Volume 3 Toward the mass gap — the questions that remain, and this book's results

# Part 18 A dictionary of two languages

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

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# How to read the dictionary — grading in two columns, and naming who judges "the same"

This chapter is a guide (how to read the dictionary).

**What this chapter shows** — The two grades of each row of the dictionary: (a) the correspondence as definitions, and (b) existence in four-dimensional Yang–Mills. That the inconsistencies of a one-column grade are resolved by two columns (Figure 1).
Who judges "the same" (theorem, definition, convention, conjecture, analogy), and how to read the Conditions entry (Figure 2).

**Prerequisite chapters** — 0-03 Reading the problem statement word by word. § 03 and § 08 are easier to read after 4-10 and 4-11. Words not in 0-03 get a one-phrase explanation where they first appear, and the details are left to the linked chapters. Within Volume 3, this part is read after Part 17 (From the side of physics).

**Said first** — This chapter only fixes how to read. The grades in Table 2 are this book's reading, and the grounds are in 18-02–18-04. This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is read in Volume 2).

The order of this chapter

1. Two vocabularies pointing at the same thing
2. The entries of a row — nine entries (including the measurement column)
3. The definition column: the correspondence as definitions — Settled, Conditional, Unsettled
4. The existence column: existence in four dimensions — Constructed, Conditional, None (not yet constructed)
5. Two columns resolve the inconsistencies — Figure 1, Table 2
6. Which vocabulary to put first
   - Detour: false friends
7. Who judges "the same" — five judges
8. How to read the Conditions entry — Figure 2, Table 3

9. What comes along if the existence column moves

01

## Two vocabularies pointing at the same thing — why a dictionary is needed

The "path integral", "vacuum" and "glueball" of physics are assigned in mathematics to "a probability measure", "a unit vector in the reconstructed Hilbert space (1-10)" and "an isolated eigenvalue of the mass operator $M = \sqrt{H^2 - P^2}$ (the operator that reads the rest mass from the energy $H$ and the momentum $P$)". Whether words of different origins point at the same thing, and how certain the correspondence is, differs from row to row, and can only be checked row by row. That is the subject of this chapter.

There are three answers: it can be said by a theorem to be the same, it can be said only under assumptions, or it cannot be said. Moreover, in four dimensions the object on the side of mathematics has often not yet been made (0-03 § 04). So we give a row the following form.

$$\underbrace{\text{physical word}}_{\text{origin}} \longleftrightarrow \underbrace{\text{mathematical word}}_{\text{origin}} \qquad [\ (a)\ \text{correspondence as definitions}\ ;\ (b)\ \text{existen}$$

$\longleftrightarrow$ only places the two side by side as a pair; whether they are the same is stated by the two grades. Example: "instanton $\longleftrightarrow$ anti-self-dual connection" is (a) Settled, (b) Constructed (classical). For classical fields there is the precedent of Wu–Yang (1975) (abstract: 'identification with the mathematical concept of connections'; 1-07). An attempt to read the assumptions of this book's record in the words of physics is in 18-05.

02

## The entries of a row — nine entries and a sample

There are nine entries: concept, the physical word and its origin, the mathematical word and its origin, the definition column, the existence column, the measurement column, judge, conditions (the assumptions needed for the identification and the gap that remains), and chapter.

The sample is the Feynman–Kac formula. The imaginary-time evolution of $H = -\frac{1}{2}\Delta + V$ ($\Delta$ is the Laplacian, not the $\Delta$ of the mass gap) can be written as an expectation over Brownian motion $B_s$ started from $x$ (a random path made by piling up independent small Gaussian displacements; 1-08) Known .

$$(e^{-tH}f)(x) = \mathbb{E}_x\Big[\exp\Big(-\int_0^t V(B_s)\,ds\Big) f(B_t)\Big] \quad (2)$$

Filling in the entries: the concept is "writing $e^{-tH}$ as an integral over paths", the physical word is the path integral (Feynman 1948), the mathematical word is the Feynman–Kac formula (Kac 1949), (a) Settled, (b) Not applicable (a row of quantum mechanics with finitely many degrees of freedom, so not asked; § 04), the judge is a theorem, the conditions are "$V$ bounded below, etc.; the real-time $e^{iS}$ does not become a measure", and the chapter is 2-07. Two kinds of thing go into the Conditions entry. "$V$ bounded below, etc." is a hypothesis of the theorem, and "the real-time $e^{iS}$ does not become a measure" is a gap that remains even after the identification. Among the arrows, $\Longleftrightarrow$ is a theorem in both directions and $\Longrightarrow$ a theorem in one direction, while $\longleftrightarrow$, as in (1), only places the two side by side as a pair.

**The measurement column: the direction reachable by finite measurement**. Besides (a) and (b) we attach one more grade: for a quantity on the side of mathematics, which direction follows as a firm inequality from numbers measured with finitely many observables, finite time $t$ and finitely many lattices. There are four values: **Upper bound only**, **Lower bound only**, **Both** and **Out of reach** (statements about limits, uniformity or all observables, which finite measurement settles in neither direction). Rows that name no quantity (rows that are only a correspondence of definitions or theorems) get "—". The grounds for the judgement are (8) and (9) of § 08. A sum of exponentials with weights $w_n \geq 0$ is log-convex, so the effective mass $m_{\mathrm{eff}}(t)$ decreases in $t$ and approaches $\mu_O$ from above, and at every finite $t$, $m \leq \mu_O \leq m_{\mathrm{eff}}(t)$ Known ((3) in 4-11). So the gap and its relatives (positions of eigenvalues, decay rates) fall under "Upper bound only", and the lengths and times that are their inverses (the correlation length $\xi$, the Poincaré constant $C_P$) under "Lower bound only". This is the box "The asymmetry of upper and lower bounds" in 1-10 made into an entry of the dictionary. The row of the sample, the Feynman–Kac formula, gets "—".

03

## The definition column: the correspondence as definitions — Settled, Conditional, Unsettled

The definition column asks: "**if both objects existed**, would they be the same?" Whether they exist is the job of the existence column.

| Settled | Conditional | Unsettled |
| --- | --- | --- |
| There is a theorem giving the same object, or the side of | The same only under assumptions that are in neither definition (the | They seem to point at the same thing, but no theorem says so. |

| | | |
|---|---|---|
| mathematics defines it so. | assumptions go into the Conditions entry). | |

Even when Settled, the arrow has a direction. The sample is the Osterwalder–Schrader (OS) reconstruction theorem (4-10). We call the axioms of real-time quantum field theory the Wightman axioms (4-09), and the five conditions imposed on imaginary-time correlations (Schwinger functions) the Osterwalder–Schrader axioms: regularity (E0), Euclidean covariance (E1), reflection positivity (E2; correlations with the quantity reflected in the mirror of time are not negative), permutation symmetry (E3) and clustering (E4). The **linear-growth** condition (E0′) is a strengthening of regularity, requiring the estimates of the $n$-point functions to stay within roughly factorial growth in $n$ (4-10 § 03) Known .

$$\textbf{there: Wightman theory} \implies (\mathrm{E0}) \wedge \cdots \wedge (\mathrm{E4}), \qquad \textbf{back: } (\mathrm{E0}') \wedge (\mathrm{E1}) \wedge \cdots \wedge (\mathrm{E4}) \implies \mathbf{W}$$

On the way there, analytically continuing the correlations of a Wightman theory to imaginary time, using that the energy is positive, gives the five axioms. On the way back the linear-growth condition is needed, and whether this condition follows from a Wightman theory is not known (Summers § 3, 4-10 § 03). Since the conditions differ on the way there and the way back, even though (a) is Settled we write the arrow as two $\Longrightarrow$, not as $\Longleftrightarrow$. How the construction published on 2026-11-10 used this way back to reconstruct a Wightman theory is read in 12-08 Reconstruction and the Wightman axioms.

The sample of Conditional is the mass gap and the decay of correlations. § 5 of the problem statement writes in one direction: 'An important consequence of the existence of a mass gap is clustering'. The converse needs a dense family of observables to decay at a uniform rate ( § 08).

Unsettled comes in three types. (i) No definition on the side of mathematics: the four-dimensional $\Lambda$. (ii) Definitions on both sides, but within the range we searched no theorem saying they are the same: centre symmetry (Chatterjee writes that the mathematical physics literature has no mathematical definition, and then that his definition on the lattice was inspired by the physical definition through the Polyakov loop). (iii) Correspondences inside conjectures or analogies: the radial direction of holography and the scale of renormalisation (17-08).

Read as two groups, "Settled" are pairs that **are settled to point at the same thing**, and "Conditional" and "Unsettled" are pairs that **mathematical certainty has not been able to assert**. The dictionary of mass and length (18-03 § 01) adds a fourth grade, "Does not correspond", for pairs known to be different quantities. Row 9 of Table 2 (a false friend, the detour in § 06) is graded at this level in the entry "gap in physical units" of 18-03.

04

## The existence column: existence in four dimensions — Constructed, Conditional, None

The existence column writes, with a **level**, whether the object the mathematical word points at has been made in four dimensions. There are four levels: classical (smooth connections), formal power series (each order of perturbation), lattice (spacing $a$ fixed), and continuum quantum theory (a measure and a Hilbert space on $\mathbb{R}^4$); the problem statement asks for the last (0-03 § 03).

| Constructed | Conditional | None |
|---|---|---|
| Made at the level the row names (the level is added). | Only versions of limited range (strong coupling, lattice fields satisfying an admissibility condition, and so on). | Not yet made at the named level. It has not been shown not to exist (versions at lower levels go into the Conditions entry). |

These are the three grades. For rows that are not objects of four-dimensional Yang–Mills (quantum mechanics with finitely many degrees of freedom, extra dimensions, and so on) (b) is not asked, and we write "Not applicable" (the sample in § 02).

The last rule is the key. "None" as the name of a grade means "not yet constructed"; it has not been shown not to exist. The problem asks exactly about that existence. The measure on a finite lattice exists for every $\beta$ (the coefficient of the coupling in the lattice action) (5-01), but the continuum measure is its limit, and for four-dimensional non-abelian groups it has not yet been constructed Known .

$$d\mu_a(U) = \frac{1}{Z}\, e^{-S_W(U)} \prod_{\ell} dU_\ell \quad \text{(exists)}, \qquad \mu = \lim_{a \to 0} \mu_a \quad \text{(not constructed)} \ (4)$$

$U_\ell$ is the link variable and $S_W$ the Wilson action. Example: instantons are Constructed classically (ADHM 1978), and the transfer matrix (the operator that advances imaginary time by one step on the lattice, § 08) is Constructed on the lattice (Lüscher 1977). Rows 10 and 11 of Table 2 point at the same geometry, but 10 names the classical level and 11 the lattice level. The second Chern number of a smooth connection exists classically, so the (b) of 11 is Conditional because assigning an integer to a lattice field needs an admissibility condition ($U_p$ close to 1 on each plaquette). On the strongly coupled lattice an isolated one-particle state has also been shown (Schor 1984), but at the lattice level, and on the opposite side from the continuum limit (5-09).

05

## Two columns resolve the inconsistencies — fourteen samples

With a one-column grade, "are they the same as definitions?" and "do they exist in four dimensions?" are mixed into one grade, and which of the two decided the grade changes from row to row.

$$\text{one-column grade} = f\big((a),(b)\big), \qquad f \text{ chosen implicitly row by row } (5)$$

Example: rows 4 and 5 of Table 2 are the same pair, yet asking about definitions gives Conditional, and asking about a four-dimensional instance gives Unsettled. The same holds for 6 and 7, and 1 (the path integral) is Settled as a definition but Conditional in one column. In two columns these fall into the same cell. With a one-column grade that "takes the weaker", the rows of continuum quantum theory almost all line up at "None", and the correspondences of definitions become invisible.

Figure 1 The slider gives the one-column grade (left) and the two columns (right; vertical (a), horizontal (b)). The numbers are rows of Table 2; pressing a point or a row shows its grades. Dashed lines join the same pairs




The positions are the grades of Table 2 (this book's reading). With JavaScript disabled, the still figure and the table can be read.

**Table 2** Fourteen samples (details in 18-02–18-04). "One column" is the grade given without separating the two questions. (c) is the measurement column of § 02 (the direction reachable by finite measurement).

| # | Physical word ⟷ mathematical word | One column | (a) | (b) | (c) | Judge |
|---|---|---|---|---|---|---|
| 1 | Path integral ⟷ Euclidean probability measure | Conditional | Settled | None | — | Definiti |
| 2 | Wick rotation ⟷ OS reconstruction theorem | Settled | Settled (⇒) | None | — | Theorer |
| 3 | Existence of the lattice Hamiltonian ($T = e^{-aH}$ positive and self-adjoint) ⟷ reflection positivity of the Wilson action | Settled | Settled | Constructed (lattice) | — | Theorer |
| 4 | One-particle state ⟷ isolated eigenvalue of the mass operator | Conditional | Conditional | None | Upper bound only | Theorer conditic |
| 5 | Glueball ⟷ isolated eigenvalue of the mass operator | Unsettled | Conditional | None | Upper bound only | Theorer conditic convent |
| 6 | Spacetime and lattice ⟷ the $a \to 0$ limit of $a\mathbb{Z}^4$ | Unsettled | Conditional | None | Out of reach | Definiti + conditic |
| 7 | Continuum limit ⟷ scaling limit | Conditional | Conditional | None | Out of reach | Definiti + conditic |
| 8 | Mass gap ⟷ spectral gap, exponential decay of correlations | Conditional | Conditional | None | Upper bound only | Theorer conditic |
| 9 | Gap in physical units ⟷ uniform gap in lattice units | Unsettled | Unsettled (false friend) | Conditional (strong coupling, etc.) | Out of reach | Convent |
| 10 | Instanton ⟷ anti-self-dual connection | Settled | Settled | Constructed (classical) | — | Theorer |
| 11 | Topological charge (lattice) ⟷ second Chern number of a lattice field | Conditional | Conditional | Conditional (lattice: fields with $U_p$ close to 1 on each plaquette) | — | Theorer conditic |
| 12 | Renormalisability ⟷ BPHZ, Epstein–Glaser (for YM, BRS and flow equations) | Conditional | Conditional | Constructed (formal power series) | — | Theorer (for YM see the note below) |

| # | Physical word ⟷ mathematical word | One column | (a) | (b) | (c) | Judge |
|---|---|---|---|---|---|---|
| 13 | $\Lambda$, "a single scale" ⟷ the diffusive-relaxation-time assumption: $C_P \asymp \xi^2$ | Unsettled | Unsettled | None | Lower bound only ($C_P$; the relaxation-time upper-bound assumption is out of reach) | Analogy |
| 14 | Radial direction of holography ⟷ Fefferman–Graham expansion | Unsettled | Unsettled | None | — | Conject |

Note: column (b) of Table 2 is the reading from before the published construction (2026-11-10). Rows 2, 6 and 7 moved with the published construction (6-01, §09).

Notes on the words of Table 2. Row 11 names the level of the lattice ( § 04). The "theorem" in the judge of row 12 is Hepp 1966 for scalar fields, Becchi–Rouet–Stora 1976 for gauge theories (for a semisimple group, with a choice of gauge in which every field gets a mass, showing the identities of gauge symmetry to all orders of perturbation), and Efremov–Guida–Kopper 2017 for massless $SU(2)$ pure Yang–Mills (by the flow equations of the renormalisation group). The source for pure Yang–Mills with a general compact group has not been confirmed. In row 13, $\Lambda$ is the mass scale the theory produces by itself (4-07), $C_P$ is the Poincaré constant (how slowly configurations return to equilibrium under the stochastic gradient flow, that is, Langevin dynamics; 1-10, 18-03), $\xi$ is the correlation length (the length over which correlations decrease as $e^{-r/\xi}$, the inverse of the gap in lattice units), and the diffusive-relaxation-time assumption is the assumption $C_P \asymp \xi^2$ made by the record of this book's study (that the two are proportional within constant factors; 23-01). The Fefferman–Graham expansion of row 14 is the formula expanding the interior metric in powers of the depth from the boundary (18-04).

06

## Which vocabulary to put first — look up by the physical word, judge by the mathematical statement

The dictionary is ordered so that it can be looked up from the physical words that many readers know first, and the grades are judged after translating into mathematical statements (17-13 § 01). When a physical word has two meanings, the row is split. The criterion for splitting is whether the grade in the definition column differs between the two meanings. If it is the same, they stay in one row; if it differs, they are split. The sample is "unitarity" Known .

| | | |
|---|---|---|
| positive norm | $\Longleftrightarrow$ reflection positivity | (Settled; within tl |
| positive norm $\wedge$ asymptotic completeness | $\Longrightarrow$ unitarity of the S-matrix | (Conditional; und |

Positive norm means that the inner product of states is positive definite (there are no states of negative norm). Asymptotic completeness means that the Hilbert space is exhausted by states of free particles in the far past and far future (asymptotic states) (18-02 § 05). The upper row is a statement within the correspondence between OS and Wightman: from reflection positivity come a positive-definite Hilbert space and $H \geq 0$ (reconstruction). The converse is the way-there direction of (3): the positivity of a Wightman theory becomes the reflection positivity of the Schwinger functions (4-10). The lower row is a statement under the Haag–Ruelle scattering theory, which builds asymptotic states by assuming an isolated one-particle mass shell.

Put into one row, the Settled grade would read as if it extended to the S-matrix. "Confinement" is another example of a split row. The three criteria are $V(R) \to \infty$ ($V(R)$ is the energy of a heavy quark pair placed at distance $R$, read from how Wilson loops decrease), the area law $\sigma > 0$ ($\sigma = \lim_{R\to\infty} V(R)/R$ is the string tension), and unbroken centre symmetry (invariance under transformations by the centre of the group; in physics, the expectation of the Polyakov loop at finite temperature is 0). In physics they mean almost the same thing, but the mathematical theorems are one-directional and on the lattice. Chatterjee's Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) shows, on the lattice $\mathbb{Z}^d$ at fixed coupling, "centre symmetry unbroken $\Rightarrow V(R) \to \infty$" (for Wilson loops in representations acting non-trivially on the centre; 18-03 § 03).

### Detour: false friends

A subsection you may skip.

In learning foreign languages, words that look alike but differ in meaning are called "false friends". The typical one is "gap". A uniform gap in lattice units is, in physical units, a property of the side where the mass goes to infinity, the opposite side from the continuum limit, so we make the definition column Unsettled and write "false friend" in the Conditions entry (row 9 of Table 2). Precisely, the same word points in the two languages (physics and mathematics) at objects with different conditions, and one holding does not imply the other holding (it lies rather on the opposite side). "Continuum limit" is not quite a false friend, but it is an example where the conditions included differ. The scaling limit of probability theory does not include "the limit is not Gaussian", which the physical word implicitly includes (for the four-dimensional Ising type and $\phi^4$ it is Gaussian: Aizenman–Duminil-Copin 2021, 7-08). This difference goes into the Conditions entry of row 7 of Table 2. For the difference between "the mass of a particle" and the gap, see 16-08 and 16-05.

07

## Who judges "the same" — five judges

**Theorem**: there is a theorem in both directions or in one ((2), (3), (6)). **Definition**: the side of mathematics has fixed an object as the meaning of the physical word. Fixing the path integral as a countably additive probability measure is such a case. § 4 of Jaffe's review lists as a "problem" of the time that the path integral had no definition as 'a true integral, given by a countably-additive Borel measure'. This way of fixing it does not go as far as agreement with other calculations. **Convention**: the habit of calling things by the same name. Calling the lowest state measured in each channel on the lattice a "glueball", and calling uniform exponential decay in lattice units a 'strictly positive mass gap' (Shen–Zhu–Zhu). **Conjecture**: for the correspondence between the radial direction and scale, the side of physics too writes that 'what cutoff on the field theory corresponds to a radial cutoff in the bulk?' is 'left unanswered' (Heemskerk–Polchinski). **Analogy**: pointing out a similarity of form. "A single scale" and the relaxation-time assumption (17-12).

$$(a) = \text{Settled} \;\Rightarrow\; \text{the judge includes a theorem or a definition,} \qquad \text{judge only convention, cc}$$

Example: row 5 of Table 2 is "Theorem + conditions, convention". The correspondence with an isolated eigenvalue is a conditional theorem (1-10), and calling the lowest state a "particle" is a convention. When they coexist we write both, and grade the definition column by the stronger.

08

## How to read the Conditions entry — a single correlation gives only an upper bound on the gap

The Conditions entry is where the hypotheses of theorems are kept. We read row 8 of Table 2 with a small model. The transfer matrix is the operator $T = e^{-H}$ that advances imaginary time by one step on the lattice (taking the step as 1; 4-10, 5-03). Let its levels be $0 = E_0 < E_1 < E_2 < \cdots$; the state $\Omega$ with $E_0 = 0$ is the vacuum. The correlation at imaginary-time separation $t$ can be written with the levels and the overlaps $w_n \geq 0$ of the observable $O$ with each level (if the levels are discrete) Known .

$$C_O(t) = \langle O(t)O(0)\rangle - \langle O\rangle^2 = \sum_{n\geq 1} w_n e^{-E_n t}, \qquad w_n = \left|\langle n|O|\Omega\rangle\right|^2 \quad (8)$$

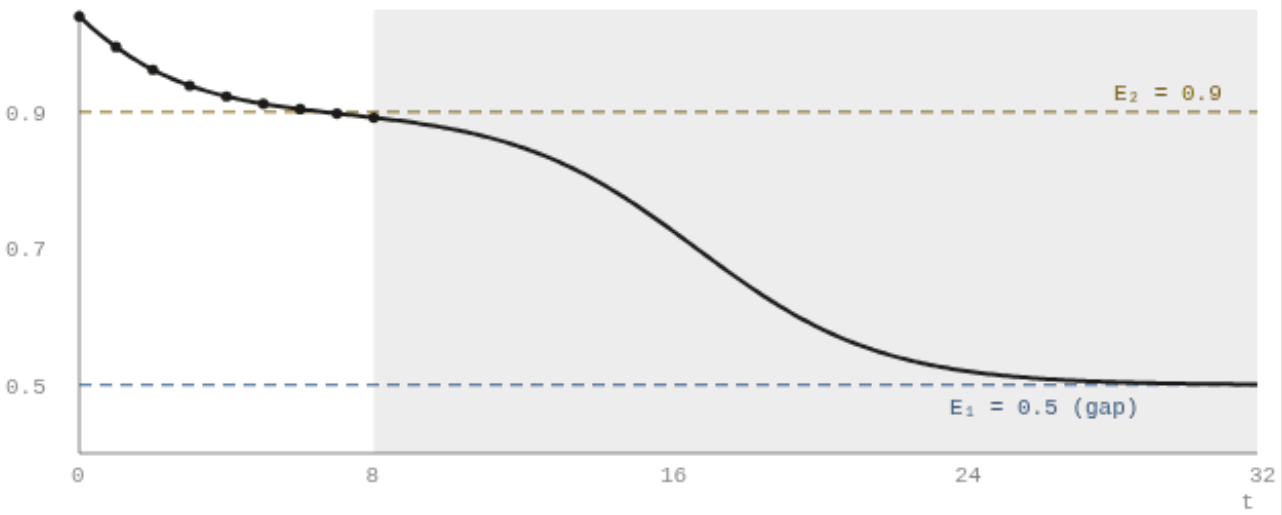
The gap is $m = E_1$. The rate at which the correlation decreases is fixed by the first level with $w_n > 0$.

$$\mu_O = \min\{E_n : w_n > 0\} \;\geq\; m, \qquad m_{\text{eff}}(t) = \log \frac{C_O(t)}{C_O(t+1)} \;\xrightarrow[t\to\infty]{}\; \mu_O \quad (9)$$

What a single correlation can say is only the upper bound $m \leq \mu_O$. One observable suffices for the upper bound, and equality needs $w_1 > 0$. The lower bound $m \geq c$ needs every member of a family of observables dense in the part orthogonal to the vacuum to decrease at rate at least $c$ (uniformity). Which observables overlap with $E_1$ is not known in advance, so a family is needed. This is the content of the conditions of row 8 (4-11 § 02, 18-03. Easy from above and hard from below is the same shape as the variational principle: the box "The asymmetry of upper and lower bounds" in 1-10). The measurement column of the dictionary ( § 02) attaches this asymmetry to each row.

Example: if $E = (0.5, 0.9, 1.4)$, $w_2 = 1$, $w_3 = 0.5$ and $w_1 = 10^{-3}$, then $m_{\text{eff}}(8) = 0.892$, 1.78 times the gap, and it comes within 5% only from $t = 24$ Computed . For $t \leq 8$ the difference between $w_1 = 10^{-3}$ and $w_1 = 0$ is at most 1.3% Computed ; at finite $t$ a small overlap cannot be told apart from no overlap.

