)	Computed	Python on this machine: plaquette_mc.py (numpy; 6^4, Kennedy–Pendleton heat-bath method, 100 sweeps of thermalisation and 300 of measurement, fixed seed, bin averages), shores.py (scipy), one_plaquette.py (scipy; numerical integration of $u = I_2/I_1$). The only input for the masses is Table 1 of Lucini–Teper

Next chapter to read: 16-06 Candidates for new concepts, which looks from the side of concepts at what seems to be needed to cross the seams. The rearrangement as tasks is in 16-07.

Candidates for new concepts — what plays the part of "Ricci flow"

> This chapter is a map of open questions. § 02, § 03 and § 04 are established knowledge.

What this chapter shows — The three components of Ricci flow (flow, monotone quantity, surgery). That in the two-dimensional σ model the renormalisation group at one loop is Ricci flow itself, and the scale at which the sphere shrinks away is Λ. Where the candidates for each component stop in four dimensions.

Prerequisite chapters — 7-09. We also use 7-05, 4-07 and 7-11. Geometric tools such as Ricci curvature and scalar curvature are defined on the spot with the example of the sphere.

Said first — This chapter lines up candidates; it cannot say which one will work. "What plays the part of Ricci flow is the renormalisation group" and "what plays the part of $R = T$ is a duality transformation" are analogies (only the one loop of the σ model is not an analogy but an agreement). This site shows neither the mass gap nor the construction of the continuum limit. The label Assessment means "according to the committee's assessment report" (“Labels” in 6-00).

The order of this chapter

1. Where the setters asked for "new ideas"
2. Ricci flow: moving a metric by its curvature
3. Perelman's components: flow, monotone quantity, surgery
4. Where the renormalisation group becomes Ricci flow: the two-dimensional σ model — the scale at which it shrinks away is Λ. Figure 1, Table 1. Detour: a flat circle does not shrink
5. What flows in four dimensions — the flow of the coupling and the gradient flow of the field
6. Monotone quantities: c in two dimensions and a in four — monotonicity comes from reflection positivity. Figure 2, Table 2
7. What plays the part of surgery

8. Another type: identification
9. The candidates on one sheet

01

Where the setters asked for "new ideas"

The Clay problem statement (Jaffe–Witten 2000), § 6, takes the construction on T^4 to be within reach as an extension of known methods, and then writes: "no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume. Nor do present methods suggest how to obtain the existence of the infinite volume limit $T^4 \to \mathbb{R}^4$". § 6.5 reads "New ideas are needed". The same § 6, however, also treats the construction on T^4 as unfinished. What can be said is only "the side where the setters asked for new ideas is the infrared" (7-10 § 01). Known

Later, the construction published in 2026 gave the continuum limit in a periodic box and an infinite-volume theory along a subsequence (satisfying the Wightman axioms). The committee's assessment is that this is “proved in Lean” Assessment (Part 6). A mass gap uniform in the volume and the uniqueness of the infinite-volume limit remain among the open questions listed in the assessment report (16-02). The infrared side, where the setters asked for new ideas, is still open.

We write "uniform in the volume" as a formula. For the gap Δ_L of a box of side L,

$$\Delta_L > 0 \ \ (\text{for every } L) \qquad \text{and} \qquad \inf_L \Delta_L > 0 \ \ (1)$$

are different claims. Example: for a free field of mass m the gap is at least m for every L, and the right-hand one also holds. If $\Delta_L = 1/L$ held, only the left-hand one would hold.

Among the new concepts that worked on other great problems, two types can be seen: a flow with a monotone quantity (Perelman, the Poincaré conjecture), and the identification of two objects (Wiles, Fermat's Last Theorem) (this book's arrangement).

02

Ricci flow: moving a metric by its curvature

A Riemannian metric g is an inner product on the tangent space at each point (1-05), with components written g_{ij}. The Ricci curvature Ric (components R_{ij}) is, for a direction v of length 1, the sum of the bending (sectional curvatures) of $n-1$ mutually

orthogonal planes containing v (n is the dimension of the manifold). Equivalently, it is $n-1$ times the average. Precisely, it is a contraction of the curvature tensor. We use only the following example.

$$\text{sphere of radius } r \text{ in } n \text{ dimensions}: \qquad \mathrm{Ric} = \frac{n-1}{r^2}\, g \quad (2)$$

On a sphere of radius r, the bending of every plane is $1/r^2$. Adding $n-1$ of them gives the coefficient $n-1$. The trace of Ric (the sum of the diagonal components in an orthonormal basis) is called the scalar curvature R. On this sphere $R = n(n-1)/r^2$.

Ricci flow is the equation that moves a metric by its Ricci curvature, and Perelman's paper starts from this equation, introduced by Hamilton (1982). Known

$$\frac{\partial g}{\partial t} = -2\,\mathrm{Ric}(g) \quad (3)$$

Positively curved directions shrink. Along the flow the scalar curvature obeys $\partial_t R = \Delta R + 2|\mathrm{Ric}|^2$. Δ is the Laplacian on the manifold (not the Δ of the mass gap), which in flat coordinates is $\sum_i \partial^2/\partial x_i^2$ (a different symbol from the gap Δ_L of § 01). $|\mathrm{Ric}|^2$ is the sum of squares of the components of Ric in an orthonormal basis.

If the right-hand side were ΔR alone it would be the heat equation, and just as heat evens out temperature, it would shave the peaks of R and fill its valleys. Precisely, the non-negative term $2|\mathrm{Ric}|^2$ is added upward everywhere. For equations of heat-equation type, a solution cannot create a new minimum in the interior (the maximum principle). So the minimum of R does not decrease along the flow. Perelman's introduction, presenting Hamilton's work, writes this equation and conclusion in this form. Known

Example: a sphere of initial radius $r(0)$ stays round and only its radius changes. From $g = r^2 \hat{g}$ ($\hat{g}$ is the metric of the unit sphere) and (2),

$$\frac{d(r^2)}{dt} = -2(n-1), \qquad r(t)^2 = r(0)^2 - 2(n-1)\,t, \qquad t_* = \frac{r(0)^2}{2(n-1)} \quad (4)$$

and at time t_* (the extinction time) it shrinks to a point. For the unit two-sphere $t_* = 1/2$. For a general shape the curvature becomes infinite somewhere. Perelman writes "The most natural way of forming a singularity in finite time is by pinching an (almost) round cylindrical neck", and continues the flow by surgery that cuts the thin neck and puts small caps on it.

03

Perelman's components: flow, monotone quantity, surgery

We split Perelman's proof into three components. The first, the flow, is (3) of § 02. The second is a quantity that increases monotonically along the flow. We define the tools first. M is a closed manifold (a compact manifold without boundary), f an auxiliary function on M, and dV the volume element determined by the metric ($\int_M dV$ is the volume of M). ∇f is the gradient of f, and $|\nabla f|^2$ the square of its length. $\nabla_i \nabla_j f$ is the second derivative of f, which in flat coordinates is $\partial_i \partial_j f$.

$$\mathcal{F}(g, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV$$

When the metric is moved by (3), f is moved at the same time by $\partial_t f = -\Delta f + |\nabla f|^2 - R$. This way of moving preserves the measure $e^{-f} dV$ up to a change of coordinates, and the total integral $\int_M e^{-f} dV$ does not change. Then

$$\frac{d\mathcal{F}}{dt} = 2 \int_M \left| R_{ij} + \nabla_i \nabla_j f \right|^2 e^{-f}\, dV \;\geq 0 \quad (5)$$

holds (Perelman § 1.2). The right-hand side is the integral of a square, so it is not negative. Known

The flow behaves as if it climbs the slope of $\mathcal{F}$. Precisely, the flow $\partial_t g_{ij} = -2(R_{ij} + \nabla_i \nabla_j f)$ with the measure $e^{-f} dV$ held fixed is a gradient flow in the direction that increases $\mathcal{F}$, and a change of coordinates takes it to the pair of (3) and the equation for f above (Perelman § 1.1–1.2). On the choice of the measure held fixed, Perelman writes "the choice of m is analogous to the choice of gauge". The m of the original is this measure, not the mass of § 06. Whatever the choice, the flow is the same up to a change of coordinates. Known

Further, seen up to changes of coordinates and of scale, the flow has no non-trivial periodic orbits (Perelman § 2, § 3). The trivial ones are called solitons (solutions that keep their shape under changes of coordinates and scale alone), and in this view they correspond to fixed points. Known Example: on a flat torus $R_{ij} = 0$, and taking f constant, the right-hand side of (5) is 0 and the flow stands still (a fixed point).

The third component is to classify in advance the shapes of the regions of large curvature, and to remove by surgery only what has been classified. What could play the same roles for Yang–Mills: for the flow, the renormalisation group and gradient flow (§ 05); for the monotone quantity, c and a (§ 06); for classification and surgery, large fields, instantons and topological sectors (§ 07). As another type there is identification (§ 08). Every one of these correspondences is an analogy.

04

Where the renormalisation group becomes Ricci flow: the two-dimensional σ model

Within one-loop perturbation theory, the place where the correspondence stops being an analogy is the two-dimensional non-linear σ model. The field ϕ is a map from the plane to a manifold M. In this section we write the metric of M as h (components h_{ij}).

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^2} h_{ij}(\phi)\,\partial_\mu \phi^i\,\partial_\mu \phi^j\,d^2x \quad (6)$$

The coupling constant is not a number but the metric h itself. The abstract of Friedan (1980) writes "the renormalization group acting on the infinite-dimensional space of Riemannian metrics". At one loop, for the scale $\ell = \ln(\mu_0/\mu)$ (larger means seen more coarsely),

$$\frac{dh_{ij}}{d\ell} = -\frac{1}{2\pi}\,R_{ij}(h) + \cdots \quad (7)$$

and setting $t = \ell/4\pi$ gives (3) itself. Physics "$\cdots$" stands for corrections at two loops and beyond. From two loops a term quadratic in the curvature is added (of the form $R_{iklm}\,R_j{}^{klm}$, where R_{iklm} are the components of the curvature tensor), and the flow departs from Ricci flow. The abstract of Friedan (1985) says the coefficients of the renormalisation group were computed in the two-loop approximation. Physics

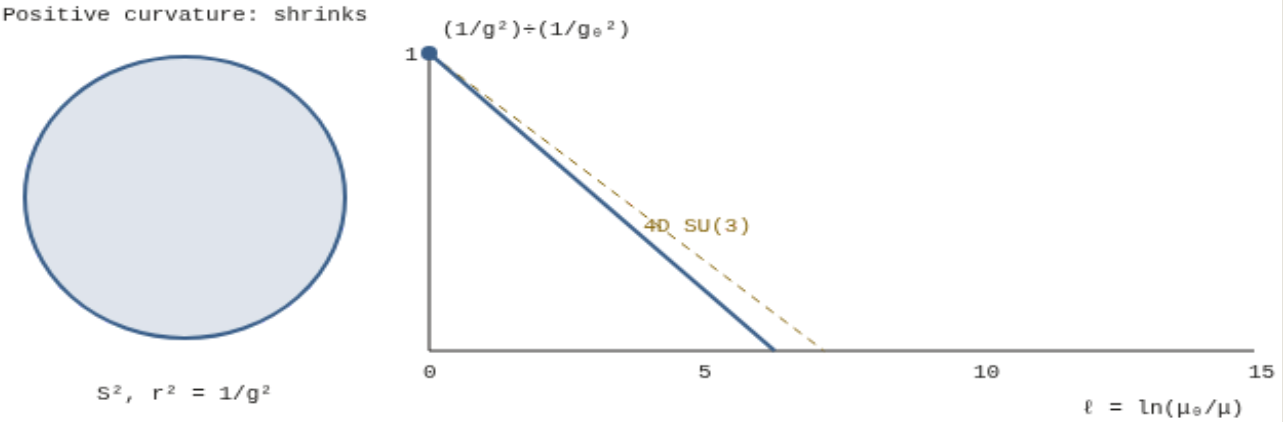
Example: taking M to be the sphere S^{N-1} of radius r gives the $O(N)$ model, with $r^2 = 1/g^2$. With $n = N-1$ in (2),

$$\frac{d}{d\ell}\Big(\frac{1}{g^2}\Big) = -\frac{N-2}{2\pi}, \qquad \ell_* = \frac{2\pi}{(N-2)\,g_0^2}, \qquad \Lambda = \mu_0\,e^{-\ell_*} \quad (8)$$

The more coarsely one looks, the more the sphere shrinks and the stronger the coupling becomes. This is the "asymptotic freedom" of the abstract of Polyakov (1975). The scale ℓ_* at which it shrinks away is 4π times the extinction time t_* of (4), and it determines the one-loop Λ (4-07). For four-dimensional $SU(N)$ too, $1/g^2$ decreases along the straight line $1/g_0^2 - 2b_0\ell$.

Perelman touches on this relation in section 2 of his introduction, writing "this connection between the Ricci flow and the RG flow suggests that Ricci flow must be gradient-like; the present work confirms this expectation". That the flow is gradient-like (moves along the slope of some quantity) was expected from the intuition of the renormalisation group, and (5) confirmed it: that is how Perelman puts it. Known

Figure 1 Change N and g_0^2 with the knobs, and advance ℓ with "Play". Left: a cross-section of the sphere whose radius squared is $1/g^2$. Right: the ratio of $1/g^2$. The dotted line is four-dimensional $SU(3)$



N = 3, g_0^2 = 1.0, ℓ = 0.00, 1/g² = 1.000, ℓ* = 6.28, Λ/μ₀ = 1.87×10⁻³, SU(3): ℓ* = 7.18

Computed The straight lines of the figure are formula (8) and $b_0 = 11N/(48\pi^2)$ drawn on the spot. Table 1 was computed on this machine.

Model	g_0^2	ℓ_*	$\Lambda/\mu 0$	Note
$O(2)$	Any	∞	–	The circle is flat and does not flow
$O(3)$	1	6.283	1.87×10⁻³	Ricci flow $t_* = 0.5$
Four-dimensional $SU(2)$	1 / 2	10.77 / 5.38	2.11×10⁻⁵ / 4.59×10⁻³	$b_0 = 0.04644$
Four-dimensional $SU(3)$	1 / 2	7.178 / 3.589	7.63×10⁻⁴ / 2.76×10⁻²	$b_0 = 0.06966$

Table 1 The scale at which the one-loop $1/g^2$ reaches 0. $\ell_* = 4\pi t_*$ was checked against formula (4). Computed

Before the straight line reaches 0 the coupling becomes large and the one-loop formula can no longer be used. As near a singularity of Ricci flow, what matters is what happens after the coupling becomes $O(1)$. The gap of the two-dimensional $O(N)$ model ($N \geq 3$) is a conclusion at the level of physics. Physics Douglas's report (2004) goes no further than citing an example in which a gap was shown for a regularised version of the $O(N)$ model (Kopper 1999). Its later status as a theorem has not been checked in this chapter.

Detour: a flat circle does not shrink

A subsection you may skip.

For $N = 2$ the target is a circle with curvature 0, and the coefficient $N - 2$ vanishes. The two-dimensional $O(2)$ model has a massless phase at low temperature. At high temperature correlations decay exponentially with distance (there is a mass); at low temperature they decay only as a power of the distance (massless). The transition between these two phases is called

the Kosterlitz–Thouless (KT) transition. Fröhlich–Spencer (1981) proved that this transition exists, for the two-dimensional lattice rotator model ($O(2)$), the Coulomb lattice gas and others. Known Set beside four-dimensional $U(1)$ (5-06), the picture "it shrinks because it is curved (non-abelian)" comes to mind, but it is an analogy.

05

What flows in four dimensions: the flow of the coupling and the gradient flow of the field

In four dimensions there are two different "flows". The **flow on the space of theories** (the renormalisation group, 7-05) has no picture of a sphere, but the one-loop $1/g^2$ is a straight line of the same form (Table 1). Bałaban's ultraviolet stability on T^4 comes with a hypothesis, and the limit is only along subsequences (7-07). The construction published in 2026 stacked one step of a coupling-dependent probability kernel into many steps and carried it as far as the continuum limit in a periodic box Assessment (Part 9 to Part 11). It carries every step within the weak-coupling band, however, and stops at the step with lattice spacing 1 (where the coupling has a large reference value) (16-03). Where the method of Bauerschmidt–Brydges–Slade reached $d = 4$ is the side where the coupling flows to 0 in the infrared (the $|\phi|^4$ model and the weakly self-avoiding walk), the exact opposite of the infrared of Yang–Mills. What is missing is "a controlled step after the coupling has reached $O(1)$" (this book's arrangement).

The **flow on the space of fields** (gradient flow) smooths a single configuration along the slope of the action. For an abelian field, following the gradient of the action $S = \frac{1}{4}\int F_{\mu\nu}F_{\mu\nu}\,d^4x$ backwards gives

$$\partial_t A_\mu = -\frac{\delta S}{\delta A_\mu} = \Delta A_\mu - \partial_\mu(\partial_\nu A_\nu), \qquad \frac{dS}{dt} = -\int \left|\frac{\delta S}{\delta A}\right|^2 \le 0 \quad (9)$$

(for a non-abelian field, replace ∂ by the covariant derivative). Since it is a gradient flow, the action always decreases. Example: after a Fourier transform, the component perpendicular to the wave number k decays as $e^{-k^2 t}$, and fluctuations with $|k| \gg 1/\sqrt{t}$ disappear. The component parallel to k (the gauge direction) does not move. In Ricci flow the same thing happens through invariance under changes of coordinates, and DeTurck (1983) fixed it by adding a term. The same device appears in the stochastic quantisation of Yang–Mills (7-09 § 07).

The abstract of Lüscher (2010) says that theory and numerics "suggest" that the flowed field is "a smooth renormalized field", and Carosso–Hasenfratz–Neil (2018) used it numerically as a continuous real-space renormalisation group. Waldron (2019) showed that, for smooth initial data on a closed four-dimensional manifold, Yang–Mills flow

has no finite-time singularity. Known The relation between the two flows is at the level of suggestion and numerics.

06

Monotone quantities: c in two dimensions and a in four

We prepare two tools. The stress tensor (energy–momentum tensor) $T_{\mu\nu}$ is the field that describes the flow of energy and momentum. Its trace $\Theta = T^{\mu}{}_{\mu}$ (in two dimensions $T11 + T22$) measures how much the action changes when the scale is changed slightly. For a theory that does not change under a change of scale (a fixed point), $\Theta = 0$. The central charge c is a number carried by the theory at a fixed point, and roughly counts the massless degrees of freedom (1 for one free scalar field). In this chapter we use only the name, and see its value through the sum rule (10) below.

In two dimensions there is the c-theorem of Zamolodchikov (1986), which constructs a function that decreases along the flow and equals the central charge at fixed points. Physics As a form we can compute, we look at the sum rule for correlations of Θ, cut off at distance r_0.

$$c(r_0) = 3\pi \int_{|x|>r_0} |x|^2 \, \langle \Theta(x)\Theta(0)\rangle \, d^2x \qquad (10)$$

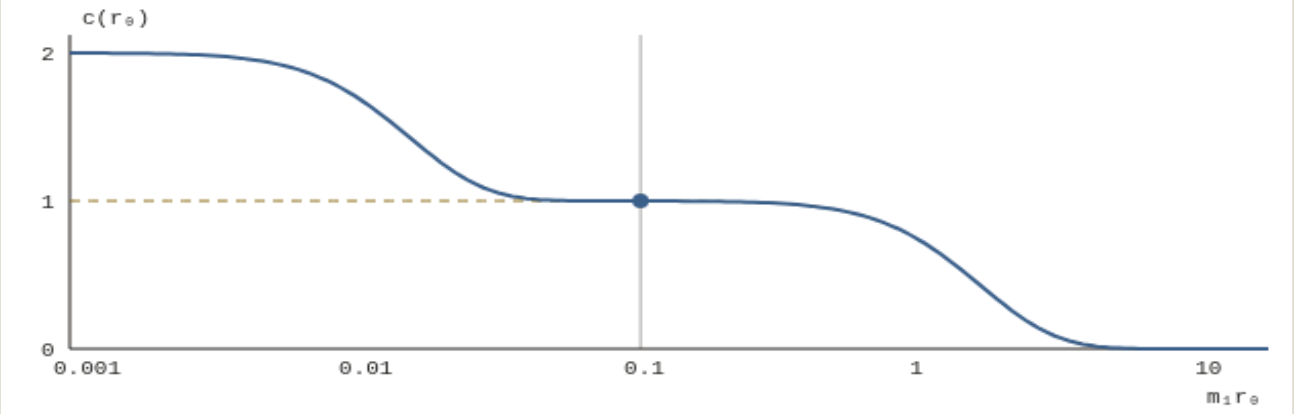
Here Θ is the trace of the stress tensor made directly from the action (in the normalisation $\Theta = 2\pi T^{\mu}{}_{\mu}$, the prefactor becomes $3/4\pi$). The content of the sum rule is that the value at $r_0 \to 0$ is the difference between the ultraviolet and infrared central charges. This sum rule is attributed to Cardy (1988). Physics

For a free scalar field of mass m, $\Theta = -m^2\phi^2$ and $\langle\Theta(x)\Theta(0)\rangle = 2m^4 G(x)^2$. $G(x) = K_0(m|x|)/2\pi$ is the propagator (1-11), and K_0 is a modified Bessel function (decaying like $e^{-m|x|}$ far away). Therefore

$$c(r_0) = 3\int_{mr_0}^{\infty} u^3 K_0(u)^2 \, du, \qquad c(0) = 1, \quad c(\infty) = 0 \qquad (11)$$

We checked $c(0) = 1$ by numerical integration as $1 - 8\times 10^{-12}$. Computed Since the integrand is never negative, $c(r_0)$ decreases. In general, $\langle\Theta(x)\Theta(0)\rangle \geq 0$ for $x \neq 0$ follows from translation and rotation invariance and reflection positivity (4-10). A property that is easily lost under regularisation (7-11 the obstacle of positive regularisation) is where the monotonicity comes from. Known

Figure 2 Change the mass ratio m_2/m_1 and the scale r_0 with the knobs, and stretch r_0 with "Play". The solid line is $c(r_0)$ for two free fields, the dotted line for one



m_2/m_1 = 100 $m_1 r_0$ = 0.10 $c(r_0)$ = 0.999 (one field: 0.999)

Computed The curve is an interpolation of a table of 85 points from numerical integration of formula (11) on this machine (embedded with the trailing 0s omitted).

$mr0$	0.1	0.5	1	2	3	5
$c(r0)$	0.99946	0.93639	0.70823	0.25841	0.06861	0.00313

Table 2 $c(r_0)$ for one field. For two fields with mass ratio 100, at $m_1 r_0 = 10^{-3}, 10^{-2}, 0.1, 1$ it is 1.999, 1.708, 0.999, 0.708, staying on the step at 1 in between. Computed

The step shows that the flow passes near an intermediate fixed point (the scales at which only the light field looks massless); in an infrared with a gap, $c = 0$.

When a four-dimensional fixed point is placed on a curved background, the Euler density (a quadratic expression in the curvature whose integral is proportional to the Euler number of the manifold) appears with a coefficient in the expectation value of Θ. That coefficient is the Euler anomaly a (not the lattice spacing a; in this chapter only the name). The abstract of Komargodski–Schwimmer (2011) derives a monotonically decreasing function joining the ultraviolet and infrared a, "establishing the a-theorem". Physics The a of free fields is positive, and the contributions of a real scalar, a Weyl fermion and a gauge field are in the ratio $1 : \frac{11}{2} : 62$ (appendix A of the same paper). Known

The ultraviolet of pure Yang–Mills approaches free gluons, so $a > 0$. If there is a gap in the infrared, no massless degrees of freedom remain, so $a = 0$. Physics The two do not contradict each other, but $a_{\mathrm{UV}} > a_{\mathrm{IR}}$ does not force a gap. That there is a gap in the infrared is itself the unproved part of this book's subject. A separate "classification of infrared endpoints", corresponding to the classification of singularities, is needed (an analogy).

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07
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What plays the part of surgery: large fields and topological sectors

In four-dimensional Yang–Mills, the fact that the action does not change under changes of scale creates the same kind of difficulty as singularities in which curvature concentrates at a point. If $A_\lambda(x) = \lambda A(\lambda x)$, then $F_\lambda(x) = \lambda^2 F(\lambda x)$, and

$$\int_{\mathbb{R}^d} |F_\lambda|^2 \, d^d x = \lambda^{4-d} \int_{\mathbb{R}^d} |F|^2 \, d^d x \quad (12)$$

For $d = 4$ this does not depend on λ. Example: an instanton of size ρ has action $8\pi^2/g^2$ whatever ρ is (3-05), and as $\rho \to 0$ the action does not decrease even when the curvature concentrates at a point.

As a ruler for the size of the curvature we use $\|F\|_{L^p} = \left(\int |F|^p \, d^d x\right)^{1/p}$ ($p = 2$ is the L^2 of 1-10, and squared it is the action). The compactness of Uhlenbeck (1982) (from a sequence of connections whose curvature is bounded in L^p, one can extract, after changes of gauge, a subsequence converging in a weak sense) works with L^p estimates for $p > d/2$, and the L^2 of $d = 4$ (the action itself) lies exactly on the borderline. On the borderline, curvature can concentrate at finitely many points and small instantons can pinch off like bubbles (bubbling; in this chapter only the name). Known

On the quantum side, what could play the part of surgery are the large-field regions treated separately in Bałaban's method (7-05 § 04), small instantons, and topological sectors (7-11 Detour). None of them has reached the stage of "classify, then remove".

08

Another type: identification with an object in which the gap is visible

The Wiles type shows that two objects are the same, and brings in from one of them a property not visible in the other ($R = T$; a summary at the level of common knowledge). Jaffe–Witten § 6.6 writes that, since the gap is not visible classically, "a nonclassical change of variables or “duality transformation”" may be needed. Known

Examples where a gap was shown by this type start from an identity of Gaussian integrals. For a Gaussian field φ with covariance V,

$$\mathbb{E}\left[e^{\,i \sum_j q_j \varphi(x_j)}\right] = \exp\Big(-\tfrac{1}{2} \sum_{j,k} q_j q_k \, V(x_j - x_k)\Big) \quad (13)$$

The right-hand side is the weight of particles with charges q_j interacting through the potential V; the left-hand side is an expectation value of the field. Summing over the gas of charges, the left-hand side becomes a field theory containing $\cos\varphi$. This is called the sine-Gordon model (a field theory with a $\cos\varphi$ term in its action; in this chapter

only the name). Brydges–Federbush (1980) used this transformation to prove exponential decay of correlations (Debye screening) in the Coulomb gas at low density or high temperature. Known It is an abelian model of statistical mechanics.

