Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 20 Firm theorems — the vessel and strong coupling

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Entrance — what can be said at one lattice spacing

This chapter is a guide (the entrance to this part).

What this chapter shows — The list of theorems of Part 20 (one line per chapter) and the scope that applies to the whole part. The theorems of this part are statements on a finite lattice with the lattice spacing fixed, or on its infinite-volume limit: the parts that set up the vessel of reconstruction (the Hilbert space, the transfer operator, the vacuum), and the mass gap inside the strong-coupling window.

Prerequisite chapters — 19-00 (how to read objects, firm theorems and premises), and 19-01 and 19-03 (the lattice, the Wilson measure, reflection, the vessel of reconstruction).

Said first — The lattice spacing is fixed. The strong-coupling theorems hold only inside the window $18 \tanh |\beta| < 1$ or $24(1 - e^{-4|\beta|}) < 1$, and say nothing about the continuum limit or weak coupling. Every theorem is a Lean formalisation of a result of a known type.

The order of this chapter

1. The theorems of this part — one line per chapter (Table 1)
2. The scope that applies to the whole part
3. Reading order

01

The theorems of this part — one line per chapter (Table 1)

Chapter	What is said (one line)	Main Lean names
20-01 Reflection positivity	Reflection positivity of the Wilson measure on $(\mathbb{Z}/2M)^4$. For the plane through lattice sites, for all real β; for the plane through the middle of links, for gauge-invariant functions with $\beta \geq 0$, $M \geq 2$; for the planes through lattice sites in all four directions, also for all β. The reflection pairing is symmetric	`TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir`, `TorusRP.wilson_refl_symm`
20-02 Decay of correlations at strong coupling	On $(\mathbb{Z}/N)^4$, the covariance of functions separated in the time direction is exponentially small. Windows $18 \tanh \lvert\beta\rvert < 1$ (the form measured by the change link by link) and $24(1 - e^{-4\lvert\beta\rvert}) < 1$ (the form with separated supports). The constants contain no N	`LinkSpec.wilson_cov_dist`, `TorusRP.wilson_strong_coupling`
20-03 The infinite-volume limit	In the window $18 \tanh \lvert\beta\rvert < 1$ ($\beta \neq 0$), the periodic lifts of the normalised Wilson measures on $(\mathbb{Z}/2(n+1))^4$ converge weakly, without passing to a subsequence, to a single probability measure ν_β on $\mathbb{Z}^4$ (the sequence over all sizes $(\mathbb{Z}/(m+1))^4$ has the same limit), and ν_β is invariant under unit translations in the four directions. At limit points, clustering in the four directions when $e^{\lambda} \cdot 18 \tanh \lvert\beta\rvert < 1$	`InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN`, `TransI.wilson_infvol_translation`, `WilsonLimit.cluster_limit`

Chapter	What is said (one line)	Main Lean names
20-04 Reconstruction and the mass gap	In the reconstruction of the limit measure in the window $18\tanh\lvert\beta\rvert < 1$: $\lVert T - P\Omega\rVert \le (18\tanh\lvert\beta\rvert)^{1/2}$ (a lower bound on the mass) and the complement of the vacuum is not 0 ($\beta \neq 0$; the same form also in the gauge-invariant sector), and $T \ge 0$ ($\beta \ge 0$). A statement bundling these into one for $0 < \beta$. That the range of T lies in the gauge-invariant sector is a separate theorem	`InfMain2.wilson_infvol_main2`, `WilsonLimit.wilson_massGap_dob_exists`, `GaugeZ.wilson_massGap_gauge_exists`, `FullT.wilson_Tpos_full`, `Bind.T_range_gauge_all`
20-05 The vessel with fermions	On space $(\mathbb{Z}/N)^3$ with a time circle $\mathbb{Z}/2h$, the measure whose weight is the product of the Wilson gauge weight and the Wilson–Dirac determinant weight $\mathrm{Re}[(\det D)^{2j}]$ is positive under reflection in a time plane through lattice sites (all β, mass $m > -1$); the partition function is positive (all β, m, j); the determinant is invariant under global gauge transformations; and the normalised measure is invariant under the generators of the lattice symmetries	`WilsonGauge.site_reflection_positive_wilson_full`, `WilsonGauge.partition_pos_wilson_full`, `GaugeDet.physDirac_det_conj`, `CubicSym.wfProb_lattice_generators`

Chapter	What is said (one line)	Main Lean names
20-06 The area law at strong coupling, and an upper bound on the vortex free energy	On a torus with the same size in all four directions, inside the strong-coupling window, the expectation of the Wilson loop on the boundary of a rectangle is exponentially small in the area (in a finite box, and for the infinite-volume limit measure). On a torus of transverse size N and longitudinal size M, the difference of the twisted partition functions is exponentially small in the area of the longitudinal plane, and the rate per unit area does not depend on the transverse size (the transverse area in the prefactor remains)	`AreaLaw.area_law_wilson`, `AreaLaw.area_law_infvol`, `VortexStrong.vortex_floor`, `VortexStrongMN.vortex_floor_MN_sharp`
20-07 What this part says, and what it does not	The scope of the theorems of this part, and a table of what is not said	—

Table 1. The theorems of Part 20. All the assumptions of each row's statement are in the corresponding chapter. Every row is a formalisation of a result of a known type Cited (reflection positivity, strong-coupling decay and the mass gap: Osterwalder–Seiler; the infinite-volume limit (convergence without a subsequence): of Dobrushin's comparison type; the area law: Osterwalder–Seiler; the upper bound on the vortex free energy: of the type of 't Hooft and Münster).

02

The scope that applies to the whole part

- **The lattice spacing is fixed.** Every theorem is a statement at one lattice spacing (taken to be 1). About the limit in which the lattice spacing goes to 0 (the continuum limit), this part says nothing.
- **Only inside the strong-coupling window.** The theorems on decay, the infinite-volume limit and the mass gap hold only inside the window $18 \tanh |\beta| < 1$ ($|\beta| < 0.0556$ Reading) or $24(1 - e^{-4|\beta|}) < 1$ ($|\beta| < 0.0106$ Reading) (the windows for the area law and for the upper bound on the vortex free energy are in 20-06, for each version of the constants). Depending on the theorem, $\beta \neq 0$ or $0 < \beta$ is added, and $T \geq 0$ needs $\beta \geq 0$. Outside the window we say neither that they hold nor that they fail.

- **The area law is a statement about lattice Wilson loops.** Continuum Wilson loops are present neither in this part nor in the published construction (14-06). The strong-coupling theorems of this part are statements at the opposite end (strong coupling, lattice spacing fixed) from the question the published construction left open.
- **Nothing is said about weak coupling.** What the continuum limit needs is the side $\beta \to \infty$ (weak coupling), the opposite end from the window. The mass gap of this part is the gap of the lattice transfer operator inside the strong-coupling window; it is not a claim about the mass gap of four-dimensional Yang–Mills.
- **Reflection positivity and the vessel with fermions do not depend on the window.** The positivity in 20-01 and 20-05 holds for all real β (the middle plane for $\beta \geq 0$; 20-05 for mass $m > -1$), and is a statement for each finite torus.
- **Only a lower bound on the mass.** The upper bound on $\|T - P\Omega\|$ is a lower bound on the mass; neither the true value of the mass nor an upper bound on it is stated.
- **Formalisations of results of known types.** Every theorem is a result of a type known in the literature, machine-checked in Lean 4 on this book's objects (19-01). We do not present them as new mathematical results.

03

Reading order

The vessel rests on three pillars, read in this order. Reflection positivity (20-01) turns the pairing into an inner product (the middle-plane form, with a gauge average in between, is also used for $T \geq 0$); strong-coupling decay (20-02) makes distant correlations small; and the infinite-volume limit (20-03) produces a single measure independent of the volume. 20-04 assembles these three into the vessel of reconstruction and obtains the mass gap inside the window. 20-05 is the part that sets up the same vessel with a measure carrying a fermion weight, and can be read with only 20-01 as a prerequisite. 20-06 gives the area law and the upper bound on the vortex free energy inside the strong-coupling window, and can be read after 20-03. 20-07 is a chapter of tables only.

After this part, Part 21 gives theorems on a lattice with twisted boundary conditions, and Part 22 theorems on the rarity of large plaquettes at weak coupling; both are likewise statements at one lattice spacing. What would need to be said to reach the mass gap in the continuum limit is in Part 28 (28-00).

What this chapter can say

What is said	What is not said
Cited Each row of Table 1 (the statement of each chapter, machine-checked in Lean)	The continuum limit, weak coupling, outside the window, the true value of the mass
Reading The window values $\lvert\beta\rvert < 0.0556$ and $\lvert\beta\rvert < 0.0106$	These are floating-point values and not part of the theorems

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity of lattice gauge theory, the strong-coupling cluster expansion	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. 5-03, 5-02
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10
The Lean declarations of Table 1	Lean	The source tables of each chapter and 19-06. sorry 0; the axioms are the three standard ones

Words used in this chapter — vessel of reconstruction, reflection positivity, strong-coupling window, infinite-volume limit, transfer operator, mass gap, lattice spacing. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-01 Reflection positivity — three forms — the first pillar of the vessel.

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Reflection positivity — three forms

> This chapter is this book's own investigation. § 01 is a summary of established knowledge.

What this chapter shows — For the SU(2) Wilson measure on the finite torus $(\mathbb{Z}/2M)^4$, the fact that the integral of a function times its mirror image is never negative (reflection positivity) has been machine-checked in Lean in three forms. It holds for all real β for the plane through sites, for $\beta \geq 0$ and gauge-invariant functions for the plane through the middle of links, and for all β for the site planes in the four directions. In addition, the reflection pairing is symmetric. This chapter gives the four statements, the route of proof (split the action into three parts; fixing the variables on the plane turns the integral into a square), and when to use which.

Prerequisite chapters — 19-01 (the lattice, the Wilson measure) and 19-03 (reflection, the positive side, the pairing). For an introduction, 5-03 (reflection positivity and the transfer matrix) and 4-10.

Said first — Each is a statement on one finite torus, with the lattice spacing fixed. Nothing is said about the continuum limit. Positivity for the infinite-volume limit measure, and the positivity $T \geq 0$ of the transfer operator of the vessel, are in 20-04; the middle-plane statement, with an average over gauge transformations in between, is the source of $T \geq 0$. Each is a formalisation of a result of a known type (reflection positivity: Osterwalder–Seiler).

The order of this chapter

1. Why reflection positivity is needed — from the pairing to a Hilbert space
2. Two mirrors and the closed positive half (Figure 1)
3. Reflection positivity for the plane through sites — the statement
4. The route of proof — split the action, fix the plane, and get a square
5. The plane through the middle of links — the roles of gauge invariance and β ≥ 0
6. Reflection positivity in four directions, and the symmetry of the pairing
7. When to use which of the three forms

8. What is not said

01

Why reflection positivity is needed — from the pairing to a Hilbert space

For an introduction, see 5-03 § 01 and 4-10. A theory on the lattice is written only as a probability measure; there is no Hilbert space and no operator that advances time. If the pairing $\langle F, G\rangle := \int F(\Theta U)\, G(U)\, d\mu$ of functions depending only on the positive side of the mirror is symmetric and $\langle F, F\rangle \geq 0$, then one can set up the Hilbert space obtained by dividing out the null functions and completing, the transfer operator T coming from time translation, and the vacuum Ω. That this positivity holds in lattice gauge theory is due to Osterwalder–Seiler Cited . This book's definitions of the pairing, the vessel and the mass gap are in 19-03, the pairing and the vessel of reconstruction.

Reflection positivity is the first pillar of the vessel, and the four statements of this chapter establish it in Lean for the Wilson measure on a finite torus. The remaining pillars and the assembly are in 20-02–20-04.

02

Two mirrors and the closed positive half (Figure 1)

The lattice is $(\mathbb{Z}/N)^4$ with $N = 2M$. In this part the directions are $\mu = 0, 1, 2, 3$, with 0 as time. This follows the index convention of Lean on the strong-coupling side and shifts the indices relative to the directions 1–4 of the text of 19-01 (the conventions of 19-01). A link $\ell = (x, \mu)$ goes from x to $x + e_\mu$, and $U_\ell \in SU(2)$ is a unit quaternion (19-01). The weight is $w_\beta(U) = \exp\bigl(\beta \sum_p \operatorname{Re} U_p\bigr)$, and the statements of this chapter are written with the unnormalised Wilson measure $w_\beta \prod_\ell dU_\ell$ (the $\tilde{\mu}_{N,\beta}$ of 19-01). It differs from the normalised measure $\mu_{N,\beta}$ only by the positive constant factor $Z_N(\beta)$, so the sign of the inequalities is the same (19-01).

The reflection in the plane through sites Θ is $x_0 \mapsto -x_0$. Setting $\vartheta x := (-x_0, x_1, x_2, x_3)$,

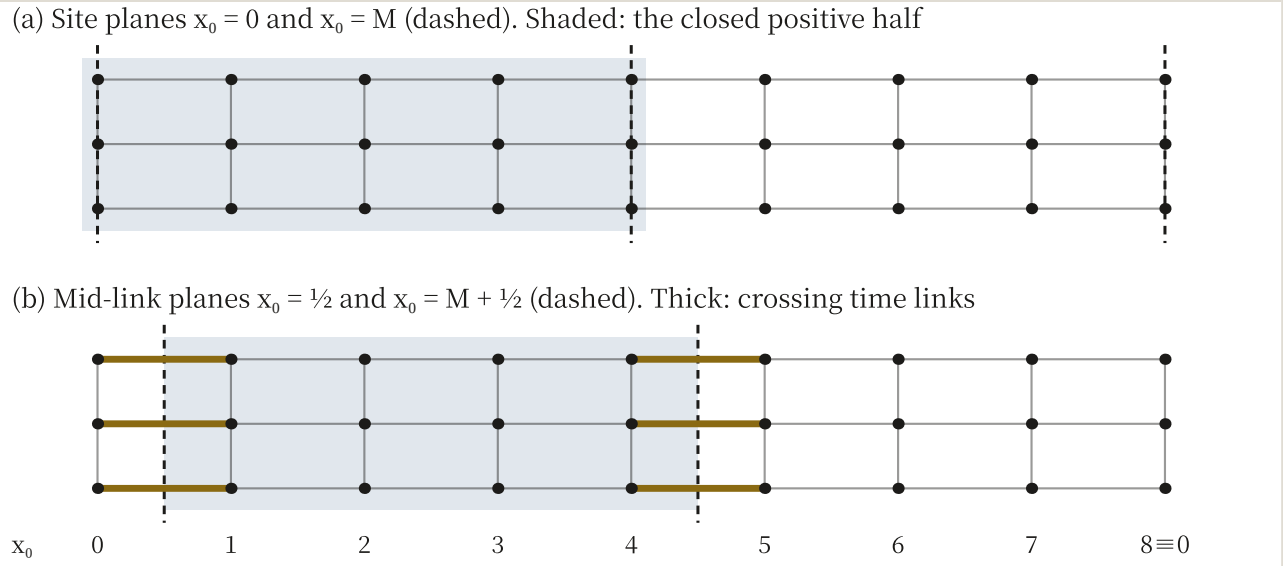
$$(\Theta U)_{(x,\mu)} = \begin{cases} U^*_{(\vartheta x - e_0,\, 0)} & (\mu = 0) \\ U_{(\vartheta x,\, \mu)} & (\mu \neq 0) \end{cases} \quad (1)$$

Time links reverse their direction, so they acquire an inverse (the quaternion conjugate $*$). Since $-M \equiv M$, the planes $x_0 = 0$ and $x_0 = M$ stay fixed. The **closed**

positive half Λ_+ consists of the spatial links at times $0, \dots, M$ and the time links whose base time is $0, \dots, M-1$.

The reflection in the plane through the middle of links Θ' is $x_0 \mapsto 1 - x_0$; its planes are $x_0 = \frac{1}{2}$ and $x_0 = M + \frac{1}{2}$. Time links with base time 0 or M **cross** the plane. The closed positive half Λ'_+ consists of the crossing links, the spatial links at times $1, \dots, M$, and the time links with base time $1, \dots, M-1$ (the positive side in 19-03).

Figure 1 A torus with $N = 8$ ($M = 4$) in the time direction, cut along time (horizontal) and one spatial direction (vertical, only 3 points). The shaded region is the closed positive half. (a) The spatial links on the plane belong to the positive half. (b) The thick crossing links straddle the plane; the spatial links at times 0 and 5 are on the negative side.



Proved here The figure draws (1) and the definitions in the text as they stand. Time 8 is the same point as 0 by periodicity.

03

Reflection positivity for the plane through sites — the statement

Statement (reflection positivity for the plane through sites)

Let $N = 2M$ and let β be any real number. If a bounded measurable real-valued function f depends only on the variables of the links of Λ_+, then

$$0 \;\le\; \int f(\Theta U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell . \qquad (2)$$

Cited A formalisation of Osterwalder–Seiler's reflection positivity for the plane through sites. Lean: `TorusRP.wilson_rp` (hypotheses: $N = 2M$; f measurable, bounded and depending only on Λ_+. β is any real number).

Meaning. The sign of β does not matter, because the sign is used nowhere in the proof (§ 04).

Example. Let C be a closed path in Λ_+, and take the real part of the Wilson loop, $f(U) = \operatorname{Re}\prod_{\ell\in C} U_\ell^{\pm 1}$ $(= \frac{1}{2}\operatorname{tr})$. f depends only on the links of Λ_+, satisfies $|f| \le 1$ and is continuous, so it meets the hypotheses. $f(\Theta U)$ is the real part of the Wilson loop of the mirrored path ϑC. Hence

$$\langle \operatorname{Re} W_{\vartheta C} \cdot \operatorname{Re} W_C \rangle_{N,\beta} \;\ge\; 0 \qquad (\text{all real } \beta) \quad (3)$$

Proved here (an application of (2); $\langle\cdot\rangle_{N,\beta}$ is the expectation of 19-01).

04

The route of proof — split the action, fix the plane, and get a square

The idea is the same as the "site reflection" of 5-03 § 04; in Lean it is divided into an abstract lemma and a concrete count.

Step 1: split the action into the positive side, the negative side and the plane

Give each plaquette p a weight $w(p)$: 1 for time plaquettes (plaquettes containing direction 0) with base time $0,\dots,M-1$ and for spatial plaquettes at times $1,\dots,M-1$; $\frac{1}{2}$ for spatial plaquettes at time 0 or M (plaquettes lying on the mirror plane); and 0 for the rest. Setting $S_+(U) := \sum_p w(p)\operatorname{Re} U_p$,

$$\sum_p \operatorname{Re} U_p \;=\; S_+(U) + S_+(\Theta U). \quad (4)$$

The mirror maps plaquettes to plaquettes, and the holonomy of a mapped plaquette is a cyclic shift of the conjugate, so its real part does not change. Plaquettes on the positive side are counted once, those on the negative side once as images, and those on the plane twice with a half each. Lean: `TorusRP.weight_factor` ($w_\beta = a(U)\,a(\Theta U)$, $a = e^{\beta S_+}$).

Step 2: make the integrand a pairing of a single function

$g := a\cdot f$ depends only on Λ_+ and is bounded. By (4), $f(\Theta U)\,f(U)\,w_\beta(U) = g(\Theta U)\,g(U)$; the weight disappears and only the product Haar measure remains. The sign of β has not been used so far.

Step 3: fixing the variables on the plane gives a perfect square

Divide the variables into the links on the plane U_0, the interior of the positive side $U_>$, and the negative side $U_<$. Writing $g(U) = G(U_0, U_>)$, we have $g(\Theta U) = G(U_0, V)$, where V is the negative-side variables carried to the positive side by (1). Haar measure is invariant under inversion, so V has the same distribution as $U_>$ and is independent of $U_>$. Therefore

$$\int g(\Theta U)\, g(U) \prod_{\ell} dU_\ell \;=\; \int dU_0 \left(\int G(U_0, U_>)\, dU_> \right)^2 \;\geq\; 0. \quad (5)$$

Lean: `LinkReflection.rp_core` (the abstract form) and `LinkReflection.rp_weighted` (the form with the weight factorised).