Figure 2 Changing the overlap $w_1$ and the measured range t_max changes the effective mass (line) and the values that can be read (points). Dashed lines are $E_1$ (the gap) and $E_2$; the shading is the unmeasured range




w₁ = 1.0e-3  m_eff(8) = 0.892 (1.78 times the gap), within 5%: t = 24 onwards

Computed The numbers in this figure were computed on this machine ((8) and (9) computed on the spot).

Computed **Table 3** $m_{\text{eff}}(t)$ (with the same $E$, $w_2$ and $w_3$ as above). The right end is the first integer $t$ with $m_{\text{eff}} \leq 1.05 E_1$.

| $w_1$ | $t = 1$ | 8 | 16 | 32 | Within 5% |
|---|---|---|---|---|---|
| 1 | 0.7013 | 0.5132 | 0.5005 | 0.5000 | 7 |
| 0.1 | 0.9322 | 0.6018 | 0.5054 | 0.5000 | 13 |
| 0.01 | 0.9888 | 0.8108 | 0.5481 | 0.5001 | 18 |
| 0.001 | 0.9953 | 0.8918 | 0.7305 | 0.5009 | 24 |

| $w_1$ | $t = 1$ | 8 | 16 | 32 | Within 5% |
|---|---|---|---|---|---|
| 0 | 0.9960 | 0.9036 | 0.9001 | 0.9000 | – |

09

## What comes along if the existence column moves — reading the problem from the dictionary

What the problem asks for is to move the existence column from "None" to "Constructed" at the level of continuum quantum theory. "None" here means, as defined in § 04, not that it has been shown not to exist, but that it has not yet been made. What then comes along is shown by the definition column.

$$(b):\ \text{None} \to \text{Constructed} \ \text{ and } \ (a) = \text{Settled} \implies \text{the row can be used as it stands}$$ (10)

In Table 2 the existence column is "None" in 9 rows, and the "Constructed" rows 3, 10 and 12 are also at levels below continuum quantum theory. If a four-dimensional measure satisfying the conditions of the way back in (3), $(\mathrm{E0}')$–$(\mathrm{E4})$, were made, the Settled rows 1 and 2 could be used as they stand, and the reconstruction theorem would give the Hilbert space, the Hamiltonian and the vacuum. For the Conditional rows 4 and 5 the upper gap ( § 5 of the problem statement counts 'an upper gap, in addition to the mass gap' as an extension), and for row 8 a uniform decay rate, that is the gap itself, must be checked separately. The Unsettled rows 13 and 14 and the false friend of row 9 are not settled even when the measure is made. In this book's reading, this is also why existence and the gap stand side by side in one sentence of the problem statement (0-03 § 01).

The construction published on 2026-11-10 moved part of this to the level of continuum quantum theory (the committee's assessment is that it is proved in Lean): a Wightman theory for each component of the phases, with its Hilbert space, Hamiltonian and vacuum, and local operators corresponding to polynomials in the curvature (6-01). The probability measure (row 1 of Table 2), one-particle states (4, 5) and uniform decay (8) remain None (6-04). The existence columns of Table 2 and of 18-02 to 18-04 are left as the reading from before publication.

## What this chapter can say

| | What is said | What is not said |
|---|---|---|
| Known | (2), the way there and the way back of (3), (6) (under the premises attached), (8), (9). The constructions of ADHM, Lüscher 1977 and Schor 1984 | That these hold in four-dimensional continuum Yang–Mills theory |
| This book's reading | How the two-column grades, levels and judges are fixed, how the measurement column is fixed, (7) and (10), the grades of Table 2 | That the grades are unique (the grounds are in 18-02–18-04) |
| Computed | $m_{\mathrm{eff}}(t)$ of the three-level model (Table 3, Figure 2) | The actual spectrum of Yang–Mills |
| Overall | — | This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The problem statement, the definitions in § 07 | Text | A. Jaffe, E. Witten, “Quantum Yang–Mills Theory” (the official problem statement of the Clay Mathematics Institute; the problem was announced in 2000). In: J. Carlson, A. Jaffe, A. Wiles (eds.), *The Millennium Prize Problems*, CMI / AMS, pp. 129–152 (year of publication not checked) / A. Jaffe, “Constructive Quantum Field Theory”, in *Mathematical Physics 2000* (Imperial College Press, 2000) 111–127, § 4 “Quantum Theory as Statistical Physics” (venue checked in Crossref) |
| (2), row 12 of Table 2 | Bibliographic data only | Feynman, RMP 20 (1948) 367 / Kac, Trans. AMS 65 (1949) 1 / 't Hooft, NPB 33 (1971) 173 / Hepp, CMP 2 (1966) 301 |
| Row 12 of Table 2 (proofs for YM) | Abstract | C. Becchi, A. Rouet, R. Stora, “Renormalization of gauge theories”, Ann. Phys. 98 (1976) 287–321 / A. N. Efremov, R. Guida, C. Kopper, “Renormalization of SU(2) Yang–Mills theory with flow equations”, J. Math. Phys. 58 (2017) 093503, arXiv:1704.06799 |
| (3) | Abstract, text | Osterwalder–Schrader, CMP 31 (1973) 83, CMP 42 (1975) 281 (abstracts; the abstract of the second paper puts necessity in parentheses: '(necessary and) sufficient') / Glaser, CMP 37 (1974) 257 (abstract: points out and corrects an error in the first paper, and suggests that temperedness near Minkowski points needs an additional assumption) / S. J. Summers, “A Perspective on Constructive Quantum Field Theory”, arXiv:1203.3991, § 3 and footnote 15 (text) / A. Jaffe, “Reflection Positivity Then and Now”, arXiv:1802.07880 (text) |

| Item | Kind | Source / tools |
|---|---|---|
| (6) | Text, bibliographic data only | Osterwalder–Schrader 1973/75 (above) / Jaffe's review (above) § 4, § 7.1 (text: that reflection positivity gives a Hilbert space and a positive Hamiltonian, and that the Haag–Ruelle theory assumes an isolated one-particle mass shell) / R. Haag, Phys. Rev. 112 (1958) 669, D. Ruelle, Helv. Phys. Acta 35 (1962) (bibliographic data only) |
| Rows 3, 4 and 5 of Table 2 | Abstract | Lüscher, CMP 54 (1977) 283 / Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 / Schor, NPB 231 (1984) 321 |
| 6, 7, the definition of centre symmetry, the Theorem from Centre Symmetry to Confinement (Theorem 2.2) | Text | Chatterjee, arXiv:1803.01950 § 5 / arXiv:2006.16229 (CMP 385 (2021) 1007) |
| 9, 10, 11, 14, § 07 | Abstract | Shen–Zhu–Zhu, arXiv:2204.12737 / ADHM, Phys. Lett. A 65 (1978) 185 / Lüscher, CMP 85 (1982) 39 / Heemskerk–Polchinski, arXiv:1010.1264 |
| § 01, detour | Abstract | Wu–Yang, PRD 12 (1975) 3845 / Aizenman–Duminil-Copin, Ann. Math. 194 (2021) 163 |
| Table 2, Figure 1 | This book's reading | The primary sources are in 18-02–18-04. The layout is `fig1_layout.py` |
| Table 3, Figure 2 | Computed | Python on this machine (`meff.py`) |

Next chapter to read: 18-02 A dictionary of the theory itself.

# A dictionary of the theory itself — the Settled pairs line up in front of a single measure

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — Nineteen pairs of words for the space, the measure and the states on which the theory rests, with their two-column grades and the grounds for them.
That many of the Settled pairs lie inside a single premise: "if a measure satisfying the conditions exists".
A calculation that sees that local symmetry does not break (Elitzur's theorem) through the order in which the external field is removed.

**Prerequisite chapters** — 4-02 and 4-10. The two columns of grades are from 18-01. We also draw on tools from 4-11 (particles, spectrum), 5-03 (the transfer matrix), 3-04 (orbit space, Singer's theorem), 2-08 (the order of removing the external field) and 5-01 (Haar measure). Words not in the prerequisite chapters are defined where they first appear.

**Said first** — We only line up pairs and grade them; we make neither the measure nor the Hilbert space of four-dimensional Yang–Mills theory. "None" in the existence column means, as in 18-01 § 04, "not yet made"; it has not been shown not to exist. The exception is the entry "gauge fixing": that there is no global continuous gauge fixing (on the four-dimensional sphere) is Singer's theorem ("ruled out" in Table 0).

The order of this chapter

1. The overall picture — nineteen pairs and one premise (Table 0)
2. The container — the path integral, the measure, the space of states
3. Real time and imaginary time — three faces of a correlation
4. The contents — the vacuum, the Hamiltonian, particles
5. Scattering and positivity — two kinds of "unitarity"
6. Approximation — spacetime, regularisation, renormalisation
7. Fields have no values at points — operator-valued distributions

8. Gauge — slices and physical states
9. Local symmetry does not break — Elitzur's theorem (the entry "local symmetry does not break")
    - Detour: gauge theory at $\beta = \infty$ is the Ising model

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01

## The overall picture — nineteen pairs and one premise

The grades are the two columns of 18-01. (a) is "if both existed, would they be the same?", and (b) is whether the mathematical object exists in four-dimensional Yang–Mills theory (level in parentheses). The judge is what supports "the same" (theorem, definition, convention, and so on; 18-01 § 07), and the chapter is the chapter that treats it in detail.

Table 0 A dictionary of the theory itself

| # | Physical word | Mathematical word | (a) Definitions | (b) 4D YM | (c) Finite measurement | Jud |
|---|---|---|---|---|---|---|
| 1 | Path integral (quantum mechanics) | Feynman–Kac formula | Settled | Not applicable* | — | The |
| 2 | Functional integral | Euclidean probability measure | Settled | None (Constructed at the lattice level) | — | Def |
| 3 | Space of states | Reconstructed Hilbert space | Conditional | None (Constructed at the lattice level) | — | The cor |
| 4 | Wick rotation | OS reconstruction theorem | Settled (⇒) | None | — | The |
| 5 | Green functions | Wightman and Schwinger functions | Settled (with direction) | None | — | The |
| 6 | Källén–Lehmann representation | Spectral measure | Settled | None (Constructed at the lattice level) | Upper bound only | The |

| # | Physical word | Mathematical word | (a) Definitions | (b) 4D YM | (c) Finite measurement | Jud |
|---|---|---|---|---|---|---|
| 7 | Vacuum and uniqueness | $\Omega$ and clustering | Settled | None (Constructed at the lattice level) | Out of reach | The |
| 8 | Hamiltonian | Logarithm of the transfer matrix | Conditional | Constructed (lattice) | — | The cor |
| 9 | One-particle state | Isolated eigenvalue of the mass operator | Conditional | None | Upper bound only | The cor |
| 10 | S-matrix | Haag–Ruelle scattering theory | Conditional | None | — | The cor |
| 11 | Positive norm | Reflection positivity | Settled | Constructed (lattice) | — | The |
| 12 | Unitarity of the S-matrix | Positive norm + asymptotic completeness | Conditional | None | — | The cor |
| 13 | Spacetime and lattice | $a\mathbb{Z}^4 \to \mathbb{R}^4$ | Conditional | None | Out of reach | Def + cor |
| 14 | Regularisation | Family of measures with lattice spacing $a$ | Conditional | Constructed (lattice) | — | Def + cor |
| 15 | Renormalisability | BPHZ, Epstein–Glaser | Conditional | Constructed (formal power series) | — | The |
| 16 | Field $\phi(x)$ | Operator-valued distribution | Conditional | None | — | Def + cor |
| 17 | Gauge fixing | Section of the orbit space | Settled | None (ruled out by Singer's theorem) | — | Def |
| 18 | Physical states | Reconstruction from gauge invariants | Unsettled | Constructed (lattice) | — | Cor (a t on latt |
| 19 | Local symmetry does not break | Elitzur's theorem | Settled | Constructed (lattice) | — | The |

Note: column (b) of Table 0 is the reading from before the published construction (2026-11-10). As at the end of §01, rows 3, 4, 5 and 8 leave room to be regraded at the level of the published construction.

Notes on the words of Table 0. (c) is the measurement column of 18-01 § 02 (the direction reachable by finite measurement), given only for rows that name a quantity. The entries "Källén–Lehmann representation" and "one-particle state" are rows where a single correlation gives an upper bound on the lowest point (18-01 § 08), and the entries "vacuum and uniqueness" and "spacetime and lattice" are statements about limits. "None" in (b) means not yet constructed at the continuum level, and "None (Constructed at the lattice level)" means rows made only at the lattice level. Only "ruled out" in the entry "gauge fixing" has the opposite sense: it is a theorem that a global section does not exist. * "Not applicable" in the entry "path integral (quantum mechanics)" means that, being a pair from quantum mechanics with finitely many degrees of freedom, it has no four-dimensional entry (18-01 § 04).

Many of the Settled pairs lie inside a single premise. If on $\mathbb{R}^4$ there is a measure $\mu$ satisfying the Osterwalder–Schrader (OS) conditions, the reconstruction theorem gives the rest (4-10 § 05) Known .

$$\mu \text{ satisfies } (\mathrm{E0}')\text{–}(\mathrm{E4}) \implies (\mathcal{H},\ H \geq 0,\ \Omega), \quad \{S_n\} \to \{W_n\}, \quad d\rho_O \geq 0 \tag{1}$$

On the left are the OS conditions (growth, Euclidean covariance, reflection positivity, symmetry, clustering; 4-10 § 03), and on the right the Hilbert space, Hamiltonian and vacuum, the passage from the Schwinger functions $S_n$ to the Wightman functions $W_n$, and the positivity of the spectral measure. Before the publication of 2026, the left was missing in four-dimensional Yang–Mills theory, so at the continuum level the existence column lined up almost entirely at "None" (Table 0 is that reading). The exceptions are rows for which the premise is not enough even if it holds (the entries "one-particle state", "S-matrix" and "unitarity of the S-matrix"), and rows decided outside the premise (Singer's theorem in the entry "gauge fixing", Elitzur's theorem in the entry "local symmetry does not break"). Example: for the four-dimensional free field the left of (1) exists, and the objects of every row except the gauge rows are present.

The committee's assessment is that the construction published on 2026-11-10 showed a variant of the OS axioms for each component of the phases, starting not from a measure but from the side of correlations (Schwinger functions), and reconstructed the Hilbert space, Hamiltonian and vacuum on the right (6-01, 12-08). The existence column of the entries "space of states", "Wick rotation", "Green functions" and "Hamiltonian" leaves room to be regraded at that level (18-01 § 09). The measure, one-particle states and the S-matrix remain None (6-04).

02

## The container — the path integral, the measure, the space of states

"Container" is a metaphor. More precisely, it means the space on which the theory rests (the totality of field configurations), the measure on it, and the Hilbert space made from them.

### Path integral (quantum mechanics) ⟷ Feynman–Kac formula

The physical word is "a weighted sum over all the paths a particle can take", and the mathematical word is an expectation over Brownian motion. They can be said to be the same after moving to imaginary time and putting conditions such as the potential being bounded below.

Writing the $e^{-tH}$ of quantum mechanics in this form is the Feynman–Kac formula (Kac 1949; equation (2) of 18-01, 2-07 § 05) Known . Feynman's (1948) sum has the real-time weight $e^{iS/\hbar}$, and moving it to imaginary time gives this formula. The real-time weight does not become a measure.

### Functional integral ⟷ Euclidean probability measure

The physical word is a sum over all field configurations with weight $e^{-S}$, and the mathematical word is a probability measure on the space of configurations. They can be said to be the same because the side of mathematics defined $\int \mathcal{D}A\, e^{-S}$ as a countably additive probability measure (the judge is a definition; § 4 of Jaffe's review).

$$\frac{1}{Z}\int \mathcal{D}\phi\; e^{-S(\phi)} F(\phi) \;\rightsquigarrow\; \int F\, d\mu, \qquad \mu:\ \mathcal{S}'(\mathbb{R}^4) \text{ etc.: a probability measure on it} \quad (2)$$

Example: the free field is the Gaussian measure with covariance $(-\Delta + m^2)^{-1}$ ($\Delta$ is the Laplacian, not the $\Delta$ of the mass gap) (4-10 § 02). Two-dimensional Yang–Mills theory has also been constructed (Driver 1989, 7-02). On the lattice, equation (9) of 4-02 is a genuine measure. The measure of four-dimensional Yang–Mills has not been constructed (the construction published on 2026-11-10 does not make the measure either; 6-04), and the space it rests on (a completion of the space of connections divided by gauge) has not yet been fixed either (3-04).

### Space of states ⟷ reconstructed Hilbert space

The physical word is Fock space (the space that lines up states by number of particles, 2-06), and the mathematical word is the Hilbert space (3) made from the measure. They can be said to be the same for free fields, and, with interactions, when asymptotic completeness ( § 05) holds.

Let $\mathcal{E}_+$ be the totality of functions determined only by fields at positive times, and $\Theta$ time reversal (4-10 § 05).

$$\mathcal{H} = \mathcal{E}_+ / \mathcal{N}, \qquad \mathcal{N} = \{F : \mathbb{E}[\Theta F\, F] = 0\} \text{ (3)}$$

$\mathcal{N}$ is the totality of functions of norm 0, and dividing by it and completing gives $\mathcal{H}$. (i) For free fields it agrees with Fock space (4-10 § 07). (ii) With interactions, the representation of the canonical commutation relations at time 0 becomes inequivalent to the Fock representation ($\phi^4_3$, Jaffe § 6). The Fock space of physics is the space of asymptotic particles, so agreement needs asymptotic completeness.

03

## Real time and imaginary time — three faces of a correlation

"Three faces" is a metaphor. More precisely, it means the values of the same correlation read at different places (real time, imaginary time, time-ordered) of a single analytic function (the entry "Green functions").

### Wick rotation ⟷ OS reconstruction theorem

The physical word is the computational procedure of setting $t = -i\tau$ and rotating the contour, and the mathematical word is the theorem that builds a Wightman theory from Schwinger functions. They can be said to be the same under the conditions (E0′)–(E4), in the direction from imaginary time to real time.

Wick rotation is Wick's (1954) procedure (4-02 § 02), and its mathematical partner is the reconstruction theorem, which has a direction Known .

$$\{S_n\} \text{ satisfy (E0′)–(E4)} \implies \text{a Wightman theory}, \qquad W_n(\dots,(-i\tau_j, x_j),\dots) = S_n(\dots,(\tau$$

A condition on the ordering of times is needed because $S_n$ does not change when its arguments are exchanged, while $W_n$ changes with the order in which the operators stand (the example in the entry "Green functions"). Care is needed in the reverse direction. The Schwinger functions obtained from a Wightman theory need not satisfy (E0′). The first OS paper (1973) claimed an equivalence, but the proof had an error, which Glaser (1974) fixed. The second (1975) states it in the form of a sufficient condition with (E0′) added (the abstract has '(necessary and) sufficient', with parentheses).

### Green functions ⟷ Wightman functions and Schwinger functions

The physical word "Green function" refers to any of the time-ordered product $G_T$, the Wightman function $W$ and the Schwinger function $S$, while mathematics defines these

three as separate functions. They can be said to be the same in the sense of the theorem that the three are values of one analytic function. Example: for an oscillator of angular frequency $\omega$ ($\hbar = m = 1$),

$$F(z) = \frac{e^{-\omega z}}{2\omega} \ (\mathrm{Re}\, z \geq 0): \qquad S(\tau) = F(|\tau|), \qquad W(t) = F(it), \qquad G_T(t) = F(i|t|) \tag{5}$$

$W$ is the boundary value on the imaginary axis, $S$ is the restriction to the real axis folded by $|\tau|$, and $G_T$ is glued together by sign (Wightman 1956, Symanzik 1966). For $\tau > 0$, $W(-i\tau) = F(\tau) = S(\tau)$, which is the two-point case of (4). For $\tau < 0$, extending the formula for $W$ as it is grows as $e^{\omega|\tau|}$, disagreeing with $S(\tau)$. $W \to S$ follows from positive energy alone, but $S \to W$ needs (E0′).

### Källén–Lehmann representation ⟷ spectral measure

The physical word is the formula that writes a two-point correlation as a superposition over energies (masses), and the mathematical word is the spectral measure of the Hamiltonian. They can be said to be the same when $H$ exists by reconstruction (under the premise of (1)). The representation of Källén (1952) and Lehmann (1954) can be written, in the form of equation (6) of 4-11, as

$$\langle O\Omega,\ e^{-\tau H} O\Omega\rangle = \int_{[0,\infty)} e^{-E\tau}\, d\rho_O(E), \qquad d\rho_O(E) = d\langle O\Omega,\ P_H(E)\, O\Omega\rangle \geq 0 \tag{6}$$

$P_H(E)$ is the projection onto the part of $H$ with energy at most $E$ (the spectral theorem of 1-10 § 05). The positivity of the weights comes from reflection positivity (Table 2 of 4-10). Example: (5) is the case where $\rho$ has weight at the single point $E = \omega$, and on the lattice $\rho_O = \sum_n |\langle n|O|\Omega\rangle|^2 \delta_{E_n}$ (equation (8) of 18-01). The breaking of positivity of the gauge-fixed gluon propagator is treated in 17-02.

## 04 The contents — the vacuum, the Hamiltonian, particles

"Contents" is also a metaphor. More precisely, it means the particular vector (the vacuum) and operators (the Hamiltonian, the mass operator) fixed on the Hilbert space of § 02.

### Vacuum and uniqueness ⟷ Ω and clustering

The physical word is "the state with nothing in it", and the mathematical word is $\Omega = [1]$, obtained in the reconstruction as the class of the constant 1. They can be said to be the same when the vacuum of physics is read as "the state of lowest energy" (a theorem under the premise of (1)).

Uniqueness means that the eigenvalue 0 of $H$ is simple (Jaffe § 2), and on the Euclidean side it is paired with clustering Known .

$$\Omega \text{ unique} \iff \lim_{|a|\to\infty} S_{n+m}\big(x_1,\dots,x_n,\, y_1+a,\dots,y_m+a\big) = S_n(x)\,S_m(y), \qquad a=(0,$$

Non-uniqueness of the infinite-volume limit, degeneracy of the ground state and phase transitions are the same phenomenon (Jaffe § 7.2). Example: on the low-temperature side of the two-dimensional Ising model, the difference between the lowest two levels closes with the width (2-09 § 08). The identification with the procedure of physics that builds the vacuum by switching on interactions slowly from the vacuum without interactions (attributed to Gell-Mann and Low) is not mathematically justified, since Haag's theorem (attributed to Haag 1955) denies in general the existence of the interaction picture (the way of writing that evolves in time by the interaction alone on the space of free fields).

### Hamiltonian ⟷ logarithm of the transfer matrix

The physical word is the operator that generates the advance of time (the energy), and the mathematical word is the logarithm of the transfer matrix $T$ that advances imaginary time by one step on the lattice. They can be said to be the same at the lattice level; agreement with the continuum $H$ is conditional on the continuum limit.

$$H_a = -\frac{1}{a}\log T, \qquad T = T^{*} > 0 \quad (8)$$

$a$ is the lattice spacing in the time direction. A positive self-adjoint $T$ was made by Lüscher (1977) (5-03 § 05), and it connects to the $H$ of Kogut–Susskind (1975) by sending the time step to 0 (Creutz 1977, 5-03 § 07) Known . It is a different thing from the generator $L$ of stochastic quantisation (18-03 § 05).

### One-particle state ⟷ isolated eigenvalue of the mass operator

The physical word is a particle classified as an irreducible representation of the Poincaré group (Wigner 1939), and the mathematical word is an isolated eigenvalue of the mass operator $M$. They can be said to be the same in the reconstructed theory, when that eigenvalue is isolated in the spectrum (4-11 § 05).

$$M = \sqrt{H^2 - \vec{P}^{\,2}}, \qquad m \text{ an eigenvalue of } M \text{ with } (m-\delta, m+\delta)\cap\sigma\big(M|_{\Omega^{\perp}}\big) = \{m\} \quad (\delta >$$

$\vec{P}$ is the momentum, $\sigma(\cdot)$ the spectrum, and $\Omega^{\perp}$ the part orthogonal to the vacuum. Jaffe writes that scattering theory assumes this and 'It entails both a lower gap and an upper gap' ( § 7.1; § 5 of the problem statement also counts the upper gap among the extensions) Known . It has been shown for weakly coupled $P(\phi)_2$ (Glimm–Jaffe–

Spencer 1974), and there is also a one-particle state at strong coupling on the lattice (Schor 1984). Example: for the free field the continuous spectrum above $m$ starts at $2m$. The glueball row is in 18-03 § 04.

