Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 21 Firm theorems — the vortex floor and Polyakov loops

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Entrance — vortices in a twisted box

> This chapter is a guide.

What this chapter shows — a list of the theorems in the chapters of Part 21, the scope that applies to the whole part (what it does not say), and the direction of the word "floor".

Prerequisite chapters — 19-02 the torus of width N and length M, the vortex ratio, Polyakov loops.

Said first — every theorem in this part is a statement on a finite torus at a fixed lattice spacing. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, none of the rates of the floors of this part (upper bounds on θ) is uniform in the transverse size N. The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ). None reaches the band or the mass gap (the statements to be shown are in 19-07). Each is a formalisation of one side of a known picture.

The order of this chapter

1. The theorems of this part
2. The scope that applies to the whole part
3. What a "floor" is
4. A guide to the record of the study

01

The theorems of this part

The objects throughout are the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N (directions 1 and 2) and length M (directions 3 and 4), its twisted partition

functions $Z_0,\ Z_1$, and the vortex ratio $\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ (definitions in 19-02).

Chapter	Theorem (one line)
21-01 Three results on sign systems	The covering inequality, $\mathbb{Z}_2$ monotonicity, and comparison with $\mathbb{Z}_2$ (abstract implications, with all assumptions shown), and their application to the $SU(2)$ Wilson measure. A representation of θ_M as an average of the sign η of the (3,4) plaquettes
21-02 The thin-vortex floor	For all real β and $N, M \geq 1$, $\lvert\theta_M\rvert \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$. If $\beta \neq 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$ with $\sigma_0 = -\ln\tanh(N^2\lvert\beta\rvert) > 0$
21-03 The thick-vortex floor — statement and route	For all $N \geq 1$ and any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M, $\lvert\theta_{M+1}\rvert \leq \exp\bigl(-(M+1)^2 e^{-\beta(S_N+\varepsilon)}\bigr)$ (S_N is the minimal action of a vortex; the thresholds are not explicit)
21-04 The thick-vortex floor — parts of the proof	The estimate $\lvert\theta_M\rvert \leq \tanh(K_{\mathrm{eff}})^{k^2}$ ($mk \leq M$) obtained by dividing the longitudinal directions into blocks of side m, and the parts that assemble the thick-vortex floor
21-05 The minimal action of a vortex	For all $N \geq 1$, $S_N > 0$ (a lower bound uniform in the block size m; the value is not explicit). For $N = 2, 3, 4$, $S_N < 2N^2$ (a computer certificate)
21-06 The sign of θ and pairs of Polyakov loops	If M is even, $\theta_M \geq 0$ for all real β ($Z_1 \leq Z_0$). Upper bounds on pairs of Polyakov loops, for example for $M \equiv 0 \pmod 4$, $\lvert\langle P_b(0) P_{b'}(T)\rangle_0\rvert \leq 2\tanh(N^2\lvert\beta\rvert)^{M\,\min(T, M-T)}$
21-07 The vortex ratio — continuity, limit, matching	With the size of the lattice held fixed, the vortex ratio is continuous in the coupling, is 0 at coupling 0, and approaches 1 as the coupling grows. For a chosen target value (at least 0 and less than 1) there is a coupling matched to it (the largest solution). A box with two twists has no flat configuration, and the deficit in a symmetric box is at least 1/2, independent of the transverse size
21-08 What this part says, and what it does not	A table of the scope of each theorem

Every theorem is machine-checked in Lean, and in addition the definitions and statements have been compared against the text by a separate pair of eyes (the computer certificates for $N = 2, 3, 4$ are compared on the side of the corollaries that use them). The names of the declarations and their scope are in each chapter.

02

The scope that applies to the whole part

- **The lattice spacing is fixed.** These are equalities and inequalities about unnormalised partition functions on a finite torus. The continuum limit is not treated.
- **Outside the window, the rate of the floor is not uniform in the transverse size N.** The rates ($\sigma_0 \approx 2e^{-2N^2\lvert\beta\rvert}$ and $e^{-\beta(S_N+\varepsilon)}$) approach 0 as N grows. These are

estimates for a slab with the transverse size held fixed, effectively two-dimensional. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ), not a rate of a floor.

- **Some thresholds are not explicit.** For the thick-vortex floor and the corollaries that rest on it, the thresholds in β and M are given only as existing; no values come out. The positive lower bound on the minimal action of a vortex also has no explicit value.
- **Reaches neither the band nor the mass gap.** Neither the statement of the band, nor the four-dimensional area law, nor the mass gap follows from the theorems of this part. The statements to be shown about the band and the mass gap are in 19-07.
- **A formalisation of one side of a known picture.** Of the known picture of a gas of vortices on a thin twisted slab (said to be the line of Mack–Petkova, Fröhlich and others; bibliography in the table at the end of the chapter, the texts not checked in this book), this part puts the side of the upper estimates into Lean as theorems about the concrete $SU(2)$ Wilson measure. The floor theorems of this part are on the side of the upper estimates; on the opposite side, 21-07 gives only one value, in the limit $\beta \to \infty$ with the lattice held fixed. This book claims no novelty.

03

What a "floor" is

The vortex ratio is tied to 't Hooft's flux free energy $\Delta F = \ln(Z_0/Z_1)$ (the same quantity as in 25-01 and 26-03; 26-03 also calls it the vortex free energy) by $\theta_M = \tanh(\Delta F/2)$. In the picture of confinement, ΔF approaches 0 exponentially in the (3,4) area M^2, and the speed (the rate) corresponds to the tension. The theorems of this part bound θ_M, and hence ΔF, **from above**. That is the same as bounding the rate **from below**, and this lower estimate is what we call a "floor". It is a statement in the direction "the rate is at least this much"; the floor theorems say nothing about an upper limit on the rate (a lower bound on ΔF). On the opposite side, 21-07 gives only one value, in the limit $\beta \to \infty$ with the lattice held fixed (that the slope of the flux free energy converges to the minimal deficit). "Thin" and "thick" differ in whether the rate of the floor is taken from the cost $2N^2$ of the configuration that gathers the vortex onto a single plane, or from the cost S_N of a cheaper thick vortex.

04

A guide to the record of the study

The record of the study of twisted tori, vortices and the band, up to the theorems of this part, is in Part 25 (from the direction of space — a small twisted T^2 and confinement) and Part 26 (the crossover band) of the present book. The view that tools built on structure alone yield no more than the floor is in 26-04. This part does not rewrite them; it points to the chapters where needed.

What this chapter can say

Can say	Cannot say
Proved here The theorems of this part (listed in § 01; statements, assumptions and routes of proof in each chapter)	Anything that moves the lattice spacing; rates of the floor uniform in N; explicit thresholds; the band; the mass gap
Computer-assisted $S_N < 2N^2$ for $N = 2, 3, 4$ (a computer certificate supporting the analytic proof, 21-05)	$S_N < 2N^2$ for general N

Sources and reproduction

Item	Kind	Source / tool
The picture of vortices on a twisted slab, and the comparison of $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked). Twisted boundary conditions: G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (25-01)
The theorems of each chapter	Proved here	The routes of proof in the text of each chapter. Machine-checked in Lean 4 (the source table of each chapter)

Words used in this chapter — vortex ratio, flux free energy, floor, thin vortex, thick vortex, minimal action of a vortex, Polyakov loop. Their meanings are in A-02 Glossary.

Next chapter to read: 21-01 Three results on sign systems — we start with the comparison tools that produce the floor.

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Three results on sign systems — the covering inequality, monotonicity, comparison

This chapter is this book's own investigation. The abstract theorems of § 02– § 05 are written as conditional implications.

What this chapter shows — three abstract inequalities (the covering inequality, Z_2 monotonicity, comparison with Z_2) for bounding the SU(2) vortex ratio from above by the vortex ratio of a Z_2 sign system, and their application to the SU(2) Wilson measure. We also set out a representation that writes the vortex ratio θ_M as an average of the signs η of the (3,4) planes.

Prerequisite chapters — 19-01 the product of Haar measures, plaquettes and holonomy; 19-02 the twist, the vortex ratio. We use the language of group characters (homomorphisms, values ± 1).

Said first — § 02– § 05 are abstract theorems in "if ... then" form with every assumption shown. As concrete objects satisfying the assumptions, the only results this chapter counts are the application to the SU(2) Wilson measure (§ 06) and the η representation (§ 07). All three are one-sided comparisons "from the Z_2 side upward"; no lower bound on the SU(2) vortex ratio (no lower bound on the flux free energy) comes out. Nothing here shows monotonicity or a covering inequality for a non-abelian group. The lattice spacing is fixed.

The order of this chapter

1. What the three results are for — the direction of the floor
2. Sign systems — characters, partition functions, twists, base systems
3. The covering inequality
4. Z_2 monotonicity
5. Comparison with Z_2 — couplings of indefinite sign and invariant measures
6. Application to the SU(2) Wilson measure
7. Writing the vortex ratio as an average of signs — the η representation
8. What is not said

9. Detour: reading the covering inequality on the lattice

01

What the three results are for — the direction of the floor

The object of Part 21 is the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N and length M (19-02). From the ratio of the partition function Z_1 with a twist inserted in one column of (3,4) planes and the partition function Z_0 without it, we form the vortex ratio

$$\theta_M = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \tanh \frac{\Delta F}{2}, \qquad \Delta F := -\ln \frac{Z_1}{Z_0} \quad (1)$$

(definition in 19-02). ΔF is 't Hooft's flux free energy (the same quantity as in 25-01 and 26-03; 26-03 also calls it the vortex free energy). In the picture of confinement, ΔF approaches 0 exponentially in the (3,4) area M^2, and its speed is the tension Cited (the framework of twisted boundary conditions of G. 't Hooft, Nucl. Phys. B 153 (1979) 141).

The three results of this chapter are tools for bounding the ratio Z_1/Z_0 of the $SU(2)$ system **from below** by the ratio of a $\mathbb{Z}_2$ system whose only variables are the signs of the plaquettes. Restated through (1), they bound θ_M and ΔF **from above**, and bound the speed (the rate) at which ΔF decreases in M^2 **from below**. In this part we call this lower estimate a "floor". The $\mathbb{Z}_2$ system we compare with is finally brought down to a system computable by a closed two-dimensional formula (21-02).

The roles of the three are as follows.

Name	What is compared	Use in this part
The covering inequality	A large $\mathbb{Z}_2$ system and a small (base) $\mathbb{Z}_2$ system mapped into it	The pattern for bringing a four-dimensional sign system down to a two-dimensional one
$\mathbb{Z}_2$ monotonicity	Two versions of the same $\mathbb{Z}_2$ system with couplings of different size	That increasing the coupling does not decrease the vortex ratio (a check of the comparison pattern)
Comparison with $\mathbb{Z}_2$	A system on a measure invariant under the action of the centre $\{\pm 1\}$ ($SU(2)$ and others), and a base $\mathbb{Z}_2$ system	Bounding the $SU(2)$ vortex ratio by the vortex ratio of a $\mathbb{Z}_2$ system (§ 05)

Each is a combination of well-known tools of the correlation-inequality type. The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$ is said to be the line of Mack–Petkova Cited

(bibliography in the source table; the text not checked in this book). This book claims no novelty.

02

Sign systems — characters, partition functions, twists, base systems

The three results are written in an abstract form that forgets the lattice. Four things appear.

The group of configurations and the plaquette index. G is a finite group (on the lattice, "a field of signs, one per link", with pointwise product), and ι is a finite set (on the lattice, the plaquettes).

Sign characters. A function $\psi : G \to \mathbb{R}$ is a sign character if it is multiplicative, $\psi(gh) = \psi(g)\psi(h)$, and takes values ± 1 (Lean: `CoveringInequality.IsSignChar`). We take one per plaquette, a family of characters $\chi = (\chi_B)_{B \in \iota}$. On the lattice, $\chi_B(g)$ is the product of the signs of the four links of plaquette B.

Partition functions and twists. For a coupling $c : \iota \to \mathbb{R}$ we set

$$Z_\chi(c) = \sum_{g \in G} \exp\Big(\sum_{B \in \iota} c_B \, \chi_B(g) \Big) \quad (2)$$

(Lean: `CoveringInequality.Z`). A twist is $\zeta : \iota \to \{\pm 1\}$, which replaces the coupling c by $\zeta c = (\zeta_B c_B)_B$. The $-\ln$ of the twisted ratio $Z_\chi(\zeta K)/Z_\chi(K)$ plays the role of the flux free energy of this abstract system.

Base systems and pull-back. Another finite group G_0, a family of characters $\chi^0 = (\chi^0_B)_{B \in \iota}$ (with the **same** index ι), and a map $p : G_0 \to G$ satisfying

$$\chi_B\big(p(t)\big) = \chi^0_B(t) \qquad (\text{for all } B \in \iota,\ t \in G_0) \quad (3)$$

are together called a "base system". p need not be a homomorphism (the proof uses only the multiplicativity of χ_B). On the lattice, p is the map that carries a sign field on a small torus periodically onto a large torus.

The starting lemma is an inequality of the first Griffiths type. For a non-negative coupling $J \geq 0$ and a twist ζ, $Z_\chi(\zeta J) \leq Z_\chi(J)$ Proved here (Lean: `CoveringInequality.Z_flip_le`; the assumption fields are "each χ_B is a sign character", "$J_B \geq 0$", "$\zeta_B = \pm 1$"). The proof expands the product using $e^{c\chi} = \cosh c + \chi \sinh c$ ($\chi = \pm 1$) and shows, by induction on the set of plaquettes, that "the absolute value of the twisted sum does not exceed the untwisted sum". Underneath lies the fact that a sum of a sign character $\sum_g \psi(g)$ is 0 or $|G|$, and never negative.

03

The covering inequality

The covering inequality (Lean: `CoveringInequality.covering_ineq`). We assume the following five.

- (hχ) Each χ_B is a sign character on G.
- (hχ_0) Each χ^0_B is a sign character on G_0.
- (hp) $p : G_0 \to G$ satisfies (3).
- (hK) The coupling is non-negative: $K_B \geq 0$.
- (hζ) $\zeta_B = \pm 1$.