Detour: the six conditions of the abstract lemma

May be skipped. `rp_core` assumes the following about an involution θ on the index set, value maps σ_ℓ, and two predicates "on the plane" and "positive". (i) θ is an involution. (ii) Links on the plane are fixed by θ, and σ is the identity on them. (iii) Positive links are not on the plane. (iv) The image of a positive link is not positive. (v) The image of a link that is neither on the plane nor positive is positive. (vi) σ_ℓ preserves the measure on values. From these six, the space of variables is measure-preservingly isomorphic to "on the plane $\times$ positive $\times$ positive (images)", and (5) follows. On the torus, (ii), (iv) and (v) are where $N = 2M$ is used, and (vi) is the invariance of the Haar measure of SU(2) under inversion.

05

The plane through the middle of links — the roles of gauge invariance and β ≥ 0

Gauge transformations g and gauge-invariant functions are as in gauge transformations in 19-01 ($g(x) \in SU(2)$ at each site, $(gU)_{(x,\mu)} = g(x)\, U_{(x,\mu)}\, g(x + e_\mu)^*$).

Statement (reflection positivity for the plane through the middle of links)

Let $N = 2M$, $M \geq 2$ and $\beta \geq 0$. If a bounded measurable real-valued function F depends only on the variables of the links of Λ'_+ and is gauge invariant, then

$$0 \;\leq\; \int F(U)\, F(\Theta' U)\, w_\beta(U) \prod_{\ell} dU_\ell. \quad (6)$$

Cited A formalisation, in the form for gauge-invariant functions, of Osterwalder–Seiler's reflection positivity for link planes. Lean: `MidRP.wilson_rp_mid`

(hypotheses: $N = 2M$, $M \geq 2$, $\beta \geq 0$; F measurable, bounded, depending only on Λ'_+, and invariant under every g with unit values).

The proof has four steps.

Step 1: set the crossing links to 1 by a gauge transformation. The crossing links enter mutually distinct points at times 1 and $M+1$, and no crossing link leaves those points. Taking $g(x)$ to be the value of the crossing link entering that point makes every crossing link 1. For gauge-invariant H, $\int H(U) \prod dU = \int H(U^\circ) \prod dU$ (U° is the configuration with the crossing links set to 1). The change of variables is a skew product, and it preserves the measure because $V \mapsto aVb^*$ (a, b units) preserves Haar on each link. $M \geq 2$ is needed so that the starting points of the crossing links (times $0, M$) and the receiving points (times $1, M+1$) do not overlap. Lean: `GaugeFix.integral_gaugeFix`, `TorusGauge.wilson_gaugeFix`. This is applied to $H = F \cdot (F \circ \Theta') \cdot w_\beta$ (Θ' carries gauge transformations to gauge transformations of a different form, and w_β is gauge invariant).

Step 2: split the action into three parts. $\sum_p \operatorname{Re} U_p = A(U) + A(\Theta' U) + C(U)$, where A is the sum over plaquettes on the positive side and C the sum over crossing plaquettes (time plaquettes having a crossing link as an edge).

Step 3: put the crossing plaquettes in kernel form. If the crossing links are 1, the holonomy of a crossing plaquette with base time 0 is VW^*, involving only the spatial link V at time 1 (positive side) and the spatial link W at time 0 (at the position of the image of V). For a crossing plaquette with base time M, it has the same form with the link at time M (positive side) and the link at time $M+1$. Viewing unit quaternions as points of $\mathbb{R}^4$, $\operatorname{Re}(VW^*) = \sum_{c=0}^{3} V_c W_c$, so

$$C(U^\circ) \;=\; \sum_j \psi_j(U)\, \psi_j(\Theta' U) \tag{7}$$

can be written as a sum of pairs of positive-side coordinates ψ_j (the quaternion components of the spatial links at times 1 and M).

Step 4: a positive-definite kernel. If $\beta \geq 0$, then

$$e^{\beta \sum_j \psi_j(U)\psi_j(\Theta' U)} \;=\; \sum_{n \geq 0} \frac{\beta^n}{n!} \Big(\sum_j \psi_j(U)\psi_j(\Theta' U) \Big)^n \tag{8}$$

is a sum, with coefficients ≥ 0, of terms of the form $h(U)\, h(\Theta' U)$, and each term becomes the integral of a square as in (5) of § 04. Lean: `LinkReflection.rp_kernel`

(abstract reflection positivity with a kernel, $\beta \geq 0$).

The role of gauge invariance. It is for Step 1. A crossing link is mapped to itself by the mirror, so it cannot be fixed and squared as in § 04; instead it is removed by setting it to 1 with a gauge transformation. For a function that does not depend on the crossing links (a function depending only on the interior of the positive side), averaging over two independent gauge transformations and then applying (6) gives a form that does not require gauge invariance (Lean `FullT.full_rp_torus`: normalised measure, $\beta \geq 0$, $M \geq 2$).

The role of $\beta \geq 0$. Step 4 uses that the coefficients $\beta^n/n!$ are ≥ 0. For $\beta < 0$ the kernel $e^{\beta\langle V,W\rangle}$ is not positive definite on S^3: the matrix $\begin{pmatrix} e^{\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta} \end{pmatrix}$ built from the two points V, $-V$ has determinant $e^{2\beta} - e^{-2\beta}$, which is negative Proved here . This means the tool of the proof cannot be used; (6) says neither that the statement holds nor that it fails for $\beta < 0$.

06

Reflection positivity in four directions, and the symmetry of the pairing

Four directions. Let τ be the swap of directions 0 and ρ, and consider the relabelling of configurations $(P_\tau U)_{(x,\mu)} := U_{(x\circ\tau,\,\tau\mu)}$. The weight w_β is invariant under P_τ (for plaquettes whose orientation is reversed the holonomy is conjugated, and the real part is the same). $\Theta_\rho := P_\tau\,\Theta\,P_\tau$ is the reflection $x_\rho \mapsto -x_\rho$ in the site plane of direction ρ.

Statement (reflection positivity for site planes in four directions)

Let $N = 2M$, $\rho \in \{0,1,2,3\}$, and let β be any real number. If a bounded measurable real-valued f depends only on the closed positive half of direction ρ (links of direction ρ whose base has x_ρ in $0,\dots,M-1$, and links of other directions with x_ρ in $0,\dots,M$), then

$$0 \;\leq\; \int f(\Theta_\rho U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell\,. \qquad (9)$$

Cited A formalisation, extended to four directions, of reflection positivity for the plane through sites (of Osterwalder–Seiler type). Lean: `PeierlsPerm.wilson_rp_dir` (all ρ, all real β; the invariance of the measure is `PeierlsPerm.wilson_perm_invariant`).

The proof just applies (2) to $f \circ P_\tau$ and returns by the invariance of the measure. Four directions are needed for the chessboard estimate, which carries a function of one block to all blocks by the mirrors of the four directions and stacks Cauchy–Schwarz direction by direction (22-01).

Statement (symmetry of the reflection pairing)

Let $N = 2M$, let β be any real number, and let f, g be measurable real-valued functions.

$$\int f(\Theta U)\, g(U)\, w_\beta(U) \prod_\ell dU_\ell \;=\; \int g(\Theta U)\, f(U)\, w_\beta(U) \prod_\ell dU_\ell . \quad (10)$$

Cited A formalisation of a known identity following from reflection invariance. Lean: `TorusRP.wilson_refl_symm` (neither boundedness nor a support condition is needed; when not integrable, both sides are 0 by Lean's convention).

There are three ingredients: Θ is an involution, Θ preserves the product Haar measure, and w_β is invariant under Θ (using (4) twice). (10) is the symmetry of the pairing and corresponds to a field of the vessel of reconstruction (19-03).

07

When to use which of the three forms

Form	Range	Class of functions	What it is used for
Plane through sites (time)	All real β, $N = 2M$	Bounded measurable functions on Λ_+	Positive semi-definiteness of the pairing → a Hilbert space. The source of positivity for the infinite-volume limit measure and of the assembly of the vessel (20-04)
Plane through the middle of links	$\beta \geq 0$, $M \geq 2$	Gauge-invariant bounded measurable functions on Λ'_+	The source of the positivity $T \geq 0$ of the transfer operator (20-04; with a gauge average in between)
Site planes in four directions	All real β, all directions	Functions on the closed positive half of direction ρ	The chessboard estimate (22-01)
Symmetry of the pairing	All real β	Measurable functions	Symmetry of the pairing (a field of the vessel)

The site-plane form has the lightest conditions and cares about neither the sign of β nor gauge invariance. In exchange, on its own it gets the transfer operator only as far as the type $T^2 \geq 0$, and cannot give positivity of T itself (the two kinds of “positive” in 5-03 § 03). The middle-plane form pays with $\beta \geq 0$ and gauge invariance and gives

positivity of the type $T \geq 0$. The $T \geq 0$ inside the vessel (20-04, Lean `FullT.wilson_Tpos_full`) is obtained by inserting into (6) the Haar average over all gauge transformations of the torus, extending it to functions on the positive side that do not depend on the crossing links and need not be gauge invariant (`FullT.full_rp_torus`), and passing to the limit.

On the lattice with twisted boundary conditions, the same type of splitting (fix the plane and take a square) is used with the plane through sites (21-06). In Part 21 the role of time is played by direction 4, and that is a separate statement on the lattice of Part 21.

08

What is not said

- **The lattice spacing is fixed and the volume is finite.** Every statement is an equality or inequality on one $(\mathbb{Z}/2M)^4$. Positivity is a property that holds exactly at each volume, and no form uniform in the volume is needed. Nothing is said about the continuum limit.
- **Unnormalised measure.** It differs by a positive constant factor, so the sign is the same. Positivity for the infinite-volume limit measure is in the bundled statement of 20-04 and in the axioms of the vessel (20-04).
- **Real-valued functions only.** Complex-valued functions are outside the statements.
- **Middle plane:** nothing is said about $\beta < 0$, $M = 1$, or F that depends on the crossing links and is not gauge invariant.
- **Lean's convention for values.** In Lean the link values are taken in the closed unit ball of the quaternions, and the reference measure is the image of the Haar measure of SU(2) (the unit sphere). As statements about bounded measurable functions they are the same as statements on SU(2) (the conventions of 19-01).
- **Not a new result.** Reflection positivity of lattice gauge theory is a result of Osterwalder–Seiler; this chapter is its formalisation for the Wilson measure on a finite torus.

What this chapter can say

What is said	What is not said
Cited Reflection positivity for the plane through sites (2): $N = 2M$, all real β, bounded measurable f on Λ_+. Lean `TorusRP.wilson_rp`	The continuum limit or a change of lattice spacing; complex-valued f

What is said	What is not said
Cited Reflection positivity for the plane through the middle of links (6): $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F. Lean `MidRP.wilson_rp_mid`	$\beta < 0$, $M = 1$, F depending on the crossing links and not gauge invariant
Cited Reflection positivity for site planes in four directions (9): all directions ρ, all real β. Lean `PeierlsPerm.wilson_rp_dir`	A four-direction form of the middle plane
Cited Symmetry of the reflection pairing (10): all real β, measurable f, g. Lean `TorusRP.wilson_refl_symm`	—
Proved here Positivity of pairs of Wilson loops (3): an application of (2)	Estimates in terms of the size of the loop

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity of lattice gauge theory (site planes and link planes) and the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-03; references in A-03
Osterwalder–Schrader reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10
(2), (10) and their parts (splitting the action, abstract reflection positivity)	Lean	`TorusRP.wilson_rp`, `TorusRP.wilson_refl_symm`, `TorusRP.weight_factor`, `LinkReflection.rp_core`, `LinkReflection.rp_weighted`. sorry 0; the axioms are the three standard ones. The list of declarations is in 19-06
(6) and its parts (gauge fixing, reflection positivity with a kernel), the route to $T \geq 0$	Lean	`MidRP.wilson_rp_mid`, `TorusGauge.wilson_gaugeFix`, `GaugeFix.integral_gaugeFix`, `LinkReflection.rp_kernel`; `FullT.full_rp_torus`, `FullT.wilson_Tpos_full`
(9)	Lean	`PeierlsPerm.wilson_rp_dir`, `PeierlsPerm.wilson_perm_invariant`
Checks of the Gram matrix (Z_2 and Z_4 in two and three dimensions, sum over all configurations: positive semi-definite; as a positive control, removing the conjugation of the time links for Z_4 gives a relative smallest eigenvalue of −0.39 to −0.50), Metropolis for the middle plane $((\mathbb{Z}/4)^4$, SU(2); no sign of a negative covariance for pairs of gauge-invariant plaquette averages)	Reading	The numerical record of this book's study (computations copying the Lean definitions; not a substitute for proof)

Words used in this chapter — reflection positivity, plane through sites, plane through the middle of links, closed positive half, crossing link, gauge invariance, gauge fixing, positive-definite kernel, pairing, transfer operator, chessboard estimate. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-02 Decay of correlations at strong coupling — the second pillar of the vessel: it shows, inside the strong-coupling window, that the covariance of distant functions is small.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling Strong coupling ·
Kiichi, Shiori and Rin

Decay of correlations at strong coupling

> This chapter is this book's own investigation.

What this chapter shows — When $|\beta|$ is small enough (strong coupling), the covariance of two functions separated in the time direction, under the SU(2) Wilson measure on the torus $(\mathbb{Z}/N)^4$, becomes exponentially small with the distance. This estimate is machine-checked in Lean in two ways, and in neither do the constants contain the volume N. One is the Dobrushin window $18\tanh|\beta| < 1$, which comes from how much the conditional distribution of one link depends on the surrounding links (coefficient $\tanh|\beta|$, row sum at most $18\tanh|\beta|$). The other is the window $24(1 - e^{-4|\beta|}) < 1$, obtained by counting chains of connected plaquettes.

Prerequisite chapters — 19-01 (the Wilson measure), 20-01 (notation). For an introduction, 5-02 (strong coupling) and 5-04 (infinite volume and clustering).

Said first — The lattice spacing is fixed, and the statements hold only inside the strong-coupling windows. The windows are $|\beta| < 0.0556$ Reading and $|\beta| < 0.0106$ Reading, the opposite end from the weak coupling ($\beta \to \infty$) that the continuum limit needs. Nothing is said outside the windows (not even that there is no decay). The distance is only the distance in the time direction, and the statements are in finite volume for each N. Each is a formalisation of a known result of Osterwalder–Seiler or Dobrushin type.

The order of this chapter

1. Two windows, two statements — an overview
2. Expectations and the time distance — the setting
3. The influence one link receives from its surroundings — the coefficient tanh|β| and the count 18
4. Decay with distance in the Dobrushin window — the statement
5. The route of proof — washing out the oscillation of G with heat-bath sweeps
6. The other window — swapping two replicas, and chains of plaquettes
7. Comparing the two windows (Table 1)

8. What is not said

01

Two windows, two statements — an overview

For an introduction, see 5-02 § 01 and 5-04. At $\beta = 0$ the Wilson measure is a product of Haar measures, and different links are independent. When $|\beta|$ is small, the coupling between links is weak, and the correlation of distant links pays a small factor each time it "crosses one coupling", so it decreases exponentially with the distance. This is the decay of correlations at strong coupling; in lattice gauge theory it is a result of Osterwalder–Seiler via the cluster expansion Cited .

The two statements of this chapter give this picture form through two different accountings.

- **Decay with distance** (§ 04): count how much the conditional distribution of one link moves when one surrounding link is changed (Dobrushin's influence coefficients). The window is $18 \tanh |\beta| < 1$.
- **Finite-volume decay** (§ 06): prepare two independent replicas, expand the plaquette weights, and count chains of plaquettes connecting the two supports. The window is $24(1 - e^{-4|\beta|}) < 1$.

Both are finite-volume statements for each N, and the constants on the right-hand side do not contain N. This form, uniform in the volume, is the source of the infinite-volume limit (20-03) and of the mass gap of the vessel of reconstruction (20-04).

02

Expectations and the time distance — the setting

The lattice is the torus $(\mathbb{Z}/N)^4$ ($N \geq 1$); the link variables $U_\ell \in SU(2)$, the weight w_β and the normalised measure $\mu_{N,\beta}$ are as in 19-01. This chapter uses the expectation of 19-01,

$$\langle H \rangle_{N,\beta} := \int H \, d\mu_{N,\beta} = \frac{\int H \, w_\beta \prod_\ell dU_\ell}{\int w_\beta \prod_\ell dU_\ell} \quad (1)$$

and writes the covariance as $\mathrm{Cov}(F, G) := \langle FG \rangle_{N,\beta} - \langle F \rangle_{N,\beta} \langle G \rangle_{N,\beta}$. β is real and its sign does not matter (the windows are written in terms of $|\beta|$).

Time distance. Fix a time $t_0 \in \mathbb{Z}/N$, and use the cyclic distance between the base time x_0 of a link $\ell = (x, \mu)$ and t_0,

$$d(\ell) := \min_{k \in \mathbb{Z}} \left| x_0 - t_0 + kN \right| \qquad (0 \le d(\ell) \le N/2) \tag{2}$$

However far apart things are in the spatial directions, this does not enter d. Two links sharing a plaquette differ in d by at most 1 (because the base times of the four links of a plaquette fit in two consecutive times). This fact, that "the distance changes by at most 1 between neighbours", is where both proofs turn distance into an exponent.

Support of a function. A function F depends only on the links of a finite set A if F takes equal values on any two configurations that agree on A.

03

The influence one link receives from its surroundings — the coefficient tanh|β| and the count 18

With all links other than x fixed at a configuration ω, the conditional distribution of x (the heat-bath kernel of one link) is

$$\gamma_x(dU \mid \omega) = \frac{\exp\bigl(\beta \sum_{p \ni x} \operatorname{Re} U_p(\omega[x \mapsto U])\bigr)\, dU}{\int \exp\bigl(\beta \sum_{p \ni x} \operatorname{Re} U_p(\omega[x \mapsto U'])\bigr)\, dU'} \tag{3}$$

The sum runs only over plaquettes containing x. In four dimensions one link is contained in 6 plaquettes (two, one on each side, for each of the remaining three directions).

Influence coefficients. For two configurations ω, ω' differing only at another link y, measure the difference of the conditional distributions of x. Let n_{xy} be the number of plaquettes containing both x and y. The difference of the local actions $h(U)$ satisfies $|h| \le 2|\beta|\, n_{xy}$, and the difference of two probability measures whose densities are tilted by e^h is bounded by

$$\sup_S \left| \gamma_x(S \mid \omega') - \gamma_x(S \mid \omega) \right| \;\le\; \tanh\bigl(|\beta|\, n_{xy}\bigr) \tag{4}$$

Lean: `LinkSpec.spec_coeff` ($\tanh$ in the form $(e^{2a}-1)/(e^{2a}+1)$).

The count 18. From $\tanh(na) \le n \tanh a$ ($a \ge 0$, Lean `LinkSpec.tanhForm_le`) and the fact that each of the 6 plaquettes containing x has 3 links other than x, $\sum_{y \neq x} n_{xy} \le 18$. Hence the row sum satisfies

$$\alpha_0 := \sum_{y \not= x} \sup\bigl|\gamma_x(\cdot \mid \omega') - \gamma_x(\cdot \mid \omega)\bigr| \;\le\; 18 \tanh|\beta| \quad (5)$$

for every x and every N. Lean: `LinkSpec.dobrushin_row`. For $N \ge 3$ there are 18 neighbours, all with $n_{xy} = 1$; for $N = 2$ there are 15 neighbours with $\sum n_{xy} = 18$ Reading (a check of the count).

Meaning. $\alpha_0 < 1$ says that "the distribution of one link is moved, by all the surrounding links together, by less than 1 in total variation". This is the form of Dobrushin's uniqueness condition, and the window $18 \tanh|\beta| < 1$ is $|\beta| < \operatorname{artanh}(1/18) \approx 0.0556$ Reading . As an explicit sufficient condition, $|\beta| < 1/19$ is in Lean (`WilsonLimit.dob_window_strict`).

04

Decay with distance in the Dobrushin window — the statement

Statement (Dobrushin decay of covariance with distance)

Let $N \ge 1$, β real, and $\lambda \ge 0$ with $e^{\lambda} \cdot 18 \tanh|\beta| < 1$. Let $t_0 \in \mathbb{Z}/N$, $r \in \mathbb{N}$, and let the finite set of links A satisfy $d(\ell) \le r$ for all its links. Let F be bounded ($|F| \le c_F$), measurable and depending only on A, and G bounded and measurable. Let $\delta_y \ge 0$ be an upper bound on "the change in G when only link y is changed", that is, $|G(\omega) - G(\omega')| \le \delta_y$ for all ω, ω' agreeing outside y. Then

$$\bigl|\operatorname{Cov}(F, G)\bigr| \;\le\; 2\, c_F\, e^{\lambda r} \sum_{y} e^{-\lambda\, d(y)}\, \delta_y. \quad (6)$$

Cited A formalisation, in sweep form, of Dobrushin's comparison type. Lean: `LinkSpec.wilson_cov_dist` (hypotheses as above; the constants on the right-hand side do not contain N).