05

## Scattering and positivity — two kinds of "unitarity"

"Unitarity" in physics is used in two senses, so we split it into the entry "positive norm" (no states of negative norm) and the entry "unitarity of the S-matrix" (probability conserved in scattering) (18-01 § 06). We put the tool of scattering (the entry "S-matrix") first.

### S-matrix ⟷ Haag–Ruelle scattering theory

The physical word is the amplitude from a set of particles in the far past to a set of particles in the far future, and the mathematical word is an operator built from two isometries. They can be said to be the same when there is an isolated one-particle mass (the entry "one-particle state").

The mass hyperboloid is the surface $p^2 = m^2$ in momentum space (on the side $p^0 = \sqrt{\vec{p}^{\,2} + m^2}$), and the one-particle states of mass $m$ lie on it. An isometry is a map that preserves inner products, and $\mathrm{Ran}$ is the range (the totality of values of the map). $\mathcal{F}$ is the Fock space of asymptotic particles, the space that lines up, by number of particles, the states of particles flying freely in the far past (or future). The scattering theory of Haag (1958) and Ruelle (attributed to 1962) builds the isometries $W^{\mathrm{in}}, W^{\mathrm{out}} : \mathcal{F} \to \mathcal{H}$ from an isolated mass hyperboloid.

$$S = (W^{\mathrm{out}})^* W^{\mathrm{in}}, \qquad \mathrm{Ran}\, W^{\mathrm{in}} = \mathrm{Ran}\, W^{\mathrm{out}} \implies S^*S = SS^* = 1 \quad (10)$$

The rightward direction of (10) is elementary. For an isometric $W$, $W^*W = 1$, and $WW^*$ is the projection onto the range; if the ranges are the same, $W^{\mathrm{out}}(W^{\mathrm{out}})^*$ does not change the values of $W^{\mathrm{in}}$, so $S^*S = (W^{\mathrm{in}})^* W^{\mathrm{out}} (W^{\mathrm{out}})^* W^{\mathrm{in}} = (W^{\mathrm{in}})^* W^{\mathrm{in}} = 1$, and $SS^* = 1$ likewise Paper . Example: for the free field $W^{\mathrm{in}} = W^{\mathrm{out}}$ and $S = 1$. Asymptotic completeness is the claim that the two ranges are the whole of $\mathcal{H}$. Within the range we searched, we found no theorem deriving asymptotic completeness in the usual sense in general. Dybalski–Gérard (2014) showed a "generalised" form under lower and upper mass gaps Known . There are partial results for individual models (unitarity of the S-matrix for weakly coupled $P(\phi)_2$ on even states with energy below about $4m$: Spencer–Zirilli 1976).

### Positive norm ⟷ reflection positivity

The physical word is that there are no states of negative norm (Wightman positivity), and the mathematical word is the reflection positivity of the Schwinger functions. They can be said to be the same within the correspondence between OS and Wightman, and this is a theorem.

Reflection positivity gives the Hilbert space and $H \geq 0$, and is preserved in limits (Jaffe § 4), so if a limit exists it can be carried over from the lattice. It holds for Wilson's lattice (Osterwalder–Seiler 1978, Lüscher 1977) Known . Example: negative weights of Pauli–Villars type break it (4-10 § 08).

### Unitarity of the S-matrix ⟷ positive norm + asymptotic completeness

The physical word is that probability is conserved in scattering ($S^*S = 1$), and the side of mathematics is the entry "positive norm" with the agreement of the ranges in (10) added. They can be said to be the same when asymptotic completeness holds.

For gauge theories, the abstract of Kugo–Ojima (1979) writes that "unitarity of the physical S-matrix" is shown in general ('quite generally') on the basis of representations of the algebra of the BRS charge (the $Q_B$ of § 08). This is a general theory assuming representations of the algebra and asymptotic fields, and its stage is a space with indefinite metric (an inner product that allows negative norms). Those assumptions have been checked within perturbation theory.

06

## Approximation — spacetime, regularisation, renormalisation

### Spacetime and lattice ⟷ the $a \to 0$ limit of $a\mathbb{Z}^4$

The physical word is four-dimensional spacetime with Minkowski signature ($\mathbb{R}^4$ after Euclideanisation), and the side of mathematics is the $a \to 0$ limit of measures on the lattice $a\mathbb{Z}^4$ of spacing $a$. They can be said to be the same when the limit exists and the rotational symmetry $O(4)$ comes back.

The rotational invariance of Euclideanised spacetime is the $O(4)$ invariance of Euclidean covariance (Osterwalder–Schrader axiom E1) (for the signature, the existing article One sign is different). What remains on the lattice is rotations by 90° and reflections, and the abstract of Wilson (1974) writes of the strong-coupling limit 'no Lorentz (or Euclidean) invariance' Known .

symmetries of the hypercube (order $2^4 \cdot 4! = 384$) $\subset O(4)$, $\qquad \mu_a \xrightarrow[a\to 0]{?} \mu$ ($O(4)$ -invaria

Example: for the free field on the lattice, the difference between correlations along an axis and along a diagonal vanishes as $a^2$ (Table 2 of 4-02). For pure gauge theories, the continuum limit of the lattice has been shown for non-abelian groups in two dimensions and for U(1) in three and four dimensions (Driver 1989, Gross 1983, Driver 1987, among others). The abstract for three dimensions writes that with the Villain action it converges to the free electromagnetic field. The free electromagnetic field is invariant under rotations, so rotational symmetry has come back too.

### Regularisation ⟷ family of measures with lattice spacing $a$

The physical word covers all devices that temporarily stop infinities, and what the side of mathematics can treat as a family of measures is mainly the lattice. They can be said to be the same when the limit with the device removed does not depend on the choice of device (universality).

§ 6.5 of the problem statement writes that there are few ways of recovering the reflection positivity lost in dimensional regularisation, Pauli–Villars and the like. What Reisz (1989) shows is that the continuum limit of the lattice is finite at each order of perturbation.

$$\lim_{a\to 0} \mu_a^{\text{Wilson}} \stackrel{?}{=} \lim_{a\to 0} \mu_a^{\text{Villain}} \stackrel{?}{=} \cdots \quad (12)$$

The non-perturbative (12) has been shown in two dimensions. Chevyrev–Shen (2023, arXiv:2302.12160) showed on the two-dimensional torus, for connected compact groups, that lattice gauge theories with the Wilson, Villain and Manton actions give the same limit. Convergence of the two-dimensional lattice approximation is also in Driver (1989).

### Renormalisability ⟷ BPHZ, Epstein–Glaser

The physical word is that infinities vanish at each order of perturbation after adjusting finitely many constants, and the mathematical word is the finiteness theorems of a subtraction rule (BPHZ) and of an inductive construction from causality (Epstein–Glaser). They can be said to be the same at the level of formal power series (4-05).

In physics, Dyson (1949), and for Yang–Mills 't Hooft (1971) and Becchi–Rouet–Stora (1976). In mathematics, Bogoliubov–Parasiuk (1957), Hepp (1966), Zimmermann (1969) and Epstein–Glaser (1973).

$$S_n^{\text{ren}}(x;g) = \sum_{k\geq 0} c_{n,k}(x)\, g^k \in \mathbb{C}[[g]] \qquad (\text{each } c_{n,k} \text{ finite}) \quad (13)$$

The abstract of Epstein–Glaser writes that the inductive construction gives 'at each order, finite terms' and is equivalent to BPHZ Known . It does not include convergence

of the sum (the series is thought to diverge, 4-12).

07

## Fields have no values at points — operator-valued distributions

### Field $\phi(x)$ ⟷ operator-valued distribution

The physical word is an operator $\phi(x)$ at each point $x$, and the mathematical word is an operator-valued distribution, in which only $\phi(f)$, smoothed with a test function $f$, is an operator. They can be said to be the same when $\phi(x)$ is read as "a symbol for the smoothed value, just before the smoothing width is sent to 0".

Distributions are in 1-11 § 04 (the reading of fields is Jaffe § 3) Known . For the free field of mass $m$ ($\omega_p = \sqrt{p^2 + m^2}$),

$$\phi(f) = \int \phi(x) f(x)\, d^4x, \qquad \|\phi(f)\Omega\|^2 = \int \frac{d^3p}{(2\pi)^3} \frac{|\hat{f}(\omega_p, p)|^2}{2\omega_p} < \infty \quad (14)$$

Example: taking $f$ to be a bump of integral 1 and width $\varepsilon$, $f_\varepsilon(x) = \varepsilon^{-4} g(x/\varepsilon)$, we have $\hat{f}_\varepsilon(p) = \hat{g}(\varepsilon p)$, so in four dimensions the right-hand side diverges as $\varepsilon^{-2}$ (1-09). The point value $\phi(x)$ is the limit as $f$ approaches $\delta$, so as an operator it does not exist. The point fields of Yang–Mills are $\mathrm{Tr}\, F_{ij}F_{kl}(x)$, and § 4 of the problem statement requires those operators. The local-field footnote of the problem statement (footnote 1 of Jaffe–Witten) says that a natural one-to-one correspondence between classical differential polynomials and operators 'does not exist', and expects a correspondence between the spaces of dimension $\le d$ (0-03 § 08). Dimension here is the power under a change of the length scale (2 for $F$, 4 for $\mathrm{Tr}\, F^2$; 1-11 § 06). A Wilson loop is a quantity on a curve, outside the axioms of point fields (4-10 § 09).

08

## Gauge — slices and physical states

### Gauge fixing ⟷ section of the orbit space

The physical word is choosing one representative from each gauge orbit by a condition $F(A) = 0$, and the mathematical word is a section of the projection to the orbit space $\mathcal{A}/\mathcal{G}$ (a map that assigns continuously one point to each orbit); as definitions they are the same. On the four-dimensional sphere, however, it is a theorem that there is no global section.

The geometry of slices is in 3-04 § 05. The source of the Faddeev–Popov (1967) rule is

$$1 = \Delta_{\mathrm{FP}}(A) \int_{\mathcal{G}} dg\ \delta\big(F(A^g)\big), \qquad \Delta_{\mathrm{FP}}(A) = \Big|\det \frac{\delta F(A^g)}{\delta g}\Big|_{g=1} \quad (15)$$

(15) holds when the condition surface $F = 0$ meets each orbit exactly once. (15) evaluates the determinant at the single point $g = 1$ and takes it outside the integral, so if it meets many times the right-hand side becomes a sum of ratios of the determinants at each intersection, and is neither 1 nor the number of intersections. Putting the determinant inside the integral gives the number of intersections $N(A)$.

$$N(A) = \int_{\mathcal{G}} dg\ \delta\big(F(A^g)\big) \Big|\det \frac{\delta F(A^g)}{\delta g}\Big| \quad (15a)$$

The actual Faddeev–Popov rule uses $\det$ without absolute value, so it counts intersections with signs (the detour in 4-08 § 05). Gribov (1978) pointed out that "exactly once" breaks down in non-abelian theories, and the abstract of Singer (1978) writes, on $S^4$, 'no continuous choice of exactly one connection on each orbit can be made' Known . The lattice integrates with Haar measure (the uniform probability measure on the group, unchanged by multiplication), so no gauge fixing is needed and it avoids this row (5-01, 4-08).

### Physical states ⟷ reconstruction from gauge invariants

The physical word is states satisfying the Gauss constraint, or BRST cohomology, and the mathematical word is the space made by carrying out (3) with gauge-invariant quantities only. They can be said to be the same up to the lattice level; in the continuum it is Unsettled.

The Gauss constraint is the non-abelian version of Gauss's law $\nabla \cdot \boldsymbol{E} = 0$, and on the side of states it is the condition "the state does not change under gauge transformations at fixed time" (Kogut–Susskind 1975, 4-01). On the lattice, $\psi(U^g) = \psi(U)$ for gauge transformations $g$ at fixed time. The BRST $Q_B$ is an operator satisfying $Q_B^2 = 0$ (the BRST charge), and $\mathbf{Ker}\, Q_B / \mathbf{Im}\, Q_B$ is "the states annihilated by $Q_B$, identified up to the range of $Q_B$". An indefinite metric is an inner product that allows negative norms.

$$\mathcal{H}^{\mathrm{BRST}}_{\mathrm{phys}} = \mathbf{Ker}\, Q_B / \mathbf{Im}\, Q_B \ \ \text{(indefinite metric)}, \qquad \mathcal{H}^{\mathrm{OS}}_{\mathrm{inv}} = \mathcal{E}^{\mathrm{inv}}_{+} / \mathcal{N} \ \ \text{(positive definite)} \quad (16$$

On the lattice, transition probabilities between gauge-invariant states are not negative (Lüscher 1977), and gauge fixing does not change the Green functions of gauge singlets (operators unchanged by gauge transformations) (Creutz 1977) Known . In the continuum it has not been shown that the two sides of (16) are the same, and non-perturbative BRST remains hard to define because of Gribov copies.

09

## Local symmetry does not break — Elitzur's theorem

### Local symmetry does not break ⟷ Elitzur's theorem

The physical word is the claim "gauge symmetry (local symmetry) does not break spontaneously", and the mathematical word is a theorem stating it as a limit in an external field. They can be said to be the same when compared in the order "put in an external field, make the volume infinite, then remove the field".

The abstract of Elitzur (1975) argues, in the form 'It is argued that', that spontaneous breaking of local symmetry is impossible in a symmetric gauge theory without gauge fixing, and says it is shown in a simple system of abelian gauge fields on a lattice Known . Mathematical successors include a proof, for an abelian Higgs model on the lattice, that the magnetisation becomes 0 when the external field is removed (De Angelis–de Falco–Guerra 1978) Known . If the external field is 0, the Haar average makes the expectation 0 (equation (9) of 17-02). Spontaneous breaking is a question of the order in which the external field is removed (2-08).

### Comparing in the order of removing the external field

We write Elitzur's argument out for the weight $\exp(\beta \sum_p U_p + h \sum_\ell U_\ell)$ of $\mathbb{Z}_2$ lattice gauge theory in $d$ dimensions (link variables $U_\ell = \pm 1$, with $U_p$ the product of the four links round a plaquette). A gauge transformation at one point $x$ flips the $2d$ links touching $x$, and does not change $\sum_p U_p$. So for $\ell$ touching $x$,

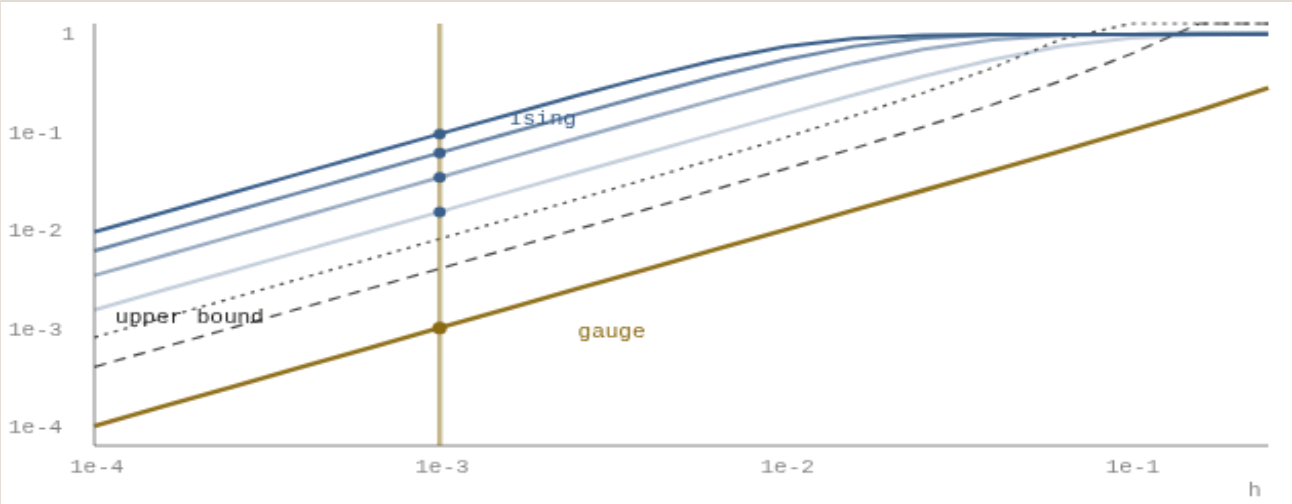
$$\langle U_\ell \rangle_h = -\langle U_\ell \, e^{-2hQ_x} \rangle_h, \quad Q_x = \sum_{\ell' \ni x} U_{\ell'} \quad \Longrightarrow \quad |\langle U_\ell \rangle_h| = \tfrac{1}{2} \big| \langle U_\ell (1 - e^{-2hQ_x}) \rangle_h \big| \le \tfrac{1}{2}$$

Paper . The right-hand side depends neither on the volume nor on $\beta$. In the Ising model (global $\mathbb{Z}_2$) all spins must be flipped, and the upper bound from the same argument is $\frac{1}{2}(e^{2h|\Lambda|} - 1)$, which loses its meaning as the volume grows. The only difference is whether one symmetry transformation touches finitely many variables or all of them. The slope at $h = 0$ is

$$\frac{d\langle U_\ell \rangle_h}{dh}\Big|_0 = \sum_{\ell'} \langle U_\ell U_{\ell'} \rangle_0 = 1, \qquad \frac{d\langle \sigma_0 \rangle_h}{dh}\Big|_0 = \sum_y \langle \sigma_0 \sigma_y \rangle_0 \approx M_0^2 \, |\Lambda| \quad (18)$$

On the left, the term with $\ell' = \ell$ is 1. The terms with $\ell' \neq \ell$ are 0 because, gauge-transforming at the endpoint of $\ell$ that does not touch $\ell'$, only $U_\ell$ of the two changes sign and $U_{\ell'}$ does not (in a box of side at least 3 such an endpoint always exists). The right is on the low-temperature side, where even distant spins keep the correlation $M_0^2$, so it is proportional to the volume.

Figure 1 Choosing the external field h (log scale) shows below ⟨σ⟩ (Ising, blue, darker in the order L = 4 to 10) and ⟨U⟩ ($\mathbb{Z}_2$ gauge, ochre) for the two-dimensional L×L periodic box. "Play" lowers h towards 0 (the direction of removing the external field). The dashed and dotted lines are the upper bound (17) (d = 2, 4)




h = 1.0e−3 ⟨σ⟩ : L=4 0.0152, L=6 0.0342, L=8 0.0607, L=10 0.0946 ⟨U⟩ 0.00100, upper bound 0.00402

Computed The numbers in this figure were computed on this machine (exact values by diagonalising the transfer matrix).

Computed Table 1 The two-dimensional $L \times L$ periodic box. Ising at $\beta = 0.6$ (low-temperature side), gauge at $\beta = 2$

| $h$ | $\langle\sigma\rangle$ $L=4$ | **6** | **8** | **10** | $\langle U\rangle$ $L=4$–**10** | **(17)** $d=2$ |
|---|---|---|---|---|---|---|
| $10^{-4}$ | 0.0015 | 0.0034 | 0.0061 | 0.0095 | 0.000100 | 0.00040 |
| $10^{-3}$ | 0.0152 | 0.0342 | 0.0607 | 0.0946 | 0.001000 | 0.00402 |
| $10^{-2}$ | 0.1512 | 0.3288 | 0.5396 | 0.7314 | 0.010002 | 0.04164 |
| $10^{-1}$ | 0.8979 | 0.9792 | 0.9810 | 0.9810 | 0.1017–0.1016 | 0.61277 |

The Ising columns rise with $L$, and $\langle\sigma\rangle/(hL^2)$ at $h = 10^{-4}$ is 0.9529, 0.9505, 0.9494, 0.9489, approaching 0.9479, the square of the formula $M_0 = (1 - \sinh^{-4} 2\beta)^{1/8}$ cited for the spontaneous magnetisation (attributed to Yang 1952) Computed . In the gauge column, for $h \le 10^{-2}$, $L = 4$–10 agree to 6 decimal places, and the slope is the 1 of (18). The "breaking of gauge symmetry" of physics is said to be reread as, for example, the breaking of a global symmetry after gauge fixing Physics ; taken literally it is a false friend (18-01).

### Detour: gauge theory at β = ∞ is the Ising model

A subsection you may skip.

As $\beta \to \infty$ all plaquettes line up at $U_p = +1$ (a flat field). On a simply connected region one can write $U_{xy} = g_x g_y$ ($g_x = \pm 1$), so the external-field term becomes

$$h \sum_{\langle xy \rangle} U_{xy} = h \sum_{\langle xy \rangle} g_x g_y \qquad (19)$$

the Ising model with coupling $K = h$. The expectation of a link is its nearest-neighbour correlation (about $\tanh h$ for small $h$), and does not become a magnetisation. The periodic box (torus) of Table 1 is not simply connected. The products of links once round the box (holonomies, 1-07) fall into four classes, $\pm 1$ horizontally and vertically, and the $\beta \to \infty$ limit is the sum of four Ising models with periodic and antiperiodic boundary conditions. At the computed $h = 0.1$ and 0.3 (the high-temperature side of Ising), the difference from the periodic Ising model alone becomes exponentially small with $L$.

Computed Table 2 $\langle U \rangle$ in the $L \times L$ periodic box and the nearest-neighbour correlation of Ising ($K =$ difference between the periodic Ising model alone and the sum of the four classe

| $h$ | Gauge $\beta = 4$, 8 | Sum of the four classes | Periodic Ising | Difference $L = 4$ | |
|---|---|---|---|---|---|
| 0.1 | 0.101689 | 0.101689 | 0.101689 | 1.2e−3 | 1.4e |
| 0.3 | 0.352256, 0.352257 | 0.352257 | 0.356234 | 6.9e−2 | 1.7e |

The first three columns are $L = 8$. At $h = 0.1$ they agree to 6 decimal places; at $h = 0.3$ a difference is visible between the gauge value 0.35226 and periodic Ising 0.35623, and the sum of the four classes agrees Computed .

## What this chapter can say

| | What is said | What is not said |
|---|---|---|
| Known | (4)(6)(7)(8)(15)(15a). The claims of the primary sources cited | That these hold in four-dimensional continuum Yang–Mills theory (the objects do not yet exist) |
| Paper The derivations of this chapter (elementary; (17)(18) follow the argument of Elitzur 1975) | The rightward direction of (10) (if the ranges are equal, $S$ is unitary), the upper bound (17), the left of (18) | Estimates of the same form for non-abelian groups or continuum theories |
| This book's reading | The grades and judges of Table 0, the reading of (1) | That the grades are unique |
| Computed | Tables 1 and 2, Figure 1 (two dimensions, $L \leq 10$) | Four dimensions, non-abelian groups, $L \to \infty$ |
| Overall | — | This book itself shows neither the existence of the mass gap nor the construction of the continuum limit (the published construction of the continuum limit is in Volume 2) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Quotations from Jaffe and the problem statement | Text | Jaffe, *Constructive Quantum Field Theory* (2000, public PDF) § 2– § 7 / Jaffe–Witten, *Quantum Yang–Mills Theory* (Clay Mathematics Institute, 2000) § 3– § 6.5, the local-field footnote and the weak-existence footnote |
| Abstract checked | Known | Wick, PR 96 (1954) 1124 / Wightman, PR 101 (1956) 860 / Symanzik, JMP 7 (1966) 510 / Osterwalder–Schrader, CMP 31 (1973) 83, 42 (1975) 281 / Glaser, CMP 37 (1974) 257 / Lüscher, CMP 54 (1977) 283 / Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 / Creutz, PRD 15 (1977) 1128 / Kogut–Susskind, PRD 11 (1975) 395 / Schor, NPB 231 (1984) 321 / Dybalski–Gérard, CMP 332 (2014) 1167 / Spencer–Zirilli, CMP 49 (1976) 1 / Kugo–Ojima, PTPS 66 (1979) 1 / Wilson, PRD 10 (1974) 2445 / Gross, CMP 92 (1983) 137 / Driver, CMP 123 (1989) 575 / Reisz, NPB 318 (1989) 417 / Chevyrev–Shen, arXiv:2302.12160 (2023) / Epstein–Glaser, AIHP A 19 (1973) 211 / Faddeev–Popov, PLB 25 (1967) 29 / Gribov, NPB 139 (1978) 1 / Singer, CMP 60 (1978) 7 / Neuberger, PLB 183 (1987) 337 / Elitzur, PRD 12 (1975) 3978 / De Angelis–de Falco–Guerra, PRD 17 (1978) 1624 / Yang, PR 85 (1952) 808 (the formula for $M_0$: main text not checked) |
| Bibliographic data only | Known | Feynman, RMP 20 (1948) 367 / Kac, Trans. AMS 65 (1949) 1 / Driver, CMP 110 (1987) 479 / Gell-Mann–Low, PR 84 (1951) 350 (attribution from secondary sources) / Haag, Kong. Dan. Vid. Selsk. Mat.-Fys. Medd. 29 (12) (1955), PR 112 (1958) 669 / Ruelle, Helv. Phys. Acta 35 (1962) 147, as attributed (bibliographic data not checked) / Källén, Helv. Phys. Acta 25 (1952) 417 / Lehmann, Nuovo Cim. 11 (1954) 342 / Wigner, Ann. Math. 40 (1939) 149 / Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585 / Dyson, PR 75 (1949) 486, 1736 / 't Hooft, NPB 33 (1971) 173 / Becchi–Rouet–Stora, Ann. Phys. 98 (1976) 287 / Bogoliubov–Parasiuk, Acta Math. 97 (1957) 227 / Hepp, CMP 2 (1966) 301 / Zimmermann, CMP 15 (1969) 208 |
| Table 0 | This book's reading | The primary sources are the rows above and 18-01 |
| Tables 1 and 2, Figure 1, the slope of (18) | Computed | Python 3 on this machine (numpy): `elitzur.py`, `table.py`, `detour_beta_inf.py`, `holonomy_check.py` (the four classes), `holonomy_brute.py` (full enumeration of the $3\times3$ box) |

Next chapter to read: 18-03 A dictionary of mass and length.