Then

$$Z_\chi(K)\, Z_{\chi^0}(\zeta K) \;\leq\; Z_\chi(\zeta K)\, Z_{\chi^0}(K) \quad (4)$$

The ratio form (`covering_ratio`) and the $-\ln$ form (`vfe_cover_le`) are

$$\frac{Z_{\chi^0}(\zeta K)}{Z_{\chi^0}(K)} \leq \frac{Z_\chi(\zeta K)}{Z_\chi(K)}, \qquad -\ln \frac{Z_\chi(\zeta K)}{Z_\chi(K)} \leq -\ln \frac{Z_{\chi^0}(\zeta K)}{Z_{\chi^0}(K)} \quad (5)$$

with the same five assumptions. Proved here The proof is the two steps below; all three are machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. The tools are of well-known types, Griffiths-type correlation inequalities and change of configuration variables; this book has not checked where in the literature a comparison of this form appears.

The route of proof. Two steps.

1. **The key identity** (Lean: `Z_mul_Z`). In the double sum $\sum_s \sum_t$, replace the configuration on the covering side by $s = u \cdot p(t)$. Since $\chi_B(u\,p(t)) = \chi_B(u)\chi^0_B(t)$, for arbitrary couplings J, J'

$$Z_\chi(J)\, Z_{\chi^0}(J') = \sum_{u \in G} Z_{\chi^0}\left(J\chi(u) + J'\right) \quad (6)$$

   The product of two systems becomes a sum, over u, of single base systems Proved here .

2. **Comparison for each u.** The left side of (4) is $(J, J') = (K, \zeta K)$, the right side $(\zeta K, K)$. The base couplings are $K\chi(u) + \zeta K = \zeta\, a(u)$ and $a(u)$ respectively, where $a_B(u) = \zeta_B K_B \chi_B(u) + K_B \geq 0$. Applying the first Griffiths-type inequality $Z_{\chi^0}(\zeta a) \leq Z_{\chi^0}(a)$ for each u and summing gives (4) Proved here .

No limits and no size conditions are used. It holds for every finite system and every non-negative coupling.

A lattice instance (Lean: `TwistedWilson.z2_four_covers_two`, ratio form `z2_four_ratio`; the only assumption is $K \geq 0$). Take as the covering side the four-dimensional $\mathbb{Z}_2$ gauge theory on the torus of width N and length M (uniform coupling K), and as the base the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus of the length directions (on the (3,4) planes the base character is the sign of a two-dimensional plaquette, not identically 1). With ζ the (3,4) twist,

$$\frac{1-\tanh(N^2K)^{M^2}}{1+\tanh(N^2K)^{M^2}} \;\leq\; \frac{Z_\chi(\zeta K)}{Z_\chi(K)}$$

Proved here . This brings the four-dimensional sign system down to the two-dimensional system of the length directions; it is not itself the detour's "covering that stretches the width n-fold". It is read within the Lean chain, but the proof of the main theorem of the $SU(2)$ thin-vortex floor does not use it; it goes directly from the application of § 06 to the two-dimensional formula.

04

Z_2 monotonicity

$\mathbb{Z}_2$ monotonicity (Lean: `CoveringInequality.mon_ineq`). There are four assumptions.

- (hχ) Each χ_B is a sign character.
- (hK) $K_B \geq 0$.
- (hKK') $K_B \leq K'_B$ (a coupling larger term by term).
- (hζ) $\zeta_B = \pm 1$.

Then

$$Z_\chi(\zeta K')\,Z_\chi(K) \;\leq\; Z_\chi(K')\,Z_\chi(\zeta K), \qquad \text{i.e.} \quad \frac{Z_\chi(\zeta K')}{Z_\chi(K')} \leq \frac{Z_\chi(\zeta K)}{Z_\chi(K)} \quad (7)$$

(the ratio form is `mon_ratio`, with the same assumptions). Increasing the coupling term by term does not increase the twisted ratio, and does not decrease the flux free energy. Proved here The proof follows the route below; it is machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. The pattern is a correlation inequality for ferromagnetic sign systems.

The route of proof. Use (6) with base = cover ($G_0 = G$, p the identity). Bring the couplings that appear into the form $\chi(u)\cdot\big(K' + \zeta\chi(u)K\big)$, remove $\chi(u)$ by the character

shift $Z_\chi(\chi(u)c) = Z_\chi(c)$ (the change of configuration $g \mapsto gu$; Lean: `Z_char_shift`), and apply the first Griffiths-type inequality to $K'_B + \zeta_B \chi_B(u) K_B \geq 0$ Proved here .

This monotonicity is a statement about $\mathbb{Z}_2$ sign systems (non-negative couplings, no twisted background). Whether the same monotonicity in the coupling β holds for the $SU(2)$ Wilson measure is not shown in this chapter (§ 08).

05

Comparison with Z_2 — couplings of indefinite sign and invariant measures

Relaxing the assumption (hK) of the covering inequality, the same form comes out for couplings of indefinite sign.

The form with dominated couplings (Lean: `CoveringInequality.dominated_covering_ineq`). The assumptions are (hχ)(hχ₀)(hp)(hζ) and

- (hJ) $|J_B| \leq K_B$ (the sign of J is arbitrary).

Then

$$Z_\chi(J)\, Z_{\chi^0}(\zeta K) \;\leq\; Z_\chi(\zeta J)\, Z_{\chi^0}(K) \quad (8)$$

The proof is the same as in § 03; the only change is that the base coupling $K_B + \zeta_B J_B$ $\chi_B(u)$ is non-negative by (hJ) Proved here . $J = K$ is the covering inequality, and base = cover with $J = K \leq K'$ is $\mathbb{Z}_2$ monotonicity.

The form on an invariant measure (comparison with $\mathbb{Z}_2$) (Lean: `CoveringInequality.invariant_dominated`). For a configuration space Ω with a measure μ, an action $g \cdot \omega$ of the sign group G on Ω, and functions $J_B : \Omega \to \mathbb{R}$, one per plaquette, we assume the following.

- (hinv) The integral is invariant under the action: for all $g \in G$ and all $f : \Omega \to \mathbb{R}$, $\int f(g \cdot \omega)\, d\mu = \int f(\omega)\, d\mu$.
- (hχ)(hχ₀)(hp) as in § 03.
- (hJ) The coupling changes sign by the character: $J_B(g \cdot \omega) = \chi_B(g)\, J_B(\omega)$.
- (hK) $|J_B(\omega)| \leq K_B$ (for all ω).
- (hζ) $\zeta_B = \pm 1$.
- (hint) For all $c : \iota \to \mathbb{R}$, $\omega \mapsto \exp\left(\sum_B c_B J_B(\omega)\right)$ is μ-integrable.

Then

$$\left(\int e^{\sum_B J_B(\omega)}\, d\mu\right) Z_\chi^0(\zeta K) \;\le\; \left(\int e^{\sum_B \zeta_B J_B(\omega)}\, d\mu\right) Z_\chi^0(K) \quad (9)$$

Cited The pattern is a comparison of a system of a group with centre against a system of the centre group, said to be the line of Mack–Petkova (bibliography in the source table; the text not checked in this book). It is machine-checked in Lean, and the comparison by a separate pair of eyes judged it to hold as an implication.

The route of proof. The key is averaging over the centre (Lean: `card_mul_integral_exp`). From (hinv) and (hJ), for each g, $\int e^{\sum J(\omega)} d\mu = \int e^{\sum \chi(g) J(\omega)} d\mu$, so averaging over g gives

$$|G| \int e^{\sum_B J_B(\omega)}\, d\mu = \int Z_\chi\big(J(\omega)\big)\, d\mu \quad (10)$$

Proved here . The integrand on the right is the partition function of a **sign system** with the configuration ω held fixed, and the coupling $J(\omega)$ is bounded by (hK). So apply (8) for each ω and integrate to obtain (9) (integrability comes from (hint)). In other words, the system on the measure is "a superposition of $\mathbb{Z}_2$ systems with couplings of indefinite sign", and each member of the superposition can be compared with the base $\mathbb{Z}_2$ system.

06

Application to the SU(2) Wilson measure

$SU(2)$ lattice gauge theory satisfies the assumptions of § 05. The correspondence is as follows (notation in 19-01 and 19-02).

Abstract field	The $SU(2)$ Wilson measure	Lean lemma that discharges it
Ω, μ	Link configurations U : links $\to SU(2)$, the product of normalised Haar measures	—
G and its action	A sign per link $g_\ell \in \{\pm 1\}$, $(g \cdot U)_\ell = g_\ell U_\ell$ (multiplying by the centre ± 1)	`TwistedWilson.signAction`
(hinv)	Haar measure is invariant under left multiplication by an element of the centre	`TwistedWilson.haarCfg_inv`
χ_B, (hχ)	The product of the signs of the four links of plaquette B (overlaps kept as overlaps)	`TwistedWilson.chiB_isSignChar`
J_B, (hJ)	$J_B(U) = \beta_B \operatorname{Re} U_B$. Multiplying by signs multiplies $\operatorname{Re} U_B$ by $\chi_B(g)$	`TwistedWilson.hol_smul`
(hK)	$\lvert\operatorname{Re} U_B\rvert \le 1$, so $\lvert J_B\rvert \le \lvert\beta_B\rvert$	`TwistedWilson.abs_re_le_one`
(hint)	The integrand is continuous and bounded	`TwistedWilson.integrable_exp`

The only remaining assumptions are on the side of the base system.

Comparison with $\mathbb{Z}_2$ ($SU(2)$, couplings per plaquette, any base system) (Lean: `TwistedWilson.su2_dominated_general`). If $N, M \geq 1$, β_B are real couplings per plaquette (of arbitrary sign), $\zeta_B = \pm 1$, and the base system (G_0, χ^0, p) satisfies

- each χ^0_B is a sign character on G_0,
- $p : G_0 \to \{\pm 1\}^{\text{links}}$ satisfies $\chi_B(p(t)) = \chi^0_B(t)$,

then

$$\left(\int e^{\sum_B \beta_B \operatorname{Re} U_B}\, dU\right) Z_{\chi^0}\left(\zeta|\beta|\right) \;\leq\; \left(\int e^{\sum_B \zeta_B \beta_B \operatorname{Re} U_B}\, dU\right) Z_{\chi^0}\left(|\beta|\right) \quad (11)$$

(dU is the product of Haar measures, $|\beta| = (|\beta_B|)_B$). Cited This is a formalisation of the comparison said to be the line of Mack–Petkova (the text not checked in this book). It is machine-checked in Lean, and the comparison by a separate pair of eyes judged that it holds conditionally on the three fields of the base system (being characters, the pull-back formula, $\zeta = \pm 1$), and that instances satisfying them exist.

An instance (Lean: `TwistedWilson.su2_dominated_general_instance`, no assumption fields). Take as the base the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions. p is the map that sends a sign t of a two-dimensional link to the links in directions 3 and 4 at (x_3, x_4) (independently of the transverse position (x_1, x_2)), and puts $+1$ on the links in directions 1 and 2 (`TwistedWilson.pull`). Taking the coupling $\beta_B = \beta\, z_B$ (z is the sign of the background twist n_{12}) and ζ the sign of the (3,4) twist, the two integrals in (11) become exactly the partition functions $Z_{N,M}(\beta; n_{12}, 0)$ and $Z_{N,M}(\beta; n_{12}, 1)$ of 19-02 (the case $n_{12} = 1$ gives this part's Z_0, Z_1), and

$$Z_{N,M}(\beta; n_{12}, 0) \cdot Z_{\text{base}}\left(\zeta|\beta|\right) \;\leq\; Z_{N,M}(\beta; n_{12}, 1) \cdot Z_{\text{base}}\left(|\beta|\right) \qquad \text{(for all real } \beta,\ N, M \geq$$

follows (the Lean name of the same formula is `TwistedWilson.su2_dominated`). The ratio on the base side is computable by a closed two-dimensional formula, and the thin-vortex floor follows from it (21-02). The background twist n_{12} is absorbed into the sign of J, so the bound $|J_B| \leq |\beta|$ of (hK) is unchanged even with a background. This contrasts with the $\mathbb{Z}_2$ system itself, where the background twist becomes a negative coupling and (hK) of the covering inequality fails.

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07
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Writing the vortex ratio as an average of signs — the η representation

The twist $n34$ was "reverse the sign of one column of (3,4) planes". In fact neither the position of the reversed column nor the way of reversing carries any meaning; only the product of the signs does. The η representation puts this into a formula.

For $\eta : (\mathbb{Z}/M)^2 \to \mathbb{R}$, write $Z[\eta]$ for the partition function in which the coupling of the (3,4) plaquette $((x_1,x_2),(x_3,x_4))$ is multiplied by $\eta(x_3,x_4)$ (the same factor for all N^2 transverse copies; the background n_{12} is left as it is; Lean: `TwistedWilson.ZEta`).

Parity of η (Lean: `TwistedWilson.ZEta_parity`). The only assumption is $\eta_y = \pm 1$ (for all y). Then

$$Z[\eta] = \begin{cases} Z_0 & \prod_y \eta_y = 1 \\ Z_1 & \prod_y \eta_y = -1 \end{cases} \quad (13)$$

(for all real β and $N, M \geq 1$. Here we wrote it in the notation of this part's convention $n_{12} = 1$. For $n_{12} = 0$ the same formula holds with Z_0, Z_1 replaced by $Z_{N,M}(\beta;0,0)$, $Z_{N,M}(\beta;0,1)$. The same goes for (14), whose left side is then $\theta_M^{(n_{12})}$ of 19-02).