Two functions with separated supports. If G depends only on the links of a finite set A', $|G| \le c_G$, and $d \ge r + 2 + k$ for all links of A', one can take $\delta_y = 2c_G$ ($y \in A'$) and $\delta_y = 0$ (otherwise), and

$$\bigl|\operatorname{Cov}(F, G)\bigr| \;\le\; 4\, c_F\, c_G\, |A'|\, e^{-\lambda(k+2)} \quad (7)$$

Proved here (an application of (6); an application of the same type is inside Lean's `TorusRP.wilson_hdec_torus_dob`). Any λ below $-\ln(18 \tanh|\beta|)$ will do; for example, for $\beta = 0.001,\ 0.005,\ 0.01$ that upper limit is $4.02,\ 2.41,\ 1.71$ Reading .

Points to note in the statement. δ is constrained by hypothesis to be at least the true change of G, so it cannot be taken small to make a false claim. If G is constant, $\delta \equiv 0$ and both sides are 0. On the right-hand side, N appears only in the range of the sum and inside the cyclic distance; if the support of δ does not depend on N, the estimate is uniform in N.

05

The route of proof — washing out the oscillation of G with heat-bath sweeps

The idea is "flatten G without touching F". It is divided into five steps.

Step 1: the heat bath preserves the measure. For the heat-bath operator $P_x H(\omega) := \int H(\omega[x \mapsto U])\, \gamma_x(dU \mid \omega)$, $\langle P_x H\rangle_{N,\beta} = \langle H\rangle_{N,\beta}$ (because the conditional distribution of one link is (3); Lean `LinkSpec.hbSweep_invariant` and its iteration `LinkSpec.iter_invariant`).

Step 2: sweep while avoiding A. Let L be a sequence listing every link outside A once, and let $Q := \prod_{x \in L} P_x$ (a sweep). Since F depends only on A, P_x ($x \notin A$) passes through with F still multiplied, and for all K

$$\langle F \cdot Q^K G\rangle_{N,\beta} = \langle FG\rangle_{N,\beta}, \qquad \langle Q^K G\rangle_{N,\beta} = \langle G\rangle_{N,\beta}. \tag{8}$$

Lean: `LinkSpec.iter_mul_invariant`, `iter_invariant`.

Step 3: the flow of oscillation. The upper bounds δ on the link-by-link change of H move, under one application of P_x, as $\delta_x \mapsto 0$, $\delta_y \mapsto \delta_y + \delta_x C_{xy}$ (C_{xy} is the coefficient of (4)). Attaching the weights $\rho_y := e^{-\lambda d(y)}$, since d differs by only 1 between neighbours,

$$\sum_{y \neq x} \rho_y\, C_{xy} \;\le\; \left(e^{\lambda} \cdot 18 \tanh|\beta|\right) \rho_x =: \alpha\, \rho_x \tag{9}$$

(Lean `LinkSpec.row_weighted`). With $a := \sum_{y \in A} \rho_y \delta_y$ and $b := \sum_{y \notin A} \rho_y \delta_y$, one sweep shrinks b by a factor of at most α, and δ on A does not decrease. After K sweeps, $b_K \le \alpha^K b_0$ (Lean `Sweep.iter_bound`). The key point is to start the sweep with A treated as "already processed"; then b does shrink even though a grows by what flows into A.

Step 4: from oscillation to covariance. If $|H - c| \le S$ and $|F| \le c_F$, then $|\operatorname{Cov}(F,H)| \le 2 c_F S$ (Lean `LinkSpec.cov_le`). The total oscillation of $H = Q^K G$ is bounded by $\sum_y \delta_y^{(K)}$, and $\rho \ge e^{-\lambda r}$ on A, so

$$\sum_y \delta_y^{(K)} \;\le\; e^{\lambda r} \sum_y \rho_y \delta_y \;+\; \Big(\sum_y \rho_y^{-1}\Big)\, \alpha^K b_0 . \tag{10}$$

Step 5: $K \to \infty$. Since $\alpha < 1$ and N is fixed, the second term goes to 0, and together with (8) this gives (6). $\sum_y \rho_y^{-1}$ depends on N, but it vanishes in the limit and does not remain on the right-hand side.

Dobrushin's uniqueness theorem itself is not used as a tool. The heat-bath sweeps follow its comparison argument on the finite torus.

A numerical check Reading On a two-dimensional torus of Z_2 gauge theory (time $6 \times$ space 2, sum over all configurations), the ratio of the covariance of pairs of spatial Polyakov loops to the estimate of the form (6) was 6.2×10^{-4} (separation 1) to 3.1×10^{-8} (separation 3) at $\beta = 0.03$, and 3.1×10^{-3} to 3.8×10^{-6} at $\beta = 0.15$, all within the estimate (the numerical record of this book's study). This is not a check on four-dimensional SU(2) itself, and whether the estimate is sharp was not examined.

06

The other window — swapping two replicas, and chains of plaquettes

Statement (finite-volume decay of correlations at strong coupling)

Let $N \ge 1$ and $c := 1 - e^{-4|\beta|}$ with $24c < 1$. Let $t_0 \in \mathbb{Z}/N$, $r, k \in \mathbb{N}$, and let the finite sets of links A, A' satisfy $d \le r$ for all links of A and $d \ge r + 2 + k$ for all links of A'. Let F ($|F| \le c_F$) depend only on A and G ($|G| \le c_G$) only on A', both measurable. Then

$$\big|\mathrm{Cov}(F, G)\big| \;\le\; 2\, c_F\, c_G \cdot 6|A| \cdot \frac{c\,(24c)^k}{1 - 24c} . \tag{11}$$

Cited A formalisation of the type of Osterwalder–Seiler's strong-coupling cluster expansion. Lean: `TorusRP.wilson_strong_coupling` (the window literally as $24(1 - (e^{2|\beta|})^{-2}) < 1$; the constants $2c_F c_G$, $6|A|$, $c/(1 - 24c)$ and the rate $24c$ do not contain N).

The window is $|\beta| < \frac{1}{4} \ln \frac{24}{23} \approx 0.0106$ Reading . The proof has four steps.

Step 1: shift the plaquette weights to be at least 1. With $w_p := e^{\beta \,\mathrm{Re}\, U_p + |\beta|} \in [1, e^{2|\beta|}]$, $\prod_p w_p$ is a positive constant multiple of the Wilson weight, and the expectation (1) does not change.

Step 2: two replicas. Taking two independent configurations U, V, $2Z^2\,\mathrm{Cov}(F,G) = \iint (F(U)-F(V))(G(U)-G(V))\,W(U)W(V)$. Expand $W(U)W(V) = \prod_p (1+\sigma_p)$, $\sigma_p := w_p(U)w_p(V) - 1 \geq 0$, into a sum over sets of plaquettes Γ.

Step 3: separable sets vanish. Γ is called separable if there is a set of links T containing A and disjoint from A' such that every plaquette of Γ lies either inside T or outside T (does not straddle the edge of T). Then swapping the two replicas on T reverses the sign of that term, and these cancel in the sum. What remains are terms containing a minimal set of plaquettes π connecting A and A', and from $\sigma_p/(1+\sigma_p) \leq 1 - e^{-4|\beta|} = c$,

$$\big|\mathrm{Cov}(F,G)\big| \;\leq\; 2c_F c_G \sum_{\pi} c^{|\pi|} \tag{12}$$

This step holds for all β and all N (Lean `SwapDecay.covBound_minimal`, applied in `TorusRP.wilson_covariance`). The sum on the right is not uniform in the volume from this step alone; making it uniform is the counting of the next step.

Step 4: count the chains. A minimal π contains a sequence of plaquettes, consecutive ones sharing a link, from a plaquette touching A to a plaquette touching A' (the counting lemma derives an upper bound from the length of this sequence and the degree of plaquettes). Since 6 plaquettes contain a given link, at most 24 plaquettes (including itself) share support with a given plaquette, and at most $6|A|$ plaquettes touch A. The time distance of neighbouring plaquettes differs by only 1, so a chain has at least $k+1$ plaquettes. There are at most $6|A|\cdot 24^m$ chains of length $m+1$, so $\sum_\pi c^{|\pi|} \leq 6|A| \sum_{m\geq k} c^{m+1} 24^m = 6|A|\, c(24c)^k/(1-24c)$, which is (11) (Lean `TorusRP.card_link_plaqs`, `plaq_degree`, `card_touch_le`, `height_lip`, `SwapDecay.minimalNonsep_sum_le`).

The number 4 comes from twice (two replicas) the range $2|\beta|$ of $\beta\,\mathrm{Re}\,U_p$, and 24 from 4×6. 24 is an upper bound; the true maximum number of plaquettes sharing support with one plaquette is 21 for $N \geq 3$ and 19 for $N = 2$ Reading .

07

Comparing the two windows (Table 1)

	Decay with distance (6)	Finite-volume decay (11)
Window	$e^{\lambda} \cdot 18 \tanh|\beta| < 1$ ($\lvert\beta\rvert < 0.0556$ as $\lambda \to 0$ Reading)	$24(1-e^{-4\lvert\beta\rvert}) < 1$ ($\lvert\beta\rvert < 0.0106$ Reading)
Basis of the accounting	Row sum of one link's influence, $18 \tanh\lvert\beta\rvert$	Number of chains of plaquettes 24^m and weight c^{m+1}

	Decay with distance (6)	Finite-volume decay (11)
Assumption on G	Link-by-link change δ_y (no support needed)	Support A' at time distance at least $r + 2 + k$
Rate (per unit of separation)	Any λ below $-\ln(18 \tanh \lvert\beta\rvert)$	$-\ln(24c)$
Rate at $\beta = 0.001$ Reading	Upper limit 4.02	2.35
Rate at $\beta = 0.005$ Reading	Upper limit 2.41	0.74
Rate at $\beta = 0.01$ Reading	Upper limit 1.71	0.061
Lean	`LinkSpec.wilson_cov_dist`	`TorusRP.wilson_strong_coupling`

Table 1 Comparison of the two statements. The rates are floating-point computations.

The window is about 5.2 times wider on the Dobrushin side Reading , and the upper limit of the rate at the same β is also larger. The advantage of the finite-volume decay is that it handles two functions with separated supports directly, without measuring the link-by-link change of G.

A sense of strength Reading : (11) becomes smaller than the trivial upper bound $c_F c_G$ (Cauchy–Schwarz), with $|A| = 1$, for $k \geq 2$ at $\beta = 0.008$ and for $k \geq 35$ at $\beta = 0.01$. At the edge of the window the estimate has almost no content. On small Z_2 lattices, the true covariance was 10^{8} to 10^{22} times smaller than (11) (the numerical record of this book's study).

In the later chapters of this part, the infinite-volume limit (20-03) and the mass gap (20-04) mainly use the Dobrushin window.

08

What is not said

- **Nothing is said outside the windows.** Not even that the decay stops. The windows are only the small end of $|\beta|$, the opposite of the weak coupling ($\beta \to \infty$) needed for the continuum limit. The lattice spacing is fixed.
- **Only distance in the time direction.** The d in (6) and (11) is the cyclic distance of the time coordinate. Clustering in four directions is a separate statement about the infinite-volume limit measure (20-03).
- **Finite volume for each N.** We say that the constants do not contain N, but the infinite-volume measure itself is not in this chapter.

- **The estimates are not sharp.** (11) contains $|A|$ and is almost trivial at the edge of the window. In (6) one supplies δ oneself.
- **This is not the mass gap itself.** Going from decay of covariance to the mass gap of the vessel of reconstruction needs reflection positivity (20-01) and the assembly of the limit (20-04).
- **Not a new result.** Decay of correlations at strong coupling is a known result of the type of Osterwalder–Seiler's cluster expansion and Dobrushin's uniqueness; this chapter is its formalisation for the SU(2) Wilson measure on a finite torus.

What this chapter can say

What is said	What is not said
Cited Dobrushin decay with distance (6): $e^{\lambda} \cdot 18 \tanh \lvert\beta\rvert < 1$, no N in the constants. Lean `LinkSpec.wilson_cov_dist`	Outside the window, distance in spatial directions, the infinite-volume measure itself
Proved here The form for functions with separated supports (7): an application of (6)	—
Cited Finite-volume decay (11): $24(1 - e^{-4\lvert\beta\rvert}) < 1$, time distance $r + 2 + k$, no N in the constants. Lean `TorusRP.wilson_strong_coupling`	Outside the window, distance in spatial directions, a form independent of $\lvert A\rvert$
Reading The window values 0.0556 and 0.0106, the rates of Table 1, the separations below the trivial bound, the comparison on Z_2	Not proofs of upper or lower bounds. The true value of the decay for four-dimensional SU(2)

Sources and reproduction

Item	Kind	Source / tools
The strong-coupling cluster expansion, exponential decay of correlations	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-02, 5-04, 7-04
Dobrushin's uniqueness and comparison conditions	Cited	R. L. Dobrushin, Theory Probab. Appl. 13 (1968) 197 (bibliographic data only, not checked). The place of the tool is in 16-10
(4), (5), (6) and their parts (sweeps, the flow of oscillation, the covariance bound)	Lean	`LinkSpec.spec_coeff`, `tanhForm_le`, `dobrushin_row`, `row_weighted`, `cov_le`, `iter_mul_invariant`, `iter_invariant`, `wilson_cov_dist`; `Sweep.iter_bound`; `WilsonLimit.dob_window_strict`. sorry 0; the axioms are the three standard ones. The list of declarations is in 19-06

Item	Kind	Source / tools
(11), (12) and their parts (two replicas, counting plaquettes)	Lean	`SwapDecay.covBound_minimal`, `minimalNonsep_sum_le`; `TorusRP.wilson_covariance`, `card_link_plaqs`, `plaq_degree`, `card_touch_le`, `height_lip`, `wilson_strong_coupling`
Values of the windows and rates, the plaquette counts (19, 21), the separations below the trivial bound	Reading	The numerical record of this book's study (floating point; the plaquette counts by exhaustive enumeration copying the Lean definitions)
Comparison with the true covariance on small Z_2 tori	Reading	The numerical record of this book's study (sum over all configurations; two and three dimensions; no violations)

Words used in this chapter — strong coupling, covariance, time distance, heat bath, influence coefficient, Dobrushin's condition, sweep, two replicas, chain of plaquettes, cluster expansion. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-03 The infinite-volume limit — no subsequence, translation invariance, clustering — from the decay of this chapter, uniform in N, it builds the limit measure without passing to a subsequence.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling The infinite-volume limit · Kiichi, Shiori and Rin

The infinite-volume limit — no subsequence, translation invariance, clustering

> This chapter is this book's own investigation. The explanation of Dobrushin's condition in § 03 introduces established knowledge.

What this chapter shows — In the strong-coupling window $18 \tanh |\beta| < 1$, $\beta \neq 0$: the sequence of Wilson measures on the tori $(\mathbb{Z}/2(n+1))^4$, lifted periodically to $\mathbb{Z}^4$, converges weakly to a single probability measure ν_β without passing to a subsequence. The sequence of all sizes $(\mathbb{Z}/(m+1))^4$ has the same limit. ν_β is invariant under unit translations in the four directions. The limit measure has the clustering property in the four directions (correlations of distant observables decrease exponentially). Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-01 The Wilson measure, 19-03 The infinite-volume configuration space and lifts, 20-02 Decay of correlations at strong coupling. The infinite-volume limit and the cluster expansion as established knowledge are in 5-04.

Said first — The lattice spacing is fixed at 1. The window is only $|\beta| < 0.0556$ Reading , and nothing is said about the continuum limit or weak coupling. The limit is a statement about “the sequence of periodic lifts”; nothing is said about other boundary conditions or about uniqueness of the Gibbs measure in the DLR sense. The “unique” part of the statement is trivial from the topology; the content is the convergence without a subsequence.

The order of this chapter

1. The setting — configuration space, periodic lifts, weak convergence: which sequence converges, and in what sense.
2. Four theorems: the theorem on the limit without a subsequence, the theorem on the limit independent of size, translation invariance of the limit, clustering at limit points.
3. Dobrushin's condition — the influence of one link on its neighbours: where the number 18 and the tanh come from.

4. Comparing two sizes — the difference is exponentially small in the distance to the boundary: the core of the proof.
5. From a Cauchy sequence to the limit: cylinder functions determine the measure, and compactness.
6. Translation invariance and clustering: carrying properties of finite boxes over to the limit.
7. Detour: what “unique” says: the uniqueness of $\exists!$, and the content of the convergence.
8. What is not said: the window, boundary conditions, the continuum limit.

01

The setting — configuration space, periodic lifts, weak convergence

The objects are those defined in 19-03. Only the main points are repeated here.

Configuration space. Let $\Lambda = \mathbb{Z}^4 \times \{0,1,2,3\}$ be the set of all links of $\mathbb{Z}^4$; placing one element of SU(2) (a unit quaternion) on each link gives a configuration. The configuration space $\mathcal{C}$ carries the product topology. Each factor is compact, so $\mathcal{C}$ is compact and metrisable. On the Lean side the values are taken in the closed ball $\{q : |q| \le 1\}$, but the measures live on the unit sphere (SU(2)) (the conventions of 19-01).

Periodic lift. From a configuration ω on the torus $(\mathbb{Z}/N)^4$, build the configuration $x_\ell = \omega_{\ell \bmod N}$ on $\mathbb{Z}^4$. Write the probability measure obtained by pushing the normalised Wilson measure (19-01) forward by this map as

$$\mu_n = \mathrm{lift}_*\, P_{N_n,\beta}, \qquad N_n = 2(n+1) \qquad (n = 0,1,2,\dots) \quad (1)$$

Its Lean name is `liftProb β n`. We also use the lift of the torus of every size $m+1$, `liftS β m`.

Weak convergence. A sequence of probability measures μ_n converges weakly to ν if $\int F\, d\mu_n \to \int F\, d\nu$ for every bounded continuous function F on $\mathcal{C}$. In Lean this corresponds to `Tendsto … atTop (𝓝 ν)` in the topology of Mathlib's space of probability measures.

Cylinder functions and supports. A function determined only by the values on a finite set of links S is called a cylinder function with support S. The proof is first carried out for cylinder functions and is finally transferred to measures.

The window and the constant. Throughout this chapter we set

$$q(\beta) = 18 \tanh |\beta| \qquad (\text{in Lean, } \texttt{dobConst}\ \beta)$$ (2)

The window $q(\beta) < 1$ is the same as $|\beta| < \operatorname{artanh}(1/18) = 0.055613$ Reading . The explicit sufficient condition $|\beta| < 1/19 \Rightarrow q(\beta) < 1$ is in Lean as `WilsonLimit.dob_window_strict`.

02

Four theorems

The theorem on the limit without a subsequence

Cited A formalisation of a result of Dobrushin-uniqueness type. Lean: `InfVol.wilson_infvol_unique` ($\beta \neq 0$, $q(\beta) < 1$).

Let $\beta \neq 0$ and $18 \tanh |\beta| < 1$. Then there is a unique probability measure $\nu\beta$ such that the sequence μ_n (all n, without passing to a subsequence) converges weakly to $\nu\beta$.

The theorem on the limit independent of size

Cited A formalisation of a result of the same type. Lean: `AllN.wilson_infvol_allN` ($\beta \neq 0$, $q(\beta) < 1$).

Under the same hypotheses, the limit ν of the sequence μ_n of even sizes is also the limit of the sequence of all sizes `liftS β m` (the tori $(\mathbb{Z}/(m+1))^4$). Moreover, for any sequence of natural numbers s with $s(k) \to \infty$, `liftS β (s k)` also converges weakly to the same ν.

Translation invariance of the limit

Cited A formalisation. Lean: `TransI.wilson_infvol_translation` ($\beta \neq 0$, $q(\beta) < 1$).

