Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 19 Objects, and the statements to be shown

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2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Entrance · Kiichi, Shiori and Rin

Entrance — how to read objects, firm theorems and premises

> This chapter is a guide.

What this chapter shows — What Parts 19 to 22 and Part 28 contain and what they do not. In which part the definitions of the objects, the statements to be shown, the machine-checked theorems and the list of premises are each found. Three ways of choosing a reading order.

Prerequisite chapters — 0-03 (reading the problem statement word by word) and 16-09 (the stages a proof must contain). The language of the lattice is in Part 5.

Said first — These five parts do not show the mass gap. For the existence of four-dimensional Yang–Mills theory (that a theory satisfying the Wightman axioms can be made as a limit of Wilson's lattice gauge theory), the committee's assessment is that the construction published on 2026-11-10 proves it in Lean. That is read in Parts 6 to 13 of Volume 2. Every theorem of Parts 20 to 22 is a statement about a single scale with the lattice spacing fixed: results of known types, machine-checked from the definitions. Most are statements inside the strong-coupling window, at the opposite end from the questions the published construction left open. Holding uniformly across scales is on the side of the premises in Part 28. A form whose rate does not depend on the transverse size exists only for the upper bound on the vortex free energy inside the strong-coupling window (20-06), and even there the prefactor (the transverse area) remains.

The order of this chapter

1. Dividing into three — objects, theorems, premises
2. A map of the parts (Table 1)
3. Reading orders — three ways
4. How to read the labels
5. The relation to Parts 23 to 27 and Part 15 (Table 2)
6. What these five parts do not say

01

Dividing into three

Parts 23 to 27 and Part 15 are written in the order in which the study proceeded. There, statements that have been machine-checked, statements placed as assumptions, and estimates only seen in numbers sit side by side in a single chapter. Parts 19 to 22 and Part 28 take the same material and lay it out again, divided into three.

Division	Contents	Where
Objects	Things defined on the lattice: the measure, twists, the vortex ratio, reflections, the block map, the pushed-forward measure. Definitions only, with no claims	Part 19
Theorems	Equalities and inequalities about the defined objects. The assumptions are only explicit ranges (the range of the coupling, the parity of sizes and so on), and they are machine-checked in Lean 4	Parts 20 to 22
Premises	Statements on the side of "it would suffice if this could be said". Those whose form is settled and those whose form is not yet settled are listed with labels	The last chapter of Part 19, and Part 28

The line between theorems and premises is whether the objects are defined from the lattice. The chapters of theorems carry only what has its objects' definitions written in the chapter, has assumptions that are explicit ranges only, and has been machine-checked and cross-checked by another pair of eyes.

The side of premises divides in two. The statements to be shown on the ultraviolet side (matching, convergence of the full sequence, non-degeneracy, equicontinuity) and the tree of premises supporting them are statements about this book's design (the measure pushed forward by a fixed block map), and are kept as a record of the statements to be shown and sufficient conditions in this book's design. Existence itself was shown by the published construction in a different design (the comparison of the two designs is in Part 14, and the record of this book's ultraviolet attempt in Part 15). The statements to be shown on the infrared side (the band statement and the mass gap) remain open, and they are the main subject of Volume 3.

02

A map of the parts

Table 1 Parts 19 to 22 and Part 28

Part	Title	Contents

Part	Title	Contents
Part 19	Objects, and the statements to be shown	19-01 The lattice and the Wilson measure, 19-02 Twists, the vortex ratio and Polyakov loops, 19-03 Reflections and the vessel of reconstruction, 19-04 The block map and the pushed-forward measure, 19-05 The fermion weight (detour), 19-06 Objects and their Lean definitions — a table of correspondence, 19-07 The statements to be shown
Part 20	Firm theorems — the vessel and strong coupling	Reflection positivity, decay of correlations at strong coupling, the infinite-volume limit, reconstruction and the mass gap (inside the strong-coupling window), the vessel with fermions (detour), 20-06 The area law at strong coupling, and an upper bound on the vortex free energy
Part 21	Firm theorems — the vortex floor and Polyakov loops	Three results on sign systems, the thin-vortex floor, the thick-vortex floor, the minimal action of a vortex, the sign of the vortex ratio, pairs of Polyakov loops, 21-07 The vortex ratio — continuity, limit, matching
Part 22	Firm theorems — the rarity of large plaquettes	22-00 Entrance, 22-01 Rarity of large plaquettes by the Chessboard Estimate, 22-02 What this part says, and what it does not
Part 28	The tree of premises	Three chapters of implications, The list of premises, What depends on design choices, Open places

One step and many steps of the free field, and the linearisation of the block map (one-scale theorems on the ultraviolet side), are in Part 15 (15-12, 15-13) as the record of this book's ultraviolet attempt.

03

Reading orders

- **Knowing first what is to be shown**: skim the definitions of 19-01 to 19-04, then go to 19-07 The statements to be shown. From there, go on to the tree of Part 28.
- **Reading from the firm places**: Part 20 → Part 21 → Part 22. Each chapter runs in the order: objects (with references to Part 19), theorems, the route of the proof, what is not said.
- **Following the course of the study**: Parts 23 to 27 and Part 15. The correspondence between chapters on the same topic is in Table 2.

04

How to read the labels

The labels of these five parts are the same five as in Parts 23 to 27 and Part 15: Proved here Cited Computer-assisted Reading Assumption . Their meanings are the same as in the table of symbols in 15-00. Where the published construction is cited, the labels of Volume 2 (Source, Inferred, Assessment; "Labels" in 6-00) are also used. For statements

machine-checked in Lean, the name of the declaration and its range are written on the same line, outside the labels.

Most rows of the theorems in Parts 20 to 22 carry Cited . It means that a result of a known type has been machine-checked on top of this book's definitions. What machine checking guarantees is that "the written conclusion follows from the written definitions and assumptions"; that the definitions represent the intended objects rests on a cross-check by another pair of eyes.

The premises carry another set of labels (theorem, paper, form settled but unproved, form not yet settled, depends on design, literature). The legend is in 28-00.

05

The relation to Parts 23 to 27 and Part 15

Parts 23 to 27 and Part 15 are kept as the record of the study. In them one can read the roads that stopped, and why. For the same topics, the present arrangement is found as follows.

Table 2 Correspondence of topics

Topic	Record of the study	Present arrangement
Twisted boundary conditions and vortices	Part 25	Definitions in 19-02, theorems in Part 21 and 20-06
Reflection positivity and the chessboard	26-04, 26-05	20-01, Part 22
The crossover band	Part 26	Statements in 19-07, premises in 28-04
The conditional statement and the ledger	Part 27	Part 28
Ultraviolet control	Part 15	Existence in Volume 2 (reading the published construction), the comparison of the two designs in Part 14. One step of the free field in this book's design in 15-12, premises in this book's design in 28-04

06

What these five parts do not say

- They show nothing about the mass gap. The existence of four-dimensional Yang–Mills theory is not a theorem of this book but a result of the published construction (Volume 2).

- The theorems are about a fixed lattice spacing. There is no theorem about the limit in which the lattice spacing is sent to 0.
- The strong-coupling theorems are only about the window where the coupling is small. They say nothing outside the window.
- Inside the strong-coupling window there is a form in which the rate of the upper bound on the vortex free energy does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, the vortex-floor theorems are all about a fixed transverse size, and there is no estimate uniform in the transverse size.
- The list of premises is a dated, present arrangement. Some premises do not yet have a settled form, and the list will change.

What this chapter can say

Can say	Cannot say
This chapter is a guide and contains no statements. It shows where the contents of Parts 19 to 22 and Part 28 are found	The scope of the statements of each part is in that part's chapter on "what it says and what it does not" (Part 20, Part 21, Part 22) and in 28-06. The scope of one step of the free field is in 15-14

Sources and reproduction

Item	Kind	Source / tools
The map of the parts, the correspondence of topics	Guide	Part 15 and Parts 19 to 28 of this book
The list of machine-checked theorems	Guide	A-04

Words used in this chapter — object, push-forward measure, the tree of premises. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-01 The lattice, link variables and the Wilson measure — the first of the definitions.

The lattice, link variables and the Wilson measure

> This chapter is this book's own investigation. § 01 to § 06 arrange established knowledge, and § 07 is this book's investigation (the conventions on the Lean side and the bridges).

What this chapter shows — The definitions of the most basic objects that the theorems of Parts 20 to 22 talk about: the periodic lattice $(\mathbb{Z}/N)^4$, the link variables $U_\ell \in SU(2)$, the product of Haar measures, the plaquette holonomy U_p and its real part, the Wilson measure and the partition function. We also show here, once, that the partition function is positive and finite. Finally, we gather in one table the fact that on the Lean side the same objects are written in four conventions (strong coupling, vortices, fermions, ultraviolet), and the bridges between them.

Prerequisite chapters — 5-01 Wilson's lattice (introduction). For the language of probability measures, 1-08.

Said first — This is a chapter of definitions, and it states no theorems. The lattice spacing is fixed to 1 and the volume is finite. The continuum limit and the infinite-volume limit are outside this chapter (the space of infinite-volume configurations is in 19-03).

The order of this chapter

1. The lattice and links — the periodic lattice $(\mathbb{Z}/N)^4$ and oriented edges
2. SU(2) as unit quaternions — quaternions of length 1 in place of matrices
3. The Haar measure and its product — the uniform probability on the group
4. Plaquettes and holonomy — the product of four links, and its real part
5. The Wilson measure — the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ and expectations
6. The partition function — positive and finite
7. Four conventions on the Lean side, and bridges — four ways of writing the same objects

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01

The lattice and links — the periodic lattice $(\mathbb{Z}/N)^4$ and oriented edges

The introduction is in 5-01 § 02. Here we fix the form used by the later parts.

Take a positive integer N, and let the four-dimensional periodic lattice (torus) be

$$\Lambda_N = (\mathbb{Z}/N)^4 \quad (1)$$

The lattice spacing is fixed to 1. We write e_μ for the unit shift in direction μ. A **link** is a pair $\ell = (x, \mu)$, representing the edge from the point x to the point $x + e_\mu$. The point x is called the **base** of the link, and for a link in the time direction the base time is taken as the time of that link (a different word from the "root" [the statements to be shown] of the tree of premises in Part 28). There are $4N^4$ links in all.

The numbering of directions differs between parts; there are two.

- **20-01 to 20-05 of Part 20** (the torus $(\mathbb{Z}/N)^4$ of the same size in all four directions): directions are numbered $\mu \in \{0, 1, 2, 3\}$, with 0 as time. Coordinates are $x = (x_0, x_1, x_2, x_3)$ (the same as the indices of `TorusRP.Lk` in Lean). The formulas of 19-03 are also written in this numbering.
- **Parts 21 and 22, and 20-06** (the lattice of Part 21 is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, 19-02): directions are numbered $\mu \in \{1, 2, 3, 4\}$, with $1, 2$ transverse and $3, 4$ longitudinal. Coordinates are $x = (x_1, x_2; x_3, x_4)$ (the indices 0 to 3 of `TwistedWilson.Link` in Lean are directions 1 to 4). 20-06 and Part 22 also number the directions 1 to 4 on the torus of the same size in all four directions (the Lean indices 0 to 3 are directions 1 to 4).

The formulas of this chapter are written with pairs of directions $\mu < \nu$, so that they apply to either numbering.

An edge in the reverse direction is not counted as a separate link. When a link is traversed in reverse, the inverse of its link variable is used (§ 02). Because the lattice is periodic, moving by e_i from a point with $x_i = N - 1$ brings us back to $x_i = 0$.

Part 21 uses the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, whose size differs between directions 1, 2 and directions 3, 4. Its definition is placed in 19-02. Links and plaquettes are built in the same way as in this chapter.

Item	Symbol	Number
Lattice points	$x \in (\mathbb{Z}/N)^4$	N^4
Links (oriented edges)	$\ell = (x, \mu)$	$4N^4$
Plaquettes (pairs of directions $\mu < \nu$)	$p = (x; \mu, \nu)$	$6N^4$

For $N = 1$ the same link appears twice in one plaquette. For $N = 2$ there are two links joining the same two points. Both are counted with their separate indices (the Lean definitions do the same).

Link variables and configurations

On each link ℓ we place one element U_ℓ of the group $SU(2)$. This is called the **link variable**, and an assignment to all links

$$U = (U_\ell)_\ell \ \in \ SU(2)^{\{\text{links}\}} \quad (2)$$

is called a **configuration**. The space of configurations is the product of $4N^4$ copies of the compact group $SU(2)$, and is itself compact. When moving from x to $x + e_\mu$ we multiply by $U_{x,\mu}$, and when moving in reverse by $U_{x,\mu}^{-1}$.

> 02

SU(2) as unit quaternions — quaternions of length 1 in place of matrices

This book's Lean definitions write $SU(2)$ not as 2×2 matrices but as **unit quaternions**. For a quaternion $q = a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k}$ ($a,b,c,d \in \mathbb{R}$, $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$) let the norm be $\|q\| = \sqrt{a^2+b^2+c^2+d^2}$, and identify

$$SU(2) \;\cong\; S^3 = \{q \in \mathbb{H} : \|q\| = 1\} \quad (3)$$

The correspondence is given by the following matrix Cited .

$$q = a + b\,\mathrm{i} + c\,\mathrm{j} + d\,\mathrm{k} \;\longmapsto\; \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \quad (4)$$

The determinant of this matrix is $\|q\|^2$ and its trace is $2a$. Hence the following three hold Proved here .

- The product of quaternions corresponds to the product of matrices. The product of unit quaternions is a unit quaternion ($\|pq\| = \|p\|\|q\|$).
- The inverse is the conjugate: $q^{-1} = \bar{q} = a - b\,\mathrm{i} - c\,\mathrm{j} - d\,\mathrm{k}$ (for matrices, the Hermitian conjugate).
- The real part of a quaternion is half the trace: $\mathrm{Re}\,q = a = \frac{1}{2}\,\mathrm{tr}\,q$. For unit quaternions, $|\mathrm{Re}\,q| \le 1$.

The centre is $\{+1, -1\}$. The twist of Part 21 (19-02) is the operation of putting a sign from this centre into the plaquette weight.

When this book writes $\mathrm{Re}\,U$, it means the real part of the quaternion $= \frac{1}{2}\,\mathrm{tr}\,U$. It is the same quantity as $\frac{1}{N}\,\mathrm{Re}\,\mathrm{tr}\,U$ for $SU(N)$ in 5-01 with the N of the group set to 2 (the N of 5-01 is the size of the group, a different quantity from the lattice size N of this chapter).

03

The Haar measure and its product — the uniform probability on the group

The introduction is in 5-01 § 06. On a compact group G there is exactly one probability measure invariant under multiplication from the left. It is called the normalised **Haar measure** and written dU Cited . On a compact group this measure is also invariant under multiplication from the right and under the map taking inverses Cited .

For $SU(2) \cong S^3$, the Haar measure is the uniform surface measure on $S^3 \subset \mathbb{R}^4$, normalised to total mass 1 Proved here . Reason: left multiplication by a unit quaternion is an orthogonal transformation of $\mathbb{R}^4$, so it preserves the uniform surface measure. Since there is only one left-invariant probability measure, the two coincide.

The reference measure on the space of configurations is the product of the Haar measures of the links.

$$dU = \prod_{\ell} dU_\ell \qquad (\text{there are } 4N^4 \text{ links}) \quad (5)$$

This is also a probability measure, and corresponds to choosing each link variable independently and uniformly. Taking inverses on individual links, and maps that rearrange the values of links onto other links, also preserve this product measure. The reflection (19-03) uses this property.

In Lean, the normalised Haar measure is defined as `Measure.haarMeasure ⊤` (the Haar measure taking the value 1 on the whole set). The names are `OSGap.haarProb` and `TwistedWilson.haarProb` (the same definition); that it is a probability measure is `OSGap.haarProb_isProbabilityMeasure`, and left invariance is `OSGap.haarProb_isMulLeftInvariant`.

04

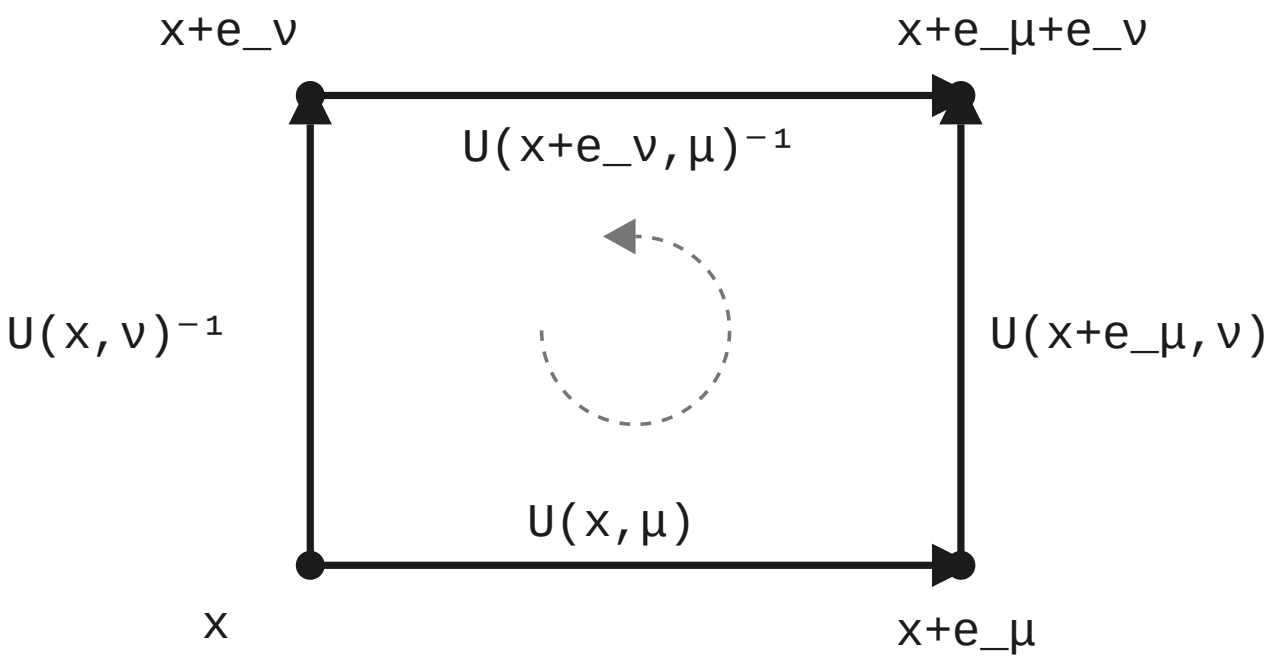
Plaquettes and holonomy — the product of four links, and its real part

The introduction is in 5-01 § 03. A **plaquette** is a pair $p = (x; \mu, \nu)$ $(\mu < \nu)$, the unit square with corners x, $x + e_\mu$, $x + e_\mu + e_\nu$, $x + e_\nu$. There are six pairs of directions ((1,2), (1,3), (1,4), (2,3), (2,4), (3,4) in the numbering of Part 21; (0,1), (0,2), (0,3), (1,2), (1,3), (2,3) in the numbering of Part 20), so there are $6N^4$ plaquettes. The orientation of a plaquette is fixed as the one turning from μ to ν, and plaquettes of the reverse orientation are not counted.

The **holonomy** of a plaquette is the product of the four link variables going once around its boundary.

$$U_p = U_{x,\mu}\, U_{x+e_\mu,\nu}\, U^{-1}_{x+e_\nu,\mu}\, U^{-1}_{x,\nu} \quad (6)$$

Figure 1 The holonomy of the plaquette $p = (x; \mu, \nu)$. Arrows show the orientation of the links. Links opposite to the direction of travel are multiplied as inverses



Proved here The dotted line in the figure is the direction of travel ($x \to x + e_\mu \to x + e_\mu + e_\nu \to x + e_\nu \to x$). The top and left edges are traversed against the orientation of their links, so inverses enter.

U_p is a unit quaternion, and its real part is $\operatorname{Re} U_p = \frac{1}{2} \operatorname{tr} U_p \in [-1, 1]$ Proved here . $\operatorname{Re} U_p = 1$ holds only when $U_p = 1$ (going once around the plaquette changes nothing).

Gauge transformations, and the invariance of the real part

A **gauge transformation** by $g_x \in SU(2)$ at each lattice point sends the link from x to $y = x + e_\mu$ to

$$U_{x,\mu} \longmapsto g_x \, U_{x,\mu} \, g_y^{-1} \quad (7)$$

The holonomy then changes by conjugation, $U_p \mapsto g_x U_p g_x^{-1}$, and the real part $\operatorname{Re} U_p$ does not change Proved here ($\operatorname{Re}(gqg^{-1}) = \operatorname{Re} q$, by cyclicity of the trace). The product of Haar measures is also invariant under gauge transformations Proved here (since each link is only multiplied from the left and the right). In Lean, the gauge transformation is written as `act` (value $g(\text{start})\, U\, g(\text{end})^*$). Gauge-invariant functions are used in Part 20 for reflection positivity through the middle plane (19-03).