θ is the average of η (Lean: `TwistedWilson.theta_eq_eta_average`, no assumption fields).

$$\theta_M = \frac{Z_0 - Z_1}{Z_0 + Z_1} = \frac{\sum_\eta \left(\prod_y \eta_y\right) Z[\eta]}{\sum_\eta Z[\eta]} \quad (14)$$

The sum runs over all $\eta : (\mathbb{Z}/M)^2 \to \{\pm 1\}$ (2^{M^2} of them), and this is a ratio of unnormalised sums. The right side is the expectation of $\prod \eta$ under the joint weight $\exp\left(\beta \sum_B z_B\, \eta_B\, \mathrm{Re}\, U_B\right)$ of the configuration (U, η). Proved here The proof follows the route below; both are machine-checked in Lean, and the definitions and statements have been compared by a separate pair of eyes. That the twist is determined only by its centre class is a standard fact about twisted boundary conditions Cited (G. 't Hooft, Nucl. Phys. B 153 (1979) 141).

The route of proof. Three parts Proved here .

1. Centre gauge invariance (Lean: `ZEta_gauge`). Reversing the links in directions 3 and 4 by a two-dimensional sign field t leaves Haar measure unchanged, and $Z[\eta] = Z[\eta \cdot \chi^{2D}(t)]$ ($\chi^{2D}_y(t)$ is the sign of the two-dimensional plaquette y).
2. Realising plaquette signs (Lean: `TwistedPlane.exists_chi2_eq`). If $\eta = \pm 1$ and $\prod \eta = 1$, there is a t with $\chi^{2D}_y(t) = \eta_y$. Shown by counting.

3. The involution $\eta \mapsto \eta \cdot e_0$ (reverse only plaquette 0) shows that the cases $\prod \eta = 1$ and -1 are equal in number.

(13) follows from 1 and 2, and (14) from (13) and 3. Independence of the position of the reversal (for every position P, "reverse only plaquette P" gives Z_1: `ZEta_flip_any`) follows from the same parts. This representation is the starting point of the thick-vortex floor (21-03).

08

What is not said

- **One side only.** All three point in the direction of bounding the $SU(2)$ ratio Z_1/Z_0 from below, from the $\mathbb{Z}_2$ side. A lower bound on ΔF (a lower bound on θ_M) does not come out of comparisons of this type.
- **Not monotonicity or a covering inequality for a non-abelian group.** § 03 and § 04 are theorems about $\mathbb{Z}_2$ sign systems. For $SU(2)$, neither monotonicity in the coupling β nor a covering inequality across torus sizes is shown. For the finite non-abelian group Q_8, without a twisted background, there are points at weak coupling where both the covering inequality and monotonicity fail Reading (floating-point values of transfer matrices on small three-dimensional lattices. Example: covering number 3, $\beta = 1.25$, $\Delta F_3(1.25) = 1.29498 > \Delta F_1(3.75) = 1.28758$; two independent implementations agree). Q_8 is a finite subgroup of $SU(2)$; whether the same happens for $SU(2)$ is not known.
- **The lattice dictionary is on paper.** The Lean statements of § 03 and § 04 are in abstract form. The dictionary (the detour) that reads a four-dimensional $\mathbb{Z}_2$ torus with the transverse directions stretched n-fold as a pair of a large torus and a base torus is on paper. The only lattice instance in Lean with a non-trivial base is the form at the end of § 03, "bringing four-dimensional $\mathbb{Z}_2$ down to two-dimensional $\mathbb{Z}_2$ of the length directions". Abstract examples whose base character is identically 1 are not counted among the results of this chapter.
- **The ratio of the base system is needed separately.** (11) and (12) go only as far as the comparison with the base $\mathbb{Z}_2$ system; the value of the base ratio is a separate computation (21-02 uses the closed two-dimensional formula).
- **The lattice spacing is fixed.** All are inequalities for finite systems on finite tori; they say nothing about the continuum limit, infinite volume or the mass gap.

The view that tools of this type yield no more than the floor is in the record of the study in 26-04 and 26-06.

Detour: reading the covering inequality on the lattice

You may skip this. In four-dimensional $\mathbb{Z}_2$ gauge theory, suppose a torus Λ covers a small torus Λ_0 n times (stretching the transverse directions periodically). With G the link signs of Λ, G_0 the link signs of Λ_0, p the periodic copy and ι the plaquettes of Λ, (3) is the obvious fact that "the plaquette signs of the copied field equal the plaquette signs of the base". The coupling on the base side is counted by the plaquette index of the cover, so if n plaquettes lie over one plaquette of the base, their couplings add up to nK. The covering inequality (5) then reads

$$\Delta F_{\Lambda}(\beta) \;\leq\; \Delta F_{\Lambda_0}(n\beta)$$

Stretching both transverse directions n-fold gives n^2 plaquettes. This reading is on paper, and has been checked by exhaustive enumeration on small lattices Reading (floating point, with a negative control: a version that miscounts the plaquettes fails). The $SU(2)$ application (§ 06) does not go through this dictionary; it builds the base system on the same lattice (the two-dimensional torus of the length directions) directly in Lean.

What this chapter can say

Can say	Cannot say
Proved here The covering inequality (4)(5), under five assumptions (two sign characters, the pull-back formula, $K \geq 0$, $\zeta = \pm 1$). Lean: `covering_ineq`, `covering_ratio`, `vfe_cover_le`	The reading on the lattice (the dictionary) is on paper. No version for non-abelian groups
Proved here A lattice instance: bringing four-dimensional $\mathbb{Z}_2$ down to two-dimensional $\mathbb{Z}_2$ of the length directions ($K \geq 0$). Lean: `z2_four_covers_two`, `z2_four_ratio`	A Lean instance of the covering that stretches the width n-fold
Proved here $\mathbb{Z}_2$ monotonicity (7): $0 \leq K \leq K'$, $\zeta = \pm 1$. Lean: `mon_ineq`, `mon_ratio`	Monotonicity in β for $SU(2)$. $\mathbb{Z}_2$ systems with a twisted background
Proved here Comparison with $\mathbb{Z}_2$ (8)(9): $\lvert J\rvert \leq K$, invariant measure, integrability. Lean: `dominated_covering_ineq`, `invariant_dominated`	The reverse direction (a lower bound on ΔF)
Proved here The $SU(2)$ application (11), under the three fields of the base system. Lean: `su2_dominated_general`. Instance (12) (no fields): `su2_dominated_general_instance`, `su2_dominated`	The value of the base ratio (21-02). Lattice spacing fixed; no continuum limit
Proved here Parity of η (13) (field $\eta = \pm 1$) and θ as the average of η (14) (no fields). Lean: `ZEta_parity`, `theta_eq_eta_average`	An estimate of the average of η itself (the thick-vortex floor is 21-03)
Reading For Q_8 there are points where the covering inequality and monotonicity fail (no background, small three-dimensional lattices)	The general form of the failure. Floating-point values, not a theorem

Sources and reproduction

Item	Kind	Source / tool
The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked)
Twisted boundary conditions; the twist being determined only by its class	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (the same bibliography as 25-01)
The three results, the application, the η representation	Proved here	Machine-checked in Lean 4 (declarations in each section; only the three standard axioms, no sorry). The definitions and statements were compared by a separate pair of eyes, which judged the three abstract results "to hold as implications", the application "conditional on the fields of the base system, with satisfying instances", and the η representation "to hold"
The failure for Q_8	Reading	The numerical record of this book's study (transfer matrices on three-dimensional lattices cross-checked by exhaustive enumeration; two independent implementations agree to 4–7 digits. D_4 gives the same values)
The check of the lattice dictionary	Reading	The numerical record of this book's study (sums over all configurations of small four-dimensional $\mathbb{Z}_2$ lattices; fails in a negative control that miscounts the plaquettes)

Words used in this chapter — sign system, sign character, base system, pull-back, covering inequality, $\mathbb{Z}_2$ monotonicity, comparison with $\mathbb{Z}_2$, averaging over the centre, vortex ratio, flux free energy, floor, η representation. Their meanings are in A-02 Glossary.

Next chapter to read: 21-02 The thin-vortex floor — putting the closed two-dimensional formula into the comparison of § 06, we obtain $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$.

The thin-vortex floor

> This chapter is this book's own investigation.

What this chapter shows — that the vortex ratio of the $SU(2)$ Wilson measure satisfies $|\theta_M| \le \tanh(N^2|\beta|)^{M^2}$ for every real β and all $N, M \ge 1$; its exponential form $|\theta_M| \le e^{-\sigma_0 M^2}$ ($\sigma_0 = -\ln\tanh(N^2|\beta|)$); the meaning of "thin"; and the five steps of the proof.

Prerequisite chapters — 19-02 the torus of width N and length M, the twist, the vortex ratio; 21-01 Three results on sign systems (the application in § 06).

Said first — the lattice spacing is fixed. The rate $\sigma_0 \approx 2e^{-2N^2|\beta|}$ approaches 0 as the width N grows and is not uniform in N. The sign of θ_M ($Z_1 \le Z_0$) is not stated in this chapter (21-06). No rate as $M \to \infty$ (a limit) is defined. Neither the band nor the mass gap is reached (the statements to be shown are in 19-07). This is a formalisation of one side of a known picture.

The order of this chapter

1. Statement — the ratio form, the two-sided form, the exponential form
2. In numbers — the size of the right side, and the gap to the actual ratio
3. What "thin" means — the cost of a vortex gathered onto one sheet
4. The route of proof — five steps
5. Part: the closed two-dimensional Z_2 formula
6. Part: moving the base partition function to two dimensions
7. Arithmetic restatements — a lower bound on the ratio and an upper bound on the flux free energy
8. What is not said

01

Statement — the ratio form, the two-sided form, the exponential form

The object is the $SU(2)$ Wilson measure on the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ of width N (directions 1 and 2) and length M (directions 3 and 4). The weight is $e^{\beta \sum_p z_p \operatorname{Re} U_p}$ ($z_p = \pm 1$ is the sign of the twist), the measure is the product of normalised Haar measures, and partition functions are not normalised (19-01). Write $Z_{N,M}(\beta; n_{12}, n_{34})$ for the partition function with background twist n_{12} and (3,4) twist n_{34}, and let the vortex ratio with background n_{12} be $\theta_M^{(n_{12})} = \big(Z_{N,M}(\beta; n_{12}, 0) - Z_{N,M}(\beta; n_{12}, 1)\big) / \big(Z_{N,M}(\beta; n_{12}, 0) + Z_{N,M}(\beta; n_{12}, 1)\big)$ (the notation of the detour in 19-02). Under this part's convention $n_{12} = 1$, these are Z_0, Z_1 and $\theta_M = \theta_M^{(1)}$. The theorems below also hold for $n_{12} = 0$.

The thin-vortex floor (ratio form) (Lean: `TwistedWilson.twisted_vortex_bound`). For every real β, all $N, M \geq 1$ and $n_{12} \in \{0, 1\}$,

$$\theta_M^{(n_{12})} \;\leq\; \tanh\big(N^2 |\beta|\big)^{M^{\angle}} \qquad (1)$$

The only assumption field is $N, M \geq 1$.

The thin-vortex floor (two-sided form) (Lean: `TwistedWilson.twisted_vortex_bound_abs`). In the same range,

$$\big|\theta_M^{(n_{12})}\big| \;\leq\; \tanh\big(N^2 |\beta|\big)^{M^{\angle}} \qquad (2)$$

For $\beta < 0$ there are cases with $\theta_M^{(n_{12})} < 0$ ($N = M = 1$, $\beta = -1.5$ in the table of § 02), and there it is (2) that has content.

The thin-vortex floor (exponential form) (Lean: `TwistedWilson.thin_vortex_floor`). If $\beta \neq 0$,

$$\sigma_0 := -\ln \tanh\big(N^2 |\beta|\big) > 0, \qquad \big|\theta_M^{(n_{12})}\big| \;\leq\; e^{-\sigma_0 M^{\angle}} \quad \text{(for all } M \geq 1) \qquad (3)$$

The only assumption is $\beta \neq 0$ (at $\beta = 0$, $\tanh 0 = 0$ and σ_0 loses its meaning; in fact $\theta_M^{(n_{12})} = 0$).

Proved here For all three, the route of proof is in § 04– § 06; they are machine-checked in Lean as theorems about the concrete $SU(2)$ Wilson measure, and the definitions and statements have been compared by a separate pair of eyes. The pattern, bounding the $SU(2)$ system by a $\mathbb{Z}_2$ system, is said to be the line of Mack–Petkova Cited (bibliography in the source table; the text not checked in this book). In exhaustive sums over finite groups and $SU(2)$ Metropolis runs, written independently during the comparison, no violation was found Reading .

In the language of this part, (3) is a lower estimate of the rate, "θ_M decreases in M^2 at least at rate σ_0", that is, a **floor** (for the direction, see 21-01 § 01).

02

In numbers — the size of the right side, and the gap to the actual ratio

We list the right side $t = \tanh(N^2|\beta|)$ and the rate $\sigma_0 = -\ln t$ for several N, β Reading (floating-point values).

N	$\lvert\beta\rvert$	$t = \tanh(N^2\lvert\beta\rvert)$	$\sigma_0 = -\ln t$	$2e^{-2N^2\lvert\beta\rvert}$
1	0.25	0.2449	1.407	1.213
1	0.5	0.4621	0.7719	0.7358
1	1	0.7616	0.2723	0.2707
2	0.25	0.7616	0.2723	0.2707
2	0.5	0.9640	0.03664	0.03663
2	1	0.99933	6.709×10^{-4}	6.709×10^{-4}
3	0.25	0.97803	0.02222	0.02222
3	0.5	0.999753	2.468×10^{-4}	2.468×10^{-4}
3	1	$1 - 3.05 \times 10^{-8}$	3.05×10^{-8}	3.05×10^{-8}

Once $N^2|\beta|$ exceeds 1, the rate is almost the same as $2e^{-2N^2|\beta|}$, and it drops by orders of magnitude each time N increases by one. The right side is determined by $N^2|\beta|$ alone, so $(N, \beta) = (1, 1)$ and $(2, 0.25)$ give the same row.

The gap to the actual ratio on small lattices is as follows Reading (for $SU(2)$, Metropolis estimates, from an implementation written independently during the comparison).