The $\nu\beta$ above is invariant under the unit translations $x \mapsto x(\cdot + e_i)$ in the four directions $i = 0, 1, 2, 3$: for every bounded continuous F,

$$\int F(x(\cdot + e_i))\, d\nu\beta(x) = \int F\, d\nu\beta.$$ (3)

Clustering at limit points

Cited A formalisation of a result of the type of exponential mixing under Dobrushin's condition. Lean: `WilsonLimit.cluster_limit` ($\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$; ν is the weak limit of some subsequence of μ_n).

Fix a direction i. Let F, G be bounded continuous cylinder functions such that, for every link of the supports S_F, S_G, the i-th coordinate of its base point lies in $[0, R]$ (the other coordinates are arbitrary). Let G_k be G shifted by k in direction i, $G_k(x) = G(x(\cdot + k e_i))$. Then for $k \geq R$

$$\left| \int F\, G_k \, d\nu - \int F \, d\nu \int G_k \, d\nu \right| \;\leq\; 4 \, \|F\|_\infty \|G\|_\infty \, |S_G| \, e^{-\lambda (k - R)}. \tag{4}$$

The constants do not depend on k, ν, or the other coordinates. The form with $\lambda = -\frac{1}{2} \log q(\beta)$ for ν_β is included in the bundled statement (20-04).

03

Dobrushin's condition — the influence of one link on its neighbours

For an introduction, see 5-04. On a compact configuration space, a sequence of measures always has a convergent subsequence. But the limit need not be unique. To pin it down, one must say that distant influences are weak. Dobrushin's condition measures this “one link at a time”.

Influence coefficients. The conditional distribution of a link x (the distribution of x when all other links are fixed) is determined only by the weights $e^{\beta \operatorname{Re} U_p}$ of the plaquettes containing x. Let C_{xy} be an upper bound on how much the conditional distribution of x moves in total variation when the value of another link y is changed. When x and y share n_{xy} plaquettes, Lean takes

$$C_{xy} = \tanh\bigl(|\beta| \, n_{xy}\bigr) \tag{5}$$

(`coef`). This is because the difference in the local action of x when y is changed lies in $[-2|\beta|, 2|\beta|]$ per shared plaquette (width $4|\beta|$), and tilting an exponential weight by a width w moves the total variation by at most $\tanh(w/4)$ Proved here .

Row sum. In four dimensions one link lies in 6 plaquettes, and each plaquette has 3 other links. So $\sum_y n_{xy} = 18$, and from $\tanh(|\beta| n) \leq n \tanh|\beta|$

$$\sum_{y \neq x} C_{xy} \;\leq\; 18 \tanh|\beta| = q(\beta) \tag{6}$$

holds for every size of torus Proved here . Dobrushin's condition is that this row sum is less than 1, and the window $q(\beta) < 1$ reads directly as “the total influence of one link on all its neighbours is less than 1”.

Weighted form. The proof uses the row sum with weights $\rho_y = e^{\lambda d(y)}$ in the distance d, $\sum_{y \neq x} \rho_y C_{xy} \leq \alpha \rho_x$, $\alpha = e^{\lambda} q(\beta)$. Since d differs by only 1 between two links sharing a plaquette, the weight changes by at most a factor e^{λ} Proved here . The role of the window is that one can take $\lambda > 0$ while keeping $\alpha < 1$.

04

Comparing two sizes — the difference is exponentially small in the distance to the boundary

The core of the proof is that "the expectations on two tori of different sizes are both close to a single number independent of N". We divide it into steps.

Step 1: heat-bath sweeps. On the torus, the operation of redrawing the value of link x from its conditional distribution is called the heat bath P_x. The Wilson measure is invariant under each P_x. Let Q be the operation of redrawing, one at a time in order, the links of a box A around the origin (links whose base point has every coordinate $|\cdot| \leq L$), and repeat Q K times. The oscillation vector δ_y of a function f (the upper bound on the change in f when only y is changed) is carried by the influence coefficients C at each sweep. Measured with the weight $e^{\lambda d(y)}$ (d is the maximum cyclic distance from the origin), the part inside the box shrinks by a factor of at most α each time, and the part leaking outside the box is divided by the weight $e^{\lambda(L+1)}$. The conclusion is that for every configuration ω

$$\left| E_N f - (Q^K f)(\omega) \right| \;\leq\; \left(\alpha^K + e^{-\lambda(L+1)} \right) \sum_y e^{\lambda d(y)} \delta_y \qquad (7)$$

(`LinkSpec.torus_box`). N does not enter the right-hand side.

Step 2: inside the box it is the same as $\mathbb{Z}^4$. If $2(L+1) < N$, the links inside the box $L+1$ map one-to-one under projection to the torus, and plaquettes containing links inside the box do not wrap around. So for F depending only on the links of the box $L+1$, the lift of the sweep on the torus agrees with the same sweep on $\mathbb{Z}^4$: $Q_N^K (F \circ \mathrm{lift}) = (Q_{\mathbb{Z}}^K F) \circ \mathrm{lift}$ (`ZHeat.iter_lift`). The function $Q_{\mathbb{Z}}^K F$ on the right does not depend on N.

Step 3: comparison with a number independent of N. Apply Steps 1 and 2 to a bounded continuous cylinder function F whose support S lies in the box R_0. If $L \geq R_0$ and $n \geq L+1$, then

$$\left| \int F \, d\mu_n - c_{K,L}(F) \right| \;\leq\; \left(\alpha^K + e^{-\lambda(L+1)} \right) \cdot 2 \|F\|_\infty \, |S| \, e^{\lambda R_0} \qquad (8)$$

holds (`InfVol.lift_close`; $\lambda \geq 0$, $\alpha \leq 1$, $\beta \neq 0$). The number compared with, $c_{K,L}(F) = (Q^K_{\mathbb{Z}} F)(\mathbf{0})$, does not depend on n. $\mathbf{0}$ is the configuration in which every link has value 0 (the centre of the closed ball). Any comparison point would do; all that is needed is that it does not depend on n. It need not even be an SU(2) configuration.

Step 4: the difference between two sizes. Applying (8) to two values $n, n' \geq L+1$, subtracting, and letting $K \to \infty$ ($\alpha < 1$) gives

$$\left| \int F \, d\mu_n - \int F \, d\mu_{n'} \right| \;\leq\; 4\|F\|_\infty \, |S| \, e^{\lambda R_0} \, e^{-\lambda(L+1)} \qquad (L \geq R_0,\ L+1 \leq \min(n, n'))$$

Proved here . L is "the distance from the support to where the two tori begin to look different". The difference is exponentially small in this distance; this is what the heading means. This step is a corollary for reading and is not in the Lean chain. The proof of convergence (Step 5) goes directly from (8).

05

From a Cauchy sequence to the limit

Step 5: a Cauchy sequence. Taking $\lambda = -\frac{1}{2}\log q(\beta)$ gives $\alpha = q(\beta)^{1/2} < 1$. For a given ε, make α^K and $e^{-\lambda j}$ small and set $L = \max(R_0, j)$; then by (8) the sequence of expectations $\int F \, d\mu_n$ of a cylinder function is a Cauchy sequence (`InfVol.cauchy_cyl`). The real numbers are complete, so each cylinder function has a limit.

Step 6: cylinder functions determine the measure. If two finite measures on $\mathcal{C}$ give the same integral for every bounded continuous cylinder function, they are equal (`CylDet.ext_of_cyl`). The marginals on finitely many links agree, and by the uniqueness of projective limits (Mathlib) the whole measures agree.

Step 7: convergence without a subsequence. $\mathcal{C}$ is compact and metrisable, so the space of probability measures is also compact. Every subsequence has a convergent subsequence, whose limit agrees with the limit of Step 5 on cylinder functions, and so is the same measure by Step 6. Since every subsequence has a sub-subsequence converging to the same limit, the sequence itself converges (`CylDet.tendsto_of_cyl_cauchy`).

All sizes. The estimate (8) does not use the parity of the size, and holds for the lifts of every size $m+1$ with the same number $c_{K,L}(F)$ in between (`AllN.lift_closeS`). So the sequence of all sizes, and any partial sequence of sizes, have the same limit Proved here .

Step	What it says	Lean name
1	The estimate for box sweeps (7), no N in the constants	`LinkSpec.torus_box`

Step	What it says	Lean name
2	For $2(L+1) < N$, the sweep inside the box agrees with $\mathbb{Z}^4$	`ZHeat.iter_lift`
3	Comparison with a number independent of N (8)	`InfVol.lift_close`
5	Expectations of cylinder functions are Cauchy	`InfVol.cauchy_cyl`
6	Cylinder functions determine the measure	`CylDet.ext_of_cyl`
7	From Cauchy to convergence of the sequence	`CylDet.tendsto_of_cyl_cauchy`

As an example showing that the hypotheses can be met, the case $\beta = 1/20$ is in Lean as `InfVol.infvol_example` ($18 \tanh(1/20) = 0.8993$ Reading).

06

Translation invariance and clustering

Translation invariance. The Wilson measure on each torus is invariant under unit translations of the torus. The lift commutes with translations, so each μ_n is invariant under unit translations of $\mathbb{Z}^4$ (`TransI.lift_shiftI_invariant`). If F is bounded continuous, so is $F(x(\cdot + e_i))$, and taking the limit of both sides under weak convergence gives (3) (`TransI.limit_shiftI_invariant`). The reverse translations are also bijections, so ν_β is invariant under all translations of $\mathbb{Z}^4$ Proved here .

Clustering. The covariance estimate on the torus (the Dobrushin distance form of 20-02) uses about the distance d only that "it changes by at most 1 between two links sharing a plaquette". So replace d by the cyclic distance of the i-th coordinate of the base point (`LinkSpec.coordLip`). The support of F is in $d \le R$ and that of the shifted G_k in $d \ge k$, and if the torus is large enough ($n \ge k + R$)

$$\left|\mathrm{Cov}_{P_N}(F, G_k)\right| \;\le\; 2\|F\|\, e^{\lambda R}\, |S_G|\, e^{-\lambda k} \cdot 2\|G\|$$

Since $F \cdot G_k$, F and G_k are all bounded continuous, the estimate passes to the weak limit and becomes (4) Proved here . Whether the limit point is unique is not used.

Example. For $\beta = 1/20$, $q = 0.8993$ and $\lambda = -\frac{1}{2}\log q = 0.0531$ Reading . The exponential factor $e^{-\lambda(k-R)}$ on the right-hand side of (4) becomes 0.1 when $k - R \approx 43$ Reading . The closer to the edge of the window, the slower the decrease. There is no record of the ratio to the true decrease of correlations.

Detour: what "unique" says

May be skipped. The Lean statement has the form $\exists!\, \nu,\ \mu_n \to \nu$. The topology of weak convergence on the space of probability measures is Hausdorff, so the limit of one sequence, if it exists, is always unique. So the "unique" part is trivial from the topology and adds no

information. The content of the statement is that “the sequence itself converges, without passing to a subsequence”.
Compactness alone only gives “there is a convergent subsequence”. In that case different subsequences could have different limits. The theorem on the limit without a subsequence closes off that possibility for this sequence (if the sequence converges, the limit of every subsequence is the same ν_β; Lean's `CylDet.subseq_limit_eq`).
The sequence is not constant. The lift for $n = 0$ (the 2^4 torus) has period 2, and $x_{\ell+2e_1} = x_\ell$ in every configuration. For $n = 1$ (4^4) these two are different links, and the probability that they agree is 0 Proved here . So $\mu_0 \neq \mu_1$.
Do not read “unique” as uniqueness of the Gibbs measure including limits coming from other boundary conditions. That is outside this theorem (§ 07).

07

What is not said

- **The lattice spacing is fixed at 1.** Nothing is said about the continuum limit, the operation of sending the lattice spacing to 0.
- **The window is narrow.** Only $|\beta| < 0.0556$ Reading . Nothing is said about weak coupling.
- $\beta = 0$ **is outside the statement.** The hypothesis is $\beta \neq 0$ ($\beta = 0$ is the product of Haar measures, and its limit is trivial).
- **Periodic boundary conditions only.** This is convergence of the sequence of periodic lifts; nothing is said about sequences with free or fixed boundary conditions, or about uniqueness of measures satisfying the DLR equations.
- **Translations are the generators, the unit translations.** Invariance under rotations and other lattice symmetries is not in the theorems of this chapter. The invariance under generators in 20-05 is a theorem about a different measure (the measure with fermions on a finite lattice).
- **The clustering estimate is an upper bound only.** (4) is an upper bound on the correlations (a lower bound on the rate of decrease) and does not say how fast the true correlations decrease. λ is chosen within $e^\lambda q(\beta) < 1$, and the $\frac{1}{2}$ in $-\frac{1}{2}\log q$ is a matter of choice.
- **Not a new result.** These are known results of Dobrushin-uniqueness type and Osterwalder–Seiler strong-coupling type, written in Lean on the concrete SU(2) Wilson measure.

What this chapter can say

What is said	What is not said
Cited The theorem on the limit without a subsequence: for $\beta \neq 0$, $18 \tanh \lvert\beta\rvert < 1$, μ_n (size $2(n+1)$) converges weakly, as a sequence, to a single ν_β (`InfVol.wilson_infvol_unique`)	Outside the window, $\beta = 0$, other boundary conditions, uniqueness of the Gibbs measure. "Unique" is trivial from the topology
Cited The theorem on the limit independent of size: the same limit for all sizes $m+1$ and for any $s(k) \to \infty$ (`AllN.wilson_infvol_allN`)	An explicit form of the rate of convergence (an estimate of the form (9) exists, but is not in the statement)
Cited Translation invariance of the limit: invariant under unit translations in four directions (`TransI.wilson_infvol_translation`)	Invariance under rotations and reflections
Cited Clustering at limit points: (4), four directions, at every limit point (`WilsonLimit.cluster_limit`, $\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$)	The true rate of decay, outside the window
Proved here The estimate (9) of the difference between two sizes (a one-line consequence of (8) in Lean)	(9) itself is not a Lean declaration
Reading The window $\lvert\beta\rvert < 0.055613$; for $\beta = 1/20$, $q = 0.8993$ and $\lambda = 0.0531$	Floating-point values (not interval estimates)

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity, the transfer matrix and the strong-coupling type for lattice gauge theory at strong coupling	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-04
Dobrushin's uniqueness condition (row sum of influence coefficients less than 1)	Cited	The original source of R. L. Dobrushin's uniqueness condition (bibliographic data only, not checked; the year and journal have not been verified in this book)
Definition of lattice Yang–Mills in the language of probability, and an overview of strong-coupling results	Cited	S. Chatterjee, arXiv:1803.01950 (full text)
The four theorems and the steps of proof	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). Declarations in the tables of the text, the table of 20-07, and A-04
The edge of the window, the rates, $18 \tanh(1/20)$	Reading	The numerical record of this book's study (double-precision floating point)

Words used in this chapter — periodic lift, weak convergence, cylinder function, Dobrushin's condition, influence coefficient, heat bath, sweep, clustering, strong-coupling window. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-04 Reconstruction and the mass gap — inside the strong-coupling window — from the $\nu\beta$ of this chapter it assembles the Hilbert space and the transfer operator, and reads off that there are states outside the vacuum and a lower bound on the gap.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling Reconstruction and the mass gap · Kiichi, Shiori and Rin

Reconstruction and the mass gap — inside the strong-coupling window

> This chapter is this book's own investigation.

What this chapter shows — In the strong-coupling window: the transfer operator T on the Hilbert space reconstructed from the infinite-volume measure ν_β satisfies $\|T\| \leq (18 \tanh|\beta|)^{1/2}$ on the orthogonal complement of the vacuum. That complement is not 0 (non-degeneracy). The same holds in the sector of gauge-invariant observables. For $\beta \geq 0$, $T \geq 0$. The range of T lies in the gauge-invariant sector. And a statement bundling these into one. Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-03 The vessel of reconstruction, the definition of the mass gap, 20-03 The infinite-volume limit, 20-01 Reflection positivity. Reconstruction and the definition of the mass gap as established knowledge are in 4-10 and 4-11.

Said first — The "mass gap" here is only the upper bound $\|T - P_\Omega\| \leq e^{-m}$, that is, a **lower bound** on the mass. It does not say that the mass is finite ($T \neq P_\Omega$). The lattice spacing is fixed at 1, the window is only $|\beta| < 0.0556$ Reading , and nothing is shown about a mass gap in the continuum limit.

The order of this chapter

1. The vessel — building the Hilbert space and T from the pairing, the translation and the constant: the three data the statements pin down.
2. The mass gap with non-degeneracy: two theorems for two windows.
3. Why non-degeneracy is put in the statement: avoiding a form that holds even for a one-point model.
4. The route of proof: from decay of correlations to the operator norm.
5. The gauge-invariant sector and the range of T: the plaquette observable, and F_0 falling into the kernel.
6. T ≥ 0: reflection in the middle plane and the gauge average.
7. The bundled statement: $0 < \beta$, everything inside the window.

8. What is not said: lower bound only, the window, the lattice spacing.

01

The vessel — building the Hilbert space and T from the pairing, the translation and the constant

The definition of the vessel is in 19-03. For an introduction, see 4-10. Here we write only the form we use.

The space of observables W. The set of bounded continuous functions depending only on finitely many links whose base points have time coordinate from 0 to R (R finite for each function). Equivalently, the cylinder functions on the closed positive half. Its Lean name is `Wsub`.

Three data. Let ν be a probability measure, and set

$$B(F,G)=\int F(\Theta x)\,G(x)\,d\nu(x),\qquad (\tau F)(x)=F(x(\cdot+e_0)),\qquad \mathbf{1}=\text{the constant } 1 \quad (1)$$

Θ is the time reflection $x_0\mapsto -x_0$ (time links are shifted by one and reversed in direction; 19-03). When B is positive semi-definite and symmetric, τ is symmetric with respect to B, $\tau\mathbf{1}=\mathbf{1}$, $B(\mathbf{1},\mathbf{1})=1$, and the orbits under translation are bounded, the triple $(B,\tau,\mathbf{1})$ is called a **vessel of reconstruction** (Lean's `OSGap.Datum`).

What is built. Put the semi-inner product B on W, divide out the null elements and complete to get $\mathcal{H}$; write $[F]$ for the image of an element of W, the vacuum $\Omega=[\mathbf{1}]$, T for the continuous extension of τ, and $P_\Omega v=\langle\Omega,v\rangle\Omega$ for the projection onto the vacuum.

Definition of the mass gap. The definition is in 19-03.

$$\texttt{HasMassGap}\ m \quad :\Longleftrightarrow \quad \|T-P_\Omega\|\le e^{-m}. \quad (2)$$

Since $T\Omega=\Omega$ and T is symmetric, this is the same as "$\|T\|\le e^{-m}$ on the orthogonal complement of Ω" Proved here . In one time step, the components other than the vacuum shrink by a factor of at least e^{-m}. m is a lower bound on the mass (the energy gap).

The statements pin down the data. All the theorems below have the form "there is a vessel D such that the B of D is the integral (1), its τ is the translation by one time step, its $\mathbf{1}$ is the constant 1, and ...". All three data fields are fixed in the conclusion, so D is determined by ν Proved here . One cannot satisfy the conclusion by swapping T for an arbitrary operator.

02

The mass gap with non-degeneracy

Set $F_0(x) = \operatorname{Re} x\ell_0$, where ℓ_0 is the link of direction 1 (spatial) based at the origin; $F_0 \in W$. Write $q(\beta) = 18 \tanh|\beta|$ (20-03).

The mass gap with non-degeneracy — the Dobrushin window

Cited A formalisation of results of Dobrushin and Osterwalder–Seiler strong-coupling type. Lean: `WilsonLimit.wilson_massGap_dob_exists` ($\beta \neq 0$, $q(\beta) < 1$).

Let $\beta \neq 0$ and $18 \tanh|\beta| < 1$. Then there are a weak limit ν of some subsequence of μ_n and a vessel D built from ν by (1) such that

$$\|T - P_\Omega\| \le \big(18 \tanh|\beta|\big)^{1/2}, \qquad B(F_0, F_0) - B(\mathbf{1}, F_0)^2 \;\ge\; \tfrac{1}{4} e^{-12|\beta|}, \qquad \exists\, v \perp \Omega,\ \|v\| =$$

The first is **HasMassGap** with $m = -\frac{1}{2}\log q(\beta)$; the second and third are non-degeneracy.