05

The Wilson measure — the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ and expectations

The introduction is in 5-01 § 04. Take a real number β. In Parts 19 to 22 and Part 28 of this book, the **weight** of a configuration U is

$$w_\beta(U) = \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \quad (8)$$

The sum runs over all $6N^4$ plaquettes. It is related to the Wilson action $S_W(U) = \beta \sum_p \big(1 - \tfrac{1}{2} \operatorname{tr} U_p\big)$ of 5-01 by

$$e^{-S_W(U)} = e^{-6N^4 \beta} \, w_\beta(U) \quad (9)$$

holds Proved here. The constant $e^{-6N^4\beta}$ disappears on normalisation, so the two ways of writing give the same probability measure. β is the same quantity as in 5-01; for $SU(2)$, $\beta = 4/g^2$ Cited.

The **Wilson measure** is the probability measure obtained by multiplying the product of Haar measures by the weight w_β and normalising the whole to 1.

$$d\mu_{N,\beta}(U) = \frac{1}{Z_N(\beta)} \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (10)$$

We fix the notation. We write $\tilde{\mu}_{N,\beta}$ for the unnormalised measure $w_\beta(U) \prod_\ell dU_\ell$ (`TorusRP.wilsonMeasure N β` in Lean), and $\mu_{N,\beta}$ for the normalised (10) (`TorusRP.wilsonProb N β` in Lean). $\tilde{\mu}_{N,\beta} = Z_N(\beta)\, \mu_{N,\beta}$.

The **expectation** of a bounded measurable function F is

$$\langle F \rangle_{N,\beta} = \int F \, d\mu_{N,\beta} = \frac{1}{Z_N(\beta)} \int F(U)\, w_\beta(U) \prod_\ell dU_\ell \quad (11)$$

The sign of β is free as far as the definition goes. Physics usually uses $\beta > 0$ (the more the plaquettes line up at 1, the heavier the weight), but some theorems of Parts 20 and 21 hold including $\beta < 0$. The range is written for each theorem.

The weight (8) is gauge invariant (§ 04). Hence the Wilson measure is also invariant under gauge transformations Proved here. Even without fixing a gauge, the integral is an integral over a finite-

dimensional, compact space.

> May be skipped. In this book the unnormalised measure $w_\beta \prod dU$ is called the "unnormalised Wilson measure". In theorems that say "a certain integral is at least 0", such as reflection positivity, multiplication by a positive constant does not change the conclusion, so many of the Lean statements are written with the unnormalised measure (§ 07). For quantities that take ratios, such as expectations and variances, the normalised measure (10) is used.

06

The partition function — positive and finite

The **partition function** is the normalising constant.

$$Z_N(\beta) = \int \exp\Big(\beta \sum_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (12)$$

For the definition (10) to make sense we need $0 < Z_N(\beta) < \infty$. This follows from the following estimate Proved here .

$$e^{-6N^4|\beta|} \;\le\; Z_N(\beta) \;\le\; e^{6N^4|\beta|} \quad (13)$$

Proof: on each plaquette $|\operatorname{Re} U_p| \le 1$, so the absolute value of the exponent is at most $6N^4|\beta|$. Hence the integrand lies between $e^{-6N^4|\beta|}$ and $e^{6N^4|\beta|}$, and the reference measure has total mass 1. (End)

This estimate gets rapidly worse with the volume. It says nothing uniform in the volume. In the infinite-volume limit (Part 20), one deals not with Z itself but with ratios of expectations.

In Lean, the two-sided bounds on the integrand are `TorusRP.wilsonWeight_le` and `TorusRP.wilsonWeight_ge` (the exponent is the number of plaquettes $\times$ $|\beta|$; that the number of plaquettes is $6N^4$ was counted by hand); that the total mass of the unnormalised Wilson measure obtained by integrating them, $\tilde{\mu}_{N,\beta}(\text{whole}) = Z_N(\beta)$, is positive and finite is `TorusRP.wilsonMeasure_univ_pos` and `TorusRP.wilsonMeasure_univ_lt_top`; and that the partition function of the twisted lattice is positive is stated by `TwistedWilson.ZW_pos` (all real β, all pairs of twists).

07

Four conventions on the Lean side, and bridges — four ways of writing the same objects

This book's Lean theorems were written separately in four lines (strong coupling, vortices, fermions, ultraviolet). The object is the same: configurations of the $SU(2)$ lattice and their measure. But the indexing of the lattice, the space of link values and the way of writing the sign of the weight differ slightly. When reading the theorems of any part, check the conventions in this table.

	Strong-coupling side (Part 20)	Vortex side (Part 21)	Fermion side (19-05)	Ultraviolet side (19-04; tl
Lattice	$(\mathbb{Z}/N)^4$. Links `Lk N` = (lattice point) $\times$ (direction). Direction index 0 is time (the same as the numbering 0 to 3 of Part 20; reflection x $0 \mapsto -x0$)	$(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. `Site N M`, `Link`, `Plaq`. Indices 0 to 3 are directions 1 to 4 (the numbering of Part 21). **Time is index 3 (x_4)** (the reflection $x4 \mapsto -x$ 4 is in 19-03)	Time circle $\mathbb{Z}/2h \times$ space $(\mathbb{Z}/N)^3$. `PhysLink N h`. Direction index 3 is time. The bridge is for $N = 2h$	The box $(\mathbb{Z}/2^m L)^4$ at st `BoxWalk.Site L m`, `Box` (lattice point $\times$ `Fin 4`
Link values	The closed unit ball of quaternions `QBall` ($\|q\| \leq 1$)	The unit sphere `SU2` ($\|q\| = 1$)	The unit sphere `SU2`. Inside the action, mapped to the matrix (4)	The closed unit ball of `ChainBlockMap.Ball`. C `ChainBlockMap.Cfg` (th block map can fall insi domain and codomain the ball; 19-04)
Reference measure	`haarBall`: the Haar measure of the sphere pushed forward to the ball. The product is `Measure.pi`	`haarCfg`: the product of Haar measures	The product of Haar measures	Not fixed on the side of push-forward measure `ChainBlockMap.vP` take probability measures o configuration spaces of each step
Weight	e^{β} action, `action` = $\sum_{\mu<\nu}$ $\mathrm{Re}\, U_p$	$\exp(\beta \sum_p z_p \mathrm{Re}\, U_p$), where $z_p = \pm 1$ is the sign of the twist (19-02)	$e^{-\beta S_W}$, $S_W = \sum_p (1 - \frac{1}{2}$ $\mathrm{Re}\,\mathrm{tr}\, U_p)$	No weight enters the de Wilson measure or ano placed on the box type
Normalisation	Not normalised (`TorusRP.wilsonMeasure`). The normalised form is `wilsonProb`	No name is given to the measure; written with the partition function `TwistedWilson.ZW` and its ratios	Not normalised (`WilsonGauge.gaugeMeasure`)	The measures taken in measures (normalised)
Notation in this book	Unnormalised $\tilde{\mu}_{N,\beta}$, normalised $\mu_{N,\beta}$ ((10)), Z_N (β) ((12))	Z_0, Z_1, θ_M (19-02)	The notation of 19-05	The notation of 19-04

We explain the meaning of the differences in convention in turn.

The closed ball and the sphere. The strong-coupling side places link values in the closed unit ball $\{\|q\| \leq 1\}$, and takes as reference measure "the Haar measure of the sphere pushed forward to the ball" (`haarBall`). Using the ball is a choice made for the construction in Lean. Since the reference measure lives on the sphere, almost every configuration takes values in $SU(2)$, and the values of integrals are the same as under the sphere convention Proved here . The conjugate (`star`) is written in place of the inverse, but the two coincide on the sphere.

The sign of the weight. The weight on the fermion side is $e^{-\beta S_W}$, which by (9) is a positive constant multiple of $e^{\beta \sum \mathrm{Re}\, U_p}$.

Bridges — identifications shown in Lean

Bridge	What it says	Lean
Sum of the actions	On the torus of the same size $N = 2h$ in all four directions, for every configuration, $\sum_{\mu<\nu} \mathrm{Re}\, U_p + S_W = 6(2h)^4$ (through the re-indexing of directions and the embedding of the sphere into the ball)	`WilsonGauge.action_add_wilsonAction`
Agreement of reference measures	Pushing the product of Haar measures on the fermion side forward by `pull` gives the product of `haarBall` on the strong-coupling side	`WilsonGauge.map_pull`
Agreement of measures	`TorusRP.wilsonMeasure (2h) β` on the strong-coupling side equals $e^{6(2h)^4\beta}$ times the push-forward of `gaugeMeasure h β` on the fermion side by the re-indexing `pull` (all real β, $2h > 1$)	`WilsonGauge.wilsonMeasure_eq_map`
Agreement of reflections	Under the same re-indexing, the reflections through lattice-point planes on the two sides correspond	`WilsonGauge.refl_pull` (19-03)
Agreement of measures (strong coupling ↔ vortices)	When $M = N$ and there is no twist ($z_p \equiv 1$), for all $N \geq 1$ and all real β, `TorusRP.wilsonMeasure N β` on the strong-coupling side equals the product of Haar measures `haarCfg N N` on the vortex side multiplied by the density $\exp(\beta \sum_p \mathrm{Re}\, U_p)$ and pushed forward by the re-indexing of links (lattice point $x \mapsto ((x_0, x_1), (x_2, x_3))$, direction indices unchanged) and the inclusion `toBall` of the sphere into the ball. There is no constant factor (both sums over plaquettes run over the $6N^4$ plaquettes with $\mu < \nu$). `toBall` is injective (`WilsonBridge.toBall_injective`)	`WilsonBridge.wilsonMeasure_eq_map`
Expectations and total mass (strong coupling ↔ vortices)	For measurable F, $\int F\, d(\mathrm{wilsonMeasure}) = \int F(\mathrm{toBall}\, U)\, e^{\beta \sum_p \mathrm{Re}\, U_p}\, d(\mathrm{haarCfg})$. With $F = 1$, the total mass is `ZW N N β false false` (converted to a value in $[0, \infty]$)	`WilsonBridge.integral_wilsonMeasure`, `WilsonBridge.wilsonMeasure_univ`
Correspondence aligning time, and reflections (strong coupling ↔ vortices)	The same equality of measures holds for the re-indexing that sends time on the strong-coupling side (index 0) to time on the vortex side (index 3, x_4) and index k to $k - 1$ (lattice point $x \mapsto ((x_1, x_2), (x_3, x_0))$). The reflections through lattice-point planes on the two sides are then the same map ($x_0 \mapsto -x_0$ on the strong-coupling side and $x_4 \mapsto -x_4$ on the vortex side). Time–space plaquettes are sent to plaquettes of reverse orientation (the real part is the same; the holonomy is conjugated)	`WilsonBridgeT.wilsonMeasure_eq_mapT`, `WilsonBridgeT.refl_toBallT`, `WilsonBridgeT.action_toBallT`
The vortex ratio (no background)	With background $n_{12} = 0$ and $M = N$, $Z_{N,N}(\beta; 0, 1)/Z_{N,N}(\beta; 0, 0)$ equals the expectation of $\exp\left(-2\beta \sum \mathrm{Re}\, U_p\right)$ (over the N^2 time–space plaquettes corresponding to $\Pi_{(0,0)}$) under the normalised measure of the strong-coupling side	`WilsonBridgeV.vortex_ratio_eq`
The definition of the Haar measure	`haarProb` on the strong-coupling side and on the vortex side are defined by the same expression `Measure.haarMeasure ⊤` (two definitions in different namespaces; inside the proof of the agreement of measures above, they are unfolded and checked to agree)	`OSGap.haarProb`, `TwistedWilson.haarProb`

When $M = N$ and there is no twist, the strong-coupling side and the vortex side are tied together in Lean as one measure. Pushing forward the product of Haar measures on the vortex side, multiplied

by the weight $\exp(\beta \sum_p \operatorname{Re} U_p)$, by the re-indexing of links and the inclusion of the sphere into the ball gives exactly the Wilson measure of the strong-coupling side (there is no constant factor, and both sums over plaquettes run over $6N^4$ plaquettes). There are two re-indexings. The one that overlays the direction indices unchanged is used for the equalities of measures, expectations and total mass. The one that overlays time on the strong-coupling side (index 0) with time on the vortex side (index 3, x_4) is used in the chapters that use reflection. With the latter, the reflections through lattice-point planes on the two sides become the same map, and time–space plaquettes are sent to plaquettes of reverse orientation. Weights and actions written only in terms of real parts do not change, but when the holonomy itself is transported a conjugation enters. Overlaying the direction indices unchanged is allowed only for the equalities of measures; where reflection is used, the re-indexing that aligns time is used.

What this bridge does not say: there is no equality transporting a twisted measure (background $n_{12} = 1$, or $n_{34} = 1$) as a measure to the strong-coupling side. The θ_M of Part 21 and the expectation $\langle\cdot\rangle_0$ of Polyakov loops are quantities of the weight with background $n_{12} = 1$, not quantities on this bridge (what the bridge transports is only the form that writes the background-free ratio $Z(0,1)/Z(0,0)$ as an expectation). Lattices with $M \neq N$ have no counterpart on the strong-coupling side. These are equalities in finite volume, and they do not restate theorems about the infinite-volume limit or the mass gap.

The Lean definitions on the ultraviolet side consist only of maps and push-forwards on box lattices and ball-valued configurations, and do not contain the Wilson weight. The correspondence between the ultraviolet side and the other three sides is dealt with in 19-04, and is not listed in this chapter's table of bridges.

What this chapter can say

Can say	Cannot say
Proved here $SU(2)$ is the group of unit quaternions, and $\operatorname{Re} q = \frac{1}{2}\operatorname{tr} q \in [-1,1]$	— (definitions and elementary facts)
Cited The normalised Haar measure is unique, and invariant under left and right multiplication and inversion	The only properties named in Lean are that it is a probability measure and left invariance
Proved here $e^{-6N^4\lvert\beta\rvert} \le Z_N(\beta) \le e^{6N^4\lvert\beta\rvert}$. Lean: `TorusRP.wilsonWeight_le`, `wilsonWeight_ge` (the integrand), `TorusRP.wilsonMeasure_univ_pos`, `TorusRP.wilsonMeasure_univ_lt_top`, `TwistedWilson.ZW_pos`	Not an estimate uniform in the volume. The infinite-volume limit is outside this chapter
Proved here The Wilson measure and $\operatorname{Re} U_p$ are gauge invariant	The argument on gauge fixing (used for the reflection through the middle plane in Part 20) is not in this chapter
Lean: by `WilsonGauge.wilsonMeasure_eq_map` ($N = 2h$, all real β), the Wilson measures of the strong-coupling side and the fermion side agree up to a constant multiple	—
Lean: by `WilsonBridge.wilsonMeasure_eq_map` ($M = N$, no twist, all $N \ge 1$ and real β), the Wilson measures of the strong-coupling side and the vortex side are equal under the re-indexing (no constant factor). Under the re-indexing that aligns time, the reflections also agree (`WilsonBridgeT.refl_toBallT`)	A bridge for twisted measures ($n_{12} = 1$ or $n_{34} = 1$) as measures, and $M \neq N$. There is no formula writing the θ_M of Part 21 ($n_{12} = 1$) as an expectation on the strong-coupling side

Sources and reproduction

Item	Kind	Source / tools
Gauge theory on the lattice, the Wilson action	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445. Introduction in 5-01
Existence and uniqueness of the Haar measure, $SU(2) \cong S^3$	Cited	Standard facts. Introduction in 5-01 § 06
The form that writes reflection positivity with the unnormalised measure	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only)
Definitions of the four conventions, and the bridges	Lean	This book's Lean bundle (bridges strong coupling ↔ vortices: `WilsonBridge.wilsonMeasure_eq_map`, `integral_wilsonMeasure`, `wilsonMeasure_univ`, `toBall_injective`, `WilsonBridgeT.wilsonMeasure_eq_mapT`, `refl_toBallT`, `action_toBallT`, `WilsonBridgeV.vortex_ratio_eq`. Strong-coupling side: `TorusRP.QBall`, `haarBall`, `Lk`, `action`, `wilsonMeasure`, `wilsonProb`. Vortex side: `TwistedWilson.SU2`, `haarCfg`, `hol`, `zt`, `ZW`. Ultraviolet side: `BoxWalk.Site`, `BoxWalk.BoxLk`, `ChainBlockMap.Ball`, `ChainBlockMap.Cfg`, `ChainBlockMap.vP`. Fermion side: `WilsonGauge.wilsonAction`, `gaugeMeasure`, `action_add_wilsonAction`, `wilsonMeasure_eq_map`). The formulas of the definitions were copied by reading the source

Words used in this chapter — lattice, link, link variable, configuration, unit quaternion, Haar measure, plaquette, holonomy, gauge transformation, Wilson measure, unnormalised Wilson measure, partition function. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-02 Twisted boundary conditions, the vortex ratio θ and Polyakov loops — on the same lattice, we define partition functions with the signs of plaquettes flipped one by one, and their ratios.

2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Twist, θ, Polyakov loops ·
Kiichi, Shiori and Rin

Twisted boundary conditions, the vortex ratio θ and Polyakov loops

> This chapter is this book's own investigation. § 01 to § 04 arrange established knowledge, and § 05 to § 07 are this book's investigation (the definition of the minimal action of a vortex, and the conventions in Lean).

What this chapter shows — The definitions of the objects that the theorems of Part 21 talk about: the torus with transverse size N and longitudinal size M, the twist signs n_{12}, n_{34}, the partition functions Z_0, Z_1, the vortex ratio θ_M, the Polyakov loop $P_b(t)$, the twisted vacuum V, and the minimal action of a vortex $S_{\mathrm{v}}(N,m)$ and S_N. We also state here that the definitions are sound ($|\theta_M| < 1$, $0 \le S_N \le 2N^2$).

Prerequisite chapters — 19-01 (the lattice, plaquettes, the Wilson measure). The introduction to twists is in 25-01 § 01.

Said first — This is a chapter of definitions; it states no estimates of θ_M or of Polyakov loops (the estimates are in Part 21, and in 20-06 inside the strong-coupling window). The transverse size N is a fixed finite number, and nothing uniform in N is said. The lattice spacing is 1.

The order of this chapter

1. The torus with transverse size N and longitudinal size M — a small cross-section and a large plane
2. Twist signs — flipping the sign of plaquette weights
3. The partition functions Z_0, Z_1 and the vortex ratio θ_M — the price of inserting one twist
4. Polyakov loops — the product once around the longitudinal direction
5. The twisted vacuum — a configuration of action 0
6. The minimal action of a vortex — the price of passing one vortex through a block
7. Correspondence with Lean names — conventions on indices and truth values

01

The torus with transverse size N and longitudinal size M — a small cross-section and a large plane

The introduction is in 25-01 § 01. Take positive integers N, M, and let the lattice be

$$\Lambda_{N,M} = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2, \qquad x = (x_1, x_2;\, x_3, x_4) \quad (1)$$

Directions 1 and 2 are called **transverse** (size N), and directions 3 and 4 **longitudinal** (size M). The transverse part is a small two-dimensional torus T^2_N, and the longitudinal part is a large plane $(\mathbb{Z}/M)^2$; in Part 21, N is held fixed and M is made large.

Links (x, μ), plaquettes $p = (x; \mu, \nu)$ $(\mu < \nu)$, the holonomy U_p and the product of Haar measures are built in the same way as in 19-01. There are $4N^2M^2$ links and $6N^2M^2$ plaquettes.

Fixing one point $q = (x_3, x_4)$ of the longitudinal plane, the N^2 transverse lattice points sit above it. We write Π_q for the bundle of $(3,4)$ plaquettes based at these N^2 points. Within the plane, Π_q looks like a single point; transversely, it is a collection of plaquettes covering T^2_N (equation (3) of 25-01 § 01).

02

Twist signs — flipping the sign of plaquette weights

On the lattice, 't Hooft's twisted boundary conditions are introduced through signs $z_p \in \{+1, -1\}$ of the plaquette weights Cited . For two twists $n_{12}, n_{34} \in \{0, 1\}$ we set

$$z_p = \begin{cases} (-1)^{n_{12}} & p \text{ is a } (1,2) \text{ plaquette with base } (x_1, x_2) = (0,0) \\ (-1)^{n_{34}} & p \text{ is a } (3,4) \text{ plaquette with base } (x_3, x_4) = (0,0) \\ 1 & \text{otherwise} \end{cases} \quad (2)$$

That is, $n_{12} = 1$ flips the sign of the one plaquette based at the origin in each transverse $(1,2)$ plane (a single T^2_N with the longitudinal point fixed). There are M^2 flipped plaquettes in all. $n_{34} = 1$ flips the one plaquette based at the origin in each longitudinal $(3,4)$ plane (a single plane with the transverse point fixed), and the flipped plaquettes are the N^2 plaquettes of $\Pi_{(0,0)}$.

The twisted weight and partition function are

$$Z_{N,M}(\beta; n_{12}, n_{34}) = \int \exp\Big(\beta \sum_p z_p \operatorname{Re} U_p\Big) \prod_\ell dU_\ell \quad (3)$$

This satisfies $0 < Z < \infty$ Proved here (the same estimate as in 19-01 § 06; $|z_p \operatorname{Re} U_p| \le 1$).

When $N, M \ge 2$, where the flipped plaquettes are placed has no effect on the partition function. Replacing a link variable by $U_\ell \mapsto -U_\ell$ flips the signs of all plaquettes containing that link, and the Haar measure is preserved Proved here . What the partition function sees is only the parity of the number of flips piercing each plane (the flux). The proof is in the Moving-Surface Lemma of 25-01 § 02 (a statement for $N, M \ge 2$; the proof rests on each plaquette containing the same link at most once). 25-01 places the $(1,2)$ flip at $(x_1, x_2) = (N-1, N-1)$, but by this lemma it gives the same partition function.