(N, M)	$n12$	β	$\theta_M^{(n12)}$ (estimate)	Right side t^{M^2}
(1,1)	1	1.5	0.7033 ± 0.0018	0.9051
(1,1)	0	-1.5	-0.0785 ± 0.0026	0.9051
(2,1)	0	0.5	0.4856 ± 0.0012	0.9640
(1,2)	0	1.0	0.0061 ± 0.0013	0.3364

For the $\mathbb{Z}_2$ system (the same lattice with $SU(2)$ replaced by $\{\pm 1\}$), in the pairs tried with $N = 1$ or $M = 1$, $|\theta_M|$ coincides with the right side Reading (floating-point values of finite sums by variable elimination). Since the system we compare with is a $\mathbb{Z}_2$ system, the bound is nearly an equality where configurations close to $\mathbb{Z}_2$ dominate, and becomes loose where the continuous fluctuations of $SU(2)$ matter.

03

What "thin" means — the cost of a vortex gathered onto one sheet

In closed form the rate σ_0 is

$$\sigma_0 = -\ln\tanh x = \ln\frac{1+e^{-2x}}{1-e^{-2x}} = 2\,\mathrm{artanh}\big(e^{-2x}\big), \qquad x = N^2|\beta| \quad (4)$$

Hence $2e^{-2x} \le \sigma_0 \le 2e^{-2x}/(1-e^{-4x})$, and for large x, $\sigma_0 \approx 2e^{-2N^2|\beta|}$ Proved here (from the positivity of the coefficients of the power series of artanh).

The exponent $2N^2|\beta|$ has the following meaning. Among the (3,4) plaquettes, fix one longitudinal position (x_3, x_4) and consider the configuration in which the signs of all N^2 transverse copies are reversed at once. This is a configuration in which the vortex is gathered onto a single sheet spread out in the transverse directions; we call it a **thin vortex**. Since N^2 plaquettes have $\mathrm{Re}\,U_p$ change from 1 to -1, the cost in action is $2\beta \cdot N^2$.

In the proof, this picture appears directly as a formula. The base system (§ 04) is the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus of the length directions, and the coupling of one of its plaquettes is the sum $N^2|\beta|$ of the couplings of the N^2 transverse plaquettes lying over it. In two-dimensional $\mathbb{Z}_2$ gauge theory, the plaquette signs on a torus are independent apart from a single constraint, "the product of all is 1". Converting (7) (§ 05) into the vortex ratio gives $\theta^{2D} = t^{M^2}$ with $t = \tanh(N^2|\beta|)$; this is the mean of the product of all plaquette signs when each plaquette sign is taken to be an independent ± 1 with mean t. From the probability $q = (1-t)/2 \approx e^{-2N^2|\beta|}$ that a plaquette is reversed, $t^{M^2} = (1-2q)^{M^2} \approx e^{-2qM^2}$ Proved here .

So the right side of (1) says only this much: that the $SU(2)$ vortex decreases in M^2 at least as fast as "a dilute gas of thin vortices, each gathered onto one sheet spanning the full width". An $SU(2)$ vortex can spread out within the continuous group and carry the twist in the length directions at a lower cost. When the cost S_N of such a thick vortex is less than $2N^2$, the rate can be taken larger. That is the thick-vortex floor (21-03; for S_N, see 19-02).

04

The route of proof — five steps

The proof has five steps. Steps 1 and 2 are the application of 21-01 § 06, steps 3 and 4 the computation of the base, and step 5 arithmetic. This section and § 06 are written in the notation Z_0, Z_1 of this part's convention $n_{12} = 1$, but the same steps go through for $n_{12} = 0$.

1. **Averaging over the centre.** Haar measure is invariant under multiplying each link by a sign ± 1, and $\operatorname{Re} U_p$ changes only by the factor of the product of the four signs of the plaquette (`haarCfg_inv`, `hol_smul`). So the $SU(2)$ partition function can be written as "an integral over U of $\mathbb{Z}_2$ sign systems with couplings $J_p(U) = \beta z_p \operatorname{Re} U_p$", with $|J_p| \le |\beta|$.
2. **Comparison with $\mathbb{Z}_2$.** Take as the base system the $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions, linked by the map that sends the signs of two-dimensional links to the links in directions 3 and 4 independently of the transverse position (`pull`). From the comparison with $\mathbb{Z}_2$ in 21-01 § 05,

$$Z_0 \cdot Z_{\text{base}}(\zeta|\beta|) \le Z_1 \cdot Z_{\text{base}}(|\beta|) \quad (5)$$

   (`su2_dominated`). ζ is the sign of the (3,4) twist.
3. **The base to two dimensions.** Up to a constant factor, the partition function of the base system is that of two-dimensional $\mathbb{Z}_2$ gauge theory with coupling $N^2|\beta|$. The twisted one has the coupling with the sign of plaquette 0 alone changed (`Z_base_const`, `Z_base_flip`; § 06).
4. **The closed two-dimensional formula.** The partition function of two-dimensional $\mathbb{Z}_2$ gauge theory can be written in closed form as $|\text{configurations}| \cdot (\prod \cosh + \prod \sinh)$ (`Z_plane`; § 05).
5. **Arithmetic.** Setting $L = N^2|\beta|$, $C = \cosh(L)^{M^2}$, $S = \sinh(L)^{M^2}$, steps 3 and 4 turn (5) into $Z_0(C - S) \le Z_1(C + S)$, which rearranges to $(Z_0 - Z_1)C \le (Z_0 + Z_1)S$, that is, (1) (`bound_of_base`).

The two-sided form (2) uses step 2 with the roles swapped. Taking the twisted side $n_{34} = 1$ as the "original" system and using `su2_dominated_general` with couplings $\beta_p = \beta\, z_p^{(n_{12},1)}$, $\zeta^2 = 1$ gives $Z_1 \cdot Z_{\text{base}}(\zeta|\beta|) \le Z_0 \cdot Z_{\text{base}}(|\beta|)$ (`su2_dominated_swap`). Applying step 5 to both gives $|Z_0 - Z_1|/(Z_0 + Z_1) \le t^{M^2}$ Proved here. The exponential form (3) is a restatement using $t^{M^2} = e^{M^2 \ln t}$ and the fact that $0 < t < 1$ when $\beta \neq 0$ Proved here.

05

Part: the closed two-dimensional Z_2 formula

Consider $\mathbb{Z}_2$ gauge theory on the two-dimensional torus $(\mathbb{Z}/M)^2$ of the length directions: points y, links (y, d) $(d = 1, 2)$, a sign field on links t, and the sign $\chi_y^{2D}(t)$ of plaquette y (the product of four signs, overlaps kept as overlaps) (Lean: `TwistedPlane.chi2`).

The closed formula (Lean: `TwistedPlane.Z_plane`; the only assumption field is $M \geq 1$). For arbitrary couplings c_y, one per plaquette,

$$Z_\chi^{2D}(c) = \sum_t \exp\Big(\sum_y c_y \chi_y^{2D}(t)\Big) = |\text{configurations}| \cdot \Big(\prod_y \cosh c_y + \prod_y \sinh c_y\Big) \quad (6)$$

For a uniform coupling L and the coupling with the sign of plaquette 0 alone changed (`Z_plane_const`, `Z_plane_flip`),

$$\frac{Z_\chi^{2D}(L \text{ with plaquette 0 reversed})}{Z_\chi^{2D}(L)} = \frac{\cosh^{M^2} L - \sinh^{M^2} L}{\cosh^{M^2} L + \sinh^{M^2} L} = \frac{1 - \tanh(L)^{M^2}}{1 + \tanh(L)^{M^2}} \quad (7)$$

(`TwistedPlane.plane_flip_ratio`). Proved here That two-dimensional lattice gauge theory is solvable is a long-known fact; this is a machine check of its $\mathbb{Z}_2$ case.

Proof. Expanding the product with $e^{c\chi} = \cosh c + \chi \sinh c$,

$$Z_\chi^{2D}(c) = \sum_{A \subseteq \text{plaquettes}} \Big(\prod_{y \in A} \sinh c_y\Big)\Big(\prod_{y \notin A} \cosh c_y\Big) \sum_t \prod_{y \in A} \chi_y^{2D}(t)$$

Every link is an edge of exactly two plaquettes, so if A is neither empty nor everything, there is on the boundary of A a "link with only one side in A", and the change of variables that reverses the sign of that link changes the sign of the inner sum, which is therefore 0 (`sum_FA_eq_zero`). For $A = \emptyset$ the product is 1, and for $A =$ everything each link appears twice, so the product is again 1 (`prod_chi2_univ`). The two remaining terms are (6) Proved here .

06

Part: moving the base partition function to two dimensions

The base system of step 2 keeps the four-dimensional plaquette index and has the two-dimensional sign field t as its variable. Its plaquette character $\chi_B^0(t)$, the product of the signs pulled back to the four links of the four-dimensional plaquette B, is as follows (Lean: `TwistedWilson.chi0_eq`).

$$\chi_B^0(t) = \begin{cases} \chi^{2D}_{(x_3,x_4)}(t) & B \text{ a (3,4) plaquette at longitudinal position } (x_3, x_4) \\ 1 & \text{other plaquettes} \end{cases} \quad (8)$$

Links in directions 1 and 2 carry $+1$, and the signs of links in directions 3 and 4 do not depend on the transverse position, so on plaquettes other than (3,4) the link signs each appear twice and cancel. Hence, if the coupling on plaquettes other than (3,4) is a uniform K,

$$Z_\chi^0(c) = e^A \cdot Z_\chi^{2D}\Big(y \mapsto \sum_{(x_1,x_2)} c((x_1,x_2),y),(3,4)\Big), \qquad A = \sum_{B \in /(3,4)} K \quad (9)$$

(`Z_base`). The couplings of the N^2 transverse plaquettes lying over the longitudinal position y add up to the coupling of the two-dimensional plaquette y. For a uniform $K = |\beta|$ the two-dimensional coupling is $N^2|\beta|$ (`Z_base_const`), and multiplying by the (3,4) twist ζ gives the coupling $N^2|\beta|$ with the sign of plaquette 0 alone changed (`Z_base_flip`). The same constant e^A multiplies both, so it cancels in the ratio Proved here .

What matters here is that the set of plaquettes of the twist $n34$ (the (3,4) plaquettes at longitudinal position $(0,0)$, N^2 transverse copies) is exactly the N^2 plaquettes lying over the two-dimensional plaquette 0 (the definition of the twist in 19-02). The thin vortex of § 03 is this bundle of N^2 plaquettes.

07

Arithmetic restatements — a lower bound on the ratio and an upper bound on the flux free energy

Setting $s = \tanh(N^2|\beta|)^{M^2}$, (1) can also be written in the following two forms. In both, the only assumption field is $N, M \geq 1$, and they hold for every real β. We write them in the notation of this part's convention $n_{12} = 1$; for $n_{12} = 0$ the same formulas hold with Z_0, Z_1 replaced by $Z_{N,M}(\beta;0,0)$, $Z_{N,M}(\beta;0,1)$.

A lower bound on the ratio (Lean: `TwistedWilson.twisted_ratio_lower`).

$$\frac{Z_1}{Z_0} \;\geq\; \frac{1-s}{1+s} \quad (10)$$

This is equivalent to (1) Proved here .

An upper bound on the flux free energy (Lean: `TwistedWilson.vortex_free_energy_bound`).

$$\Delta F = -\ln\frac{Z_1}{Z_0} \;\leq\; \ln\frac{1+s}{1-s} \;\leq\; \frac{2s}{1-s} \quad (11)$$

ΔF is 't Hooft's flux free energy (the same quantity as in 25-01 and 26-03). This is arithmetic from $0 \leq s < 1$ and $\ln x \leq x - 1$; both (10) and (11) are machine-checked in Lean Proved here .

(11) is a form, for each finite M, that bounds ΔF from above by $2s/(1-s) \approx 2e^{-\sigma_0 M^2}$. It is an upper bound on ΔF, not a lower bound ($\Delta F \geq 0$ follows, when M is even, from

the sign in 21-06). The word floor is used in the direction of bounding the speed (the rate) at which ΔF decreases from below.

08

What is not said

- **The sign.** $Z_1 \le Z_0$ ($\theta_M \ge 0$) is not part of the theorems of this chapter. If M is even, $\theta_M \ge 0$ for every real β (21-06). Since there are examples with $\theta_M < 0$ at $\beta < 0$ (the table of § 02), for $\beta < 0$ it is the two-sided form (2) that has content.
- **No rate uniform in N.** $\sigma_0 \approx 2e^{-2N^2|\beta|}$ goes to 0 as $N \to \infty$. This is an estimate for a slab with the width held fixed, effectively two-dimensional, and not a positive lower bound on the four-dimensional tension.
- **The rate as $M \to \infty$ is not defined.** (3) is an inequality for each finite M. We do not have, as a Lean statement, "$\sigma_N(\beta) \ge \sigma_0$" for a tension $\sigma_N(\beta)$ defined by $\lim$ or $\liminf$ (what can be read from (3) is only the estimate for each M).
- **Weak at large $N^2|\beta|$.** The rate is the value for a thin vortex of cost $2N^2|\beta|$; the thick-vortex floor (21-03) can be better at large β ($S_N < 2N^2$ for $N = 2, 3, 4$: 21-05).
- **The lattice spacing is fixed.** Nothing is said about the continuum limit, infinite volume, the four-dimensional area law, the band or the mass gap (the statements to be shown about the band and the mass gap are in 19-07).

A discussion that places this inequality within the view "tools built on structure alone yield no more than the floor" is in the record of the study, 26-04.