Fragments for Yang–Mills itself are the gauge–string duality for $SO(N)$ at strong coupling and large N of Chatterjee (2019), the representation of Wilson loops as sums over surfaces of Cao–Park–Sheffield (17-04 § 09; the picture is in § 05), and "unbroken centre symmetry ⇒ confinement" of Chatterjee (2021) (an implication, not an equivalence). At the level of physics there are Seiberg–Witten (17-11) and holography (17-09). The problem asks for fixed N, so large N alone is not enough.

09

The candidates on one sheet

The proposal to give a symbol to $1/0$ is not included, since it carries little information (7-12 § 10).

First, how to read the table. The "Component" column says which of the three components of § 03 the candidate corresponds to. They divide into flow (i–iii), identification (v, vi) and surgery (vii); (iv) corresponds to none. "How far" is the range reached as a theorem; "Where it stops" is the first place not reached.

Lining up the places where they stop, they gather roughly in two. One is that small numbers disappear in the infrared, where the coupling becomes $O(1)$. (i) has no step there (the published construction, too, stops at the step with lattice spacing 1 just before it); the continuum limit obtained in (iii) is of a design that ties the coupling to 0; and (iv) is a power series, so it does not reach e^{-1/g^2}. The other is that they work only in a different limit, $N \to \infty$ or strong coupling: (v) and (vi) are of this kind. (i) tries to go from the ultraviolet side to the infrared and stops; (vi) has a picture of the infrared but stops without being able to return to finite N. They stop facing in opposite directions (this book's arrangement).

	Candidate	What it makes new	How far	Where it stops	Component
i	Constructive renormalisation group (Bałaban, MRS, BBS, the construction published in 2026)	Treat one step as a map on the space of theories	Published construction: the continuum limit in a periodic box and infinite volume along a subsequence, the Wightman axioms Assessment . Bałaban: ultraviolet stability on T^4 (with a hypothesis)	A step after the coupling has reached $O(1)$. The published construction stops at the step with lattice spacing 1 (where the coupling is large) (16-03)	Flow
ii	Stochastic quantisation and regularity structures (Hairer, CCHS)	The measure as the invariant measure of a dynamics, the space of orbits as the state space	The measure on T^2 is the unique invariant measure of the dynamics (Chevyrev–Shen). Local solutions for three-dimensional YMH	Subcritical $d < 4$ (7-09)	Flow
iii	New ways of taking limits (Chatterjee and others)	Decide first the space in which the limit lives	The gap of the strong-coupling lattice (in lattice units). The Gaussian limit of YMH (of a design that ties the coupling to 0)	A non-Gaussian limit	Flow
iv	Algebraic formulations (Connes–Kreimer, Costello)	Write renormalisation as an algebraic structure	A formulation within perturbation theory (at the level of formal power series)	Does not reach quantities invisible to perturbation theory, such as e^{-1/g^2}	—
v	Duality, change of variables (JW § 6.6)	Identify with an object in which the gap is visible	Debye screening (abelian). Fragments of gauge–string duality	No duality transformation that works for pure Yang–Mills was found, as far as we looked	Identification

	Candidate	What it makes new	How far	Where it stops	Component
vi	Large N and holography	See the gap in the geometry of extra dimensions	Level of physics Physics	$N \to \infty$, strong coupling	Identification
vii	Classical YM in geometric analysis (Uhlenbeck, Waldron)	Analysis in the critical dimension (bubbling, § 07)	The classical theory is within reach of existing methods (JW § 2)	Quantum configurations are expected not to be smooth	Surgery

This book's study treated in Part 23 (the diffusive-relaxation-time assumption, 23-01) is closest to type (i): it cuts out the physical premise "a single scale" as an inequality on the lattice (the judgement of closeness is this book's arrangement).

What this chapter can say

	Content
Said	What the setters asked new ideas for is the uniform gap and the infinite-volume limit. Perelman's monotonicity formula (5) (the formula when f is moved at the same time), and the statement that the gradient-like nature of the flow had been expected from the intuition of the renormalisation group (both from the original texts) Known
Said	The one-loop flow (8) of the $O(N)$ model agrees with the Ricci flow (4) of the sphere under $t = \ell/4\pi$, and the scale at which it shrinks away is the one-loop Λ (Table 1). From two loops it departs from Ricci flow Computed Physics
Said	For a free scalar field of mass m, $c(r_0)$ decreases monotonically from 1 to 0, and the monotonicity comes from reflection positivity Computed Known
Not said	Which candidate will work (the correspondences are analogies). The status as a theorem, after 2004, of the gap of the two-dimensional $O(N)$ model. Whether the infrared of pure Yang–Mills has a gap ($a = 0$ is under that condition). Whether a monotone quantity exists in four dimensions as a theorem. Exhaustive coverage from 2025 on

Sources and reproduction

Item	Kind	Source / tools
Quotations in § 01 and § 08	Text checked (verbatim comparison by the checked survey)	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000)

Item	Kind	Source / tools
Formulas (3) and (5), the evolution equation of f, the evolution equation and minimum of R, section 2 of the introduction, § 1.1 and § 1.2, the periodic orbits and solitons of § 2 and § 3, the cutting of the neck	Text checked	G. Perelman, arXiv:math/0211159 (2002)
Introduction of Ricci flow, DeTurck, Uhlenbeck, the Wiles type	Bibliographic data only	R. S. Hamilton, J. Differential Geom. 17 (1982) 255–306. D. DeTurck, ibid. 18 (1983). K. Uhlenbeck, CMP 83 (1982) 11–29, 31–42. A. Wiles, Ann. of Math. 141 (1995) 443–551. Taylor–Wiles, ibid. 553–572
Renormalisation group of the σ model, asymptotic freedom	Abstract checked (the coefficient $(N-2)/2\pi$ is the standard one-loop count)	D. Friedan, PRL 45 (1980) 1057. A. M. Polyakov, Phys. Lett. B 59 (1975) 79
Two-loop coefficients (departure from Ricci flow)	Abstract checked (up to the two-loop calculation; the form of the term $R_{iklm}R_j{}^{klm}$ is the standard two-loop count)	D. Friedan, Ann. Phys. 163 (1985) 318
Status of the gap of the two-dimensional $O(N)$ model	Checked survey (Douglas verbatim from the text, Kopper bibliographic data only)	M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem, Clay Math. Inst. (2004). C. Kopper, CMP 202 (1999) 89
KT transition, Debye screening, the a-theorem, gradient flow	Abstract checked	Fröhlich–Spencer, CMP 81 (1981) 527. Brydges–Federbush, CMP 73 (1980) 197. Komargodski–Schwimmer, JHEP 12 (2011) 099, arXiv:1107.3987. Lüscher, JHEP 08 (2010) 071. Carosso–Hasenfratz–Neil, arXiv:1806.01385. Waldron, Invent. Math. 217 (2019) 1069
The values of a for free fields ($1 : \frac{11}{2} : 62$, and that they are positive)	Text checked (appendix A)	Z. Komargodski, A. Schwimmer, arXiv:1107.3987

Item	Kind	Source / tools
The c-theorem, sum rule (10)	Bibliographic data only (sum rule (10) is attributed to Cardy 1988; the abstract goes only as far as a universal quantity of the second moment, and the form of the sum rule is unchecked. The normalisations of the coefficients 3π and $3/4\pi$, and $c = 1$ for a free field, were checked by calculation)	A. B. Zamolodchikov, JETP Lett. 43 (1986) 730. J. L. Cardy, PRL 60 (1988) 2709
Bauerschmidt–Brydges–Slade in § 05	Abstract checked	R. Bauerschmidt, D. C. Brydges, G. Slade, CMP 338 (2015) 169–193, arXiv:1403.7268 (the four-dimensional weakly self-avoiding walk). Same authors, arXiv:1403.7424 (the four-dimensional n-component $\lvert\phi\rvert^4$). Same authors, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474. D. C. Brydges, G. Slade, J. Stat. Phys. 159 (2015) 530, arXiv:1403.7255 (part IV of the method)
§ 09 table (ii), (iii)	Checked survey (Chevyrev–Shen from the text, Chatterjee from the abstract)	I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026), arXiv:2302.12160. S. Chatterjee, A scaling limit of SU(2) lattice Yang–Mills–Higgs theory, Prob. Math. Phys. 7 (2026) 339–381, arXiv:2401.10507
The published construction in § 01, § 05 and row (i) of the table	Assessment	The assessment report `yang-mills-group/YM`, `docs/assessment-1.pdf` (2026-11-10), opening and § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). For how to read it, see Part 6 and 16-03
Candidates in § 05, § 08, § 09	Checked survey (abstracts; Chevyrev from the text)	Bałaban, CMP 122 (1989). Magnen–Rivasseau–Sénéor, CMP 155 (1993). Hairer, arXiv:1303.5113. CCHS, arXiv:2006.04987, 2201.03487. Chevyrev, arXiv:2202.13359. Connes–Kreimer, CMP 210 (2000). Costello, arXiv:0706.1533. Shen–Zhu–Zhu, CMP 400 (2023). Chatterjee, CMP 366 (2019), CMP 385 (2021). Cao–Park–Sheffield, arXiv:2307.06790
Table 1, Figure 1	Computed	Python on this machine: `sigma_rg.py` (formula (8) and four-dimensional one loop, the check of $\ell_* = 4\pi t_*$)

Item	Kind	Source / tools
Table 2, Figure 2, formula (11)	Computed	Python on this machine: `c_function.py` (scipy's quad; $\int_0^\infty u^3 K_0^2 du = 1/3$ checked to the order of 10^{-11})

Next chapter to read: 16-07 The list of open tasks (what would move if what were known).

The list of open tasks — joining the open questions by arrows of "if this were known, what would move"

In this chapter, § 01– § 08 are a map of open questions and § 09 is guide material.

What this chapter shows — Twelve open tasks and the closest results there are now. The three strengths of the arrows that join the tasks (theorem, roadmap, physical expectation). Checking, with a figure and a count, which requirements of the problem statement move when a task is assumed known.

Prerequisite chapters — 16-05 Where the two ways of digging part. We use the vocabulary of 7-10 (what is reached at each stage in four dimensions), 7-11 (the six obstacles) and 16-01 (the board).

Said first — This chapter lines up and joins the tasks; it settles none of them. The choice of arrows and the way of counting are this book's arrangement, and only arrows checked in primary sources are used. This site shows neither the existence of the mass gap nor the construction of the continuum limit. The tasks touched by the construction published in 2026 are written in the "closest result" column of Table 1 and at the end of § 02 (the committee's assessment Assessment ; for the meaning of the label Assessment , see "Labels" in 6-00). The list of questions that remain after the published construction is in 16-02.

The order of this chapter

1. Writing the tasks with "if ... then" — requirements and arrows
2. Twelve tasks — continuum and lattice (Table 1)
3. Arrows joined by theorems — reflection positivity, the uniform gap, the centre
4. Roadmap arrows — the order the setters and surveys write
5. Suppose it were known — a map of the tasks (Figure 1, Table 2)
6. Counting the map — tasks needed on every road (Table 3, computed)
7. For every group — arrows that vanish when the group is changed
8. Tasks outside the problem statement — one-particle states, general manifolds
9. The book's coordinate system — dictionary, seams, tasks and walls in nine cells (Figure 2, Table 4)

01

Writing the tasks with "if … then"

A **task** in this chapter is a statement that has not yet become a theorem. What moves when a task becomes known can be written as an arrow "if A is known, B can be said", and we divide its strength into three. **Theorem** (a proved implication, or a definition), **roadmap** (a prospect that the problem statement or a survey writes as "once this is known, the next step should be possible"), and **expectation** (an implication at the level of physics). As a metaphor it is a table matching keys to doors; precisely, it is a list of rules of the form "if all the premises are in place, the conclusion follows".

$$X_1 \wedge X_2 \wedge \cdots \wedge X_k \overset{\ell}{\Longrightarrow} Y, \qquad \ell \in \{0\ (\text{theorem}),\ 1\ (\text{roadmap}),\ 2\ (\text{expectation})\} \tag{1}$$

The destination is the problem statement (Jaffe–Witten 2000, below JW; 0-03). Fix one group G, and divide the demand into four **requirements** Known (a summary of JW § 4; the division into four is this book's arrangement).

$$\mathsf{P} = \bigwedge_{G} \mathsf{P}_G, \qquad \mathsf{P}_G = \mathsf{E} \wedge \mathsf{A} \wedge \mathsf{N} \wedge \mathsf{M} \tag{2}$$

The requirement of existence (E) is the existence of the theory on $\mathbb{R}^4$; the requirement of the axioms (A) is the axioms (at least as strong as those of the book of Streater–Wightman [45] and the papers of Osterwalder–Schrader [35] cited by JW); the requirement of non-triviality (N) is non-triviality, local fields and agreement with asymptotic freedom at short distances (JW § 4); and the requirement of the gap (M) is the gap $\Delta > 0$ and $m < \infty$.

We add one more requirement, outside the problem statement: the requirement of the string tension (C). The requirement of the string tension is that the string tension d of the continuum theory is not 0. d is the constant of the continuum string-tension problem of Chatterjee 2018 (Chatterjee's Problem 5.2), written σ in 7-10 § 04. The physical expectation is that d is positive.

Example: the Theorem from Uniform Decay to the Area Law of Chatterjee (2021) (Theorem 2.4 of Chatterjee 2021) is a theorem arrow. If a lattice gauge theory has uniform exponential decay under arbitrary boundary conditions (the definition of strong decay (Definition 2.3 of Chatterjee 2021); the constants depend only on the group, β (the coefficient of the coupling in the lattice action; not an inverse temperature) and the dimension of the lattice, not on the box or the boundary conditions), then centre symmetry is not broken. Further, rectangular Wilson loops in an irreducible representation that acts non-trivially on the centre obey the area law in every Gibbs measure (a distribution on the infinite lattice that follows the lattice

weights inside any box once the outside is fixed)[Known] (16-08 § 06). An example of a roadmap is JW § 6.5's 'Such a result presumably would enable the study of the limit as T4 → R4' (from the uniform gap to the infinite-volume limit).

02

Twelve tasks — six in the continuum and six on the lattice

The upper six of Table 1 (from the task of the T⁴ construction to the task of local fields and short distances) are tasks for the continuum theory, corresponding to the ultraviolet-limit step through the uniform-gap step of the six steps of construction (7-01) and to the additional conditions of the problem statement. The lower six (from the lattice correlation-length problem to the task of the three-dimensional measure) are lattice tasks, and tasks that turn lower dimensions and physical premises into mathematics. In this chapter limits are required along the whole sequence (this book's arrangement). JW's weak-existence footnote (footnote 2 of Jaffe–Witten; 0-03 § 08) writes that weak existence by compactness alone is not accepted as a solution of the existence part unless the properties of the limit (the gap and the axioms) are shown by other methods[Known] . The first three (in order, the task of the T⁴ construction, the task toward infinite volume and the task of a uniform gap) have the following form (notation as in 7-10).

$$\lim_{a\to 0}\langle W_{\gamma a}\rangle^{\mathrm{ren}}_{a,L}\ \text{exists along the whole sequence,}\qquad \lim_{L\to\infty}\mu_L\ \text{exists,}\qquad \inf_L \Delta_L > 0 \tag{3}$$

$W_{\gamma a}$ is the Wilson loop of a lattice path, $\langle\cdot\rangle^{\mathrm{ren}}$ is the expectation with a term proportional to the perimeter removed from its logarithm (formula (5.2) of Chatterjee 2018; 7-10 § 04), μ_L is the measure on the box T^4 of side L, and Δ_L its gap. Requiring the limit along the whole sequence in the task of the T⁴ construction and the task toward infinite volume is this book's arrangement.

[Known] Table 1 Twelve tasks

	Task	Statement	Closest result	What moves if known
1	Task of the T⁴ construction	Expectation values of gauge-invariant quantities converge as $a \to 0$ along the whole sequence (the whole sequence is this book's arrangement)	For correlation functions of local fields, the construction published in 2026 showed convergence along the whole sequence of cutoffs in a periodic box[Assessment] (8-05). Not in the form of this row's statement (renormalised Wilson loops) (16-02 § 05). Before that, ultraviolet stability with a hypothesis and subsequences (Bałaban), an outline (MRS)	The ultraviolet-limit step, half of the requirement of existence

	Task	Statement	Closest result	What moves if known
2	Task toward infinite volume	The limit $L \to \infty$ exists along the whole sequence (likewise this book's arrangement)	What the published construction has goes only as far as limits along subsequences and phase decomposition Assessment . Convergence of the whole sequence and uniqueness are open questions of the assessment report (16-02 § 02). At the time of the problem statement, 'Nor do present methods suggest how to obtain the existence of the infinite volume limit' (JW § 6), a stage at which § 6.5 writes that the task of a uniform gap would open the way. On the lattice it is shown at strong coupling	The infinite-volume step, the rest of the requirement of existence
3	Task of a uniform gap	$\inf_L \Delta_L > 0$	'New ideas are needed' (JW § 6.5). What the published construction has goes only as far as polynomial estimates uniform in the phase; there is no exponential decay (16-02 § 04)	The uniform-gap step; through the roadmap, the task toward infinite volume
4	Task of the remaining axioms	Recovery of rotation symmetry, and the regularity needed for reconstruction	In the published construction, the Wightman axioms (including Euclidean invariance) for each phase Assessment (12-06, 12-08). Reflection positivity survives from the lattice. Rotations: Driver's Corollary 6.7 in two dimensions	The axiom step, the seam of spacetime and space
5	Task of non-triviality	The limit is not Gaussian	The scaling limit of lattice-cut four-dimensional ϕ^4 (including the Ising type) is Gaussian (Aizenman–Duminil-Copin 2021; a result in the opposite direction from non-triviality). In the published construction, the short-distance law of the two-point function of the trace field excludes descriptions of the type "polynomials in free fields of finitely many masses" Assessment (13-02). It does not exclude every description built from Gaussian fields	Half of the requirement of non-triviality

	Task	Statement	Closest result	What moves if known
6	Task of local fields and short distances	Local fields such as $\mathrm{Tr}\, F_{ij}F_{kl}(x)$. Agreement with asymptotic freedom at short distances	The level of perturbation theory. There is no natural one-to-one correspondence (JW's local-field footnote). The published construction has labels for local fields (8-03), and at short distances checks the one-loop running and the trace coefficient. Agreement with perturbation theory at all orders is not shown Assessment (6-04 § 06)	The rest of the requirement of non-triviality
7	Lattice correlation-length problem (Chatterjee's Problem 5.1)	$\xi(\beta) \in (0, \infty)$ (all β), $\xi \to \infty$	At strong coupling exponential decay (an upper bound on ξ) has been shown. For finite groups, exponential decay even at weak coupling (Adhikari–Cao; outside the scope of the lattice correlation-length problem, and without $\xi \to \infty$)	Only cells of the lattice
8	Continuum string-tension problem (Chatterjee's Problem 5.2)	String tension d $/= 0$ in the continuum limit of Wilson loops	Not yet. Whether a perimeter term is needed is also open. The published construction has no continuum Wilson loops either (16-02 § 05)	The requirement of the string tension
9	Task of strong decay	Uniform exponential decay under arbitrary boundary conditions (all β)	For small β it can be shown by the method of Dobrushin–Shlosman, as Chatterjee 2021 remarks. Whether the strong version is expected at large β is not clear (ibid.)	The lattice area-law problem
10	Lattice area-law problem (Chatterjee's Problem 4.1)	The area law for all β	Small β for every group (Osterwalder–Seiler 1978). Three-dimensional $U(1)$ for all β (Göpfert–Mack 1982)	Only lattice confinement
11	Task of one scale	$c\,\sigma(\beta) \le m(\beta)^2 \le C\,\sigma(\beta)$ for $\beta \ge \beta_0$ (constants $c, C > 0$ independent of β; m, σ are the mass and string tension in lattice units, and neither $m > 0$ nor $\sigma > 0$ is assumed)	No theorem. The record of this book's study is at the stage of placing it as an assumption in the form of 17-12 formula (3) (the diffusive-relaxation-time assumption is a separate formula (6), 17-12 § 05)	Joins the requirement of the string tension and the requirement of the gap at the level of expectation

	Task	Statement	Closest result	What moves if known
12	Task of the three-dimensional measure	Construction of the measure on T^3	'remains open' (Chevyrev 2022)	Only the 3D ultraviolet-limit step

For details: the task of the T^4 construction is in 7-07 (the published construction in 6-01), the task of non-triviality in 7-08, the lattice correlation-length problem and the lattice area-law problem in 16-01 § 07, and the task of the three-dimensional measure in 7-06. Example: what moves the four-dimensional cells of the board (16-01 Table 3) is the upper four of Table 1 (from the task of the T^4 construction to the task of the remaining axioms) and the lattice correlation-length problem; the three-dimensional row is moved by the task of the three-dimensional measure. The continuum string-tension problem, the task of strong decay, the lattice area-law problem and the task of one scale have no cells; they are tasks on the side of the seams. In the ledger of conditions (27-02), the lattice side of the task of a uniform gap corresponds to the row of uniform decay of block observables (correlations of block observables decay at a rate uniform in L and the lattice spacing; the premises are ultraviolet control, the small-T^2 gap and the band comparison (in the language of the gap)) Paper . The ultraviolet-control row of the ledger was shown, as a continuum limit in a periodic cubic box, by the published construction, while the form of this book's road remains open (27-02 Table 1). What remains on this book's road is the seam that puts the ultraviolet result into a form the infrared arguments can use (16-03, 16-04) Inferred . The arrow from the area law (the lattice area-law problem) to the task of a uniform gap does not hold in general. The area law is about the gap in the charged sector (electric flux $e \neq 0$), the task of a uniform gap about the gap in the neutral sector ($e = 0$), and the transfer matrix preserves this decomposition Paper . The known theorem goes the other way (uniform exponential decay ⇒ the area law; the Theorem from Uniform Decay to the Area Law of Chatterjee 2021 in § 01) Known .

After the published construction. Of the upper six of Table 1, the construction published in 2026 meets the task of the T^4 construction for correlation functions of local fields and the task of the remaining axioms for each phase, and touches the task toward infinite volume in the form of subsequences, and the task of non-triviality and the task of local fields and short distances in the form of checks at short distances Assessment . In the words of this chapter, the requirement of existence is not met in the form that requires the limit along the whole sequence (the volume side is along a subsequence), and nothing is given towards the requirement of the gap (the mass gap) Inferred . The figure of § 05 and the count of § 06 are a map of the logic between the tasks, and the rules do not change because of the published construction. The list of which tasks remain is placed in 16-02, following the five open questions of the assessment report, and where the construction stops is in 16-03.

03

Arrows joined by theorems

Not many arrows have a proof. We look at three in turn.

Reflection positivity survives in the limit

The first is reflection positivity. It holds on Wilson's lattice (Osterwalder–Seiler 1978), and since it is an inequality, if the sequence of measures μ_n **converges weakly** it survives in the limit Known (4-10). Converging weakly means that the expectation values of all bounded continuous quantities converge (0-03 § 08).

$$\langle \Theta F\, F\rangle_{\mu} = \lim_{n\to\infty} \langle \Theta F\, F\rangle_{\mu_n} \;\geq\; 0 \quad (4)$$

Θ is time reflection, and F a bounded continuous quantity determined by the side of positive times alone. Together with the task of the remaining axioms, the Osterwalder–Schrader reconstruction sets up a Hilbert space and $H \geq 0$ (the rule of the axioms; JW also writes 'Establishing a quantum mechanical Hilbert space is part of the solution to this Millennium problem').

A uniform gap survives in the limit

The second is that a uniform gap survives in the limit (the rule of the gap). Let τ_t be the operation of shifting by t in the time direction, and write ν_F for the spectral measure of the vector made by F with its vacuum component subtracted. ν_F is the same thing as the μ_F of 4-11 § 02. In this chapter we use μ for the sequence of measures, so we change the symbol.

$$\langle \Theta F\, \tau_t F\rangle - \left|\langle F\rangle\right|^2 = \int_{[0,\infty)} e^{-tE}\, \mathrm{d}\nu_F(E) \quad (5)$$

That the left-hand side is at most a constant times $e^{-\Delta t}$ for every bounded F is the same as $H \geq \Delta$ in the directions orthogonal to the vacuum Known (4-11 § 02). The integral also includes the point $E = 0$, so uniqueness of the vacuum comes out at the same time.

In the estimate box by box (7-10 formula (7)), if the rate of decay $\Delta_L \geq \Delta_*$ does not depend on L, the estimate survives in the limit and becomes a gap (the standard argument). The prefactor $\|A^*\Omega_L\|\,\|B\,\Omega_L\|$ in (7) is at most 1 if $|F| \leq 1$. This argument needs three premises. (i) The expectation values of the F used converge along the sequence $L \to \infty$ (part of the requirement of existence). (ii) The vectors of those F are dense in the reconstructed Hilbert space. (iii) Since T^4 is also periodic in the time direction, the term $e^{-\Delta_L(L-t)}$ added to the correlations in the box vanishes as $L \to \infty$

(it does if $\Delta_L \geq \Delta_*$). What the rule of the gap gives is only $\Delta > 0$; $m < \infty$ comes separately from the theory not being the vacuum alone (7-10 § 06).

Example: the decay of a single F tells us only the lower end of the support of ν_F. The $\xi(\beta)$ of the lattice correlation-length problem is the decay of plaquette correlations. Chatterjee defines ξ by the limit $|x| \to \infty$ taken in every direction, but what connects to the spectrum of the transfer matrix (5-03) is the decay in the time direction. Read through the decay in the time direction, what is learned is the lower end of H on the subspace made by the plaquette (this chapter's reading).