It is related to the partition function $Z[z] = \int e^{-\beta S^z} \prod dU$ of 25-01 ($S^z = \sum_p (1 - \frac{1}{2} z_p \operatorname{tr} U_p)$) by $Z[z] = e^{-6N^2M^2\beta} Z_{N,M}(\beta; n_{12}, n_{34})$ Proved here . This constant disappears in the ratio of the next section.

03

The partition functions Z_0, Z_1 and the vortex ratio θ_M — the price of inserting one twist

In Part 21 the transverse twist is held fixed as a background at $n_{12} = 1$, and two partition functions are compared according to whether the longitudinal twist n_{34} is present.

$$Z_0 = Z_{N,M}(\beta; 1, 0), \qquad Z_1 = Z_{N,M}(\beta; 1, 1) \quad (4)$$

The **vortex ratio** is defined as

$$\theta_M = \theta_{N,M}(\beta) = \frac{Z_0 - Z_1}{Z_0 + Z_1} \quad (5)$$

The transverse N is held fixed in the notation; when needed we write $\theta_{N,M}$. Since Z_0, $Z_1 > 0$, θ_M is well defined and $|\theta_M| < 1$ Proved here . In terms of the **flux free energy** $\Delta F = \ln(Z_0/Z_1)$ (the term of "the definition of the sector and flux free energies" in 25-01; the same quantity as the vortex free energy F_v of 26-03), $\theta_M = \tanh(\Delta F/2)$ Proved here .

Meaning: flipping the N^2 plaquettes of $\Pi_{(0,0)}$ imposes the condition that one centre vortex (flux -1) pierces the longitudinal plane. θ_M close to 0 means that inserting one vortex hardly changes the partition function, that is, that vortices arise cheaply in the

plane. That θ_M decreases in M at the rate e^{-cM^2} is, through the Tomboulis–Yaffe inequality, the entrance to the argument for confinement Cited (25-01 § 02, 26-01).

$\theta_M \geq 0$ ($Z_1 \leq Z_0$) does not follow from the definition. It is a theorem of Part 21 (for even M), and is not assumed here. The continuity of the vortex ratio in the coupling, its limits, and matching it to a chosen value are in 21-07.

May be skipped. The thin-vortex floor (Part 21) is written in Lean in a form that holds whether the background n_{12} is set to 0 or 1. We write $\theta_M^{(n_{12})}$ for the ratio $(Z_{N,M}(\beta; n_{12}, 0) - Z_{N,M}(\beta; n_{12}, 1))/(Z_{N,M}(\beta; n_{12}, 0) + Z_{N,M}(\beta; n_{12}, 1))$ in that case. θ_M means $\theta_M^{(1)}$.

04

Polyakov loops — the product once around the longitudinal direction

Fix a transverse position $b \in (\mathbb{Z}/N)^2$ and a position $t \in \mathbb{Z}/M$ in direction 4, and multiply in order the M link variables going once around direction 3.

$$P_b(t) = \mathrm{Re}\Big(U_{(b;0,t),3}\, U_{(b;1,t),3} \cdots U_{(b;M-1,t),3} \Big) \quad (6)$$

This is called the **Polyakov loop**. Since the real part is half the trace, $P_b(t) = \frac{1}{2}\,\mathrm{tr}(\cdots) \in [-1, 1]$ Proved here . Being the trace of a product along a closed path, it does not change under gauge transformations Proved here .

We write $\langle \cdot \rangle_0$ for the expectation normalised with the weight of the twist background $n_{12} = 1$, $n_{34} = 0$.

$$\langle F \rangle_0 = \frac{1}{Z_0} \int F(U) \exp\Big(\beta \sum_p z_p \,\mathrm{Re}\, U_p \Big) \prod_\ell dU_\ell \qquad (n_{12} = 1,\ n_{34} = 0) \quad (7)$$

What Part 21 estimates is the pair of two Polyakov loops separated by T in direction 4, $\langle P_b(0)\, P_{b'}(T) \rangle_0$.

The expectation of a single loop is 0 Proved here : multiplying all direction-3 links at $x_3 = 0$ by -1, every plaquette contains 0 or 2 such links (counting repetitions), so the weight does not change, the Haar measure is preserved, and only $P_b(t)$ changes sign. What carries meaning is therefore the correlation of a pair.

05

The twisted vacuum — a configuration of action 0

Using the unit quaternions i, j, define the configuration V by

$$V_{x,1} = \begin{cases} \mathrm{i} & x_1 = 0 \\ 1 & x_1 \neq 0 \end{cases}, \qquad V_{x,2} = \begin{cases} \mathrm{j} & x_2 = 0 \\ 1 & x_2 \neq 0 \end{cases}, \qquad V_{x,3} = V_{x,4} = 1 \tag{8}$$

This is called the **twisted vacuum**. For the signs (2) with background $n_{12} = 1$, $n_{34} = 0$, on every plaquette

$$z_p \operatorname{Re} V_p = 1 \tag{9}$$

holds Proved here . For example, on the $(1,2)$ plaquette with base $(x_1, x_2) = (0,0)$, $V_p = \mathrm{i}\mathrm{j}\,\mathrm{i}^{-1}\mathrm{j}^{-1} = -1$, which cancels against $z_p = -1$. On the other plaquettes the holonomy is 1. Since the action $1 - z_p \operatorname{Re} U_p$ of each plaquette is at least 0, V gives the minimum (0) of the twisted action. Lean: `TwistBlocks.vac`; (9) is `TwistBlocks.vac_term`.

That i and j anticommute ($\mathrm{i}\mathrm{j} = -\mathrm{j}\mathrm{i}$) is the mechanism by which the transverse twist is absorbed at action 0. That the twisted vacuum is unique up to gauge transformations and global transformations is in 25-01 § 05.

06

The minimal action of a vortex — the price of passing one vortex through a block

We define the number S_N that appears in the exponent of the thick-vortex floor (Part 21). Take a positive integer m, and let the longitudinal size be $M = 2m$. The lines $x_3 \in \{0, m\}$ and $x_4 \in \{0, m\}$ divide the longitudinal plane $(\mathbb{Z}/2m)^2$ into four **blocks** of side m; we use the block at the origin, $b_0 = \{0, \dots, m-1\}^2$.

- **Frame**: the links lying on the boundary lines. Direction-3 links whose x_4 is on a line, direction-4 links whose x_3 is on a line, and links in the transverse directions 1 and 2 whose longitudinal point (x_3, x_4) is on a line.
- **Interior**: the links that are not in the frame and whose longitudinal point lies in b_0.
- **Plaquettes of the block**: all plaquettes whose base has its longitudinal point in b_0 (all six orientations, all transverse positions).

Fix the frame to the twisted vacuum V, let the interior move freely, and flip the sign only of the $(3,4)$ plaquettes above the corner point $(x_3, x_4) = (0,0)$ (the N^2 plaquettes of $\Pi_{(0,0)}$); the infimum of the resulting action is

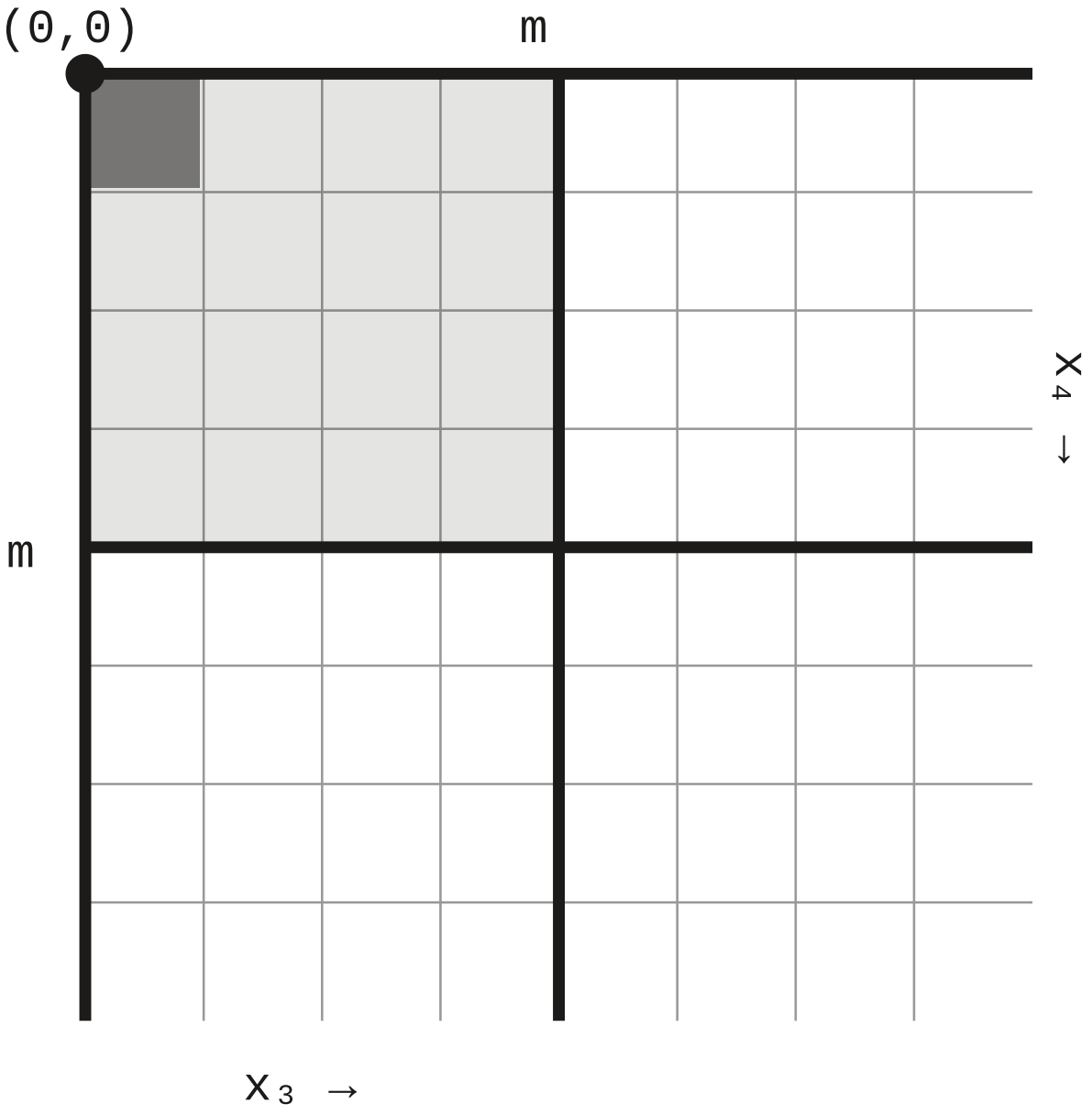
$$S_{\mathrm{v}}(N,m) = \inf_{\text{interior}} \sum_{p \in b_0} \left(1 - z_p\, \eta_p \operatorname{Re} U_p\right), \qquad \eta_p = \begin{cases} -1 & p \in \Pi_{(0,0)} \\ 1 & \text{otherwise} \end{cases} \tag{10}$$

Here z_p are the signs of the background $n_{12}=1$, $n_{34}=0$. The flipped $\Pi_{(0,0)}$ are the same plaquettes as for the longitudinal twist $n_{34}=1$, so $S_{\mathrm{v}}(N,m)$ is "the minimal action of passing one vortex through a block of side m whose boundary is fixed to the twisted vacuum". The infimum over m,

$$S_N = \inf_{m\geq 1} S_{\mathrm{v}}(N,m) \tag{11}$$

is called the **minimal action of a vortex**.

Figure 1 The longitudinal plane $(\mathbb{Z}/2m)^2$ ($m=4$). Thick lines are the frame (fixed to the twisted vacuum), the grey area is the block $b0$ at the origin, and the black dot is the corner $(0,0)$ whose signs are flipped. Above each point sits the transverse T^2_N



Proved here The figure shows one period of the periodic plane (the right and bottom edges coincide with the left and top edges). The dark square is the square based at the corner point $(0,0)$, above which $\Pi_{(0,0)}$ sits (it refers to the same thing as the black dot in the figure).

We check that the definition is sound. Each term $1 - z_p \eta_p \operatorname{Re} U_p$ lies between 0 and 2, so the infimum is finite and $S_{\mathrm{v}}(N,m) \geq 0$. Taking the interior also to be V, all unflipped plaquettes give 0 ((9)) and each of the N^2 flipped ones gives 2, so $S_{\mathrm{v}}(N,m) \leq 2N^2$. Therefore

$$0 \leq S_N \leq 2N^2 \quad (12)$$

Proved here . Lean: `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`. The infimum in (10) is attained by an interior configuration (because the space of interior configurations is compact; Lean: `TwistBlocks.Svortex_attained`). The upper bound $2N^2$ is the same number as the exponent of the thin-vortex floor; whether S_N is strictly smaller than it, and whether $S_N > 0$, do not follow from the definition. The latter is a theorem of 21-05, and the former for $N = 2, 3, 4$ is a computer certificate in the same chapter.

07

Correspondence with Lean names — conventions on indices and truth values

The Lean definitions on the vortex side correspond to the notation of this chapter as follows. Direction indices start at 0, and 0, 1, 2, 3 are directions 1, 2, 3, 4 of this chapter. Twists are truth values, and `true` is $n = 1$.

This chapter	Lean	Note
$\Lambda_{N,M}$, links, plaquettes	`TwistedWilson.Site N M`, `Link`, `Plaq`	`Site N M = (ZMod N × ZMod N) × (ZMod M × ZMod M)`. Plaquettes are pairs of directions $\mu < \nu$
U_p	`TwistedWilson.hol`	The quaternion product $U_{x,\mu} U_{x+e_\mu,\nu} U^*_{x+e_\nu,\mu} U^*_{x,\nu}$
z_p (equation (2))	`TwistedWilson.zt n12 n34`	The flips are placed at bases $(x_1,x_2) = (0,0)$ and $(x_3,x_4) = (0,0)$
$Z_{N,M}(\beta; n_{12}, n_{34})$	`TwistedWilson.ZW N M β n12 n34`	Not normalised. The reference measure is `haarCfg`
Z_0, Z_1	`ZW N M β true false`, `ZW N M β true true`	
θ_M	`TwistBlocks.thetaM N M β`	The same expression as equation (5)
$P_b(t)$	`HalfSplit.polyLoop N M b t`	Direction 3 is index 2. The product is in the order $k = 0, 1, \ldots, M-1$

This chapter	Lean	Note
$\langle P_b(s) P_{b'}(t) \rangle_0$	`PolyakovT.polyExp N M β b b' s t`	`PolyakovT.corr / ZW N M β true false`. The weight `HalfSplit.wE N M β (fun _ => 1)` of `corr` is the same expression as the weight of Z_0 (`PolyakovT.wE_one`)
V	`TwistBlocks.vac`	
$S_{\mathrm{v}}(N,m)$, S_N	`TwistBlocks.Svortex N m`, `TwistBlocks.SN N`	`Svortex N m = Smin N (2m) m 2 true (0,0) (V on the frame) true`. `SN N` is the infimum over positive integers m

The block action of `Smin` in Lean also includes in its sum the plaquettes lying only on the frame. The difference is a function of the frame alone and has no effect on the infimum over the interior. The "plaquettes of the block" in equation (10) are matched to this convention.

What this chapter can say

Can say	Cannot say
Proved here $0 < Z_{N,M}(\beta; n_{12}, n_{34}) < \infty$, hence $\lvert\theta_M\rvert < 1$. Lean: `TwistedWilson.ZW_pos`	Estimates of the size and sign of θ_M (Part 21)
Proved here For $N, M \geq 2$, the placement of the flips has no effect on the partition function; only the parity of the flux matters (proof in the Moving-Surface Lemma of 25-01 § 02)	The Lean definitions of this chapter fix the placement at the origin. The lemma that moves the placement is not listed as a Lean theorem
Proved here The twisted vacuum V has $z_p \operatorname{Re} V_p = 1$ on every plaquette. Lean: `TwistBlocks.vac_term`	The uniqueness of the vacuum (25-01 § 05) is not dealt with in this chapter
Proved here $\langle P_b(t) \rangle_0 = 0$ (centre transformation)	Not listed as a Lean theorem. Estimates of the pair correlation are in Part 21
Proved here $0 \leq S_N \leq 2N^2$, and the infimum in (10) is attained. Lean: `TwistBlocks.SN_bounds`, `Svortex_attained`	$S_N > 0$ and $S_N < 2N^2$ do not follow from the definition (21-05). Nothing uniform in N is said

Sources and reproduction

Item	Kind	Source / tools
Twisted boundary conditions	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. Introduction in 25-01
The lattice form with a field of plaquette signs	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117

Item	Kind	Source / tools
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. 26-01
Polyakov loops	Cited	A standard quantity. Its treatment as an order variable at finite temperature is Svetitsky–Yaffe, NPB 210 (1982) 423 (the bibliographic data follow the citation in this book)
Definitions (lattice, twist, Z, θ_M, Polyakov loops, twisted vacuum, S_{v}, S_N) and (9), (12)	Lean	This book's Lean bundle (`TwistedWilson.ZW`, `zt`, `hol`, `haarCfg`, `TwistBlocks.thetaM`, `vac`, `vac_term`, `Svortex`, `SN`, `Svortex_le`, `SN_bounds`, `Svortex_attained`, `HalfSplit.polyLoop`, `PolyakovT.polyExp`, `PolyakovT.corr`). The formulas of the definitions were copied by reading the source

Words used in this chapter — twisted boundary conditions, twist sign, flux, partition function, vortex ratio, flux free energy, Polyakov loop, twisted vacuum, block, frame, minimal action of a vortex. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-03 Reflection, the transfer operator and the vessel of reconstruction — defines the reflection and the vessel that builds from it a Hilbert space and a transfer operator.

2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Reflection, the vessel, the definition of the mass gap · Kiichi, Shiori and Rin

Reflection, the transfer operator and the vessel of reconstruction

> This chapter is this book's own investigation. § 01 to § 04 and § 06 arrange established knowledge, and § 05, § 07 and § 08 are this book's investigation (the conventions in Lean for the vessel and for the infinite-volume space).

What this chapter shows — The definitions of the objects that the theorems of Part 20 talk about: the reflection Θ (through lattice-point planes and through middle planes), functions on the positive side, the reflection pairing $\langle F, G\rangle = \int F(\Theta\omega)G(\omega)\,d\mu$, time translation, the vessel of reconstruction (the triple of pairing, translation and constant) and the Hilbert space, vacuum Ω and transfer operator T built from it, the definition of the mass gap $\|T - P\Omega\| \le e^{-m}$, and the space of infinite-volume configurations with the periodic lift.

Prerequisite chapters — 19-01 (the lattice and the Wilson measure). Introductions in 4-10 (reflection positivity and reconstruction), 5-03 (reflection on the lattice and the transfer matrix) and 4-11 (the definition of the mass gap).

Said first — This is a chapter of definitions. Reflection positivity (that the pairing is positive semidefinite), the existence of the infinite-volume limit and estimates of the mass gap are all theorems of Part 20, and are not even assumed here. The definition of the mass gap is a quantity at a fixed lattice spacing, in units of one lattice step. It is not a definition of the mass gap of the continuum limit.

The order of this chapter

1. Reflection — lattice-point planes, and planes through the middle of links
2. The positive side — functions depending on one side of the mirror only
3. The reflection pairing — the integral of a product with the mirrored function
4. Time translation — moving the positive side one step further in
5. The vessel of reconstruction — from pairing, translation and constant to a Hilbert space and a transfer operator
6. The definition of the mass gap — T contracts by at most e^{−m} outside the vacuum

7. The space of infinite-volume configurations — configurations on $\mathbb{Z}^4$ and the periodic lift
8. Correspondence with Lean names — the three mirrors and the names of the vessel

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01

Reflection — lattice-point planes, and planes through the middle of links

The introduction is in 5-03 § 02. The formulas of this chapter are written in the **numbering of Part 20** of 19-01 § 01 (directions $0, 1, 2, 3$, with 0 as time), and the time coordinate x_0 is written t. The lattice is periodic with even size $2M$ in the time direction. The reflections in Lean (`TorusRP.thL`, `MidRP.thM`) are defined only on $(\mathbb{Z}/2M)^4$, of the same size $N = 2M$ in all four directions.

There are two kinds of mirror.

- **Reflection through lattice-point planes**: $t \mapsto -t$. The mirror is the two planes $t = 0$ and $t = M$ passing through lattice points (by periodicity $-M \equiv M$).
- **Reflection through middle planes**: $t \mapsto 1 - t$. The mirror is the two planes $t = \frac{1}{2}$ and $t = M + \frac{1}{2}$ cutting across the middle of the time-direction links.