The mass gap with non-degeneracy — at every limit point

Cited A formalisation of a result of the same type. Lean: `WilsonLimit.wilson_massGap_nondegenerate` ($\beta \neq 0$, $24(1 - e^{-4|\beta|}) < 1$; ν is the weak limit of any subsequence of μ_n).

In the window $24(1 - e^{-4|\beta|}) < 1$ ($|\beta| < 0.0106$ Reading), for **every** limit point ν, the vessel D satisfies **HasMassGap**$\big(-\log 24(1 - e^{-4|\beta|})\big)$ and the two non-degeneracy conditions of (3). The estimate in this window comes from the finite-volume decay of 20-02.

The two theorems differ in window and rate. The Dobrushin-window form picks one limit point in its statement, but the same estimate holds at every limit point (`WilsonLimit.wilson_massGap_dob`, § 04 Step 4). Moreover, inside the window the limit point is unique, ν_β (20-03). The narrow-window form reaches a conclusion of the same type by a different route, from the finite-volume decay (20-02), with rate $-\log 24(1 - e^{-4|\beta|})$. § 07 bundles the Dobrushin-window form for the limit ν_β without a subsequence. The narrow-window rate is not in § 07.

An example. For $\beta = 1/20$, $q = 0.8993$, $\|T - P_\Omega\| \le 0.948$, and the lower bound on the mass is $m = 0.053$ Reading. The lower bound for non-degeneracy is $\frac{1}{4}e^{-0.6} = 0.137$ Reading. The closer to the edge of the window, the closer the lower bound on the gap gets to 0.

03

Why non-degeneracy is put in the statement

(2) alone can have thin content. If $\mathcal{H}$ is the one-dimensional space spanned by Ω, then $T = P_\Omega$ and $\|T - P_\Omega\| = 0$. Then (2) holds for every m. This happens, for example, if ν is a point mass. So unless "there is a gap" is paired with "there are states outside the vacuum", it is a statement that holds even for a one-point model. For this reason, this book does not present theorems in the form without non-degeneracy.

Two forms of non-degeneracy. The second item of (3) is "the variance of F_0 measured by the reflection pairing is positive"; the third is "there is a unit vector orthogonal to Ω". The third follows from the second: $y = F_0 - B(\mathbf{1}, F_0)\mathbf{1}$ has $B(\mathbf{1}, y) = 0$ and $B(y, y) > 0$, so $v = [y]/\|[y]\|$ is a unit vector (`OSGap.Datum.exists_unit_orth`). The third rules out a one-dimensional $\mathcal{H}$ at the level of types.

Where the lower bound on the variance comes from. In steps:

1. ℓ_0 is a spatial link at time 0, so it is fixed by Θ, and $F_0 \circ \Theta = F_0$. Hence $B(F_0, F_0) - B(\mathbf{1}, F_0)^2 = \mathrm{Var}_\nu(F_0)$ Proved here .
2. For the Haar measure on SU(2), left invariance under $q \mapsto -q,\ iq,\ jq,\ kq$ gives $E\,\mathrm{Re}\,U = 0$ and $E(\mathrm{Re}\,U)^2 = \frac{1}{4}$. Hence $E(\mathrm{Re}\,U - c)^2 \geq \frac{1}{4}$ for every c (`HaarRe.integral_sq_sub_ge`).
3. The density of the conditional distribution of one link is given by the weights of the 6 plaquettes containing that link, and lies between $e^{\pm 6|\beta|}$. So the conditional variance is at least $\frac{1}{4}e^{-12|\beta|}$, and integrating gives $\mathrm{Var}(F_0) \geq \frac{1}{4}e^{-12|\beta|}$ on every torus (`LinkSpec.wilsonProb_sq_sub_ge`).
4. F_0 and F_0^2 are bounded continuous, so the lower bound passes to the weak limit (`WilsonLimit.var_limit_ge`).

04

The route of proof

The mass-gap part combines a general lemma that turns decay of correlations into an operator norm with the decay estimate for the Wilson measure.

Step 1: the axioms of the vessel. Positive semi-definiteness of B comes from carrying reflection positivity for the plane through sites on each torus (20-01) over to the weak limit. Symmetry of τ comes from translation invariance and its relation to Θ; boundedness of orbits from $|F| \leq \|F\|_\infty$.

Step 2: from decay to the operator norm. On the vessel, if for every $F \in W$ there is a constant C_F with

$$B(\tau^k F, \tau^k F) - B(\mathbf{1}, F)^2 \;\le\; C_F\, r^{2k} \qquad (k = 0, 1, 2, \dots) \quad (4)$$

then $\|T - P_\Omega\| \le r$ (`OSGap.Datum.opNorm_gap`). The left-hand side is $\|T^k[F] - P_\Omega[F]\|^2$ Proved here, so this is an argument that turns an estimate on a dense set into a spectral estimate for a symmetric operator.

Step 3: decay for the Wilson measure. $B(\tau^k F, \tau^k F) = \int F(\Theta\tau^k x) F(\tau^k x)\, d\nu$ is the correlation of two observables about $2k$ apart in time. Applying the Dobrushin distance covariance estimate on the torus (20-02, `LinkSpec.wilson_cov_dist`) gives $C_F\, e^{-2\lambda k}$ with a constant independent of N (`TorusRP.wilson_hdec_torus_dob`). λ is any number with $\lambda \ge 0$ and $e^{\lambda} q(\beta) < 1$ (in Step 4, $\lambda > 0$). Decay of this form is `WilsonLimit.dec_eventually_dob`. The estimate passes to limit points.

Step 4: choice of rate. Applying Step 2 with $r = e^{-\lambda}$ gives `HasMassGap` λ (`WilsonLimit.wilson_massGap_dob`). The $\lambda = -\frac{1}{2}\log q(\beta)$ in (3) is one choice satisfying $e^{\lambda} q = q^{1/2} < 1$, and the $\frac{1}{2}$ has no physical meaning. From the same theorem one can take a rate arbitrarily close to $-\log q(\beta)$.

Step 5: non-degeneracy. The four steps of § 03.

05

The gauge-invariant sector and the range of T

For an introduction, see 4-11. A gauge transformation $g : \mathbb{Z}^4 \to SU(2)$ acts as $(g \cdot x)_\ell = g(b)\, x_\ell\, g(b + e_\mu)^*$ (b the base point of ℓ, μ its direction). Let W_{gi} be the set of elements of W invariant under every g. $F_p = \mathrm{Re}\, U_p$ (the real part of the holonomy of the $(1,2)$ plaquette at the origin) belongs to W_{gi}.

The mass gap in the gauge-invariant sector

Cited A formalisation. Lean: `GaugeZ.wilson_massGap_gauge_exists` ($\beta \neq 0$, $q(\beta) < 1$).

The same form as (3) holds with W replaced by W_{gi} and F_0 by F_p. For some limit point ν, on the vessel over W_{gi}, $\|T - P_\Omega\| \le q(\beta)^{1/2}$, $\mathrm{Var}_\nu(F_p) \ge \frac{1}{4} e^{-12|\beta|}$, and there is a unit vector orthogonal to the vacuum. The lower bound on the variance comes from the fact that when one link U of the plaquette is varied, $U_p = UV$ (V, the product of the other three, is a unit), and the Haar variance of $\mathrm{Re}(UV)$ is the same as that of $\mathrm{Re}\, U$.

Why separate the sector. The witness F_0 of non-degeneracy for the whole W is not gauge invariant. The bundled statement (§ 07) contains $T[F_0] = 0$. That is, $[F_0]$ is a nonzero vector orthogonal to the vacuum, but it is killed by T. This is because a gauge

transformation placing the central element -1 at the site e_0 gives $\int G(\Theta x) F_0(\tau x)\, d\nu = 0$ for every $G \in W$ Proved here . This is the same mechanism as Elitzur's theorem (local gauge symmetry cannot be broken) Cited . Non-degeneracy in the gauge-invariant sector (witness F_p) is the non-degeneracy with content. F_p is not a vector known to be killed by T. However, $T[F_p] \;/= 0$ (that T is not 0 on the complement in this sector) is not in the Lean statement either.

The range of T lies in the gauge-invariant sector

Cited A formalisation. Lean: `Bind.T_range_gauge_all` ($\lambda > 0$, $e^{\lambda} q(\beta) < 1$; ν any limit point).

For the vessel D over W and the vessel D' over W_{gi} (both with data pinned down by (1)), both satisfy **HasMassGap** λ, and there is an isometric linear map $\iota : \mathcal{H}' \to \mathcal{H}$ with

$$\iota \Omega' = \Omega, \qquad \iota T' = T \iota, \qquad T u \in \iota(\mathcal{H}') \text{ for all } u \in \mathcal{H}. \quad (5)$$

The gauge-invariant sector embeds isometrically into the whole Hilbert space, commutes with T, and the range of T lies entirely inside it. The proof uses that taking the Haar average of gauge transformations over the end points of the support of τF gives an element of W_{gi} without changing the values of the pairing. The image of ι is closed and T is continuous, so the statement extends from a dense set to all u.

06

T ≥ 0

Cited A formalisation. Lean: `FullT.wilson_Tpos_full` ($\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$; ν any limit point).

Let $\beta \geq 0$. There is a vessel D over W (with data pinned down by (1)) for which **HasMassGap** λ, the two non-degeneracy conditions of (3), and

$$\langle v, T v \rangle \;\geq\; 0 \qquad (\text{for all } v \in \mathcal{H}) \quad (6)$$

all hold.

Route. $\langle [F], T[F] \rangle = B(F, \tau F) = \int F(\Theta x) F(\tau x)\, d\nu$. Since $\tau \circ \Theta' = \Theta$ (Θ' is the reflection $x_0 \mapsto 1 - x_0$ in the plane through the middle of links), this is the pairing for reflection in the middle plane. Reflection positivity for the middle plane (20-01) is about gauge-invariant functions and uses $\beta \geq 0$. So we insert the Haar average A over gauge transformations on all sites of the torus (G is a bounded measurable function depending only on the positive side). The measure is invariant under gauge

transformations, so $\int G(\Theta'\omega)G(\omega) = \int (AG)(\Theta'\omega)(AG)(\omega)$, and AG is gauge invariant. Thus it is positive on each torus, positive in the limit, and extends from a dense set to all v.

Reading. Combining $T \geq 0$, $T\Omega = \Omega$ and $\|T - P\Omega\| \leq e^{-m}$, the spectrum of T on the complement of Ω lies in $[0, e^{-m}]$ Proved here . Writing $-\log T$, we have $-\log T \geq m$ on the complement (read as $+\infty$ on the kernel of T). Nothing is said about $\beta < 0$.

07

The bundled statement

Cited A formalisation of results of Dobrushin-uniqueness type and Osterwalder–Seiler strong-coupling type. Lean: `InfMain2.wilson_infvol_main2` ($0 < \beta$, $18 \tanh \beta < 1$).

Let $0 < \beta$ and $18 \tanh \beta < 1$, and set $m = -\frac{1}{2}\log(18 \tanh \beta)$. There is a unique probability measure ν_β for which all of the following hold.

1. $\mu_n \to \nu_\beta$ (weak convergence without a subsequence; 20-03).
2. ν_β is invariant under unit translations in the four directions.
3. $\int F(\Theta x)F(x)\, d\nu_\beta \geq 0$ for every $F \in W$ (reflection positivity).
4. On the vessel over W (data pinned down by (1)): **HasMassGap** m, $\mathrm{Var}(F_0) \geq \frac{1}{4}e^{-12\beta}$, existence of a unit vector orthogonal to Ω, $T[F_0] = 0$, and $T \geq 0$.
5. On the vessel over W_{gi} (pinned down likewise): **HasMassGap** m, $\mathrm{Var}(F_p) \geq \frac{1}{4}e^{-12\beta}$, existence of a unit vector orthogonal to Ω, and $T \geq 0$.
6. Clustering in four directions: (4) of 20-03 with rate m.

The "unique" part is trivial from the topology; the content is the convergence in item 1 (the detour in 20-03). That the range of T lies in the gauge-invariant sector ((5) of § 05) is not in this statement. It is the separate theorem `Bind.T_range_gauge_all`.

08

What is not said

- **Only a lower bound on the mass.** $\|T - P\Omega\| \leq e^{-m}$ also holds if T is 0 on the complement ("infinite mass"). It is not said that the mass is finite, or that $T \neq P\Omega$. The number $-\frac{1}{2}\log(18 \tanh \beta)$ is a lower bound, and the $\frac{1}{2}$ is a matter of choice. There is no record of its ratio to the true mass.

- **The window is narrow.** $|\beta| < 0.0556$ Reading (the form for every limit point: $|\beta| < 0.0106$ Reading). Nothing is said about weak coupling or the physical range of couplings.
- **The lattice spacing is fixed at 1.** This is a lower bound on the mass measured in lattice units. Whether a mass gap survives in the continuum limit is outside the theorems of this part.
- **Sign.** The bundled statement needs $0 < \beta$, and $T \geq 0$ needs $\beta \geq 0$. The gap and non-degeneracy alone hold for $\beta \neq 0$ of either sign.
- **Reading of the sector.** W_{gi} is the set of functions that are "cylinder functions and gauge invariant". Whether its closure coincides with the physical space of gauge-invariant states is not said.
- **Not a new result.** These are known results of Osterwalder–Seiler strong-coupling type, written in Lean on the concrete SU(2) Wilson measure.

What this chapter can say

What is said	What is not said
Cited The mass gap with non-degeneracy (Dobrushin window, some limit point): (3) (`WilsonLimit.wilson_massGap_dob_exists`, $\beta \neq 0$)	That the mass is finite; outside the window
Cited The mass gap with non-degeneracy (every limit point, window $24(1 - e^{-4\|\beta\|}) < 1$) (`WilsonLimit.wilson_massGap_nondegenerate`)	As above; the window is narrower still
Cited The mass gap and non-degeneracy in the gauge-invariant sector (witness F_p) (`GaugeZ.wilson_massGap_gauge_exists`)	That the closure of W_{gi} coincides with the physical space of states
Cited $T \geq 0$ (on the whole W) (`FullT.wilson_Tpos_full`, $\beta \geq 0$)	$\beta < 0$
Cited The range of T lies in the gauge-invariant sector, (5) (`Bind.T_range_gauge_all`)	Not included in the bundled statement
Cited The bundled statement (`InfMain2.wilson_infvol_main2`, $0 < \beta$, $18 \tanh \beta < 1$)	The continuum limit, changing the lattice spacing, an upper bound on the mass
Proved here The reading, from $T \geq 0$ and the gap, that the spectrum on the complement lies in $[0, e^{-m}]$	The reading itself is not a Lean declaration
Reading For $\beta = 1/20$: $\|T - P_\Omega\| \leq 0.948$, $m = 0.053$, lower bound for non-degeneracy 0.137	Floating-point values

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity, the transfer matrix and the strong-coupling mass-gap type for lattice gauge theory	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-03, 5-04
Euclidean axioms and reconstruction	Cited	K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281 (abstracts). 4-10
Local gauge symmetry cannot break spontaneously (the reading of $T[F_0] = 0$)	Cited	S. Elitzur, Phys. Rev. D 12 (1975) 3978 (abstract)
Dobrushin's uniqueness condition	Cited	Original source: bibliographic data only, not checked (the year and journal have not been verified in this book)
The theorems of this chapter and their parts	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). The list of declarations is in 20-07 and A-04
The numbers for $\beta = 1/20$, the edge of the window	Reading	The numerical record of this book's study (double-precision floating point)

Words used in this chapter — vessel of reconstruction, transfer operator, vacuum, mass gap (lower bound), non-degeneracy, gauge-invariant sector, Elitzur's theorem, reflection positivity, strong-coupling window. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-05 The vessel with fermions — on a finite lattice whose weight includes the Wilson–Dirac determinant, it reads reflection positivity, Z > 0 and the symmetries.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling The vessel with fermions · Kiichi, Shiori and Rin

The vessel with fermions

> This chapter is this book's own investigation.

What this chapter shows — On the finite lattice $(\mathbb{Z}/N)^3 \times \mathbb{Z}/2h$, the measure given by the Wilson gauge weight $e^{-\beta S_W}$ multiplied by the determinant of the Wilson–Dirac operator, $\mathrm{Re}\left[(\det D)^{2j}\right]$, is positive with respect to reflection in a time plane through sites (mass $m > -1$, all β). The partition function is positive (all β, m, j). $\det D$ is invariant under global gauge transformations. The normalised measure is invariant under the generators of the lattice symmetries. Each is a formalisation, machine-checked in Lean, of a result of a known type.

Prerequisite chapters — 19-05 The Wilson–Dirac operator, the measure with fermions, 20-01 Reflection positivity. Reflection positivity on the lattice and the transfer matrix as established knowledge are in 5-03.

Said first — All are statements on one finite lattice. Nothing is said about the infinite-volume limit, the continuum limit, chiral symmetry, or mass renormalisation. The symmetries are invariance under each generator; closure as a group is outside the statements. There is no mass gap with fermions in this chapter either.

The order of this chapter

1. The objects — the lattice, the action, the determinant weight: which measure.
2. Reflection positivity for the plane through sites: $m > -1$, all β.
3. The route of proof of reflection positivity: how the action splits, and positivity of the plane matrix.
4. Positivity of the partition function: normalisation can be defined.
5. Global gauge invariance of the determinant: conjugation by a constant g.
6. Invariance under the generators of the lattice symmetries: translations, rotations, reflections, time reflection.
7. What is not said: a finite lattice, the continuum limit, chiral symmetry.

01

The objects — the lattice, the action, the determinant weight

The definitions are in 19-05. Here we write only what is needed to read the theorems.

Lattice and links. Space is the three-dimensional torus $(\mathbb{Z}/N)^3$, and time is the circle $\mathbb{Z}/2h$ ($h \geq 1$). The links are the spatial links $x \to x + e_i$ ($i = 1, 2, 3$) at time t, and the time links $t \to t + 1$. Placing an element of SU(2) on each link gives a configuration ω, and the configuration space carries the product of Haar measures over links.

Gauge action. Counting unoriented plaquettes once each, set

$$S_W(\omega) = \sum_p \left(1 - \tfrac{1}{2} \operatorname{Re} \operatorname{tr} U_p\right) \quad (1)$$

and take the weight $e^{-\beta S_W}$. Since $\frac{1}{2} \operatorname{tr} U$ equals the real part of the quaternion, this is a constant multiple of the weight $e^{\beta \sum_p \operatorname{Re} U_p}$ of 19-01 Proved here. The orientation of β is also the same.

The Wilson–Dirac operator. $D = D(\omega)$ is the operator with Wilson-term coefficient $r = 1$ and mass m, with antiperiodic boundary conditions in all four directions (Lean's `WilsonGauge.physDirac`).

$$D = (4 + m)\,\delta - \tfrac{1}{2} \sum_\mu \left[(1 - \gamma_\mu)\, U_\mu(x)\, \delta_{x+\hat{\mu}} + (1 + \gamma_\mu)\, U_\mu(x - \hat{\mu})^\dagger\, \delta_{x-\hat{\mu}}\right] \quad (2)$$

The weight with fermions. Fix a natural number j, and set

$$w(\omega) = e^{-\beta S_W(\omega)} \cdot \operatorname{Re}\left[(\det D(\omega))^{2j}\right] \quad (3)$$

(`wfWeight`). As in 19-05, $(\det D)^{2j}$ is a non-negative real number, and Re is a notational convention. $\det D^{2j}$ corresponds to integrating out, with Grassmann variables, $2j$ flavours of fermions of the same mass. In Lean no Grassmann variables are introduced; the determinant itself is defined as the weight. The normalised measure $P = w\, d\omega / Z$, $Z = \int w\, d\omega$, is `wfProb` (19-05).

02

Reflection positivity for the plane through sites

For an introduction, see 5-03. The time reflection $\Theta : t \mapsto -t \pmod{2h}$ fixes the two site planes at times $t = 0$ and $t = h$. Divide the configuration into three parts: the spatial links on the two planes, c; the positive side a_+ (spatial links with $0 < t < h$, and time

links leaving $0 \le t < h$); and the negative side a_- (the rest, carried by Θ into the form of the positive side, with time links inverted).

Reflection positivity of the Wilson–fermion weight

Cited A formalisation of a result of Osterwalder–Seiler type. Lean: `WilsonGauge.site_reflection_positive_wilson_full` ($h > 0$, $m > -1$, all $\beta \in \mathbb{R}$, all j).