What this chapter can say

Can say	Cannot say
Proved here The thin-vortex floor (1): $\theta_M^{(n_{12})} \le \tanh(N^2\|\beta\|)^{M^2}$, every real β, $N, M \ge 1$, $n_{12} \in \{0,1\}$, no assumption fields. Lean: `TwistedWilson.twisted_vortex_bound`	$Z_1 \le Z_0$ (21-06). For $\beta < 0$ the one-sided form may have no content
Proved here The two-sided form (2): $\|\theta_M^{(n_{12})}\| \le \tanh(N^2\|\beta\|)^{M^2}$, same range. Lean: `TwistedWilson.twisted_vortex_bound_abs`	A lower bound on θ_M
Proved here The exponential form (3): for $\beta \neq 0$, $\sigma_0 > 0$ and $\|\theta_M^{(n_{12})}\| \le e^{-\sigma_0 M^2}$. Lean: `TwistedWilson.thin_vortex_floor`	A rate uniform in N. A definition of the rate as $M \to \infty$, and a statement of its lower bound

Can say	Cannot say
Proved here The lower bound on the ratio (10) and the upper bound on the flux free energy (11). Lean: `twisted_ratio_lower`, `vortex_free_energy_bound`	A lower bound on ΔF
Proved here The closed two-dimensional $\mathbb{Z}_2$ formula (6)(7). Lean: `TwistedPlane.Z_plane`, `plane_flip_ratio`	The two-dimensional ratio for non-abelian groups (not used in this chapter)
Proved here $\sigma 0 = 2\,\mathrm{artanh}(e^{-2N^2\lvert\beta\rvert}) \approx 2e^{-2N^2\lvert\beta\rvert}$	—
Reading The gap between θ_M and the right side on small lattices (§ 02)	The strength on large lattices

Sources and reproduction

Item	Kind	Source / tool
The pattern of comparing $SU(2)$ with $\mathbb{Z}_2$	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the same bibliography as 26-01; the texts not checked in this book). J. Fröhlich, Phys. Lett. B 83 (1979) 195, bibliography only (not checked)
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (the same bibliography as 25-01)
(1)(2)(3)(10)(11) and the parts	Proved here	The route of proof in the text. Machine-checked in Lean 4 (declarations in each section; only the three standard axioms, no sorry). The definitions and statements were compared by a separate pair of eyes and judged "to hold" (reading the definitions line by line)
Numerical checks during the comparison	Reading	Zero violations in independently written exhaustive sums over finite groups and $SU(2)$ Metropolis runs
The tables of § 02	Reading	The numerical record of this book's study (floating-point values of $\tanh$. Metropolis for $SU(2)$ and Q_8: 2000 chains × 250 sweeps, estimating Z_1/Z_0 under the measure of Z_0. Sums by variable elimination for $\mathbb{Z}_2$, Q_8 and the binary tetrahedral group)

Words used in this chapter — thin-vortex floor, thin vortex, vortex ratio, flux free energy, floor, rate, base system, closed two-dimensional Z_2 formula, averaging over the centre. Their meanings are in A-02 Glossary.

Next chapter to read: 21-03 The thick-vortex floor — statement and route — instead of gathering the vortex onto one sheet, we take the rate again from the cost S_N of a cheaper thick vortex.

The thick-vortex floor — statement and route

This chapter is this book's own investigation.

What this chapter shows — that for the SU(2) Wilson measure on a twisted torus of width N and length M, the upper bound on the vortex ratio θ_M, "for every $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, $|\theta_{M+1}(\beta)| \le \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$" (the thick-vortex floor), is machine-checked in Lean for all $N \ge 1$ with no assumption fields. How to read the order of quantifiers, the difference from the thin-vortex floor (exponent $2N^2$), and the route of the seven steps of the proof.

Prerequisite chapters — 19-02 (the twisted torus, the vortex ratio θ, the minimal action of a vortex S_N), 19-01 (the Wilson measure), 21-02 The thin-vortex floor. We use the language of limits ("sufficiently large") and the minimum of a continuous function on a compact set.

Said first — this is an upper bound **at a fixed lattice spacing, with the width N held fixed**. Neither the threshold in β nor the threshold in M for each β is explicit. The value of S_N for general N is not in Lean. The reverse direction (a lower bound on the rate) is not shown. It does not reach the band (the statement of 19-07), the four-dimensional area law, the mass gap or the continuum limit. In form it is a formalisation of a result of a known type, belonging to the picture of vortices on a thin twisted slab (the line of Mack–Petkova; conditional independence of blocks with the frame held fixed).

The order of this chapter

1. The objects, restated — the twisted torus, θ_M, how action is counted, S_v(N,m) and S_N
2. Statement — the thick-vortex floor, and how to read the order of quantifiers
3. Meaning — thin vortices and thick vortices — the exponent $2N^2$ drops to S_N. Figure 1
4. A map of the route — the seven steps, and which chapter holds which step
5. First half — dividing into blocks, turning into differences of action — the block-by-block estimate and the Laplace step

6. Second half — swapping the vortex into a cheap window — pigeonhole, compactness, the local model, rigidity, transplanting
7. Assembly — where the order of quantifiers comes from — from ε to m, β, M. Detour: why the twisted background is needed
8. What is not said — thresholds, the value of S_N, the reverse direction, N = 1, the band

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01

The objects, restated

The definitions are in Part 19. Here we restate, one line each, only what the statement needs.

- The lattice is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 are the width (side N), directions 3 and 4 the length (side M) (19-02).
- The background twist $n_{12} = 1$ is always present. Z_0 and Z_1 are the partition functions with twist $n_{34} = 0$ and 1 on the (3,4) planes (the sign of the twist). The measure is the unnormalised Wilson weight $e^{\beta \sum_p z_p \operatorname{Re} U_p}$ times the product of Haar measures.
- The vortex ratio is $\theta_M(\beta) = (Z_0 - Z_1)/(Z_0 + Z_1)$ (19-02). The denominator is positive.
- Action is counted in the form before multiplying by β: $S(U;\eta) = \sum_p \big(1 - z_p\, \eta_p \operatorname{Re} U_p\big) \geq 0$. $\eta_p = \pm 1$ is a sign that multiplies only the (3,4) plaquettes.
- The twisted vacuum V is the configuration with i on the direction-1 links at $x_1 = 0$, j on the direction-2 links at $x_2 = 0$, and 1 elsewhere. Since $ij\,i^{-1}j^{-1} = -1$, even on the twisted (1,2) plaquettes $z_p \operatorname{Re} U_p = 1$, and the action is 0.
- $S_{\mathrm{v}}(N,m)$ is the infimum of the action of the block when, on the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/2m)^2$, the frame of a block of side m is held fixed at V and the sign of the corner (3,4) plaquette (N^2 transverse copies) is reversed. $S_N = \inf_{m \geq 1} S_{\mathrm{v}}(N,m)$ (19-02).

Leaving the interior at V and reversing the corner plaquette raises the action of each of the N^2 transverse plaquettes from 0 to 2. So $S_{\mathrm{v}}(N,m) \leq 2N^2$ and $S_N \leq 2N^2$.
Proved here Lean: `TwistBlocks.SN_bounds` ($0 \leq S_N \leq 2N^2$, for all $N \geq 1$).

02

Statement

The thick-vortex floor. Fix $N \geq 1$. For any $\varepsilon > 0$, for all sufficiently large β, for all sufficiently large M,

$$|\theta_{M+1}(\beta)| \;\leq\; \exp\Big(-(M+1)^2\, e^{-\beta(S_N+\varepsilon)}\Big). \quad (1)$$

Lean: `TwistBlocks.thick_vortex_floor` (all $N \geq 1$, no assumption fields; the statement is `ThickVortexFloorAbs N`). The form with definitions unfolded is `TwistBlocks.thick_vortex_floor_unfolded`. It is a formalisation of a result of a known type; the type is said to be the line of Mack–Petkova Cited (bibliography in the table at the end of the chapter; the text not checked in this book).

In the language of filters, the Lean statement has the order $\forall \varepsilon > 0,\ \forall^{\mathrm{f}} \beta \in \mathrm{atTop},\ \forall^{\mathrm{f}} M \in \mathrm{atTop}$. Unfolded into quantifiers, the order is as follows.

$$\forall \varepsilon > 0 \;\; \exists \beta_0 \;\; \forall \beta \geq \beta_0 \;\; \exists M_0(\beta) \;\; \forall M \geq M_0(\beta) : \quad (1). \quad (2)$$

Four points to note when reading it.

1. It is of the form in which β_0 **may depend on** ε**, and** M_0 **on** ε **and** β. The values of neither threshold appear in the statement (§ 07).
2. It is **with absolute value**. It bounds $|\theta|$ from above without using the sign of θ. The sign ($\theta \geq 0$ for even M) is a separate theorem (21-06).
3. Writing $M+1$ is only to avoid $M = 0$. Reading it as "for all sufficiently large M" is the same.
4. Only the large side of β. Nothing is said about $\beta < 0$ or about finite β.

Rewriting (1) in terms of the rate, for M with $\theta_M \neq 0$, $-\frac{1}{M^2} \ln |\theta_M| \geq e^{-\beta(S_N+\varepsilon)}$. Setting the slab tension to $\sigma_N(\beta) = \liminf_M -\frac{1}{M^2} \ln |\theta_M|$, we get $\limsup_{\beta\to\infty} \frac{1}{\beta} \ln \frac{1}{\sigma_N(\beta)} \leq S_N$. Proved here This restatement is one line of arithmetic outside Lean.

03

Meaning — thin vortices and thick vortices

The thin-vortex floor (21-02) says $|\theta_M| \leq \tanh(N^2|\beta|)^{M^2}$ for every real β and all $N, M \geq 1$. For large β, $-\ln \tanh(N^2\beta) \approx 2e^{-2N^2\beta}$, so the rate in the exponent is of order $e^{-2N^2\beta}$. **The exponent of the rate is** $2N^2$.

In the thick-vortex floor (1), the rate is $e^{-\beta(S_N+\varepsilon)}$. **The exponent of the rate drops from** $2N^2$ **to** S_N. If $S_N < 2N^2$, the thick-vortex floor is the stronger upper bound

where β is large.

In pictures: a vortex is a single reversal of the sign η of a (3,4) plaquette, threaded through the whole transverse T^2_N.

- **A thin vortex**: the reversal is absorbed at the position of a single (3,4) plaquette. Each of the N^2 transverse plaquettes pays an action of 2, so the cost is $2N^2$.
- **A thick vortex**: the reversal is spread within a block of side m in the directions of the (3,4) planes, and also over the transverse T^2_N. The links are rotated little by little to thin out the burden on each plaquette. The minimal cost is $S_{\mathrm{v}}(N, m)$, and its infimum over $m \to \infty$ is S_N.

Precisely, S_N is the infimum of "the minimal action for threading one vortex through a block whose boundary is held at the twisted vacuum". It is not the minimal action over the whole torus.

Figure 1 Dividing the longitudinal (3,4) plane T^2_M into blocks of side m (left). A single block enlarged (right): the frame (thick lines) is held fixed, and the sign is reversed on the corner plaquette (dark square). The cost of the vortex is paid by spreading this one sheet within the block. Above each point sits the transverse T^2_N.

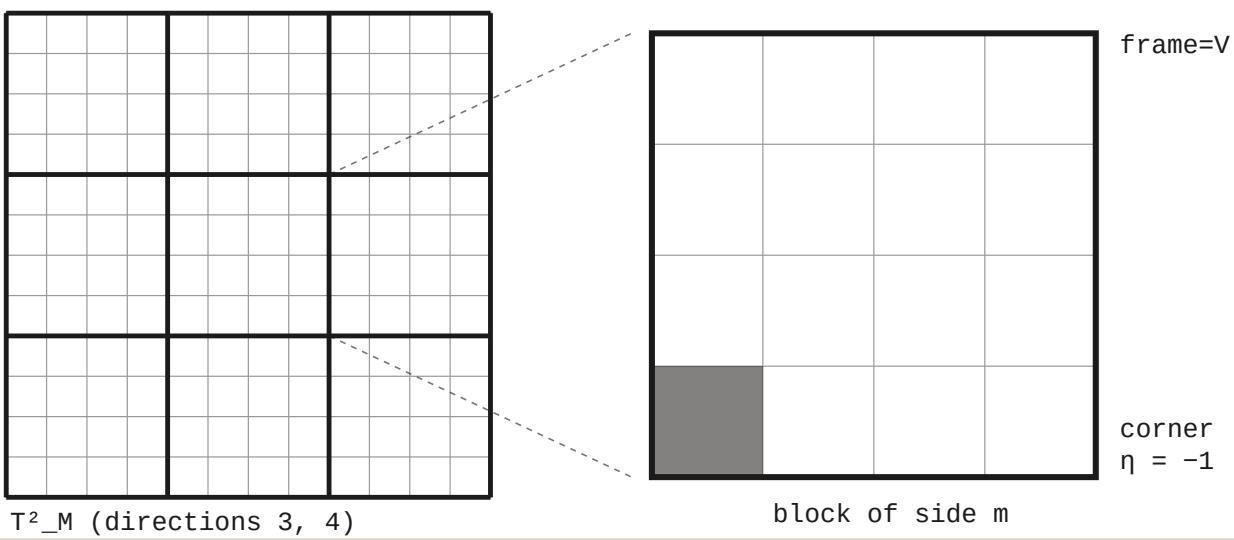

Figure 1 is schematic. No numbers are drawn.

The numbers for S_N are as follows.

N	Thin-vortex exponent $2N^2$	What can be said about S_N	Label / Lean
1	2	$S_1 = 2$ (numerical cooling). No gap	Reading Inside Lean only $S_1 \leq 2$

N	Thin-vortex exponent $2N^2$	What can be said about S_N	Label / Lean
2	8	$S_2 \le 6.67401$, $S_2 < 8$	Computer-assisted 21-05 (`VortexCert.SN_two_le`, `SN_two_lt`)
3	18	$S_3 < 18$ (upper bound 9.09321)	Computer-assisted 21-05 (`VortexCert.SN_three_lt`)
4	32	$S_4 < 32$ (upper bound 12.02057)	Computer-assisted 21-05 (`VortexCert.SN_four_lt`)
General	$2N^2$	$0 < S_N \le 2N^2$	Lean: `VortexPos.SN_pos`, `TwistBlocks.SN_bounds` (21-05)

$S_2 \le 6.67401$ for $N = 2$ is a computer certificate by a configuration with rational entries, checked in the Lean kernel. We use it as support for the analytic proof. For $N = 2$, the rate of the thick-vortex floor $e^{-6.675\beta}$ exceeds the rate of the thin-vortex floor $-\ln\tanh(4\beta)$ (the upper bound becomes smaller) from about $\beta > 0.523$, comparing down to the coefficients Reading . However, since the thresholds of the thick-vortex floor are not explicit, one cannot say that the thick upper bound is smaller at any concrete (β, M). Only the rates as $\beta \to \infty$ can be compared.