# A dictionary of mass and length — pairs that overlap on the lattice and come apart on the side of the limit

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — We pair 20 physical words about "mass" and "length" with mathematical words, and see two layers: an inner layer where they overlap once the lattice is fixed, and an outer layer, on the side where the lattice spacing is sent to 0, where the conditions gather. We also fix here the distinction between "two times" and two $\eta$.

**Prerequisite chapters** — 4-11, 5-08 and 2-08. The Poincaré constant is in 1-10, and the two columns of grades in 18-01. Words not in the prerequisite chapters (asymptotic freedom, stochastic differential equations, log-Sobolev, mixing time, the Polyakov loop, central charge, and so on) are defined in a line each where they are used.

**Said first** — We only line up correspondences; we show neither the mass gap, nor confinement, nor the continuum limit. The assumptions of the record of this book's study (the diffusive-relaxation-time assumption and others) remain assumptions.

The order of this chapter

1. The overall picture — two layers, two-column grades, how to read Table 0
2. The mass gap and the correlation length — lattice units and physical units
3. Potential, string tension, confinement — only one direction is a theorem
4. $\Lambda$, one scale, glueballs — the distinction between a gap and a particle
5. Two times — $H_T$ and $L$, the unit of time
6. Autocorrelation times and the Poincaré constant — times that split into three
7. Critical slowing down — $z$, the conventional theory, the variational lower bound, two $\eta$. Detour: for the free field the relaxation time equals the susceptibility
8. Fluctuations of the tube and topological freezing — effective strings, Wulff, bottlenecks. Detour: sectors of topological charge and narrow passages

01

## The overall picture — two layers and two-column grades

In physics, mass and length are two sides of one coin. More precisely, in units with $\hbar = c = 1$ the inverse of a mass has the dimension of length; the force carried by a particle of mass $m$ stops reaching beyond a distance of about $1/m$ (Yukawa type, 3-01), and correlations fade as $e^{-r/\xi}$ with $\xi = 1/m$. In mathematics this one number splits into separate quantities: the spectral gap, the decay rate of correlations, and the rate of relaxation of a stochastic dynamics.

The pairs of Table 0 fall into two layers. The inner layer is pairs that become the same thing by the spectral theorem alone (1-10) once the lattice spacing $a$ and the dynamics are fixed (the gap of the transfer matrix and the decay rate of correlations, for example). The outer layer is pairs that appear where $\xi \to \infty$ and $a \to 0$ are joined, and the conditions and the Unsettled grades gather there.

The two columns of the table are the grades of 18-01. (a) is "if both objects existed, would they be the same?": to the three grades Settled, Conditional and Unsettled we add "Does not correspond" for pairs known to be different quantities (the "false friend" of 18-01). (b) is the existence column of 18-01 given at the level "lattice ($a$ fixed)", read as: Constructed = exists at every $\beta$ in finite volume; Conditional = only versions of limited range, such as strong coupling; None = nothing shown at this level (including things known only numerically). At the continuum four-dimensional level every row is "None" (the construction published on 2026-11-10 does not make the mass gap, Wilson loops or one-particle states either; 6-04). Notation: $\beta = 2N/g^2$, lengths in lattice units, $m_{\mathrm{lat}} = 1/\xi$, $\Delta = m_{\mathrm{lat}}/a$. $A \asymp B$ means that the ratio $A/B$ is bounded above and below by positive constants (A-01).

Table 0 The list of the dictionary of mass and length (from top, in the order of § 02– § 08)

| # | Physical word | Mathematical word | (a) Definitions | (b) Lattice ($a$ fixed) | (c) Finite measurement |
|---|---|---|---|---|---|
| 1 | Mass gap | Spectral gap, uniform decay | Conditional | Conditional (strong coupling) | Upper bound only |
| 2 | Correlation length $\xi$ | Inverse of the gap of the transfer matrix | Conditional | Conditional (strong coupling) | Lower bound only |
| 3 | Gap in physical units | Uniform lower bound in lattice units | Does not correspond (false friend; settled as a different quantity) | Conditional (strong coupling) | Out of reach |

| # | Physical word | Mathematical word | (a) Definitions | (b) Lattice ($a$ fixed) | (c) Finite measurement |
|---|---|---|---|---|---|
| 4 | $V(R)$ | Decay rate of Wilson loops | Conditional | Conditional | Upper bound only |
| 5 | String tension $\sigma$ | Coefficient of the area law, the $d$ of the continuum string-tension problem (Chatterjee's Problem 5.2) | Conditional | Conditional ($\sigma > 0$ at strong coupling) | — |
| 6 | Confinement | One-directional theorems | Conditional | Conditional | — |
| 7 | Centre symmetry | Centre invariance on a slab | Unsettled | None (whether it holds is unproved) | — |
| 8 | $\Lambda$, one scale | No counterpart | Unsettled | None | — |
| 9 | $m/\sqrt{\sigma}$ | The limit of the pair of the lattice correlation-length problem and the continuum string-tension problem | Unsettled | None (numerical only) | Out of reach |
| 10 | Glueball | Isolated eigenvalue of the mass operator | Conditional | Conditional (strong coupling; see note) | Upper bound only |
| 11 | Langevin relaxation | Generator $L$ | Settled | Constructed (finite volume) | Upper bound only ($\mathrm{gap}(L)$) |
| 12 | $z = 2$ | $\mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$ | Unsettled | None (a statement about a family; see note) | Out of reach |
| 13 | Exponential autocorrelation time | Poincaré constant $C_P$ | Settled in continuous time | Constructed (finite volume) | Lower bound only |
| 14 | Thermalisation time | Mixing time | Conditional | Constructed (finite volume) | Lower bound only |
| 15 | Dynamical exponent $z$ | $C_P \asymp \xi^z$ | Unsettled | None (a statement about a family; see note) | Out of reach |
| 16 | $z = 2 - \eta_s$ | Variational lower bound $C_P \geq \chi$ | Conditional | Constructed (lower bound only) | Lower bound only |

| # | Physical word | Mathematical word | (a) Definitions | (b) Lattice ($a$ fixed) | (c) Finite measurement |
|---|---|---|---|---|---|
| 17 | No slowing down away from criticality | Uniform mixing ⟺ uniform log-Sobolev | Conditional | Conditional (strong coupling) | Out of reach |
| 18 | Lüscher term | Casimir energy of a strip | Unsettled | None (numerical only) | — |
| 19 | Roughening, shape of the tube | Wulff construction | Unsettled | None (a picture of physics) | — |
| 20 | Topological freezing | Bottleneck in state space | Conditional | None (freezing numerical only) | Lower bound only |

Note: (c) is the measurement column of 18-01 §02 (the direction reachable by finite measurement). Relatives of the gap (the entries "mass gap", "V(R)", "glueball" and "Langevin relaxation") reach an upper bound from a single correlation or test function, and the lengths and times that are their inverses (the entries "correlation length ξ", "exponential autocorrelation time", "thermalisation time", "z=2-η_s" and "topological freezing") swap direction and reach a lower bound. Statements about uniformity, families or limits (the entries "gap in physical units", "m/√σ", "z=2", "dynamical exponent z" and "no slowing down away from criticality") are out of reach. The ground for the entry "topological freezing" is (14) (choosing one set gives a lower bound on $C_P$; §08), and for the entry "thermalisation time" the definition of mixing time (§06; if the total variation distance exceeds $\varepsilon$ on a single event, that time is before the mixing time). The entry "string tension σ" (the string tension) gets "—", since the main text of this book does not treat a form that pins down one side from finite measurements. The (b) of the entry "glueball" has the form that for each momentum there is a dispersion curve isolated from the energy spectrum, since the lattice has no Lorentz symmetry and the mass operator $M$ is not defined as it is (Schor 1984; §04). Both sides of the entries "z=2" and "dynamical exponent z" are defined on a fixed lattice, but $\asymp$ is a statement about a family with the lattice spacing sent to 0, so on a single lattice it has no content. "None" in (b) means that nothing has been shown about that family.

02

## The mass gap and the correlation length — lattice units and physical units

The problem statement (Jaffe–Witten) defines the mass gap as the Hamiltonian $H$ having no spectrum in $(0, \Delta)$ ( § 4), and lists equal-time clustering as a consequence ( § 5) Known . After OS reconstruction (4-10), if the vacuum $\Omega$ is unique, the spectral theorem gives

$$\langle O\Omega, e^{-tH} O\Omega\rangle - |\langle \Omega, O\Omega\rangle|^2 = \int_{[\Delta,\infty)} e^{-tE}\, d\rho_O(E) \;\le\; e^{-t\Delta}\, \|O\Omega\|^2 \quad (1)$$

($\rho_O$ is the spectral measure of $O\Omega$ with 0 removed). The converse of (1) holds only when a dense family of observables decreases at a *uniform* rate (the entry "mass gap"). On the lattice, reflection positivity lets one build a positive self-adjoint transfer matrix $T = e^{-HT}$ (Lüscher 1977, 5-03), and from its eigenvalues $\lambda_0 > \lambda_1 \ge \cdots$,

$$m_{\mathrm{lat}} = -\log \frac{\lambda_1}{\lambda_0}, \qquad \xi = \frac{1}{m_{\mathrm{lat}}} \quad (2)$$

The correlation length of the plaquette correlation of Chatterjee (2018) is in general $\xi_{\mathrm{plaq}} \le 1/m_{\mathrm{lat}}$ (equality needs overlap with the lowest state). Example: for the free field $\xi = 1/(2\,\mathrm{arsinh}(m/2))$ (Table 1).

### Two quantities under one word

Many of the results called "the mass gap is positive" in mathematics are uniform lower bounds in lattice units, $m_{\mathrm{lat}} \ge c > 0$. An example that can be checked is Shen–Zhu–Zhu (2023), whose abstract states a 'strictly positive mass gap' for strongly coupled lattice Yang–Mills Known . Then $\Delta = m_{\mathrm{lat}}/a \ge c/a \to \infty$, which in physical units is a property of the side where the mass goes to infinity, the opposite side from the continuum limit. A gap in physical units needs $\xi \to \infty$ to cancel $a \to 0$, and the lattice correlation-length problem (Chatterjee's Problem 5.1) also asks for $\xi(\beta) \to \infty$ as $\beta \to \infty$ (5-09, 16-08). This is why the entry "gap in physical units" in Table 0 is "Does not correspond".

03

## Potential, string tension, confinement — only arrows in one direction are theorems

The static potential $V(R)$ is the energy of an infinitely heavy quark pair placed at distance $R$, read, following Wilson (1974), from the $R \times T$ Wilson loop (5-01) as

$$V(R) = -\lim_{T\to\infty} \frac{1}{T} \log\langle W_{R,T}\rangle, \qquad \sigma = \lim_{R\to\infty} \frac{V(R)}{R} \quad (3)$$

Taking the left as the definition it is Settled, but identifying it with the ground-state energy of static charges needs the $T \to \infty$ limit and overlap with the ground state (the entry "V(R)"). From the monotonicity and concavity of $V$ shown by Bachas (1986), $\sigma \in [0, \infty)$ exists Known . The area law holds at strong coupling for every group (Osterwalder–Seiler 1978, 5-02). The continuum string tension is a proposal to define it as the $d$ of Chatterjee's continuum string-tension problem.

$$\log\langle W_{R/\varepsilon,\,T/\varepsilon}\rangle = -c\,(R+T) - d\,RT + o(1), \qquad \varepsilon(\beta)\to 0,\ c(\beta)\to\infty,\ d\ /=0 \quad (4)$$

He himself writes that it is not clear whether the perimeter term is needed. The physical rule of thumb is $\sqrt{\sigma} \approx 485(6)$ MeV. This is a conversion that tentatively applies to the $r_0\sqrt{\sigma} = 1.160(6)$ of SU(3) pure gauge theory the length scale $r_0 = 0.472$ fm fixed by lattice calculations of QCD, and the authors themselves note that it is a conversion that cannot be made exactly, since pure gauge theory and QCD are different theories (Athenodorou–Teper 2020) Physics .

### Three ways of saying confinement

The three are (a) $V(R) \to \infty$, (b) $\sigma > 0$ and (c) unbroken centre symmetry. The physical definition of (c) is at finite temperature. On a lattice whose time direction is a ring of length $1/T$ ($T$ is the temperature), the trace of the product of links once round the time direction from a point is called the Polyakov loop, and its expectation being 0 is taken as the sign of confinement (Polyakov 1978, Susskind 1979; the universality argument through an order parameter transforming under the centre is Svetitsky–Yaffe 1982) Physics . A centre transformation multiplies all the time-direction links crossing the surface at some time by an element of the centre of the gauge group (for $SU(N)$, $e^{2\pi ik/N}$ times the identity matrix). The action does not change and only the Polyakov loop is multiplied by $e^{2\pi ik/N}$, so if this symmetry is unbroken the expectation is 0.

What has been shown in mathematics is only the one-directional implications of Table A (Chatterjee 2021) Known . The definition of unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) is that for some $N \geq 1$ the theory restricted to the slab $\{0, \dots, N\} \times \mathbb{Z}^{d-1}$ is invariant under centre transformations for every boundary condition, and the definition of strong decay (Definition 2.3 of Chatterjee 2021) is uniform exponential decay for every cube and boundary condition (strong decay). The Wilson loops of the Theorem from Centre Symmetry to Confinement (Theorem 2.2 of Chatterjee 2021) are those in representations acting non-trivially on the centre.

Table A Implications between the ways of saying confinement (lattice)

| Direction, example | Status |
|---|---|
| Strong exponential decay (Definition 2.3 of Chatterjee 2021) $\Longrightarrow$ unbroken $\wedge$ area law | Theorem (the Theorem from Uniform Decay to the Area Law) |
| Unbroken centre symmetry (Definition 2.1 of Chatterjee 2021) $\Longrightarrow$ $V(R) \to \infty$ | Theorem (the Theorem from Centre Symmetry to Confinement) |
| Area law $\Longrightarrow$ $V(R) \to \infty$ | Trivial |
| $V \to \infty \Longrightarrow \sigma > 0$ | Not shown |
| Weak decay (for particular boundary conditions only) $\Longrightarrow$ confinement | Not shown |
| Finite groups, weak coupling: decay of two-point correlations for particular boundary conditions / no confinement | Theorem (Adhikari–Cao 2025) / belief |

The gap and confinement are separate claims (17-02, 16-08, 5-10). On centre symmetry, Chatterjee proposes the definition of unbroken centre symmetry on the grounds that there is "to my knowledge" no mathematical definition, and cites Borgs–Seiler (1983) for the finite-temperature analogue. Whether the two definitions separate the same phases is unproved (the entry "centre symmetry").

> 04

## Λ, one scale, glueballs — the distinction between a gap and a particle

The value of the dimensionless coupling $g$ changes with the scale $\mu$ at which it is measured. Asymptotic freedom means that $g$ becomes smaller the larger $\mu$ is (the shorter the distance) (4-06), and the one-loop coefficient that fixes the rate is $b_0$. Solving the one-loop formula,

$$\Lambda = \mu \exp\!\Big(-\frac{1}{2b_0\, g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{3}\cdot\frac{1}{16\pi^2} \quad (5)$$

becomes a mass scale independent of $\mu$ (dimensional transmutation, 4-07). In physics every mass is a constant multiple of $\Lambda$, and $m/\sqrt{\sigma}$ is read as a pure number Physics . The side of mathematics has no definition of the four-dimensional $\Lambda$ (the lattice $\Lambda_L$ is in 18-04). The record of this book's study writes "a single scale" as chain (ii) $m^2 \asymp \sigma$, and calls it, by *analogy*, the sentence that plays the role of an equivalence principle (17-12). There is no proof. If the continuum string-tension problem holds with $\varepsilon(\beta)$ and $\varepsilon\xi \to \ell \in (0,\infty)$, then

$$\frac{m}{\sqrt{\sigma}} = \lim_{\beta\to\infty} \frac{1/(\varepsilon\,\xi)}{\sqrt{d}} = \frac{1}{\ell\sqrt{d}} \quad (6)$$

and this pairing is a reading not in Chatterjee's text. The numbers are 3.781(23) for SU(2), 3.405(21) for SU(3) and 3.072(14) for SU(∞) ($0^{++}$, Athenodorou–Teper 2020, 2021) Physics . There are also opposite examples. In three-dimensional compact U(1) (Villain action) the string tension is positive at every coupling, the continuum limit is a free field of mass $m_D$, and $\sigma/m_D^2$ in physical units diverges to infinity (Göpfert–Mack, abstract) Known . It is an example where there is confinement but the scale does not become one (17-12).

Glueballs are measured as the lowest decay rate in each $J^{PC}$ (spin, parity, charge conjugation) channel (5-08; the SU(3) $0^{++}$ is $r_0 m = 4.21(11)(4)$, Morningstar–Peardon 1999) Physics . The rotational symmetry of the lattice is only the 48 rotations and reflections that carry the cube onto itself (the group $O_h$), so the channels are divided by the irreducible representations of $O_h$. The correspondence with continuum spin presupposes the recovery of rotational symmetry. A one-particle state in mathematics is an isolated eigenvalue of the mass operator

$$M = \sqrt{H^2 - P^2} \;\geq 0 \quad (7)$$

($P$ is the momentum; § 6 of the problem statement), and in § 5 the problem statement counts "an isolated one-particle state (the upper gap)" among the extensions Known . A gap and a particle are separate claims. The lattice has no Lorentz symmetry and $M$ is not defined as it is, so what has been shown at strong coupling on the lattice has the

form that for each momentum there is a dispersion curve isolated from the energy spectrum (Schor 1983, 1984) Known .

05

## Two times — the $H_T$ of the transfer matrix and the $L$ of Langevin

There are two times. One is the imaginary time of the path integral: correlations are carried by $T = e^{-H_T}$, and the gap is $1/\xi$. The other is the "fictitious fifth time" $s$ of stochastic quantisation (Parisi–Wu 1981, 7-09) (the abstract of Zwanziger 1981): the whole configuration moves by the following stochastic differential equation and settles to $e^{-S}/Z$ (the time of Monte Carlo methods is of the same kind).

$$d\phi_x = -\frac{\partial S}{\partial \phi_x}\, ds + \sqrt{2}\, dW_x(s) \quad (8)$$

$W_x(s)$ is a Brownian motion independent at each point $x$ (1-09), and $dW_x$ is the Gaussian increment over time $ds$, with mean 0 and variance $ds$. The first term moves in the direction that lowers the action, and the second adds fluctuations. On a computer one advances with step $h$ as $\phi_x \leftarrow \phi_x - h\,\partial S/\partial\phi_x + \sqrt{2h}\,Z_x$ ($Z_x$ independent standard normals), which is called the Euler–Maruyama method. The operator that generates the time change of the expectation of a function $f$ (the generator) is

$$Lf = -\Delta f + \nabla S \cdot \nabla f, \qquad \langle f, Lf\rangle_\mu = \mathcal{E}(f) = \int |\nabla f|^2 \, d\mu \quad (9)$$

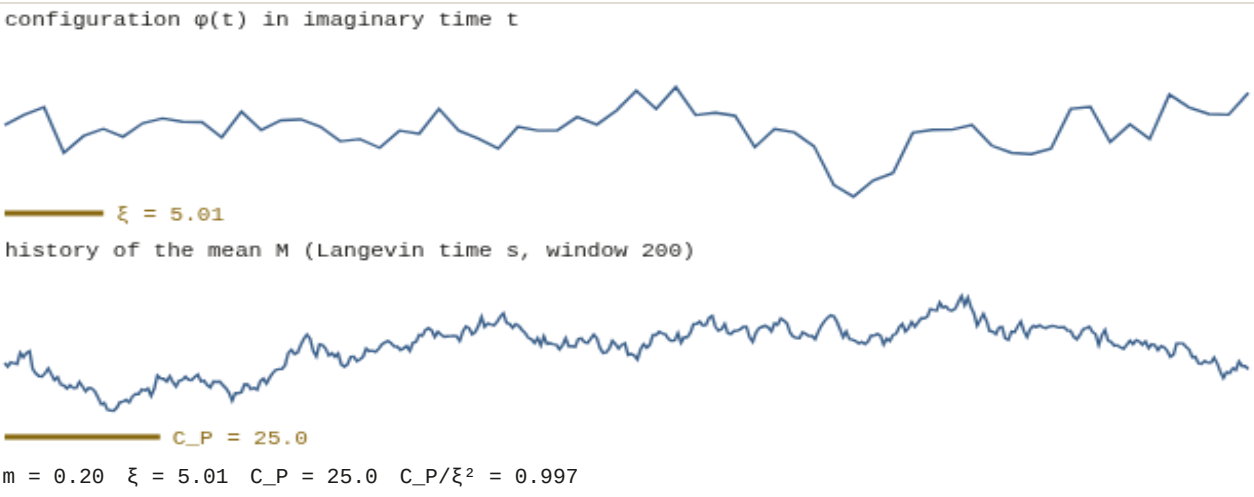
($\mu = e^{-S}/Z$, and $\mathcal{E}$ is the Dirichlet form; the $\Delta$ in (9) is the Laplacian, not the $\Delta$ of the mass gap), and the inverse of the gap above 0 is the Poincaré constant $C_P$ (1-10). On a finite lattice both are defined (the entry "Langevin relaxation"), but $H_T$ acts on the Hilbert space of a time slice, while $L$ acts on the space $L^2(\mu)$ of functions of the whole configuration (functions square-integrable with respect to $\mu$), and no theorem joins the two in general. For the free field $\mathrm{gap}(H_T) = 2\,\mathrm{arsinh}(m/2) \approx m$ and $\mathrm{gap}(L) = m^2$, so $C_P \approx \xi^2$ (1-10).

### The diffusive-relaxation-time assumption and the unit of time

The relaxation-time assumption of the record of this book's study is the assumption that $\mathrm{gap}(L) \asymp \mathrm{gap}(H_T)^2$ holds uniformly in the lattice spacing (23-01, 18-05). However, the unit of time of $L$ is fixed by the normalisation of $S$ and of the noise. Replacing $S$ by $\beta S$ multiplies the gap of $L$, in the Gaussian case, by exactly $\beta$. In gauge theory with $S = \beta\times$ (Wilson action), in the free approximation $C_P \approx \xi^2/\beta$; that is, $C_P$ is proportional to $g^2(a)\,\xi^2$, and this factor moves with the lattice spacing. Written at the level of a band (constant factors), the relaxation-time assumption is a statement made after

fixing this normalisation once (23-01). By the convention of the record, the relaxation-time assumption is put at the level of powers, and the band level is split off as a separate assumption that names the normalisation (18-05 § 03).

Figure 1 Changing m with the knob, the bar ξ at the top (the configuration φ(t) in imaginary time) grows as about 1/m, and the bar C_P at the bottom (the history of the mean M along the Langevin time s) as $1/m^2$. "Play" runs equation (8)




Computed The numbers in this figure were computed on this machine. The fluctuations are Euler–Maruyama with step $h = 0.05$. At this step the mean $M$ is multiplied by $1 - hm^2$ each step and noise is added, a first-order recurrence whose integrated autocorrelation time $1/m^2 - h/2$ agrees with $1/m^2$ to within 2.5% over the whole range of the knob ($0.1 \le m \le 1$). Measured on the histories run, it was checked at $m = 0.3,\ 0.5,\ 1$ and agrees to within 10%. Without JavaScript it can be read from Table 1.