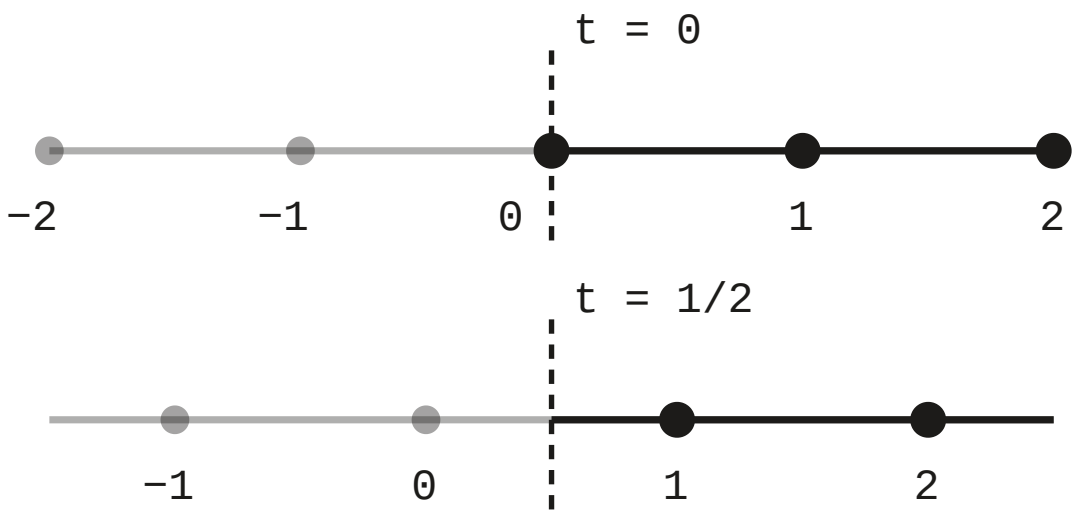
From the map of lattice points $x \mapsto \vartheta x$, the map of link variables is defined as follows. A space-direction link (x, ν) goes to $(\vartheta x, \nu)$ with its value unchanged. A time-direction link has its orientation reversed, so we take the **inverse** of the link it is mirrored to.

$$(\Theta U)_{x,\nu} = U_{\vartheta x,\nu} \quad (\nu \text{ spatial}), \qquad (\Theta U)_{x,0} = \left(U_{\vartheta(x+e_0),0}\right)^{-1} \quad (1)$$

For the reflection through lattice-point planes, $\vartheta(x + e_0) = \vartheta x - e_0$; that is, the link from t to $t + 1$ goes to the link from $-t - 1$ to $-t$. For the reflection through middle planes, the link from t to $t + 1$ goes to the link from $-t$ to $1 - t$, and the time links at $t = 0$ and $t = M$ (the links crossing the mirror) go to their own inverses.

Either Θ is a map on the space of configurations; applied twice it returns to the original, and it preserves the product of Haar measures Proved here (a rearrangement of links and an inverse on each link; 19-01 § 03). Plaquettes go to plaquettes and the value of $\mathrm{Re}\, U_p$ equals the value on the mirrored plaquette, so the Wilson weight is also preserved Proved here (for the lattice-point mirror, Lean: `TorusRP.holQ_refl`. The middle-plane mirror is the lattice-point mirror combined with a translation by one step, and the same follows from translation invariance).

Figure 1 The two mirrors seen along one row in the time direction. Dots are lattice points (space links sit on them), segments are time links. Top: the lattice-point plane $t = 0$. Bottom: the middle plane $t = \frac{1}{2}$. The dark part is the positive side



Proved here In the top row, the space links at the point $t = 0$ on the mirror do not move, and are counted on both the positive and the negative side. In the bottom row there is no lattice point on the mirror, and the time link from 0 to 1 crosses the mirror.

On the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of the vortex side (Part 21), the reflection through lattice-point planes in direction 4, $x4 \mapsto -x4$ (M even, mirrors at $x4 = 0$ and $x4 = M/2$), is used with the same rule (1) (in the numbering of Part 21, direction 4 plays the role of time). When $M = N$, under the re-indexing that overlays time on the strong-coupling side with direction 4 on the vortex side, the reflections through lattice-point planes on the two sides are the same map (Lean: `WilsonBridgeT.refl_toBallT`, 19-01 § 07).

02

The positive side — functions depending on one side of the mirror only

One side of the mirror is called the **positive side**, and we consider functions that depend only on links on the positive side. Precisely: fix the set L_+ of links on the positive side; a function F whose values agree whenever two configurations agree on L_+ is said to "depend only on L_+".

Mirror	Links on the positive side L_+
Lattice-point planes ($t = 0, M$)	Space links at times $0 \le t \le M$, and time links whose base time is $0 \le t \le M - 1$ (the closed positive half)

Mirror	Links on the positive side L_+
Middle planes ($t = \frac{1}{2}, M + \frac{1}{2}$)	Space links at times $1 \le t \le M$, and time links whose base time is $0 \le t \le M$ (including the links at $t = 0$ and $t = M$ crossing the mirror)

Θ sends the positive side to the negative side. So if F depends only on the positive side, $F \circ \Theta$ depends only on the negative side (and the mirror) Proved here .

The class of functions used by the theorems of Part 20 is real-valued functions depending only on the positive side that are **bounded and measurable**. For the middle-plane mirror, some theorems use the class with **gauge invariance** (19-01 § 04) added. Which class is used is written for each theorem.

03

The reflection pairing — the integral of a product with the mirrored function

The introduction is in 4-10 § 04. For a finite measure μ on the space of configurations (in Part 20, the Wilson measure) and a reflection Θ, the **reflection pairing** of bounded functions F, G on the positive side is defined as

$$\langle F, G \rangle_\mu = \int F(\Theta\omega)\, G(\omega)\, d\mu(\omega) \quad (2)$$

It is bilinear in F, G. If μ is invariant under Θ, it is symmetric Proved here (change variables by $\omega \mapsto \Theta\omega$ and use $\Theta^2 = \mathrm{id}$; for the Wilson measure, Lean: `TorusRP.wilson_refl_symm`).

Reflection positivity is the property that $\langle F, F \rangle_\mu \ge 0$ for every F on the positive side. This is not a definition but a property of the measure, and that it holds for the Wilson measure is a theorem of Part 20 (20-01). It is not assumed in this chapter.

For the measure $c\mu$ multiplied by a positive constant c, the pairing is also multiplied by c, so whether it is positive semidefinite does not depend on the normalisation. This is why reflection positivity in Lean is written with the unnormalised Wilson measure (19-01 § 05).

04

Time translation — moving the positive side one step further in

Let S be the map shifting a configuration by one step in the time direction, acting on functions by $(\tau F)(\omega) = F(S\omega)$. The direction is chosen so that if F depends on links at times $[0, R]$, then τF depends on links at times $[1, R+1]$. Then τ sends functions on the positive side to functions on the positive side.

When the measure is translation invariant and $\Theta S = S^{-1}\Theta$ holds, τ is symmetric with respect to the pairing Proved here .

$$\langle \tau F, G\rangle_\mu = \langle F, \tau G\rangle_\mu \quad (3)$$

The operator that this τ induces on the lattice is the **transfer operator** (transfer matrix). 5-03 writes it directly in the temporal gauge (5-03 § 05). Part 20 of this book builds it from the pairing and τ alone, through the vessel of the next section.

05

The vessel of reconstruction — from pairing, translation and constant to a Hilbert space and a transfer operator

The introduction is in 4-10 § 05. What is extracted from the Osterwalder–Schrader reconstruction as the part that does not depend on the measure is called the **vessel of reconstruction**. A vessel is the following tuple on a real vector space V.

Field	Condition	When built from a measure
Pairing B (bilinear form)	Positive semidefinite $B(x,x) \geq 0$, symmetric $B(x,y) = B(y,x)$	The reflection pairing (2)
Translation τ (linear map)	Symmetric with respect to B: $B(\tau x, y) = B(x, \tau y)$	Time translation (§ 04)
Constant $\mathbf{1} \in V$	$B(\mathbf{1},\mathbf{1}) = 1$, $\tau\mathbf{1} = \mathbf{1}$	The constant function 1 (probability measure)
Bounded orbits	For each x there is C with $B(\tau^k x, \tau^k x) \leq C$ for all k	For bounded functions, $C = \|F\|_\infty^2$

From a vessel we build the following.

1. **Hilbert space $\mathcal{H}$**: the completion of V with the semi-inner product B. Vectors with $B(x,x) = 0$ collapse to 0 in the completion. We write $[x]$ for the image of an element x of V.
2. **Vacuum $\Omega = [\mathbf{1}]$**. $\|\Omega\| = 1$.
3. **Transfer operator T**: the continuous extension of $T[x] = [\tau x]$. It can be extended because $B(\tau x, \tau x) \leq B(x,x)$ (τ contracts). This follows from symmetry, positive semidefiniteness and bounded orbits Proved here . Route: by Cauchy–Schwarz, $a_k = B(\tau^k x, \tau^k x)$ is log-convex ($a_{k+1}^2 \leq a_k a_{k+2}$), so the ratio a_{k+1}/a_k increases. If $a_1 > a_0$, then a_k grows exponentially, contradicting boundedness (Lean: `OrbitGrowth.contraction_of_bounded`).

The T built in this way is self-adjoint, $\|T\| \leq 1$, and $T\Omega = \Omega$ Proved here . In Lean, the vessel is written as the structure `OSGap.Datum`, the Hilbert space as `Datum.Hilb`, T as `Datum.T` and Ω as `Datum.Ω`, and the properties are stated by `Datum.contraction`, `Datum.T_symm`, `Datum.T_Ω` and `Datum.norm_Ω`.

The assembly that builds a vessel from a measure is `OSGap.measureDatum`. For a probability measure ν, a reflection and a translation, it takes three fields as assumptions — the pairing is positive semidefinite (reflection positivity), the pairing is symmetric, the translation is symmetric — and fills the remaining fields (constant, bounded orbits) itself. The theorems of Part 20 discharge these three assumptions for the limit of Wilson measures.

> May be skipped. Examples satisfying all the fields of a vessel can be made even small. Lean's `OSGap.exampleDatum` is a vessel on the two-dimensional $V = \mathbb{R}^2$ in which, for $|r| \leq 1$, T acts as multiplication by r on the complement of the vacuum (`example_opNorm_eq`). It checks that the fields of the vessel are consistent (non-empty).

06

The definition of the mass gap — T contracts by at most e^{−m} outside the vacuum

The introduction is in 4-11 § 01. Let the projection onto the vacuum be $P_\Omega v = \langle \Omega, v \rangle \, \Omega$. For a real number m, a vessel D is said to **have mass gap** m if

$$\|T - P_\Omega\| \leq e^{-m} \quad (4)$$

(operator norm). Lean: `OSGap.Datum.HasMassGap D m`. The unit is one lattice step, and the lattice spacing is fixed. Since T is a self-adjoint contraction with $T\Omega = \Omega$, $\|T - P_\Omega\| \leq 1$ always holds, and for $m \leq 0$, (4) holds automatically Proved here . It carries meaning when $m > 0$.

Meaning: T does not move Ω and is self-adjoint, so it preserves the orthogonal complement of the vacuum. (4) is the same as $\|T\| \leq e^{-m}$ on that complement Proved here . If $T > 0$ ($T \geq 0$ with no kernel), we can write $T = e^{-H}$, and (4) becomes the statement that the spectrum of H on the complement of the vacuum lies in $[m, \infty)$, that is, that the gap in the sense of 4-11 is at least m. The T of Part 20 has a kernel (20-04). In that case it cannot be written as $T = e^{-H}$, and (4) is read in the form $\|T\| \leq e^{-m}$ on the complement of the vacuum.

(4) is equivalent to truncated correlations decaying at a uniform rate for every x on the positive side.

$$\|T - P_\Omega\| \le e^{-m} \iff \forall x \in V\ \exists C_x\ \forall k:\ B(\tau^k x, \tau^k x) - B(\mathbf{1}, x)^2 \le C_x\, e^{-2mk} \quad (5)$$

Lean: right to left is `OSGap.Datum.hasMassGap_of_decay`, left to right is `OSGap.Datum.decay_of_hasMassGap` (the constant is $C_x = B(x,x) - B(\mathbf{1},x)^2$). This is the form on the vessel of "the gap is equivalent to all correlations decaying uniformly" in 4-11 § 02.

There are two things this definition alone cannot say.

- **Non-degeneracy**: if $\mathcal{H}$ is only the one dimension spanned by Ω, then $T - P_\Omega = 0$ and (4) holds for every m Proved here . So the theorems of Part 20 write separately in their conclusion that there is a vector of length 1 orthogonal to the vacuum (that the complement of the vacuum is not 0).
- $T \ge 0$: the definition does not forbid negative spectrum of T. $T \ge 0$ is also a separate conclusion of Part 20.

07

The space of infinite-volume configurations — configurations on $\mathbb{Z}^4$ and the periodic lift

The introduction is in 5-04 § 01. To compare measures on tori of different sizes, we place them all on one space. Let the space of configurations on all links (x,μ) ($x \in \mathbb{Z}^4$) of $\mathbb{Z}^4$ be

$$\mathcal{C} = \prod_{\ell \text{ a link of } \mathbb{Z}^4} \{q \in \mathbb{H} : \|q\| \le 1\} \quad (6)$$

with the product topology. The closed unit ball is compact, so $\mathcal{C}$ is compact Cited (Tychonoff's theorem). Taking values in the closed ball here is the convention of 19-01 § 07.

The periodic lift: a configuration ω on the torus $(\mathbb{Z}/L)^4$ is made into a point of $\mathcal{C}$ by reading the coordinates of the links of $\mathbb{Z}^4$ modulo L. We write the probability measure on $\mathcal{C}$ obtained by pushing forward the normalised Wilson measure $\mu_{L,\beta}$ (19-01 (10)) by this map, for $L = 2(n+1)$, as

$$\mu^{\text{lift}}_{n,\beta} = (\text{periodic lift})_*\, \mu_{2(n+1),\beta} \qquad (n = 0, 1, 2, \dots) \quad (7)$$

Lean: `WilsonLimit.liftProb β n`. The sequence over all sizes L is `AllN.liftS`.

Convergence of probability measures on $\mathcal{C}$ is measured by **weak convergence** ($\int F$ converges for every bounded continuous F) Cited . That the sequence (7) converges to

one probability measure ν_β in the strong-coupling window is a theorem of Part 20 (20-03); the existence of the limit is not assumed here.

On $\mathcal{C}$, too, the reflection (lattice-point planes $t \mapsto -t$, rule (1)) and time translation are defined by the same formulas. The space of functions on the positive side is

$$W = \{F : \mathcal{C} \to \mathbb{R} \text{ bounded and continuous } : \ F \text{ depends only on finitely many links with bas}$$

W contains the constant 1 and is closed under translation. The subspace W_{gi} restricted to gauge-invariant functions is also used. The vessel of Part 20 has $V = W$ (or W_{gi}), the pairing (2) with respect to ν_β, the translation of § 04, and the constant 1.

08

Correspondence with Lean names — the three mirrors and the names of the vessel

This chapter	Lean	Note
The reflected configuration $\Theta\omega$	`LinkReflection.linkRefl θ σ ω`	$(\Theta\omega)_\ell = \sigma_\ell(\omega_{\theta\ell})$. θ is a rearrangement of links, σ a map on values
The lattice-point mirror $((\mathbb{Z}/2M)^4)$	`TorusRP.thL`, `TorusRP.sig`	Conjugate (`star`) on time links. The positive side is `TorusRP.fixP` (space links on the mirror) and `TorusRP.posP`
The middle-plane mirror	`MidRP.thM`, `MidRP.sigT`	`thM` is the lattice-point mirror combined with a translation by one step. The positive side is `TorusGauge.cr` (crossing links) and `MidRP.posM`
The mirror on the vortex side (direction 4 of Part 21)	`HalfSplit.Th` (`HalfSplit.thL`, `HalfSplit.sgL`)	M even. Links on the mirror are `HalfSplit.fixL`. `HalfSplit.thL` has the same name as `TorusRP.thL` but is a different definition
The mirror on the fermion side	`WilsonGauge.thetaPhys`	Corresponds to the mirror on the strong-coupling side through `WilsonGauge.refl_pull` (19-05)
Vessel, Hilbert space, T, Ω	`OSGap.Datum`, `Datum.Hilb`, `Datum.T`, `Datum.Ω`	The fields of the vessel are `B`, `τ`, `one` and six conditions (`nonneg`, `symm`, `τ_symm`, `one_one`, `τ_one`, `bounded`)
Vessel built from a measure	`OSGap.measureDatum`	Assumption fields: reflection positivity, symmetry of the pairing, symmetry of the translation
Mass gap	`OSGap.Datum.HasMassGap D m`	`‖D.T - vacProj D.Ω‖ ≤ Real.exp (-m)`
$\mathcal{C}$, mirror, translation	`WilsonLimit.Cfg`, `zRefl`, `zShift`	`Cfg` is a map from the links of $\mathbb{Z}^4$ to `QBall`

This chapter	Lean	Note
Periodic lift, (7)	`WilsonLimit.perLift`, `WilsonLimit.liftProb`	The torus size is `WilsonLimit.Nn n = 2(n+1)`
W, W_{gi}	`WilsonLimit.Wsub`, `GaugeZ.Wgi`	`Wgi = Wsub ⊓ Winv` (gauge invariant)

The theorems of Part 20 are stated in a form in which the fields `B`, `τ`, `one` of the vessel are tied, within the conclusion, to formula (2), § 04 and the constant 1. In a form that only says a vessel exists, the fields could be chosen trivially so as to satisfy the mass gap (the same reason as the remark on non-degeneracy in § 06).

What this chapter can say

Can say	Cannot say
Proved here The two reflections are involutions, and preserve the product of Haar measures and the Wilson weight. Lean: `TorusRP.holQ_refl` (lattice-point planes)	Reflection positivity (a theorem of 20-01) is not even assumed in this chapter
Proved here If the measure is invariant under Θ, the pairing is symmetric. For the Wilson measure, Lean: `TorusRP.wilson_refl_symm`	—
Lean: from a vessel, T can be built, and it is self-adjoint, a contraction, and $T\Omega = \Omega$ (`Datum.contraction`, `T_symm`, `T_Ω`)	That the assumptions of the vessel are satisfied for a concrete measure (Part 20)
Lean: the mass gap (4) and the uniform decay of correlations (5) are equivalent (`hasMassGap_of_decay`, `decay_of_hasMassGap`)	(4) includes neither non-degeneracy nor $T \geq 0$, and holds automatically for $m \leq 0$. The lattice spacing is fixed; it is not a definition of the mass gap of the continuum limit
Cited $\mathcal{C}$ is compact	The convergence of the sequence (7) (a theorem of 20-03) is not assumed

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and reconstruction	Cited	K. Osterwalder, R. Schrader, CMP 31 (1973) 83–112; 42 (1975) 281–305. Introduction in 4-10
Reflections of lattice gauge theory (lattice-point planes, link planes) and the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract only). Introduction in 5-03

Item	Kind	Source / tools
The definition of the mass gap	Cited	The same definition as 4-11, written on the vessel in units of one lattice step
Tychonoff's theorem, weak convergence of probability measures	Cited	Standard facts
Definitions of the vessel, the mirrors and the infinite-volume space, and the properties of § 05 and § 06	Lean	This book's Lean bundle (`OSGap.Datum` , `measureDatum` , `exampleDatum` , `HasMassGap` , `hasMassGap_of_decay` , `decay_of_hasMassGap` , `LinkReflection.linkRefl` , `TorusRP.thL` , `sig` , `fixP` , `posP` , `WilsonLimit.Cfg` , `liftProb` , `Wsub` , `MidRP.thM` , `posM` , `HalfSplit.Th`). The formulas of the definitions were copied by reading the source

Words used in this chapter — reflection, lattice-point plane, middle plane, positive side, reflection pairing, reflection positivity, time translation, transfer operator, vessel of reconstruction, vacuum, mass gap, non-degeneracy, periodic lift, weak convergence. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-04 The block map and one step of renormalisation — on the same lattice, defines the map to a coarser lattice and the push-forward measure.

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The block map and one step of renormalisation — families of paths, averaging, push-forward

> This chapter is this book's own investigation. § 01 is established knowledge.

What this chapter shows — We define the map B from the fine lattice $(\mathbb{Z}/2^{m+1}L)^4$ to the coarse lattice $(\mathbb{Z}/2^{m}L)^4$ through a family of paths, weights, averaging and an operation returning to the group, and set up its linear part C, the push-forward $B_*\mu$, and the multi-step family of the free field $\mathcal{M}$, G_k. The theorems of the second half of Part 15 (15-12, 15-13) talk about these.

Prerequisite chapters — 19-01 Link variables, SU(2) as unit quaternions, 7-05 § 02 The one-step transformation.

Said first — This is a chapter of definitions, and it states no theorems (only the minimum of facts is added, with their Lean names). B is one of many possible constructions, and the weights and the threshold r_0 are design choices (§ 08). The limit of lattice spacing to 0 is not dealt with. This block map is this book's design. The one step of renormalisation in the published construction is of a different design, and it is that construction which showed existence (the comparison of the two designs is in 14-02).

The order of this chapter

1. The idea of one step of renormalisation
2. Two lattices and coarse links
3. The family of paths — weights $(1/8, 3/4, 1/8)$ and six orders
4. The non-linear block map B — returning the average to the ball by a retract
5. The path kernel C — the linear part
6. The push-forward of measures
7. The multi-step family on the free-field side
8. The weights are a design choice

01

The idea of one step of renormalisation

The introduction is in 7-05 § 02 and § 05. One step of renormalisation is the operation of making coarse variables from fine variables by one rule, and carrying the measure to the coarse lattice by that rule Cited . For gauge fields, a coarse link variable is made from the "product along a path" of fine links (under a gauge transformation only the two ends move). However, group elements cannot be added and divided, and the average of path products in general leaves $SU(2)$, so a procedure that returns the average to the group is needed Cited .