From centre symmetry to the area law

The third is the Theorem from Uniform Decay to the Area Law of Chatterjee (2021) from § 01 (the rule from decay to the area law). Its introduction also cites Durhuus–Fröhlich (1980), which derives confinement in a gauge theory from exponential decay in a non-linear σ model one dimension lower, and Fröhlich (1979), which carries confinement from a $\mathbb{Z}_n$ theory to an $SU(n)$ theory Known (bibliographic data only). By the title of Fröhlich's paper the destination is the lattice Higgs theory of $SU(n)$; the introduction of Chatterjee 2021 presents it as "the $SU(n)$ theory".

04

Roadmap arrows — the order the setters and surveys write

JW writes prospects for the order of the tasks. For the task of the T[4] construction it says there are ample signs that extending known methods would reach it, writing 'One presumably needs to revisit known results at a deep level, simplify the methods, and extend them'. For the task of a uniform gap it writes 'New ideas are needed', and goes on that this would make the task toward infinite volume possible. § 5 also writes that a uniform gap 'may play a fundamental role' in the proof of the infinite-volume limit Known . This is the rule of the problem setters' route.

$$\text{task of a uniform gap} \overset{1}{\Longrightarrow} \text{task toward infinite volume}: \qquad \inf_L \Delta_L \geq \Delta_* > 0 \quad \rightsquigarrow \quad \text{correla}$$

Precisely, the right-hand side of (6) is not the existence of the limit but the material of a uniform estimate for taking the limit, and how to assemble it is not written ('presumably').

Example: the two things that the rule of the problem setters' route joins (uniform decay and the infinite-volume limit) were obtained together from the cluster expansion in models with a small parameter. Weak-coupling ϕ^4_2 and ϕ^4_3 are examples Known (7-04 § 08). They did not pass in the order of deriving the infinite-volume limit from a uniform gap. Four-dimensional Yang–Mills has no such parameter at long

distances (7-11 the obstacle of the infrared parameter). On the lattice side, Chatterjee (2018) places the continuum string-tension problem as 'the most basic question that needs to be solved before making any further progress' on the road through Wilson loops Known . This is a sentence in the direction of necessity, "needed first", and not an arrow of sufficiency from the continuum string-tension problem to the requirements.

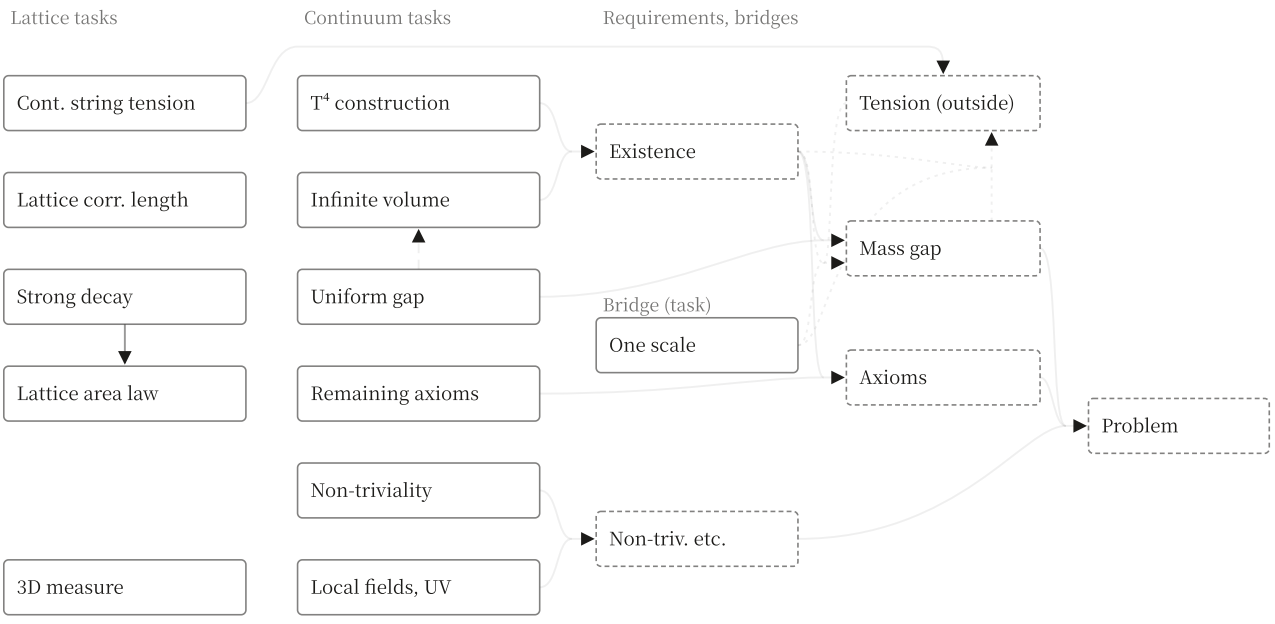
05

Suppose it were known — a map of the tasks

We put the rules of Table 2 into one figure. Pressing the box of a task makes that task "assumed known", and the conclusions of rules whose premises are all in place light up. What lights up is the smallest set that contains the conclusion whenever it contains all the premises (the closure).

$$\mathrm{cl}_\ell(S) = \bigcap \{ X \supseteq S : \text{ for every rule of level } \le \ell \text{, premises} \subseteq X \Rightarrow \text{conclusion} \in X \} \quad (7)$$

Figure 1 Pressing the box of a task makes it "assumed known", and the requirements that move and the cells of the board light up. The knob selects the strength of the arrows (theorems only / + roadmap / + expectation), and the button selects a group with trivial centre. Solid lines are theorems, dashed lines roadmaps, dotted lines expectations. G is fixed



Assumed known: none. To the problem statement: still T⁴ construction, Uniform gap, Remaining axioms, Non-triviality, Local fields, UV. Cells moved on the board: none (arrows: theorems, roadmaps, physical expectations)

Computed How this figure lights up is the closure (7) of the rules of Table 2, computed on the spot (the choice of arrows is this book's arrangement).

Known Table 2 Rules (if all the premises are in place, the conclusion). The upper seven (from the rule of existence to the rule of the string tension) are theorems or definitions, the rule of the problem setters' route is a roadmap, and the rule from the

string tension to the gap and the rule from the gap to the string tension are physical expectations Physics . The choice of rules is this book's arrangement. The requirement of the string tension is that of § 01: the continuum string tension d is not 0 (outside the problem statement)

Rule	Premises ⇒ conclusion	Strength	Grounds
Rule of existence	Task of the T^4 construction ∧ task toward infinite volume ⇒ requirement of existence	Definition	JW § 6.5
Rule of the axioms	Requirement of existence ∧ task of the remaining axioms ⇒ requirement of the axioms	Theorem	Formula (4) and OS reconstruction
Rule of the gap	Requirement of existence ∧ task of a uniform gap ⇒ requirement of the gap	Theorem	Formula (5) and 7-10 formula (7) (the standard argument; premises in § 03)
Rule of non-triviality	Task of non-triviality ∧ task of local fields and short distances ⇒ requirement of non-triviality	Definition	JW § 4
Rule of the problem statement	Requirement of the axioms ∧ requirement of non-triviality ∧ requirement of the gap ⇒ requirement of the problem statement (P_G)	Definition	Formula (2)
Rule from decay to the area law	Task of strong decay ⇒ lattice area-law problem	Theorem	Chatterjee 2021, Theorem from Uniform Decay to the Area Law
Rule of the string tension	Continuum string-tension problem ⇒ requirement of the string tension	Definition	Chatterjee 2018 § 5 (the requirement of the string tension is exactly the $d \neq 0$ of the continuum string-tension problem)
Rule of the problem setters' route	Task of a uniform gap ⇒ task toward infinite volume	Roadmap	JW § 5, § 6.5
Rule from the string tension to the gap	Requirement of existence ∧ requirement of the string tension ∧ task of one scale ⇒ requirement of the gap	Expectation	A single scale (17-12)
Rule from the gap to the string tension	Requirement of existence ∧ requirement of the gap ∧ task of one scale ⇒ requirement of the string tension	Expectation	The same sentence in the reverse direction

Example: even if the "Lattice tasks" button makes the lattice correlation-length problem, the continuum string-tension problem, the task of strong decay, the lattice area-law problem and the task of the three-dimensional measure assumed known,

only the requirement of the string tension lights up, and not a single requirement. This restates in the language of arrows the "showing it on the lattice is enough" of 16-08 § 08.

06

Counting the map — tasks needed on every road

Among the sets of tasks that reach the requirement of the problem statement, we call those that no longer reach it when any one task is removed **minimal sets**. We examined the 4096 subsets at three levels and counted all the minimal sets Computed .

$$\mathcal{S}\ell = \{ S : \ \mathsf{P}G \in \mathrm{cl}\ell(S), \quad S \text{ minus any one element gives } \mathsf{P}G \notin \mathrm{cl}\ell \} \tag{8}$$

With theorems only, there is one set, the upper six of Table 1 (from the task of the T^4 construction to the task of local fields and short distances); adding roadmaps, one set, {the task of the T^4 construction, the task of a uniform gap, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances}; adding physical expectations too, {the task of the T^4 construction, the task toward infinite volume, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances, the continuum string-tension problem, the task of one scale} is added, making two sets. The same held for groups with trivial centre.

Computed Table 3 For each task, whether it is in every minimal set for the requirement of the problem statement (required), in some, or in none

Task	Theorems only	+ Roadmap	+ Physical expectation	What it moves alone (up to physical
Task of the T^4 construction, task of the remaining axioms, task of non-triviality, task of local fields and short distances	Required	Required	Required	Nothing
Task toward infinite volume	Required	—	In some	Nothing
Task of a uniform gap	Required	Required	In some	Task toward infinite volume (roadr

Task	Theorems only	+ Roadmap	+ Physical expectation	What it moves alone (up to physical
Continuum string-tension problem	—	—	In some	Requirement of the string tension
Task of one scale	—	—	In some	Nothing
Lattice correlation-length problem, task of strong decay, lattice area-law problem, task of the three-dimensional measure	—	—	—	The task of strong decay alone give with non-trivial centre)

Example: (a) the task of the T^4 construction, the task of the remaining axioms, the task of non-triviality and the task of local fields and short distances are needed on every road. (b) At the level of physical expectation, instead of the task of a uniform gap, "the continuum string-tension problem and the task of one scale" reach the mass gap. The physical sentence that sees mass and string tension as two faces of one scale (formula (3) of 17-12) takes over the role of the task of a uniform gap. (c) The lattice correlation-length problem, the task of strong decay, the lattice area-law problem and the task of the three-dimensional measure are in no minimal set. The count uses only arrows checked in primary sources, so if an arrow from the lattice to a continuum requirement becomes a theorem, it must be redone.

07

For every group — arrows that vanish when the group is changed

The problem statement asks for the requirement of the problem statement for all compact simple groups G (formula (2)). The candidate Lie algebras are four series and five exceptions, and with each Lie algebra come groups in forms divided by the centre Known (1-04 Table 6). No general theorem carrying a result for one group to another was found in what this chapter read. Some arrows vanish depending on the group. The rule from decay to the area law uses that a finite-dimensional irreducible unitary representation π acts non-trivially on the centre, that is, that there is an element z of the centre with $\pi(z) \neq I$. An element of the centre commutes with every $\pi(g)$, so in an irreducible π it acts as a scalar by Schur's lemma (1-04 § 06).

$$\exists\, z \in Z(G): \qquad \pi(z) = \omega I, \quad \omega \neq 1 \qquad (\pi \text{ irreducible}) \ (9)$$

Example: in the fundamental representation of $SU(N)$, $z = e^{2\pi i/N} I$ and $\omega = e^{2\pi i/N}$ (1-04 § 07). For a group with trivial centre there is no representation satisfying (9). Besides the simply connected G_2, F_4 and E_8, this includes the forms divided by the centre $SO(3) = SU(2)/\mathbb{Z}_2$ and $PSU(N) = SU(N)/\mathbb{Z}_N$; every group divided by its centre has trivial centre Known . The reason in one line: for a connected Lie group, the centre is the kernel of the adjoint representation (1-04 § 06). $G/Z(G)$ is the same group as the image $\mathrm{Ad}(G) \subset GL(\mathfrak{g})$ of the adjoint representation, and its adjoint representation is $\mathrm{Ad}(g)$ itself on $\mathfrak{g}$ (for a simple Lie algebra ad is injective). The kernel is the elements with $\mathrm{Ad}(g) = I$, that is, the identity alone. The "group with trivial centre" of Figure 1 is this case, and the arrow from the task of strong decay to the lattice area-law problem vanishes. Chatterjee notes that for groups with trivial centre (for example $SO(3)$) it is believed they may not confine at weak coupling Physics . The problem statement asks for a gap for such groups too, so a road through confinement cannot be used for every group. On the other hand, the one-loop coefficient $16\pi^2 b_0 = \frac{11}{3} C_A$ is positive for every group (4-06 § 02), and perturbative asymptotic freedom does not depend on the group Known (0-03).

Replacing by SU(N). Moving the rows of this book's ledger of conditions (27-02) one by one to $SU(N)$, they fall into three kinds Paper . (i) Rows independent of the group: the uniform gap of the double well (the double well), the positivity of the 't Hooft tension and the reflection positivity of z (algebra), the uniform acceptance of insertion (an identity), and the limit to R^4. (ii) Rows that move as they are into the language of the centre $\mathbb{Z}_N$: in the twisted part of the small-T^2 gap, the Rigidity Lemma for Anticommuting Pairs of 25-01 (anticommuting pairs) is replaced by the uniqueness of the clock and shift pair (if the twist is primitive it is unique up to conjugation and its centraliser is $\mathbb{Z}_N$; González-Arroyo's review hep-th/9807108 § 4.2–4.3 Known), and the adjoint gap of the twisted T^2 becomes $4\sin^2(\pi/(NL))$ (for $SU(3)$, $L = 3$–6, it agrees with the computation of eigenvalues, and the value is determined by NL alone Computed). The source of the base point at small T^2 is for $SU(N)$, and the reflection positivity of z and frame decorrelation are in the general-group form of Osterwalder–Seiler. The one-loop $\beta - \beta'$ is $11N^2 \ln 2/(12\pi^2)$ (0.2575 for $N = 2$, 0.5794 for $N = 3$) Paper . The lemma of the colour average goes through with the $n = N^2 - 1$ colours of $\mathfrak{su}(N)$, and the constant 5 becomes $n(n+2)/3$ (including the existence of a weighted spherical 4-design, Lean: ColorAverageN (colorAverage, design_identity); the application to $\mathfrak{su}(N) \cong \mathbb{R}^n$ is Paper). The $SU(N)$ version of the Tomboulis–Yaffe inequality is Theorem 2 of Kanazawa arXiv:0808.3442 (unpublished) Known (its application to the slabs of this book is not done). (iii) Rows that need redoing: the ultraviolet parts that use properties of $SU(2) = S^3$ and $-1 \in$ the Weyl group (the

evenness of class functions) — that the conditional law of one link is one-parameter (15-07 (5)), the upper bound that drops the negative terms of the image sum of the heat kernel (the Heat-Kernel Fluctuation Rarity Lemma of 15-08), and the evenness step (in the hierarchical model, for $SU(3)$ it breaks through the cubic invariant $\operatorname{tr} X^3$) — and the numerics of the band (26-09 has only $SU(2)$ and $U(1)$). So to move this book's road to $SU(N)$, one must add, to the rows open for $SU(2)$ (the renormalisation comparison and others), the group versions of these three lemmas and the redoing of the numerics. Existence on the ultraviolet side is supplied by the published construction for any compact simple group Assessment . This book's ultraviolet parts are read not as parts for existence but as parts for the seam of this book's road Inferred (16-04). For groups with trivial centre (G_2, F_4, E_8 and groups divided by their centre) there is no tool of twists, and for $Sp(n)$ ($n \geq 2$) the centraliser of the pair that absorbs the twist is continuous, so a single twist of T^2 does not remove the zero modes Paper (one example). The road to all compact simple groups is open within the scope of this book.

08

Tasks outside the problem statement

JW § 5 lists 'natural extensions': isolated one-particle states (an upper gap in addition to the gap), confinement, other four-dimensional gauge theories with fields added that keep asymptotic freedom and their dynamics, and the extension from $\mathbb{R}^4$ to general four-dimensional manifolds Known . None of these is a requirement, and in Figure 1 only the requirement of the string tension was placed outside. Isolated one-particle states can be written through the spectrum of the mass operator $\sqrt{H^2 - \mathbf{P}^2}$ ($\mathbf{P}$ is the momentum) (16-05 the seam of the word "particle").

$$\operatorname{spec} \sqrt{H^2 - \mathbf{P}^2} \cap (0,\, m + \delta) = \{m\} \qquad (\text{some } \delta > 0) \ (10)$$

Example: for a free field of mass m, the one-particle states of Fock space (2-06) sit at m and two or more particles fill $[2m, \infty)$; the spectrum is $\{0\} \cup \{m\} \cup [2m, \infty)$, and the upper gap is m Known . JW § 5 also writes that the clustering that follows from the gap (7-10 § 08) may carry results on $\mathbb{R}^4$ to general four-dimensional manifolds ('may make it possible').

09

The book's coordinate system — dictionary, seams, tasks and walls on one grid

We use the two-column grades of the dictionary (18-01 § 05) as common coordinates, and plot four lists on nine cells. (a) is "if both exist, are they the same", (b) is "does it

exist in four dimensions". The grades of the dictionary rows are as in 18-01; the grades of seams, tasks and walls are this book's arrangement. The grounds are as follows.

Seams: take over the grade of the dictionary row corresponding to the pair of sentences in 16-05 Table 1 (1→1, 2→13, 3→9, 4→8, 5→2, 6→5, 7→14). Tasks: (a) is the strength of the rules leaving the task in Table 2 (theorem or definition = Settled, conditional on the group = Conditional, expectation = Unsettled; the lattice correlation-length problem, the lattice area-law problem and the task of the three-dimensional measure, which have no rule, are unrated), (b) is the "Closest result" of Table 1 (Conditional if there is a version with limited scope). Walls: the grade of the dictionary row named by the column of 23-02 Roads tried on the four-dimensional lattice, and walls (the relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption, η, chain → 13; the physical-unit gap statement → 8; limit → 7. One step and the summability-of-shortfalls assumption have no row in the dictionary and are unrated).

Example. The thirteen walls that bear on the diffusive-relaxation-time assumption gather in the same "Unsettled, None" cell as dictionary row 13 (Λ, "a single scale"), the seam of "there is one scale" and the task of one scale. Although the types of wall (23-02 Roads tried on the four-dimensional lattice, and walls) divide into principle, structure, tool, assumption and measurement, on the dictionary they all bear on the same single pair. Only tasks enter "Settled, Conditional": those with a version of limited scope whose arrows to the requirements are theorems or definitions (the task of the T^4 construction, the task toward infinite volume, the task of the remaining axioms).

Figure 2 Choose a list with the buttons, and its rows are plotted on the nine cells of (a)×(b). "Overlay" plots all four at once. Below the frame: unrated

(a)	(b) Exists in four dimensions: Constructed	Cond.	None
Settled	#3 #10		#1 #2
Cond.	#12	#11	#4 #5 #6 #7 #8
Unsettled		#9	#13 #14

Unrated

```
Dictionary : Settled / Constructed 2, Settled / None 2, Conditional / Constructed 1, Conditional / Conditional 1, Conditional / None 5, Unsettled / Conditional 1, Unsettled / None 2
```

Computed The assignment of grades was made on this machine by the rules above from 18-01 Table 2, 16-05 Table 1 and Tables 1 and 2 of this chapter, and 23-02 Table 3, and the counts (14, 7, 12, 21) were checked.

Table 4 The rows that fall in each of the nine cells and in unrated (numbers are the rows of each table; walls are numbered as in 23-02 Table 3; in the figure, dictionary rows carry # and seams carry S).

(a) / (b)	Dictionary (18-01 Table 2)	Seams (16-05 Table 1)	Tasks (Table 1)	Walls (23-02 Table 3)
Settled / Constructed	3, 10			
Settled / Conditional			T^4 construction, infinite volume, remaining axioms	
Settled / None	1, 2	1, 5	uniform gap, non-triviality, local fields and short distances, continuum string tension	
Conditional / Constructed	12			
Conditional / Conditional	11		strong decay	
Conditional / None	4, 5, 6, 7, 8	4, 6		2, 9, 21
Unsettled / Constructed				
Unsettled / Conditional	9	3		
Unsettled / None	13, 14	2, 7	one scale	1, 3, 4, 5, 7, 10, 11, 12, 14, 16, 17, 18, 20
Unrated			lattice correlation length, lattice area law, three-dimensional measure	6, 8, 13, 15, 19

What this chapter can say

	Said	Not said
Known	The statements and current status of the tasks (within the primary sources). The implications of the rule of the axioms, the rule of the gap and the rule from decay to the area law (the rule of the gap is the standard argument, with the premises (i)–(iii) of § 03) and JW's roadmap sentences	That twelve exhausts them. The choice of arrows, and requiring limits along the whole sequence (this book's arrangement). Arrows not checked are not counted
Computed	The minimal sets under the rules of Table 2, and Table 3	Counts under other rules. Writing the rule of the gap out in detail for observables of four-dimensional Yang–Mills
Physics	That, if the rule from the string tension to the gap and the rule from the gap to the string tension are admitted, the continuum string-tension problem and the task of one scale take over the role of the task of a uniform gap	That the rule from the string tension to the gap and the rule from the gap to the string tension are theorems
Arrangement	The plots of Figure 2 and Table 4 come out mechanically from each table under the rules of § 09	The choice of rules itself (the grades of seams, tasks and walls are not theorems)
Overall	—	This site has settled none of the tasks. It does not show the existence of the mass gap. What the published construction reached, as written in Table 1, is at the level of the committee's assessment Assessment ; whether that proof is correct is outside this chapter

Sources and reproduction

Item	Kind	Source / tools
Quotations from the problem statement (§ 4, § 5, § 6, § 6.5, footnotes 1 and 2), current status of the task toward infinite volume, the task of a uniform gap and the task of local fields and short distances	Text	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Math. Inst. (2000)
Reference [45] for the axioms (the requirement of the axioms in formula (2))	Bibliographic data only	R. Streater, A. Wightman, PCT, Spin and Statistics and All That, W. A. Benjamin (1964) (bibliographic data from the reference list of JW)

Item	Kind	Source / tools
Problems 4.1, 5.1, 5.2 (the lattice area-law problem, the lattice correlation-length problem, the continuum string-tension problem) and the sentences of §4, §5, §6, the definition of the requirement of the string tension	Text	S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (2018)
Theorems 2.2 and 2.4, the definition of strong decay and its remark (the task of strong decay), the remark on $SO(3)$, the presentation in the introduction	Text	S. Chatterjee, A probabilistic mechanism for quark confinement, arXiv:2006.16229, Comm. Math. Phys. 385 (2021) 1007–1039
Source of the remark on $SO(3)$ / small β for the task of strong decay	Bibliographic data only	J. Greensite, B. Lautrup, Phys. Rev. Lett. 47 (1981) 9–11 / R. L. Dobrushin, S. B. Shlosman, Constructive criterion for the uniqueness of Gibbs field, in Statistical Physics and Dynamical Systems, Birkhäuser (1985) 347–370 (bibliographic data for both from the reference list of Chatterjee 2021)
Confinement from the σ model / $\mathbb{Z}_n$ to $SU(n)$ Higgs	Bibliographic data only	B. Durhuus, J. Fröhlich, CMP 75 (1980) 103–151 / J. Fröhlich, Phys. Lett. B 83 (1979) 195–198 (titles and the presentation in Chatterjee 2021)
Reflection positivity, reconstruction, small β for the lattice area-law problem	Abstract, bibliographic data only	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (abstract: existence and analyticity of the infinite-volume limit at strong coupling, Wilson's confinement bound) / K. Osterwalder, R. Schrader, CMP 31 (1973) 83–112; 42 (1975) 281–305
What the published construction reached, in Table 1	Assessment	Assessment report `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), opening, §5, §7, §8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). How to read it: Parts 6, 8, 12 and 13
Current status of the task of the T^4 construction	Abstract and preface / text pp.325–328 (7-07)	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 / J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383
The task toward infinite volume and the lattice correlation-length problem at strong coupling, the lattice correlation-length problem for finite groups	Text §1 (16-01)	H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737, Comm. Math. Phys. 400 (2023) 805–851 / A. Adhikari, S. Cao, arXiv:2202.10375, Ann. Probab. 53 (2025) 140–174
Driver's Corollary 6.7 for the task of the remaining axioms	Introduction of the text	B. K. Driver, CMP 123 (1989) 575–616

Item	Kind	Source / tools
Current status of the task of non-triviality	Text (7-08)	M. Aizenman, H. Duminil-Copin, Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models, Ann. of Math. 194 (2021) 163, arXiv:1912.07973
Three-dimensional $U(1)$ for the lattice area-law problem	Abstract	M. Göpfert, G. Mack, Proof of confinement of static quarks in 3-dimensional U(1) lattice gauge theory for all values of the coupling constant, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit with m_D fixed is a free scalar field. Text not read. The year is Crossref's year of issue; INSPIRE gives 1981)
Statement and current status of the task of one scale	Record of this book's study	Formulas (3) and (6) of 17-12 and its source table
Current status of the task of the three-dimensional measure	Text § 1.1 (7-06)	I. Chevyrev, Stochastic quantisation of Yang–Mills, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359
Chapters of this book cited in the text	Known	0-03 § 08 (the local-field footnote, the weak-existence footnote, weak convergence), 1-04 § 06, § 07 (Schur's lemma, the centre), 4-06 § 02 (b_0), 4-10, 4-11 § 02 (reconstruction and μ_F), 7-04 § 08 (the cluster expansion), 7-10 § 04, § 06, § 07 (ren, $m < \infty$, formula (7))
Formulas (5) and (10)	Known	The spectral theorem and reconstruction (4-10, 4-11), Fock space (2-06)
Table 3, Figure 1	Computed	Python on this machine: graph.py (closures and minimal sets over 4096 subsets × three levels × with and without centre). The figure uses the same rules
Table 4, Figure 2	Computed	Python (coord.py; reads 18-01 Table 2, 16-05 Table 1, Tables 1 and 2, and 23-02 Table 3, makes the assignment and checks the number of rows)

Next chapter to read: 16-08 Common misconceptions, which takes ways of speaking that mistake the tasks back to the words of the problem statement. The tasks on the side of the record of this book's study are in 23-15.