Computed Table 1 The two gaps of the free field on the lattice (zero-momentum channel)

| $m$ | $\mathrm{gap}(H_T)$ | $\xi$ | $C_P = 1/m^2$ | $C_P/\xi^2$ |
|---|---|---|---|---|
| 1.00 | 0.962424 | 1.039 | 1.00 | 0.9263 |
| 0.10 | 0.099958 | 10.004 | 100 | 0.9992 |
| 0.02 | 0.020000 | 50.001 | 2500 | 1.0000 |

The gap agrees to 6 digits with the slope of the correlation on a ring of length 8192, and $CP$ with the smallest eigenvalue of $-\Delta_{\mathrm{lat}} + m^2$ on a ring of 256 points Computed . $CP/\xi^2 \to 1$ is the $z = 2$ of the free field.

06

## Autocorrelation times and the Poincaré constant — equality only in continuous time

When the dynamics is run from equilibrium, the correlation between values of an observable $f$ separated by time $t$, divided by the variance, is called the autocorrelation $\rho_f(t)$. The time of its slowest decay is $\tau_{\mathrm{exp}}$, $\tau_{\mathrm{int},f} = \frac{1}{2} + \sum_{t \ge 1} \rho_f(t)$ is the integrated autocorrelation time, and the variance of the sample mean is about $2\tau_{\mathrm{int},f}\,\mathrm{Var}(f)/n$

(Wolff 2004) Known . In continuous time the sum is replaced by an integral, $\tau_{\mathrm{int},f} = \int_0^\infty \rho_f(s)\,ds$. For reversible dynamics in continuous time $\rho_f(s) = \int e^{-s\lambda} d\nu_f(\lambda)$, with $\nu_f$ supported on $[\mathrm{gap}, \infty)$, so

$$\tau_{\exp} = \frac{1}{\mathrm{gap}(L)} = C_P, \qquad \tau_{\mathrm{int},f} = \int \frac{d\nu_f(\lambda)}{\lambda} \;\le\; C_P \quad (10)$$

For a reversible chain in discrete time (eigenvalues $1 > \lambda_2 \ge \cdots > -1$) they split into three (Levin–Peres–Wilmer § 12.2) Known .

$$C_P = \frac{1}{1-\lambda_2}, \qquad t_{\mathrm{rel}} = \frac{1}{1-\lambda_\star}, \qquad \tau_{\exp} = \frac{-1}{\log \lambda_\star}, \qquad \lambda_\star = \max_{\lambda \neq 1} |\lambda| \quad (11)$$

Example: for the AR(1) chain $X_{n+1} = \rho X_n + \sqrt{1-\rho^2}\,Z$ (eigenvalues $\rho^k$) with $\rho = -0.9$, $C_P = 5.263$, $t_{\mathrm{rel}} = 10$ and $\tau_{\exp} = 9.491$, all three different, and $\tau_{\mathrm{int}}$ is 0.026 for $x$ and 4.763 for $x^2$. In continuous time (Ornstein–Uhlenbeck) all three line up at 1 Computed . For a lazy chain (one that stays put with probability 1/2 each time, so all eigenvalues are at least 0), $\lambda_\star = \lambda_2$ and $C_P = t_{\mathrm{rel}}$ (LPW § 12.2, Exercise 12.3) Known .

### Differences of dynamics, and mixing time

The heat-bath method (redrawing variables one at a time from their conditional distributions), Langevin and HMC (hybrid Monte Carlo, Duane et al. 1987: add fictitious momenta, move by classical mechanics for a fixed time, and correct by acceptance or rejection) have different generators, and regarding the time of HMC as the $C_P$ of Langevin is an assumption (5-07). Thermalisation time and mixing time are also a Conditional pair. The mixing time $t_{\mathrm{mix}}(\varepsilon)$ is the number of steps until, from any initial state, the total variation distance (the largest difference between the probabilities two distributions give to the same event) between the distribution and the equilibrium distribution $\pi$ is at most $\varepsilon$. The $\log(1/\pi_{\min})$ in the upper bound $t_{\mathrm{mix}}(\varepsilon) \le t_{\mathrm{rel}} \log(1/(\varepsilon\pi_{\min}))$ (LPW Theorem 12.4; $\pi_{\min}$ is the smallest probability of $\pi$) grows with the volume. This is a theorem for reversible chains with finitely many states; for gauge fields taking values in a continuous group $\pi_{\min}$ has no meaning, and other tools are needed, such as the log-Sobolev constant ( § 07).

07

## Critical slowing down — $z$, the conventional theory, two $\eta$

As $\xi$ grows, the relaxation of the dynamics also becomes slower (critical slowing down). Physics writes this with the dynamic scaling hypothesis $\tau \propto \xi^z$ and calls $z$ the dynamical exponent (Halperin–Hohenberg 1969, Hohenberg–Halperin 1977). $z$ changes with the dynamics and the observable. (8) is the relaxational dynamics

without conserved quantities called model A in the classification of Hohenberg–Halperin. In mathematics it is $C_P \asymp \xi^z$, but the existence of a power is itself a hypothesis, and there is no theorem fixing as a power the $\xi$-dependence of $C_P$ in Yang–Mills (the entry "dynamical exponent z").

### The conventional theory and RG

The static exponents are those of 2-08: $\xi \sim |t|^{-\nu}$, $\chi \sim |t|^{-\gamma}$ ($t$ is the distance in temperature from the critical point), and $\eta_s$ is the amount by which the critical correlation $G(x) \sim |x|^{-(d-2+\eta_s)}$ deviates from the power of the free field (the anomalous dimension). From the scaling relation $\gamma = \nu(2-\eta_s)$, $\chi \propto \xi^{\gamma/\nu} = \xi^{2-\eta_s}$ Physics . The conventional theory (called by van Hove's name) is summarised as making the relaxation time proportional to $\chi$, giving $z = 2 - \eta_s$ Physics . The abstract of Halperin–Hohenberg–Ma (1972), who applied Wilson's expansion method to a continuum version of the kinetic Ising model, writes that the results 'disagree with the mode-mode coupling approach and the conventional (Van Hove) theory'. RG (the $\varepsilon$ expansion) and numerics both suggest $z > 2 - \eta_s$ (Table 2) Physics .

Table 2 The dynamical exponent $z$ and $2-\eta_s$ ($\eta_s$ is the static anomalous dimension)

| Model, method | $2-\eta_s$ | $z$ | $\eta_{\mathrm{dyn}} = 2 - z$ |
|---|---|---|---|
| 2D Ising, Glauber numerics | 1.75 | 2.1665(12) | −0.1665 |
| 2D Ising, 5 loops ($\varepsilon$ expansion) | 1.75 | 2.14(2) | −0.14 |
| 3D Ising, Glauber numerics | 1.9637 | 2.032(3) | −0.032 |
| 3D Ising, 5 loops ($\varepsilon$ expansion) | 1.9637 | 2.0235(8) | −0.0235 |
| Free field, Langevin | 2 | 2 | 0 |

Computed $2-\eta_s$ ($\eta_s = 1/4$ in two dimensions, $\eta_s = 2\Delta_\sigma - 1$ in three, with the scaling dimension of the spin $\Delta_\sigma = 0.5181489(10)$) and $\eta_{\mathrm{dyn}}$ computed on this machine. $z$ and $\Delta_\sigma$ are values from primary sources Physics . Glauber is a dynamics that updates spins one at a time at random.

### The variational lower bound

The counterpart on the side of mathematics is the variational lower bound. Putting $f = M = \sum_x \phi_x$ into (9), $|\nabla M|^2 = |\Lambda|$ (the number of points) and $\mathrm{Var}(M) = |\Lambda|\chi\Lambda$, so

$$C_P \;\geq\; \frac{\mathrm{Var}(M)}{\mathcal{E}(M)} = \chi\Lambda \quad (12)$$

which is a theorem holding on every lattice of finite volume Known . For the free field $\chi = 1/m^2 = C_P$, with equality (Table 1). Moving from here to a lower bound on the exponent needs two hypotheses: dynamic scaling $C_P \asymp \xi^z$ and static scaling $\chi \asymp \xi^{\gamma/\nu}$, under which $z \geq \gamma/\nu = 2 - \eta_s$ Physics (a lower bound regarded as standard in the

Monte Carlo literature). An example of a firm lower bound from an inequality of the same type is $z_{\mathrm{SW}} \geq \alpha/\nu$ for the Swendsen–Wang method ($\alpha$ is the specific-heat exponent; the abstract of Li–Sokal 1989) Known . The value of the conventional theory sits exactly at this lower bound, and the $z$ of Table 2 all lie above it.

In gauge theory this form does not reach $\xi^2$ Physics . Measuring the dimension $\Delta$ of an operator by the power when correlations behave at short distances like $|x|^{-2\Delta}$, the lowest dimension of a gauge-invariant local operator is 4. The two-point correlation of a dimension-4 operator decreases as $|x|^{-8}$, so its sum $\chi$ in four dimensions is decided at short distances and does not grow with $\xi$ (23-01).

### Two $\eta$

We split $\eta$ into two. The static $\eta_s$ is the anomalous dimension above, the dynamic one is $\eta_{\mathrm{dyn}} := 2 - z$, and they are equal only within the conventional theory (in Table 2 they differ even in sign). This book always attaches the subscript, and which one the record's $\eta$ is is fixed in 18-05.

### No slowing down away from criticality

Conversely, the premise of physics is that if $\xi$ is finite there is no slowing down (the entry "no slowing down away from criticality"). For the mathematical form we use two tools. The log-Sobolev inequality bounds the entropy, instead of the variance, by $\mathcal{E}$:

$$\int f^2 \log \frac{f^2}{\int f^2 \, d\mu} \, d\mu \;\leq\; 2C_{LS}\, \mathcal{E}(f)$$

It is stronger than the Poincaré inequality and gives $C_P \leq C_{LS}$ Known . The Dobrushin–Shlosman mixing condition is that, whatever boundary condition is imposed on a finite box, correlations inside the box decrease exponentially with distance at a uniform rate. The mathematical form of "no slowing down away from criticality" is, for discrete spin systems (finite spin space, finite range), the equivalence theorem "uniform mixing independent of boundary conditions (Dobrushin–Shlosman) $\Longleftrightarrow$ uniform log-Sobolev" (Stroock–Zegarlinski 1992; bibliographic data only) Known . A finite correlation length alone is not enough of a hypothesis. For gauge fields taking values in a continuous group the equivalence is not found within the material of this chapter. For lattice Yang–Mills, Shen–Zhu–Zhu (2023) show Poincaré, log-Sobolev and exponential decay at strong coupling. Neither result fixes the power on the side $\xi \to \infty$.

### Detour: for the free field the relaxation time equals the susceptibility

A subsection you may skip.

For the free field $S = \frac{1}{2}\sum_x[(\phi_{x+1} - \phi_x)^2 + m^2\phi_x^2]$, summing (8) over points, the nearest-neighbour coupling terms cancel, and the mean $M$ obeys

$$dM = -m^2 M\, ds + \sqrt{2/N}\, dW(s)$$

($N$ is the number of points and $W$ a single Brownian motion). $M$ is a single Ornstein–Uhlenbeck process, with relaxation time $1/m^2$, the same value as the susceptibility $\chi = \sum_x \langle \phi_x \phi_0 \rangle = 1/m^2$ (in the calculation on this machine too, $\chi$ and $C_P$ agree for every $m$ in Table 1) Computed . The "relaxation time $\propto \chi$" of the conventional theory has the same form as this relation, exact for the free field. With interactions the two are said to drift apart, and the $z > 2 - \eta_s$ of Table 2 corresponds to that drift Physics .

08

## Fluctuations of the tube and topological freezing — effective strings, Wulff, bottlenecks

### Effective strings and the Lüscher term

A long flux tube spontaneously breaks transverse translation, and Goldstone bosons live in 1+1 dimensions (Lüscher 1981). A Goldstone boson is a massless fluctuation that appears when a continuous symmetry is spontaneously broken; here it is the fluctuation of the transverse position of the tube. These fluctuations add a universal $1/R$ term to $V(R)$ (Lüscher–Symanzik–Weisz 1980, 17-03).

$$V(R) = \sigma R + \mu + \frac{\gamma}{R} + O(R^{-2}), \qquad \gamma = -\frac{\pi(d-2)}{24} \quad (13)$$

A two-dimensional conformal field theory (a two-dimensional field theory with a symmetry that includes scaling of distances) has a number called the central charge $c$, and one free boson has $c = 1$. The part of the ground-state energy of fluctuations confined to a strip of width $R$ that depends on the width (the Casimir energy) is proportional to $c$ (Blöte–Cardy–Nightingale 1986, Affleck 1986): $-\pi c/(24R)$ for an open strip and $-\pi c/(6L)$ for a periodic strip of circumference $L$. Setting $c = d - 2$, the former is $\gamma$ and the latter at $d = 4$ is $-\pi/(3L)$, which agrees with the correction subtracted when extracting the string tension on the lattice (Lucini–Teper 2001) Computed . The lattice $\sigma$ contains the physical assumption "the central charge of the tube is $d - 2$" (the entry "Lüscher term").

### Rigid tubes and roughening

At strong coupling, where the area law has been shown, the tube is said to be rigid. More precisely, each shift of the surface by one lattice step costs a small weight, so transverse fluctuations are exponentially suppressed (17-03). The shape of the tube at strong coupling can be read by analogy with the Wulff construction from a direction-dependent tension. The Wulff construction is a drawing that gives, under a tension $\tau(\mathbf{n})$ depending on direction, the shape that minimises the total tension of the boundary for a fixed enclosed area (17-03). It is said to be separated from the region of fluctuating tubes by the roughening transition. This is a picture of physics found in the title (abstract) of Lüscher 1981, and no theorem showing its existence in lattice gauge theory is found in the material of this chapter Physics (the entry "roughening, shape of the tube").

### Topological freezing and bottlenecks

In HMC for pure gauge theory, the effective dynamical exponent of the square of the topological charge is about 5, and Wilson loops decouple from this slow mode (the abstract of Schaefer–Sommer–Virotta 2011) Physics . The mathematical word is a bottleneck: putting into the variational principle a smoothed version $f$ of the indicator function of a set $A$ of configurations (configurations with topological charge close to 1, say),

$$C_P \;\geq\; \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \;\approx\; \frac{\mu(A)\big(1-\mu(A)\big)}{\mathcal{E}(f)} \qquad (14)$$

and the narrower the passage joining $A$ and its outside, the larger $C_P$ Known (the two-valley model, 1-10). More precisely, the configurations near the boundary of $A$ have small measure, so $\mathcal{E}(f)$ becomes small. (14) is a theorem, but that it is the cause of freezing, and carrying the phenomenon of HMC over to the $C_P$ of Langevin, have not been checked (the entry "topological freezing", 23-02).

### Detour: sectors of topological charge and narrow passages

A subsection you may skip.

For continuum fields the topological charge (the second Chern number, 3-05, 18-04) is an integer, and sectors of configurations with different values are not joined by smooth deformations. On the lattice the topological charge is an integer only approximately, and the sectors are joined through rough configurations at the scale of the lattice spacing. These passages are expected to become narrower as $a \to 0$ Physics . Taking the $A$ of (14) to be "configurations with topological charge close to 1", this expectation translates into the lower

bound on $CP$ becoming larger. There is no theorem showing how fast the passages become narrow.

## What this chapter can say

| What is said | What is not said |
|---|---|
| Known One direction of (1), (2), the existence of $\sigma$, the Theorem from Centre Symmetry to Confinement and the Theorem from Uniform Decay to the Area Law (Chatterjee 2021), (10)(11), $CP = t_{\mathrm{rel}}$ for lazy chains, the variational lower bounds (12)(14), "uniform mixing $\Longleftrightarrow$ uniform log-Sobolev" for discrete spin systems, the results for three-dimensional compact U(1) (abstract). | The converse theorems for every pair. That these quantities exist in the four-dimensional continuum theory. Which way the "Unsettled" pairs will move. The equivalence of mixing and log-Sobolev for gauge fields taking values in a continuous group. |
| Computed Table 1, the differences in Table 2, Figure 1 and the autocorrelation time of the mean $M$, the example in § 06, that for the free field the relaxation time agrees with $\chi$, $\pi/(3L)$. | The $z$ of Table 2 are not measurements on this machine. |
| Physics The numbers for $m/\sqrt{\sigma}$, $\sqrt{\sigma}$, glueballs and $z$, $z \geq \gamma/\nu$ under the two scaling hypotheses, the rigidity and roughening of the tube, the passages between topological sectors. | Not theorems. The form of extrapolation and $c = d - 2$ are assumptions. The conversion of $\sqrt{\sigma}$ into MeV is borrowed from another theory. |
| Which pairs the relaxation-time assumption and chain (ii) correspond to (the entries "Λ, one scale" and "z=2"), and that the relaxation-time assumption is a statement with the normalisation of time fixed. | Their proofs. Nothing is shown about the mass gap, confinement or the continuum limit. |

## Sources and reproduction

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| Gap, clustering, one-particle states, transfer matrix | Known | Jaffe–Witten (the Clay problem statement) § 4– § 6 (text). Lüscher, CMP 54 (1977) 283 (abstract only); Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (abstract: transfer matrix, infinite-volume limit and analyticity at strong coupling, Wilson's confinement estimate; exponential clustering not checked in the main text); Schor, NPB 222 (1983) 71, 231 (1984) 321 (abstracts only); Shen–Zhu–Zhu, CMP 400 (2023) 805 (abstract only; 'strictly positive mass gap'). |

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| $\xi$, $V$, $\sigma$, confinement, centre symmetry | Known | Chatterjee, arXiv:1803.01950; CMP 385 (2021) 1007, arXiv:2006.16229 (text). Bachas, PRD 33 (1986) 2723; Wilson, PRD 10 (1974) 2445; Polyakov, PLB 72 (1978) 477; Susskind, PRD 20 (1979) 2610; Svetitsky–Yaffe, NPB 210 (1982) 423; Adhikari–Cao, Ann. Probab. 53 (2025) (abstract only). Borgs–Seiler, CMP 91 (1983) 329 (bibliographic data only). Göpfert–Mack, CMP 82 (1982) 545–606, doi:10.1007/BF01961240 (checked in the abstract; INSPIRE; main text not read; the year is Crossref's year of publication, INSPIRE gives 1981). |
| $m/\sqrt{\sigma}$, $\sqrt{\sigma}$, $0^{++}$, $\pi/(3L)$ | Physics | Athenodorou–Teper, JHEP 11 (2020) 172 (the conversion of $\sqrt{\sigma}$ and the caveat are in § 5.2), JHEP 12 (2021) 082; Lucini–Teper, JHEP 06 (2001) 050; Morningstar–Peardon, PRD 60 (1999) 034509 (text). |
| Two times, autocorrelation, mixing time, HMC | Known, Physics | Levin–Peres–Wilmer, *Markov Chains and Mixing Times* 2nd ed. § 12.2, Exercise 12.3, Theorem 12.4 (text). Zwanziger, NPB 192 (1981) 259; Wolff, CPC 156 (2004) 143; Duane et al., PLB 195 (1987) 216 (abstracts only). Parisi–Wu, Sci. Sin. 24 (1981) 483 (bibliographic data only). |
| Critical slowing down, Table 2, the variational lower bound, the equivalence away from criticality | Physics, Known | Halperin–Hohenberg–Ma, PRL 29 (1972) 1548 (abstract). Nightingale–Blöte, PRB 62 (2000) 1089 (bibliographic data only; the source of the two-dimensional 2.1665(12)); Liu et al., PRE 108 (2023) 034118 (abstract: reconfirmation of 2.1665(12) and the three-dimensional value); Adzhemyan et al., PLA 425 (2022) 127870, arXiv:2111.04719 (abstract: the 5-loop values); Kos et al., JHEP 08 (2016) 036 (abstract); Li–Sokal, PRL 63 (1989) 827 (abstract). Halperin–Hohenberg, Phys. Rev. 177 (1969) 952, RMP 49 (1977) 435 (abstracts only). Stroock–Zegarlinski, CMP 144 (1992) 303 (title: the equivalence of the log-Sobolev inequality and the Dobrushin–Shlosman mixing condition), CMP 149 (1992) 175 (title: the log-Sobolev inequality for discrete spin systems) (bibliographic data only). The authority for van Hove's form $z = 2 - \eta_S$ and for the lower bound $z \geq \gamma/\nu$ not checked. |
| Effective strings, central charge, topological freezing | Physics | Lüscher–Symanzik–Weisz, NPB 173 (1980) 365; Lüscher, NPB 180 (1981) 317 (roughening in the title); Blöte–Cardy–Nightingale, Affleck, PRL 56 (1986) 742, 746; Schaefer–Sommer–Virotta, NPB 845 (2011) 93 (abstract only). |
| Wulff construction, roughening | Physics | Sources not checked in this chapter (Dobrushin–Kotecký–Shlosman: bibliographic data not checked). The construction and examples are in 17-03 and its table of sources. |
| The relaxation-time assumption, chain (ii) | Assumption, analogy | The arrangement of the record of this book's study (23-01, 17-12). |

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| Table 1, the differences in Table 2, Figure 1, the detour in § 06 and § 07, the numbers in § 08 | Computed | Python 3 on this machine (numpy): `two_times.py`, `zero_mode.py`, `ar1.py`, `exponents.py`. |

Next chapter to read: 18-04 A dictionary of scales and limits. For the correspondence with the assumptions of the record, see 18-05.

# A dictionary of scales and limits — words of flow translate into maps, words of limits into questions of existence

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — We pair 23 physical words about scale and limits with mathematical words, and grade them in the two columns of 18-01. We also see that a change of scheme is a one-off shift (Table 1), and the flow of the approximate Migdal–Kadanoff recursion (Figure 1).

**Prerequisite chapters** — 4-06, 7-05 and 17-08. The two columns are in 18-01, and mass and length in 18-03. The characters and Bessel functions of § 04 are in 7-02, and the minimum is supplied in § 04.

**Said first** — We only line up correspondences; we construct neither a flow nor a continuum limit. Figure 1 is a calculation of an approximate recursion, not a claim about the four-dimensional lattice theory.

The order of this chapter

1. The overall picture — Table 0
2. The renormalisation group and the β function — sequences of steps and Polchinski's equation
3. Flow on the lattice — Table 1
4. Migdal–Kadanoff and hierarchical lattices — Figure 1 / Detour: why the decrease per step is $(1-b^{-2})/3$
5. The continuum limit and a(β) — two kinds of scaling
6. Universality, Symanzik, marginality — the price of improvement
7. Large N, the master field, holography, the KK tower
8. Topology, instantons, the Wilson flow, the Gribov region
9. What had no word on the side of mathematics — the trace anomaly

01

## The overall picture — flow, limits, geometric scales

The words fall into three groups. Words of **flow** (renormalisation group, β function, $\Lambda$) translate into properties of the maps of the steps. Example: "the β function is negative" becomes "a step that doubles the lattice spacing lowers $\beta$ by an almost constant amount on the weak-coupling side" ( § 02, § 03).

Words of **limits** (continuum limit, $a(\beta)$, universality) translate into the question "does the limit exist, and is it unique?". Example: "universality" is the uniqueness claim "changing the details of the action leads to the same limiting measure" ( § 05, § 06).

Words of **geometric scale** ($N$, an extra direction, flow time, topology) look at the theory by adding another scale. Classically, many of them agree by theorems. Example: an "instanton" is exactly an anti-self-dual connection ( § 07, § 08).