If φ is a bounded measurable real-valued function of (c, a_+), then

$$\int \varphi(c, a_+)\, \varphi(c, a_-)\, e^{-\beta S_W(\omega)}\, \mathrm{Re}\big[(\det D(\omega))^{2j}\big]\, d\omega \;\ge\; 0. \tag{4}$$

$\varphi(c, a_-)$ is φ read on the reflected configuration, that is, $(\Theta\varphi)$. (4) is reflection positivity of the form "$\langle \Theta\varphi \cdot \varphi \rangle \ge 0$", stated for the unnormalised measure. In Lean the integral is written in the split coordinates (c, a_+, a_-), and `WilsonGauge.physDirac_eq_split` says that the operator assembled in those coordinates is the same as the D of the physical configuration. Since $Z > 0$ (§ 04), the sign is the same after normalisation Proved here .

Why the sign of β does not matter. For reflection in the plane through sites, no plaquette straddles the plane. Every plaquette lies on one of the two fixed planes, on the positive side, or on the negative side. So $e^{-\beta S_W}$ splits as "a factor on the planes only $\times$ a factor on the positive side $\times$ a factor on the negative side", and no sign condition on the factors is needed Proved here . For reflection in the plane through the middle of links, $\beta \ge 0$ is needed to expand the factor of the straddling plaquettes as a positive kernel. The theorems of this chapter do not treat that case.

Why $m > -1$. On the fermion side, we use that the block of the operator on the fixed planes (the plane matrix) is positive definite (`FermionSliceBlock.wilsonPlane_posDef`). The plane matrix has the form $(m+1) + \sum_i (1 - \frac{1}{2}(T_i + T_i^\dagger))$ (T_i unitary hops), and each term $1 - \frac{1}{2}(T_i + T_i^\dagger)$ is positive semi-definite, so it is at least $m + 1$ Proved here . $m > -1$ is a sufficient condition for this. Whether it is necessary cannot be told from this theorem.

03 The route of proof of reflection positivity

Step 1: how the action splits. $S_W(\omega) = K(c) + H(c, a_+) + H(c, a_-)$, where H is **the same function** on the positive and negative sides (`WilsonGauge.wilsonAction_split`). A time plaquette on the negative side takes the form of one on the positive side when its time links are inverted and its two times are swapped, and $\mathrm{Re}\,\mathrm{tr}$ is unchanged by the cyclicity of tr and $\mathrm{tr}\, M^\dagger = \mathrm{tr}\, M$ (`WilsonGauge.plaqE_reflect`).

Step 2: abstract reflection positivity. If the weight has the form $k(c)\,w_+(c,a_+)\,w_+(c,a_-)$ ($k \geq 0$, w_+ real and bounded), and the fermion determinant side satisfies operator conditions such as positivity of the plane matrix, then an integral of the form (4) is non-negative (`FermionSliceReflection.site_reflection_positive_phys`). By Step 1 the gauge weight fits the form $k = e^{-\beta K}$, $w_+ = e^{-\beta H}$.

Step 3: the operators agree. The D written in the physical configuration (equation (2)) agrees with the operator assembled in the split configuration (`WilsonGauge.physDirac_eq_split`). This lets the operator conditions of Step 2 be discharged for the concrete D.

An example showing the hypotheses can be met: with $h = 1$, $m = 0$, $\varphi = 1$, all hypotheses hold. The statement is not vacuous.

04

Positivity of the partition function

Cited A formalisation. Lean: `WilsonGauge.partition_pos_wilson_full` ($h > 0$, all β, m, j).

$$Z = \int e^{-\beta S W(\omega)}\ \mathrm{Re}\big[(\det D(\omega))^{2j}\big]\,d\omega\ >\ 0. \quad (5)$$

There is no condition on the mass m. Since $Z > 0$, the normalised probability measure `wfProb` can be defined. $Z > 0$ is not trivial: it contains the content that $\mathrm{Re}\big[(\det D)^{2j}\big]$ does not vanish on every configuration.

05

Global gauge invariance of the determinant

Proved here A Lean formalisation of a known fact (the proof is the sentence below). Lean: `GaugeDet.physDirac_det_conj` (all N, h, m, g, ω; no hypotheses).

For the configuration $(g\cdot\omega)_\ell = g\,\omega_\ell\,g^{-1}$ obtained by conjugating every link by a constant $g \in SU(2)$,

$$\det D(g\cdot\omega) = \det D(\omega). \quad (6)$$

$D(g\cdot\omega)$ is similar to $D(\omega)$, sandwiched by the matrix that multiplies the colour index by the constant g, so the determinant does not change. g is a constant independent of the site. Local gauge transformations, which take a different $g(x)$ at each site, are outside this declaration.

06

Invariance under the generators of the lattice symmetries

Proved here A Lean formalisation of known facts (the route of proof is below). Lean: `CubicSym.wfProb_lattice_generators` ($h > 0$, all N, j, m, β).

The normalised measure P (`wfProb`) is invariant under push-forward by each of the following generators R: $R_* P = P$.

Generator	Action on configurations	Lean name
Global gauge transformation	Conjugate every link by a constant g	`conjPhys g`
Spatial unit translation (three directions)	$x \mapsto x + e_k$	`transPhys k`
Time unit translation	$t \mapsto t + 1$	`tshiftPhys`
90° rotation in a spatial plane	$e_a \mapsto e_b \mapsto -e_a$ ($a \neq b$, six oriented cases)	`rotPhys a b`
Reflection of a spatial axis	$x_k \mapsto N - 1 - x_k$	`reflPhys k`
Time reflection	$t \mapsto 2h - 1 - t$, time links inverted (the reversal about the mid-link plane $t = h - \frac{1}{2}$, different from the site-plane Θ of § 02. What Lean has is invariance of the measure under this reversal, not reflection positivity)	`treflPhys`

Proved here Likewise a formalisation. Lean: `CubicSym.wfProb_hypercubic_generators` (isotropic lattice $N = 2h$, $h > 0$). On a lattice whose spatial and time sizes are equal, in addition to all the above, the measure is also invariant under 90° rotations in a (space, time) plane (`trotPhys a`).

Route. For every generator, invariance of the measure splits into three parts. (i) The product of Haar measures over links is preserved by relabelling links and by inverting some links, $g \mapsto g^{-1}$. (ii) S_W is invariant. (iii) $\det D$ is invariant. (iii) is shown in Clifford form. Write D as $1 \otimes M + \sum_\mu \gamma_\mu \otimes A_\mu$; with a lattice permutation W and a spin matrix S (for rotations $S = 1 + \gamma_b \gamma_a$; for the reflection of axis k, the product of the other three γ), it follows from $(S \otimes W)\, D(\omega)\, (S \otimes W)^{-1} = D(R\omega)$ (`CubicSym.cliffOp_conj`, `CubicSym.det_cliffOp_perm`).

Time reflection and the determinant. For the time reflection too, the determinant itself is invariant: $\det D(\Theta_t \omega) = \det D(\omega)$ (`CubicSym.physDirac_det_trefl`; all N, h, m, ω; isotropy not needed). The spin matrix is $\gamma_1\gamma_2\gamma_3$ Proved here .

The antiperiodic seam. If the coordinate reversal is taken as $z \mapsto N-1-z$, the link carrying the antiperiodic sign ($N-1 \to 0$) is mapped to itself (with its direction reversed). So the lattice permutation W needs no sign correction Proved here . If it is taken as $z \mapsto -z$, the seam moves and a signed permutation is needed.

07

What is not said

- **One finite lattice.** All theorems are statements on fixed N and h. Neither the infinite-volume limit nor reflection positivity there is in this chapter.
- **No continuum limit.** The lattice spacing is fixed. Nothing is said about the continuum limit or mass renormalisation (how to tune m).
- **No chiral symmetry.** The Wilson term breaks chiral symmetry explicitly. Nothing is said about its restoration or the treatment of fermion doubling.
- **Symmetries generator by generator.** Closure as the cubic or hypercubic group (that every element of the group arises as a product of generators) is outside the statements. Rotations involving time are only for the isotropic lattice $N = 2h$. Gauge invariance is for a global constant g; invariance of $\det D$ under local gauge transformations is not among the declarations.
- **Reflection in site planes only.** Reflection positivity for the plane through the middle of links (with fermions) is not in Lean. $m \leq -1$ is outside the range.
- **No gap.** A mass gap for the transfer operator of the measure with fermions is not among the theorems of this chapter.
- **Not a new result.** This writes, in Lean on the concrete operator, reflection positivity of Osterwalder–Seiler type and known facts about lattice symmetries.

What this chapter can say

What is said	What is not said
Cited Reflection positivity of the Wilson–fermion weight (4) (`WilsonGauge.site_reflection_positive_wilson_full`, $m > -1$, all β)	$m \leq -1$, reflection in the plane through the middle of links, infinite volume
Cited Positivity of the partition function (5) (`WilsonGauge.partition_pos_wilson_full`, all β, m, j)	Estimates of the size of Z
Proved here Global gauge invariance of the determinant (6) (`GaugeDet.physDirac_det_conj`, unconditional)	Local gauge transformations

What is said	What is not said
Proved here Invariance of the measure under the generators of the lattice symmetries (`CubicSym.wfProb_lattice_generators`)	Closure as a group, rotations involving time (anisotropic lattices), lifting to the block level
Proved here Invariance under the hypercubic generators on the isotropic lattice (`CubicSym.wfProb_hypercubic_generators`, $N = 2h$)	Closure as a group, lifting to the block level
Proved here Invariance of the determinant under time reflection (`CubicSym.physDirac_det_trefl`)	Reflection positivity for this reversal
—	The continuum limit, chiral symmetry, mass renormalisation, a mass gap with fermions

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and the transfer matrix for lattice gauge theory (including fermions)	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). 5-03
Wilson's lattice formulation	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445 (abstract). The original source of the Dirac operator with the Wilson term: bibliographic data only, not checked
The theorems of this chapter and their parts	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). The list of declarations is in 20-07 and A-04
Agreement of the operator formula (2) with the Lean definition, the geometry of the generators' actions	Reading	The numerical record of this book's study (on small lattices, the difference between the operator copying the Lean definition and the textbook formula was compared in double precision; the difference was of rounding size)

Words used in this chapter — Wilson–Dirac operator, Wilson term, antiperiodic boundary conditions, reflection in a site plane, reflection positivity, partition function, global gauge transformation, hypercubic group, generator, chiral symmetry. Their meanings are in the A-02 Glossary.

Next chapter to read: 20-06 The area law at strong coupling, and an upper bound on the vortex free energy — estimates of Wilson loops and of the twisted partition function inside the strong-coupling window.

2026-11-13 · chapter Yang–Mills and the mass gap Part 20 Firm theorems — the vessel and strong coupling The area law ·
Kiichi, Shiori and Rin

The area law at strong coupling, and an upper bound on the vortex free energy

This chapter is this book's study.

What this chapter shows — In the window where the coupling $\beta \geq 0$ is small (strong coupling), we read two estimates for the SU(2) Wilson measure on the four-dimensional torus $(\mathbb{Z}/N)^4$ that are machine-checked in Lean. (i) **The area law**: the expectation of the Wilson loop along the boundary of a rectangle with sides R, T is $\leq C\,(Kq)^{\mathrm{area}}$ ($Kq < 1$, K is 441 or 4096, $q = 1 - e^{-2\beta}$), and the constants contain no N, R or T. It holds for rectangles with any corner and orientation, and for the standard rectangle it also holds for the infinite-volume limit measure. (ii) **An upper bound on the vortex free energy**: the difference between the partition function Z_1 with a 't Hooft twist by the centre element -1 and the untwisted Z_0 is small at a rate per unit area independent of N. The prefactor N^2 remains. The proof is made of four parts: the positive mixture, the terms that vanish under reversal of the central sign, the domination lemma, and the connectedness, area lower bound and counting of minimal sets of plaquettes.

Prerequisite chapters — 19-01 The Wilson measure, plaquettes and holonomy, the product of Haar measures; 19-02 the sign of the twist, the vortex ratio; 20-03 The infinite-volume limit. The introductions are 5-02 (strong coupling and the area law) and 25-01 (twisted tori and flux).

Said first — The lattice spacing is fixed at 1, and these are statements only inside the strong-coupling window. Depending on the version of the constants, the window is $\beta < 1.22 \cdot 10^{-4}$ or $\beta < 1.135 \cdot 10^{-3}$ Reading . Nothing is said about the continuum limit, weak coupling ($\beta \to \infty$) or the band. These are statements on the lattice, distinct from "continuum Wilson loops and the area law" Assessment , which the assessment of the published construction lists as an open question. The constants are coarse, and the rates are on the side much smaller than the true values. Each is a formalisation of known results of the type of Osterwalder–Seiler (area law) and of 't Hooft and Münster (vortex free energy) Cited .

The order of this chapter

1. Setting — Wilson loops and the twisted partition function — the closed path on the boundary of a rectangle, q and the window
2. Statement (i) — the area law in finite volume — five versions (Table 1)
3. Statement (ii) — for the infinite-volume limit measure — the limit of the whole sequence, and every limit point
4. Statement (iii) — an upper bound on the vortex free energy — the rate per unit area does not depend on the transverse size
5. Route of the proof — four parts and one sum — the overall flow
6. Parts of the proof (1) — the positive mixture and the domination lemma — a one-step inequality that needs no positivity
7. Parts of the proof (2) — terms that vanish under reversal of the central sign — spanning sets, good sets and bad sets
8. Parts of the proof (3) — connectedness, area lower bound and counting of minimal sets of plaquettes — root, $R\min(T, N-T)$, N^2, $\Delta^{2(n-1)}$
9. Detour: from the parity of row cochains to the area lower bound — a difference equation column by column
10. In numbers — the window, the non-trivial range, comparison with the thin-vortex floor
11. What is said, and what is not

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book. The names of Lean declarations are written outside the labels, on the same line as the scope.

01

Setting — Wilson loops and the twisted partition function

The introduction is 5-02. At strong coupling (small β), the expectation of a Wilson loop starts from terms that tile a surface bounded by the loop one plaquette at a time, and is exponentially small in the area. This is the strong-coupling area law, the starting point of the picture of confinement in lattice gauge theory Cited . This chapter reads that estimate as a theorem about the SU(2) Wilson measure on a finite torus.

The measure. The lattice is $(\mathbb{Z}/N)^4$; the link variables $U_\ell \in SU(2)$ (unit quaternions), the plaquette holonomy U_p and its real part $s_p := \operatorname{Re} U_p = \frac{1}{2}\operatorname{tr} U_p \in [-1,1]$ are as in 19-01. The plaquettes are the $6N^4$ unoriented ones with $i<j$. The Wilson measure is the product of Haar probability measures multiplied by the density $e^{\beta\sum_p s_p}$, and expectations are

$$\langle F\rangle_{N,\beta} := \frac{\int F\, d\mu_{\mathrm{W}}}{\int 1\, d\mu_{\mathrm{W}}}, \qquad d\mu_{\mathrm{W}} = e^{\beta\sum_p s_p}\prod_\ell dU_\ell \tag{1}$$

(the ratio of Lean's `wilsonMeasure N β` to its total mass). β follows the same convention as the standard SU(2) $\beta = 4/g^2$.

The boundary of a rectangle. Fix two directions $i \neq j$, a corner a and sides R, T ($1 \le R, T < N$), and let C be the closed path that goes from a R steps in direction i, T steps in direction j, then back R steps in direction i and T steps in direction j. $W(C)$ is the ordered product of the links along this path (links traversed backwards are conjugated), and $\operatorname{Re} W(C) = \frac{1}{2}\operatorname{tr} W(C)$ is gauge invariant. The standard rectangle has its corner at the origin and lies in the plane spanned by directions 1 and 2 (0 and 1 in the Lean indices). Like Parts 21 and 22, this chapter numbers the directions 1 to 4. 20-01 to 20-05 write the directions 0 to 3 as in the Lean indices, and their directions 0 and 1 are directions 1 and 2 of this chapter. Lean: `AreaLaw.rectWord` (standard), `AreaLaw.rectWordG` (general); the product is `wordHol`. For $R = T = 1$ it coincides literally with the boundary of a single plaquette (`AreaLaw.rectWord_one_one`). Since R, $T < N$, the path does not wind around the torus.

The twisted partition function. Write Z_D for the total mass of the weight $e^{\beta\sum_p \eta_p s_p}$ in which the sign of s_p is reversed on a set D of plaquettes ($\eta_p = -1$ for $p \in D$, $\eta_p = 1$ otherwise). On $(\mathbb{Z}/N)^4$, take D to be "the plaquettes of orientation 1, 2 with $x_1 = x_2 = 0$" (x_3, x_4 arbitrary, N^2 plaquettes), and set $Z_0 := Z_\varnothing$ (the ordinary partition function) and $Z_1 := Z_D$. This is the partition function with one 't Hooft twist by the centre element -1 (the same placement of plaquettes as the twist in 19-02). Lean: `VortexStrong.Ztw`, `VortexStrong.twist`.

The window. Throughout this chapter we set

$$q := 1 - e^{-2\beta} \qquad (\beta \ge 0) \tag{2}$$

so $q \approx 2\beta$. The window has the form $Kq < 1$: for $K = 4096$ it is $\beta < 1.22085\cdot 10^{-4}$, and for $K = 441$ it is $\beta < 1.13507\cdot 10^{-3}$ Reading .

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02
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Statement (i) — the area law in finite volume

Statement (the strong-coupling area law, finite volume)

Let $N \geq 2$, $\beta \geq 0$ and $4096q < 1$. For the boundary C of a standard rectangle with $1 \leq R, T < N$,

$$\left|\langle \operatorname{Re} W(C)\rangle_{N,\beta}\right| \;\leq\; 16\,\frac{(4096\,q)^{R\cdot\min(T,\,N-T)}}{4096\,(1-4096\,q)}. \tag{3}$$

Lean A formalisation of the type of the Osterwalder–Seiler strong-coupling area law. Lean: `AreaLaw.area_law_wilson` (the only assumptions are $\beta \geq 0$, $1 \leq R, T < N$, $4096q < 1$; the constants 16 and 4096 do not depend on N, R, T).

If $2T \leq N$, then $\min(T, N-T) = T$ and the exponent is the area RT (`AreaLaw.area_law_wilson_area`). There are four more versions of the same form (Table 1).

Version	Rectangle	Window	Right-hand side	Lean
Basic	Standard	$4096q < 1$	$16\,(4096q)^{R\min(T,N-T)}/(4096(1-4096q))$	`AreaLaw.area_law_wilson`
Position and orientation	Any corner a, any $i \neq j$	$4096q < 1$	Same	`AreaLaw.area_law_wilson_general`
Tightened constants	Standard	$441q < 1$	$6\,(441q)^{R\min(T,N-T)}/(441(1-441q))$	`AreaLaw.area_law_wilson_sharp`
Tightened, position and orientation	Any corner and orientation	$441q < 1$	Same	`AreaLaw.area_law_wilson_general_sharp`
Maximum of exponents	Standard	$441q < 1$	$6\,(441q)^{\max(R\min(T,N-T),\,T\min(R,N-R))}/(441(1-441q))$	`AreaLaw.area_law_wilson_max_sharp`

Table 1. Five versions of the area law. $\langle\cdot\rangle$ is the normalised expectation of (1). All with $\beta \geq 0$, $1 \leq R, T < N$.

Meaning. The expectation of a Wilson loop is exponentially small in the area enclosed by the loop, at a rate of at least $-\ln(441q)$. The rate and the prefactor depend neither on the lattice size N nor on the size of the rectangle. The versions with position and orientation are corollaries of the invariance of the Wilson measure under translations and permutations of directions (`AreaLaw.wilson_cfgShift_invariant`, `expect_perm`).

On the exponent. If $R \le N/2$ or $T \le N/2$, the exponent of the maximum version is the area RT. When R and T both exceed $N/2$, the exponent is $\max(R(N-T), T(N-R))$, smaller than RT. In this case the $N^2 - RT$ plaquettes of the whole plane of directions 1, 2 with the rectangle removed also have the same boundary mod 2, so the number of plaquettes truly needed can also be smaller than RT (7 plaquettes on two- and three-dimensional lattices with $N = 4$, $R = T = 3$ Reading ; a record of integer programming; not checked on the four-dimensional 4^4). The exponent of the theorem does not exceed that number, and a gap remains. The $N - T$ in $\min(T, N-T)$ is a weakness coming from the tool of the proof (the detour in § 08); it does not mean that $R \times (N - T)$ plaquettes span the boundary C.