Common misconceptions — taking half-right ways of speaking back to the words of the problem statement

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

What this chapter shows — For nine commonly seen ways of speaking, the correct core and the slip. How the state of proof changes with group and dimension (Figure 1). That the decay of correlations does not go as far as saying whether there is a particle (Figure 2).

Prerequisite chapters — 0-03 (faster with 4-11, 5-02, 7-11 and 17-01; the words used are defined one line each at the end of § 01).

Said first — What is corrected goes only as far as the level of ways of speaking. This chapter shows neither the existence of the mass gap nor the construction of the continuum limit. The ways of speaking that changed with the four-dimensional construction published in 2026 (the committee's assessment Assessment ; for the meaning of the labels see "Labels" in 6-00) are written in § 02, § 03 and § 08.

The order of this chapter

1. How to read the misconceptions — taking them back to the one sentence of the problem statement, and the words used in this chapter
2. "The heart of the difficulty is infinity" — it remains even when the infinities are removed
3. "Two and three dimensions are already done" — for non-abelian groups, only two
4. "There is a gap because it is non-abelian" — the theorems divide by group and dimension (Figure 1)
5. "The mass gap is the mass of the lightest particle" — decay does not speak of particles (Figure 2)
6. "Show confinement and the gap follows" — different claims
7. "It has been confirmed by lattice numerics" — predictions that assume existence

8. "Showing it on the lattice is enough", "Build it in a finite box and the rest is technique" — the lattice correlation-length problem (Chatterjee's Problem 5.1) and T^4
9. "The mass gap comes from the uncertainty principle" — the theorem only converts a length into a mass
10. Detour: other ways of speaking one comes across — 1/0, holography, triviality, QCD

01

How to read the misconceptions

Many of the commonly seen ways of speaking are half right, and to that extent the slip is harder to see. This chapter divides each item into **the way of speaking**, **the correct core**, **the slip** and **a precise way of saying it**. The standard is the one sentence of the problem statement (0-03); lining up its conditions,

$$\forall G \;\; \exists \mathcal{T}_G : \quad \textbf{axioms} \;\wedge\; \textbf{non-trivial} \;\wedge\; \textbf{agrees with asymptotic freedom at short distances} \;\wedge$$

We read each word in one line Known .

- G: a compact simple group. All of them, from $SU(2)$ and $SU(3)$ to the exceptional group G_2 (0-03).
- $\mathcal{T}_G$: quantum Yang–Mills theory on $\mathbb{R}^4$. A theory not on a lattice, nor in a finite box, but on the infinitely extended continuous four dimensions (0-03).
- Axioms: the list of conditions a quantum field theory must satisfy (the Wightman axioms or the Osterwalder–Schrader axioms) (0-03).
- Non-trivial: not collapsed to a Gaussian theory without interaction (a free field) (0-03).
- Agrees with asymptotic freedom at short distances: the behaviour at short distances matches the prediction of perturbation theory (asymptotic freedom; (4) of § 04) (0-03).
- $\Delta_G > 0$: the spectrum of the Hamiltonian H has no points in the interval $(0, \Delta_G)$ above the vacuum energy 0 (0-03).

Most slips happen by replacing one word of (1) with a nearby word. Examples: "Δ" → "the mass of a particle" (§ 05), "existence" → "the numbers appear to converge" (§ 07), "$\mathbb{R}^4$" → "the lattice" (§ 08).

Words used in this chapter

We borrow words from Parts 4 and 5 in advance. Here we give only one-line definitions, and leave the details to the chapters on the right Known .

Word	Meaning in this chapter	Details
β (lattice coupling)	The number multiplying the lattice action (2), with $\beta \propto 1/g^2$. Small β is strong coupling, large β weak coupling, and the continuum limit lies on the side of large β	5-01
Exponential decay, correlation length	The correlation of two observables decreases with distance r like $e^{-r/\xi}$. ξ is called the correlation length. The gap measured in units of the lattice spacing corresponds to $1/\xi$	2-08, 4-11
Wilson loop	The quantity W_γ obtained by multiplying the group elements of the links along a closed path γ and taking the trace of the representation matrix	5-01
Area law	For large rectangular loops, $\langle W_\gamma \rangle$ decreases like $e^{-\sigma \times \text{area}}$. σ is called the string tension. The lattice sign of confinement	5-02
Perimeter law	$\langle W_\gamma \rangle$ decreases only to the extent of $e^{-c \times \text{perimeter}}$. Slower than the area law	5-02
Centre of a group	The set of elements that commute with every element of the group. For $SU(N)$ the N elements $e^{2\pi i k/N} I$; for $SO(3)$ and G_2 the identity alone	1-04
Casimir number C_A	A number determined by each group, entering the coefficient of (4). N for $SU(N)$, 0 for $U(1)$	1-04
Spectral measure	The ρ of (5). The weight of how much of the energy-E component the state $O\Omega$ has	1-10
Transfer matrix	The operator T that advances one step in the time direction of the lattice. With $T = e^{-H}$ it gives the Hamiltonian H on the lattice	5-03
Reflection positivity	Positivity with respect to a mirror that flips time. With it the transfer matrix is a positive operator, and the Hilbert space and the Hamiltonian come back	5-03
Boundary condition	A placement of group elements fixed in advance on the links at the edge of a finite box	(7) of § 06
Infinite-volume limit	The limit of the measures as the box is extended to the whole of $\mathbb{Z}^d$ (a subsequential limit is allowed). The limiting measure is called a Gibbs measure	5-04
Phase transition	A point at which, as β is varied, the properties of the infinite system change as if with a kink	2-08

02

"The heart of the difficulty is infinity"

The way of speaking: physics proceeds by handling the infinities along the way cleverly, but mathematics does not allow that. That is why it is hard.

The correct core: the continuum theory needs renormalisation (4-04) to control quantities that diverge as the lattice spacing $a \to 0$. Bałaban's work is aimed mainly at this ultraviolet control (7-07) Known .

The slip: having granted that a construction on T^4 can be expected from extensions of known methods, the setters write 'no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume' (problem statement § 6) Known . On a finite periodic lattice with the lattice spacing fixed, the theory is a finite-dimensional integral and there are no infinities (7-11).

$$\langle F \rangle = \frac{1}{Z} \int F(U)\, e^{-\beta S_W(U)} \prod_{\ell} dU_\ell, \qquad 0 < Z \le 1 \tag{2}$$

dU_ℓ is the Haar measure of weight 1, and $S_W \ge 0$ the Wilson action (5-01). Example: for $SU(3)$ on a lattice of side 32 there are $4 \cdot 32^4$ links, each with 8 variables (the dimension of $SU(3)$), and the integral has $4 \cdot 32^4 \times 8 = 33{,}554{,}432$ variables Computed . Many, but finite.

What is hard is the theory with the box extended to $\mathbb{Z}^4$. Lecture material (Chatterjee 2020) wrote 'In 3D and 4D, mass gap has not been established at large β for any theory.' Known . Since then, for finite groups, Adhikari–Cao have shown exponential decay at large β (§ 04) Known . For continuous non-abelian groups, the gap at large β in three and four dimensions has still not been shown Known . What is missing is not the value of an infinity but an estimate uniform in the box and in β. The gap itself is also thought to have the form $e^{-1/(2b_0 g^2)}$, invisible to perturbation theory (0-02) Physics .

The ultraviolet side now: until 2025, "the ultraviolet is already settled" was also an overstatement. Bałaban's four-dimensional theorem (1989) comes with a hypothesis, and the theorem removing the hypothesis 'has not been published yet'. The limit of gauge-invariant observables, too, was at the level of taking subsequences by compactness (preface of Magnen–Rivasseau–Sénéor 1993; 7-07 § 05) Known . The construction published in 2026 takes the continuum limit along the whole sequence of cutoffs in a periodic box, passes to infinite volume along a subsequence, and builds a theory satisfying the Wightman axioms. The committee assesses this as "proved in Lean" Assessment (Part 6). Even so, one cannot go as far as saying "the ultraviolet is settled, so only the infrared remains". The uniqueness of the infinite-volume limit has not been shown, and the construction carries every step at weak coupling and stops at the step with lattice spacing 1. The seam that would put the result of that step into a

form the infrared arguments can use (the object of the limit, local estimates uniform in the volume, the image of observables) has not yet been made into a statement (16-03) Inferred .

A precise way of saying it

The ultraviolet side was supplied by the construction published in 2026, as a continuum limit in a periodic box (the committee's assessment). What the setters asked new ideas for is the gap uniform in the volume and the limit $T^4 \to \mathbb{R}^4$, that is, the infrared side, and that is still open. The seam connecting the two also remains.

03

"Two and three dimensions are already done"

The way of speaking: lower the dimension and it is done; only four dimensions remain.

The correct core: φ^4 has been constructed in two and three dimensions (problem statement § 6.2), and the two-dimensional abelian Higgs model has been shown up to the axioms. The problem statement calls it 'the only complete example of an interacting gauge theory satisfying the axioms' Known .

The slip: for non-abelian Yang–Mills the story is different. The two-dimensional construction (7-02) relies on an exactly solvable structure, the plaquettes becoming independent once the gauge is fixed, and there are no propagating degrees of freedom. In three dimensions, 'a construction of the 3D YM measure … even on T3, remains open' (Chevyrev 2022), and Bałaban (1985) goes as far as uniform estimates of the partition function on finite lattices (7-06) Known . The core of the difference is the dimension of the coupling constant.

$$[g^2] = (\text{mass})^{4-d} \quad (3)$$

Example: for $d = 3$, $g^2\ell$ (ℓ a length) is the dimensionless coupling, and it gets smaller at shorter distances. Interaction weakening at short distances is the foothold for the construction. For $d = 4$ this foothold survives only as a slow logarithmic change. Precisely, for $d = 4$, g^2 becomes dimensionless, and the dependence on scale is only the logarithm from the one-loop running ((4) of § 04) Known (a table by dimension is in 16-01).

A precise way of saying it

For non-abelian groups, two dimensions has been constructed (relying on an exactly solvable structure). In three dimensions the continuum measure has not been constructed even in finite volume. Four dimensions is the critical dimension, where the coupling constant becomes dimensionless, and the construction published in 2026 built a theory satisfying the Wightman axioms as a limit of Wilson's lattice (the committee's assessment). Even in four dimensions, the mass gap and the uniqueness of the infinite-volume limit have not been shown.

Even in two dimensions, for the continuum limit and mass gap of asymptotically free sigma models there was no claimed proof up to October 2026 (a manuscript with a claim made public on 2026-10-06 is unverified; 16-01 Table 5).

04

"There is a gap because it is non-abelian"

The way of speaking: the photon ($U(1)$) is massless, but gluons are non-abelian and interact with themselves, so there is a gap.

The correct core: in four-dimensional $U(1)$ lattice theory, the area law breaks down at large β (Guth 1980; the complete proof is Fröhlich–Spencer 1982), and the theory is in a massless phase (Fröhlich–Spencer 1982)Known . Both are said to be results mainly for Villain-type actions (actions of a different form from the Wilson action of (2)). The one-loop coefficient is also determined by the group.

$$\mu \frac{dg}{d\mu} = -b_0\, g^3 + \cdots, \qquad b_0 = \frac{11}{3} \cdot \frac{C_A}{16\pi^2} \quad (4)$$

The Casimir number C_A of the adjoint representation is N for $SU(N)$ and 0 for $U(1)$, and a theory without matter fields is asymptotically free only when non-abelian (4-06) Known .

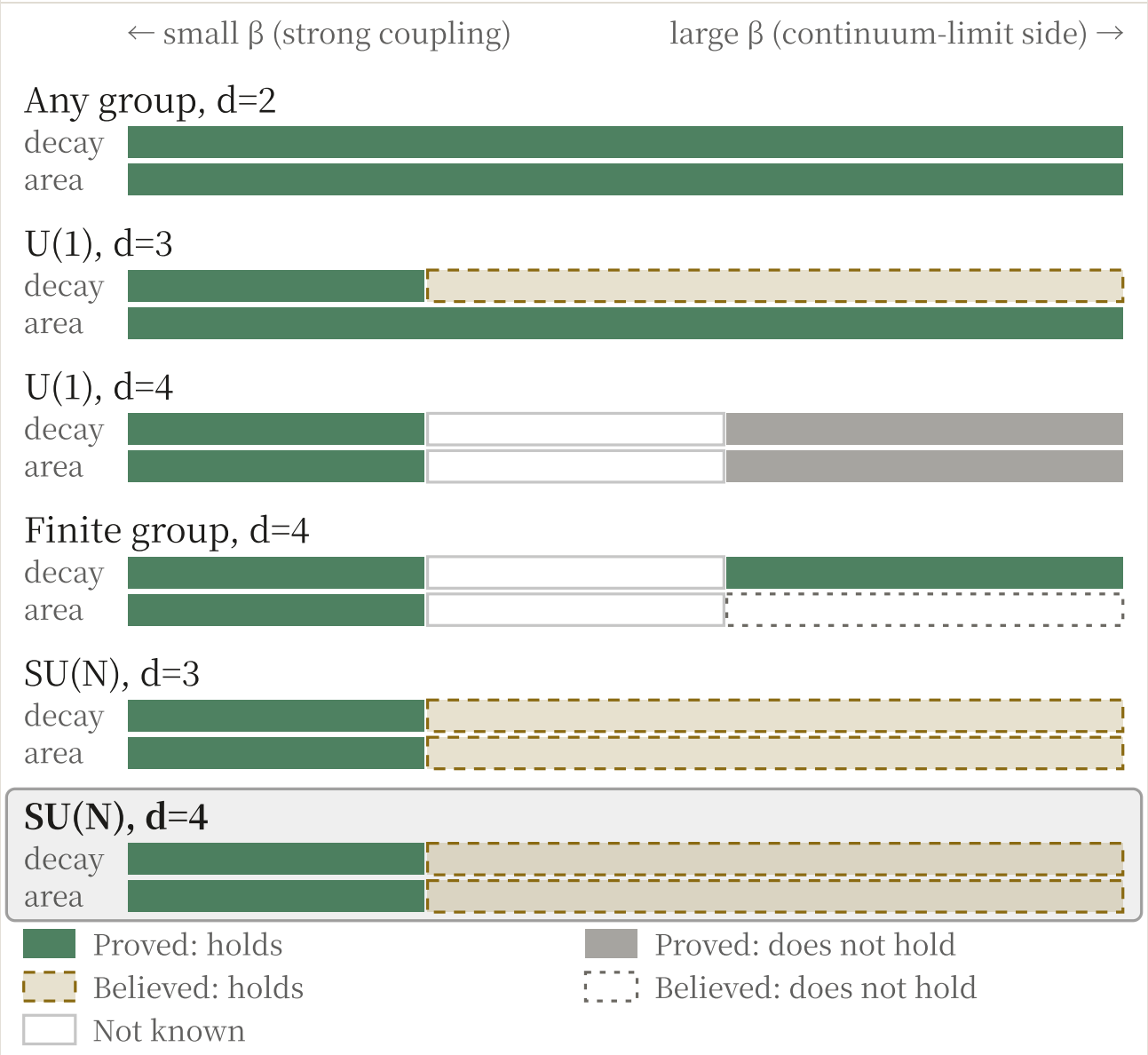
The slip: the lattice theorems divide by group and dimension (Figure 1, Table 1). At strong coupling (small β), exponential decay holds for every group, and the area law also holds for loops in representations that act non-trivially on the centre (Osterwalder–Seiler 1978)Known . For groups with trivial centre (G_2, $SO(3)$, finite groups with trivial centre), large loops are said to follow the perimeter law even at strong coupling, through screening by gluons Physics .

The area law at strong coupling is not evidence of confinement, because it holds at small β even for four-dimensional $U(1)$, which does not confine (Chatterjee 2018). Example: three-dimensional $U(1)$ is abelian, yet the area law holds for every β (Göpfert–Mack 1982; said to be for a Villain-type action)Known . Four-dimensional

finite groups (including non-abelian ones) have exponential decay even at large β (Adhikari–Cao) Known , but are believed not to confine Physics .

What matters is the following. For continuous non-abelian groups it is believed that there is no phase transition at finite β and that the correlation length diverges as $\beta \to \infty$ Physics . In that limit the gap measured in units of the lattice spacing, $1/\xi$, approaches 0, but physics holds that the gap in physical units (for example $m/\sqrt{\sigma}$) stays positive Physics . This is what needs to be shown (5-06, 5-10). An argument that does not distinguish groups would wrongly conclude that four-dimensional $U(1)$ also has a gap.

Figure 1 Choose a group and dimension with the buttons or a row of Table 1; that row is highlighted, and its content and sources appear below. In each row the upper band is exponential decay, the lower band the area law. Horizontal: β (the divisions are schematic)



SU(N), d=4: Theorems only on the strong-coupling side (Osterwalder–Seiler, Shen–Zhu–Zhu).

Known Table 1 The content of Figure 1 (theorems from primary sources and prospects that are "believed". The columns are for the side of large β). At small β, exponential decay is a theorem in every row, and the area law is a theorem for loops in representations that act non-trivially on the centre (in the row of finite groups, for groups and representations whose centre is non-trivial). Behaviour at intermediate β differs from group to group, and four-dimensional $\mathbb{Z}_N$ (for large N) is said to have a massless phase (note in the table of sources).

Group, dimension	Exponential decay (gap)	Area law (confinement)
Any group, $d = 2$	Yes	Yes (all β)
$U(1)$, $d = 3$	Said not to have been shown as of the lecture material (2020) (in physics it exists)	Yes (Göpfert–Mack; said to be for a Villain-type action)
$U(1)$, $d = 4$	No (massless phase; Fröhlich–Spencer)	No (the area law breaks down; Guth, Fröhlich–Spencer)
Finite group, $d = 4$	Yes (Adhikari–Cao)	Believed not to hold
$SU(N)$, $d = 3$	Unproved (believed to hold)	Same
$SU(N)$, $d = 4$	Unproved (believed to hold)	Same

A precise way of saying it

In four dimensions $U(1)$ is massless at weak coupling and non-abelian groups are asymptotically free; in that sense the group matters. But "there is a gap because it is non-abelian" is not a line of proof. For finite groups exponential decay has been shown even at weak coupling, and the weak-coupling side of continuous non-abelian groups is unproved in three and four dimensions.

05

"The mass gap is the mass of the lightest particle"

The way of speaking: the mass gap is the mass of the lightest glueball (a bound state of gluons), about 1.7 GeV.

The correct core: physics holds that there is an isolated one-particle state (the 0^{++} glueball) at the lower end of the gap, and measures it on the lattice. For pure $SU(3)$, $r_0 m = 4.21(11)(4)$ (Morningstar–Peardon 1999) (5-08) Physics .

There are three **slips**. First: what the problem statement asks for is a gap in the spectrum, and isolated one-particle states are one of the 'natural extensions' (0-03) Known . Second: what the lattice measures is the decay of correlations. Consider the

state $O\Omega$ obtained by applying a gauge-invariant operator O to the vacuum Ω. ρ is the weight (spectral measure) of how much of the energy-E component this state has. The connected correlation of O with itself separated by imaginary time τ can be written with ρ as follows (0-04 § 07, 4-11) Known .

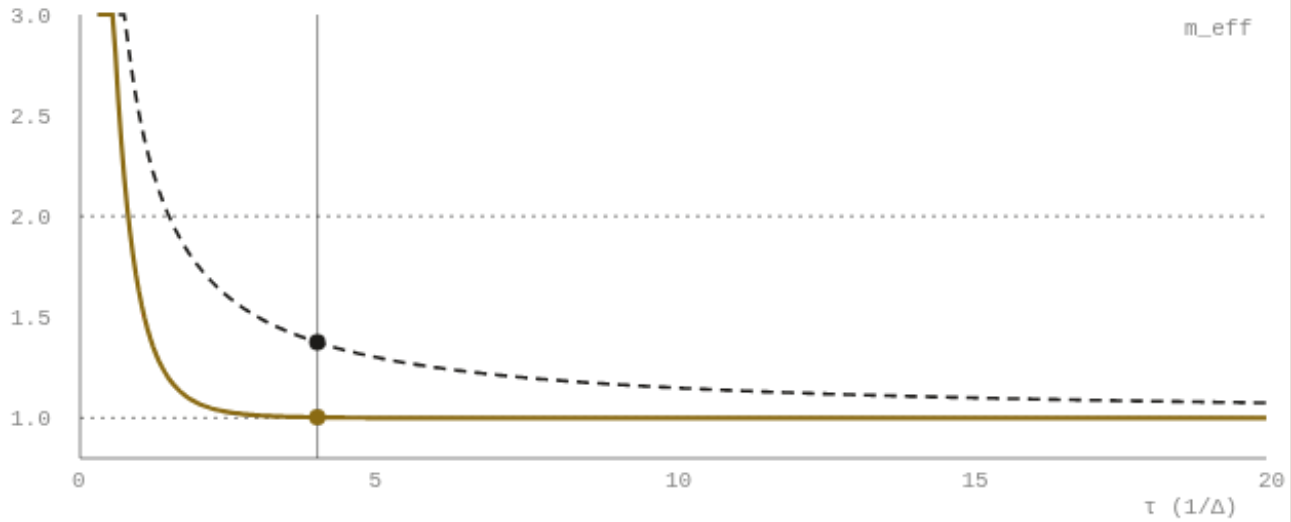
$$C(\tau) = \int_{[\Delta,\infty)} e^{-E\tau}\, d\rho(E), \qquad m_{\mathrm{eff}}(\tau) = -\frac{d}{d\tau}\log C(\tau) \quad (5)$$

m_{eff} approaches the lower end of the support of ρ as $\tau \to \infty$, but does not tell whether that lower end is an isolated point or the edge of a continuous spectrum. We compare two models ($\Delta = 1$).

$$\text{A (with particle)}: \; d\rho = Z\,\delta(E-1)\,dE + c\,(E-2)_+^{1/2}\,dE, \qquad \text{B (no particle)}: \; d\rho = (E-1$$

$(x)_+^{1/2}$ is $\sqrt{x}$ for $x > 0$ and 0 otherwise. From $\int_{E_0}^{\infty}(E-E_0)^{1/2}e^{-E\tau}dE = \Gamma(\tfrac{3}{2})e^{-E0\tau}\tau$ $^{-3/2}$, B has $m_{\mathrm{eff}} = 1 + \frac{3}{2\tau}$. Example: A comes within 1% at $\tau \approx 3.2$ ($c = 1$) to 4.2 ($c = 5$), while B only comes within it at $\tau = 150$ (Table 2). Fitting a single exponential in an intermediate window, B shows a "particle" heavier than Δ Computed . Reading the flat part as the mass of one particle is an interpretation at the level of physics Physics .

Figure 2 Changing the weight c of the continuous part of A (Z = 1) and the time τ changes the curves of the effective mass (solid A: with particle; dashed B: without particle) and the readings. Dotted lines are 1 and 2



c = 1.0, τ = 4.0: A 1.0028, B 1.3750 (Δ = 1)

Computed The numbers in this figure were computed on this machine (the closed form of (6) computed on the spot).

Computed Table 2 $m_{\mathrm{eff}}(\tau)$ ($\Delta = 1$, $Z = 1$). The last row is the first τ within 1%. The difference between the closed form and numerical integration is below 4×10^{-5} for $\tau \le 16$.

τ	A ($c=1$)	A ($c=5$)	B
1	1.6147	2.5495	2.5000
2	1.0712	1.3061	1.7500
4	1.0028	1.0138	1.3750
8	1.0000	1.0001	1.1875
16	1.0000	1.0000	1.0938
Within 1%	3.15	4.23	150

Third: the unit GeV does not come out of pure Yang–Mills itself; it borrows the scale r_0 of lattice QCD with quarks. The lattice papers also state that pure gauge theory and QCD are different theories (Athenodorou–Teper) Physics . "Gluons acquire a mass" is also a different matter: the mass term $m^2 \operatorname{tr} A_\mu A_\mu$ is not gauge-invariant Known .

A precise way of saying it

The mass gap is a gap in the spectrum. Physics holds that there is an isolated particle (a glueball) at its lower end, and measures its mass. What the decay of correlations tells is only the position of the lower end.

06

"Show confinement and the gap follows"

The way of speaking: confinement (that a quark cannot be taken out alone) and the gap are the same thing; show one and the other follows.

The correct core: physics sees the string tension σ and the gap m as two faces of one scale (17-01). Precisely, physics holds that the ratio $m/\sqrt{\sigma}$ approaches a finite positive number independent of β and the lattice spacing Physics . On the lattice there are also conditional theorems. Chatterjee (2021), for G a closed connected subgroup of $U(n)$, assumed decay in the following strong form Known .

$$\exists\, K_1, K_2 > 0 \;\; \forall B, \delta, f, g: \quad \left|\langle fg\rangle_{B,\delta} - \langle f\rangle_{B,\delta}\langle g\rangle_{B,\delta}\right| \le K_1 e^{-K_2 \operatorname{dist}(f,g)} \quad (7)$$

B is a cubic box, δ the boundary condition of the box, f, g local functions taking values in $[-1,1]$ (functions determined only by the variables near a single link), and $\operatorname{dist}(f,g)$ the distance between them. The constants K_1, K_2 do not depend on B, δ, f or g, and are determined by G, β and d alone. This order is the point: if the constants were allowed to differ from box to box and from boundary condition to boundary condition, the assumption would say almost nothing.

The Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021) is that if (7) holds, centre symmetry is not broken, and the area law holds for rectangular loops in representations that act non-trivially on the centre Known . The author writes that at small β (7) can be shown for almost any lattice gauge theory, while also writing that it is not clear whether the decay believed in four dimensions includes this strong form.

The slip: (a) the problem statement does not ask for confinement; it only lists it among the 'natural extensions' Known . (b) "Any compact simple group" includes G_2 and others with trivial centre Known , and such groups are believed possibly not to confine at weak coupling (the remark on $SO(3)$ in the same paper) Physics . The problem statement asks for a gap there too. (c) Example: for three-dimensional $U(1)$ the area law has been shown (Göpfert–Mack; said to be for a Villain-type action). On the other hand, the gap at large β is said not to have been shown as of the lecture material (2020). For finite groups, exponential decay at large β is a theorem (Adhikari–Cao) Known , and they are believed not to confine Physics (Figure 1).