Notation: $\beta = 2N/g^2$ is the coefficient of the lattice coupling (not the inverse temperature of statistical mechanics), $b$ is the factor of one step (multiplying the lattice spacing $a$ by $b$), and $\kappa = 4Nb_0$ (0.3715 for SU(2), A-01). Writing asymptotic freedom (4-06) in the lattice coupling gives the running per step. For SU(2), doubling $a$ lowers $\beta$ by about 0.2575, and halving $a$ raises it by the same amount.

$$\beta(ba) = \beta(a) - \kappa \ln b + O(1/\beta) \qquad \text{(one loop; } O(1/\beta) \text{ is two loops and beyond)} \quad (1)$$

Table 0 The list of the dictionary of scales and limits ((a) correspondence as definitions, (b) existence in four dimensions, (c) direction reachable by finite measurement; levels and conditions in the text)

| # | Physical word | Mathematical word | (a) | (b) | (c) |
|---|---|---|---|---|---|
| 1 | Renormalisation group | Sequence of steps, UV stability | Conditional | Conditional | — |
| 2 | β function | Recursion for the effective coupling | Conditional | Conditional | — |
| 3 | Polchinski's equation | Polchinski equation of a decomposition of the measure (heat flow) | Settled | None | — |
| 4 | Δβ of MCRG | The map of one step | Conditional | None | — |
| 5 | Change of scheme for Λ | Reparametrisation of the coupling | Settled | Constructed (formal power series) | — |
| 6 | Migdal–Kadanoff | Exact recursion on hierarchical lattices | Conditional | Constructed (hierarchical lattice) | — |

| # | Physical word | Mathematical word | (a) | (b) | (c) |
|---|---|---|---|---|---|
| 7 | Continuum limit | Scaling limit | Conditional | None | Out of reach |
| 8 | a(β), setting the scale | ε(β) (the continuum string-tension problem) | Unsettled | None | — |
| 9 | Scaling | Limit of dimensionless ratios | Unsettled | None | Out of reach |
| 10 | Asymptotic scaling | Asymptotic form of ε(β) | Unsettled | None | Out of reach |
| 11 | Universality | Uniqueness of the limit | Conditional | None | Out of reach |
| 12 | Symanzik's $O(a^2)$ | The shortfall per step δ(b) | Unsettled | None | — |
| 13 | Improved action | Loss of reflection positivity (contributions with negative weight) | Settled (for actions with dimension-6 operators added) | Constructed (lattice) | — |
| 14 | Marginal coupling | Critical Sobolev exponent | Settled | Constructed (classical) | — |
| 15 | Large N, planar diagrams | Planar maps, sums over strings | Conditional | Conditional | — |
| 16 | Master field | Limit of non-commutative distributions | Conditional | None | — |
| 17 | Radial direction of holography | Fefferman–Graham expansion | Unsettled | None | — |
| 18 | KK tower | $\lambda_1$ of the internal Laplacian | Conditional | Not applicable | Upper bound only |
| 19 | Topological charge | Second Chern number | Conditional | Conditional (lattice) | — |
| 20 | Instanton | Anti-self-dual connection | Settled | Constructed (classical) | — |
| 21 | Wilson flow | Yang–Mills heat flow | Settled | Constructed (classical) | — |
| 22 | Gribov region | Region where the FP operator is positive | Settled | Constructed (classical) | — |
| 23 | Trace anomaly | No counterpart | Unsettled | None | — |

Note: column (b) of Table 0 is the reading from before the published construction (2026-11-10). As in §02 and §09, the rows for the continuum limit (row 7) and the trace anomaly (row 23) are candidates to move with the published construction.

(c) is the measurement column of 18-01 §02 (the direction reachable by finite measurement). The rows about limits and uniqueness (the entries "continuum limit", "scaling", "asymptotic scaling" and "universality") are out of reach, and for the $\lambda_1$ of the entry "KK tower" a single test function gives an upper bound (1-10). "None" in (b) means, as in 18-01 §04, "not yet made".

The entries "Δβ of MCRG", "scaling" and "asymptotic scaling" have lattice numbers (at the level of physics), treated in §03 and §05.

02

## The renormalisation group and the β function — translated into sequences of maps

The renormalisation group is the operation of integrating out fluctuations at short scales and moving to an effective action at long scales. Wilson (1971) rewrote Kadanoff's theory of scaling in the form of differential equations (abstract: 'The Kadanoff theory of scaling ... is cast in differential form') Known . The existence of the map and convergence to a fixed point are premises in physics. In mathematics this becomes a **sequence of steps** that estimates the compositions of the one-step map $R_b$ uniformly (7-05).

$$\mu_a \xrightarrow{R_b} \mu_{ba} \xrightarrow{R_b} \mu_{b^2 a} \to \cdots, \qquad \text{control the effective action } S_k \text{ uniformly in } k \ (2)$$

For four-dimensional pure gauge theory, Bałaban (1987, 1989) built a sequence of effective actions and showed UV stability in finite volume. Chatterjee (2018) § 6 calls the existence of convergent subsequences UV stability Known . In Bałaban's method, convergence of the orbits and limits of observables are unproved (7-07). The construction published on 2026-11-10 takes limits of correlations of local observables with a sequence of renormalisation steps of a different design (Parts 9–11). Agreement of the sign of the β function (the entry "β function") with an exact flow is a theorem for the two-dimensional Gross–Neveu model (Gawędzki–Kupiainen 1985). In four dimensions it goes as far as Bałaban's recursion in the small-field approximation.

### Polchinski's equation

Polchinski (1984) wrote, for four-dimensional $\lambda\phi^4$, the change of the effective action as the momentum cutoff is moved continuously as an exact renormalisation group equation. On that basis he showed perturbative renormalisability Known . The same equation exists on the side of mathematics. Splitting the covariance of a Gaussian measure into a sum over scales and integrating one layer at a time from the fine scales, the remaining effective action $V_t$ obeys this equation. Rewritten for $e^{-V_t}$, it is convolution with a Gaussian kernel, that is, the flow of the heat equation Known . The pair is equation to equation, so the (a) of the entry "Polchinski's equation" is Settled.

There are also criteria standing on this equation. The log-Sobolev inequality is an inequality of the form $\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 d\mu$. It is a stronger version of the Poincaré inequality $\mathrm{Var}_\mu(f) \le C \int |\nabla f|^2 d\mu$ (1-10), with the variance replaced by the entropy $\mathrm{Ent}_\mu(g) = \int g \ln g \, d\mu - \int g \, d\mu \, \ln \int g \, d\mu$, and it guarantees how fast a stochastic dynamics

returns to equilibrium. Bauerschmidt–Bodineau (2021) extended the Bakry–Émery criterion (1-10) to multiple scales by controlling Polchinski's equation, and showed an asymptotically optimal log-Sobolev inequality for the massive continuum sine-Gordon model (a two-dimensional model, with coupling $\beta_{\mathrm{sG}} < 6\pi$) Known . The criterion is a tool standing on the pair, so it goes on the side of the conditions. There is no application to four dimensions, so (b) is "None".

03

## Flow on the lattice — MCRG and a one-off shift

MCRG (Swendsen 1979) fixes $\Delta\beta$ so that observables blocked with $b = 2$ match a lattice with coupling $\beta - \Delta\beta$. The first results for SU(3) (Gupta et al. 1984) reported no asymptotic scaling for fundamental-representation coupling $K_F < 6.1$ Known . $\Delta\beta$ is the amount by which $\beta$ is lowered when $a$ is doubled, the same quantity, measured by another method, as "the step that halves $a$" in Table 4 of 4-06. The mathematical partner is the one-step map $\beta \mapsto \beta - \Delta(b)$ ($\Delta(b)$ is the decrease per step, not the $\Delta$ of the mass gap), and with the same projection (the way of matching to the Wilson form) chosen, they are the same quantity (the entry "Δβ of MCRG").

Consider one step within the same way of fixing things (scheme). If at leading order at weak coupling ($\beta \to \infty$) $\Delta$ does not depend on $\beta$, then two steps in succession and a single multiplication by $b_1 b_2$ are the same, so

$$\Delta(b_1 b_2) = \Delta(b_1) + \Delta(b_2) \quad \Longrightarrow \quad \Delta(b) = C \ln b \ \text{(3)}$$

(Cauchy's functional equation under monotonicity). The constant $C$ is not fixed by additivity alone; comparing with (1) (one loop) gives $C = \kappa$. From two loops on, $\Delta$ keeps a $\beta$-dependence of order $O(1/\beta)$, and (3) is the leading-order formula that ignores it.

Integrating (1), the one-loop formula is $\beta = -\kappa \ln(a\Lambda)$ (the one-loop part of equation (8) of 4-06). A change of the way of fixing things, $\tilde{\beta} = \beta - c + O(1/\beta)$, only moves the map by conjugation and does not change the leading term of $\Delta$. What changes is the reading of $\Lambda$: from $\tilde{\beta} = -\kappa \ln(a\Lambda e^{c/\kappa})$, $\tilde{\Lambda} = \Lambda e^{c/\kappa}$, that is, $\Lambda$ is multiplied once only by $e^{c/\kappa}$ (the entry "change of scheme for Λ"). Dashen–Gross (1981) found this ratio to be $\Lambda_{\mathrm{MS}}/\Lambda_{\mathrm{L}} = 38.852704\, e^{-3\pi^2/(11N^2)}$ (19.823 for N = 2) (Wilson action) Known .

Computed Table 1 The Dashen–Gross ratio and the corresponding shift of $\beta$, $c_N = \kappa_N \ln(\text{ratio})$ ($\kappa_N = 11N^2/12\pi^2$). $c_N$ is the amount by which $\beta$ is lowered, $\beta_{\mathrm{MS}} = \beta_{\mathrm{L}} - c_N$ (the one-loop relation compared at $\mu = 1/a$). Number of steps $c_N/(\kappa_N \ln 2)$

| $N$ | Ratio | $c_N$ | Steps |
|---|---|---|---|

| $N$ | Ratio | $c_N$ | Steps |
|---|---|---|---|
| 2 | 19.823 | 1.1096 | 4.309 |
| 3 | 28.809 | 2.8092 | 4.848 |
| $\infty$ | 38.853 | – | 5.280 |

For SU(2), $c_2 = 1.1096$ is about 4.3 steps of the running $\kappa \ln 2 = 0.2575$ of one step with $b = 2$. Moving from the Wilson action to MS is the same as lowering $\beta$ once only by about 4.3 steps, that is, taking once only about 4.3 extra steps that double $a$. A map whose constant shifts at every step, $\Delta(b) = \kappa(\ln b - \delta(b))$, breaks (3) and moves the reading of $\Lambda$ step by step. The record of this book's study calls $\delta(b)$ in this form "the shortfall" (23-02, 18-05).

04

## Migdal–Kadanoff and hierarchical lattices — reading an approximation as an exact recursion on another lattice

Kadanoff (1976) rederived Migdal's (1975) recursion formula as an approximation to decimation, and proposed the scheme of potential (bond) moving (abstract) Known . Here we write one step as a pair of two operations. **Bond moving** gathers $b^{d-2}$ plaquettes in one place and turns the weight $f$ into $f^{\,b^{d-2}}$. **Decimation** combines $b \times b$ plaquettes of a plane into one. Decimation is exact in two dimensions (7-02).

We set down the notation a line at a time. Let the eigenvalues of an element $U$ of SU(2) be $e^{\pm i\theta}$, and write the weight of one plaquette of the Wilson action as $e^{\beta \cos\theta}$ ($\beta = 4/g^2$). $\chi_n$ is the character of the $n$-dimensional irreducible representation (spin $(n-1)/2$), $\chi_n(U) = \sin n\theta / \sin\theta$, with dimension $d_n = n$. $I_n$ is the modified Bessel function $I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta \, d\theta$. Expanding the weight in characters as $f = a_1 \sum_n d_n c_n \chi_n$, we call the coefficients $c_n$ ($c_1 = 1$) the normalised coefficients, and decimation is $c_n \mapsto c_n^{\,b^2}$. 7-02 writes the weight as $e^{\beta \,\mathrm{Re\,tr}\, U}$, so the $\beta$ of 7-02 is the $\beta/2$ of this chapter.

$$c_n \longmapsto \left[c_n\left(f^{\,b^{d-2}}\right)\right]^{b^{2}}, \qquad \text{for the SU(2) Wilson action } c_2 = \frac{I_2(\beta)}{I_1(\beta)} \quad (4)$$

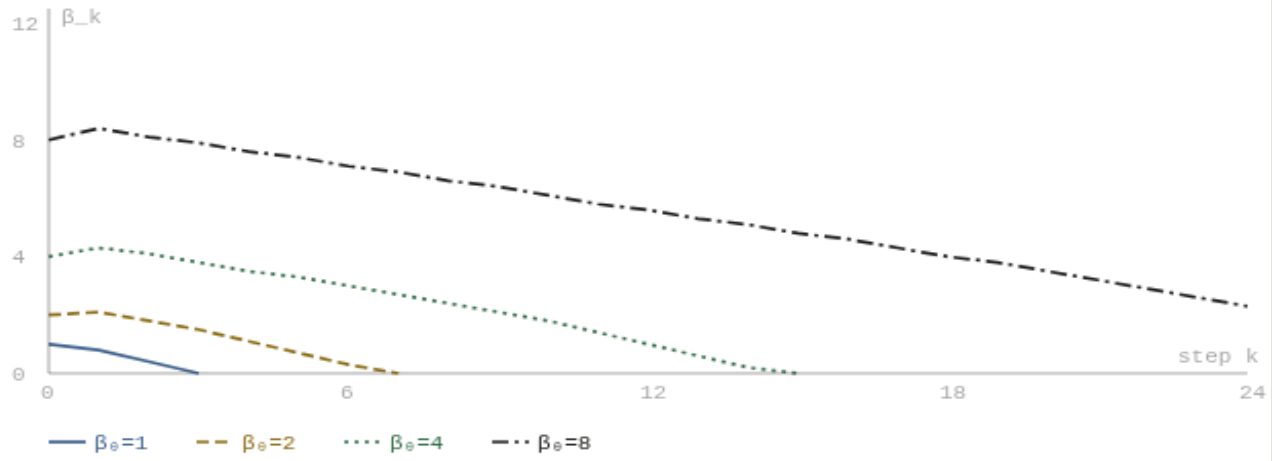
The approximation lies only in bond moving. On the lattice made by repeating the two operations in a nested way (a hierarchical lattice), (4) is exact by construction. For spin systems there are the hierarchical lattices of Griffiths–Kaufman (1982). For gauge theories Ito (1985) showed permanent confinement for groups such as SU(N) in a four-dimensional hierarchical lattice gauge theory on which the MK recursion holds exactly (abstract: 'hierarchical lattice gauge theories in which the Migdal-Kadanoff approximate renormalization recursion formulas hold exactly') Known . The

"approximation" of physics is, in mathematics, "an exact map on another lattice" (the entry "Migdal–Kadanoff"; (b) is at the level of this hierarchical lattice).

Example: at strong coupling $c_2 \approx \beta/4$, so one step with $d = 4$, $b = 2$ is $\beta' \approx 4\beta^4$, and with decimation alone $\beta^4/64$ (at $\beta = 0.1$, $3.895 \times 10^{-4}$ and $1.560 \times 10^{-6}$) Computed . The latter is the strong-coupling formula of one step of bookkeeping in the record (23-02), and the missing factor $b^{d-2}$ is known in the record too.

### Where the flow goes, and the decrease per step

Figure 1 The Migdal–Kadanoff flow ($b = 2$). Changing d with the knob and the group with the buttons changes where $\beta_k$ goes from four starting points. "Play" advances the steps. The colour and dash pattern of each line correspond to the starting point $\beta_0$ in the legend at the bottom




d = 4, SU(2): 1 → towards strong coupling, 2 → towards strong coupling, 4 → towards strong coupling, 8 → 24 steps, down to 2.3

Computed The numbers in this figure were computed on this machine (the weight function iterated with 8000 points and 400 characters; $\beta$ is read from the $c_2$ of (4), and for U(1) from $I_1/I_0$). Without JavaScript it can be read from the next two paragraphs.

For $d < 4$ the flow goes towards strong coupling, for $d > 4$ towards weak coupling, and $d = 4$ is the border. For SU(2) at $d = 4$, all the starting points examined ($\beta_0 \le 20$) go towards strong coupling ($\beta_0 = 8$ reaches $\beta < 0.05$ in 31 steps, and 20 in 79 steps) Computed . For U(1), starting points with $\beta_0 \ge 1$ stop at values of about $\beta \approx 1.4$ or more and lie on a line of fixed points (5-06). The first step goes up, since it moves from the Wilson form to the settled form of the recursion; this too is a one-off shift.

The decrease per step after the form has settled at weak coupling (SU(2) at $d = 4$, the 10th step from $\beta_0 = 60$) is 0.2503, 0.2965 and 0.3126 for $b = 2, 3, 4$ Computed . The step given by one loop, $\kappa \ln b$, is 0.2575, 0.4081 and 0.5150. The MK values match the formula $(1 - b^{-2})/3$ of the detour below (0.2500, 0.2963, 0.3125) Computed . For $b = 2$ it is 97% of the one-loop value, but two steps with $b = 2$ (0.5006) and one step with $b = 4$ (0.3126) differ, so (3) is not satisfied. The $a(\beta)$ of MK depends on $b$.

### Detour: why the decrease per step is (1−b⁻²)/3

A subsection you may skip.

There is a weight whose form is preserved by decimation $c_n \mapsto c_n^{b^2}$ : the weight with coefficients $c_n = e^{-(n^2-1)/(2\beta)}$, called the SU(2) **heat kernel** (the $Q_t$ of 7-02 § 04 with $t = 4/\beta$). Decimation only moves it as $\beta \mapsto \beta/b^2$. When $\theta$ is small, this weight can be written

$$f_\beta(\theta) \propto \frac{\theta}{\sin\theta}\, e^{-\beta\theta^2/2}, \qquad \frac{\theta}{\sin\theta} = 1 + \frac{\theta^2}{6} + \cdots \approx e^{\theta^2/6}$$

Known . It differs from the Wilson weight $e^{\beta\cos\theta} \approx e^{\beta} e^{-\beta\theta^2/2}$ only by the prefactor $\theta/\sin\theta$. Raising to the power $b^2$ by bond moving ($d = 4$), the prefactor is also multiplied $b^2$ times. The heat-kernel form needs only one, so the extra $b^2 - 1$ enter the exponent as $e^{(b^2-1)\theta^2/6}$.

$$f_\beta^{b^2} \propto \frac{\theta}{\sin\theta}\, e^{-\left(b^2\beta - \frac{b^2-1}{3}\right)\theta^2/2} : \qquad \beta \mapsto b^2\beta - \frac{b^2-1}{3} \xrightarrow{\text{decimation}} \beta - \frac{1-b^{-2}}{3}$$

The decrease per step is $(1 - b^{-2})/3$, independent of $\beta$. Unlike $\kappa \ln b$, it is not additive in $b$, because the decrease comes not from a logarithm but from the coefficient $1/6$ of $\theta^2$ in $\theta/\sin\theta$. This calculation is only the leading term for small $\theta$, so agreement with the numbers in the text (0.2500 and 0.2503, and so on) is within the range checked by computation Computed .

For how this book uses this, see 15-04 § 02 (the invariant curve and the contraction of the same single-step map).

05

## The continuum limit and a(β) — two kinds of scaling

On the continuum limit (the entry "continuum limit"), Chatterjee (2018) § 5 summarises the usage of physics: if the correlation length diverges at a critical point $\beta_c$ one says 'the model has a continuum limit' Known . The scaling limit is the operation of shrinking the lattice to $\varepsilon\mathbb{Z}^d$, normalising the field and letting it converge weakly as a distribution (2-08); once what is held fixed and the normalisation are specified, it is the same operation. The non-triviality that the physical word includes is not in the definition, and the limits of the four-dimensional Ising type and of $\phi^4$ are Gaussian (Aizenman–Duminil-Copin 2021, 7-08).

There are two ways of choosing $a(\beta)$ (the entry "a(β), setting the scale"). The explanation in § 5 fixes the correlation length, and the continuum string-tension problem quantifies existentially with Wilson loops (the same formula as equation (4) of 18-03).

$$\log\langle W_{R/\varepsilon, T/\varepsilon}\rangle = -c\,(R+T) - d\,RT + o(1), \qquad \varepsilon(\beta) \to 0,\ c(\beta) \to \infty,\ d\ /= 0 \quad (5)$$

The $d$ here is the coefficient of the area law (the number corresponding to the continuum string tension), a different symbol from the dimension $d$ in (8). The continuum string-tension problem does not presuppose the lattice correlation-length problem (Problem 5.1), and the existence of either $\varepsilon(\beta)$ is unproved. Physics fixes the scale by the string tension (Creutz 1980) or $r_0$ (Sommer 1994) Physics .

**Scaling** (the entry "scaling") is that dimensionless ratios stop depending on $a$, and **asymptotic scaling** (the entry "asymptotic scaling") is that $a(\beta)$ follows the two-loop formula.

$$\frac{m_{\mathrm{lat}}(\beta)}{\sqrt{\sigma_{\mathrm{lat}}(\beta)}} \to \text{constant} \qquad \text{and} \qquad a\Lambda_{\mathrm{L}} = (b_0 g^2)^{-b_1/(2b_0^2)} e^{-1/(2b_0 g^2)}\bigl(1 + O(g^2)\bigr) \tag{6}$$

In the lattice numbers the two behave separately. The deviation on the right shows up in the report that MCRG for SU(3) sees no asymptotic scaling for $K_F < 6.1$ (Gupta et al. 1984). In Table 4 of 4-06 too, the data for the step that halves $a$ (0.35–0.52) do not reach the two-loop value (about 0.61) Physics . The left is supported by dimensionless ratios lining up at one value in extrapolations that assume $a^2$ corrections (the $m/\sqrt{\sigma}$ of Lucini–Teper 2001, the $\sqrt{8t_0}/r_0$ of Lüscher 2010) Physics . Fits for SU($N$) with $N > 4$ do not pin down $a(\beta)$ in units of $\Lambda_{\overline{\mathrm{MS}}}$ (Allton–Teper–Trivini 2008) Physics . The mathematical partners are, for the left, the limit of the pair of the lattice correlation-length problem and the continuum string-tension problem (the entry "m/√σ" of 18-03), and for the right, the asymptotic form of $\varepsilon(\beta)$; neither has a theorem (5-05, 5-09).

06

## Universality, Symanzik, marginality — two errors in different variables

Universality (the entry "universality") is that the same limit comes out independently of the details of the action. Griffiths (1970) and others stated it as a hypothesis, and Wilson (1971) explained it by irrelevant variables. In mathematics it is uniqueness of the limit, and the theorems are in two dimensions: Ising on isoradial graphs (Chelkak–Smirnov), and the Wilson, Villain and Manton actions going to the same measure for the YM measure of the trivial bundle on the two-dimensional torus (Chevyrev–Shen) Known . In four dimensions, that the Wilson action and improved actions give the same theory is a premise.

Symanzik's (1983) improvement (the entry "Symanzik's O(a²)") is the idea of reducing the errors at finite $a$ systematically by adding irrelevant terms. Under a single action, the error of a dimensionless ratio is expanded in $a$.

$$R(a) = R(0)\bigl(1 + s\,a^2 + \cdots\bigr) \ \ (a \to 0\text{: vanishes}) \qquad \text{versus} \qquad \sum_{j<k} \kappa\,\delta(b_j) \ \ (\text{accumulates ste}$$

For Yang–Mills the logarithmic corrections to $a^2$ are also said to be weak (Husung–Marquard–Sommer 2020) Known . The record's $\delta(b) = c_2/b^2 + c_3/b^3 + O(b^{-4}\ln b)$ is an expansion in the factor $b$, a different variable, so the pairing with Symanzik remains an analogy (Unsettled). Example: over 20 steps, the left is $4^{-20}s \approx 9\times10^{-13}s$, and the right, with $b$ fixed, is $20\kappa\delta(2)$. In an example where the factor increases by one at each step, $b_j = j+2$, the leading terms alone give $2.39\kappa\delta(2)$ (the limit with the steps continued without end is $4(\pi^2/6-1)\approx 2.58$ times) Computed .

### The price of improvement (the entry "improved action")

Lüscher–Weisz (1984) wrote that adding dimension-6 operators to the Wilson action means 'physical positivity is lost in general', and showed that the transfer matrix can still be defined. It has two properties. Complex eigenvalues can appear ('may occur'). The spectral decomposition of two-point functions always has contributions with negative weight ('there are always contributions ... that come with a negative weight') Known . In the language of reflection positivity (5-03), the mathematical word of the entry "improved action" is "loss of reflection positivity (contributions with negative weight)", grounded on the 'always'. (a) is Settled for actions with dimension-6 operators added (the judge is a theorem). The kernel that sets $\delta(2)=0$ in the record also has negative weights; whether it is of the same type is treated in 18-05.

### Marginality (the entry "marginal coupling")

Shrinking as $A_\lambda(x) = \lambda A(\lambda x)$,

$$\int_{\mathbb{R}^d} \operatorname{tr} F_\lambda^2 \, d^dx = \lambda^{4-d}\int_{\mathbb{R}^d}\operatorname{tr}F^2\,d^dx, \qquad g^2 \sim (\text{length})^{d-4} \tag{8}$$

so at $d=4$ the action is scale invariant (the border in Figure 1). In mathematics this is the critical Sobolev exponent. The $L^p$ norm $\|F\|_{L^p} = (\int |F|^p\, d^dx)^{1/p}$ gives stronger control the larger $p$ is, and under shrinking $\|F_\lambda\|_{L^p}^p = \lambda^{2p-d}\|F\|_{L^p}^p$. The exponent that does not change under a change of scale, $p=d/2$, is the critical exponent (1-11, 1-12). Uhlenbeck's (1982) weak compactness (the theorem extracting a convergent subsequence from a sequence of connections) needs $p>d/2$ for boundedness of the $L^p$ norm of the curvature. The four-dimensional action has $p=2=d/2$, exactly at the threshold Known . The problem statement writes the outlook that asymptotic freedom rescues the case where the Sobolev inequality is at the borderline (0-03). The agreement goes only as far as classical dimension counting (7-09, the four-dimensions article).