The B of this chapter fixes one family of paths, averages the products along each path with weights, and shrinks the value in the radial direction to return it to the unit ball. It is one concrete form of the abstract one step T (15-03).

Below, we write $\mathbb{H}$ for the quaternions and $\mathbb{B} = \{q \in \mathbb{H} : \ |q| \le 1\}$ for their closed unit ball. $SU(2)$ is the sphere $|q| = 1$ (19-01). The conjugate q^* is the inverse on $SU(2)$.

02

Two lattices and coarse links

Take a positive integer L and a step number $m \ge 0$, and set

$$\Lambda_m = (\mathbb{Z}/2^m L)^4, \qquad \Lambda_{m+1} = (\mathbb{Z}/2^{m+1} L)^4 \quad (1)$$

Λ_{m+1} is the fine lattice and Λ_m the coarse lattice. Note that the larger number is the finer one. A coarse link is a pair $b = (y, \mu)$ ($y \in \Lambda_m$, direction $\mu \in \{1, 2, 3, 4\}$), going from y to $y + e_\mu$. In this chapter directions are written 1 to 4, corresponding in order to the Lean indices 0 to 3 (the two numbering conventions are in 19-01 § 01). Since the four directions are treated alike, which direction is called time is not specified. Fine links have the same form (x, ν), $x \in \Lambda_{m+1}$.

A coarse point y sits at the fine point $2y$. The map $y \mapsto 2y$ is well defined as $\mathbb{Z}/2^m L \to \mathbb{Z}/2^{m+1} L$ (since $2^m L \equiv 0$ goes to $2 \cdot 2^m L \equiv 0$) Proved here . On the fine lattice, a coarse link $b = (y, \mu)$ corresponds to the interval $2y \to 2y + 2e_\mu$ of length 2, whose end point $2y + 2e_\mu$ is the image $2(y + e_\mu)$ of the next coarse point $y + e_\mu$ Proved here .

A configuration is a function $U : \{\text{links}\} \to \mathbb{B}$ taking a value in the ball on each link. The Wilson measure lives on configurations with values in $SU(2)$, but since the result of averaging can fall inside the ball, the domain and codomain of the map are taken to be spaces of ball-valued configurations.

Lean: coarse links `BoxWalk.BoxLk L m`, the ball `ChainBlockMap.Ball` (other names in 19-06). A configuration takes values only on links of positive orientation, and the convention is that the value of a reversed link is read as the conjugate U^*.

03

The family of paths

For each coarse link $b = (y, \mu)$, we fix one finite family of fine paths from $2y$ to $2y + 2e_\mu$. Let n_1, n_2, n_3 be the three directions other than μ, in increasing order of number; these are called the **transverse directions**.

- **Offset** $A = (A_1, A_2, A_3)$: for each transverse direction n_t, choose one offset $d_t \in \{-1, 0, +1\}$ (27 ways).
- **Order** o: choose the order in which to move in the three transverse directions, as a permutation of three letters (6 ways).

The path $\gamma_{o,A}(b)$ joins the following three segments.

1. From $2y$, following the order o, step by d_t in the transverse direction n_t, one step at a time (no movement in directions with $d_t = 0$). Call the point reached $z = 2y + \sum_t d_t e_{n_t}$.
2. From z, take two steps in direction μ, arriving at $z + 2e_\mu$.
3. Undo the transverse offsets one step at a time in the reverse of the outgoing order, arriving at $2y + 2e_\mu$.

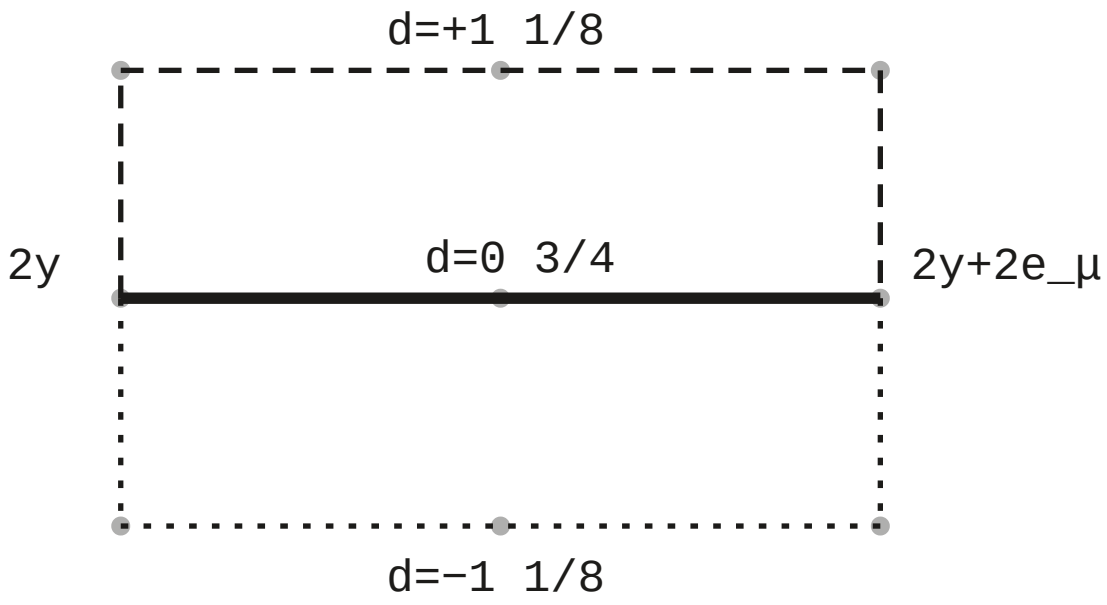
If all transverse offsets are 0, the path is the **straight path** that only takes two steps in μ. The length is $2 + 2\#\{t : d_t \neq 0\}$, at most 8. A step of a path is a pair of a link and its orientation (forward or backward), and a backward step traverses the link from its end point to its start point.

The weight of a path is the product of the per-direction weights $\mathrm{wt}(-1) = \frac{1}{8}$, $\mathrm{wt}(0) = \frac{3}{4}$, $\mathrm{wt}(+1) = \frac{1}{8}$, divided equally among the six orders.

$$w(o, A) = \frac{1}{6} \prod_{t=1}^{3} \mathrm{wt}(d_t), \qquad \sum_o \sum_A w(o, A) = 1. \quad (2)$$

Proved here The weights are non-negative and sum to 1 (the cube of $\sum_d \mathrm{wt}(d) = 1$, divided equally among the six orders). Lean: `ChainBlockMap.sixWeight_total`.

Figure 1 One coarse link ($2y \to 2y + 2e_\mu$) and three paths, drawn for one transverse direction only. There are three transverse directions; the offset is chosen independently in each and the weights are multiplied



In four dimensions this choice is made independently in three transverse directions, so the weight of the straight path is $(3/4)^3 = 27/64$ Proved here .

Detour: the same path counted twice

May be skipped. There are $6 \times 27 = 162$ indices (o, A), but changing the order of directions with offset 0 does not change the path, so the same path arises from several indices. In Lean this is handled by converting to a finite set of paths with merged weights (the sum of the weights of the indices giving the same path) (`BoxWalk.boxPaths`, `BoxWalk.boxW`). Merging does not change the value of the average. Counting on $\mathbb{Z}^4$, there are $\binom{3}{k} 2^k k!$ paths with k non-zero offsets, $1 + 6 + 24 + 48 = 79$ in all (1, 6, 24 and 48 paths of lengths 2, 4, 6 and 8 respectively) Proved here . On small tori, paths that are distinct on $\mathbb{Z}^4$ can fall onto the same path.

Averaging over the six orders makes the family symmetric under permutations of the three transverse directions. Since the offset weights are symmetric under $d \mapsto -d$, the family is also symmetric under reversal of transverse directions Proved here . The value of a reversed coarse link (from $y + e_\mu$ to y) is read as the conjugate, following the convention of § 02. Forming the average with the family of the same transverse steps, two steps towards $-\mu$ and the same return, and retracting, also gives a value equal to that conjugate, so the convention and the family of paths are compatible (Lean: `BoxWalk.boxMap_rev`).

Lean: paths `WalkCast.walk` (on $\mathbb{Z}^4$), weights `ChainBlockMap.sixWeight`.

04

The non-linear block map B

The product along a path. For a path $\gamma = (s_1, \dots, s_\ell)$ and a configuration U, assign U_s to a forward step s and U_s^* to a backward step, and multiply in the order of the path.

$$U(\gamma) = \prod_{i=1}^{\ell} U_{s_i}^{\pm} \quad \text{(from left to right)}, \qquad U_s^{+} = U_s, \;\; U_s^{-} = U_s^*. \tag{3}$$

The weighted average. The raw value of a coarse link b is the weighted average of path products.

$$q_b(U) = \sum_{o} \sum_{A} w(o, A)\, U\big(\gamma_{o,A}(b)\big) \;\in \mathbb{H}. \tag{4}$$

Proved here If all $|U_\ell| \le 1$, then $|q_b(U)| \le 1$ (the quaternion norm is multiplicative, and the weights are non-negative with sum 1). Lean: `ChainBlockMap.norm_sixAvg_le`.

The operation returning to the ball (retract). Fix a threshold $r_0 > 0$, and set

$$\rho_{r_0}(q) = \frac{q}{\max(|q|,\, r_0)} \tag{5}$$

If $|q| \ge r_0$, then $\rho_{r_0}(q) = q/|q|$ is a unit quaternion (an element of $SU(2)$); if $|q| < r_0$, then $\rho_{r_0}(q) = q/r_0$, with norm $|q|/r_0 < 1$. In either case $|\rho_{r_0}(q)| \le 1$, and ρ_{r_0} is continuous Proved here . For $r_0 = 1$, ρ_1 is the identity on the ball. The Lean definition makes sense for every $r_0 > 0$, but this book takes $0 < r_0 \le 1$ (for $r_0 > 1$, $|q_b| \le 1 < r_0$, so the value always falls inside the ball and never lands in $SU(2)$ Proved here).

Definition (block map). The map $B = B_m : \mathbb{B}^{\{\text{links of } \Lambda_{m+1}\}} \to \mathbb{B}^{\{\text{links of } \Lambda_m\}}$ that assigns to a ball-valued configuration U on the fine lattice the configuration on the coarse lattice

$$B(U)_b = \rho_{r_0}\big(q_b(U)\big) \qquad (b \text{ a coarse link of } \Lambda_m) \tag{6}$$

is called the **non-linear block map** of step m.

Proved here B is continuous. On the "small field" event $|q_b(U)| \ge r_0$, where the average $q_b(U)$ is at or above the threshold, $B(U)_b = q_b(U)/|q_b(U)| \in SU(2)$; below the threshold the value falls inside the ball. Lean: `ChainBlockMap.continuous_blockMap`, `ChainBlockMap.blockMap_of_small_field`.

The Lean definition is `ChainBlockMap.blockMap r₀ hr (BoxWalk.boxPaths L) (BoxWalk.boxW L) m` (`hr : 0 < r₀`). The composed map $B_m \circ B_{m+1} \circ \cdots \circ B_{n-1}$ (from step n to step m) is `ChainBlockMap.down`.

The event below the threshold is a large-field event and is dealt with separately (7-05 § 04, Parts 22 and 28).

05

The path kernel C — the linear part

We look at the form of B when U is near the identity 1. Take a field a of imaginary quaternions on the fine links ($\mathrm{Re}\, a_\ell = 0$; elements of the Lie algebra $\mathfrak{su}(2)$) and a real number ε, and set $U_\ell = \exp(\varepsilon a_\ell) \in SU(2)$ (Lean: `BlockLinearKernel.expCfg`). Expanding the path product in ε, the first-order term is the signed sum of the field along the path. Its weighted average is written

$$(Ca)_b = \sum_o \sum_A w(o,A) \sum_{s \in \gamma_{o,A}(b)} \pm\, a_s \qquad (\text{forward steps } +,\ \text{backward } -) \tag{7}$$

and C is called the **path kernel**. Its Lean name is `BlockLinear.sixSgn`. That for $0 < r_0 \le 1$, $\varepsilon \ge 0$, $|a_\ell| \le M$, $\varepsilon M \le 1/32$, the difference between $B(e^{\varepsilon a})_b$ and $\exp(\varepsilon (Ca)_b)$ is at most $257(\varepsilon M)^2$ is a theorem of Part 15 (15-13, Lean: `BlockLinearKernel.blockMap_exp_linear_le_one`). The retract does not change the linear part.

The form of the coefficients. C is a real linear map acting colour by colour (on the i, j, k components), and can be written with a table of coefficients. On the fine lattice $\Lambda_{m+1} = (\mathbb{Z}/2^{m+1}L)^4$,

$$(Ca)_\mu(X) = \sum_{r \in \Lambda_{m+1}} \sum_{\nu=1}^{4} c(r)_{\mu\nu}\, a_\nu(2X + r), \tag{8}$$

$$c(r)_{\mu\nu} = \sum_o \sum_A w(o,A) \sum_{\substack{s \text{ a step of } \gamma_{o,A} \\ \text{direction } \nu,\ \text{base point } \equiv r}} \pm 1. \tag{9}$$

Here $\gamma_{o,A}$ is the path on $\mathbb{Z}^4$ leaving the origin in direction μ. Lean: the coefficients `LatticeTorus.pathKernel` (in the form with side $2^{m+1}L$), the operator of the form (8) `TorusFourier.Cop` (built from the map $X \mapsto 2X$ and the table of coefficients). The real linear map for a single colour is `PushCov.blockLin`, and that it agrees with `Cop` is the first half of `PushCov.lattice_pushCov`.

Example (constant field). When the field a takes the same value a_ν at every point, a transverse step and its return traverse two links of the same direction with opposite signs and cancel, leaving only the two steps in μ. Hence $(Ca)_\mu = 2a_\mu$. A coarse link is as long as two fine links, so this is the expected form, "proportional to length". Proved here In terms of coefficients, $\sum_r c(r)_{\mu\nu} = 2\delta_{\mu\nu}$. This form of the sum is also the symbol of the path kernel taken at momentum 0 ($k = 0$ in `LatticePath.pathSym_closed`). The contribution of the straight path is $27/64$ each to $c(0)_{\mu\mu}$ and $c(e_\mu)_{\mu\mu}$.

C is only the first order about the identity (for the view of commuting configurations, see 15-05); linearisation about a background field $U \neq 1$ is not defined.

06

The push-forward of measures

For a probability measure μ on configurations of the fine lattice, the **push-forward** $B_*\mu$ is defined by

$$(B_*\mu)(E) = \mu\big(B^{-1}(E)\big) \qquad (E \text{ a measurable set of coarse configurations}) \quad (10)$$

B is continuous and hence measurable, and $B_*\mu$ is a probability measure on coarse configurations Proved here . In other words, it is the distribution of the coarse field $B(U)$ when U follows μ. In Lean it is `Measure.map` (in the form for probability measures, `ProbabilityMeasure.map`).

Multi-step push-forward. We write $\nu_{n,m}$ for the measure μ_n on the fine lattice of step n pushed by the composed map down to step $m \le n$. Pushing one step at a time or all at once gives the same measure: for $m < n$, $(B_m)_*\nu_{n,m+1} = \nu_{n,m}$, and $\nu_{n,n} = \mu_n$ Proved here . Lean: `ChainBlockMap.νP`; the agreement is `ChainBlockMap.hcons_blockMap` and `ChainBlockMap.νP_self`.

The μ_n being pushed is a probability measure on ball-valued configurations of the box Λ_n (the Lean type has the form of a box). The Wilson measure (19-01) and the measure with fermions (19-05) have different lattice types, so what is put in is the measure normalised and re-indexed onto configurations of the box lattice (on the fermion side, `BoxFermion.fermProbBox` in Lean).

The statements to be shown, written with this push-forward measure only, are in 19-07.

Covariance. For linear fields, the covariance of the push-forward is handled in the following form. For a measure μ on a real inner product space E, the **covariance form**

is

$$\operatorname{cov}_\mu(u,v) = \int \langle u,x\rangle\langle v,x\rangle \, d\mu(x) - \int \langle u,x\rangle \, d\mu(x) \int \langle v,x\rangle \, d\mu(x) \quad (11)$$

(Lean: `PushCov.covForm`; it is meaningful when second moments exist). The T with $\operatorname{cov}_\mu(u,v) = \langle u, Tv\rangle$ is called the covariance operator. That pushing by a linear C gives $CTC^\dagger$ is a theorem of 15-13; for the non-linear B it does not take this form.

07

The multi-step family on the free-field side

The "one step of the free field" in Part 15 (15-12) talks about operators obtained by repeating the one-step map $T \mapsto CTC^\dagger$ on the covariance operator of the free field. That this is the covariance of the Gaussian free field pushed by C is in Lean in the one-step, one-colour form (`PushCov.lattice_pushCov`); the multi-step form is a one-line argument repeating the push-forward by a linear map, and is on paper. We set up the objects in two forms, on the lattice side and on the momentum side.

Lattice side. Let the side of the coarse lattice be N, and consider a four-component field $\phi_\nu(X)$ on the torus $(\mathbb{Z}/N)^4$ (the N of this section is the side of the coarse lattice).

- **Free-field covariance** (Feynman gauge) $\mathrm{freeCov}_N$: the inverse of the lattice Laplacian $-\Delta$, taken component by component (0 on the constant component). Lean: `LatticePush.freeCov`.
- **One step**: $T \mapsto CTC^\dagger$ (from operators on $(\mathbb{Z}/2L)^4$ to operators on $(\mathbb{Z}/L)^4$). Lean: `LatticePush.stepOp`.
- **Covariance at step k**: the free-field covariance on $(\mathbb{Z}/2^k N)^4$ pushed down to $(\mathbb{Z}/N)^4$ by k one-step maps, $\mathrm{stageCov}_{k,N}$. Lean: `LatticePush.stageCov`.
- **Plaquette covariance**: the covariance $\mathrm{curl} \circ \mathrm{stageCov}_{k,N} \circ \mathrm{curl}^\dagger$ of the plaquette field $(\mathrm{curl}\,\phi)_{(a,b)}(X) = \big(\phi_a(X+e_b) - \phi_a(X)\big) - \big(\phi_b(X+e_a) - \phi_b(X)\big)$. Lean: `LatticeFace.faceCov`.

Momentum side. Let the lattice momentum be $q = (q_1, \dots, q_4)$, $q_\nu = 2\pi n_\nu / N$.

$$\omega(q) = \sum_\nu 4\sin^2\frac{q_\nu}{2}, \qquad s_\nu(q) = \sin\frac{q_\nu}{2}, \qquad P(q) = 1 - \frac{s(q)s(q)^{\mathsf{T}}}{|s(q)|^2}. \quad (12)$$

$\omega(q)$ is an eigenvalue of $-\Delta$, and $P(q)$ is the projection onto the directions orthogonal to $s(q)$. The symbol of the difference is $1 - e^{iq_\nu} = -2ie^{iq_\nu/2}\sin(q_\nu/2)$, so P is the

projection onto the "physical part" with the gradient (gauge) directions removed Proved here .

The momenta of the fine torus (side $2N$) that fall onto the coarse momentum q under $y \mapsto 2y$ are $r_\zeta(q) = q/2 + \pi\zeta$ ($\zeta \in \{0,1\}^4$, 16 of them); these are called **aliases**. The **one-step map** $\mathcal{M}$ assigns to a family G of 4×4 matrices, one for each momentum,

$$(\mathcal{M}G)(q) = \frac{1}{16} \sum_{\zeta\in\{0,1\}^4} A_\zeta(q)\, G\big(r_\zeta(q)\big)\, A_\zeta(q)^{\mathsf{T}} \quad (13)$$

$A_\zeta(q)$ is a 4×4 real matrix built explicitly from q and ζ. The family at step k is

$$G_0 = P/\omega, \qquad G_k = \mathcal{M}^k(P/\omega). \quad (14)$$

The domain is the momenta with $\omega(q) > 0$ ($\omega = 0$ is the constant component).

Lean holds the momentum as $\theta = q/4$ ($\sin 2\theta_\nu = \sin(q_\nu/2)$). The core names are `OneStepBridge.Mfam` for $\mathcal{M}$ and `TauSplit.Gs` for G_k; other names are in 19-06.

The relation between the two sides: that $\mathrm{stageCov}_{k,N}$ on the lattice side has a symbol of phase form for each momentum (`LatticePush.stageCov_phase`), that the physical part PXP of that symbol equals G_k (`LatticePush.projFam_stage`), and that the plaquette covariance equals the operator built from the momentum side (`LatticeFace.faceCov_eq_posC`) have been checked in Lean, and are used in Part 15 (15-12). What is pushed here is the linear C, not the non-linear B.

08

The weights are a design choice

The B of this chapter was defined with the following choices fixed.

Choice	Value in this chapter	Other possibilities (examples)
Weights per transverse direction $(\alpha, 1-2\alpha, \alpha)$	$\alpha = 1/8$	Other $\alpha \in [0, 1/2]$. $\alpha = 0$ gives only the straight path
Order of transverse moves	Equal average over six orders	Fixed to one order (the symmetry becomes weaker)
Operation returning to the ball	Retract ρ_{r_0}, $0 < r_0 \le 1$	Other thresholds, other ways of projecting

As long as one chooses a family of paths with non-negative weights summing to 1 and a continuous way of returning that maps the ball to the ball, the minimal facts of § 03 to § 06 (values stay in the ball, continuity, the push-forward is a probability measure) stand in the same form Proved here . The threshold-free $q \mapsto q/|q|$ is undefined at $q = 0$

and not continuous, so it does not satisfy this condition. The choice of the width of transverse offsets and of the blocking factor is dealt with in 28-05.