However, the author cautions that the "decay" spoken of for theories such as finite groups refers to the decay of two-point correlations under a particular boundary condition, which differs from the strong form of (7) (uniform under arbitrary boundary conditions) (the remark after Theorem 2.4 of Chatterjee 2021) Known . So the existence of theories that decay but do not confine does not contradict the Theorem from Uniform Decay to the Area Law.

A precise way of saying it

Confinement and the gap are different claims (17-02 § 02), and in the four-dimensional continuum theory neither implication is a theorem. On the lattice, for closed connected subgroups of $U(n)$, the area law for loops in representations that act non-trivially on the centre follows from the strong form of decay (7). What the problem statement asks for is the gap alone.

07

"It has been confirmed by lattice numerics"

The way of speaking: large-scale computations give glueball masses precisely, so the existence of the gap is known.

The correct core: example: for $SU(2)$, $m_{0^{++}}/\sqrt{\sigma}$ is 3.844(61) (Lucini–Teper 2001) and 3.781(23) (Athenodorou–Teper 2021), in good agreement Physics .

The slip: what is being confirmed is consistency under the assumption that it exists (17-13). In a finite box at finite β the gap is always positive (17-13) Known , so a positive number coming out does not test the claim. The value in the continuum is

$$\frac{m(a)}{\sqrt{\sigma(a)}} = \frac{m(0)}{\sqrt{\sigma(0)}} + c\,a^2\sigma \quad (8)$$

an extrapolation that assumes this form Physics , and the σ in the denominator passes through a correction that treats the flux tube as a Nambu–Goto string (17-13).

A precise way of saying it

Lattice numbers are precise predictions under the assumption that the theory exists, and strong physical evidence for the gap. They are not part of a mathematical proof, but they are a target the constructed theory must hit.

08

"Showing it on the lattice is enough", "Build it in a finite box and the rest is technique"

The way of speaking (lattice): show on the lattice that the correlation length is finite for every β, and that solves the prize problem.

The correct core: the lattice correlation-length problem of Chatterjee (2018) is a lattice version of the uniform gap: for any infinite-volume limit with a four-dimensional non-abelian compact Lie group $G \subseteq U(N)$ ($N \geq 2$), show the following about the length $\xi(\beta)$ over which plaquette correlations decay Known .

$$0 < \xi(\beta) < \infty \quad (\text{for all } \beta > 0), \qquad \lim_{\beta \to \infty} \xi(\beta) = \infty \quad (9)$$

The slip: the continuum limit, the axioms on $\mathbb{R}^4$, non-triviality and agreement at short distances are not in (9), and (9) alone does not solve the prize problem. The converse implication is not obvious either (5-09) Known .

The way of speaking (finite box): if it can be built on a finite box T^4 (the continuous four-dimensional torus), all that remains is to enlarge the box.

The slip: the problem statement says that construction on a compact space alone 'would represent a major breakthrough', and places the wall of the uniform gap beyond it Known . Example: even on a periodic lattice box (the lattice T^4), reflection positivity gives a transfer matrix, and the gap in a finite box is positive Known . The gap in a finite box is a quantity one naturally has. What is needed is $T^4 \to \mathbb{R}^4$ and a gap that does not collapse there (7-10). The construction published in 2026 gave a construction in a

periodic box and an infinite-volume theory along a subsequence Assessment , but what it has for correlations goes only as far as polynomial estimates uniform in the phase; there is no exponential decay (16-02 § 04). It is a current example of where "the rest is technique" in this way of speaking does not hold.

A precise way of saying it

(9) is a lattice version of the uniform gap, but it does not reach the requirements of the prize (continuum, $\mathbb{R}^4$, axioms, non-triviality). A construction on T^4 is also a big step, and beyond it remain the uniform gap and the infinite-volume limit.

09

"The mass gap comes from the uncertainty principle"

The way of speaking: squeeze a particle into a small region, and by the uncertainty principle its momentum spreads, so the energy does not go down. The mass gap is that lowest energy, and its origin is the uncertainty principle.

The correct core: the uncertainty relation is a theorem Known . For a state ψ in a Hilbert space and self-adjoint operators A, B,

$$\Delta_\psi A \, \Delta_\psi B \;\geq\; \tfrac{1}{2}\,\big|\langle \psi, [A,B]\psi\rangle\big| \qquad (10)$$

holds (Robertson 1929, Schrödinger 1930), and the proof is a single Cauchy–Schwarz inequality. Applied to $A = x,\ B = p$ it gives $\Delta x\,\Delta p \geq \hbar/2$ (Kennard 1927); equality holds for Gaussians, the ground state of the harmonic oscillator being an example (2-06 § 03). In the language of Fourier analysis, it is the theorem that a function and its transform cannot both be made narrow (1-11; there are also the strong forms of Hardy and Beurling). The kinetic energy of a state localised in a region of size R does not go below about $\hbar^2/(2mR^2)$, or, relativistically, about $\hbar c/R$. Physics In the picture of physics, a glueball is a lump of colour electric flux of size about $1/\Lambda$, and this count is why its mass is about Λ (17-01–17-03).

The slip: (a) what the theorem says is "given a length R, an energy $\hbar c/R$ is guaranteed"; it does not supply R itself. A free massless field (the photon, U(1)) also obeys (10), but its gap is 0. The gap of a free field of mass m is just the m put in by hand (2-06 § 08). The Yang–Mills action has neither a mass term nor a ruler of length (3-02). The length $1/\Lambda$ comes from a single scale being born out of a dimensionless coupling through a logarithmic flow (dimensional transmutation, 4-07), and the localisation comes from colour electric flux being squeezed into a closed tube (17-02, 26-03). Neither is a theorem in the four-dimensional continuum theory. (b) In the language of the Euclidean measure, uncertainty is not a principle but a duality. Exponential decay of

correlations and the absence of light modes are two faces of the same thing (4-11), and the content of (10) is already inside the definition of the gap. It is a restatement, not a cause. On this book's road the gap is the Poincaré constant of the vacuum measure (24-05), and uncertainty is dissolved inside that inequality. (c) In units with $\hbar = c = 1$, what (10) gives is $E \gtrsim 1/R$, and for the gap to be positive one needs R to be bounded, that is, that there is only one scale (17-12).

A precise way of saying it

The uncertainty relation is a theorem: given a length R, it guarantees an energy $1/R$. What the mass gap needs is the R side. That a single scale is born and that excitations are localised at that scale has not been shown for continuum non-abelian groups in four dimensions.

10

Detour: other ways of speaking one comes across

A section you may skip.

Four more commonly seen ways of speaking, corrected briefly.

- "Give infinity a symbol (1/0) and it is settled". The work of assigning values to divergences is already in order as mathematics inside perturbation theory, and the problem still remains (7-12).
- "Holography has shown the gap". What can be computed are models on the side of $N \to \infty$ and strong coupling, and the correspondence itself is a conjecture of physics (17-10) Physics .
- "φ^4_4 is trivial, so it collapses" / "It is asymptotically free, so non-triviality is automatic". Both are wrong; non-triviality is a condition to be shown separately (7-08).
- "It is a problem about QCD". The object is pure Yang–Mills (0-03).

What this chapter can say

Said	Not said
Known The separation of the correct core and the slip of nine ways of speaking, against the problem statement and theorems in primary sources	The existence of the mass gap. Unsolved, and this chapter shows nothing about it (the existence of the continuum limit was shown by the published construction and assessed by the committee Assessment ; whether that proof is correct is outside this chapter)

Said	Not said
Known That the lattice theorems divide by group and dimension (Figure 1, Table 1; the dates of the table are as in the table of sources)	Physics The gap and confinement at large β for continuous non-abelian groups in four dimensions. Unproved
Computed That the decay of correlations alone does not decide whether there is a particle (model (6), Table 2)	Physics That the continuum values of lattice numbers (values passed through the extrapolation (8)) are the values of the constructed theory
Known That the area law follows from decay in the strong form of (7) (closed connected subgroups of $U(n)$, representations acting non-trivially on the centre)	The current status of the gap of three-dimensional $U(1)$ at large β (according to the lecture material (2020)), and the phase of $\mathbb{Z}_N$ at intermediate β (unchecked)

Sources and reproduction

Item	Kind	Source / tools
Number of variables (§ 02), Table 2, Figure 2	Computed	python3 (Table 2 cross-checked by numerical integration). Figure 2 computes (6) in the browser
Problem statement, lecture material, (7), (9)	Known	Jaffe–Witten, the Clay problem description (2000). Chatterjee's slides "Yang–Mills on the lattice" (2020; the quotation in § 02 is a statement as of 2020). Chatterjee, arXiv:1803.01950 (§ 4, the lattice correlation-length problem). Chatterjee, arXiv:2006.16229, Comm. Math. Phys. 385 (2021) 1007–1039 (Definition 2.3, Theorems 2.2 and 2.4 and remarks). Text checked
Ultraviolet, three dimensions, (4)	Known	Bałaban, CMP 122 (1989) 355–392; CMP 102 (1985) 255–275. Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325–383 (the quotation from the preface follows 7-07). Chevyrev, arXiv:2202.13359 (opening checked). (4) is from the PDG 2024 review of QCD
$U(1)$	Known	Guth, Phys. Rev. D 21 (1980) 2291–2307. For Guth, bibliographic data only; the content follows Chatterjee 2018 § 4. Göpfert–Mack, CMP 82 (1982) 545–606 (abstract (INSPIRE): Villain action, positive string tension at all couplings, the continuum limit at fixed mD is a free scalar field. Text not read. The year is Crossref's year of publication; INSPIRE gives 1981). Fröhlich–Spencer, CMP 83 (1982) 411–454 (bibliographic data checked; the content follows Chatterjee 2018 § 4). The type of action (Villain type) of Guth and Fröhlich–Spencer, and the massless phase of $\mathbb{Z}_N$ at intermediate β, are unchecked

Item	Kind	Source / tools
Strong coupling, finite groups	Known	Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 (abstract: existence and analyticity of the infinite-volume limit at strong coupling, Wilson's confinement estimate. The condition attached to the strong-coupling area law ⟨representations acting non-trivially on the centre⟩ and the perimeter law for groups with trivial centre are unchecked in the text). Shen–Zhu–Zhu, CMP 400 (2023) (abstract checked). Adhikari–Cao, arXiv:2202.10375, Ann. Probab. 53(1) (2025) (Theorem 1.1 checked)
Lattice numbers	Physics	Morningstar–Peardon, PRD 60 (1999). Lucini–Teper, JHEP 06 (2001). Athenodorou–Teper, JHEP 11 (2020); 12 (2021) (tables checked)

Next chapter to read: 16-09 The stages a proof must contain. To read from the side of this book's study, 23-01 The line of construction and the diffusive-relaxation-time assumption.

The stages a proof must contain — the table of seven stages

> This chapter is a map of open questions. § 02, § 05 and § 06 are guide material.

What this chapter shows — The seven stages that a proof of the mass gap of four-dimensional Yang–Mills must contain, whichever road it starts from (the vessel, the ultraviolet, the gap on a small T^2, confinement on a small T^2, the crossover band, R^4, general groups). For each stage, what is known in the world, what this book has shown, and what has not yet been shown (within what we surveyed), set out with labels in one table. That the mass gap and confinement (the area law) are different statements. A figure of the two roads meeting at the band. An index of which cell of the table each chapter from Part 23 on fills, and the correspondence with the six continuum tasks of 16-07. The stages taken on by the construction published in 2026 (part of the vessel, existence on the ultraviolet side, existence and the axioms for R^4, existence for all groups), and the stages that remain.

Prerequisite chapters — 16-07 The list of open tasks (the twelve tasks and the three strengths of arrows) and 16-08 § 06 (confinement and the gap are different claims). The words for the stages follow 7-10 (what is reached at each stage in four dimensions). The questions that remain after the published construction are in 16-02, and where this book's holdings stand is in 16-04.

Said first — This chapter lays out stages; it settles none of them. The division into the seven stages a proof must contain is an arrangement often used in constructive field theory, cut again to fit the work of Parts 25 and 26 of this book. The marks ■◐□ in the table are marks of the state of the record, not a graduation of how much is proved (§ 02). This site shows neither the existence of the mass gap nor the construction of the continuum limit. The label Assessment means "according to the committee's assessment report" ("Labels" in 6-00).

The order of this chapter

1. The table of the seven stages a proof must contain — known, this book and unknown for each stage (Table 1)
2. How to read ■◐□ — marks of the state of the record, not a graduation

3. The mass gap and confinement are different statements — the road to each, and the missing arrow between them
4. The two roads meet at the band — the flow of the logic (Figure 1)
5. Which chapter fills which cell — an index of Parts 23–27 (Table 2)
6. Correspondence with the tasks of 16-07 — the six continuum tasks and the seven stages a proof must contain (Table 3)

01

The table of the seven stages a proof must contain — known, this book and unknown for each stage

We split the way from the lattice to the theory on $\mathbb{R}^4$ and the mass gap into seven stages. In order: (1) the vessel (the measure on the lattice, and the mechanism that assembles a quantum theory from it), (2) the ultraviolet (uniform control at all scales that does not break down as the lattice spacing a is sent to 0; as existence it was shown by the published construction, and what remains on this book's road is the seam), (3) the gap on a small T^2, (4) confinement on a small T^2, (5) the crossover band (the band for short), (6) the theory on R^4, (7) general groups.

The "small T^2" of the two small-T^2 stages (rows 3 and 4 of Table 1) is the case where four-dimensional space is written as $\mathbb{R}^2 \times T^2$ and the side L of the T^2 is taken much smaller than the physical length $1/\Lambda$ (Λ is the mass scale that the theory generates by itself, 4-07). Putting 't Hooft's twist (a boundary condition that glues opposite sides shifted by an element of the centre of the group) on the T^2 removes the zero modes (directions in which one can move without raising the action), and there is a prospect of keeping the coupling weak at scales up to L. It is a waypoint for splitting the problem into the stages up to there (the two small-T^2 stages) and the stage of widening the T^2 (the band stage). The content is treated in Part 25 (25-00).

The band of the band stage is the intermediate scale $L \sim 1/\Lambda$ at which the coupling passes from weak to strong. In four dimensions no tool has crossed this scale non-perturbatively (26-00). In the language of the space road, a statement that crosses the band takes the following form.

$$\ln \frac{1}{\theta(L,L)} \;\geq\; c\,L^2 \qquad (\text{for all } L, \text{ the same } c>0) \quad (1)$$

$\theta(L,L)$ is the sector bias when one vortex winding the T^2 is put into a container of side L (a quantity that moves away from 1 the "cheaper" the vortex is). (1) is an area law in the

form of 't Hooft's criterion; that "θ moves away from 1" alone is not enough is treated in 26-01.

After the construction published in 2026. The published construction takes on part of the vessel stage (reflection positivity of the original Wilson measure, the Chessboard Estimate, reconstruction; 12-01, 12-02, 12-08), existence for the ultraviolet stage (the continuum limit in a periodic box; Parts 8–11), existence and the axioms for the R^4 stage (infinite volume along a subsequence, phase decomposition, the Wightman axioms for each phase; 12-05, 12-07), and existence for the general-group stage (any compact simple group). The committee assesses this as "proved in Lean" Assessment . We have written what it takes on into the "known in the world" column of Table 1. What it does not take on is the two small-T^2 stages (twisted boundary conditions are not treated), the band stage, a uniform mass gap, and uniqueness of the infinite-volume limit (16-02). For the ultraviolet stage, even after existence has been shown, there remains the seam that puts the ultraviolet result into a form the infrared arguments can use (16-03) Inferred .

Table 1 The seven stages a proof must contain (to the right of the stage name is the mark: ■ closed in the record / ◐ partial / □ open. Labels are per claim within the row. "Chapters" is the same as the index of Table 2)

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
1 Vessel ◐ The lattice, reflection positivity, block observables, compactness. From decay on a dense set to a gap of the transfer matrix	Cited Reflection positivity and the transfer matrix (Osterwalder–Seiler 1978, 5-03), reconstruction (4-10). Assessment Shown by the published construction: reflection positivity of the original Wilson measure for planes through lattice points and planes through midpoints of edges, the Chessboard Estimate, reconstruction (Part 12)	Proved here Reflection positivity of z (positivity, under reflections through lattice points, of the ensemble summed over twists; the algebra of the kernel is in Lean: ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), finite sums with non-negative weights). Lean: GapFromDecay (`GapFromDecay.gap_of_decay_dense`) (an abstract lemma for a bounded self-adjoint operator, with uniform decay on a dense set placed as a hypothesis)	Writing out the whole construction of the vessel (at present a strand) Chapters: 5-03, 26-04, 26-05

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
2 Ultraviolet ◐ Multiscale analysis uniform in a, from the lattice spacing a up to the scale L (locality of the effective action, rarity of large fields, flow of the coupling). Existence is from the published construction; what remains is the seam	Assessment Shown by the published construction: in a periodic box, the continuum limit along the whole sequence of cutoffs (Parts 8–11). It carries every step at weak coupling and stops at the step with lattice spacing 1 (16-03). Cited Before that, Bałaban's series (CMP, 1980s; parts remain at the central stage that the original sources call "unpublished", 7-07)	Proved here Only parts at the one-loop level (the perturbative level). The record so far placed the whole of ultraviolet control as an assumption, Assumption	The seam: fixing the theory at the step with spacing 1 as an object, local estimates uniform in the volume, the bridge for observables (writing continuum correlations as expectations at the step with spacing 1). On this book's road, in addition: the translation into quantities fixed by the measure, the construction with twisted boundary conditions and in elongated boxes, and how reflection positivity is carried Inferred Chapters: 16-03, 7-07, 25-00, 26-11, 27-02 ultraviolet control
3 Gap on a small T^2 ◐ The twist removes the zero modes, and the coupling stays weak up to the size L of the T^2. The gap of the transfer matrix $\geq c/L$ is uniform in a	Cited 't Hooft's twisted boundary conditions and flux (1979, 1981)	Proved here The twisted part: uniqueness of the vacuum that absorbs the twist, the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, and $S_{\min} > 0$ in the (1,1) sector (SU(2), lattice, N fixed). The form uniform in a holds only under the assumption of an ultraviolet construction in a box with twisted boundary conditions	A lower bound on the gap uniform in a, without that assumption. The published construction treats only periodic cubic boxes, so the construction in a twisted box becomes the first question of this book's road Inferred Chapters: 25-00, 25-01, 27-02 small-T^2 gap
4 Confinement on a small T^2 ◐ Bounds on containers and on insertions reduce confinement to "the insertion of one vortex" and "independence between regions"	Cited The container identity (Mack–Petkova 1980)	Proved here The reduction (on a finite configuration space, with positive weights and a bijection that flips the sector. Lean: Containers (`Containers.container_bound`), InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`), the algebra of finite sums). Proved here Slab confinement (with the lattice spacing and the T^2 fixed, an area law at every β. In the continuum limit the tension is 0)	Two inputs: uniform acceptance of insertions, and frame decorrelation Assumption Chapters: 25-03, 25-04, 25-05, 25-06

Stage and content	Known in the world	Shown in this book (with labels)	Unknown, and chapters that fill it
5 Band ☐ That (1) holds with the same c as the T^2 is widened. Crossing the intermediate scale $L \sim 1/\Lambda$ non-perturbatively	Cited 'New ideas are needed' (Jaffe–Witten § 6.5, the task of a uniform gap). The published construction gives no foothold inside the band (it carries every step at weak coupling) Inferred . The inequality of Tomboulis–Yaffe (1985) (from 't Hooft's criterion to the area law)	Proved here The correct form ($\theta \leq \theta_0$ is not enough). Proved here A conditional reduction to the lateral-widening inequality (assuming three things: the base point, frame decorrelation and lateral widening. The algebra of the iteration is in Lean: Doubling (`Doubling.iterate`)). Proved here The negative lemma that structural inputs alone cannot cross. Proved here The renormalisation comparison at the one-loop level ($\beta - \beta' = 0.258$). Reading Numerical signs (26-09)	The renormalisation comparison, non-perturbatively. The band itself. Stating the two ends of the band in the same quantity and the same class of actions (the strong side is β of the Wilson action in this book's Lean window; the weak side is the coefficients of the effective action at the entrance to the one-step estimate of the published construction) Inferred Chapters: 26-01–26-07, 26-09
6 R^4 ◐ The infinite-volume limit, the OS axioms, recovery of rotation invariance, non-triviality	Cited Reflection positivity survives weak limits; OS reconstruction (16-07 § 03). Assessment Shown by the published construction: the infinite-volume limit along a subsequence, phase decomposition, the Wightman axioms for each phase (Part 12). Checks at short distances (Part 13)	Assumption Only the strand that, under the band stage, most of it is the standard argument	Uniqueness of the infinite-volume limit (probably not needed for the mass gap of a single theory Inferred). If decay is obtained with twisted boundary conditions or in a pure phase, identifying the theory built from that state Inferred Chapters: 16-07, 27-01, 27-02 separate box
7 General groups ☐ SU(N), and groups without centre (G_2 and others)	Cited For groups with trivial centre, the arrows that use the centre disappear (16-07 § 07). Assessment Existence was shown by the published construction for any compact simple group	None (Parts 25 and 26 of this book are written for SU(2))	Everything on the side of the mass gap. The tools of vortices cannot be used for groups without centre Chapters: 16-07 § 07, 27-02 separate box

What to read from Table 1. (a) The "□" stages are the band and the general-group stage, which changes the group. The "◐" of the ultraviolet stage means that the continuum limit in a periodic cubic box was shown by the published construction, but there is no construction in twisted or elongated boxes and no statement of the seam (the same mark as 27-02 Table 1). What is open now at the ultraviolet stage is not existence but the seam; the band remains a long-standing open problem of the field. (b) The "◐" of the two small-T^2 stages is in both cases a "◐" under the assumption of an ultraviolet construction in a box with twisted boundary conditions. As long as that construction is missing, the marks of these two stages do not go up. (c) Of the claims written "Proved here" in this book, most of the chapters that place the paper proofs in the book are chapters still to be written (Parts 25 and 26). Until they are placed, the rows of the ledger (27-02) do not disappear.

02

How to read ■◖□ — marks of the state of the record, not a graduation

The marks of Table 1 mark how the record of this book's work treats each stage. ■ means closed in the record (there is a paper proof, and where there is a Lean entry its arithmetic is closed too), ◐ means part is closed and the rest stays a strand or a numerical sign, and □ means open.

These marks are not a graduation of how much is proved. There are three reasons. First, ■ says only that something is closed within the conditions written in the row, not the whole stage. Second, a ◐ stage does not necessarily have "less remaining" than a □ stage. The ◐ of the small-T^2 gap stage says that the twisted part is closed, but the rest depends on the seam of the ultraviolet stage (the construction in a twisted box; open). Third, the book's labels (proved here, cited, computer-assisted, reading, assumption) attach to each claim, while the marks attach to each stage. It is normal for "Proved here" and "Assumption" to stand side by side in the same stage. We do not add the marks up and read them as "so many of the seven stages a proof must contain".

Example. The reflection positivity of z in the vessel stage is closed in the record (a claim corresponding to ■). Even so, the mark of the vessel stage is ◐. This is because writing out the whole construction of the vessel (from block observables to the gap of the transfer matrix) remains a strand.

03

The mass gap and confinement are different statements — the road to each, and the missing arrow between them

What the problem statement asks for is the mass gap (that the spectrum of the Hamiltonian H of the quantum theory is separated by $\Delta > 0$ above the vacuum, 4-11); it does not ask for confinement (the area law of Wilson loops). As in 16-08 § 06,

confinement and the gap are different claims, and in the four-dimensional continuum theory neither implication is a theorem Cited . Physics sees the two as two faces of one scale (the ratio $m/\sqrt{\sigma}$ approaches a finite positive number), but that is a statement at the level of expectation Reading (physics). The implication from the area law to the mass gap does not hold as a general implication, because the area law is a statement about a gap in the charged sector, and the mass gap about a gap in the neutral sector (16-07 § 02).

The two roads of this book arrive at different statements.

- **The road to the gap** (the small-T^2 gap stage + the transfer matrix): carry the gap of the transfer matrix on a small T^2 (the small-T^2 gap stage) through the band to $\mathbb{R}^4$ with the same constant. If it can be carried, the gap survives in the limit (the rule of the gap of 16-07 § 03, with premises (i)–(iii)). The road of Part 23 (from the direction of time) also heads to the same statement of the gap, through the diffusive-relaxation-time assumption and the one-sided bridge.
- **The road to confinement** (the small-T^2 confinement stage, the band stage, the R^4 stage): make confinement on a small T^2 uniform in the size of the T^2 through the band, and turn it into an area law in the theory on $\mathbb{R}^4$ (the R^4 stage).

There is no known arrow joining the two roads, that is, no known road "from the area law to the mass gap". Even the statement of what must be added to get a gap from the area law is not yet settled (the last arrow of 27-01, the separate box of 27-02). So even if the road to confinement were closed entirely, that alone would not reach the requirement of the problem statement. This chapter sets the two roads side by side, but does not count the arrow from confinement to the gap as a "road that joins them".

A precise way of saying it

The mass gap and confinement (the area law) are different statements. The space road of this book heads separately to the two statements: from the small-T^2 gap stage through the transfer matrix to the gap, and through the small-T^2 confinement stage, the band stage and the R^4 stage to confinement. There is no general theorem that gives a gap from the area law, and this book does not show one either. This book's strong-coupling area law (20-06) is a statement inside the strong-coupling window with the lattice spacing fixed, and sits at the opposite end from the questions that remain.