> 07

## Large N, the master field, holography, the KK tower — words that add another scale

The four words of this section look at the theory by adding one scale outside four-dimensional spacetime: the size $N$ of the group, the depth from the boundary into the interior, and the size of rolled-up extra dimensions.

### Large N and planar diagrams (the entry "large N, planar diagrams")

't Hooft (1974) showed that with $\lambda = g^2 N$ fixed and $N \to \infty$, planar diagrams dominate (perturbation theory, 17-04). Two things agree as theorems: planar maps of matrix models (Guionnet–Maurel-Segala), and Wilson loops of the strongly coupled SO($N$) lattice written as sums over trajectories of strings (Chatterjee, Basu–Ganguly) Known .

### The master field

The idea that as $N \to \infty$ fluctuations disappear and gauge invariants are fixed by a single configuration. In mathematics it corresponds to a deterministic limit of non-commutative distributions. A non-commutative distribution regards the totality of expectations of traces of products, $\frac{1}{N}\mathbb{E}\,\mathrm{tr}(X_{i_1}\cdots X_{i_k})$, for a set $X_1, X_2, \ldots$ of random $N \times N$ matrices, as the "distribution" of the matrices (the language of free probability). For planar two-dimensional Yang–Mills, Lévy showed that Wilson loops approach a deterministic limit in this sense (17-04) Known .

### The radial direction of holography

Since the infrared of the interior is mirrored in the ultraviolet of the boundary (Susskind–Witten 1998), the radial direction is read as the scale of the boundary (17-08). What is certain in mathematics is only the expansion of the metric near the boundary (Fefferman–Graham). It expands the metric of the interior in powers of the depth $z$ from the boundary, in the form $ds^2 = (dz^2 + g_z)/z^2$, $g_z = g_{(0)} + z^2 g_{(2)} + \cdots$ ($g_{(0)}$ is the metric of the boundary) Known . The side of physics too says the correspondence with the cutoff is 'left unanswered' (Heemskerk–Polchinski 2010) Physics .

### The KK tower

Rolling up extra dimensions into an internal space $K$ makes a sequence of masses appear. For a massless scalar on an unwarped product,

$$m_1^2 = \lambda_1(\Delta_K) \qquad \text{(for a circle of radius } R,\ 1/R^2\text{)} \ (9)$$

(17-06) Known . $\lambda_1(\Delta_K)$ is the smallest non-zero eigenvalue of the Laplacian on $K$. No construction is known that separates the KK tower from $\Lambda$ while keeping geometric calculation (17-10). Pure Yang–Mills has no extra dimensions, so (b) is Not applicable.

08

## Topology, instantons, the Wilson flow, the Gribov region — words that are Settled classically

The four words of this section agree with mathematical words by theorems for classical fields. Where they are carried into quantum fields, they become conditional or at the level of physics.

### Instantons

The conventions of this section are the same as A-01: $g$ is put in front of the action and not included in $F$: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$ ($A_\mu = A_\mu^a T^a$). The Hermitian generators are normalised as $\operatorname{tr} T^a T^b = \frac{1}{2}\delta^{ab}$, so the action is $S = \frac{1}{2g^2}\int \operatorname{tr} F_{\mu\nu}F_{\mu\nu} = \frac{1}{4g^2}\int F^a_{\mu\nu}F^a_{\mu\nu}$. Setting the dual to $\tilde F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$ ($\varepsilon$ totally antisymmetric with $\varepsilon_{1234} = 1$), $\int (F^a \mp \tilde F^a)^2 \geq 0$ bounds the action below by the topological charge.

$$S \geq \frac{8\pi^2}{g^2}|Q|, \qquad Q = \frac{1}{32\pi^2}\int F^a_{\mu\nu}\tilde F^a_{\mu\nu}\, d^4x \in \mathbb{Z} \quad (10)$$

Equality holds only for (anti-)self-dual fields (3-03). (12) of 3-03 is written in the canonical normalisation $D_\mu = \partial_\mu - igA_\mu$, and rewriting with $A = gA_{\text{3-03}}$ (and so $F = gF_{\text{3-03}}$) agrees with (10). With the solution of BPST (1975), the construction of all solutions by ADHM (1978), and Uhlenbeck's (1982) "finite-action fields on $\mathbb{R}^4$ come from $S^4$", they are Settled as classical objects Known . Approximating the path integral by sums over instantons is a separate, quantum claim.

### Topological charge

The topological charge is the second Chern number. The second Chern number is an integer determined by $\frac{1}{8\pi^2}\int \operatorname{tr} F \wedge F$ for a connection on a closed four-dimensional manifold (for example $S^4$) (in the convention with $A$ anti-Hermitian, A-01), unchanged when the connection is varied continuously (the four-dimensional version of the first Chern number of 1-07). It differs from the $Q$ of (10) only by a sign convention (the detour in 3-05). On the lattice, from fields with each plaquette $U_p$ close enough to the identity (action density below a threshold: admissible) a bundle can be built, and it is an integer (Lüscher 1982, 3-05) Known . Otherwise the sectors are 'a dynamical property of the theory' (Lüscher 2010, SU(3)) Physics .

### The Wilson flow

The Wilson flow is the equation $\dot{B}_\mu = D_\nu G_{\nu\mu}$, $B_\mu|_{t=0} = A_\mu$, which smooths the field by adding a flow time $t$. $G_{\mu\nu}$ is the field strength of $B$, and $D_\nu$ the covariant derivative by $B$. It is steepest descent of the action, and the equation is exactly the Yang–Mills heat flow. The radius $\sqrt{8t}$ is the value at $d = 4$ of the spread $\langle |x|^2 \rangle = 2dt$ of the heat kernel $e^{t\Delta}$. Its action on quantum fields goes as far as the three-dimensional case under a GFF assumption (Cao–Chatterjee) and two-dimensional YM and three-dimensional YMH (a model including a Higgs field, CCHS) through stochastic quantisation Known . Four dimensions is at the stage of Lüscher's 'suggest' Physics .

### The Gribov region

Gribov (1978), faced with the ambiguity of the Landau gauge $\partial_\mu A_\mu = 0$, restricted the integral to fields where the FP determinant is positive (3-04, 4-08). The FP operator $M(A) = -\partial_\mu D_\mu(A)$ is the operator expressing the change in the gauge condition when the field is moved a little by a gauge transformation, and its determinant is the Faddeev–Popov determinant that multiplies the path integral. $D_\mu(A)X = \partial_\mu X - i[A_\mu, X]$ is linear in $A$, so

$$\Omega = \{A : \ \partial_\mu A_\mu = 0,\ M(A) \geq 0\}, \qquad M\big(tA_1 + (1-t)A_2\big) = tM(A_1) + (1-t)M(A_2) \quad (11)$$

and $\Omega$ is convex ($M \geq 0$ in the sense of positive semi-definite as an operator). Zwanziger (1982) showed 'bounded in every direction', and Dell'Antonio–Zwanziger (1991) showed that every orbit enters this region Known . The picture that the measure on it produces confinement is at the level of physics (17-02).

09

## What had no word on the side of mathematics — the trace anomaly

In classical four-dimensional Yang–Mills, the scale invariance of (8) makes the trace $T^\mu{}_\mu$ of the energy-momentum tensor vanish (3-03). The trace is a quantity that measures locally the change of the action when the field is shrunk by $x \mapsto \lambda x$, and it is 0 if the action does not change under a change of scale. In the quantum theory, integrating the fluctuations needs a cutoff (a lattice spacing, say), and a length enters there. The coupling $g(\mu)$ runs with respect to that length, so scale invariance breaks. Writing the way it breaks locally is the trace anomaly.

Here we write $F$ in the canonical normalisation ($D_\mu = \partial_\mu - igA_\mu$, the convention of 3-03). It is related to the field of the convention of (10) by $F_{(10)} = gF$, and writing (12) in the convention of (10), the coefficient becomes $\frac{\mu\, dg/d\mu}{2g^3}$.

$$T^{\mu}{}_{\mu} = \frac{\mu\, dg/d\mu}{2g}\, F^{a}_{\mu\nu} F^{a\,\mu\nu}, \qquad \mu\frac{dg}{d\mu} = -b_0 g^3 + O(g^5) \quad (12)$$

This is a formula cited in standard form (the normalisation differs between references; Collins–Duncan–Joglekar 1977) Physics , and the coefficient is the β function itself. Under asymptotic freedom ($\mu\, dg/d\mu < 0$) the trace does not become 0. This is the local mechanism of dimensional transmutation (4-07). Before the publication of 2026 there was no counterpart on the side of mathematics, because there was no four-dimensional continuum theory and no place to put this operator (as with the Λ of 18-03). The committee's assessment is that the construction published on 2026-11-10 defined the stress tensor and its trace as renormalised local operators and showed their properties (13-02 The trace anomaly, and not being free). Agreement with perturbation theory at all orders has not been shown (6-04). The 7 rows of Table 0 whose (a) is Settled are agreements of equations, definitions or classical objects, and in the reading from before publication no row has (b) "exists" in four-dimensional continuum quantum theory (if the rows are regraded, the entry "trace anomaly" is a candidate to move; 18-01 § 09).

## What this chapter can say

| What is said | What is not said |
|---|---|
| Known The leading term of (3) (excluding $O(1/\beta)$) and the change of scheme by conjugation, the Dashen–Gross ratio, (8) (10)(11), that MK is exact on hierarchical lattices (Ito 1985), the rows whose (a) is Settled. | The existence of the objects in the rows whose (b) is "None". Where the Conditional and Unsettled pairs will go. |
| Computed Table 1, Figure 1, the numbers in § 04 and § 06. Physics (12), the numbers for asymptotic scaling and scaling, the radial direction. | MK and the Physics labels are not theorems. |
| That the record's $\delta(b)$ and Symanzik differ in the variable of expansion. | The summability of $\delta$ (18-05). Nothing is shown about the flow, the continuum limit or the mass gap. |

## Sources and reproduction

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| Flow ( § 02, § 03) | Known | Checked in abstracts: Wilson, PRB 4 (1971) 3174; Bałaban, CMP 109 (1987) 249, 122 (1989) 355; Gawędzki–Kupiainen, PRL 54 (1985) 2191; Polchinski, NPB 231 (1984) 269; Bauerschmidt–Bodineau, CPAM 74 (2021) 2064; Swendsen, PRL 42 (1979) 859; Gupta et al., PRL 53 (1984) 1721; Dashen–Gross, PRD 23 (1981) 2340. Text: Chatterjee, arXiv:1803.01950. |

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| MK, hierarchical lattices ( § 04) | Known | Checked in abstracts: Kadanoff, Ann. Phys. 100 (1976) 359; Ito, PRL 55 (1985) 558. Bibliographic data only: Migdal, JETP 42 (1975) 413; Griffiths–Kaufman, PRB 26 (1982) 5022 (hierarchical lattices for spin systems). |
| Limits, universality, improvement ( § 05, § 06) | Known, Physics | Checked in abstracts: Aizenman–Duminil-Copin, Ann. Math. 194 (2021) 163; Creutz, PRD 21 (1980) 2308; Sommer, NPB 411 (1994) 839; Allton–Teper–Trivini, JHEP 07 (2008) 021; Griffiths, PRL 24 (1970) 1479; Chelkak–Smirnov, arXiv:0910.2045; Chevyrev–Shen, arXiv:2302.12160; Symanzik, NPB 226 (1983) 187; Lüscher–Weisz, NPB 240 (1984) 349; Husung–Marquard–Sommer, EPJC 80 (2020) 200; Uhlenbeck, CMP 83 (1982) 11, 31. Text: Lucini–Teper, JHEP 06 (2001) 050; Lüscher, JHEP 08 (2010) 071; Jaffe–Witten (the Clay problem statement). |
| § 07 | Known, Physics | Checked in abstracts: 't Hooft, NPB 72 (1974) 461; Guionnet–Maurel-Segala, arXiv:math/0503064; Chatterjee, arXiv:1502.07719; Basu–Ganguly, arXiv:1608.04379; Lévy, arXiv:1112.2452; Susskind–Witten, hep-th/9805114; Heemskerk–Polchinski, arXiv:1010.1264; Fefferman–Graham, arXiv:0710.0919. The original Fefferman–Graham is said to be from 1985 (bibliographic data only; not cross-checked). |
| § 08, § 09 | Known, Physics | Checked in abstracts: Lüscher, CMP 85 (1982) 39; BPST, PLB 59 (1975) 85; ADHM, Phys. Lett. A 65 (1978) 185; Cao–Chatterjee, arXiv:2111.12813; CCHS, arXiv:2006.04987, 2201.03487; Gribov, NPB 139 (1978) 1; Zwanziger, NPB 209 (1982) 336; Dell'Antonio–Zwanziger, CMP 138 (1991) 291; Collins–Duncan–Joglekar, PRD 16 (1977) 438 (the abstract has no formula, and the normalisation of (12) is not cross-checked; (12) carries the Physics label as a standard form). Text: Lüscher, JHEP 08 (2010) 071. |
| $\delta(b)$, one step of bookkeeping | Record | The arrangement of the record of this book's study (23-02, 18-05). |
| Table 1, Figure 1, numbers | Computed | Python 3 on this machine: `mk_flow.py`, `mk_steady.py`, `mk_long.py`, `dg_shift.py`, `sym_delta.py` (the 2.39, 2.58 and $9 \times 10^{-13}$ of § 06). |

Next chapter to read: 18-05 The physical counterparts of our assumptions.

# The physical counterparts of our assumptions — are the record's assumptions stronger than the hypotheses of physics?

This chapter is this book's own investigation. The established knowledge it builds on is in 18-03 and 18-04.

**What this chapter shows** — We set the assumptions that the record of this book's study put in the language of mathematics against the hypotheses of physics, one row at a time, and attach "the same", "a conditional restatement" or "an analogy" in the form of questions. The record's $\eta$ is fixed to $\eta_{\mathrm{dyn}} = 2 - z$.

**Prerequisite chapters** — 18-03 (two times, critical slowing down), 18-04 (change of scheme) and 17-12 (a single scale, the diffusive-relaxation-time assumption, the three chains). The two-column grades are in 18-01. The detailed treatment of the relaxation-time assumption is in 23-01, in a later part.

**Said first** — The correspondences are put as questions and analogies, and none of the assumptions has been proved. No row shows anything about the mass gap or the continuum limit.

The order of this chapter

1. The overall picture — setting the assumptions against the hypotheses of physics, and dividing them into the same, conditional, analogy (Table 0)
2. Fixing the record's η to one — the η in the power form is the dynamic η_dyn = 2 − z
3. The diffusive-relaxation-time assumption and z = 2 — the same power, but uniformity up to constant factors breaks logarithmically depending on the normalisation (Table 1). Detour: why diffusion takes $\xi^2$
4. The relaxation-time upper-bound assumption and z ≤ 2 — three conditions for the restatement to hold (Figure 1)
5. The relaxation-time lower-bound assumption and the lower bound of critical slowing down — the same inequality, and dimension-4 operators do not reach $\xi^2$ (Table 2)

6. Summability of the shortfall — step-by-step shifts and changes of scheme, MCRG, kernels with negative weights
7. The equivalence of the physical-unit gap statement — readable as the product of the invariance of mass and z = 2, with cancellation when the premise is dropped (Figure 2, Table 3)
8. A single scale and extra dimensions — dimensional analysis inside the models, which stays an analogy
9. What is known and settled — five rows not counted as results

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01

## The overall picture — setting the assumptions against the hypotheses of physics, one row at a time

The record of this book's study wrote the line towards the continuum limit as a sequence of "bookkeeping of steps" (23-01), and put assumptions in the language of mathematics along the way. The answers given against the old hypotheses of physics are of three kinds: **the same** (reducing to the same inequality or definition), **a conditional restatement** (the same once the power form or the choice of dynamics is added), and **an analogy** (similar in form, but no theorem joins them). They correspond to Settled, Conditional and Unsettled in the definition column of 18-01 (the existence column is "None" in every row). The central relaxation-time assumption splits into an upper and a lower half.

$$(\xi^2)\ (C_P \asymp \xi^2)\ =\ (\xi^2)\uparrow\,(C_P \lesssim \xi^2)\ \wedge\ (\xi^2)\downarrow\,(C_P \gtrsim \xi^2)$$

In the formula, (ξ²) is the diffusive-relaxation-time assumption, (ξ²)↑ the upper-bound assumption and (ξ²)↓ the lower-bound assumption.

Table 0 The record's assumptions and the candidates on the side of physics ("stronger" means that the record's assumption is the stronger; $\gamma$ and $\nu$ are the exponents of the susceptibility and the correlation length, 18-03 § 07)

| The record's assumption | Candidate on the side of physics | (a), strength |
|---|---|---|
| The relaxation-time assumption: $C_P \asymp \xi^2$ | Dynamical exponent $z = 2$ (model A) | Unsettled, stronger |
| The upper-bound assumption: $C_P \lesssim \xi^2$ | $z \le 2$ under the power form | Conditional, the same |
| The lower-bound assumption: $C_P \gtrsim \xi^2$ | Lower bound $z \ge \gamma/\nu$ | Conditional, stronger |
| $\sum_k \delta(b_k) < \infty$ | Change of scheme, MCRG, error of the projection | Unsettled |

| The record's assumption | Candidate on the side of physics | (a), strength |
|---|---|---|
| Physical-unit gap statement $\Longleftrightarrow \sum \delta < \infty \wedge \eta_{\mathrm{dyn}} = 0$ | Renormalisation-group invariance of mass + $z = 2$ | Unsettled |
| A single scale $\Longleftrightarrow$ extra dimensions are visible | Non-separation of KK | Unsettled |

02

## Fixing the record's η to one — $\eta_{\mathrm{dyn}} = 2 - z$

The record's $\eta$ pointed at two quantities (18-03 § 07). The **static** $\eta_s$ is the power of correlations fixed by the scaling dimension $\Delta$ of an operator $O$ (not the $\Delta$ of the mass gap) as $\eta_s(O) = 2\Delta - (d-2)$, and the record's Lean theorems that deal with static exponents (theorem names and hypotheses in 23-13) are about this one. What enters the power form of $C_P$ and the correlation length $\xi$ is the **dynamic** exponent.

$$C_P \asymp \xi^{\,2-\eta} \quad \text{and} \quad C_P \asymp \xi^{\,z} \quad \Longrightarrow \quad \eta = 2 - z =: \eta_{\mathrm{dyn}} \ (1)$$

This book fixes the $\eta$ of the power form and of the equivalence of the physical-unit gap statement ( § 07) to $\eta_{\mathrm{dyn}}$, and uses $\eta_s$ only for the test-function lower bound ( § 05). The two are equal only within the conventional theory $z = 2 - \eta_s$.

There is one price. The record had put the premise $\eta \geq 0$ of the equivalence as an assumption, grounded on Källén–Lehmann positivity (18-02). What follows from that positivity is $\eta_s \geq 0$ (the unitarity bound) Physics . For the Glauber dynamics of the two-dimensional Ising model, $\eta_s = 1/4$ yet $\eta_{\mathrm{dyn}} = -0.1665$ (18-03 Table 2). The premise $\eta_{\mathrm{dyn}} \geq 0$ is $z \leq 2$, that is, it includes the power form of the upper-bound assumption ( § 04) among the premises.

03

## The relaxation-time assumption and $z = 2$ — the same power, but different uniformity up to constant factors

The relaxation-time assumption is the assumption that the slowest relaxation time $C_P$ of Langevin dynamics lines up with the square of the correlation length up to constant factors independent of the lattice spacing (18-03 § 05, 23-01).

$$c\,\xi^2 \ \leq \ C_P \ \leq \ C\,\xi^2 \qquad (0 < c \leq C \text{ independent of the lattice spacing}) \ (2)$$

(2) is the way of writing at the level of a band, and depends on the normalisation of $\mathcal{E}$ ((3) below). By the record's convention the relaxation-time assumption is put at the

level of powers, $\lim \ln C_P / \ln \xi = 2$, and the band level is split off as a separate assumption naming the normalisation (end of § 03, 23-01).

The closest thing in physics is the dynamic scaling hypothesis $\tau \propto \xi^z$ (Halperin–Hohenberg 1969, Hohenberg–Halperin 1977), with the $z$ of Langevin dynamics (model A in that classification) set to 2 Physics . For the free field in the normalisation of the field $A$ ($s^2 = 1$ below), (2) holds exactly (18-03 Table 1).

Question: is the relaxation-time assumption the same as "$z = 2$"? Answer: the relaxation-time assumption is stronger. The hypothesis says only the power, not uniformity up to constant factors. Moreover, (2) includes the normalisation of the Dirichlet form $\mathcal{E}$. For the Gaussian measure $N(0, s^2(K + m^2)^{-1})$ ($K$ is the lattice Laplace operator $-\Delta$ [here $\Delta$ is the Laplacian], $s^2$ the amplitude, $m$ the mass), taking $\mathcal{E}(f) = \mathbb{E}|\nabla f|^2$ makes $C_P$ the largest eigenvalue of the covariance, $s^2/m^2$, while the correlation length does not depend on $s$ (1-10) Known . Written in the link angle $\vartheta$ ($U = e^{i\vartheta \cdot \sigma}$), $s^2 \propto g^2$; written in the field $A = \sqrt{\beta}\,\vartheta$ ($A = 2\vartheta/g$ in this chapter's convention $\beta = 4/g^2$), $s^2 = 1$. The $\beta$ of this chapter is the SU(2) lattice inverse coupling $\beta = 4/g^2$ (the $2N/g^2$ of the table of notation), not an inverse temperature.

The free approximation here is a model that substitutes a Gaussian field with the mass $1/\xi_{\mathrm{RG}}$ put in by hand, and does not include the gauge directions (zero modes). The bookkeeping length is the correlation length in lattice units fixed by the one-loop running (the record sets the lattice spacing to $a := 1/\xi_{\mathrm{RG}}$, § 07).

$$C_P^{(\vartheta)} = \frac{g^2}{4} C_P^{(A)}, \qquad C_P^{(A)} = \xi^2 \quad \text{(free approximation)} \tag{3}$$

The record's $\mathcal{E}$ is written in link angles. Putting in the one-loop $\ln \xi_{\mathrm{RG}} = 3\pi^2\beta/11 + \text{const.}$ (SU(2), 4-06), $C_P/\xi^2 \propto 4/\beta$ falls to 0 like $1/\ln \xi$, and the effective exponent is $z_{\mathrm{eff}} = 2 - \kappa/\beta$ ($\kappa = 4Nb_0 = 11/(3\pi^2)$).

Computed Table 1 Free approximation, SU(2), link-angle normalisation

| $\beta$ | $C_P/\xi^2$ (1 at $\beta = 2.5$) | $z_{\mathrm{eff}}$ |
|---|---|---|
| 2.5 | 1.000 | 1.8514 |
| 4 | 0.625 | 1.9071 |
| 100 | 0.025 | 1.9963 |

In powers $z = 2$, but in the link-angle normalisation the lower side of (2) breaks logarithmically. To set up the relaxation-time assumption one must first decide either to write $\mathcal{E}$ in the normalisation of the field $A$ ($\mathcal{E}_A = \mathcal{E}/\beta$, that is, multiply the angle $\mathcal{E}$ by $g^2/4$; $C_P$ is conversely multiplied by $\beta$), or to put the relaxation-time assumption at the level of powers ($\lim \ln C_P / \ln \xi = 2$) and split the band-level statement off as a

separate assumption naming the normalisation. In interacting theories the scale of stochastic time is also renormalised [Known] (at the level of perturbation theory; 23-00 Entrance — the staircase, and the random walk, the ruler and the clock, 23-01 The line of construction and the diffusive-relaxation-time assumption; there is also an example checked at one loop in supersymmetric Yang–Mills [Physics]), so there is no guarantee that the logarithm disappears in either normalisation.

In terms of the measurement column of the dictionary (the direction reachable by finite measurement; 18-01 § 02), the lower-bound assumption is on the "Lower bound only" side: exhibiting one test function gives a lower bound on $C_P$ (§ 05). **The upper-bound assumption is on the "Out of reach" side**. An upper bound on $C_P$ is an inequality about every $f$, so no number of added test functions, and no amount of autocorrelation measurement up to finite times, reaches it (the box "The asymmetry of upper and lower bounds" in 1-10). The upper-bound assumption and the mass gap are not the same statement, but they are of the same type in that both stand on the side of lower bounds.

### Detour: why diffusion takes $\xi^2$

A subsection you may skip.