04

A map of the route

The proof splits into two halves: "bound $|\theta|$ by a quantity defined block by block, and bound the rate of that quantity as $\beta \to \infty$ by the minimal action of a vortex". Divided into steps, it is as follows.

```
Step 1 block-by-block estimate  |θ| ≤ tanh(K_eff)^{k²}            (first hal
  |                       K_eff = sup of log of the (+)/(−) ratio of a block
Step 2 Laplace step        K_eff ≤ ½(β(κ_c + 2δ) + D)             (first hal
  |                       κ_c = sup of the (+)/(−) difference of the block's
  |
Step 3 canonical signs     the block's sign field goes to a standard form, ac
Step 4 pigeonhole          the block contains a window of small action
Step 5 compactness         "small action" reduced to a claim about "action 0"
Step 6 local model, rigidity  action-0 configurations are gauge transforms of
Step 7 transplant          put a minimising configuration of S_v(N, m₀) into
  |                       ⇒ κ_c ≤ S_N + ε for sufficiently large m
  ▼
Assembly                   take the thresholds in the order ε → m → β₀ → M₀
```

The output of steps 3–7 is "for sufficiently large block side m, $\kappa_c \le S_N + \varepsilon$ for every frame". Inside Lean there is a named intermediate form of this, and from there to (1) is

arithmetic (21-04). The statements and scope of the parts are placed in 21-04. This chapter writes only the flow.

05

First half — dividing into blocks, turning into differences of action

Step 1: the block-by-block estimate

Divide the longitudinal plane T_M^2 into $k \times k$ blocks of side m ($mk \le M$; only the last row and column are made wider). The links lying on the boundary lines of the blocks are called the **frame**. Every plaquette contains only frame links and interior links of at most one block. So once the frame is held fixed, the weight splits into a product over blocks.

In each block, compare the partition functions $Z_B^F(\pm)$ of the class where the product of the signs of the (3,4) plaquettes is $+1$ and the class where it is -1. Letting K_{eff} be half the supremum, over blocks and frames, of the logarithm of that ratio,

$$|\theta_M| \le \tanh(K_{\mathrm{eff}})^{k^2}. \quad (3)$$

Lean: `TwistBlocks.block_bound_abs` (every real β, $mk \le M$, arbitrary background n_{12}). The ratio lies within $e^{\pm 2N^2|\beta|}$, so $K_{\mathrm{eff}} \le N^2|\beta|$ (`TwistBlocks.Keff_le`). At $m = 1$ this returns to the form of the thin-vortex floor.

Step 2: the Laplace step

For large β, $Z_B^F(\pm)$ is nearly determined by the minimum $S_{\min}^F(\pm)$ of the block's action. From above it is trivially bounded by $e^{-\beta S_{\min}}$. From below it is bounded by the Haar measure of a small ball around the minimiser. In this way, for $\beta \ge 0$ and $\delta > 0$,

$$K_{\mathrm{eff}} \le \tfrac{1}{2}\big(\beta(\kappa_C + 2\delta) + D(\delta)\big), \qquad \kappa_C = \sup_{b,F} |S_{\min}^F(+) - S_{\min}^F(-)|. \quad (4)$$

Lean: `TwistBlocks.Keff_le_laplace` ($\beta \ge 0$, $\delta > 0$). $D(\delta)$ does not depend on β and is bounded above by a quantity determined by the block size alone (uniform in M). Combined with $-\ln\tanh K \ge e^{-2K}$, (3) and (4) give $|\theta_M| \le \exp(-k^2 e^{-\beta(\kappa_C + 2\delta) - D})$. What remains is to bring κ_C close to S_N.

06

Second half — swapping the vortex into a cheap window

What we want to show is that for sufficiently large m, **for every frame** F, $S^F_{\min}(-) \le S^F_{\min}(+) + S_N + \varepsilon$ (and the reverse) holds. That is, "from a configuration close to the minimum of (+), the class can be reversed at a cost of at most $S_N + \varepsilon$". We show it in five steps.

Step 3: canonical form of the signs

There are many sign fields η within a block. But applying reversals of links by the central $\mathbb{Z}_2$ in the interior, η can be moved to a standard form (all $+1$, or -1 on the corner plaquette alone) without changing the action. The class is determined by the product $\prod \eta$ alone. Lean: `TwistBlocks.canon_holds` (all $N, M \ge 1$, $mk \le M$).

Step 4: choosing a cheap window by pigeonhole

In either class, the block action of a configuration close to the minimum is at most $144N^2m$, since only the plaquettes touching the frame contribute. Inside the block, place T^2 windows of side $m_0 + 2$ so that they share no plaquettes. The sum of the actions of the windows is at most the action of the block, so some window has action at most $144N^2m/T^2$. Taking T large makes the action of this window as small as we like.

Step 5: compactness

The space of configurations around the window is compact, and the action is continuous. So if one can say "for configurations of action 0, the class can be reversed at a cost of at most c", then "for configurations of sufficiently small action, it can be reversed at a cost less than $c + \varepsilon$" follows. This is phrased in terms of cost, without using a distance. The smallness γ has no explicit value (it is a different quantity from the δ of step 2). Lean: `ThickFloor.small_action_cost`.

Step 6: the local model, and rigidity of the twisted torus

Lift the neighbourhood of the window to a single finite model independent of M (the transverse $T^2_N \times$ a square of side $m_0 + 2$). In this model, a configuration of action 0 is, **on the links that appear in plaquettes**, a gauge transform of the local twisted vacuum. The core is a problem on the transverse T^2_N. A flat configuration on T^2_N with one twisted plaquette is determined by two generators a, b, and the twist gives $ab = -ba$. A pair of anticommuting unit quaternions is mapped to (i, j) by conjugation. So the configuration is a gauge transform of the vacuum. Lean: `TwistBlocks.rigidT_all` (all $N \ge 1$), `ThickFloor.anticomm_conj`.

Step 7: transplanting the minimising configuration

Put a minimising configuration of $S_{\mathrm{v}}(N, m_0)$ into the window, matched by a gauge transformation. The plaquettes outside the window do not change. Inside the window the class is reversed once, at a cost of at most $S_{\mathrm{v}}(N, m_0)$. Choosing m_0 so that $S_{\mathrm{v}}(N, m_0) < S_N + \varepsilon/4$, the cost approaches S_N. Lean: `TwistBlocks.thick_vortex_floor_zero_of_rigid` (from rigidity to the zero-action claim; 21-04).

07

Assembly — where the order of quantifiers comes from

The thresholds are fixed in the following order.

1. Give ε.
2. Choose m_0 with $S_{\mathrm{v}}(N, m_0) < S_N + \varepsilon/4$ (existence only, from the definition of the infimum).
3. Obtain $\gamma > 0$ from the compactness of step 5. **Its value is not explicit**.
4. Take a natural number T with $\gamma T > 144N^2(m_0 + 3) + \varepsilon/4$, and set the block side to $m = (m_0 + 2)T + 1$. The action of the window of step 4 falls below γ, and for all $M \geq m$, κ_c comes within about ε of S_N.
5. Take β_0 large enough that D and $\ln(4m^2)$ can be absorbed by $\beta\varepsilon/2$. Since D contains the Haar measure of a small ball, **this too is not explicit**.
6. Take M_0. Within this proof $M \geq m$ suffices. But statement (2) is of the form in which M_0 may depend on β, and says no more than that.

So the statement takes the form "$\forall \varepsilon\ \exists \beta_0\ \forall \beta \geq \beta_0\ \exists M_0\ \forall M \geq M_0$", and none of the thresholds can be written as numbers.

Detour: why the twisted background is needed

You may skip this. The background twist $n_{12} = 1$ matters in one place, step 6. Because there is exactly one twisted plaquette, the two generators anticommute. Every pair of anticommuting unit quaternions is mapped to (i, j) by conjugation, and the zero-action configuration is unique up to gauge. Without the twist, the generators of a flat configuration may commute. For example, the configuration with every link equal to 1 and the flat configuration whose holonomy around the direction-1 loop is i have holonomies in different conjugacy classes, so they are not related by a gauge transformation. Proved here In that case the rigidity of step 6 fails, and this route does not go through as it stands.

08

What is not said

- **The thresholds are not explicit.** Neither $\beta_0(\varepsilon)$ nor $M_0(\varepsilon, \beta)$ can be written as numbers. They cannot be compared with numerical values at finite β and M.
- **The value of S_N for general N is not in Lean.** What Lean has is $0 < S_N \leq 2N^2$. $S_N < 2N^2$ exists only as certificates for $N = 2, 3, 4$ (21-05). No upper or lower bound uniform in N is shown.
- **For $N = 1$ it is no better than the thin-vortex floor.** $S_1 = 2 = 2N^2$ Reading , the same exponent as the thin-vortex floor.
- **No reverse direction.** A lower bound on $|\theta_M|$ (an upper bound on the rate, $\sigma_N \leq e^{-\beta(S_N - \varepsilon)}$) is not shown.
- **Only the rate as $\beta \to \infty$.** The prefactor affects β with a power of the order of the number of degrees of freedom of a block. It cannot be used for the band (the statement of 19-07).
- **The lattice spacing is fixed.** It does not reach the continuum limit, the area law for four-dimensional Wilson loops, or the mass gap.

What this chapter can say

Can say	Cannot say
Proved here The thick-vortex floor (1): all $N \geq 1$, quantifiers in the order of (2). Lean: `TwistBlocks.thick_vortex_floor` (no assumption fields)	The values of β_0 and M_0. Comparisons at finite β. A lower bound on the rate
Proved here $S_N \leq 2N^2$. Lean: `TwistBlocks.SN_bounds`	$S_N < 2N^2$ for general N. Estimates uniform in N
Computer-assisted $S_N < 2N^2$ for $N = 2, 3, 4$ (21-05)	That the thick bound is smaller at a concrete (β, M)
Proved here The seven steps of the route (§ 05, § 06). The parts of each step, with Lean names and scope, are in 21-04	A sharp form of the prefactor of step 2 (the degree of the power of β)
Reading $S_1 = 2$; for $N = 2$, the thick rate exceeds the thin rate from about $\beta > 0.523$	Proofs of these numbers

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions and flux	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
Conditional independence of blocks with the frame held fixed; vortex containers	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the texts not checked in this book; the type is the same as the containers of 25-03). T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017
The thick-vortex floor and the steps of the route (machine-checked in Lean 4)	Proved here	A formalisation of the type above. Lean: `TwistBlocks.thick_vortex_floor` and its chain (parts in 21-04). The three standard axioms, no sorry
$S_1 = 2$, the threshold $\beta > 0.523$	Reading	The numerical record of this book's study (numerical values of S_{v} by cooling, comparison of coefficients with the thin-vortex floor, floating point)
Upper bounds on S_N for $N = 2, 3, 4$	Computer-assisted	21-05 (certificates by configurations with rational entries, checked in the Lean kernel)

Words used in this chapter — thick-vortex floor, thin-vortex floor, vortex ratio θ, twisted vacuum, minimal action of a vortex S_N, block, frame, window, effective coupling K_eff, rigidity, order of quantifiers. Their meanings are in A-02 Glossary.

Next chapter to read: 21-04 The thick-vortex floor — parts of the proof — the Lean statements and scope step by step, and theorems that show the limits of the method.

2026-11-13 · chapter Yang–Mills and the mass gap Part 21 Firm theorems — the vortex floor and Polyakov loops The thick-vortex floor · Kiichi, Shiori and Rin

The thick-vortex floor — parts of the proof

> This chapter is this book's own investigation.

What this chapter shows — the statements and scope of the parts used in the proof of the thick-vortex floor (21-03): the block estimate and the effective coupling K_{eff}, the Laplace step, the lemma on the canonical form of signs, the compactness step, conjugation of anticommuting pairs and rigidity of the twisted torus. The form of the intermediate implications that link the parts, and the fact that all their assumptions are discharged in the final theorem. And a theorem showing that the rate of this method cannot be better than S_N.

Prerequisite chapters — 21-03 § 04 (the map of the route), 19-02 (the minimal action of a vortex), 19-01 (the product of Haar measures). We use the product of quaternions and the minimum of a continuous function on a compact set.

Said first — every part is a statement on a finite lattice with the lattice spacing and the width N held fixed. The intermediate implications are in "if ... then" form, with all assumptions shown. Those assumptions are discharged in the final theorem, but the intermediate implications themselves do not show content. The theorem on the limits of the method says only that an intermediate statement of this proof is false for $S < S_N$. It does not deny the floor itself, nor does it say that the true rate is S_N.

The order of this chapter

1. Blocks, frames, and the partition functions of classes — the conventions for dividing into blocks, and the definitions of $Z_B^F(\pm)$ and K_{eff}
2. The block estimate — $|\theta| \le \tanh(K_{\mathrm{eff}})^{k^2}$ and $K_{\mathrm{eff}} \le N^2|\beta|$
3. The Laplace step — the Haar measure of a small ball, and an upper bound on K_{eff} by κ_c
4. The lemma on the canonical form of signs — bringing a block's sign field to a standard form without changing the action
5. The compactness step — from "if the action is 0" to "if the action is small"
6. Rigidity — conjugation of anticommuting pairs, and zero-action configurations on the twisted torus

7. The form of the intermediate implications, and how the assumptions are discharged — the six links of the chain
8. A theorem showing the limits of the method — for $S < S_N$ the intermediate statement is false

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01

Blocks, frames, and the partition functions of classes

The lattice is the twisted torus of width N and length M (19-02). Divide the longitudinal plane T^2_M into $k \times k$ blocks. The only condition is $mk \le M$. Blocks have side m; only the blocks in the last row and column have width $M - (k-1)m$.