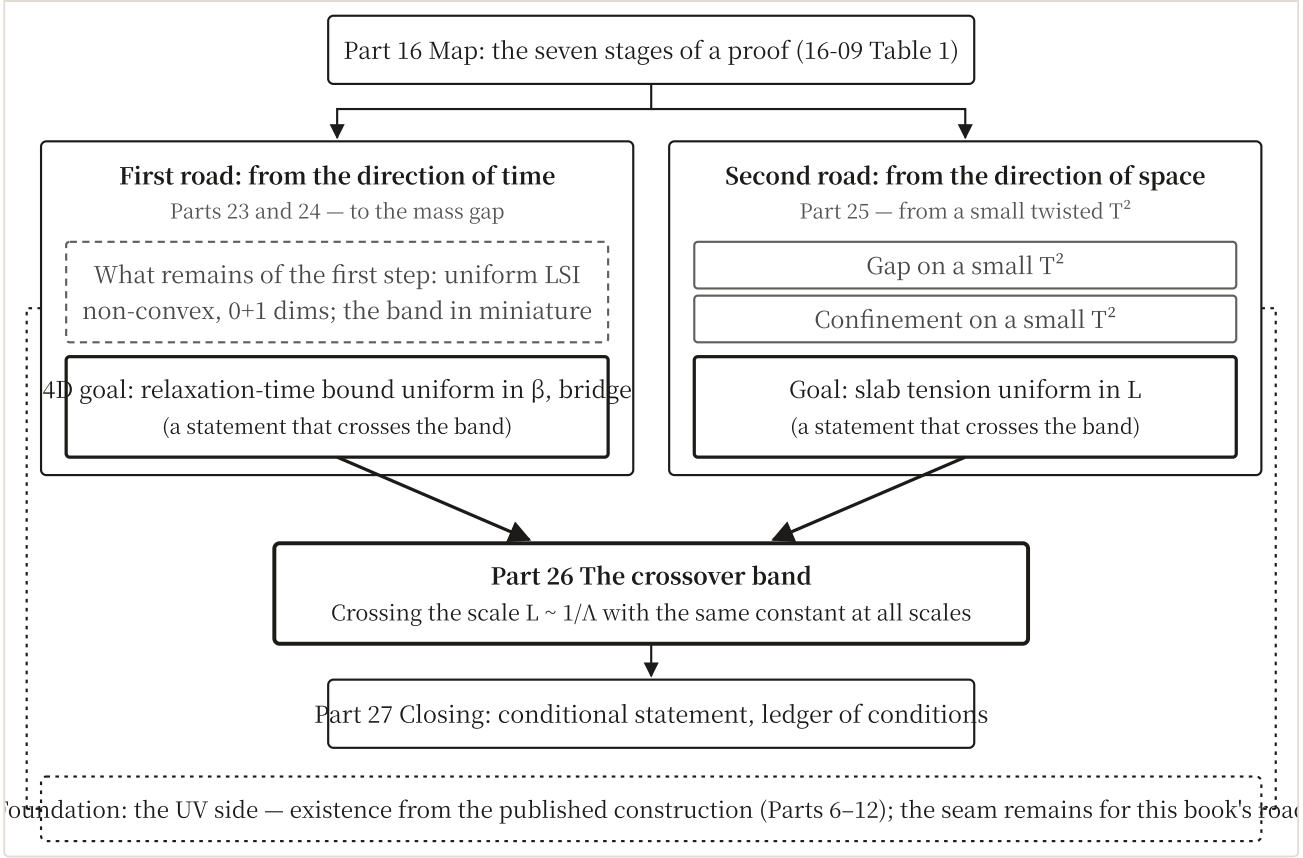
04

The two roads meet at the band — the flow of the logic

Figure 1 is the flow of the logic from Part 16 to Part 27. The first road comes from the direction of time (Parts 23 and 24), the second from the direction of space (Part 25), and at the level of the four-dimensional goal both arrive at a statement of one type: "the same

constant at all scales". That statement is the statement that crosses the band, and Part 26 treats it.

Figure 1 The two roads meet at the band. The column of the first road is split into two rows, and the arrow flowing into the band comes only from the "four-dimensional goal" row (what remains of the first step is a miniature of the band, not the band itself). The ultraviolet side at the bottom (existence from the published construction; what remains on this book's road is the seam) lies under both roads



What to read from Figure 1. (a) "What remains of the first step" on the first road (a uniform logarithmic Sobolev inequality for a measure that contains non-convex parts, the path measure of the 0+1-dimensional double well, 23-15 Table 1) is a wall of the same type as the band (there are different tools at the two ends, and the bound on the worst configuration stops in the middle), but it is not the band. The length of the crossover in the first step is fixed by a classical quantity, the first conjugate point of the Jacobi equation at the top; it does not depend on $\hbar$ and is finite in lattice units. The length of the crossover in four dimensions, $1/\Lambda$, is generated quantum mechanically (4-07) and diverges in lattice units. This difference is treated in 26-07 together with the proof of the computation of the conjugate point on the first-step side (until the proof is placed, Assumption). (b) What flows into the band is, on both roads, the "four-dimensional goal" row. On the first road it is the upper side of the diffusive-relaxation-time assumption uniform in $\beta \to \infty$ (the relaxation-time upper-bound assumption) and the one-sided bridge (the physical-unit target level of 23-15 § 07); on the second road it is the slab tension uniform in L. (c) The ultraviolet side lies outside the band. Existence was supplied by the published construction Assessment . This book's two roads proceed by

placing as an assumption the seam that puts the ultraviolet result into a form usable for their own objects (twisted boxes, slab tension, path measures)Inferred .

The meeting in Figure 1 means "arriving at a statement of the same type at the level of the goal"; it does not mean that the present walls of the two roads are in the same place. The "band" of the band part and the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window) are different things (the note on words in 26-00).

05

Which chapter fills which cell — an index of Parts 23–27

Table 2 is an index of which cell of which stage of Table 1 the chapters from Part 23 on supply material to. The stages taken on by the published construction are read in Parts 6–13 of Volume 2, and this book's firm theorems are in Parts 19–22 (16-04 § 04). A "cell" is a column of Table 1 (known, this book, unknown). Parts 25–27 have chapters still being written, and the links are made by chapter number.

Table 2 The chapters of Parts 23–27, and the stages and cells of Table 1 (this book's arrangement)

Chapter	Stage	Cell	In a word
23-00–23-17	5 (time side)	This book, unknown	The first road. The statement of the four-dimensional goal (the physical-unit target level), and the list of rows of the first step (0+1 dimensions, a miniature of the band)
24-00–24-08	5 (time side), 6	This book, unknown	The first road (second half). Repairing the form of the assumption, and making time continuous. 24-07 is cited from the band part
25-00	2, 3	Known, unknown	Why go by way of a small twisted T^2
25-01	3	Known, this book	Twist and flux, and the three facts of the twisted part
25-02	4, 5	Known, this book	Vortices and fractional instantons, the form of the string tension
25-03, 25-04	4	Known, this book	The Container Identity and slab confinement
25-05, 25-06	4	This book	Insertion identities, driven insertion and the Flow Lemma
25-07	5	This book	Why fixed maps fail (at the one-loop level)
25-08	4	This book (Reading)	The measured record of the space road
25-09	5	Unknown	Where this road stops = widening the T^2
26-00–26-02	5	This book, unknown	Confluence, the correct form of the band, reduction to a single inequality
26-03, 26-04, 26-05	1, 5	This book	Where non-abelianness enters, structure alone cannot cross, the chessboard

Chapter	Stage	Cell	In a word
26-06, 26-07	5	This book, unknown	The renormalisation comparison, the band seen from the side of time
26-08, 26-09, 26-10	5	Known, this book (Reading), unknown	Roads tried and stopped, the measured record, candidate viewpoints
26-11	2	Known, this book, unknown	Ultraviolet control: what closing in one step means, the five stages of the series and where they stopped, a draft statement
27-00–27-04	All	Unknown (the column of conditions)	The conditional statement, the ledger of conditions, the index of questions and of what is not claimed
A-06, A-07	All	—	The index of names, the drafts of papers that came out of this book

06

Correspondence with the tasks of 16-07 — the six continuum tasks and the seven stages a proof must contain

The six continuum tasks of 16-07 Table 1 split the tasks needed for the continuum theory according to the requirements of the problem statement. The seven stages a proof must contain in Table 1 split the stages of one road that starts from the lattice. Since the axes of the splitting differ, they do not correspond one to one. Table 3 is the correspondence, and it is this book's arrangement.

Table 3 The seven stages a proof must contain and the tasks of 16-07 (this book's arrangement. The strengths of the arrows are as in 16-07)

Stage	Tasks of 16-07	Note
1 Vessel	Part of the task of the remaining axioms	Reflection positivity and the transfer matrix. The rest of the task of the remaining axioms (recovery of rotations) is in the R^4 stage. The published construction satisfied them for each phase Assessment
2 Ultraviolet	The task of the T^4 construction, the task of non-triviality, the task of local fields and short distances	Existence of the continuum limit, non-triviality, and agreement with asymptotic freedom at short distances are all tasks on the ultraviolet side. Existence, for local fields, was shown by the published construction; non-triviality and short distances are at the level of checks (Part 13) Assessment
3, 4 Small T^2	A partial problem of the task of the T^4 construction	Construction and bounds on a small volume. The task of the T^4 construction is a task on T^4, while the two small-T^2 stages are tasks on the small T^2 of $\mathbb{R}^2 \times T^2$, so they are not the same task
5 Band	The task of a uniform gap	The uniform gap $\inf_L \Delta_L > 0$. The task for which the setters write 'New ideas are needed'

Stage	Tasks of 16-07	Note
6 R^4	The task toward infinite volume, the task of the remaining axioms	The infinite-volume limit, and the remaining axioms
7 General groups	16-07 § 07	For groups with trivial centre, the arrows that use the centre disappear
(confinement side)	The continuum string-tension problem, the lattice area-law problem	The continuum string tension and the lattice area law. Outside the requirements of the problem statement (C of 16-07); it corresponds to where the small-T^2 confinement stage and the band stage arrive

What to read from Table 3. In the counting of 16-07 (16-07 § 06), if the arrows of the roadmap are admitted, the minimal set needed was one: {the task of the T^4 construction, the task of a uniform gap, the task of the remaining axioms, the task of non-triviality, the task of local fields and short distances}. In the language of the seven stages a proof must contain, it is the ultraviolet stage (the task of the T^4 construction, the task of non-triviality, the task of local fields and short distances), the band stage (the task of a uniform gap), and the axioms of the vessel stage and the R^4 stage (the task of the remaining axioms). After the published construction, the ultraviolet side has been filled as existence (for local fields), and the task of the remaining axioms has also been met for each phase Assessment . In either way of splitting, what remains is the band (the task of a uniform gap) and the seam that hands the ultraviolet result over to the infrared arguments Inferred (16-02, 16-03).

The list of premises is in 28-04 (a list written before the publication; the premises on the ultraviolet side have, as existence, been taken up by the published construction).

What this chapter can say

What is said	What is not said
Cited The "known in the world" column of each stage (the extent of the original sources is in each chapter's source table). That confinement and the gap are different claims	That the seven stages a proof must contain exhaust it. The division into stages is this book's arrangement
Proved here The claims in the "this book" column of Table 1 (within the conditions written in the row. The chapters that place the paper proofs are in Table 2)	Any of: the seam of the ultraviolet stage, the band stage, the mass gap of the general-group stage. The form of the two small-T^2 stages uniform in a (outside the assumption of an ultraviolet construction in a twisted box)
The arithmetic of the theorem names in the Lean entries (finite sums and lemmas on abstract operators. The physical measure is not formalised)	That the Lean entries checked the claims of the stages themselves
The meeting in Figure 1, in the sense of arriving at a statement of the same type at the level of the four-dimensional goal	That the present walls of the two roads are the same. The arrow from the area law to the gap

What is said	What is not said
Assessment The "shown by the published construction" entries of Table 1 are at the level of the committee's assessment	Whether the proof of the published construction is correct (outside this chapter)
Overall	This site shows neither the existence of the mass gap nor the construction of the continuum limit (the existence of the continuum limit belongs to the published construction)

Sources and reproduction

Item	Kind	Source / tools
Division into stages	This book	The stages of constructive field theory (7-01, 7-10), cut again to fit the material of Parts 25 and 26 (the table of stages, the list of tasks up to a proof, the list of assumptions)
Known for the vessel stage	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (abstract; source table of 16-07)
Known for the ultraviolet-control stage	Cited	T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 and the rest of the series (source table of 7-07)
The twist of the small-T^2 gap stage, the Container Identity of the small-T^2 confinement stage	Cited	't Hooft's twisted boundary conditions and flux (1979, 1981), the Container Identity of Mack–Petkova (1980). Bibliographic data are in the source tables of 25-01 and 25-03
Known for the band stage	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) § 6.5. E. T. Tomboulis, L. G. Yaffe, Comm. Math. Phys. 100 (1985) (bibliographic data in the source table of 26-01)
"Shown by the published construction" in Table 1	Assessment	Assessment report `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), opening, § 5, § 6, § 8. The code is `yang-mills-group/OpenAI-YM-20261028` (submitted 2026-10-28). The ultraviolet seam is in 16-03
Known for the R^4 stage and the general-group stage, the tasks of Table 3	This book	16-07 Table 1, Table 2, § 07
Confinement and the gap	This book	16-08 § 06, 17-02

Item	Kind	Source / tools
The "this book" column of Table 1	Proved here	The chapters of Parts 25 and 26 (Table 2), and the draft papers 'Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds', 'Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics', 'Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus' (all drafts of 2026-10, A-07)
Lean	Lean	GapFromDecay (`GapFromDecay.gap_of_decay_dense`), ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), Containers (`Containers.container_bound`), InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`), Doubling (`Doubling.iterate`). What the bundles guarantee is only the algebra of finite sums and inequalities for abstract operators (A-04)
Figure 1	This book	A hand-drawn figure (no numerical computation)

Words used in this chapter — the seven stages a proof must contain (vessel, ultraviolet, gap on a small T^2, confinement on a small T^2, band, R^4, general groups), the crossover band (the band), 't Hooft's twist, the sector bias θ, the marks of the state of the record (■◐□), the road to the gap and the road to confinement, the four-dimensional goal, a miniature of the band. Their meanings are in A-02 Glossary.

Next chapter to read: 16-10 A map of the tools. To read from the side of this book's study, 23-00 Entrance — the first road (from the direction of time). If you read from the list of conditions, 27-02 The ledger of conditions.

A map of the tools — where the known tools used by this book live

> This chapter is a guide (a map of the tools). It lists the homes of the known tools used by this book's study, and where each stops.

What this chapter shows — The twenty-two tools that the work from Part 23 on uses (the Poincaré inequality, Bakry–Émery, log-Sobolev, reflection positivity, the strong-coupling expansion, the renormalisation step, Migdal–Kadanoff, one-loop perturbation theory, interval arithmetic, Lean and others) in one table. For each tool, with labels: (1) its home — the chapter where its definition and known results are first set out (Parts 1–5 and Parts 7, 17 and 18), and the source; (2) how this book uses it — in which chapter from Part 23 on, and for what; (3) where it stopped when carried over to the group, and cautions. Also which chapter to read first when the same topic appears in two or more parts, three reading orders, the dependencies between the tools, and a list of the limits of the tools.

Prerequisite chapters — 16-09 The stages a proof must contain — the table of seven stages (the stages and the labels) and 0-01 § 03, the key to the labels.

Said first — This is a guide chapter and makes no new claims. The claims in each cell of the table are copied from the chapters listed there, with those chapters' labels (the conditions are in the text of each chapter). The column "where it stopped" is not a claim that the tool cannot be used; it records where the way this book used it did not reach, within this book's study. This site shows neither the existence of the mass gap nor the construction of the continuum limit. Where the tools used by the construction published in 2026 live is written in (6) (the label Assessment means "according to the committee's assessment report"; see "Labels" in 6-00).

The order of this chapter

1. The role of this chapter — in five lines
2. The table of tools — home, use, where it stopped (Table 1), and how topics found in two places are divided (Table 2)
3. Three reading orders — from the tools, from the results, from the map
4. Dependencies between the tools — tools needed first, and tools used later (Table 3)
5. What cannot be said — a list of the limits of the tools

00

The role of this chapter

(1) The chapters from Part 23 on take tools from Parts 1–5 and Parts 7, 17 and 18 into "lattice gauge theory on a group" and use them there. This chapter lets you look up, in one table, where the definition and the known results of each tool are.

(2) When the same tool appears in two or more parts (for example, Bakry–Émery in 5-11, 24-06 and Part 23; Migdal–Kadanoff in 18-04, 26-08 and 15-04), it divides them into "home" and "use", and fixes one chapter to read first.

(3) It lists, from the side of the tools, where they stopped when carried over to the group. In which part's chapter each stopped can be looked up in the third column.

(4) The labels are the same five classes as in 16-09. Proved here = this book has set out a proof on paper, Cited = known, checked in the source, Computer-assisted = computer-assisted (interval arithmetic), Reading = a reading of numbers or a physical picture, Assumption = an assumption. The correspondence with the labels of the chapters of Parts 1–5 and Parts 7, 17 and 18 (Lean, paper, computed, known, physics) is in 0-01 § 03.

(5) There are three ways to read (§ 02). Whether you enter from the tools, from the results or from the map, coming back to this table takes you to the home chapter.

(6) The construction published in 2026 assembled, from Table 1, reflection positivity (row 5), the renormalisation-group step (row 10), the one-loop coefficient (row 12) and Lean (row 21), together with tools not in the table (a coupling-dependent probability kernel, matching of the coupling by a shooting method, a multi-step ledger, Arzelà–Ascoli compactness, phase decomposition, reconstruction), and built a continuum limit in a periodic box and an infinite-volume theory along a subsequence. The committee assesses this as "proved in Lean" Assessment . Where those tools live is Volume 2: the kernel of one step in Part 9, the matching of the coupling in 10-04, the multi-step estimates in Part 11, and reflection positivity, the chessboard, compactness, phase decomposition and reconstruction in Part 12. The comparison with this book's tools is in Part 14.

01

The table of tools — home, use, where it stopped

Table 1 has one tool per row. The first column, "home", is the chapter where the definition and known results of the tool are first set out; it is the place a second-year student reads first. The second column gives the chapters from Part 23 on that use the tool, and what for. The third column gives where the tool stopped when carried over to a lattice on the group $SU(2)$, and cautions for reading. Numbers and conditions are written on the same row; the detailed conditions are in the chapters listed.

Table 1 The tools used by this book's work (home, use, where it stopped; labels are per claim within a cell)

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
1 The Poincaré inequality and the gap The constant C_P that bounds the variance by the squared gradient. The gap is $1/C_P$	1-10 § 08 (on the line, $V'' \geq \kappa > 0$ gives $C_P \leq 1/\kappa$) Cited . The reading as a relaxation time is in 2-10	23-06 (the condition of local Poincaré within a phase). 24-05 (on the lattice with continuous time, the gap $\Delta = \hbar^2/(2C_P(\nu))$, with ν the vacuum measure) Proved here	The gap in Langevin time and the gap of the transfer matrix are different quantities (18-03). The identity of 24-05 is a rewording; there is no tool that bounds C_P of the vacuum measure uniformly in the coupling
2 The Bakry–Émery criterion Curvature + second derivative of the potential $\geq K > 0$ gives a gap $\geq K$	5-11 § 02 (the criterion and the outline of its proof) Cited . 5-04 § 08 uses only the conclusion	5-11 § 05– § 07 (integrate a perfect family first; curvature lower bound $2 - 27\beta^2$) Proved here . 23-02 (families on the four-dimensional lattice). 24-06 § 02	On one link the criterion speaks only for $\kappa < 2$, while the true gap grows with κ Reading . On the whole lattice $\beta < 1/12$ Cited ; integrating first, $\beta < \sqrt{2/27} \approx 0.272$ (the strong-coupling side, lattice spacing fixed). The curvature of the group does not move with β, so the threshold cannot exceed $O(1)$. At weak-coupling scales convexity on the group itself fails (the lemma on the ceiling of convexity on the group, in 24-06 § 03) Proved here

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
3 The log-Sobolev inequality Bounds the entropy by the gradient. Stronger than Poincaré	5-11 § 02 (the criterion also gives log-Sobolev) Cited . The language of measures is in 1-08	24-06 § 06 (tools that pay with rarity instead of convexity; the entropy currency). 23-10 the type of mistaking the currency	The arrangement "Poincaré is the currency of exit speed, and only the entropy currency can pay for rarity" is a picture Reading . "The class of measures that pay with rarity" is a candidate rule and promises no prospect of closing the third step Assumption
4 Brascamp–Lieb and the hierarchical induction An induction whose entry is that the conditional measure at each scale is uniformly convex	No home in Parts 1–5 or Parts 7, 17 and 18. The definition and the conditions of the source are in 24-06 § 02 (Bauerschmidt–Bodineau) Cited	24-06 § 05 (carrying the induction over to the group: what applies, what changes, what does not)	The convexity assumption cannot hold, in principle, on the group at weak-coupling scales Proved here . The non-convex region is the reverse of flat ϕ^4 (on the group, the hemisphere far from the identity), and the wall is only pushed into a region of probability $e^{-c\beta}$ Proved here . Results for the hierarchical model imply nothing for four-dimensional Yang–Mills
5 Reflection positivity and the transfer matrix	5-03 (Osterwalder–Seiler), 4-10 (reconstruction) Cited . On the side of the published construction, 12-01 (two kinds of planes), 12-02 (the chessboard) Assessment	26-04 (positivity of the twist-summed ensemble z under reflections through lattice points) Proved here . 26-05 (input to the chessboard estimate). 16-09 the vessel stage	Transverse reflections break with the twist (the lemma on non-positivity of transverse reflections in 26-04) Proved here . The structural inputs (reflection positivity, duality, the transfer matrix, the centre identities) hold unchanged for $\mathbb{Z}_2$ and $U(1)$, and this set alone does not give the widening inequality (the negative lemma; its last step inherits the Reading of 26-03)
6 The Kogut–Susskind Hamiltonian lattice Only space is a lattice; time is continuous	5-03, 4-01 Cited	24-05 § 04 (a road that treats the gap directly). In the colour singlet the Poincaré constant of the vacuum measure halves at leading order Proved here	Local-to-global gap theorems do not apply as they stand on this lattice (reasons in 24-05 § 04). For the Kogut–Susskind form used in 24-05, read 5-03 first
7 The strong-coupling and cluster expansions Add weak couplings as corrections, and guarantee convergence by a criterion	5-02, 5-04, 7-04 (the Kotecký–Preiss, Dobrushin and Fernández–Procacci conditions) Cited	26-06 (the strong-coupling end). 15-04 § 05 (the exit of the chain in the hierarchical model = the strong-coupling decision region) Computer-assisted	They reach only the strong-coupling shore, with the lattice spacing fixed. They do not reach the band (intermediate coupling). How the strong-coupling ranges compare is in 5-11 § 07. This book's Lean theorems at strong coupling (the infinite-volume limit, the mass gap, the area law) are in 20-02–20-06, and are statements inside the strong-coupling window with the lattice spacing fixed. They sit at the opposite end from the questions that remain
8 The character expansion and class functions Expand a weight on the group in characters of irreducible representations. Conjugation-invariant functions = class functions	5-02 (the character expansion), 7-02 (the heat kernel in two dimensions), 1-04 Cited	15-04 § 01 (the one-link heat-kernel class is a semigroup and closes with one parameter) Proved here . 15-06 (couplings by loop shape and the attracting class)	It closes because in this model one weight returns to one weight in one step. Multi-link couplings, gauge re-fixing and large-field remainders are absent by construction, so nothing is implied for one step of the four-dimensional hypercubic lattice ("Said first" of 15-04)

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
9 Dobrushin's influence coefficient A criterion of the type: if the total influence of the neighbours on one site is below 1, correlations decay	7-04 (Dobrushin's condition, $4/27$ in the gas example) Cited	26-07 § 03 (one site at the top in the first step). 26-05 § 04. 26-10 mixing of the box fluxes	At the top in the first step, the conditional one-site measure is $\frac{15}{4}$-convex and the coupling to a neighbour is 2, so the influence from two neighbours is $\frac{16}{15} > 1$, back to the deficit $\frac{15}{4}$ versus 4 Proved here . With the frame fixed, the continuous field follows the boundary and the influence coefficient is close to 1 Reading (physics; the original papers not checked)
10 The renormalisation-group step (blocking) Integrate out the fine variables to make the action at the coarse scale	7-05 (the sequence of steps, blocking, small and large fields) Cited . What has been reached is in 7-07. The one step of the published construction (a coupling-dependent probability kernel) is in 9-01–9-07, and the many steps in Part 11 Assessment	15-03 (the decomposition statement, the ledger of transverse contraction). 15-05. 15-07 (background-dependent fluctuations η)	The numbers of 15-05 are for a step seen in the marginal of one coarse plaquette, not a true blocking step. The true two steps do not match two iterated steps (the contraction λ is 3.4–3.7 times larger) Reading . Fixing all coarse plaquettes at once makes the contraction a matrix, exceeding $1/16$ in coarse directions (Bundle Network Theorem; abelian configurations, tree level) Proved here . The multi-step induction for η has not been written Assumption
11 Migdal–Kadanoff and the hierarchical model A recursion that approximates blocking by a map on one weight	18-04 Figure 1 (the approximate recursion and its character) Cited	15-04 § 02– § 05 (pulled to an invariant curve at rate $1/16$ per step, its position set by the balance of source and contraction, $1/3600$: leading saddle-point term Proved here ; invariance of the tube for $\beta \in [2.46, 64]$ Computer-assisted). 26-08 road 30	The MK flow goes to strong coupling whether the group is abelian or not, and does not distinguish $U(1)$ from $SU(2)$ (Kanazawa) Cited . Tomboulis's interpolation relies on a common t^* for the untwisted and twisted sectors; this reduces to inequality (3.5) of Ito–Seiler, not shown outside strong coupling Cited . The hierarchical model confines even for $U(1)$ and is no test of non-abelianness
12 One-loop perturbation theory and the coefficient 11/3 Gaussian fluctuations around a background field. $\frac{11}{3} = 4 - \frac{1}{3}$	4-03, 4-06 (asymptotic freedom and the split of the coefficient) Cited	25-07 (isometric maps give only $\beta' \geq \beta + 0.023$; maps that reshape the fluctuations reproduce $\beta - \beta' = 0.258$) Proved here . 26-03, 26-06, 15-06	All are statements at the one-loop (Gaussian) level. Controlling the error of the reshaping maps at intermediate coupling and large fields is open. The $\frac{11}{3}$ at the head of those tables repeats 4-06; read 4-06 first for the split
13 Dimensional transmutation and Λ A mass scale Λ arises from a dimensionless coupling	4-07 Cited	24-07 (because it is four dimensions). 26-00, 26-07 (the position of the band, $L \sim 1/\Lambda$)	Placing the band at $L \sim 1/\Lambda$ is a physical reading of scales Reading . Most of 24-07 is an arrangement of known material and can be read with Parts 4 and 7 alone