Smoothing out a distance $\xi$ by a random walk takes $\xi^2$ steps. The Langevin dynamics of the free field is independent for each Fourier component.

$$d\hat{\phi}_k = -\left(\hat{k}^2 + m^2\right)\hat{\phi}_k \, ds + \sqrt{2}\, d\hat{W}_k(s) \quad (4)$$

$s$ is the Langevin time, not the amplitude $s^2$ above. $\hat{k}^2 = \sum_\mu 4\sin^2(k_\mu/2)$ is the lattice version of $|k|^2$. The slowest is the rate $m^2$ at $k = 0$, which as a time is $\xi^2$. As a metaphor, the longer the wave, the more slowly it is smoothed out. Precisely, this is an equality for the free field; with interactions there is no guarantee they line up, and putting that they do is the relaxation-time assumption.

04

## The relaxation-time upper-bound assumption and $z \leq 2$ — three conditions needed for the restatement

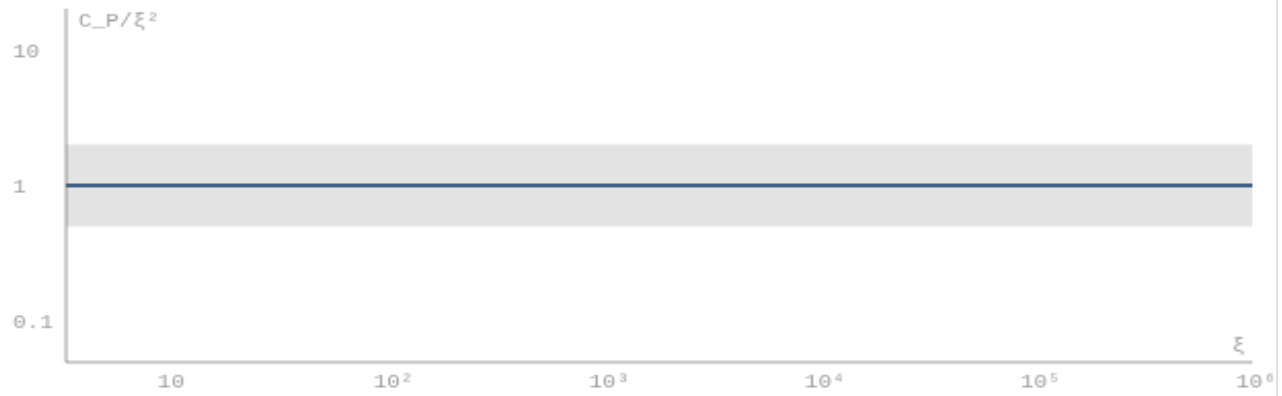
The upper-bound assumption ($C_P \lesssim \xi^2$) is, in the record's reading, the side of "the mass gap is positive" (17-12 § 10). The record restates it as $z \leq 2$. Question: is this a restatement under the dynamic scaling hypothesis? Answer: under three conditions. (i) The power form $C_P \asymp \xi^z$ exists. (ii) The dynamics is reversible Langevin in continuous time ($C_P = \tau_{\text{exp}}$ is an equality only there, 18-03 § 06). (iii) The

normalisation of $\mathcal{E}$ has been decided ( § 03). Writing $q$ for the power of a logarithmic factor, the judgement is as follows.

$$C_P = \xi^{\,z}(\ln \xi)^q \implies \Big((\xi^2)\uparrow \iff z < 2 \text{ or } (z = 2,\ q \le 0)\Big) \quad (5)$$

The hypothesis of Hohenberg–Halperin only assumes the existence of $z$, and does not claim $z \le 2$. The Glauber dynamics of the Ising model in 18-03 Table 2 has $z > 2$ in both two and three dimensions Physics . Reading the upper-bound assumption as a conjecture of physics needs the condition "$\eta_s = 0$ at the relevant fixed point", and we have not checked any primary source stating it for four-dimensional Yang–Mills.

Figure 1 $C_P/\xi^2$ when $C_P = \xi^z(\ln\xi)^q$ (the ratio normalised to 1 at $\xi = 10$). Changing $z$ and $q$ with the knobs changes the curve and the judgements of the upper-bound and lower-bound assumptions. The band is constant factors $[1/2, 2]$




z = 2.0000, q = 0.00: C_P/ξ² at ξ = 10⁶ (1 at ξ = 10) = 1.00, pure power form: yes, z ≤ 2: yes, upper-bound assumption holds, lower-bound assumption holds, relaxation-time assumption holds

Computed A schematic drawing of equation (5), drawn on the spot. Without JavaScript it can be read from the next paragraph.

Examples ($\xi \to \infty$): $\xi^2$ satisfies both; $\xi^2/\ln\xi$ (the angle normalisation) and $\xi^{1.9}$ satisfy the upper-bound assumption only; $\xi^2 \ln\xi$ and $\xi^{2.1665}$ satisfy the lower-bound assumption only. For $\xi^2 \ln\xi$ the power is 2 and yet the upper-bound assumption breaks, so dropping (i) leaves only the one direction "upper-bound assumption $\implies z \le 2$".

05

## The relaxation-time lower-bound assumption and the lower bound of critical slowing down — the same inequality, operators that do not reach

The lower-bound assumption ($C_P \gtrsim \xi^2$) is the side that can be shown with test functions: for any $f$, $C_P \ge \mathrm{Var}(f)/\mathcal{E}(f)$ (1-10). Physics says "local dynamics cannot relax faster than the susceptibility". Taking the magnetisation as $f$ gives $C_P \ge \chi$

(equation (12) of 18-03) Known . Under the two hypotheses of dynamic and static scaling, $z \geq \gamma/\nu = 2 - \eta_S$ ($\gamma$ is the susceptibility exponent and $\nu$ the correlation-length exponent; 18-03) Physics . It is a lower bound regarded as standard in the Monte Carlo literature, and we have not checked its authority.

Question: is the lower-bound assumption the same as this lower bound? Answer: the inequality is the same, but where it reaches differs. Taking $f$ to be a sum of local operators of scaling dimension $\Delta$, the lower bound is fixed by the susceptibility $\chi_O$.

$$\frac{\mathrm{Var}(\sum_x O_x)}{\mathcal{E}(\sum_x O_x)} \asymp \chi_O \sim \begin{cases} \xi^{\,2-\eta_S(O)} & 2\Delta < d \\ \ln \xi & 2\Delta = d \\ \text{bounded} & 2\Delta > d \end{cases} \tag{6}$$

It reaches $\xi^2$ only when there is an operator with $\eta_S(O) = 0$, that is, of canonical dimension $\Delta = (d-2)/2$. We check this with the four-dimensional free field.

Computed Table 2 Lower bounds on $C_P$ given by test functions for the free field on a four-dimensional lattice

| $m$ | $\xi$ | **Dimension 1:** $\sum \phi$ **(**$= C_P$**)** | **Dimension 2:** $\sum \phi^2$ | **Dimension 4:** $\sum (\nabla\phi)^2$ |
|---|---|---|---|---|
| 1.0 | 1.04 | 1 | 0.0783 | 0.0537 |
| 0.1 | 10.0 | 100 | 0.1719 | 0.0624 |
| 0.01 | 100.0 | 10000 | 0.2657 | 0.0625 |

Dimension 2 grows with slope 0.0408 in $\ln \xi$ (continuum estimate $1/(16\pi^2 G(0)) = 0.0409$), and dimension 4 stops at $1/16$ Computed . The infinite-volume integrals were reduced to one dimension and computed, and matched at $m = 0.5$ to 5 significant figures with sums on a $24^4$ lattice.

The gauge-invariant local operators of Yang–Mills are of the $\mathrm{tr}\, F^2$ type, of dimension 4, so they fall into the third step of (6) (23-01). The link angle has dimension 1, but under the measure without gauge fixing the correlations between angles of links not sharing an endpoint are exactly 0 after averaging over gauge transformations (the same mechanism as Elitzur's theorem, 18-02). The lower-bound assumption is the demand to push the same inequality up to $\xi^2$ in a theory with no local order parameter, and is stronger by that much. The record checked that three families of test functions (some by U(1) calculations) do not reach it, but it does not say that the lower-bound assumption is false (23-02).

The lower-bound assumption is, in the record's reading, the side of "the mass in physical units is finite", a condition different from "non-trivial" (there are interactions) in the Clay problem statement. It is satisfied even by a free massive field.

> 06

## Summability of the shortfall — do step-by-step shifts fit into a single change of scheme?

The record writes one step that coarsens by a factor $b$ as $\beta \mapsto \beta - \kappa(\ln b - \delta(b))$, and calls $\delta(b)$ the shortfall (23-02, 18-04 § 03). $\delta$ is a number measured in the one-loop kernel model, and in the record's measurements in four dimensions it has the form $c_2/b^2 + c_3/b^3 + \cdots$ Computed (that there is no $\ln b/b^2$ term is within the detection limit). Advancing $k$ steps with factors $b_1, b_2, \ldots,$

$$\beta_k = \beta_0 - \kappa \ln\Big(\prod_{j=1}^{k} b_j\Big) + \kappa \sum_{j=1}^{k} \delta(b_j) \quad (7)$$

For an exact step within the same scheme the third term is absent (equation (3) of 18-04), and a change of scheme shifts $\beta$ once only by $c$ (Dashen–Gross, 18-04 Table 1). If the third term converges, the step-by-step shifts add up to a one-off shift $c = \kappa \sum \delta$, indistinguishable from a single change of scheme.

$$\sum_j \delta(b_j) < \infty \iff \text{the step-by-step shifts fit into a single change of scheme} \quad \text{(within the re}$$

Rough guide: for SU(2), $\Lambda_{\overline{\text{MS}}}/\Lambda_{\text{L}} = 19.82$ corresponds to $\ln 19.82 = 2.987$ in $\sum \delta$ Computed. With $b$ fixed, it is bounded only when $\delta(b) = 0$ Lean (`fixed_step_bounded_iff`. Hypothesis: the error of a step is the same at every step).

Question: which thing in physics is (8)? There are three candidates, and all are analogies. (i) It is not Symanzik's $O(a^2)$. The variables of expansion differ, $a$ against $b$, and the former does not accumulate step by step (18-04 § 06). (ii) The first SU(3) result measuring the $\Delta\beta$ of gauge theory by the method of MCRG (the Monte Carlo renormalisation group, introduced by Swendsen 1979 for spin systems) was 'no asymptotic scaling' for $K_F < 6.1$ ($K_F$ is the coupling of the Wilson action) (Gupta et al. 1984) Known. This $\Delta\beta$ and the record's decrease per step are, with the same projection, quantities that correspond by definition (the identification is this dictionary's reading). (iii) The error of truncating to the Wilson form: for untruncated perfect actions Hasenfratz–Niedermayer (1994) write 'There exist lattice actions which give cut-off independent physical predictions even on coarse grained lattices'. One reading is that $\delta$ is close to this error.

### Kernels with negative weights and reflection positivity

The blocking kernel that gave $\delta(2) = 0$ in the record's calculation had a negative weight on the neighbours Computed. If the weights are non-negative, $c_2 > 0$ Lean

(`c2_ge_of_nonneg_weights`. Hypothesis: the measurement that $c_2$ is an increasing affine function of the amount of weight, positive at weight 0). The abstract of Lüscher–Weisz (1984) writes 'When operators of dimension 6 are added to the standard Wilson action in lattice gauge theories, physical positivity is lost in general.' Known . The pattern, a negative coefficient that removes the leading term taking away positivity, is the same. The difference is the object: theirs is the action (5-03), the record's is the blocking kernel. Whether a kernel with negative weights breaks reflection positivity has not been checked.

07

### The equivalence of the physical-unit gap statement — is it the product of "mass is invariant under the renormalisation group" and "$z = 2$"?

The record treats the physical-unit gap statement (the mass gap in physical units) in the form "$a^2 C_P$ lies in a positive band independent of the lattice spacing $a = 1/\xi_{\mathrm{RG}}$", and puts two powers.

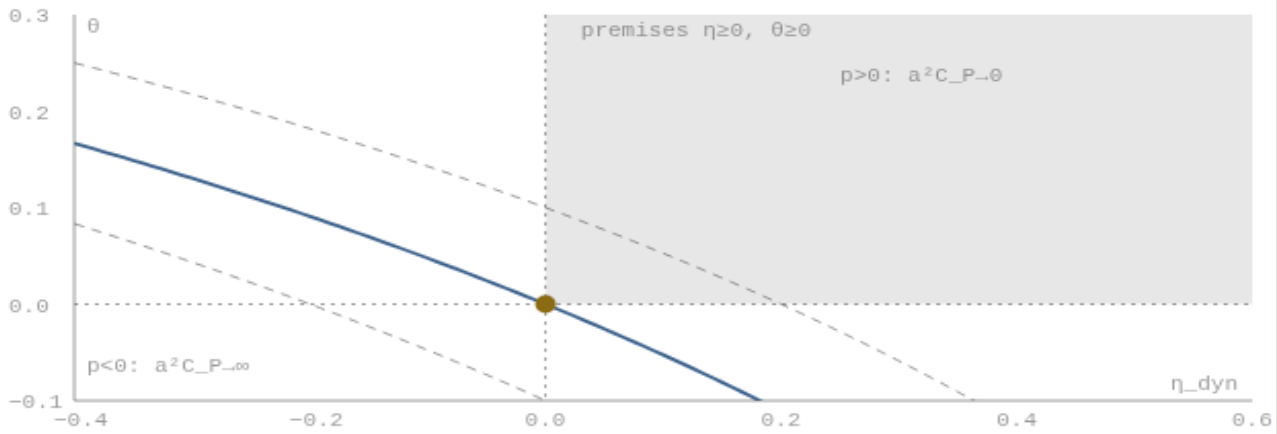
$$C_P \asymp \xi^{\,2-\eta_{\mathrm{dyn}}}, \quad \xi_{\mathrm{phys}} = a\,\xi \propto a^{\theta} \implies a^2 C_P \propto a^{p}, \quad p = \eta_{\mathrm{dyn}} + (2-\eta_{\mathrm{dyn}})\,\theta \ (9)$$

$\theta$ is the exponent of accumulation of the shortfall; in the model with $b$ fixed, $\theta = \delta(b)/\ln b$, and if $\sum \delta < \infty$, then $\theta = 0$. If $\eta_{\mathrm{dyn}} \geq 0$ and $\theta \geq 0$, then $p = 0$ is equivalent to both being 0, and this is the content of the record's "physical-unit gap statement $\iff \sum \delta < \infty \wedge \eta = 0$" (the record's $\eta$ is $\eta_{\mathrm{dyn}}$, § 02) Lean (`drain_exponent`, `drainExp_eq_zero_iff`. Hypotheses: the power form of (9) and the sign conditions). That the error of the steps is additive and bounded, and that $\delta$ is the value in the one-loop kernel model, are also premises.

Question: is this the product of two statements of physics? Answer: within the algebra of the model, it can be read so. $\theta = 0$ is the statement "physical mass is invariant under the renormalisation group, and $m_{\mathrm{lat}}(\beta) \propto a(\beta)\Lambda$". Physics treats $\theta = 0$ as following, if a continuum theory is presupposed, from the Callan–Symanzik equation (4-04) for quantities with no anomalous dimension (dimensional transmutation is in 4-07) Physics . Physics has not made this into a named hypothesis. In mathematics, the lattice correlation-length problem of Chatterjee (2018) (Chatterjee's Problem 5.1; $\xi(\beta) \in (0, \infty)$, and $\xi \to \infty$ as $\beta \to \infty$) asks for the same direction Known . $\eta_{\mathrm{dyn}} = 0$ is $z = 2$, the expectation of § 03 and § 04.

Dropping $\eta_{\mathrm{dyn}} \geq 0$, $\theta^* = -\eta_{\mathrm{dyn}}/(2-\eta_{\mathrm{dyn}})$ gives $p = 0$, and $a^2 C_P$ lies in a constant band, yet neither the relaxation-time assumption nor $\theta = 0$ holds (the same type as Figure 2 of 17-12).

Figure 2 The $(\eta_{\mathrm{dyn}}, \theta)$ plane. Moving the point with the knobs changes where $a^2 C_P$ goes and the judgements. "Play" moves the point left along the solid line $p = 0$, and cancellation happens outside the light shading




η_dyn = 0.000 (z = 2.0000), θ = 0.000: p = 0.000, a²C_P → a constant band, relaxation-time assumption holds, θ = 0 holds

Computed A schematic drawing of equation (9), drawn on the spot. The light shading is the region of the premises, and the dashed lines are $p = \pm 0.2$. Without JavaScript it can be read from Table 3.

Computed Table 3 Examples of cancellation ($z$ from 18-03 Table 2)

| Example | $z$ | $\eta_{\mathrm{dyn}}$ | Cancelling $\theta^*$ |
|---|---|---|---|
| 2D Ising, Glauber | 2.1665 | −0.1665 | 0.0769 |
| Free field, Langevin | 2 | 0 | 0 |

Repeating the step that halves $a$ 20 times, $a^2 C_P$ grows by about 10 times at $p = -0.1665$, and stays at 1 times at the point of cancellation Computed . The numerical tables alone cannot tell a breaking of the relaxation-time assumption from the exponent of accumulation of the shortfall.

08

## A single scale and extra dimensions — an analogy with the non-separation of KK?

This section is easier to follow after reading 17-09 and 17-10 (the cigar, $M_{\mathrm{KK}}$, supergravity). The record has the sentence "a single scale ⟺ extra dimensions are visible (radius $R \sim 1/\Lambda$)". Question: is it the same thing as the non-separation of KK? In the holographic model (Witten's 1998 cigar, 17-09) the background has a single length $R = 1/M_{\mathrm{KK}}$, and multiplying the metric by $s^2$ multiplies all eigenvalues of the fluctuations by $s^{-2}$. So $m_n = c_n M_{\mathrm{KK}}$. Since there is only one length scale $R$, every quantity with the dimension of length squared (both $\xi^2$ and $C_P$ measured in Langevin time) is proportional to $R^2$, and the relaxation-time assumption becomes an identity by dimensional analysis inside the model Physics . Precisely, this is counting dimensions after defining $C_P$ in the model. For the same reason the KK modes do not

separate from the glueballs. Sakai–Sugimoto (2005), a model of QCD with quarks, also writes about mesons: 'our D4/D8 model deviates from QCD at the energy scale of M_KK, which is, unfortunately, the same energy scale of the mass of the vector mesons'.

$\Lambda \approx M_{\mathrm{KK}}\, e^{-24\pi^2/(11\lambda)}$ ($\lambda$ is the 't Hooft coupling at $M_{\mathrm{KK}}$ ; the coefficient depends o

At $\lambda = 30,\ 16.6,\ 10,\ 5$, $\Lambda/M_{\mathrm{KK}} = 0.49,\ 0.27,\ 0.12,\ 0.013$ Computed (16.6 is the Sakai–Sugimoto fit converted to $N_c = 3$, the $\lambda$ in that convention. Putting it into the one-loop formula (10) is a rough guide, and can be off by a factor of 2 by convention). Supergravity can be used on the side of large $\lambda$, and Witten writes of the small side 'This looks like a tall order' (17-10).

Answer: an analogy. "One length $\Longrightarrow$ one scale" is obvious inside the model, but there is neither the converse nor a passage to $R \to 0$. The physical name for one scale in pure Yang–Mills is dimensional transmutation (4-07), and the $\Longleftrightarrow$ is the formulation of the record of this book's study.

09

## What is known and settled — rows not counted as results

The following rows are treated as known, and are not listed as results in the later parts (this book's study). (1) The correction of direction, that a uniform gap in lattice units is on the trivial side $m_{\mathrm{phys}} \to \infty$, is just the definition that takes the continuum limit where the correlation length diverges, and the direction of the lattice correlation-length problem (Settled, 5-09). (2) $\kappa = 11/(3\pi^2)$ is $4Nb_0$ (Gross–Wilczek, Politzer 1973), and the 5-digit agreement with measurement is agreement within the one-loop model (Settled, 4-06). (3) In the record's one step of bookkeeping, the strong-coupling $\beta' = \beta^4/64$ corresponds only to the decimation step of Migdal–Kadanoff (one MK step including the bond-moving factor $b^{d-2}$ is $\beta' \approx 4\beta^4$, 18-04 § 04), and the weak-coupling $\beta' = \beta\, b^{d-4}$ is dimension counting (Conditional). (4) Reflection positivity and the transfer operator in the reflection-positivity preservation statement (one of the eight statements of construction in 23-01), and the strong-coupling area law, are Osterwalder–Seiler 1978, Lüscher 1977 and Wilson 1974 (Settled, 5-03). (5) The infinite-volume uniqueness statement is the lattice version of uniqueness of the vacuum, and $R \le \tau_{\mathrm{int}} \le C_P$ is error analysis within the same reversible dynamics (Conditional, 18-02, 18-03).

What could not be asserted to be the same are the uniformity of the relaxation-time assumption up to constant factors ( § 03) and that step-by-step accumulation fits into a single change of scheme ( § 06); both go only as far as analogy.

## What this chapter can say

| What is said | What is not said |
|---|---|
| Known The type of the variational lower bound (6), the algebra of (3)(7)(9), that the rows of § 09 are known. | A theorem that the record's assumptions are the same as hypotheses of physics. None of the assumptions has been proved. |
| Computed Tables 1–3, Figures 1 and 2, the numbers in § 06 and § 08 (for the free field and within the model). | Whether four-dimensional Yang–Mills has $z = 2$, or a logarithmic factor. Whether kernels with negative weights break reflection positivity. |
| Physics The numbers for $z$ (18-03 Table 2), the estimate (10), the testimony on the non-separation of KK. | This chapter shows nothing about the mass gap, the continuum limit or one scale. |

## Sources and reproduction

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| Dynamic scaling, critical slowing down, lower bound | Physics, Known | Halperin–Hohenberg, Phys. Rev. 177 (1969) 952; Hohenberg–Halperin, RMP 49 (1977) 435 (both abstracts only). Sokal, "Monte Carlo Methods in Statistical Mechanics: Foundations and New Algorithms", in DeWitt-Morette, Cartier, Folacci (eds.), *Functional Integration: Basics and Applications* (NATO ASI Series, 1997) 131–192, doi:10.1007/978-1-4899-0319-8_6 (bibliographic data only; the authority for the lower bound $z \geq \gamma/\nu$ not checked). The numbers for $z$ are from the sources of 18-03 Table 2 (by the checks in 18-03). The unitarity bound $\eta_S \geq 0$ ( § 02): original not cross-checked. |
| Change of scheme, MCRG, perfect actions, improved actions | Known | Dashen–Gross, PRD 23 (1981) 2340 (abstract); Swendsen, PRL 42 (1979) 859 (abstract); Gupta et al., PRL 53 (1984) 1721 (abstract; the quotation checked verbatim in INSPIRE); Hasenfratz–Niedermayer, NPB 414 (1994) 785 (abstract; quotation checked verbatim); Lüscher–Weisz, NPB 240 (1984) 349 (abstract; quotation checked verbatim). |
| Renormalisation-group invariance of mass, the lattice correlation-length problem | Known, Physics | Callan, PRD 2 (1970) 1541; Symanzik, CMP 18 (1970) 227 (both abstracts; by the checks in 4-04). Chatterjee, arXiv:1803.01950 (text of § 5; by the checks in 18-04). |
| KK, holography | Physics | Witten, ATMP 2 (1998) 505; Sakai–Sugimoto, PTP 113 (2005) 843 (verbatim by the checks in 17-09 and 17-10). |

| Item | Kind | Source / tools (level of checking) |
|---|---|---|
| The rows of § 09, Elitzur's theorem | Known | Gross–Wilczek, PRL 30 (1973) 1343; Politzer, PRL 30 (1973) 1346 (both bibliographic data only). Osterwalder–Seiler, Ann. Phys. 110 (1978) 440; Lüscher, CMP 54 (1977) 283; Wilson, PRD 10 (1974) 2445 (all abstracts; by the checks in other chapters). Kadanoff, Ann. Phys. 100 (1976) 359 (abstract by the checks in 18-04; main text not obtained, so not cross-checked at the level of formulas); Migdal, Sov. Phys. JETP 42 (1975) 413 (bibliographic data only; not cross-checked at the level of formulas). Elitzur, PRD 12 (1975) 3978 (abstract; by the checks in 18-02). |
| The record's assumptions | Assumption, analogy | The arrangement of the record of this book's study (23-01, 23-02, 23-13, 17-12). |
| Tables 1–3, the numbers in § 06 and § 08 | Computed | Python 3 on this machine (numpy, scipy): `free_log.py`, `testfn_dim.py`, `cancel.py`. |

Next chapter to read: 19-00 Entrance — how to read objects, firm theorems and premises (within Volume 3, from here we enter the objects and the firm theorems). For reading where things stand from a timeline and tables by dimension, see 16-01 How far the world has come. The conventions for $\eta_s$ and $\eta_{\mathrm{dyn}}$ are in A-01 Table of notation.