What depends on the choices is the coefficients (9) of the path kernel C, and A_ζ, $\mathcal{M}$, G_k built from them. The concrete constants appearing in the theorems of the second half of Part 15 (the one-step contraction rate, upper bounds on ratios of plaquette covariances, and so on) were computed with the choices of this chapter ($\alpha = 1/8$, factor 2), and cannot be used as they are with other choices. What changes when the weights are changed, and which choice makes the premises easier to satisfy, are left to 28-05 What depends on design choices.

What this chapter can say

Can say	Cannot say
Proved here The weights are non-negative and sum to 1. Lean: `ChainBlockMap.sixWeight_total`	That these weights are better than others (28-05)
Proved here B is a continuous map between ball-valued configurations, and takes values in $SU(2)$ on the event at or above the threshold. Lean: `ChainBlockMap.continuous_blockMap`, `ChainBlockMap.blockMap_of_small_field`	That B maps $SU(2)$ configurations to $SU(2)$ configurations. An estimate of the probability of the event below the threshold
Proved here The convention of reading reversed links as conjugates is compatible with the family of paths. Lean: `BoxWalk.boxMap_rev`	All forms of compatibility with local gauge transformations
Proved here The multi-step push-forward agrees with step-by-step push-forwards ($m < n$). Lean: `ChainBlockMap.hcons_blockMap`	Decay and uniformity of the push-forward measure (19-07, Part 28)
Proved here For a constant field, $(Ca)_\mu = 2a_\mu$	Linearisation about a background field $U \neq 1$

Sources and reproduction

Item	Kind	Source / tools
The one-step transformation; that averages leave the group for gauge fields	Cited	The sources of 7-05 § 05 (T. Bałaban, Comm. Math. Phys. 98 (1985) 17–51, bibliographic data only; J. Dimock, arXiv:1108.1335)
The objects of § 02 to § 07	Lean	Lean definitions from this book's investigation (names in the text and in 19-06). Theorems in 15-12 and 15-13
The number of paths 79, 27/64, the sum of coefficients	Proved here	Counted by hand (formulas in the detour of the text). A script enumerating the paths on $\mathbb{Z}^4$ from the definition gives the same numbers

Words used in this chapter — block map, family of paths, transverse direction, retract, threshold, small-field event, path kernel, push-forward, covariance form, alias, one-step map, covariance at a step, plaquette covariance. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-05 The fermion weight — defines one of the measures to be pushed: the measure whose weight is the determinant of the Wilson–Dirac operator.

2026-11-13 · chapter Yang–Mills and the mass gap Part 19 Objects, and the statements to be shown Objects · Kiichi, Shiori and Rin

The fermion weight — the Wilson–Dirac operator and the measure with a determinant

> This chapter is this book's own investigation. § 01 is established knowledge.

What this chapter shows — On the lattice of the time circle $\mathbb{Z}/2h$ and the spatial torus $(\mathbb{Z}/N)^3$, we define the Wilson–Dirac operator $D_m(U)$ and set up the measure $\mu^{\mathrm{wf}}_{\beta,m,j}$ obtained by multiplying the gauge weight $e^{-\beta S_W}$ by $\mathrm{Re}\det D_m(U)^{2j}$. The theorems on the vessel with fermions in Part 20 (20-05) talk about this measure.

Prerequisite chapters — 19-01 Plaquettes and holonomy, the Wilson measure, the Lean conventions, 5-01, the Wilson action.

Said first — This is a chapter of definitions. Fermions appear only in the form of the determinant left after integrating them as Grassmann variables; the Grassmann integral itself is not defined. The mass m and the coupling β are bare parameters on the lattice, and their meaning in the continuum limit (mass renormalisation, chiral symmetry) is not dealt with.

The order of this chapter

1. Why a determinant weight — integrating out fermions leaves a determinant for each gauge configuration
2. The lattice and physical links — a time circle $2h$ and space N^3; the shape of this chapter's lattice
3. The convention for the gauge action — $S_W = \sum_p (1 - \frac{1}{2}\,\mathrm{Re}\,\mathrm{tr}\,U_p)$, and its relation to the weight of 19-01
4. The Wilson–Dirac operator — $r = 1$, antiperiodic in all four directions, the representation of γ
5. The determinant weight — $\mathrm{Re}\det D^{2j}$ is non-negative
6. The measure with fermions — the unnormalised measure, and the probability measure μ^{wf}
7. The range of the mass — the definitions for all m, the reflection-positivity theorem for $m > -1$

8. The relation to the conventions for gauge-field measures — five differences, and the return to the Wilson measure at $j = 0$

01

Why a determinant weight

A quark field on the lattice is represented by Grassmann variables $\psi, \bar{\psi}$ carrying at each point spin (4 components) and colour (2 components for $SU(2)$) Cited . If the action is a quadratic form $\bar{\psi} D(U) \psi$, the integral over the Grassmann variables gives, for each configuration U, the determinant $\det D(U)$. With n_f species (flavours) it is $\det D(U)^{n_f}$ Cited .

The measure after integrating out the fermions in this way is a measure on gauge configurations, with weight $e^{-\beta S_W(U)} \det D(U)^{n_f}$. The determinant can be complex, so making it a probability measure requires the weight to be a non-negative real number. In this chapter the number of flavours is taken even, $n_f = 2j$, and the real part of the weight, $\mathrm{Re} \det D^{2j}$, is used (§ 05 shows that it is non-negative).

The naive difference form of the Dirac operator on the lattice produces extra particles (doublers). Adding a second-order difference term, named after Wilson, makes them heavy Cited . That operator is D. Reflection positivity for lattice gauge theory with fermions was treated by Osterwalder–Seiler Cited (5-03, reflection positivity of the Wilson action).

02

The lattice and physical links

The lattice of this chapter, with time circle of length $2h$ ($h \geq 1$) and spatial side $N \geq 1$, is

$$\Lambda_{N,h}^{\mathrm{wf}} = (\mathbb{Z}/2h) \times (\mathbb{Z}/N)^3 \quad (1)$$

Points are written (t, x), $t \in \mathbb{Z}/2h$, $x \in (\mathbb{Z}/N)^3$. Links are (t, x, μ): for $\mu = 1, 2, 3$, the space link $x \to x + e_\mu$ within the slice at time t; for $\mu = 4$, the time link $(t, x) \to (t + 1, x)$ ($t + 1$ modulo $2h$). These are called **physical links**, and their set is written $\mathcal{L}_{N,h}$.

A configuration is $U : \mathcal{L}_{N,h} \to SU(2)$, with $SU(2)$ the unit quaternions (19-01). The reference measure on the space of configurations is the product $\prod_\ell dU_\ell$ of the Haar probability measures of the links (19-01).

The time direction is given even length $2h$ so that the planes of the reflection $t \mapsto -t$ are the two planes $t = 0$ and $t = h$ (19-03). Taking $2h = N$ gives a torus of the same size in all four directions.

Lean: the type of physical links `FermionSU2Config.PhysLink N hT` (`Fin (2 * hT) × (Fin 3 → Fin N) × Fin 4`; Lean's direction numbers 0, 1, 2 are space and 3 is time), $SU(2)$ is `FermionSU2Config.SU2` (the unit sphere of the quaternions), the reference measure is `Measure.pi fun _ => Measure.haarMeasure ⊤`, and the next time $t \mapsto t + 1 \bmod 2h$ is `FermionSliceBlock.sc`.

03

The convention for the gauge action

The Wilson action of this chapter, counting each unoriented plaquette once, is

$$S_W(U) = \sum_p \Big(1 - \tfrac{1}{2}\operatorname{Re}\operatorname{tr} U_p\Big), \qquad U_p = U_\mu(z)\,U_\nu(z+e_\mu)\,U_\mu(z+e_\nu)^\dagger\,U_\nu(z)^\dagger \tag{2}$$

and the gauge weight is $e^{-\beta S_W(U)}$ ($\beta \in \mathbb{R}$). The plaquettes are the space plaquettes (μ, ν) ($\mu < \nu \le 3$) within each time slice, and the time plaquettes $(\mu, 4)$ joining the slices at times t and $t+1$. The trace is taken by representing quaternions as 2×2 matrices, $q = a + bi + cj + dk \mapsto \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$.

Proved here This representation preserves products, and $\frac{1}{2}\operatorname{tr}$ is the real part of the quaternion. Hence $\frac{1}{2}\operatorname{Re}\operatorname{tr} U_p = \operatorname{Re} U_p$, and with $|\mathcal{P}|$ the number of plaquettes,

$$e^{-\beta S_W(U)} = e^{-\beta|\mathcal{P}|}\, e^{\beta \sum_p \operatorname{Re} U_p}. \tag{3}$$

The constant $e^{-\beta|\mathcal{P}|}$ disappears on normalisation, so the normalised measure is the same as under the convention $e^{\beta\sum_p \operatorname{Re} U_p}$ of the Wilson measure of 19-01.

Lean: the action `WilsonGauge.wilsonAction N hT`, the per-plaquette quantity `WilsonGauge.plaqE` ($1 - \frac{1}{2}\operatorname{Re}\operatorname{tr}(ABC^\dagger D^\dagger)$), the quaternion matrix `FermionSU2Config.quatMat`, and links as matrices on the configuration space `WilsonGauge.physSp` (space) and `WilsonGauge.physT` (time).

04

The Wilson–Dirac operator

The fermion field $\psi(t, x)$ is a complex vector carrying at each point 4 spin components and 2 colour components. The operator $D_m(U)$ is a square matrix on $\mathbb{C}^{2h\cdot N^3\cdot 4\cdot 2}$, and

we take Wilson's form with $r = 1$,

$$(D_m(U)\psi)(z) = (m+4)\,\psi(z) - \frac{1}{2}\sum_{\mu=1}^{4}\Big[(1-\gamma_\mu)\,U_\mu(z)\,\psi(z+e_\mu) + (1+\gamma_\mu)\,U_\mu(z-e_\mu)^\dagger\,\psi($$

$U_\mu(z)$ acts on the colour index as a 2×2 matrix, and γ_μ acts on the spin index. $m \in \mathbb{R}$ is the bare mass.

The γ matrices. The spin index is written as a pair $(s,r) \in \{0,1\}^2$. s is ordered by the eigenvalues of γ_4, $+1$ ($s=1$) then -1 ($s=0$), and with the Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma 2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma 3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix} \quad (i = 1,2,3) \tag{5}$$

(blocks in the order $s = 1, 0$, each block 2×2 in r). This is the arrangement exactly as in the Lean definition. They satisfy $\gamma_\mu^2 = 1$, $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ ($\mu \neq \nu$) Proved here .

Boundary conditions. Antiperiodic in all four directions. In space direction i, on the links leaving $x_i = N-1$, and in the time direction, on the link from $t = 2h-1$ to 0, the hopping term is multiplied by the sign -1. The gauge link variables themselves are periodic; the sign is attached to the hopping of the operator.

Lean: the operator `WilsonGauge.physDirac N hT m ω`. Its content is the per-time-slice form `FermionSliceBlock.std4` (the operator (4) with $r = 1$; within a slice, `FermionSliceBlock.stdSlice`), with spatial hopping `FermionSliceBlock.torusX` (boundary signs in `FermionSliceBlock.torusLink`) and temporal hopping (boundary sign `FermionSliceBlock.eps`) inserted. γ is `FermionSliceBlock.gammaT` (γ_4) and `FermionSliceBlock.gammaSp` (γ_i).

The component formula is in Lean: `WilsonGauge.physDirac_apply` says that each component of physDirac has the form (4).

Reading On random configurations, with $m \in \{0.3, -0.7, -4.0\}$ for $2^3 \times 2$ and $m \in \{0.3, -0.7\}$ for $2^3 \times 4$, a literal transcription of `physDirac` was compared with an operator written independently from (4) in another representation of γ (mapped by the change-of-representation matrix), and the component differences were at most 3.6×10^{-15} (numerical record of this book's investigation. Removing the boundary sign in time makes the difference 2.0, which confirms that the comparison is effective).

```
05
```

The determinant weight

With $2j$ flavours ($j = 0, 1, 2, \ldots$), the fermion weight is

$$F_{m,j}(U) = \mathrm{Re}\big(\det D_m(U)\big)^{2j} \quad (6)$$

For $j = 0$, $F \equiv 1$ Proved here .

Proved here For all $m \in \mathbb{R}$, all j and all configurations, the imaginary part of $(\det D_m(U))^{2j}$ is 0 and its real part is at least 0. Lean: `FermionSliceReflection.wilson_weight_nonneg_std4` (the torus form `FermionSliceReflection.wilson_weight_nonneg_torus`); the form on configurations is `WilsonGauge.fermWeight_nonneg`. Hence (6) is a non-negative real number even before taking the real part, and Re is a matter of notation.

$F_{m,j}$ is a continuous function of the configuration (since the matrix entries are polynomials in the components of U) Proved here . Lean's `WilsonGauge.continuous_fermWeight` is continuity on the split coordinates (next paragraph); continuity on configurations is read together with the continuity of the splitting map. Invariance of the weight under global gauge transformations and under lattice symmetries are theorems of Part 20 (20-05).

Lean's `WilsonGauge.fermWeight` writes (6) on coordinates that split a configuration at the reflection plane into "plane, positive side, negative side", and is the same as the form written with the operator on unsplit configurations (`WilsonGauge.physDirac_eq_split`). The splitting is matched to the definition of the positive side in 19-03.

06

The measure with fermions

With the product of the gauge weight and the fermion weight

$$W_{\beta,m,j}(U) = e^{-\beta S_W(U)}\, F_{m,j}(U) \;\geq 0 \quad (7)$$

as density, we set the measure on the space of configurations to be

$$\mu^{\mathrm{wf},0}_{\beta,m,j}(dU) = W_{\beta,m,j}(U) \prod_{\ell \in \mathcal{L}_{N,h}} dU_\ell, \qquad Z^{\mathrm{wf}}_{\beta,m,j} = \int W_{\beta,m,j}(U) \prod_{\ell} dU_\ell \quad (8)$$

(unnormalised). The partition function Z^{wf} is positive for all β, m, j. This is a theorem of Part 20 (20-05, Lean: `WilsonGauge.partition_pos_wilson_full`; the form with density W is `WilsonGauge.wf_partition_pos`). So

$$\mu^{\mathrm{wf}}_{\beta,m,j} = \left(Z^{\mathrm{wf}}_{\beta,m,j}\right)^{-1} \mu^{\mathrm{wf},0}_{\beta,m,j} \quad (9)$$

is called the **Wilson–fermion measure**. It is a probability measure Proved here .

Lean: the density `WilsonGauge.wfWeight N hT j m β`, the unnormalised measure `WilsonGauge.wfMeasure` (the density enters through the truncation `toNNReal` to non-negative reals; since (7) is non-negative the value does not change: `WilsonGauge.wfWeight_nonneg`), the normalised measure `WilsonGauge.wfProb N hT j m β`, and that it is a probability measure is `WilsonGauge.wfProb_isProbability` ($h \geq 1$).

This measure is also one of the measures to be pushed by the block map (19-04 § 06).

07 The range of the mass

The definitions (4) to (9) make sense for every real m. Non-negativity of the weight and $Z^{\mathrm{wf}} > 0$ also hold for every m. The range narrows on the side of the theorems.

Object or property	Range of the mass m	Where
The operator D_m, the weight $F_{m,j} \geq 0$, the measure μ^{wf}	All $m \in \mathbb{R}$	This chapter
$Z^{\mathrm{wf}} > 0$, invariance under lattice symmetries	All m	20-05
Reflection positivity through lattice-point time planes	$m > -1$	20-05

$m > -1$ includes $m = 0$ and small negative masses. Reflection positivity for $m \leq -1$ is outside the Lean statements of this book. m is a bare parameter, and its relation to the physical mass (tuning to the critical mass) is not defined in this chapter.

08 The relation to the conventions for gauge-field measures

We summarise the differences between the measure of the other chapters of Part 19 (19-01) and the measure of this chapter.

Item	The Wilson measure of 19-01	The measure of this chapter
Lattice	$(\mathbb{Z}/N)^4$, or the torus with transverse size N and longitudinal size M	$(\mathbb{Z}/2h) \times (\mathbb{Z}/N)^3$
Gauge weight	$e^{\beta \sum_p \mathrm{Re}\, U_p}$	$e^{-\beta S_W}$. Differs only by a constant multiple (equation (3))
Haar measure	On the strong-coupling side, the product of the Haar measure of the sphere pushed forward to the closed ball (19-01, the Lean conventions)	The product of Haar probability measures on the unit sphere of $SU(2)$

Item	The Wilson measure of 19-01	The measure of this chapter
Normalisation	Unnormalised $\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure`) and normalised $\mu_{N,\beta}$ (`TorusRP.wilsonProb`). The vortex side is written with ratios of partition functions	Both the unnormalised (8) and the normalised (9) are set up
Boundary	Gauge links periodic (twists in 19-02)	Gauge links periodic, fermions antiperiodic in all four directions

Proved here For $j = 0$, $F \equiv 1$, so $\mu^{\mathrm{wf}}_{\beta,m,0}$ is the normalised Wilson measure on the lattice $\Lambda^{\mathrm{wf}}_{N,h}$, and does not depend on m.

The declarations of this chapter are in the namespaces `WilsonGauge`, `FermionSliceBlock` and `FermionSU2Config`, and are definitions separate from `TorusRP.wilsonMeasure` and `TwistedWilson.ZW` of 19-01. When all four directions have the same size (spatial side $N = 2h$, $2h > 1$), it is shown in Lean that pushing forward the pure-gauge measure `WilsonGauge.gaugeMeasure` (a definition without j, whose density is the same as $\mu^{\mathrm{wf},0}$ at $j = 0$ Proved here) by the re-indexing gives a constant multiple of the Wilson measure of the strong-coupling side (`WilsonGauge.wilsonMeasure_eq_map`, 19-01, bridges). There is no theorem connecting the twisted vortex side with the measure of this chapter.

What this chapter can say

Can say	Cannot say
Proved here $\frac{1}{2}\operatorname{Re}\operatorname{tr} U_p = \operatorname{Re} U_p$, and the gauge weight differs from the convention of 19-01 only by a constant multiple (the Lean bridge for $N = 2h$ is in 19-01)	Connecting by Lean theorems to lattices with $N \neq 2h$ or to twisted measures
Proved here For all m and j, $(\det D_m)^{2j}$ is a non-negative real number. Lean: `FermionSliceReflection.wilson_weight_nonneg_std4`	The sign of $\det D_m$ itself (before raising to the power $2j$)
Proved here The normalised μ^{wf} is a probability measure ($Z > 0$ is in 20-05). Lean: `WilsonGauge.wfProb_isProbability`	An odd number of flavours; the Grassmann integral itself
Reading Difference between `physDirac` and an independent transcription of (4) $\leq 3.6 \times 10^{-15}$ (small lattices)	The numerical comparison is for small lattices only. The component formula for general N, h is in Lean (`WilsonGauge.physDirac_apply`)
Proved here At $j = 0$ it returns to the Wilson measure	Mass renormalisation, chiral symmetry, the continuum limit

Sources and reproduction

Item	Kind	Source / tools
The Wilson action	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445 (bibliography of the present book). 5-01
The Wilson term (the second-order difference that makes doublers heavy)	Cited	Bibliographic data only (unverified)
Reflection positivity of lattice gauge theory with fermions	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only). 5-03
That the Grassmann integral gives a determinant; doublers	Cited	Standard facts of lattice field theory. Textbook bibliographic data unverified
Lattice, action, operator, weight, measure (§ 02 to § 06)	Lean	Lean definitions from this book's investigation (declarations of `WilsonGauge`, `FermionSliceBlock`, `FermionSU2Config`, `FermionSliceReflection`; names in the text). Table of correspondence in 19-06
Numerical cross-check of the operator (§ 04)	Reading	Numerical record of this book's investigation: comparison of the operator transcribed from the Lean text with the operator written independently from (4) (removing the boundary sign in time gives a difference of 2.0, confirming that the comparison is effective)

Words used in this chapter — Wilson–Dirac operator, Wilson term, antiperiodic, physical link, bare mass, flavour, fermion weight, Wilson–fermion measure. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-06 Objects and their Lean definitions — a table of correspondence — makes every object of Part 19 searchable in tables of symbol, meaning, Lean name and the parts that use it.

Objects and their Lean definitions — a table of correspondence

This chapter is an index.

What this chapter shows — The objects of Part 19 can be looked up from the book's notation to the names of Lean declarations, and from Lean names to the book's notation. Each row gives the anchor of the chapter where the definition is placed, and the parts that state theorems about the object.

Prerequisite chapters — None. Follow the anchor of each row to the defining chapter.

Said first — These tables list only the correspondence of definitions. The names of theorems are in the chapters of each part. Lean names are declaration names; locations (files) are not listed. Where the same object is defined separately in several lines (four: strong coupling, vortices, fermions, ultraviolet), the rows are split by line (the differences in convention are in 19-01 § 07).