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
14 The Gibbs variational principle and relative entropy Transporting a measure by a map, the remainder in the bound on the free-energy difference is a relative entropy. Crooks and Jarzynski are of this type	2-08 (free energy, relative entropy) Cited	25-06 (dynamically driven insertion). 25-07 the transport identity (the principle of direction) Proved here	In a sandwich by fixed maps, the remaining relative entropy grows with the area of the vortex sheet, and the form $e^{-\sigma A}$ does not come out Reading (physics; the stopped road "the deterministic sandwich" of 26-08). What can be used is the qualitative target $\Delta F(L,L)\to 0$
15 Twisted boundary conditions, the centre and 't Hooft flux	3-05 (topology), 5-01 (the lattice), 1-04 (the centre) Cited . The definition of the twist itself is in 25-01 Cited	25-01 (three facts, including uniqueness of the vacuum that absorbs the twist) Proved here . 25-04, 26-01, 26-04	For groups with trivial centre (G_2 and others) the arrows that use the centre disappear (16-09 the general-group stage) Cited . Parts 25 and 26 are written for $SU(2)$
16 Centre vortices and fractional instantons	17-02 (pictures of confinement), 3-05 (instantons) Cited	25-02 (definition and action, an upper bound on the minimal lattice action). 26-03 (a picture of where non-abelianness acts)	The dilute-gas count is a physical reading Reading . For the vortices of 26-03, read the definitions of 25-02 first
17 The container identity and the Tomboulis–Yaffe inequality Fixing the frame makes containers independent. From 't Hooft's criterion to the area law	The basis is 5-02, 5-04 (sums over surfaces). The identity is in 25-03 § 03 (Mack–Petkova), the inequality in 26-01 Cited	25-03–25-05 (rewriting in the sector-summed ensemble, nesting) Proved here . 26-02 (reduction to a single inequality)	Placing containers of fixed length independently gives, in four dimensions, only $V(R)\gtrsim R/\ln R$ (Kovács–Tomboulis) Cited . The area law needs one container of the size of the loop, which is the same condition as crossing the band. That the sector bias θ moves away from 1 is not enough (26-01) Proved here
18 The flux network — Thomson's principle, Kirchhoff's laws, orthogonal projection	1-01 (linear maps and projections) Cited . Standard facts of electrical circuits are in the source table of 15-05	15-05 Unit Bundle Theorem (the minimiser is fixed by a projection; contraction $\lambda\le\max f^2$, $1/(16d-8)\le\lambda\le 1/16$) Proved here	Statements at tree level (leading term as $\beta\to\infty$) restricted to abelian configurations. They do not include non-abelian configurations, the one-loop source, large fields or gauge re-fixing
19 The BCH formula and quaternions The commutator series of $\log(e^X e^Y)$. $SU(2)$ is the quaternions of length 1	1-03 (BCH up to third order), 1-02, 1-04 Cited	5-11 (the quaternion inner product and one-link integration). 15-06 (commutators as images under BCH). 15-07 (the group-geometry part = the third-order BCH coefficient $1/9$; leading Gaussian term, two links) Proved here	For $SU(2)$ the group-geometry part rides on the background itself and does not contract, and its budget diverges whatever threshold is chosen (arithmetic treating the one-loop flow as exact) Proved here . For $U(1)$ the third-order BCH term is identically 0

Tool	Home (Parts 1–5 and Parts 7, 17 and 18, source)	Use in this book (Part 23 on)	Where it stopped on the group, cautions
20 Computer assistance — interval arithmetic, certificates, the discipline of double precision Hold every number as an interval containing the true value, rounding outward	No home in Parts 1–5 or Parts 7, 17 and 18. The general theory is in 23-08 § 01– § 03 (rounding conventions, holding intervals, positive and negative controls) Cited	23-07, 23-08 (the chain condition $q < 1$ at every point of the $\hbar$ window) Computer-assisted . 15-04 § 05 (the tube of the hierarchical model) Computer-assisted . 23-15	Inequalities have been enclosed by interval arithmetic in the first-step window and in the hierarchical model, a map on one weight (plus the upper bounds on the action of explicit configurations in 25-02 Computer-assisted , upper bounds only). Even in the hierarchical model, $\beta > 64$ (the limit of zero lattice spacing) is not closed. The road of checking finitely many L by interval arithmetic is out of computational reach in four dimensions (26-08 road 3) Reading . Computer assistance says nothing outside the window (23-08 § 10)
21 Formalisation in Lean Statements as types, proofs as terms, checked by a small kernel	A-04 (kernel checking, the criterion of the three standard axioms) Cited . How to read the Lean of the published construction is in 6-00	5-11 (the curvature lower bound and the combinatorics of families). 23-13 (the bundles of the first step). The Lean column of Table 1 of 16-09	What is guaranteed is only the arrow from the stated hypotheses to the stated conclusion. Physical measures are not formalised. The interval-arithmetic layer lies outside Lean (23-13 § 04). The general theory is A-04; the list of bundles and examples is 23-13
22 Reading Monte Carlo Generate lattice configurations at random and measure averages	5-07, 5-08 Cited	25-08, 26-09 (the measured record) Reading . The numbers of 15-05 and 15-07 Reading	All measured numbers carry the "Reading" label and are not used as grounds for theorems. Lattice size, β, seeds and errors are on the same row in each chapter. There are examples where the numbers move when the way of measuring changes (the true two steps and two iterated steps in 15-05)

What to read from Table 1. (a) Two tools have no home in Parts 1–5 or Parts 7, 17 and 18 (the hierarchical induction and computer assistance). For both, chapters from Part 23 on (24-06 § 02, 23-08 § 01– § 03) double as the home. (b) The ways of stopping in the third column fall into types. The curvature and convexity tools (rows 2–4, 9) stop at the closed shape of the group and at the single site at the top; the renormalisation tools (rows 10, 11, 19) stop where one step is stacked into many; the structural tools (rows 5, 17) stop where they cannot be told apart from $U(1)$. (c) Computer assistance and Lean (rows 20, 21) both say nothing outside what was written.

How topics found in two or more places are divided

When the same tool appears in several parts, read the "home" of Table 2 first and the "use" after the home. The home chapter alone has the definitions and the known results.

Table 2 Topics found in two or more places (home = read first, use = read later)

Topic	Home	Use	Row of Table 1
The Bakry–Émery criterion	5-11 (the criterion and the one-link example)	24-06 § 02– § 03 (why it fails on the group at weak coupling), 23-02, 23-06	2
Migdal–Kadanoff	18-04 Figure 1 (the character of the approximation)	15-04 (invariant curve and interval arithmetic), 26-08 road 30 (Tomboulis's claim and its critiques)	11
The Kogut–Susskind lattice	5-03	24-05 (the gap and the vacuum measure)	6
Reflection positivity and the transfer matrix	5-03	26-04, 26-05 (checked for $\mathbb{Z}_2$ and $U(1)$)	5
The one-loop coefficient $\frac{11}{3}$	4-06	25-07, 26-03, 26-06	12

Topic	Home	Use	Row of Table 1
Dimensional transmutation	4-07	24-07, 26-00, 26-07	13
Vortices (centre vortices, fractional instantons)	The picture in 17-02; definition and action in 25-02	26-03 (a picture)	16
Renormalisation steps and Bałaban's series	7-05, 7-07 (what has been reached)	15-02 (inside the series and the BBS chain), 15-03	10
Interval arithmetic	23-08 § 01– § 03	23-07, 23-15, 15-04	20
Lean	A-04 (the general theory)	23-13 (the bundles of the first step and two examples of the types)	21

02

Three reading orders — from the tools, from the results, from the map

In all three, going back to Table 1 along the way to look up the home chapter lets you read without missing prerequisites. They can be used alongside the reading orders of 0-01 § 07.

Three reading orders (read in the order of the arrows; [ ] are rows of Table 1)

Way in	Reading order	What you know when you finish
From the tools readers from the mathematical side	1-10 → 5-03 → 5-11 → 7-04 → 7-05 → 18-04 → 23-08 § 01– § 03 → A-04 → 24-06 → 15-04 → 15-05 [1, 2, 5, 7, 10, 11, 20, 21, 3, 4, 18]	What each tool needs, and where on the lattice on the group it stopped. The three types of stopping ((b) after Table 1)
From the results from this book's conditional statement	16-09 → 27-01 → 27-02 → 25-04 → 26-02 → 15-03, going back from the "use" column of Table 1 to the home chapters as you read	Which tools each arrow of the conditional statement relies on. How the rows of the ledger of conditions correspond to the tools
From the map seeing the whole of what is open first	0-02 → 16-05 → 16-07 → 16-09 → this chapter → 26-08 → 15-08 → 27-04	Which of the seven stages are open, which tools stopped where, and what this book does not claim

03

Dependencies between the tools — tools needed first, and tools used later

Table 3 connects the tools of Table 1 by "tools needed first". The numbers are rows of Table 1. A tool needed first may have a later row number, but there are no cycles. Following this column from the left fixes the order in which to read the home chapters.

Table 3 Dependencies of the tools (this book's arrangement; "used together" is where they are combined from Part 23 on)

Tool (row of Table 1)	Tools needed first	Where used together
1 Poincaré inequality	— (linear algebra and spectra of 1-10)	The form of the conclusions of 2 and 3. The identity of 24-05
2 Bakry–Émery	1, 19 (the curvature of the group comes from $SU(2)$ being the three-sphere)	Gives 3. Entry to 4. Combined with 21 in 5-11
3 log-Sobolev	2	The tools of 24-06 that pay with rarity
4 Hierarchical induction	2, 10	Compared with 11 in 24-06 § 05
5 Reflection positivity	— (the axioms of 4-10)	The chessboard of 26-05. With 15 and 17, the negative lemma of 26-04
6 Kogut–Susskind lattice	5	Combined with 1 in 24-05
7 Strong-coupling expansion	8	The exit of the chain in 15-04. A criterion of the same type as 9

Tool (row of Table 1)	Tools needed first	Where used together
8 Character expansion and class functions	19	Combined with 10 and 11 in 15-04 and 15-06
9 Dobrushin's influence coefficient	7	Set beside 2 in 26-07 (back to the same deficit)
10 Renormalisation step	12	The frame for 11 and 18. The decomposition of 15-03
11 Migdal–Kadanoff	10, 8	Combined with 20 in 15-04. Road 30 of 26-08
12 One loop	— (perturbation theory of 4-03)	Combined with 14 in 25-07. 26-06, 15-06
13 Dimensional transmutation	12	The reading of the band's position (26-07)
14 Gibbs variational principle	— (2-08)	Combined with 12 in 25-06 and 25-07
15 Twist, centre, flux	— (3-05, 5-01)	The stage for 16 and 17. 26-04
16 Vortices	15	25-02, 26-03
17 Containers and Tomboulis–Yaffe	15, 7	25-03–25-05, 26-02
18 Flux network	10, 11 (comparison with $1/16$)	the Unit Bundle Theorem and the Bundle Network Theorem of 15-05
19 BCH and quaternions	— (1-03, 1-04)	The group-geometry part of 15-07
20 Interval arithmetic	— (23-08 § 01– § 03)	In 23-07, 23-08 and 15-04, encloses inequalities for maps such as 11
21 Lean	— (A-04)	Checks the algebraic steps of 2, 5, 17, 19 and others. The output of 20 is placed as a hypothesis
22 Monte Carlo	— (5-07)	The numbers of 25-08, 26-09, 15-05, 15-07 (Reading)

What to read from Table 3. The rows with no tool needed first (1, 5, 12, 14, 15, 19, 20, 21, 22) are the homes a second-year student can read first. Of these, 20 and 21 have their homes in Part 23 and the appendices, so read them first following the division of Table 2.

04

What cannot be said — a list of the limits of the tools

The limits of each tool, taken one line at a time from the third column of Table 1. What is not claimed, part by part, is in 27-04 What is not claimed (a cross-index); the limits of the ultraviolet tools of Part 15 are in 15-09 What cannot be said.

1. The curvature tools (Bakry–Émery, Brascamp–Lieb, Dobrushin) work only on the strong-coupling side with the lattice spacing fixed, and in the first-step window. At weak-coupling scales convexity on the group itself fails (24-06 § 03), and at the top in the first step they return to the same deficit $\frac{15}{4}$ versus 4 (26-07 § 03).
2. The strong-coupling and cluster expansions are tools of the strong-coupling shore and do not cross the band (intermediate coupling).
3. The statements about one renormalisation step (15-04, 15-05, 15-06) are each restricted to a map on one weight, the marginal of one coarse plaquette, or tree level on abelian configurations; none of them stacks a true step of the four-dimensional hypercubic lattice into many (15-09).
4. The Migdal–Kadanoff flow and the hierarchical model do not distinguish $U(1)$ from $SU(2)$. Closing there is no test of non-abelianness.
5. The structural inputs alone (reflection positivity, duality, the transfer matrix, the centre identities) do not give the widening inequality (the negative lemma of 26-04, in the form "this set of inputs alone").

6. Comparisons by one-loop tools and by the Gibbs variational principle are statements at the one-loop (Gaussian) level, not non-perturbative bounds.
7. Interval arithmetic says nothing outside the intervals written. Lean guarantees only the arrow from the stated hypotheses to the stated conclusion, and physical measures are not formalised (23-13, A-04).
8. Monte Carlo numbers are readings and are not used as grounds for theorems.
9. Whatever tools in this table are combined, this book's study shows neither the existence of the mass gap nor the construction of the continuum limit (27-04). The existence of the continuum limit was shown by the published construction, with the set of tools in (6)Assessment . The existence of the mass gap is in neither (16-02).

What this chapter can say

Can say	Cannot say
Cited The known results in the "home" column of Table 1 (the extent of the sources is in each chapter's source table)	That the twenty-two of Table 1 exhaust the tools of this book. The division into tools is this book's arrangement
Proved here Computer-assisted Reading Assumption The claims in the second and third columns of Table 1 (with the labels and conditions of the chapters listed)	That "where it stopped" in the third column is a limit of the tool in principle (it records the range of this book's use)
The division of Table 2 and the dependencies of Table 3 (this book's arrangement)	That the dependency table exhausts all logical implications
Overall	This site shows neither the existence of the mass gap nor the construction of the continuum limit

Sources and reproduction

Item	Kind	Source, tools
The claims and numbers in each row of Table 1	As labelled in each chapter	The text and source tables of the chapters listed on the row (this chapter copies them and does no new computation)
The Bakry–Émery criterion	Cited	D. Bakry, M. Émery, Diffusions hypercontractives, Séminaire de Probabilités XIX (1985) 177–206 (bibliographic data in the source table of 5-11)
The whole-lattice threshold $1/12$	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851 (the source table of 5-11)
The assumption of the hierarchical induction	Cited	Bauerschmidt–Bodineau (the source table of 24-06)
Tomboulis's interpolation and its critiques	Cited	E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930 ((3.5)). T. Kanazawa, arXiv:0805.2742 (the source table of 26-08)
The container identity and $R/\ln R$	Cited	Mack–Petkova (1980), Kovács–Tomboulis hep-lat/0108017 (the source table of 25-03)
Thomson's principle, Kirchhoff's laws	Cited	Standard facts of electrical circuits (the source table of 15-05)
Dobrushin's condition	Cited	The source table of 7-04. The original papers of the criteria in 26-05 are not checked
Table 2, Table 3, the three reading orders	This book	Arranged from the prerequisite paragraphs and the cross-references in the text of the chapters (no numerical computation)

Words used in this chapter — home (the chapter where a tool's definition and known results are first set out), use, where it stopped, the Bakry–Émery criterion, the log-Sobolev inequality, the hierarchical model, reflection positivity, class function, influence coefficient, blocking, Migdal–Kadanoff, the flux network, BCH, interval arithmetic, certificate. Their meanings are in A-02 Glossary.

Next chapter to read: 16-11 A map of ideas for crossing the band — living ideas, broken ideas, mathematics that seems to be needed. To read alongside it, 23-00 Entrance — the first road (from the direction of time). If you read from the tools, 5-11 Detour: Bakry–Émery and one-link integration.

A map of ideas for crossing the band — living ideas, broken ideas, mathematics that seems to be needed

> This chapter is a map of what is unsolved. The derivations of § 02 and § 03 are this book's own study.

What this chapter shows — The ideas that this book's study tried for crossing the crossover band (26-00), on a single map. First we set out the two ends of the band and a rough order of magnitude for the distance between them, and split the core into two jobs (extending the control of the published construction, keeping the sum of errors finite, to where the coupling is of order a number; and a statement for the last stretch from there to strong coupling). Next we derive why a chain of lossy inequalities cannot give a conclusion uniform in the cutoff. Then we see that the facts specific to SU(2) in this book's Lean are all facts about the action of configurations (facts on the side of energy), that what the band needs is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale (the shortfall on the side of entropy), and which direction the bounds that can be written in variational form go. On that basis we sort the ideas into three tables, living, broken and undecidable, put the smallest next step in the form of questions, and list by name the mathematics that seems to be needed. Widening the test opponents to seven, and the small-lattice computations, are in 26-12.

Prerequisite chapters — 16-03 The infrared edge of the published construction, 16-04 Where this book stands, 26-00 Entrance — confluence, 26-06 The renormalisation comparison, A-08 § 05 The infrared side.

Said first — This chapter does not give a road across the band. There is no proof. What there is: a map of ideas, facts checked by hand counts and small-lattice computations, and a sketch of "this type of argument cannot cross". The observations written here (the two-way split of the core, the lossy chain, the shortfall on the side of entropy) may be known to specialists. This book writes them not as new results but as a sketch map used by this book's study. The numbers for the distance (the number of doublings, the ratio of scales) are rough orders of magnitude, not quantitative claims.

The order of this chapter

1. The two ends of the band, and the core split in two — a rough order of magnitude for the distance; the ultraviolet extension and the statement for the last stretch to strong coupling
2. A lossy chain gives no conclusion uniform in the cutoff — the rate is uniform, the prefactor grows
3. Facts on the side of energy, and the shortfall on the side of entropy — the direction of variational bounds — the facts specific to SU(2) in this book, what the vortex floors count, what is missing
4. The list of ideas — living (Table 1), broken (Table 2), undecidable (Table 3)
5. The smallest next step — in the form of questions
6. Mathematics that seems to be needed — seven items
7. What this chapter does not say

Labels — Proved here = a hand count whose derivation is placed in this chapter / Cited = a fact from the literature, checked at the relevant place in the original or a review / Reading = a reading of this book's small-lattice numbers (lattice, sweeps and seeds given) / Assumption = a conjecture or an expectation / Lean = a theorem in this book's Lean (only inside the strong-coupling window, with the lattice spacing fixed). For details, see "Labels" in 6-00. Statements in the literature that we have not checked are written as "a statement in the literature (not checked by this book)", and are not used as grounds for claims.

Notation follows this book's conventions. The one-loop slope is $s = 11C_g/(24\pi^2)$, for SU(2) $s = 11/(3\pi^2) \approx 0.3715$ (in Parts 25 and 26 the same quantity is written $8b_0$; the $s(N)$ of 26-06 is a different quantity). We write $\bar{\beta}$ for the reference coupling of the published construction (a coefficient of the effective action), and $\beta = 4/g^2$ for the coupling of the SU(2) Wilson action.

01

The two ends of the band, and the core split in two

The two ends. The strong end is this book's Lean window: the infinite-volume limit and the mass gap for $|\beta| < 1/19$ (an explicit sufficient condition for $18 \tanh |\beta| < 1$; the window itself is $|\beta| < 0.0556$; 20-03), and the area law for $\beta < 1.1 \times 10^{-3}$ in a finite box and $\beta < 1.2 \times 10^{-4}$ in infinite volume Lean (20-06; lattice spacing fixed, Wilson action

only). The weak end is the threshold required by the one-step estimate of the published construction; it is a lower bound on a coefficient of the effective action, not a Wilson β. The value of the threshold itself is not written as a number in the source; from just one of the conditions imposed on it, the threshold comes out as at least 9×10^5 ($e^{13.7065}$) (16-03 § 01). The two ends are stated in different quantities and in different classes of actions.

A rough order of magnitude for the distance. In the one-loop formula, each doubling of the lattice spacing lowers the coupling by $s \ln 2 \approx 0.2575$. If this one-step pace is extended straight into the region where it cannot be trusted (coupling of order 1), and moreover the coefficient of the effective action and the Wilson β are put on the same axis, the number of doublings from 9×10^5 to $1/19$ is

$$k \;\approx\; \frac{9 \times 10^5 - 1/19}{0.2575} \;\approx\; 3.5 \times 10^6, \qquad 2^k \;\approx\; 10^{1.05 \times 10^6} \quad (1)$$

Proved here (the division and the computation of $k \log_{10} 2$). This is a **rough order of magnitude**. The one-loop formula cannot be used where the coupling is of order 1, and the two axes are coefficients of different actions, so (1) is not a quantitative claim. What can be said goes only as far as the expectation that the weak end and the strong end are separated by an astronomical ratio of scales in units of step 0 Assumption .

Splitting the core in two. We view the core of the band as split into the following two jobs Assumption .

1. **The ultraviolet extension.** The job of extending the control of the published construction from the present threshold (at least 9×10^5) down to the value β_* where the coupling is of order a number, while keeping the sum of the errors over the steps finite. It is the same kind of job as the published construction, but the whole amount remains (the order of magnitude of (1)).
2. **The statement for the last stretch to strong coupling.** A statement with few steps, from β_* to the strong-coupling window. It is a non-perturbative statement where the coupling is of order 1 (it is different from the "Corollary Crossing the Band in Finitely Many Steps" of 26-06).

Where the boundary β_* is placed changes how the difficulty is divided. If the threshold is left at its present value, the "box of physical size of order $1/\Lambda$" that the second statement treats has a side of the order of (1) in units of step 0, and is no longer a statement about a small box. The second becomes a statement about a small box only when the first job has brought the threshold down to order a number Assumption . Note that in a periodic box, the published construction has the result that, with the volume fixed, the effective density at step 0 converges along the whole sequence of cutoffs (16-

03 § 02; the limiting measure itself is not defined as an object). There is no quantitative information at the size $1/\Lambda$.

This view comes with another reading. On a line of fixed physical size (fixed ratio of the side of the box to the correlation length), one step of the renormalisation comparison (26-06) is a statement that "compares boxes of the same physical size with the lattice spacing halved". So the sequence of doubling steps is at the same time a sequence of steps running into the infrared and a sequence of statements of ultraviolet uniformity (control of the continuum limit) Assumption . The conclusion of § 02 agrees with this reading.

02

A lossy chain gives no conclusion uniform in the cutoff

Consider the renormalisation comparison (26-06) in a form that allows a loss of a factor $K \geq 1$ at each step. With $\Delta F(N, M, \beta)$ the vortex free energy of a slab of width N and length M (19-02) and $f(\beta) = \beta - c$ ($c > 0$), suppose that

$$\Delta F(2N, 2M, \beta) \;\leq\; K\, \Delta F(N, M, f(\beta)) \qquad (N \geq N_1,\; M \geq M_0,\; \beta_s \leq \beta \leq \beta_0) \quad (2)$$

holds at every β the chain passes through (from the edge β_s of the strong-coupling window up to the starting β_0) (an assumption; it has not been shown). Here β_s is the edge of the window in which the estimate exponential in the area, with prefactor the transverse area, can be used (the finite-box window of 20-06, $\beta < 1.1 \times 10^{-3}$). It is a different window from the $1/19$ of § 01 (the window for the infinite-volume limit and the mass gap).

Derivation Proved here . Chain (2) k_0 times. k_0 is the number of steps, starting from β, until the coupling reaches the edge β_s of the strong-coupling window, and $k_0 \approx (\beta - \beta_s)/c$. For any $N' \geq N_1$, $M' \geq M_0$,

$$\Delta F(2^{k_0} N',\, 2^{k_0} M',\, \beta) \;\leq\; K^{k_0}\, \Delta F(N', M', \beta_s).$$

Insert the area-exponential estimate at the edge of the window, $\Delta F(N', M', \beta_s) \leq C\, N'^2 e^{-\sigma_s M'^2}$ (of the type of 20-06; the prefactor is the transverse area and σ_s the rate at the edge of the window), and set $L := 2^{k_0} M'$, $\xi := 2^{k_0}$. Then

$$\Delta F(\xi N',\, L,\, \beta) \;\leq\; C\, N'^2\, K^{k_0}\, \exp\!\Big(-\sigma_s\, \frac{L^2}{\xi^2}\Big) \quad (3)$$

Three things can be read from this.

- **The rate is uniform.** The rate per unit area is σ_s/ξ^2 and does not depend on K. If ξ is read as "the ratio of scales between the lattice at β and the lattice at the edge of the window" (of the order of the correlation length), the rate in physical units is σ_s and does not depend on the cutoff.
- **The prefactor grows.** $K^{k0} = \xi^{\log_2 K}$ (an equality), so the prefactor diverges as a power of the correlation length (the cutoff seen in lattice units). The right-hand side of (3) becomes smaller than 1 only from $\sigma_s (L/\xi)^2 \gtrsim k_0 \ln K + \ln(CN'^2)$, which in physical length L/ξ is of order $\sqrt{k_0 \ln K/\sigma_s}$; since k_0 is proportional to β, this stretches like $\sqrt{\beta}$ (of order $\sqrt{\ln \xi}$).
- **The condition for a uniform statement.** For the prefactor not to depend on the cutoff, the product of the per-step losses K_i must be bounded, that is, $\sum_i \ln K_i < \infty$ ($K_i = 1 + \varepsilon_i$, $\sum_i \varepsilon_i < \infty$).

Reading. A chain whose loss is a constant $K > 1$ uniform over the steps can carry **confinement at each** β (exponential in the area at fixed lattice spacing), but cannot carry **constants uniform in the cutoff** Proved here . A uniform statement needs the sum of the losses to be finite, and this is not a matter of cleverness with inequalities but a job of the kind of controlling the continuum limit (the first job of § 01) Assumption .