- **Frame**: the links lying on the boundary lines of the blocks. A direction-3 link is in the frame if its $x4$ is on a line, a direction-4 link if its $x3$ is on a line, and a direction-1 or 2 link if either is on a line.
- **Interior**: the non-frame links belonging to the block.
- **Class**: whether the product $\prod \eta$ of the signs η of the (3,4) plaquettes in the block is $+1$ or -1.

Every plaquette contains only frame links and interior links of at most one block. So once the frame F is held fixed, the weight splits into a product over blocks. Write $Z^F_B(s)$ for the partition function of block b with frame F held fixed, summed over the class $s = \pm 1$ (integrating over the interior links with the product of Haar measures).

$$K_{\text{eff}} = \tfrac{1}{2} \sup_{b,\,F} \left| \ln \frac{Z^F_B(+)}{Z^F_B(-)} \right| . \quad (1)$$

The supremum is taken over all blocks b and all frame values F. Lean: `TwistBlocks.Keff`, `TwistBlocks.ZB`. In Lean, $Z^F_B(s)$ is the integral of the integrand $C_b + sS_b$ obtained by first summing over the signs plaquette by plaquette. It is the same as the sum within the class.

02

The block estimate

The block estimate. For $mk \le M$, every real β, and both backgrounds n_{12},

$$\left|\frac{Z_0 - Z_1}{Z_0 + Z_1}\right| \le \tanh(K_{\mathrm{eff}})^{k^2}, \qquad 0 \le K_{\mathrm{eff}} \le N^2|\beta|. \tag{2}$$

Proved here Lean: `TwistBlocks.block_bound_abs` (no assumption besides $mk \le M$), `TwistBlocks.Keff_le` ($mk \le M$), `TwistBlocks.Keff_nonneg`. This is the form of conditional independence of blocks with the frame held fixed, together with a product of block-by-block suprema.

Cited The type is a formalisation of conditional independence of blocks with the frame held fixed (the line of Mack–Petkova; sources in the table at the end of the chapter).

The proof runs in three lines. (i) With the frame held fixed, the weight of the sum over classes of η becomes a product over blocks. (ii) In each block, the ratio of the weights of classes $\pm$ lies within $e^{\pm 2K_{\mathrm{eff}}}$. Two positive numbers $Z_\pm$ whose ratio lies within $e^{\pm 2K}$ satisfy $|Z_+ - Z_-| \le \tanh K\,(Z_+ + Z_-)$. (iii) By the triangle inequality for integrals, $\tanh$ is multiplied in once for each of the k^2 blocks. $K_{\mathrm{eff}} \le N^2|\beta|$ holds because reversing one class changes only the signs of the N^2 transverse plaquettes.

From (2) and $-\ln\tanh K \ge e^{-2K}$ $(K > 0)$, $|\theta_M| \le \exp(-k^2 e^{-2K_{\mathrm{eff}}})$. Lean: `TwistBlocks.abs_block_bound_exp`.

What is not said. Since K_{eff} also takes the supremum over blocks, when block shapes are mixed it is weaker than the block-by-block product $\prod_b \tanh K_b$. For example, in a small $\mathbb{Z}_2$ model, the right side is $\tanh(1)^4 = 0.3364$ while the block-by-block product is 0.0862 Reading. It does not affect the rate.

03

The Laplace step

A coarse Laplace lower bound. Let ι be a finite set and ν the product of Haar probability measures on $SU(2)^\iota$. The distance is the maximum of the componentwise distances. If f is continuous and L-Lipschitz $(L > 0)$, then for all x_0, $\beta \ge 0$ and $\delta > 0$,

$$e^{-\beta(f(x_0)+\delta)}\,\mu_1(\delta/L)^{|\iota|} \le \int e^{-\beta f}\,d\nu, \qquad \mu_1(r) = \mathrm{Haar}\big(\{g : |g - 1| < r\}\big) > 0. \tag{3}$$

Proved here Lean: `CoarseLaplace.laplace_lower` ($\beta \ge 0$, $\delta > 0$, $L > 0$). The proof only uses $f \le f(x_0) + \delta$ on the ball of radius δ/L around x_0. The measure of the ball does not depend on its centre because left multiplication is an isometry and Haar measure is left-invariant. The prefactor does not depend on β. This is a weak form of Laplace's method that does not make the constants explicit.

We apply this to the action of a block. The action S_B is Lipschitz in the interior links, and from the number n_b of plaquettes in the block the constant can be taken as $L_b = 4n_b + 1$. Let $S^F_{\min}(\pm)$ be the infimum of the action over class $\pm$ and the interior, and set

$$\kappa_C = \sup_{b,F} \left| S^F_{\min}(+) - S^F_{\min}(-) \right|, \qquad D(\delta) = \max_b \Big(\ln \#\{\text{sign fields of block } b\} - \#\{\text{interior li}$$

Then, for $\beta \geq 0$ and $\delta > 0$,

$$K_{\text{eff}} \leq \tfrac{1}{2} \big(\beta(\kappa_C + 2\delta) + D(\delta) \big). \quad (4)$$

Proved here Lean: `TwistBlocks.Keff_le_laplace` ($mk \leq M$, $\beta \geq 0$, $\delta > 0$). When $M < (k+1)m$, $D(\delta)$ is bounded by a quantity $D^*(N, m, \delta)$ determined by N, m, δ alone (`TwistBlocks.Dmax_le`). That is, it is uniform in M. As a trivial upper bound on κ_C there is also $\kappa_C \leq 2N^2$ (`TwistBlocks.kappaC_le_two`, for all m).

What is not said. The value of $\mu_1(r)$ (that it is of order r^3) is not used, and D is not explicit. A sharp form of the prefactor (the degree of the power of β) is not treated. $\beta < 0$ is not treated.

04

The lemma on the canonical form of signs

The class infimum $S^F_{\min}(\pm)$ is the infimum over all sign fields σ in the class and all interior values. The following lemma makes it equal to "the infimum with η held at one value".

The lemma on the canonical form of signs. Let $mk \leq M$. For a block b, a frame F, interior values y, and a sign field σ of the block ($t = 0$ if $\prod \sigma = +1$, $t = 1$ if -1), there are interior values y' such that the action of the block in the standard field (all $+1$ if $t = 0$; -1 on the corner plaquette alone if $t = 1$) equals the action with σ.

Proved here Lean: `TwistBlocks.canon_holds` (statement `TwistBlocks.Canon N`; all $N, M \geq 1$ and $mk \leq M$; no condition on the shape of the block). The construction is a comb-shaped reversal. Let $\rho = \sigma \cdot (\text{standard field})$. In each column, accumulate the product of ρ from bottom to top to determine the signs κ_3 of the direction-3 links, and in the corner row only, accumulate the total product of each column from right to left to determine the direction-4 signs κ_4. Then on each plaquette the product of κ is ρ, and since the product of ρ within the block is 1, $\kappa = 1$ on the frame. On plaquettes spanning directions 1, 2 and directions 3, 4, the same κ appears twice and cancels.

With this lemma, Lean's κ_C equals the version κ with η held fixed (`TwistBlocks.kappaC_eq_kappa`, $mk \le M$).

05

The compactness step

Compactness in the form of cost. Let X be a compact space and $S, \Phi : X \to \mathbb{R}$ continuous. If $\Phi(x) \le c$ at points with $S(x) \le 0$, then for every $\varepsilon > 0$ there is $\gamma > 0$ such that $\Phi(x) < c + \varepsilon$ at points with $S(x) < \gamma$.

Proved here Lean: `ThickFloor.small_action_cost` (closes using Mathlib alone). Proof: $K = \{\Phi \ge c + \varepsilon\}$ is closed and hence compact. If it is empty, $\gamma = 1$ works. If not, S attains a minimum $S(x_0)$ on K, and by assumption $S(x_0) > 0$. Take $\gamma = S(x_0)$.

How it is used: let X be the configuration space of the local model around the window, S the local action, and Φ the infimum of "the action after replacing the interior and reversing one plaquette". There are finitely many positions for the reversal, so Φ is continuous. This reduces "in a window of sufficiently small action, the class can be reversed at cost $S_{\mathrm{V}} + \varepsilon$" to "for a local configuration of action 0, it can be reversed at cost S_{V}" (`TwistBlocks.localVortexL_of_zero`).

What is not said. The value of γ (a different quantity from the δ of § 03). γ depends only on N, m_0 and ε, but is not explicit. This is the main reason the thresholds of the thick-vortex floor are not explicit.

06

Rigidity

Conjugation of anticommuting pairs

Anticommuting pairs of unit quaternions. If unit quaternions A, B satisfy $AB = -BA$, there is a quaternion $g \neq 0$ with $gAg^{-1} = i$ and $gBg^{-1} = j$.

Proved here Lean: `ThickFloor.anticomm_conj`. First, $\mathrm{Re}\, A = \mathrm{Re}\, B = 0$ and the imaginary parts are orthogonal (`ThickFloor.anticomm_unit`). Next, setting $L(h) = h - ihA - jhB - kh(AB)$, from $A^2 = B^2 = -1$ and $AB = -BA$ alone, $L(h)A = iL(h)$ and $L(h)B = jL(h)$. For $e \in \{1, i, j, k\}$, $\sum_e \mathrm{Re}(\bar{e}\, L(e)) = 4$, so $L(e) \neq 0$ for some e. Take $g = L(e)$. Replacing g by $g/|g|$ gives conjugation by a unit quaternion.

Rigidity of the twisted torus

Consider configurations (A, B) on the transverse $T^2_N = (\mathbb{Z}/N)^2$ (A in direction 1, B in direction 2) whose plaquette products equal the sign of the twist (-1 at the origin, $+1$ elsewhere).

Rigidity of the twisted torus. For all $N \geq 1$, every such (A, B) is a gauge transform of the vacuum of T^2_N ($A = i$ at $a_1 = 0$, $B = j$ at $a_2 = 0$, 1 elsewhere). That is, there is $R : T^2_N \to SU(2)$ such that for all a, $R(a)A(a)R(a + e_1)^{-1}$ and $R(a)B(a)R(a + e_2)^{-1}$ equal the values of the vacuum.

Proved here Lean: `TwistBlocks.rigidT_all` (statement `TwistBlocks.RigidT N`, all $N \geq 1$). This is the form "a twisted flat connection is unique up to gauge". Proof: fixing the gauge on a tree of products taken from the front makes the tree links equal to 1. By induction on the flatness of plaquettes, the remaining links are determined by two generators $a = A(N-1, 0)$, $b = B(0, N-1)$ (`TwistBlocks.cfA`, `cfB`). Since there is exactly one twisted plaquette, $ab = -ba$ (`TwistBlocks.cf_comm`). Take the constant u that maps (a, b) to (i, j) by conjugation of anticommuting pairs, and compare with the vacuum gauge-fixed on the same tree (`TwistBlocks.uniqT`). $N = 1$ is a single point, so it is direct (`TwistBlocks.rigidT_one`).

Lifting to the local model

The local model around the window is the product of the transverse T^2_N and a square of side $V + 1$. This model has **links that appear in no plaquette** (the direction-3 and 4 links at the edges of the square, and the direction-1 and 2 links at the corners). They do not affect the action, so they can take arbitrary values even in a configuration of action 0. Local rigidity is therefore stated in the following scope.

Local rigidity. A configuration of local action 0 equals, **on all links that appear in plaquettes**, a gauge transform of the local twisted vacuum.

Proved here Lean: `TwistBlocks.rigidL_of_rigidT` (RigidT $N \Rightarrow$ RigidL N $(V + 1)$, for all V). Fixing the gauge on a maximal tree in directions 3 and 4 makes all direction-3 and 4 links equal to 1, and the direction-1 and 2 links no longer depend on the longitudinal position. What remains is the problem on T^2_N, which closes by the rigidity above. As an equality of functions on all links it is false, so restricting the scope to "links that appear in plaquettes" is the correct form.

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07
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The form of the intermediate implications, and how the assumptions are discharged

The Lean proof is built as a chain of implications that replaces the assumptions one by one with weaker claims. Each link has the form "if ... then the thick-vortex floor". Here we list them with all assumptions shown. $\mathrm{Floor}(S)$ is the statement (1) of 21-03 with S_N replaced by S.

Link	Lean	Assumption (if ...)	Conclusion
1	`thick_vortex_floor_abs_of_H4`	Upper bound on the flipping cost: for some S: for every $\varepsilon > 0$ there is m such that for all $M \geq m$, $\kappa_c(N, M, m, \lfloor M/m \rfloor) \leq S + \varepsilon$	$\mathrm{Floor}(S)$
2	`H4_of_local`	The lemma on the canonical form of signs (`Canon N`), and for all m_0, "if the action of the window is small, the class can be reversed at cost $S_{\mathrm{v}}(N, m_0) + \varepsilon$" (`LocalVortex`)	The upper bound on the flipping cost with $S = S_N$
1+2	`thick_vortex_floor_abs_of_local`	Of the assumptions of link 2, `LocalVortex` for all m_0 (`Canon N` is discharged inside by `canon_holds`)	$\mathrm{Floor}(S_N)$
3	`thick_vortex_floor_abs_of_localL`	The same claim, on the local model independent of M (`LocalVortexL`)	$\mathrm{Floor}(S_N)$
4	`thick_vortex_floor_abs_of_zero`	For all m_0, configurations of local action 0 can be reversed at cost at most $S_{\mathrm{v}}(N, m_0)$ (`ZeroVortexL`)	$\mathrm{Floor}(S_N)$
5	`thick_vortex_floor_abs_of_rigid`	Local rigidity for all m_0 (`RigidL N (m₀+2)`)	$\mathrm{Floor}(S_N)$
6	`thick_vortex_floor_abs_of_rigidT`	Rigidity of the twisted torus (`RigidT N`)	$\mathrm{Floor}(S_N)$

(All names are under `TwistBlocks.`. The `H4` remaining in the Lean names is the name from the source; in the text we call this intermediate statement the "upper bound on the flipping cost".) The theorems connecting the links are as follows.