03

Statement (ii) — for the infinite-volume limit measure

The space of infinite-volume configurations, the sequence of periodic lifts μ_n (the Wilson measure on the torus $(\mathbb{Z}/2(n+1))^4$ lifted to $\mathbb{Z}^4$, Lean's `liftProb β n`) and weak convergence are as in 20-03. Write $C_{R,T}$ for the boundary of the standard rectangle in $\mathbb{Z}^4$ (Lean's `zRectWord`).

Statement (the strong-coupling area law, infinite volume)

Let $\beta > 0$, $18 \tanh\beta < 1$, $4096q < 1$. Then there is a probability measure ν_β such that the whole sequence μ_n converges weakly to ν_β, and for all $R, T \ge 1$

$$\left| \int \operatorname{Re} W(C_{R,T}) \, d\nu_\beta \right| \;\le\; 16 \, \frac{(4096\,q)^{RT}}{4096\,(1 - 4096\,q)} \cdot \tag{4}$$

Lean A formalisation of the type of the strong-coupling area law. Lean: `AreaLaw.area_law_infvol` (the existence of ν_β is taken from the strong-coupling infinite-volume theorem `InfMain2.wilson_infvol_main2` of 20-03 and 20-04).

The intersection of the two windows is $0 < \beta < 1.22085 \cdot 10^{-4}$, and the window of the area law lies entirely inside the Dobrushin window ($\beta < 0.0556$) Reading .

At every limit point. There is also a form that does not use the existence of the limit. If $\beta \ge 0$, $4096q < 1$, and $\mu_{\varphi(n)} \to \nu$ (weakly) for a strictly increasing φ, then (4) holds for ν (`AreaLaw.area_law_limit_sub`; the case $\varphi = \mathrm{id}$ of the whole sequence is `AreaLaw.area_law_limit`). The proof uses, for $n \ge R + T$, the $2T \le N$ version of the finite-volume (3), and simply keeps the inequality in the limit of the integrals of the bounded continuous function $\operatorname{Re} W(C_{R,T})$.

The infinite-volume version is stated only with the constants 16 and 4096 and the standard rectangle (origin, directions 1, 2). The versions with tightened constants or with arbitrary corner and orientation are in finite volume only.

04

Statement (iii) — an upper bound on the vortex free energy

Statement (the vortex floor at strong coupling, isotropic torus)

Let $N \geq 1$, $\beta \geq 0$, $4096q < 1$. For Z_0, Z_1 on $(\mathbb{Z}/N)^4$ (§ 01),

$$|Z_0 - Z_1| \;\leq\; N^2 \,\frac{(4096\,q)^{N^4}}{4096\,(1-4096\,q)}\,(Z_0 + Z_1). \quad (5)$$

Lean A formalisation of the type of the strong-coupling estimate of 't Hooft's vortex free energy (Münster and others). Lean: `VortexStrong.vortex_floor` (the only assumptions are $\beta \geq 0$, $4096q < 1$). The version with tightened constants `VortexStrong.vortex_floor_sharp` (with $441q < 1$, 4096 replaced by 441); the 441 version with the denominator written as the total mass of the Wilson measure, `VortexStrong.vortex_floor_wilson` ($441q < 1$).

If the coefficient on the right-hand side is ε and $\varepsilon < 1$, then $|Z_0 - Z_1| \leq \frac{2\varepsilon}{1-\varepsilon} Z_0$ (`VortexStrong.vortex_floor_rel`; an arithmetic rewriting).

The same form also exists on the lattice of Part 21 (the torus of transverse size N and longitudinal size M, 19-02), and deals with the same quantity as the vortex ratio θ_M (19-02).

Statement (the vortex floor at strong coupling, torus of transverse size N and longitudinal size M)

Let $N, M \geq 1$, $\beta \geq 0$, $441q < 1$, and let the background twist $n_{12} \in \{0, 1\}$ be arbitrary. For $Z_k := Z_{N,M}(\beta; n_{12}, k)$ $(k = 0, 1)$,

$$|Z_0 - Z_1| \;\leq\; N^2 \,\frac{(441\,q)^{M^4}}{441\,(1-441\,q)}\,(Z_0 + Z_1). \quad (6)$$

Lean A formalisation of the same type. Lean: `VortexStrongMN.vortex_floor_MN_sharp` (the version with constant 9216 is `VortexStrongMN.vortex_floor_MN`).

Since $|Z_0 - Z_1|/(Z_0 + Z_1) = |\theta_M^{(n12)}|$, (6) is the same as

$$|\theta_M^{(n12)}| \;\le\; N^2 \, \frac{(441\,q)^{M^2}}{441\,(1 - 441\,q)} \qquad (7)$$

Proved here .

Meaning. Inserting one vortex hardly changes the partition function, and the difference is exponentially small in the area M^2 of the longitudinal plane. **The rate per unit area $-\ln(441q)$ depends neither on the transverse size N nor on the longitudinal size M.** The prefactor N^2 (the number of plaquettes of the twist) remains. The thin-vortex floor (21-02), $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$, holds for all β, but its rate per unit area $-\ln\tanh(N^2|\beta|)$ approaches 0 as $N \to \infty$. (7) holds only inside the strong-coupling window, but differs in that its rate does not depend on N (comparison in § 09).

The statements of this section are upper bounds on the difference $|Z_0 - Z_1|$, and do not give the direction $Z_1 \le Z_0$ ($\theta_M \ge 0$). That direction is a separate theorem for even longitudinal size M (21-06).

05

Route of the proof — four parts and one sum

The area law (3) and the vortex floors (5), (6) are shown along the same line. We write it for the area law.

1. **Positive mixture (§ 06).** Write the plaquette weight as $e^{\beta s_p} = e^{-\beta}(1 + \sigma_p)$ with $0 \le \sigma_p \le t := e^{2\beta} - 1$, and expand $\prod_p(1 + \sigma_p)$ as a sum $\sum_X \prod_{p \in X} \sigma_p$ over sets X of plaquettes. Numerator and denominator become sums over all X, $\sum_X \int W \prod_X \sigma$ and $\sum_X Z_X$ ($Z_X := \int \prod_{p \in X} \sigma_p \, dU \ge 0$).
2. **Vanishing terms (§ 07).** The transformation that reverses the sign $U_\ell \mapsto -U_\ell$ on a set of links preserves the product of Haar measures. A term in which the boundaries of the plaquettes of X do not span C mod 2 changes sign under this transformation, and is 0.
3. **Domination (§ 06).** From $Z_{X \cup \{p\}} \le t\, Z_X$, the sum of the weights of the X containing a set Y of plaquettes is at most $q^{|Y|} \sum_X Z_X$.
4. **One sum.** Since $|W| \le 1$, the remaining terms are bounded by $q^{|Y|}$ for each "minimal set Y of plaquettes spanning C", giving

$$\big|\langle \operatorname{Re} W(C)\rangle\big| \;\le\; \sum_{Y:\ C\text{-spanning minimal set}} q^{|Y|} \qquad (8)$$

This holds for all $\beta \ge 0$ and all N (Lean `AreaLaw.abs_wordExpect_le`; the family of minimal sets `minSpanC` is placed concretely inside the definition, and is not an assumption).

5. **Geometry of minimal sets of plaquettes (§ 08).** A minimal Y is connected in the sense of sharing links, contains a plaquette touching a fixed link, and has $|Y| \ge R \min(T, N-T)$. The number of connected sets of size n touching the root is at most $16 \cdot 64^{2(n-1)}$, so the right-hand side of (8) is at most $16 \sum_{n \ge A} 4096^{n-1} q^n$ ($A = R \min(T, N-T)$), and summing the geometric series gives (3).

For the vortex floor, step 2 is replaced by "on good sets the twisted and untwisted weights coincide", step 4 by "the sum over bad sets", and the area lower bound of step 5 by "a bad set has at least N^2 plaquettes" (§ 07, § 08).

This line does not use the many-body sums of the cluster expansion. The only inequalities used are $0 \le \sigma_p \le t$ at each point and induction over finite sets. It is accounting of the same family as the "two copies and chains of plaquettes" of strong-coupling decay of correlations (20-02 § 06).

06

Parts of the proof (1) — the positive mixture and the domination lemma

Positive mixture. For $s \in [-1, 1]$,

$$e^{\beta s} = e^{-\beta}\big(1 + \sigma\big), \qquad \sigma := e^{\beta(s+1)} - 1 \in [0,\, e^{2\beta} - 1] \qquad (9)$$

($\beta \ge 0$). Lean: `AreaLaw.exp_eq_mix`, `mixVar_mem`. A product over a finite set is $\prod_p (1 + a_p) = \sum_X \prod_{p \in X} a_p$ (`AreaLaw.prod_one_add_univ`), and the form expanded inside an integral is `AreaLaw.integral_mul_prod_one_add`. That each σ in the expanded terms is at least 0 is the source of treating the weights, in the domination lemma below, as quantities that "increase as more are added".

The one-step inequality. For $Z_X := \int \prod_{p \in X} \sigma_p \prod_\ell dU_\ell$, using $\sigma_p \le t$ pointwise gives

$$Z_{X \cup \{p\}} \;\le\; t\, Z_X \qquad (p \notin X) \qquad (10)$$

(`AreaLaw.ZX_insert_le`).

The domination lemma (Lean: `AreaLaw.sum_superset_le`). If a real-valued function Z on the subsets of a finite set P and $t \geq 0$ satisfy $Z(S \cup \{p\}) \leq t\, Z(S)$ for all S and $p \notin S$, then for all Y

$$\sum_{X \supseteq Y} Z_X \;\leq\; \Big(\frac{t}{1+t}\Big)^{|Y|} \sum_{X} Z_X \qquad (11)$$

Here $t/(1+t) = 1 - e^{-2\beta} = q$ (`AreaLaw.q_eq`).

The core of the proof is one line. Let $a_Y := \sum_{X \supseteq Y} Z_X$; for $p \notin Y$, splitting a_Y into "X containing p" and "X not containing it" gives $a_Y = a_{Y \cup \{p\}} + b$ (b the side not containing it), and using (10) term by term gives $a_{Y \cup \{p\}} \leq t\, b$. Together, $(1+t)\, a_{Y \cup \{p\}} \leq t\, a_Y$, that is, $a_{Y \cup \{p\}} \leq q\, a_Y$. Induction on the size of Y gives (11). $Z \geq 0$ **is not used.** It follows from the splitting and the one-step inequality alone.

07

Parts of the proof (2) — terms that vanish under reversal of the central sign

Reversal. For an $\mathbb{F}_2$-valued function (cochain) T on links, let flip_T be the transformation $U_\ell \mapsto -U_\ell$ on the links with $T_\ell = 1$. -1 is in the centre of SU(2), and $-U$ has the same distribution as U, so flip_T is a measurable involution preserving the product of Haar measures (`AreaLaw.flipZ_preserving`). The product along a path w changes as

$$W(\mathrm{flip}_T\, U) = (-1)^{\langle T,\, \partial w \rangle}\, W(U) \qquad (12)$$

where ∂w is the number of times the path w passes each link, mod 2, and $\langle T, \partial w \rangle$ is its sum (`AreaLaw.wordHol_flip`; no condition on the lattice size is needed). The plaquette holonomy is the product along the path once around the boundary of the plaquette (`hol_eq_wordHol`), so s_p is also multiplied by $(-1)^{\langle T, \partial p \rangle}$.

Vanishing terms (Lean: `AreaLaw.integral_word_mul_prod_eq_zero_of_not_mem_span`). If ∂C is not in the $\mathbb{F}_2$-span of $\{\partial p : p \in X\}$, then for any function g

$$\int \mathrm{Re}\, W(C) \prod_{p \in X} g(s_p) \prod_{\ell} dU_\ell \;=\; 0. \qquad (13)$$

By duality in $\mathbb{F}_2$ linear algebra (`AreaLaw.not_mem_span_iff_exists_sep`), not being in the span is equivalent to "there is a T with $\langle T, \partial p \rangle = 0$ on every plaquette of X and $\langle T, \partial C \rangle = 1$ on C". Reversing by that T, the factor $\prod g(s_p)$ of the integrand is unchanged

and only W changes sign, so the integral equals its own negative and is 0. g is arbitrary, and not even measurability is needed.

The correct criterion is "lies in the span", not "X contains the plaquettes of C". For example, on a periodic torus the boundary of one plaquette can equal, mod 2, the sum of the boundaries of all the remaining plaquettes. For the quaternion group Q_8, summing over all configurations of a two-dimensional 2×2 torus, the terms of the 7 non-spanning X were exactly 0, those of the 9 spanning ones were not 0, and $\{1, 2, 3\}$, which does not contain plaquette 0, was not 0 either Reading .

On the vortex-floor side: good sets and bad sets. The twisted weight can be written $e^{\beta \sum_p (-1)^{m_p} s_p}$ using a mask of signs per plaquette $m : \text{plaquettes} \to \mathbb{F}_2$ (1 on D). The reversal flip_T changes the mask as $m \mapsto m + \delta T$ (δT the coboundary of T, $(\delta T)_p = \langle T, \partial p \rangle$) (`VortexStrong.ZM_gauge`). When there is a T for which the mask equals a coboundary on the set X of plaquettes, that is $m_p = \langle T, \partial p \rangle$ ($p \in X$), we call X a **good** set (Lean `VortexStrong.Good`). On good sets a single reversal makes the twisted and untwisted terms coincide (`VortexStrong.ZM_good`), and only the terms of **bad** sets remain in $Z_0 - Z_1$. Using the domination lemma for both weights,

$$|Z_0 - Z_1| \;\le\; \Big(\sum_{Y\text{: minimal bad set}} q^{|Y|} \Big) (Z_0 + Z_1) \qquad (14)$$

(`VortexStrong.abs_Z_sub_le`). The vortex floor does not use duality, because the definition of good and bad is itself written in terms of the existence of a coboundary.

08

Parts of the proof (3) — connectedness, area lower bound and counting of minimal sets of plaquettes

Connectedness (area law). Call two plaquettes neighbours when they share a link. Split a minimal set Y spanning C into two parts and suppose there is no pair of neighbours between them; then the sum of the boundaries of one part is closed ($\partial\partial = 0$, `AreaLaw.div_plaqBd`) and its support lies in the boundary of the rectangle. The boundary of the rectangle is a simple closed cycle, so such a closed chain is only 0 or ∂C itself (`AreaLaw.rect_closed_cases`, $1 \le R, T < N$). Either contradicts the minimality of Y, so a minimal Y is connected (`AreaLaw.minSpan_rect_conn`). Moreover Y contains a plaquette touching a fixed link of the rectangle (the root).

Connectedness (vortex floor). If a minimal bad set splits into two parts sharing no neighbours, one can glue the cochains of the two sides, "on the support of one" and "outside it", into a cochain for the whole, making the whole good, a contradiction (`VortexStrong.minBad_conn`).

Area lower bound (area law). A set of plaquettes spanning C has at least $R \min(T, N - T)$ plaquettes (`AreaLaw.card_ge_of_spans_rect`). The proof is placed in the detour. The same argument with directions 1 and 2 exchanged gives $T \min(R, N - R)$ as a lower bound too (`AreaLaw.wordBd_swap_rect`), and their maximum is the last version in Table 1.

Area lower bound (vortex floor). The mask of the translate $D_{a,b}$ of D (plaquettes of orientation 1, 2 with $x_1 = a$, $x_2 = b$) differs from the mask of D only by the coboundary of row and column cochains (`VortexStrong.good_twist_iff`). Hence a bad set meets every one of the N^2 mutually disjoint $D_{a,b}$, and has at least N^2 plaquettes (`VortexStrong.card_bad_ge`). The lower bound is attained: the plane $x_3 = x_4 = 0$ of directions 1, 2 (N^2 plaquettes) is a bad set (`VortexStrong.plane_bad`). A bad set contains a plaquette of D, so the roots are the N^2 plaquettes of D. This is the prefactor N^2 in (5).

Counting. If, for a symmetric neighbour relation, each point has at most Δ neighbours, then the number of connected sets of size n containing a fixed root is at most $\Delta^{2(n-1)}$ (`AreaLaw.card_conn_le`). It follows by an induction that inserts "go to a neighbour and come back" detours into a closed walk, without building a spanning tree. In the coarse count there are at most 64 neighbours and at most 16 roots, and $64^2 = 4096$. In the tightened count, the plaquettes containing a given link number 6, the plaquettes sharing a link with a given plaquette number at most 21 including itself ($1 + 4 \cdot 5$), so $21^2 = 441$, and there are 6 roots (`AreaLaw.card_rootP_le`, `card_nbQ_le`). The geometric series

$$\sum_{n \geq A} c_0 \, \Delta^{2(n-1)} q^n = \frac{c_0}{\Delta^2} \cdot \frac{(\Delta^2 q)^A}{1 - \Delta^2 q} \qquad (\Delta^2 q < 1) \tag{15}$$

gives the right-hand sides of (3), (5) and (6) (c_0 is the number of roots).

Detour: from the parity of row cochains to the area lower bound

May be skipped. For the standard rectangle (plaquettes of orientation 1, 2), for a column $0 \leq a < R$ and a row $y \in \mathbb{Z}/N$, consider the cochain $\rho_{a,y}$ that is 1 on "the links of direction 1 whose coordinates in directions 1, 2 are (a, y)" (all values of the other coordinates). Its pairing with plaquette boundaries is $\langle \rho_{a,y}, \partial p \rangle = [p \in Q(a,y)] + [p \in Q(a, y-1)]$ ($Q(a,y)$ is the class of plaquettes of orientation 1, 2 at coordinates (a, y); plaquettes of orientation 1, 3 or 1, 4 are hit twice and cancel), and its pairing with the boundary of the rectangle is $\langle \rho_{a,y}, \partial C \rangle = [y = 0] + [y = T]$. If X spans C, then, with $n(y)$ the number of plaquettes of X in $Q(a,y)$, $n(y) + n(y-1) = [y = 0] + [y = T]$ holds mod 2 for every y. Solving this difference equation, n is odd on all of $[0, T)$ or odd on all of $[T, N)$. Either way column a needs at least $\min(T, N - T)$ plaquettes, and R

columns need at least $R \min(T, N-T)$ Proved here . The solution on the $[T, N)$ side is a solution of the parity equation; it does not mean that the plaquettes on that side span the boundary C.

09

In numbers

Quantity	Value	Label
Window $4096q < 1$	$\beta < 1.22085 \cdot 10^{-4}$	Reading
Window $441q < 1$	$\beta < 1.13507 \cdot 10^{-3}$	Reading
Window of the infinite-volume version (intersection with $18 \tanh \beta < 1$)	$0 < \beta < 1.22085 \cdot 10^{-4}$	Reading
Range in which (3) is non-trivial (right-hand side < 1) for every area ≥ 1	$\beta < 1.2161 \cdot 10^{-4}$	Reading
Range in which the coefficient of (5) is < 1 for every N (4096 version / 441 version)	$\beta < 1.2093 \cdot 10^{-4}$ / $\beta < 1.1025 \cdot 10^{-3}$	Reading
Rate per unit area $-\ln(441q)$: $\beta = 10^{-5}, 10^{-4}, 10^{-3}$	`4.73, 2.43, 0.127`	Reading
Rate of the leading term at small β, $-\ln(\beta/4)$: same β	`12.9, 10.6, 8.29`	Reading

Table 2. Windows and rates. Floating-point calculations. The rate of the leading term is a rough guide read from the average of a single plaquette, $I_2(\beta)/I_1(\beta) = \beta/4 + O(\beta^3)$, not an estimate.

The rates of the upper bounds are much smaller than the rates of the leading term. The constant 441 (or 4096) comes from the coarseness of the count of connected sets, and also determines the width of the window.

Comparison with the thin-vortex floor Reading . With $N = M$, comparing the common logarithms of the upper bounds (7) and the thin-vortex floor $\tanh(N^2\beta)^{N^2}$, (7) becomes smaller (stronger) for $N \geq 30$ at $\beta = 10^{-5}$ and 10^{-4}, and for $N \geq 38$ at $\beta = 10^{-3}$. For example, at $\beta = 10^{-5}$, $N = 100$, the thin floor gives $-1.00 \cdot 10^4$ against $-2.05 \cdot 10^4$ for (7); at $N = 10$, the thin floor gives -300 against -206 for (7). The thin floor holds for all β, whereas (7) holds only inside the window.