The order of this chapter

1. How to read these tables — the meaning of the columns, and namespaces
2. Lattice, links, measures (19-01)
3. Twist, the vortex ratio, Polyakov loops (19-02)
4. Reflection, the vessel, infinite volume (19-03)
5. The block map and the multi-step family (19-04)
6. The fermion weight (19-05)
7. Correspondence between lines (bridges) — identifications closed in Lean
8. List of namespaces — finding the line from the head of a Lean name

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01

How to read these tables

The columns of each table are "Symbol", "Meaning", "Lean declaration", "Defining chapter" and "Parts that use it". Symbols are those of the text; Lean declarations are written in the form `Namespace.name`. Arguments (`N`, `M`, `β` and so on) are omitted in places. "Parts that use it" are the parts that state theorems or premises about the object.

The Lean bundle is written in the four lines of 19-01, the Lean conventions (strong-coupling side, vortex side, fermion side, ultraviolet side). The convention for numbering directions is in 19-01 § 01 (20-01 to 20-05 use 0 to 3 with 0 as time; Parts 21 and 22 and 20-06 use 1 to 4). The lattice on the ultraviolet side is a box of side $2^m L$, and link values are in the closed unit ball (19-04 § 02). Even for the same mathematical object, different lines give different declarations. The bridges connecting the lines in Lean are gathered in 19-01, bridges.

The letter m means different things in different chapters: in 19-02 the side of a vortex block, in 19-04 the step number (the m of Lean's `stageCov m L` is also a step), and in 19-05 the bare mass.

02

Lattice, links, measures (19-01)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Links of $(\mathbb{Z}/N)^4$	Links of the torus of the same size in all four directions (point × direction)	`TorusRP.Lk`	19-01 #lattice	20
$(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$	Points of the torus with transverse size N and longitudinal size M	`TwistedWilson.Site`	19-01 #lattice, 19-02	20, 21
$U_\ell \in SU(2)$	Link variables. Unit quaternions	`OSGap.SU2`, `TwistedWilson.SU2`, `FermionSU2Config.SU2`	19-01 #su2	15, 20, 21, 22
$\{\lvert q\rvert \le 1\}$	The closed unit ball of quaternions (link values on the strong-coupling and ultraviolet sides)	`TorusRP.QBall`, `ChainBlockMap.Ball`	19-01 #conventions	15, 20

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
dU	The normalised Haar measure and its product	`OSGap.haarProb`, `TwistedWilson.haarProb`, `TorusRP.haarBall`, `TwistedWilson.haarCfg`	19-01 #haar	15, 20, 21, 22
U_p, Re U_p	Plaquette holonomy and its real part	`TorusRP.hol`, `TwistedWilson.hol`	19-01 #plaquette	15, 20, 21, 22
$\sum_p$ Re U_p	The action on the strong-coupling side (positive sign)	`TorusRP.action`	19-01 #wilson-measure	20, 22
$\tilde{\mu}_{N,\beta}$, $\mu_{N,\beta}$	The Wilson measure (unnormalised $\tilde{\mu}$) and the normalised μ. $\tilde{\mu}_{N,\beta} = Z_N(\beta)\mu_{N,\beta}$	`TorusRP.wilsonMeasure`, `TorusRP.wilsonProb`	19-01 #wilson-measure	20, 22
$Z_N(\beta)$	The partition function (total mass of $\tilde{\mu}_{N,\beta}$)	The value of `TorusRP.wilsonMeasure` on the whole space	19-01 #partition	20, 22

03

Twist, the vortex ratio, Polyakov loops (19-02)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
n_{12}, n_{34}, z_p	Presence of twists, and the sign of each plaquette	`TwistedWilson.zt`	19-02 #twist	20, 21
Z_0, Z_1	Partition functions under the background $n_{12} = 1$ ($n_{34} = 0, 1$)	`TwistedWilson.ZW`	19-02 #twist	20, 21
θ_M	The vortex ratio $(Z_0 - Z_1)/(Z_0 + Z_1)$	`TwistBlocks.thetaM`	19-02 #theta	20, 21
$P_b(t)$	The Polyakov loop once around direction 3, and the expectation of a pair	`HalfSplit.polyLoop`, `PolyakovT.polyExp`	19-02 #polyakov	21
$S_{\mathrm{v}}(N, m)$, S_N	The infimum of the vortex action, and its infimum over m (the minimal action of a vortex)	`TwistBlocks.Svortex`, `TwistBlocks.SN`	19-02 #vortex-action	21

04

Reflection, the vessel, infinite volume (19-03)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Θ	Reflection (lattice-point planes, middle planes). The general form written as a pair of a map on link indices and a map on values	`LinkReflection.linkRefl`; concrete reflections `TorusRP.thL`, `TorusRP.sig` (lattice-point planes), `MidRP.thM` (middle planes), `HalfSplit.Th` (vortex side)	19-03 #reflection	20, 22
Positive side	Links on the positive side of the reflection (lattice-point planes, middle planes), and links within the mirror plane	`TorusRP.posP`, `TorusRP.fixP`, `MidRP.posM`	19-03 #positive-side	20
Pairing	A bilinear form of the form $\int F(\Theta\omega)G(\omega)\,d\mu$	`OrbitGrowth.pairingForm` (the field `OSGap.Datum.B` of the vessel; the vessel built from a measure, `OSGap.measureDatum`)	19-03 #pairing	20
T, Ω, P_Ω	The vessel of reconstruction (pairing form, time translation, constant), and the transfer operator, vacuum and projection onto the vacuum	`OSGap.Datum` (fields `B`, `τ`, `one`); objects built: `OSGap.Datum.Hilb`, `OSGap.Datum.T`, `OSGap.Datum.Ω`; projection `OSGap.vacProj`	19-03 #datum	20
Mass gap m	$\|T - P_\Omega\| \le e^{-m}$	`OSGap.Datum.HasMassGap`	19-03 #mass-gap-def	20, 28
ν_β	The periodic lift to $\mathbb{Z}^4$ of the normalised measure on $(\mathbb{Z}/2(n+1))^4$, and its limit	`WilsonLimit.liftProb`	19-03 #infinite-volume-space	20

05

The block map and the multi-step family (19-04)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Coarse link $b = (y, \mu)$ of Λ_m	Links of the torus of side $2^m L$. To the fine lattice by $y \mapsto 2y$	`BoxWalk.BoxLk`, `BoxWalk.double`, `BoxWalk.base`	19-04 § 02	15

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\gamma_{o,A}(b)$	The family of paths (order o, transverse offset A)	`WalkCast.walk`, `BoxWalk.boxPathsIdx`, `BoxWalk.boxPaths`	19-04 #paths	15
$w(o,A)$	Path weights $\frac{1}{6}\prod_t \mathrm{wt}(d_t)$, $\mathrm{wt} = (1/8, 3/4, 1/8)$	`ChainBlockMap.wt`, `ChainBlockMap.sixWeight`, `BoxWalk.boxW`	19-04 #paths, #weights	15, 28
$U(\gamma)$, $q_b(U)$	The product along a path, and its weighted average	`ChainBlockMap.pathProd`, `ChainBlockMap.sixAvg`, `ChainBlockMap.coarseAvg`	19-04 #block-map	15
ρ_{r_0}	The retract with threshold r_0, $q/\max(\lvert q\rvert, r_0)$	`ChainBlockMap.retract`	19-04 #block-map	15
B	The non-linear block map, and the composed map over steps	`ChainBlockMap.blockMap`, `ChainBlockMap.down`	19-04 #block-map	15, 28
C	The path kernel (the linear part of B about the identity), and its coefficients and operator	`BlockLinear.sixSgn`, `LatticeTorus.pathKernel`, `TorusFourier.Cop`, `PushCov.blockLin`	19-04 #path-kernel	15
$e^{\varepsilon a}$	The exponential configuration of a field of imaginary quaternions	`BlockLinearKernel.expCfg`	19-04 #path-kernel	15
$B_*\mu$, $\nu_{n,m}$	The push-forward measure, and the multi-step push-forward	`Measure.map`, `ChainBlockMap.vP`	19-04 #pushforward	15, 28
cov_μ	The covariance form	`PushCov.covForm`	19-04 #pushforward	15
$\mathrm{freeCov}_N$, $\mathrm{stageCov}_{k,N}$	The covariance of the free field (Feynman gauge), and the covariance at a step, pushed k times by C	`LatticePush.freeCov`, `LatticePush.stepOp`, `LatticePush.pushOp`, `LatticePush.stageCov`	19-04 #family	15
curl, plaquette covariance	The plaquette field, and its covariance	`LatticeFace.curl`, `LatticeFace.faceCov`	19-04 #family	15

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\omega(q)$, $P(q)$	$\sum_\nu 4\sin^2(q_\nu/2)$, and the projection onto the physical part	`OneStepET.omF`, `OneStepBridge.Pfam`	19-04 #family	15
Momenta with $\omega > 0$, $r_\zeta(q)$	The domain (coordinates $\theta = q/4$) and the 16 aliases	`OneStepFinal.Qpos`, `OneStepBridge.thR`, `OneStepFinal.rqPos`	19-04 #family	15
$\mathcal{M}$, A_ζ	The one-step map for each momentum, and its matrices	`OneStepBridge.Mfam`, `OneStepBridge.AM`	19-04 #family	15
G_k	$\mathcal{M}^k(P/\omega)$	`TauSplit.Gs`, `TorusLam.Gst`	19-04 #family	15

06

The fermion weight (19-05)

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
$\mathcal{L}_{N,h}$	Physical links of time circle $2h \times$ space $(\mathbb{Z}/N)^3$	`FermionSU2Config.PhysLink`	19-05 § 02	20
S_W	$\sum_p (1 - \frac{1}{2}\operatorname{Re}\operatorname{tr} U_p)$ (negative-sign convention)	`WilsonGauge.wilsonAction`, `WilsonGauge.plaqE`	19-05 § 03	20
$D_m(U)$	The Wilson–Dirac operator ($r = 1$, antiperiodic in all four directions)	`WilsonGauge.physDirac`, `FermionSliceBlock.std4`	19-05 #wilson-dirac	20
γ_μ	γ_4 and γ_i	`FermionSliceBlock.gammaT`, `FermionSliceBlock.gammaSp`	19-05 #wilson-dirac	20
$F_{m,j}$	$\operatorname{Re}(\det D_m)^{2j}$	`WilsonGauge.fermWeight`	19-05 § 05	20
$W_{\beta,m,j}$, μ^{wf}	The density, and the Wilson–fermion measure (unnormalised, normalised)	`WilsonGauge.wfWeight`, `WilsonGauge.wfMeasure`, `WilsonGauge.wfProb`	19-05 #fermion-measure	15, 20
Pure-gauge measure	The measure with density $e^{-\beta S_W}$ (a definition without j, $N = 2h$)	`WilsonGauge.gaugeMeasure`	19-01 (bridges), 19-05 § 08	20

Symbol	Meaning	Lean declaration	Defining chapter	Parts that use it
Θ (fermion side)	The action of the time reflection $t \mapsto -t$ on configurations	`WilsonGauge.thetaPhys`	19-03 #reflection	20, 22

07

Correspondence between lines (bridges)

Declarations showing in Lean that two definitions from different lines are the same object. The explanation is in 19-01, bridges.

What it connects	What it says	Lean declaration	Range
Strong-coupling side ↔ vortex side (measures)	$\tilde{\mu}_{N,\beta}$ (`TorusRP.wilsonMeasure N β`) equals the measure given by the product of Haar measures on the vortex side times the weight, transported by a one-to-one correspondence given by definition. No constant factor	`WilsonBridge.wilsonMeasure_eq_map`	Only for $M = N$ with no twist. There is no bridge for twisted measures (background $n_{12} = 1$)
Same (expectations, total mass)	The expectations of measurable observables, and the total mass $Z_N(\beta)$, agree on the two sides	`WilsonBridge.integral_wilsonMeasure`, `WilsonBridge.wilsonMeasure_univ`	
Same (correspondence aligning time)	Under the correspondence that overlays time on the strong-coupling side (index 0) with index 3 on the vortex side, the measures also agree, and the reflections through lattice-point planes on the two sides become the same map	`WilsonBridgeT.wilsonMeasure_eq_mapT`, `WilsonBridgeT.refl_toBallT`	
Same (vortex ratio)	Writes the vortex ratio with no background twist ($n_{12} = 0$) as an expectation under the probability measure of the strong-coupling side. It is not the book's θ_M ($n_{12} = 1$)	`WilsonBridgeV.vortex_ratio_eq`	

What it connects	What it says	Lean declaration	Range
Strong-coupling side ↔ fermion side	$\tilde{\mu}_{2h,\beta}$ is $e^{6(2h)^4\beta}$ times the pure-gauge measure `WilsonGauge.gaugeMeasure` transported by re-indexing	`WilsonGauge.wilsonMeasure_eq_map`	Spatial side $N = 2h$, $2h > 1$

Declarations connecting the ultraviolet side with the other lines by a single theorem are not included in this table (for re-indexing of measure types, see 19-04 § 06).

08

List of namespaces

From the head of a Lean name (its namespace), one can find which line the declaration belongs to.

Namespace	Line	Main objects
`TorusRP`, `OSGap`, `LinkReflection`, `MidRP`, `OrbitGrowth`, `WilsonLimit`	Strong-coupling side	The Wilson measure on $(\mathbb{Z}/N)^4$, the vessel, infinite volume
`TwistedWilson`, `TwistBlocks`, `HalfSplit`, `PolyakovT`	Vortex side	Twisted tori, θ_M, vortex action, Polyakov loops
`WilsonBridge`, `WilsonBridgeT`, `WilsonBridgeV`	Bridges between the strong-coupling and vortex sides	Agreement of measures for $M = N$ with no twist
`WalkCast`, `ChainBlockMap`, `BoxWalk`	Ultraviolet side (non-linear)	The family of paths, the block map, push-forward
`BlockLinear`, `BlockLinearKernel`, `LatticeTorus`, `TorusFourier`, `PushCov`	Ultraviolet side (linear part)	The path kernel, the covariance form
`LatticePush`, `LatticeFace`, `OneStepBridge`, `OneStepET`, `OneStepFinal`, `TauSplit`, `TorusLam`	Ultraviolet side (free field)	Covariances at steps, families over momenta
`WilsonGauge`, `FermionSliceBlock`, `FermionSU2Config`, `FermionSliceReflection`	Fermion side	Physical links, the Wilson–Dirac operator, the Wilson–fermion measure

What this chapter can say

Can say	Cannot say
Proved here The correspondence between the objects of Part 19 and the names of Lean declarations (with anchors to the defining chapters)	Identification between lines only within the range of the bridges of § 07 (there is no bridge for twisted measures)

Sources and reproduction

Item	Kind	Source / tools
Names of Lean declarations and namespaces	Lean	Copied by reading the source of the Lean bundle of this book's investigation. The definition and explanation of each object are in 19-01 to 19-05

Words used in this chapter — object, declaration, namespace, line, bridge. Their meanings are in the A-02 Glossary.

Next chapter to read: 19-07 The statements to be shown — a list of the statements to be shown, written with only the objects defined here.

The statements to be shown — written with defined objects only

> This chapter is this book's own investigation. The reading of the problem statement in § 01 and § 08 arranges established knowledge.

What this chapter shows — Statements of "what would have to be shown" towards the continuum limit and the mass gap, written with only the objects defined in Part 19. First we fix how the sequence β_n of bare couplings is chosen (matching), and then write five statements about that sequence: convergence of the full sequence, non-degeneracy, equicontinuity, the mass gap (uniform decay), and the band. Each can be written as a one-line formula, and none uses undefined quantities (effective couplings, activities and so on). Finally, we tabulate the gap to the problem statement and the nearby forms that are already theorems.

Prerequisite chapters — The definitions of the objects are in 19-01 The Wilson measure, 19-02 The vortex ratio, 19-03 The definition of the mass gap and 19-04 The push-forward measure. The problem statement is in 0-03, and the definition of the mass gap in 4-11.

Said first — Every statement in this chapter is set down without a proof Assumption . This book shows none of them. The first four (matching, convergence of the full sequence, non-degeneracy, equicontinuity) are statements on the ultraviolet side, written for this book's design (the measure pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10, and the committee's assessment is that it is proved in Lean (Volume 2; a comparison of the two designs is in Part 14, and the record of this book's ultraviolet attempts in Part 15). The first four are therefore kept as a record of the statements to be shown in this book's design. The last two (uniform decay, with the mass gap as its consequence, and the band) are statements on the infrared side and remain open. They are the main subject of Volume 3. The five statements have content only when read for the sequence β_n fixed by matching. Read in a form where the sequence can be chosen freely, the mass-gap and band statements hold just by setting the coupling to 0 (§ 01). Choices remain in how the matching is done. Of the requirements of the problem statement, fine observables and rotation invariance do not yet have the objects needed to write statements (§ 08).

The order of this chapter

1. Setting — one box, and a sequence of finer lattices
2. Matching — how the sequence of bare couplings is fixed
3. Convergence of the full sequence — for the matched sequence, without passing to subsequences
4. Non-degeneracy — the coarse field stays in small fields
5. Equicontinuity — uniformly in physical position
6. Uniform decay — uniformly in lattice spacing and box (the consequence is the mass gap)
7. The band — the vortex ratio becoming small exponentially in the area
8. The gap to the problem statement — separate stages, and stages whose objects do not yet exist
9. Nearby forms that are already theorems (Table 2)

01

Setting — one box, and a sequence of finer lattices

What the problem statement asks for is a theory on infinitely extended continuous four-dimensional space (0-03 § 03). In this chapter, as a stage before that, we consider bringing the lattice spacing towards 0 on **one box of fixed size**. The objects used in the statements are only those defined in Part 19.

Let the size of the box be a positive integer L. Only the matching of § 02 is done on a box of one fixed **reference size** L_{ref}. The statements of § 03 to § 05 are about a box of one fixed size L (not limited to $L = L_{\mathrm{ref}}$), and in § 06 and § 07 the sizes L and L_0 vary. The lattice at step $n \geq 0$ is

$$\Lambda_n = (\mathbb{Z}/2^n L)^4 \quad (1)$$

and the lattice spacing at step n is read as $a_n = 2^{-n}$. Read this way, the lattice at every step subdivides the same box of side L. The larger the step number, the finer the lattice (the same direction as in 19-04 § 02). The direction playing the role of time is direction 0 in the numbering of Part 20 in 19-01 (direction 1 in the numbering 1 to 4 of 19-04; 19-04 itself does not specify a time direction).

Four objects are used.

Symbol	Contents	Definition
μ_n^β	The normalised Wilson measure on Λ_n (coupling β). Once the sequence of couplings (β_n) is fixed, $\mu_n := \mu_n^{\beta_n}$	19-01. Lean: `TorusRP.wilsonProb`
$\nu_{n,m}$	The measure μ_n pushed by the block map down to step $m \le n$. A probability measure on configurations of the coarse lattice Λ_m	19-04. Lean: `ChainBlockMap.vP`
$\theta_{N,M}(\beta)$	The ratio $(Z_0 - Z_1)/(Z_0 + Z_1)$ of the partition functions without and with the (3,4) twist, on the torus with transverse size N and longitudinal size M, with the background twist $n_{12} = 1$ fixed	19-02. Lean: `TwistBlocks.thetaM`
The vessel and the mass gap	The Hilbert space, transfer operator T and vacuum Ω built from a measure. $\|T - P_\Omega\| \le e^{-m}$ is called a gap of mass m	19-03. Lean: `OSGap.Datum.HasMassGap`

$\nu_{n,m}$ is the theory on the fine lattice of step n, viewed at the fixed resolution of step m. Holding step m fixed and letting $n \to \infty$, we see the limit in which the lattice spacing a_n approaches 0, on the same coarse lattice. The push-forward is a rule for making coarse fields; no decomposition (representation) into an effective action is used.

The space of configurations is compact (a finite product of balls). So at each step m, $\nu_{n,m}$ converges weakly along a subsequence, and by the diagonal argument a subsequence common to all steps can be taken Proved here . **Subsequential limits exist without any premise**, so the statements to be shown have meaning only in forms stronger than that. The following six are all of that form.

Read together with the matching. Each of the five statements of § 03 to § 07 involves the sequence of couplings (β_n). Read in a form where the sequence can be chosen freely, the statements lose their content.

- Setting $\beta_n \equiv 0$, the fine measure is the product of Haar measures. Since the block map is local, coarse variables separated in time are independent and their covariance is 0. So the mass-gap statement (§ 06) holds for this sequence Proved here . Also, at $\beta = 0$ the partition function does not change with or without the twist, $Z_0 = Z_1$, so $\theta = 0$, and the band statement (§ 07) holds too Proved here .
- Taking β_n large pushes the fine measure towards flat configurations, and the non-degeneracy statement (§ 04) is expected to become easier to satisfy.

Conversely, read as "for every sequence", there are expected to be sequences for which they are false (for instance, for a sequence alternating between the strong-coupling end and the weak-coupling end, convergence of the full sequence cannot be expected). So in this chapter we first fix one sequence (β_n) by **matching** in § 02, and write the five of § 03 to § 07 **for that sequence**. Note also that the theorems in the strong-coupling window (§ 09) lie in the direction on which the mass-gap statement holds trivially.

02

Matching — how the sequence of bare couplings is fixed

The coupling β on the lattice is, in the continuum limit, not an "input" but a number adjusted to calibrate the lattice spacing (introduction in 5-05 § 02). Here we write that adjustment with defined objects only.