Seen additively, the distinction lies in the leading term. On the weak-coupling side, the vortex free energy is semiclassically of the type $\Delta F \approx \pi^2 \beta_{\text{eff}}(L)$ (π^2 is the lower bound of the sector action of the continuum theory; 25-02). If at one loop β_{eff} decreases by $s \ln 2$ per doubling, the change per doubling is

$$\Delta F(2L) - \Delta F(L) \;\approx\; -\pi^2 s \ln 2 \;=\; -\tfrac{11}{3} \ln 2 \;\approx\; -2.54 \quad (4)$$

Proved here ($\pi^2 \cdot 11/(3\pi^2) = 11/3$; at the one-loop level). In the Coulomb phase of U(1), the quantity of the same type does not run. So "the difference on the weak side is small, $O(1/\beta)$" is a relative way of speaking; on an additive ruler, the distinction is $O(1)$ per step. Subtracting values in this book's measured record (Tables 8 and 10 of 26-09, 1–3 seeds), SU(2) $4^4 \to 8^4$ gives 1.61, 1.88, 1.80 at $\beta = 2.7, 2.9, 3.0$ Reading (with a scatter of ± 0.3–0.7 on each value). The recorded values are $\Delta F(L) - \Delta F(2L)$ (the decrease), and compared with the size 2.54 of (4) they are of the same order.

However, the K-fold loss of (2) multiplies $\Delta F \approx \pi^2 \beta$, so $(K-1)\pi^2\beta$ eats up the 2.54 of (4) in a single step. Looking at the approximate formula, the lossless additive comparison $\Delta F(2L, \beta) \le \Delta F(L, \beta - c')$ is false in small boxes because of lattice artefacts: the minimal sector action $S(L)$ seen by cooling increases with L (7.84 at 2×2, 9.80 at 4×4, approaching the continuum $\pi^2 \approx 9.87$ from below; 25-02 Table 1) Reading , and the first term of $\Delta F(2L) - \Delta F(L) \approx \beta\,(S(2L) - S(L)) - 2.54$ is positive, so

the lossless form can only be written with a lower bound of order $\sqrt{\beta}$ on the side of the box Proved here . That the recorded decrease $2^4 \to 4^4$ moves away from 2.54 as $1.83,\ 0.85,\ 0.55$ at $\beta = 2.5,\ 2.75,\ 3.0$ agrees with the direction of this artefact Reading .

03

Facts on the side of energy, and the shortfall on the side of entropy — the direction of variational bounds

1. The facts specific to SU(2) in this book Assumption (within what we read of this book's Lean theorems (A-04 Table 4g and the pages 21-05, 21-07, 20-06); we have not checked by carrying the proof of each theorem over to U(1)). The ones that can clearly be read as depending on the non-commutativity of SU(2) and not carrying over to U(1) as they stand are the following four.

- With a twist in one direction only as background, there is a configuration of action 0 (the twisted vacuum) Lean (`ThetaLimit.flat_exists_10`, 21-07). This is because the quaternions **i** and **j** anticommute. In an abelian group, a single twist cannot be satisfied with action 0 Proved here (multiplying the products over the faces of a closed surface leaves only the sign of the twist).
- The gauge transformations that fix the twisted vacuum are only ± 1 Lean (`TwistBlocks.stab_vacT`, `TwistBlocks.rigidT_all`, the detour of 21-05).
- A box with twists in two directions at once has no configuration of action 0 (`TwistFlat.no_flat_twist11`, 21-07), and the minimal action of a vortex is positive, $S_N > 0$ (`VortexPos.SN_pos`, 21-05) Lean .
- For $N = 2, 3, 4$, a thick vortex is cheaper than a thin one: $S_N < 2N^2$ Lean (computer certificates; `VortexCert.SN_two_lt` and others, 21-05).

All of these are **facts about the action of configurations (facts on the side of energy)**. They point both ways: the first and fourth say "the twist goes in for free" and "a vortex can be made cheap", while the third says "a vortex is not free".

2. What this book's vortex floors count. The thin-vortex floor and the thick-vortex floor (21-02, 21-03) are upper bounds on the vortex ratio θ, that is, bounds in the direction of confinement, with rate $-\ln\tanh(N^2\beta)$ for the thin vortex and $e^{-\beta(S_N+\varepsilon)}$ for the thick vortex Lean (the thick-vortex floor is a statement for sufficiently large β and, for each β, sufficiently large M; the threshold is not explicit). This rate can be read as coming from the action of a single vortex (its classical value) and the number of positions at which the vortex pierces the longitudinal face (the factor M^2) Assumption . It is a value with the transverse size N fixed, not uniform in N, and the decrease at large β stays the exponential of the classical action. The upper-bound certificate $S_N <$

$2N^2$ is used in the direction of improving this rate (`TwistBlocks.thick_vortex_floor_two` and others, 21-05).

3. What is missing Assumption . What the band needs is that the rate is preserved when the transverse size is widened beyond the physical length (26-01, 26-02). In the picture of physics, what acts there is that the fluctuations of vortices and the number of their arrangements (entropy) run with scale (the reading of the margin $5/3 = 11/3 - 2$ in 26-00 Figure 2). In U(1) there is no such running. The vortex floors count positions too, but that count is with the transverse size fixed. However many facts about the action of configurations are piled up, this running is not expected to come out. What is needed is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale. This book calls this **the shortfall on the side of entropy**.

4. The direction of variational bounds Proved here . Written with the partition functions Y_0, Y_1 of electric flux $e \in \{0, 1\}$, the untwisted and twisted partition functions are $Z_{n34=0} = Y_0 + Y_1$ and $Z_{n34=1} = Y_0 - Y_1$, and the vortex ratio is $\theta = Y_1/Y_0$ (26-06 § 02, 26-04). Confinement needs an upper bound on θ, that is, a **lower bound** on the energy of the flux sector. What a trial state (a variational bound) gives is an **upper bound** on the energy, that is, a lower bound on Y_1 and a lower bound on θ, the opposite of the direction confinement needs. On the magnetic side, a lower bound on the twisted partition function $Z_{n34=1}$ (an ensemble of trial configurations and Jensen's inequality) gives an upper bound on θ, which is the direction of confinement. Only on the magnetic side can the direction of confinement be written in variational form. We do not, however, say that a proof for the band is limited to this type alone.

04

The list of ideas

Each row gives the reason and the range checked. The seven test opponents, and the type of argument each one knocks out, are collected in 26-12; here we give only names and conclusions. Three terms first: a **mixed action** is the SU(2) action made by adding an adjoint-representation term to the Wilson action (the fundamental-representation term); a **first-order transition** is a transition in which the plaquette average or the like jumps as a function of the coupling; a **critical endpoint** is the point where a line of first-order transitions ends. A **flat connection** (a flat configuration, as 21-07 calls it) is a configuration in which the holonomy of every face equals the identity (or the sign of the twist when there is a twist), and a **frozen phase** is the phase of a gauge theory with a finite group, on the large-β side, that clings to the neighbourhood of flat connections.

Table 1 Living ideas (none of them crosses the band. "Living" means that there is not yet a reason to break it)

Name	Content	Reason and range checked
Seven test opponents	Test arguments about the band not only against weak-coupling U(1) but against seven opponents	With U(1) alone, the types of argument that can be knocked out are coarse. Finite non-abelian groups, the limit of the adjoint Higgs model, and the mixed action of SU(2) itself knock out arguments of other types. The counts and the traces of the first-order line are in 26-12 § 05 Reading ; the properties of the opponents checked in the literature are in the source table of 26-12 Cited
Splitting the core in two	The ultraviolet extension (bringing the threshold down to β_* while keeping the sum of errors finite), and the statement for the last stretch to strong coupling	§ 01, § 02. The count of the prefactor Proved here . Where the boundary is placed is Assumption
Mixing statement for a box of physical size	With the side of the box taken at a physical length C/Λ, the total variation of the law of gauge-invariant observables inside, when the frame (the links on the boundary of the box) is changed, is below a threshold, uniformly in β	Can be written independently of the group, the centre and the class of action, and is a form that should fail both in the Coulomb phase of U(1) and at a point of the endpoint type of the mixed action (if the gap closes at the endpoint; for SU(2) not checked by this book) Assumption . But it is a restatement of the mass gap (in a strong form), not a road. The direction of the input on the strong side (finite-box mixing ⇒ uniform exponential decay) is a combination of theorems of known types Assumption (not written out in the form for gauge theory)
Looking at the side of energy and the side of entropy separately	The facts specific to SU(2) in this book are facts about the action of configurations (the side of energy), and they point both ways. What the band needs is a bound that counts fluctuations and the number of arrangements from below, uniformly in scale (named by the picture of § 03: the shortfall on the side of entropy)	§ 03. Within the list we read Assumption ; the direction of variational bounds Proved here
Three classes by level of value	The quantity for a symmetric box with one twist: at weak coupling, proportional to β for U(1), constant in the frozen phase of finite non-abelian groups, of the type $\frac{3}{2}\ln\beta$ for SU(2)	Counts for finite groups Reading ; the power for SU(2) on 1^4 Proved here Reading (26-12 § 05). A distinction at the level of values; it does not cross the band

Name	Content	Reason and range checked
Stating the quantifiers explicitly	In the renormalisation comparison, where U(1) fails changes depending on whether the aspect ratio is fixed, or the width is fixed and the length extended	The form with fixed aspect ratio holds with a uniform constant even for U(1) on the weak side Proved here Reading ; extending the aspect ratio makes the constant grow (26-12 § 02)

Table 2 Broken ideas ("this type of argument does not go through, because ...")

Idea	Why it does not go through
Setting a measure-defined coupling by the ratio of the twisted to the untwisted partition function	The untwisted periodic box has a continuous family of flat connections, whose contribution acts logarithmically near the centre, so the denominator of the ratio is contaminated (the power on 1^4 in 26-12 § 05) Proved here Reading . This book's ΔF, a ratio of two twisted sectors, suffices
Crossing the band by an open-and-closed dichotomy ("if the vortex ratio in some box is below a threshold, it decays exponentially in the area from then on", plus connectedness of the set of β with decay)	Showing "the closed side is empty" takes in the whole of the running (the steps of the order of (1)), and the open-and-closed form returns to a restatement of "there is no transition" Proved here (logic)
Getting a conclusion uniform in the cutoff from a chain with a loss $K > 1$ uniform over the steps	The prefactor grows like $\xi^{\log 2\, K}$ (§ 02) Proved here . No harm as far as confinement at each β
Using "no transition on the β axis / no fixed point of the step map" in a class of effective actions	Within the class of SU(2) fundamental + adjoint mixed actions there is a line of first-order transitions and a critical endpoint (26-12 § 02) Cited Reading . That both sides of the line are confining phases is an inference from the line ending at the endpoint and connecting smoothly to the confining phase on the Wilson axis Assumption . It is a fact specific to the Wilson axis, and does not extend to the class of effective actions at step 0 of the published construction
Arguments whose only input is "non-commutative as an algebra", or only "the group is continuous and non-abelian, centre symmetry, positivity of the weights"	The former fails for finite non-abelian groups. The latter fails for the adjoint Higgs family: these are the same for every J in the family, and $J = \infty$ becomes the U(1) Wilson action Proved here , which has a non-confining phase Cited (Fröhlich–Spencer; shown mainly in the Villain approximation, with the text stating that all the results can be extended to the Wilson action; the extension relies on techniques of other papers) (26-12 § 03)
Using convexity at a single scale (of the Bakry–Émery type) at weak coupling	The curvature of the Haar measure is $O(1)$ and the Hessian of the action is $O(\beta)$, so the input at a single scale is relatively $O(1/\beta)$ and vanishes at weak coupling Proved here (a count of orders). The form that accumulates step by step is in Table 3

Idea	Why it does not go through
Getting the direction of confinement from a variational bound on the electric side (trial states)	What a trial state gives is an upper bound on the energy of the flux sector, the wrong direction (§ 03) Proved here
Taking a global log-Sobolev inequality on the torus as a gap quantity that works for every group	If the statement in the literature (not checked by this book) that the weak-coupling phase of Z_2 has a gap is right, then there the mixing between the holonomy sectors is slow and the global form is expected to fail, so it would not work as a gap quantity Assumption . Recast as local mixing with boundary conditions (spatial mixing) and kept in Table 1
Taking reflection positivity of the push-forward measure and a fixed block map as non-abelian-specific parts usable for the band	The push-forward identities hold just the same when the U(1) Wilson measure is pushed forward by the same map. It is a vessel, not an input that distinguishes groups Proved here
Convexity of the slab tension in the logarithm of the scale	For twisted slabs, in the semiclassical dilute region the logarithmic slope of $N^2\sigma_N$ is expected to be $11/3$, and 2 at large N, so the slope decreases Assumption (based on the picture of 26-00 Figure 2)
Using the a-theorem or trace-anomaly sum rules on the continuum side to say that the infrared is not a free photon	The value of a is consistent with decreasing from the ultraviolet (three gluons) to the infrared (one photon), so it does not exclude a free photon Assumption

Table 3 Undecidable ideas

Idea	Present state
Termwise sign inequalities in the twisted sector ($\partial\Delta F/\partial\beta_p \geq 0$ for the per-face couplings β_p)	All 48 positive for SU(2) on 1^4 Reading (26-12 § 05). For the finite group Q_8 there are points where a similar monotonicity breaks (21-01: monotonicity in β, three dimensions, no background) Reading . The lattice is a single point, so we cannot say it "holds"
A finite-box criterion for the total variation under a change of frame (on gauge-invariant observables)	The proofs of this book's Lean strong-coupling window use Dobrushin's condition on all links without fixing a gauge (a reading of the proofs of 20-02 and 20-03). A criterion that captures, outside the window, decay present only in the gauge-invariant part has no precedent in this book. In an inner region that does not touch the frame, this distinction may not be needed (item 5 of § 05, not checked by computation)
Filling the gap between physical strong coupling ($\beta \lesssim 2.2$) and the proof window ($1/19$) with finite-box criteria and certificates from the loop equations	There is a method that gets upper and lower bounds on lattice plaquette averages from the loop equations and positivity (in the large-N limit) Cited . The scale needed for small boxes of SU(2) is unknown

Idea	Present state
Seeing the generator of the heat bath as a sum of projections, and borrowing the criterion from quantum spin systems "if the gap of a finite chain exceeds a threshold, there is a gap in infinite size too"	The criterion itself exists for one-dimensional translation-invariant systems (and the two-dimensional square lattice) Cited . Whether the form of the threshold fits for projections on links in four dimensions has not been checked
Using the binary icosahedral group for a finite version of the band	There is a count of flat connections Reading . The value of β at which it freezes, and whether it imitates the behaviour of SU(2), are statements in the literature (not checked by this book)
Curvature that accumulates step by step	In the hierarchical model the one-step pace $(1 - b^{-2})/3$ (0.25 for $b = 2$) comes from the curvature factor of the group Proved here (note below). Most of the $11/3$ in four dimensions is not expected to come from curvature, so this is not expected to be the core Assumption
From tension to gap, by soft arguments within a class of actions	For SU(3) there is a report that the scalar mass in lattice units drops as the endpoint of the first-order line of the mixed action is approached Cited . For SU(2) not checked by this book, and the position of the endpoint also varies in the literature (an apparent endpoint on small lattices, and a revision on large lattices; 26-12 § 02). It concerns fixed lattice spacing and says nothing about the gap of the continuum theory
Using "the rate is uniformly positive" in a class of effective actions	Across a first-order line the tension may only jump and stay positive, so the first-order line of the mixed action does not break this. What would break it is the gap closing at the endpoint, which for SU(2) is not checked by this book
With pure gauge fields and a local action, the 't Hooft loop cannot have an area law (so for SU(2) the gap and confinement are hard to separate)	A conjecture Assumption
The trajectory of renormalised actions passes far from the endpoint of the mixed action	A statement in the literature (not checked by this book)
Vortex insertions adapted to the configuration, with relative entropy tending to 0 (a lower bound on the side of entropy)	Only a draft of the form. The first hurdle is to reproduce the one-loop $11/3$ Assumption
A monotonicity formula with the trace anomaly as its deficit	Only a draft of the form Assumption

Note (one step of curvature accumulating step by step) Proved here . Approximating the SU(2) plaquette weight by the heat kernel $K_t(\theta)$ (θ the angle of the plaquette holonomy), for small t one has $K_t(\theta) \propto \frac{\theta}{\sin\theta}\, e^{-\theta^2/(4t)}$. Bundling n plaquettes (the n-th power of the weight) gives $K_t^n \propto (\theta/\sin\theta)^{n-1} K_{t/n}$, and putting $\ln(\theta/\sin\theta) \approx \theta^2/6$ into the exponent, the Wilson β ($\beta = 1/(2t)$ from $1 - \cos\theta \approx \theta^2/2$) becomes $n\beta - (n-1)/3$.

In the four-dimensional hierarchy with $b = 2$ (bundling $n = b^2 = 4$ plaquettes, and convolving b^2 plaquettes within a face, $t \to b^2 t$), one step gives $\beta \mapsto \beta - (1 - b^{-2})/3 = \beta - 0.25$. We read its closeness to the one-loop step $s \ln 2 \approx 0.2575$ as a coincidence Assumption .

05

The smallest next step

We write it in the form of questions. None of them has been answered.

1. **Filling in the sieve table.** For each "—" cell of 26-12 Table 2: in that opponent model, is the statement true, false, or undefined? Can the candidates for the core be re-selected by the criterion of keeping only statements that are false for weak-coupling U(1) and also false at a point of the endpoint type of the mixed action (if the gap closes at the endpoint; for SU(2) not checked by this book)?
2. **The minimal form of the ultraviolet extension.** In a box with background twist $n_{12} = 1$, does $\Delta F(N, M, \beta)$ with the ratio of the side of the box to the correlation length fixed converge as $\beta \to \infty$? The first step is reading: in the published construction's proof of convergence of the whole sequence in a periodic box, where does it depend on the boundary conditions?
3. **Testing the sign inequality.** For SU(2), background $n_{12} = 1$ and non-negative per-face couplings, is $\partial \Delta F / \partial \beta_p \geq 0$? After 1^4, elongated boxes (because the breaking for finite groups appears in geometries long in the transverse direction), then the closed formula in two dimensions and 2^4. Lift the smallest example that breaks for Q_8 to SU(2), and see where the breaking disappears.
4. **The additive comparison on the weak side.** Can $\Delta F(2L, \beta) \leq \Delta F(L, \beta - c')$ $(0 < c' < 0.2575)$ for $\beta \geq \beta_0$, $L \geq N_1(\beta)$ (N_1 of order $\sqrt{\beta}$) be written as a proposition at the one-loop level? The margin per doubling is the 2.54 of (4).
5. **A model case of a finite-box criterion on the strong side** Assumption (not checked by computation). In a two-dimensional SU(2) box, is the total variation, under a change of frame, of the law of an inner region not touching the frame bounded for all β by $C(\beta)\,(I_2(\beta)/I_1(\beta))^n$, with n the number of plaquettes in between (I_ν the modified Bessel functions)? And when the law of the inner links is invariant under gauge transformations at the inner points, is its total variation the same whether measured on all links or on the gauge-invariant part? Both are proposals not checked within this book's study; if checked, they would likely give the smallest model case of "the box criterion works even at β where the single-link criterion fails".

6. **A single-plaquette ruler** Assumption (not checked by computation). Measuring disorder by the ratio of fundamental-representation coefficients of a single plaquette, SU(2)'s $I_2(\beta)/I_1(\beta)$ matches U(1)'s value at its transition point, $I_1(1.01)/I_0(1.01) \approx 0.450$, at $\beta \approx 2.105$ (a numerical value of a ratio of Bessel functions, recomputed with this book's script). Whether it is meaningful to compare the two groups with this ruler, and whether its closeness to where SU(2) crosses over is not a coincidence, has not been checked.

06

Mathematics that seems to be needed

All of these are this book's conjectures Assumption . The names were chosen to describe the content.

- **Control of the continuum limit in twisted boxes.** Comparisons of scales, with a finite sum of errors, for quantities fixed by the measure and in twisted (framed) boxes. An extension of the type of the published construction, and the tool for the first job of § 01.
- **Coupling the laws of gauge-invariant observables.** A way of coupling conditional laws of gauge-invariant functions rather than link by link, and a finite-box criterion of Dobrushin–Shlosman type on top of it.
- **A method for computing lower bounds on the side of entropy.** Insertions adapted to the configuration, with relative entropy tending to 0. The first hurdle is to reproduce the one-loop $11/3$. A tool to fill the shortfall on the side of entropy of § 03.
- **Orderings of actions in the twisted sector.** A non-abelian inequality of Ginibre type saying that ΔF increases with the per-face couplings, or an ordering between actions that would take its place.
- **Expansion-free tools where the coupling is of order 1.** The combination of finite-box criteria and computational certificates.
- **A monotonicity formula with the trace anomaly as its deficit.** A quantum version of the classical monotonicity that becomes exactly trivial in four dimensions.
- **A stand-in for the vortex free energy for groups without centre.** The response to boundary fields and the like. For G_2 the tools of vortices cannot be used (26-12 § 02).

07

What this chapter does not say

- It does not give a road across the band. None of the ideas crosses the band. There is no proof.
- The number of doublings and the ratio of scales in (1) are rough orders of magnitude, not quantitative claims. They were computed from the lower bound that the threshold is at least 9×10^5.
- It does not say that "the remaining question is a question about the lattice theory at lattice spacing 1" (16-03 § 05).
- The discussion of the endpoint of the mixed action concerns fixed lattice spacing; it does not mean that the continuum SU(2) theory has no mass gap.
- The enumeration of the facts specific to SU(2) in this book (§ 03) is within what we read of the list of Lean theorems; it was not checked by carrying the proofs over to U(1).
- "Broken" means that arguments of that type do not go through; it does not mean that the goal itself is false.
- It does not say that the observations written here are new.

What this chapter has established

Established	Not established
Lean The strong end: the infinite-volume limit, the mass gap and the area law in the strong-coupling window (20-06). Facts specific to SU(2): the vacuum with a twist in one direction, its stabiliser ± 1, no configuration of action 0 in a box with two twists, $S_N > 0$, $S_N < 2N^2$ ($N = 2, 3, 4$) (21-05, 21-07)	Outside the window. The same statements in a class of effective actions. A rate uniform in the transverse size
Proved here The division of (1) (as a rough order of magnitude). (3): the prefactor is exactly $K^{k0} = \xi^{\log 2\, K}$, the rate σ_s/ξ^2 does not depend on K, and uniformity needs $\sum \ln K_i < \infty$. The one-loop 2.54 per doubling of (4). The direction of variational bounds. The hierarchical step 0.25	The quantitative meaning of (1). That (2) holds. The existence of a comparison with a finite sum of losses
Reading The recorded decrease $4^4 \to 8^4$: 1.61, 1.88, 1.80. The minimal sector action 7.84, 9.80. Cited The literature rows of Tables 2 and 3 (the source table of 26-12)	Conclusions about large lattices from values on small lattices
Assumption The two-way split of the core, the shortfall on the side of entropy, the division into Tables 1–3, § 05 and § 06	A road across the band

Sources and reproduction

Item	Kind	Source, tool
Theorems at the strong end, facts specific to SU(2) (§ 03)	Lean	The declarations of 20-03, 20-04, 20-06, 21-05, 21-07 (`InfMain2.wilson_infvol_main2`, `AreaLaw.area_law_wilson`, `AreaLaw.area_law_infvol`, `ThetaLimit.flat_exists_10`, `TwistFlat.no_flat_twist11`, `TwistBlocks.stab_vacT`, `TwistBlocks.rigidT_all`, `VortexPos.SN_pos`, `VortexCert.SN_two_lt`, `TwistBlocks.thick_vortex_floor_two` and others). The statements were checked against A-04 Table 4g and the pages 21-05, 21-07 and 20-06
Lower bound on the threshold of the published construction	Proved here	16-03 § 01 (from one of the conditions imposed on the threshold; a floating-point computation)
The division of (1)	Proved here	A script of this book's study (floating point; rerun for this chapter)
Differences of the recorded ΔF, minimal sector action	Reading	26-09 Tables 8 and 10, 25-02 Table 1
The non-confining phase of the U(1) Wilson action	Cited	J. Fröhlich, T. Spencer, “Massless phases and symmetry restoration in abelian gauge theories and spin systems”, Commun. Math. Phys. 83 (1982) 411 (checked against the abstract and § 1 of the preprint IHES/P/81/40; shown mainly in the Villain approximation, stating that all the results can be extended to models with the Wilson action; the extension relies on techniques of other papers)
Bounds on plaquette averages from the loop equations and positivity	Cited	V. Kazakov, Z. Zheng, arXiv:2203.11360 (the large-N limit, $D = 2, 3, 4$; checked against the abstract)
Threshold for the gap of finite chains	Cited	D. Gosset, E. Mozgunov, “Local gap threshold for frustration-free spin systems”, J. Math. Phys. 57 (2016) 091901, arXiv:1512.00088 (checked against the abstract; one dimension and the two-dimensional square lattice. The earlier threshold is due to Knabe, according to the same source)
The first-order line and endpoint of the mixed action, the revision of the endpoint's position, the scalar mass near the endpoint (SU(3))	Cited	G. Bhanot, M. Creutz, Phys. Rev. D 24 (1981) 3212. R. V. Gavai, Nucl. Phys. B 474 (1996) 446. U. M. Heller, Phys. Lett. B 362 (1995) 123 (for details, see the sources of 26-12)

Item	Kind	Source, tool
Statements in the literature in Tables 1–3 that have not been checked (the gap in the weak-coupling phase of Z_2, the freezing β of the binary icosahedral group, the position of the renormalised trajectory, the scalar mass at the SU(2) endpoint, the equivalence of Stroock–Zegarlinski type)	—	Statements in the literature (not checked by this book). Not used as grounds for claims in the text

Words used in this chapter — the crossover band, the two-way split of the core, the ultraviolet extension, the statement for the last stretch to strong coupling, the lossy chain, the prefactor, facts on the side of energy, the shortfall on the side of entropy, the mixing statement for a box of physical size, the seven test opponents, the three classes by level of value, mixed action, first-order transition, critical endpoint, flat connection, frozen phase, control of the continuum limit in twisted boxes, coupling the laws of gauge-invariant observables. Their meanings are in A-02 Glossary.

Next chapter to read: 17-01 How physics sees it. The table of opponents against which arguments about the band are tested is in 26-12 (after reading Part 26).