Checks on small lattices Reading . We looked at the minimum size of bad sets by exhaustive search and integer programming in a separate script copying the Lean definitions. It was 4 on the three-dimensional 2^3, 4 on the four-dimensional 2^4, and 9 on 3^4 (all N^2); on lattices of transverse size N and longitudinal size M with $(N, M) = (2, 3), (3, 2), (2, 4)$ it was 9, 4, 16 $(= M^2)$, the same values as the Lean lower bound.

10

What is said, and what is not

- **The lattice spacing is fixed at 1.** These are statements about finite tori and their infinite-volume limit measure. Nothing is said about the continuum limit or sequences sending the lattice spacing to 0.
- **Only the strong-coupling window.** $\beta < 1.135 \cdot 10^{-3}$ (tightened constants), and $\beta < 1.22 \cdot 10^{-4}$ for the infinite-volume version and the basic version of the vortex floor. Nothing is said outside the window (nor that the area law fails there). This is the end far from the β of physical interest (the band around 2 to 3).
- **Rectangles only.** Loops along general closed paths are not treated. The infinite-volume version is only for the standard rectangle with the coarse constants.
- **A gap in the exponent.** When R and T both exceed $N/2$, the exponent stays smaller than the number of plaquettes truly needed.
- **The rates are coarse.** The rate per unit area is much smaller than the rate of the leading term at small β, and is not an estimate of the value of the tension. Estimates from below (lower bounds on Wilson loops, upper bounds on the tension) are not said.
- **The prefactor of the vortex floor.** The rates of (5) and (6) do not depend on N or M, but the prefactor N^2 remains. The direction $Z_1 \leq Z_0$ is not said.
- **Reaches neither the band nor the mass gap.** The mass gap inside the strong-coupling window is a separate theorem in 20-04. The band statement continuing to weak coupling (19-07) does not follow from the theorems of this chapter.
- **Distinct from the open question of the assessment.** The question listed by the assessment of the published construction, "construct Wilson-loop observables, connect them to local observables, and show the area law" Assessment , is a question about the continuum theory. This chapter is a strong-coupling statement on the lattice and does not answer that question.
- **Not a new result.** It is a formalisation, for the SU(2) Wilson measure, of the type of the strong-coupling area law (Osterwalder–Seiler) and of the strong-coupling estimate of the vortex free energy ('t Hooft, Münster and others).

The scope of what this chapter says

Said	Not said
Lean The area law in finite volume (3): $\beta \geq 0$, $4096q < 1$, $1 \leq R, T < N$, no N, R, T in the constants. Any corner and orientation; the version with constants 6 and 441; the version with exponent $\max(R\min(T, N-T), T\min(R, N-R))$. Lean `AreaLaw.area_law_wilson`, `area_law_wilson_general`, `area_law_wilson_sharp`, `area_law_wilson_general_sharp`, `area_law_wilson_max_sharp`	Outside the window. General loops. The true exponent when both R, $T > N/2$. Lower bounds on Wilson loops
Lean The area law in infinite volume (4): $\beta > 0$, $18\tanh\beta < 1$, $4096q < 1$, the limit ν_β of the whole sequence. Also at every limit point ($\beta \geq 0$, $4096q < 1$). Lean `AreaLaw.area_law_infvol`, `area_law_limit`, `area_law_limit_sub`	Infinite-volume versions with tightened constants or with arbitrary corner and orientation
Lean The upper bound on the vortex free energy (5): $(\mathbb{Z}/N)^4$, $\beta \geq 0$, $4096q < 1$ (tightened version $441q < 1$), the rate per unit area does not depend on N. Lean `VortexStrong.vortex_floor`, `vortex_floor_sharp`, `vortex_floor_rel`, `vortex_floor_wilson` (the 441 version)	Removal of the prefactor N^2. The direction $Z_1 \leq Z_0$. Outside the window
Lean The version (6) with transverse size N and longitudinal size M: every background n_{12}. The tightened version `vortex_floor_MN_sharp` with $441q < 1$ (the 441 window), the untightened version `vortex_floor_MN` with 9216q < 1 (the 9216 window). Lean `VortexStrongMN.vortex_floor_MN_sharp`, `vortex_floor_MN`	The band statement. The continuum limit
Proved here (7): (6) written in terms of the vortex ratio. The parity argument of the detour	—
Reading Windows, non-trivial ranges, rates (Table 2), comparison with the thin-vortex floor (crossovers at $N = 30, 38$), minimal sets on small lattices	Not proofs of upper or lower bounds. The true rate for four-dimensional SU(2)

Sources and reproduction

Item	Kind	Source / tool
The strong-coupling area law and the foundations of lattice gauge theory	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. K. G. Wilson, Phys. Rev. D 10 (1974) 2445. The introduction is in 5-02
Twisted boundary conditions and the vortex free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. Münster, Nucl. Phys. B 180 (1981) 23–60 (bibliography only; the text not checked). The introduction is in 25-01

Item	Kind	Source / tool
Open question of the published construction (Wilson loops and the area law)	Assessment	The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8, item 4. How to read it is in 16-02
(3) and the versions of Table 1; parts (mixture, domination, reversal, duality, connectedness, lower bounds, counting, translation and orientation)	Lean	`AreaLaw.area_law_wilson`, `area_law_wilson_area`, `area_law_wilson_general`, `area_law_wilson_sharp`, `area_law_wilson_general_sharp`, `area_law_wilson_max_sharp`, `abs_wordExpect_le`, `exp_eq_mix`, `mixVar_mem`, `prod_one_add_univ`, `ZX_insert_le`, `sum_superset_le`, `q_eq`, `flipZ_preserving`, `wordHol_flip`, `not_mem_span_iff_exists_sep`, `integral_word_mul_prod_eq_zero_of_not_mem_span`, `rect_closed_cases`, `minSpan_rect_conn`, `card_ge_of_spans_rect`, `wordBd_swap_rect`, `card_conn_le`, `card_rootP_le`, `card_nbQ_le`, `wilson_cfgShift_invariant`, `rectWord_one_one`. sorry 0, the three standard axioms, confirmed by re-checking with the kernel, statements checked against the wording of the source
(4)	Lean	`AreaLaw.area_law_infvol`, `area_law_limit`, `area_law_limit_sub`; the existence of the limit is `InfMain2.wilson_infvol_main2` (20-03)
(5), (6) and parts (good sets, bad sets, gluing, translation)	Lean	`VortexStrong.vortex_floor`, `vortex_floor_sharp`, `vortex_floor_rel`, `vortex_floor_wilson`, `ZM_gauge`, `Good`, `ZM_good`, `abs_Z_sub_le`, `good_twist_iff`, `card_bad_ge`, `minBad_conn`, `plane_bad`; `VortexStrongMN.vortex_floor_MN_sharp`, `vortex_floor_MN`
Windows, rates, non-trivial ranges, comparison with the thin-vortex floor	Reading	The numerical record of this book's study (floating point; recomputed with a separate script)
The spanning criterion (Q_8, 2×2); minimum sizes of bad sets and spanning sets	Reading	The numerical record of this book's study (sums over all configurations, exhaustive search, integer programming; no proof of optimality for the four-dimensional 4^4)

Terms used in this chapter — strong coupling, Wilson loop, area law, positive mixture, domination lemma, reversal of the central sign, spanning set, cochain, coboundary, good set, bad set, root, counting of connected sets, 't Hooft twist, vortex free energy. Their meanings are in A-02 Glossary.

Next chapter to read: 20-07 What this part says, and what it does not — the theorems of Part 20 in one table. Other forms of the vortex floor are in 21-02 and 21-07.

What this part says, and what it does not

> This chapter is this book's own investigation. It is a chapter of tables.

What this chapter shows — The theorems of Part 20 gathered into one table, with their scope (range of β, lattice size, window), what they do not say, and the names of their Lean declarations.

Prerequisite chapters — 20-00 Entrance. The content of each row is in 20-01 to 20-06.

Said first — All the theorems of this part are statements on a lattice with the lattice spacing fixed at 1, and are formalisations of results of known types. No row says anything about the continuum limit, weak coupling, or an upper bound on the mass.

The order of this chapter

1. How to read the tables: notation and columns.
2. Reflection positivity (20-01): the three forms and the symmetry of the pairing.
3. Strong-coupling decay (20-02): two estimates in finite volume.
4. The infinite-volume limit and reconstruction (20-03, 20-04): limit, translation, clustering, gap, non-degeneracy, $T \geq 0$.
5. The vessel with fermions (20-05): four theorems on a finite lattice.
6. The area law at strong coupling, and an upper bound on the vortex free energy (20-06): the area law in finite and infinite volume, and the upper bound on the vortex free energy.
7. What the part as a whole does not say: limits common to all rows.

01

How to read the tables

Each row is one theorem. “Scope” copies the hypotheses of the Lean statement and the main form of the conclusion; “What is not said” gives limits specific to that row. Every row is machine-checked in Lean 4. The rows from 20-01 to 20-05 (§ 02– § 05) carry the

label Cited (a formalisation of a result of a known type from the literature). The exception is the rows of § 05 on the determinant and the lattice symmetries, which carry Proved here (a formalisation of known facts whose proof is given in the text). The rows from 20-06 (§ 06) carry the label Lean (a Lean theorem of this book, as in the legend of 20-06). Notation follows Part 19.

- $q(\beta) = 18 \tanh |\beta|$. The window $q(\beta) < 1$ is $|\beta| < 0.0556$ Reading .
- The narrow window $24(1 - e^{-4|\beta|}) < 1$ is $|\beta| < 0.0106$ Reading .
- μ_n: the periodic lift to $\mathbb{Z}^4$ of the Wilson measure on the torus $(\mathbb{Z}/2(n+1))^4$. ν_β: its limit without a subsequence (20-03).
- "Limit point": the weak limit of some subsequence of μ_n.
- W: cylinder functions on the closed positive half. W_{gi}: the gauge-invariant ones among them (20-04).

02

Reflection positivity (20-01)

Theorem	Scope	What is not said	Lean declaration
Reflection positivity for the plane through sites	$(\mathbb{Z}/2M)^4$, all β, bounded measurable f depending only on the closed positive half	The plane through the middle of links; the time direction only	`TorusRP.wilson_rp`
Reflection positivity for the middle plane	$(\mathbb{Z}/2M)^4$, $M \geq 2$, $\beta \geq 0$, gauge-invariant bounded measurable F	F that is not gauge invariant; $\beta < 0$. This theorem alone does not enter the vessel (the $T \geq 0$ of the vessel is a separate theorem in 20-04)	`MidRP.wilson_rp_mid`
Reflection positivity for site planes in four directions	$(\mathbb{Z}/2M)^4$, all β, for each direction ρ	A four-direction form for the plane through the middle of links	`PeierlsPerm.wilson_rp_dir`
Symmetry of the reflection pairing	$(\mathbb{Z}/2M)^4$, all β, measurable f, g	—	`TorusRP.wilson_refl_symm`

03

Strong-coupling decay (20-02)

Theorem	Scope	What is not said	Lean declaration
Finite-volume decay of correlations at strong coupling	Narrow window, for each finite N, no N in the constants, time distance	Distance in spatial directions; outside the window	`TorusRP.wilson_strong_coupling`
Dobrushin decay of covariance with distance	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, finite N, no N in the constants	Outside the window; the true rate of decay	`LinkSpec.wilson_cov_dist`

04

The infinite-volume limit and reconstruction (20-03, 20-04)

Theorem	Scope	What is not said	Lean declaration
The theorem on the limit without a subsequence	$\beta \neq 0$, $q(\beta) < 1$	Other boundary conditions; uniqueness of the Gibbs measure. "Unique" is trivial from the topology; the content is convergence of the sequence	`InfVol.wilson_infvol_unique`
The theorem on the limit independent of size	$\beta \neq 0$, $q(\beta) < 1$, all sizes $m+1$, any $s(k) \to \infty$	The rate of convergence (outside the statement)	`AllN.wilson_infvol_allN`
Translation invariance of the limit	$\beta \neq 0$, $q(\beta) < 1$, unit translations in four directions	Rotations and reflections	`TransI.wilson_infvol_translation`
Clustering at limit points	$\lambda \geq 0$, $e^{\lambda} q(\beta) < 1$, at every limit point, four directions	The true rate of decay	`WilsonLimit.cluster_limit`
The mass gap with non-degeneracy (Dobrushin window)	$\beta \neq 0$, $q(\beta) < 1$; the statement picks one limit point (the same estimate at every limit point: `WilsonLimit.wilson_massGap_dob`); $\|T - P_\Omega\| \leq q(\beta)^{1/2}$, $\mathrm{Var}(F_0) \geq \frac{1}{4} e^{-12\|\beta\|}$	That the mass is finite (lower bound only); the $\frac{1}{2}$ is a matter of choice	`WilsonLimit.wilson_massGap_dob_exi`

Theorem	Scope	What is not said	Lean declaration
The mass gap with non-degeneracy (every limit point)	$\beta \neq 0$, narrow window, every limit point, rate $-\log 24(1-e^{-4\lvert\beta\rvert})$ (not included in the bundled statement)	As above	`WilsonLimit.wilson_massGap_nondege`
The mass gap in the gauge-invariant sector	$\beta \neq 0$, $q(\beta) < 1$; the statement picks one limit point; witness $F_p = \mathrm{Re}\, U_p$	That the closure of W_{gi} coincides with the physical space of states	`GaugeZ.wilson_massGap_gauge_exists`
$T \geq 0$ (on the whole W)	$\beta \geq 0$, $\lambda > 0$, $e^{\lambda} q(\beta) < 1$, every limit point	$\beta < 0$	`FullT.wilson_Tpos_full`
The range of T lies in the gauge-invariant sector	$\lambda > 0$, $e^{\lambda} q(\beta) < 1$, every limit point	Not included in the bundled statement	`Bind.T_range_gauge_all`
The bundled statement	$0 < \beta$, $18 \tanh \beta < 1$. Convergence, translation, reflection positivity, gap and non-degeneracy in both sectors, $T[F_0] = 0$, $T \geq 0$, clustering	The continuum limit; an upper bound on the mass; outside the window	`InfMain2.wilson_infvol_main2`

05

The vessel with fermions (20-05)

The lattice is $(\mathbb{Z}/N)^3 \times \mathbb{Z}/2h$ ($h > 0$), the weight is $e^{-\beta S_W}\, \mathrm{Re}\big[(\det D)^{2j}\big]$, and D is the Wilson–Dirac operator ($r = 1$, antiperiodic in four directions).

Theorem	Scope	What is not said	Lean declaration
Reflection positivity of the Wilson–fermion weight	$m > -1$, all β and j, time plane through sites	$m \leq -1$; the middle plane; infinite volume	`WilsonGauge.site_reflection_positive_wilson_full`
Positivity of the partition function	All β, m, j	The size of Z	`WilsonGauge.partition_pos_wilson_full`

Theorem	Scope	What is not said	Lean declaration
Global gauge invariance of the determinant	Unconditional; constant g	Local gauge transformations	`GaugeDet.physDirac_det_conj`
Invariance under the generators of the lattice symmetries	All N, j, m, β: global gauge, translations in four directions, spatial 90° rotations, reflections of spatial axes, time reflection	Closure as a group; lifting to the block level	`CubicSym.wfProb_lattice_generators`
Invariance under the hypercubic generators	Isotropic lattice $N = 2h$; the above plus (space, time) 90° rotations	Rotations involving time on anisotropic lattices	`CubicSym.wfProb_hypercubic_generators`
Invariance of the determinant under time reflection	Unconditional (all N, h, m, ω); the reversal $t \mapsto 2h - 1 - t$	Reflection positivity for this reversal	`CubicSym.physDirac_det_trefl`

06

The area law at strong coupling, and an upper bound on the vortex free energy (20-06)

Set $q = 1 - e^{-2\beta}$. The window differs for each version of the constants (details in 20-06).

Theorem	Scope	What is not said	Lean declaration
The area law in finite volume	$\beta \geq 0$, $4096q < 1$ (the tightened version: $441q < 1$), the Wilson loop on the boundary of a rectangle; the rate and the prefactor depend neither on the size of the lattice nor on the size of the rectangle	Outside the window, general loops, a lower bound on Wilson loops, continuum Wilson loops	`AreaLaw.area_law_wilson`, `AreaLaw.area_law_wilson_sharp`, `AreaLaw.area_law_wilson_max_shar`

Theorem	Scope	What is not said	Lean declaration
The area law in infinite volume	$\beta > 0$, the intersection of $18 \tanh \beta < 1$ and $4096q < 1$, the limit measure (also at every limit point)	Infinite-volume versions with the tightened constants, or with arbitrary corners and orientations	`AreaLaw.area_law_infvol`, `AreaLaw.area_law_limit_sub`
Upper bound on the vortex free energy	$\beta \geq 0$. On a torus of transverse size N and longitudinal size M, for every background, $441q < 1$ (`VortexStrongMN.vortex_floor_MN_sharp`). On an isotropic torus, $4096q < 1$ (`VortexStrong.vortex_floor`), and the tightened version $441q < 1$ (`VortexStrong.vortex_floor_sharp`). The difference is exponentially small in the area M^2 of the longitudinal plane, and the rate per unit area depends neither on the transverse size nor on the longitudinal size	Removal of the prefactor N^2 (the transverse area), the direction of the sign of the difference, outside the window, the band statement	`VortexStrongMN.vortex_floor_MN_s`, `VortexStrong.vortex_floor`, `VortexStrong.vortex_floor_sharp`

All of these are formalisations of known results of the type of Osterwalder–Seiler (area law) and of 't Hooft and Münster (vortex free energy)Cited . The area law is a statement about lattice Wilson loops; continuum Wilson loops are present neither in this part nor in the published construction.

07

What the part as a whole does not say

- **The lattice spacing is fixed at 1.** No row says anything about the continuum limit or about uniformity when the lattice spacing is changed.
- **Only the strong-coupling window.** The rows on the area law and the upper bound on the vortex free energy (§ 06) also hold only inside the strong-coupling window. The infinite-volume rows hold only for $|\beta| < 0.0556$Reading (the narrow window: $|\beta| < 0.0106$Reading), and nothing is said about weak coupling.
- **Only a lower bound on the mass.** The mass-gap rows are upper bounds on $\|T - P\Omega\|$. It is not said that the mass is finite, nor how it compares with the true mass.
- **Formalisations of known types.** These are known results of Osterwalder–Seiler strong-coupling type, of Dobrushin-uniqueness type, and of the type of the strong-coupling area law and the vortex free energy, written in Lean on the concrete SU(2) Wilson measure. They do not show anything about the mass gap problem for four-dimensional Yang–Mills. They are statements at the opposite end (strong coupling,

lattice spacing fixed) from the questions the published construction left open (convergence and uniqueness, a uniform mass gap, continuum Wilson loops and the area law) (14-06).

The next part treats the vortex floor and Polyakov loops on a twisted lattice, at the same level of “one lattice spacing”.

What this chapter can say

What is said	What is not said
Cited Each row of the tables of § 02– § 05, Lean each row of the table of § 06 (all machine-checked in Lean; scope as in each row)	Changing the lattice spacing, the continuum limit, outside the window, an upper bound on the mass
Reading The edges of the windows, $\lvert\beta\rvert < 0.055613$ and $\lvert\beta\rvert < 0.010640$	Floating-point values (not interval estimates)

Sources and reproduction

Item	Kind	Source / tools
Original sources for the type of each row	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (abstract). The original source of Dobrushin's uniqueness condition: bibliographic data only, not checked. The source tables of each chapter
The declarations of each row	Lean	Machine-checked in Lean 4 (all pass the kernel re-check; the axioms are the three standard ones). A-04
The numbers at the edges of the windows	Reading	The numerical record of this book's study (double precision)

Words used in this chapter — strong-coupling window, limit point, mass gap (lower bound), non-degeneracy, gauge-invariant sector, reflection positivity, generator. Their meanings are in the A-02 Glossary.

Next chapter to read: 21-00 Entrance — vortices in a twisted box — on to Part 21, which treats the vortex floor and Polyakov loops on a twisted lattice.