The matching rule. Choose a step m_0, a bounded continuous function Y_0 of the coarse configuration at step m_0 (for example, the average of $\mathrm{Re}\, U_p$ over coarse plaquettes), and a target value κ_0. For each step $n \ge m_0$, set

$$f_n(\beta) = \int Y_0 \, d\nu_{n,m_0}^{\beta}, \qquad \beta_n \text{ a solution of } f_n(\beta_n) = \kappa_0 \quad (2)$$

Here ν_{n,m_0}^{β} is the push-forward of the Wilson measure with coupling β on the box of reference size L_{ref}.

Existence of solutions. f_n is continuous in β Proved here . Reason: $f_n(\beta)$ is the ratio of two integrals with respect to the product of Haar measures, $\int (Y_0 \circ B)\, e^{\beta S}$ and $\int e^{\beta S}$ ($S = \sum_p \mathrm{Re}\, U_p$, B the composed map from step n to step m_0). The integrands are bounded and continuous in β, and the configuration space is compact, so both integrals are continuous by dominated convergence, and the denominator is positive. Hence if κ_0 lies between two values $f_n(\beta')$, $f_n(\beta'')$, there is a solution in $[\beta', \beta'']$ by the intermediate value theorem Proved here . Whether κ_0 falls in this range for every n depends on the choice of Y_0 and m_0.

Where choices remain. (i) How Y_0, m_0 and κ_0 are chosen. (ii) Since the solution is not necessarily unique, which solution is taken (for example, the largest solution in a fixed interval). These two are design choices, and are listed in the table of 28-05.

The statement to be shown. If solutions exist for every n, the matching rule fixes one sequence. The existence of solutions hinges on checking the end values, which has not yet been done. Beyond that, what must be shown is that the sequence so fixed falls within **a bracket on how the coupling runs**.

Assumption `Matching: there are` β_* `and` $0 < b_- \le b_+$ `such that` $\beta_* + b_- n \le \beta_n \le \beta_* + b_+ n$ `for every` n

It is the form in which β increases by roughly a fixed amount each time the lattice spacing is halved. It corresponds to the logarithmic running $\beta \sim \text{const} \times \log(1/a)$ predicted by one-loop asymptotic freedom, written in terms of $n = \log_2(1/a_n)$ (introduction in 5-05 § 03). In the conditional implication assembled in this book's study (weak ultraviolet existence in a small box), a bracket with $b_\pm$ taken as constant multiples of the one-loop coefficient (the matching bracket for the bare coupling) was placed in a form that follows from an assumption on the effective coupling (the one-step bracket assumption for the coupling) (28-01).

The treatment of the sequence of couplings differs between that implication and Lean. The implication defines and uses the fine measure as "the Wilson measure at the β_n chosen by the matching bracket for the bare coupling" (the implication's β_n is chosen by matching the effective coupling to a target value, a different matching from the rule (2) of this section). The Lean theorems take an arbitrary sequence of bare couplings and do not tie it to that choice.

The convention from here on. The matching is done once, on the box of reference size L_{ref}, and the resulting sequence β_n (the coupling at lattice spacing $a_n = 2^{-n}$ of step n) is used for boxes of every size. In § 03 to § 07, β_n is this sequence, $\mu_n = \mu_n^{\beta_n}$ is the Wilson measure of the box of the size under consideration, and $\nu_{n,m}$ is its push-forward. When the size is varied in § 06 and § 07, the sequence is the same.

03

Convergence of the full sequence — without passing to subsequences

Assumption `Convergence of the full sequence: for the sequence $\beta_n$ fixed by matching, for every step $m$ and every bounded cont`

In other words, at each step m, $\nu_{n,m}$ converges weakly to some measure ν_m without passing to subsequences. Then, since the block map is continuous, push-forward and limit commute, and ν_m satisfies the consistency relation $(B_m)_*\nu_{m+1} = \nu_m$ Proved here . The consistent family (ν_m) gives the "continuum-limit theory" on the box of size L at all resolutions at once.

As in § 01, subsequential limits exist without any premise. The content of this statement is that "every subsequence gives the same limit". If the limit depended on the subsequence, the theory would not be determined uniquely.

This statement does not constrain the sequence of couplings, so it is read for the sequence fixed by the matching of § 02. Without a fixed matching, it is not determined what theory the limit is the limit of.

The conclusion of the conditional implication assembled in this book's study goes only as far as convergence along a subsequence (28-01). That implication has an assumption related to convergence of the full sequence (the summability-of-increments assumption), but it is **not the same as this statement**. The summability-of-increments assumption says that, for a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. Being summable, it is stronger than this statement, and the family of functions is also different. The step from gauge-invariant functions to all bounded continuous functions is an argument on paper, going through the invariance of the limiting measure under gauge transformations (28-02).

04

Non-degeneracy — the coarse field stays in small fields

Choose a fixed finite region R_m of the coarse lattice at step m (not the whole box, but a region of fixed physical size). The set of configurations in which the coarse links in R_m are close to unit length and the coarse plaquettes are nearly flat is called the **small-field set** C_m. For example, choosing a threshold $y_m > 0$,

$$C_m = \{U : \ |U_b| \geq 1 - \epsilon_m \text{ for every coarse link } b \text{ of } R_m, \ \ \beta_m (1 - \operatorname{Re} U_p) \leq y_m \text{ for every coarse plaquette } p \text{ of } R_m\} \quad (3)$$

(a closed set). β_m is the m-th term of the sequence fixed by matching ($m \geq m_0$). $\epsilon_m < 1$ and y_m are fixed **before the statement** to be as small as later stages (perturbative expansion) can use (for example, for a fixed angle φ_c setting the small-field threshold, below the floor $y_m \leq \beta_m(1 - \cos\varphi_c)$ at which the angle of plaquette values is at most φ_c). Coarse link values lie in the closed unit ball and $1 - \operatorname{Re} U_p \leq 2$, so taking $\epsilon_m \geq 1$, $y_m \geq 2\beta_m$ makes C_m the whole configuration space, and the statement holds trivially Proved here . How R_m is taken is also a choice. Written for the whole box, η_m could approach 1 with the volume, so it is written in the form of a region.

Assumption `Non-degeneracy: for the sequence $\beta_n$ fixed by matching and $(R_m, \epsilon_m, y_m)$ fixed beforehand, for every step $m$ there is`

Meaning: however fine the lattice pushed from, the coarse field at step m stays in the small-field region with at least a fixed probability. The same lower bound remains for the limit ν_m (C_m is closed), so the limiting measure gives positive weight to the small-field region Proved here . It is an entrance condition for entering later stages (the range where perturbative expansion can be used).

$\eta_m < 1$ is the content of the statement. Allowing $\eta_m = 1$, it holds for any measure. Also, this statement is expected to be of the kind that becomes easier to satisfy the larger the coupling is taken (§ 01). That is why it is read for the sequence fixed by matching.

05

Equicontinuity — uniformly in physical position

The continuum-limit theory should have observables not only at lattice points but at any position in the box. This is the statement for that.

Choose one family of observables. Let D be the countable dense set of positions x in the box $(\mathbb{R}/L\mathbb{Z})^4$ that are lattice points at some step $\mathrm{lev}(x)$ (dyadic rational points), and to each $x \in D$ assign a bounded continuous function G_x of the coarse configuration at step $\mathrm{lev}(x)$ (a fixed, non-constant local observable G placed at position x). The statement is needed for every local observable G. Taking G constant makes the statement hold trivially, so a single G alone has no content.

$$s_n(x) = \int G_x \, d\nu_{n,\mathrm{lev}(x)} \qquad (n \geq \mathrm{lev}(x)) \quad (4)$$

Assumption Equicontinuity: for the sequence β_n fixed by matching, there are $\tau_n \to 0$ and a function $\Omega(r)$ tending to 0 as $r \to 0$ s

Meaning: the expectations of observables have a modulus of continuity in position that is uniform in n (apart from the error τ_n coming from the lattice subdivision). Combined with convergence of the full sequence, the limit $s(x) = \lim_n s_n(x)$ is uniformly continuous on D, and extends continuously to every position in the box Proved here .

The choice of the family of observables (D, G) follows the form of the conditional implication assembled in this book's study; a cross-check of the definition and statement by another pair of eyes has not yet been done (the family of observables and small-field set in 28-04).

06

Uniform decay — uniformly in lattice spacing and box (the consequence is the mass gap)

We write τ_t for the translation by t steps in the time direction of the lattice at step m (the convention of § 01: direction 0 in the numbering of Part 20 in 19-01). Since the box is periodic, τ_t is defined as a map on coarse configurations. In physical time it is a shift of $a_m t$.

Assumption Uniform decay: for the sequence β_n fixed by matching, there is $m_* > 0$ such that, for every size L, every step m,

The time difference t is limited to half the period. In a periodic box, τ_t returns to the identity with period $2^m L$, and when t equals the period the covariance returns to $\mathrm{Var}(F)$ Proved here . The constants are taken in the form $C_F = c_F \|F\|_\infty$, where c_F is determined only by the number of links F depends on, and does not depend on n or L. m_* is a mass in physical units, depending neither on the lattice spacing nor on the box size. This is the core of the statement.

There are choices in the form of the statement. Whether to take constants for each observable (the form of this chapter), or one constant C uniformly as $C\|F\|_\infty\|G\|_\infty$. Whether observables depend on links near time 0, or on links at exactly time 0 (whose number grows with the volume). For obtaining the mass gap of the vessel, either form suffices Proved here .

The implication that this decay gives the mass gap ($\|T - P\Omega\| \leq e^{-m}$ in 19-03) in the vessel of reconstruction of the limiting measure is written in 28-03. Building the vessel also needs reflection positivity and symmetries of the limiting measure. Reflection positivity of the fine lattice is a theorem of Part 20, but carrying it over to the push-forward measure is on the side of the premises (reflection positivity at the coarse level in 28-04, symmetries of the measure).

The mass gap of the problem statement (4-11 § 01) is a gap in the spectrum of the Hamiltonian of the infinite-volume theory. This statement takes, among the steps toward it, the part "in a box of fixed size, uniformly in the lattice spacing" and makes it uniform in the size of the box as well. The step to infinite volume is a separate row of the table in § 08.

In this chapter the statement is written with this book's push-forward measure, but the core of the question (that correlations of observables on the renormalised lattice decay at an exponential rate, in physical units, uniformly in the cutoff and the volume) is open independently of the design. If it were obtained, the mass gap and the uniqueness of the vacuum of the limiting theory would follow from known general theory. What the published construction has goes only as far as estimates at a polynomial rate at long distances (16-03).

07

The band — the vortex ratio becoming small exponentially in the area

The vortex ratio $\theta_{N,M}(\beta)$ (19-02) is the relative difference of the partition functions when one twist is inserted. Since $Z_1 > 0$, $\theta_{N,M} \leq 1$ Proved here . If M is even, $0 \leq \theta_{N,M}$ also holds (a theorem of 21-06). In the convention of 19-02, the free energy of the vortex (flux) is $\Delta F = \ln(Z_0/Z_1)$, and $\theta = \tanh(\Delta F/2)$. So when θ is small, $-\ln\theta$ is nearly equal to minus the logarithm of ΔF, $-\ln(\Delta F/2)$ (introduction in 25-01).

Subdividing a square box of physical size L_0 at the lattice spacing of step n, the number of lattice points on a side is $2^n L_0$ (even for $n \geq 1$). For the coupling, the sequence β_n matched on the reference box in § 02 is used in the same way for every L_0.

Assumption Band: for the sequence β_n fixed by matching, there is $c > 0$ such that, for every L_0 and all sufficiently large n,

It is written without the logarithm $-\ln\theta$ so that the statement does not break at $\theta = 0$ (the best case). Lowering c slightly, it is the same as the form $\liminf_n (-\ln\theta) \geq c\,L_0^2$.

Meaning: in the continuum limit, the vortex free energy ΔF becomes small exponentially in the area of the box ($-\ln\theta$ grows in proportion to the area). This is the form of confinement in the language of vortices (an exponential in the area); that writing the band in the naive form "θ stays uniformly away from 1" is not enough is in 26-01 § 04. Crossing the transition between the weak-coupling side (small boxes) and the strong-coupling side (large boxes) with a single c is the content of this statement.

The band statement can be written with θ and Z alone, and does not use the push-forward measure. The present book has no arrow from this statement to the mass gap (28-03, 28-06).

08

The gap to the problem statement — separate stages, and stages whose objects do not yet exist

Between what the problem statement (0-03) asks for and the six statements of § 02 to § 07, there are still stages.

Table 1 The requirements of the problem statement and their relation to the statements of this chapter (as of 2026-11-13)

Requirement of the problem statement	Relation to the statements of this chapter	Objects for writing the statement
On $\mathbb{R}^4$ (0-03 § 03)	This chapter uses a box of size L. Enlarging the box is a separate stage	Can be written by adding one definition (restriction to a window in the box, or periodic extension). In the strong-coupling window it can be written with the periodic lift, and is a theorem (20-03)
Local fields, Schwinger functions (4-10 § 02)	The observables of this chapter are functions of coarse configurations at step m, not point fields	**Not yet available**. Definitions of a lattice gradient flow and of flowed observables are needed
Rotation invariance (the Euclidean group)	The symmetries of the lattice are only the hypercubic group	**Not yet available**. Observables on which rotations finer than the hypercubic ones act, or a block map from a rotated lattice to the standard lattice, are needed
Non-triviality (0-03 § 05)	Not in this chapter	Can be written, but in terms of which quantity is undecided (example: when matched at step m_0, the limiting value of the same observable at another step m_1)
Mass gap (0-03 § 06)	The uniform decay of § 06 is the form on a box, uniform in lattice spacing and box	Available (the vessel is in 19-03)
Agreement at short distances (0-03 § 07)	The bracket of § 02 is only about how the coupling runs, and says nothing about the short-distance form of correlations	—
Any compact simple group (0-03 § 02)	This chapter is $SU(2)$ only	—

For the two of fine observables and rotations, the objects must be defined before the form of the statement can be discussed. The list is in 28-06.

09

Nearby forms that are already theorems

The theorems of Parts 20 to 22 are forms close to one of the six statements (forms at another scale, or for another object). None is literally a special case. All are theorems at **a single scale with the lattice spacing fixed** (or for one linearised step), and do not reach the six statements themselves. The right-hand column of each row says where they fall short.

Table 2 The six statements, and the forms that are already theorems (as of 2026-11-13)

Statement	Form that is a theorem	Chapter, Lean	Where it falls short of the statement
Uniform decay	Cited In the strong-coupling window $0 < \beta$, $18 \tanh\beta < 1$: a gap of mass $-\frac{1}{2}\log(18\tanh\beta)$ in the vessel of reconstruction of the infinite-volume limit measure, and clustering in the four directions. A formalisation of results of a known type	20-04. `InfMain2.wilson_infvol_main2`	β stays inside a small window, not a matched sequence $\beta_n \to \infty$. The lattice spacing is fixed at 1. The mass is in lattice units, and only a lower bound. Strong coupling is the direction on the side where things hold trivially (§ 01); it is an example in which the statement can be satisfied, not a check of its content
Convergence of the full sequence	Cited In the same window, the lifts of the Wilson measures on periodic boxes converge weakly to the same measure without a subsequence, for every sequence of box sizes. A formalisation of results of a known type	20-03. `AllN.wilson_infvol_allN`, `InfVol.wilson_infvol_unique`	What moves is the size of the box, not the lattice spacing (not the direction of n in this chapter). Only inside the window
The band	Cited The thin-vortex floor: for all real β and $N, M \geq 1$, $\theta_{N,M}(\beta) \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$. A formalisation of one side of a known picture	21-02. `TwistedWilson.twisted_vortex_bound`	The form with the lattice spacing and the transverse size N held fixed and M made large. The exponent $-\ln\tanh(N^2\lvert\beta\rvert)$ goes to 0 as $N \to \infty$, and does not reach the form $N = M = 2^n L_0$
The band	Cited The thick-vortex floor: for any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M, $\lvert\theta_{N,M+1}(\beta)\rvert \leq \exp\bigl(-(M+1)^2 e^{-\beta(S_N+\varepsilon)}\bigr)$ (S_N is the minimal action of a vortex, 19-02). A formalisation of one side of a known picture	21-03. `TwistBlocks.thick_vortex_floor`	Likewise the form with N and the lattice spacing held fixed. The thresholds in β and M are not explicit. The exponent goes to 0 as $\beta \to \infty$
The band	Cited The vortex floor at strong coupling (an upper bound on the vortex free energy): inside the strong-coupling window, on a torus of transverse size N and longitudinal size M, the difference of the twisted partition functions is exponentially small in the area of the longitudinal plane, and the rate per unit area depends neither on the transverse size nor on the longitudinal size. The prefactor (the transverse area) remains. A formalisation of a known type	20-06. `VortexStrongMN.vortex_floor_MN_sharp`	The coupling stays inside the strong-coupling window, while matched sequences leave the window. The lattice spacing is fixed at 1. Strong coupling is the direction on the side where things hold trivially (§ 01); it is an example in which the statement can be satisfied, not a check of its content
Non-degeneracy (a nearby form)	Cited Rarity at one scale: for $\beta \geq 81$, on every torus with even side ≥ 4 and for every set X of plaquettes, the probability that all plaquettes of X have $\mathrm{Re}\,U_p < 0$ is $\leq q_1^{\lfloor\lvert X\rvert/6\rfloor}$, with $q_1 < 1$ independent of the volume. A formalisation of the Chessboard Estimate	22-01. `FoldCap.rarity_explicit`	An estimate for the measure of the fine lattice itself (no push-forward, $m = n$), with the threshold fixed at $\mathrm{Re}\,U_p < 0$. It is not an estimate about the coarse plaquettes of $\nu_{n,m}$ ($m < n$), and is not a special case of non-degeneracy either
(none of the statements)	One step of the free field: for the field obtained by pushing the Gaussian free field m times by the linearised blocking, an estimate of the covariance of the field strength (ratio $\leq 640/9$), and a contraction $< 1/170$ over 17 steps, uniform in the step. Machine-checked in Lean	15-12. `LatticeFace.lattice_face_T1h`, `TorusLam.stage_tau_17`	The free field rather than the Wilson measure, and the linear part rather than the nonlinear block map. Depends on the weights $(1/8, 3/4, 1/8)$. Material, for the free field, toward a sufficient condition for the six statements (the premise of ultraviolet stability in 28-04)

The right-hand column of Table 2 says the same one thing in every row. **The theorems are in forms with the lattice spacing held fixed, or in free-field or linearised forms; there is not a single theorem of the form in which $n \to \infty$ under a matched sequence of couplings β_n.** The distance from the six statements to theorems is visible in the list of premises (28-04), as which premise carries which label.

What this chapter can say

Can say	Cannot say
Assumption The matching bracket, and five statements about the sequence fixed by matching (convergence of the full sequence, non-degeneracy, equicontinuity, uniform decay, the band), have been written with the objects of Part 19 only	None of the statements is shown
Proved here If the sequence is chosen freely, the uniform-decay and band statements hold for $\beta_n \equiv 0$	Whether they hold for the sequence fixed by matching
Proved here Limits along subsequences exist by compactness of the configuration space alone	That the limit does not depend on the subsequence (convergence of the full sequence)
Proved here The matching function $f_n(\beta)$ is continuous in β, and if the target value lies between the end values there is a solution	Whether the target value falls in the range for every n. Uniqueness of the solution. That the matched sequence falls within the bracket
Proved here Under convergence of the full sequence, the family of limits is consistent under push-forward	Non-triviality of the limit. How the limit depends on the choices in matching

Can say	Cannot say
Table 2: the strong-coupling window, the vortex floors with lattice spacing and transverse size fixed, rarity at one scale, and one step of the free field are theorems machine-checked in Lean	None is one of the six statements itself (they are forms with the lattice spacing fixed, or free-field or linearised forms)
The gap to the problem statement (Table 1)	For fine observables and rotation invariance, the objects for writing statements do not yet exist. Statements for the stages of infinite volume and non-triviality are not in this chapter

Sources and reproduction

Item	Kind	Source / tools
The problem statement	Cited	A. Jaffe, E. Witten (the official problem statement of the Clay Mathematics Institute, 2000). Reading in 0-03
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. Introduction in 4-10
Asymptotic scaling and the lattice β function	Cited	Introduction and sources in 5-05
The form of the six statements	Assumption	This book's study (an inventory of premises, 2026-11-13). The assumptions and conclusions of the conditional implication assembled in this book's study (weak ultraviolet existence in a small box), rewritten in the language of defined objects
Definitions of the objects used	Lean	This book's Lean bundle: `TorusRP.wilsonProb`, `ChainBlockMap.vP`, `TwistBlocks.thetaM`, `OSGap.Datum.HasMassGap`. Each definition is in the corresponding chapter of Part 19
The theorems of Table 2	Lean	The source tables of each chapter (20-03, 20-04, 21-02, 21-03, 15-12, 22-01)

Words used in this chapter — statement to be shown, matching, bare coupling, bracket on how the coupling runs, push-forward measure, convergence of the full sequence, subsequential limit, non-degeneracy, small-field set, equicontinuity, uniform decay, vortex ratio, exponential in the area, band. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-00 Entrance — what can be said at one lattice spacing — reads the theorems of Table 2 in order, from the definitions. To go to the list of premises first, see 28-00.