- `Canon N` in 2 is discharged by `canon_holds` (§ 04). Pigeonhole (`exists_win_le`) is closed within the proof of 2.
- 3 → 1+2: write the answer on the local model back into the interior of the block (`localVortex_of_local`).
- 4 → 3: the compactness step (`localVortexL_of_zero`, § 05).
- 5 → 4: pull a minimising configuration of $S_{\mathrm{v}}(N, m_0)$ back to the local model and transplant it into the window using the rigidity gauge

(`thick_vortex_floor_zero_of_rigid`). The existence of a minimising configuration is `Svortex_attained`.

- $6 \to 5$: the lift to the local model (`rigidL_of_rigidT`, § 06).

The final theorem `TwistBlocks.thick_vortex_floor` is link 6 with `rigidT_all` put in. With this, every assumption of the chain is discharged and no assumption fields remain. As a sanity check of the machinery, we have also checked that the chain contains no intermediate theorem that is vacuous because its assumptions are false. The inner field "local action ≤ 0" of links 4 and 5 is satisfied by the local vacuum (`SL_vacL`), and the inner field "a flat configuration on T^2_N" of link 6 is satisfied by the vacuum of T^2_N (`flatT_nonempty`). These check that the fields are non-empty; they are examples of satisfaction without content.

> The intermediate links 1–6 do not, on their own, show content. For example, link 1 is easily satisfied if S is large. What has content is the pair of the final theorem with its assumptions discharged and the estimates of the value of S_N (21-05).

08

A theorem showing the limits of the method

Consider whether the S in the assumption of link 1 (the upper bound on the flipping cost) can be taken smaller than S_N. That it cannot is shown in Lean.

Comparison of a block with its frame at the vacuum. For all $m \geq 1$, $S_{\mathrm{v}}(N,m) \leq \kappa_c(N, 2m, m, 2)$. Proved here Lean: `TwistBlocks.kappaC_ge_Svortex`. With $M = 2m$, block $(0,0)$, and the frame at the twisted vacuum, the infimum for (+) is 0 and the infimum for (−) is $S_{\mathrm{v}}(N,m)$.

Negation of the intermediate statement. If $S < S_N$, the upper bound on the flipping cost is false. Proved here Lean: `TwistBlocks.H4_fails_below` (assumption $S < S_N$). Applying the upper bound on the flipping cost with $\varepsilon = (S_N - S)/2$ and $M = 2m$ contradicts the comparison above and $S_N \leq S_{\mathrm{v}}(N,m)$.

Three ways to read this.

1. This is **a limit of the method of this proof**. As long as one goes through the intermediate statement, the upper bound on the flipping cost (an upper bound on κ_c), the exponent of the rate obtained cannot be better than S_N.
2. This is **not a denial of the thick-vortex floor**. The floor (1) itself might hold with an S satisfying $S < S_N$; this theorem says nothing about that.

3. This **does not say that the true rate is S_N** either. A lower bound on the rate (the reverse direction) is still missing.

What this chapter can say

Can say	Cannot say
Proved here The block estimate (2): every real β, $mk \le M$. Lean: `TwistBlocks.block_bound_abs`, `Keff_le`	The block-by-block product form when block shapes are mixed
Proved here The coarse Laplace lower bound (3) and the upper bound (4) on K_{eff}: $\beta \ge 0$, $\delta > 0$. Lean: `CoarseLaplace.laplace_lower`, `TwistBlocks.Keff_le_laplace`	The value of D. A sharp form of the prefactor. $\beta < 0$
Proved here The lemma on the canonical form of signs: $mk \le M$. Lean: `TwistBlocks.canon_holds`	—
Proved here Compactness in the form of cost. Lean: `ThickFloor.small_action_cost`	The value of γ
Proved here Rigidity of the twisted torus: all $N \ge 1$. Lean: `TwistBlocks.rigidT_all`, `ThickFloor.anticomm_conj`	Equality on all links of the local model (only on links that appear in plaquettes)
Proved here Links 1–6 (and 1+2) of the intermediate implications, and the discharge of their assumptions in the final theorem (§ 07)	That the intermediate links alone yield content
Proved here The limit of the method: for $S < S_N$, the upper bound on the flipping cost is false. Lean: `TwistBlocks.H4_fails_below`	A denial of the floor itself. That the true rate is S_N
Reading The numbers for the example with mixed block shapes (0.0862 versus 0.3364)	—

Sources and reproduction

Item	Kind	Source / tool
Conditional independence of blocks with the frame held fixed	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442; 125 (1980) 117 (the texts not checked in this book; the type is the same as the containers of 25-03). T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017
Twisted flat connections	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
The Laplace-method lower bound	Proved here	A weak form of Laplace's method that does not make the constants explicit. Proof in § 03 of the text

Item	Kind	Source / tool
The statements of each part (machine-checked in Lean 4)	Proved here	The Lean bundle (block estimate, Laplace, canonical form, compactness, quaternions, rigidity, transplanting, limits of the method). The three standard axioms, no sorry
The example with mixed block shapes	Reading	The numerical record of this book's study (exhaustive enumeration of a small $\mathbb{Z}_2$ model)

Words used in this chapter — block, frame, interior, class, effective coupling K_eff, coarse Laplace lower bound, lemma on the canonical form of signs, compactness in the form of cost, anticommuting pair, rigidity of the twisted torus, local rigidity, intermediate implication, limits of the method. Their meanings are in A-02 Glossary.

Next chapter to read: 21-05 The minimal action of a vortex — positivity, and a certificate for the upper bound — what can be said about the value of the exponent s_N of the floor.

The minimal action of a vortex — positivity, and a certificate for the upper bound

This chapter is this book's own investigation. The upper bounds of § 05 are computer-assisted certificates, placed as support for the analytic proof.

What this chapter shows — we read two facts, machine-checked in Lean, about the quantity S_N (the minimal action of a vortex) that appears in the exponent of the thick-vortex floor. One is that $S_N > 0$ for every $N \geq 1$, and moreover that there is a positive lower bound uniform in the block size m. The other is a certificate for an upper bound showing that, for $N = 2, 3, 4$, S_N is smaller than the thin-vortex value $2N^2$.

Prerequisite chapters — 19-02 the definition of the minimal action of a vortex, the vortex ratio θ_M; 21-02 The thin-vortex floor; 21-03 The thick-vortex floor

Said first — what we could say is only "positivity" and "upper bounds". The value of S_N is not known. The lower bound δ comes from compactness, so it is not an explicit number, nor is it uniform in the width N. The upper bounds are the action of a single configuration at $m = 8$ ($m = 4$ for $N = 4$), not the value of the infimum over m. The lattice spacing is fixed, and nothing is said about the continuum limit or the mass gap.

The order of this chapter

1. The minimal action of a vortex — the main points of the definition, and trivial bounds
2. S_N is positive — statement and meaning
3. The route to positivity — labels and a telescoping product
4. Detour: where the background twist matters
5. Upper bounds for N = 2, 3, 4 — statement
6. How the certificate is made — rational unit quaternions and rounding up
7. Comparison with the thin-vortex floor — comparing rates, and its limits
8. What is not said

01

The minimal action of a vortex — the main points of the definition, and trivial bounds

The definition is in 19-02. Here we repeat only the main points.

The lattice is the torus with transverse part $(\mathbb{Z}/N)^2$ and longitudinal part $(\mathbb{Z}/2m)^2$. In the longitudinal (3,4) plane take the block at the origin $b_0 = \{0,\dots,m-1\}^2$ (the set of longitudinal points at the roots of plaquettes), and consider its product with the transverse torus T^2_N (notation as in 19-02). The frame of the block (the links on the lines $x_3 \in \{0,m\}$ and $x_4 \in \{0,m\}$) is held at the **twisted vacuum** V. V is the configuration whose direction-1 links are the quaternion i at $x_1 = 0$, whose direction-2 links are j at $x_2 = 0$, and which is 1 everywhere else. Under the background twist $n_{12} = 1$, the action of V is 0.

Let $\Pi_{(0,0)}$ be the (3,4) plaquettes at the block corner $(x_3,x_4) = (0,0)$, collected over all N^2 transverse positions, and reverse their sign ($\eta_p = -1$ for $p \in \Pi_{(0,0)}$, $\eta_p = +1$ otherwise). The sum of the actions of all plaquettes whose root lies in b_0 is

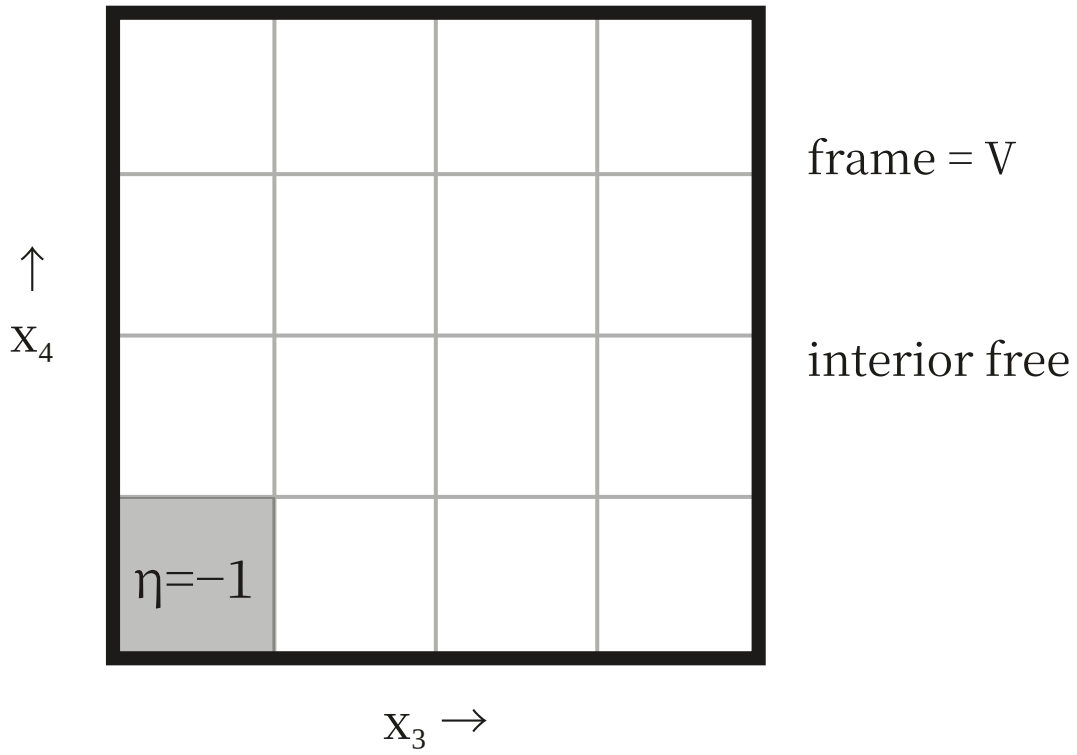
$$S(U;\eta) = \sum_{p:\ \mathrm{root}\in b_0} \left(1 - z_p\,\eta_p\,\mathrm{Re}\,U_p\right) \tag{1}$$

(the notation for action is as in 21-03). z_p is the sign of the background twist. β is not multiplied in. We write the infimum over all configurations of the interior links, and the infimum over block sizes, as

$$S_{\mathrm{v}}(N,m) = \inf_{\text{interior}} S(U;\eta), \qquad S_N = \inf_{m\ge 1} S_{\mathrm{v}}(N,m) \tag{2}$$

The Lean definitions are `TwistBlocks.Svortex` and `TwistBlocks.SN`. Since the frame is the vacuum while one corner sign is reversed, somewhere inside the block a vortex sheet must pass all the way around the transverse torus. $S_{\mathrm{v}}(N,m)$ is the minimal action for threading one such vortex.

Figure 1 A block in the (3,4) plane (example $m = 4$). On the squares of block b_0, the frame lines $x_3, x_4 \in \{0,m\}$ are drawn thick (fixed at the twisted vacuum). The shaded square is the position of the corner reversal $\Pi_{(0,0)}$. Above each point sits the transverse torus T^2_N



There are two trivial bounds. Proved here Since $|\operatorname{Re} U_p| \le 1$, each term of (1) is at least 0, and $S_{\mathrm{v}}(N,m) \ge 0$. Proved here In the configuration with the interior also at V, only the N^2 reversed plaquettes contribute, each $1-(-1)\cdot 1 = 2$, and the rest give 0, so the action is exactly $2N^2$. Hence $0 \le S_N \le 2N^2$ (Lean: `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`). This $2N^2$ is the value of the "thin vortex" that gathers the reversal onto one layer of plaquettes, and corresponds to the exponent of the thin-vortex floor. Reading For $N=1$, $S_1 = 2 = 2N^2$, and a thick vortex gains nothing (the numerical record of this book's study).

S_N appears in the exponent of the thick-vortex floor in the form $e^{-\beta(S_N+\varepsilon)}$. The two facts of this chapter are what can be said about the number in that exponent from below (positivity) and from above (the certificates for $N=2,3,4$). The Lean proof of the thick-vortex floor does not use the two facts of this chapter.

02

S_N is positive — statement and meaning

Positivity of the minimal action of a vortex. For every $N \ge 1$ there is $\delta > 0$, depending only on N, such that

$$\delta \le S_{\mathrm{v}}(N,m) \quad (\text{all } m \ge 1), \qquad \text{hence} \qquad S_N > 0. \ (3)$$

Lean: `VortexPos.Svortex_ge_unif` ($\exists\,\delta>0,\ \forall m\geq 1,\ \delta\leq S_{\mathrm{V}}(N,m)$; all $N\geq 1$, no assumption fields) and `VortexPos.SN_pos` ($0<S_N$).

The order of quantifiers is the point. δ is fixed first and does not depend on m. However large the block is made and however the vortex is spread out, the action does not fall below δ. If it were merely positive for each m, it might approach 0 as $m\to\infty$, and the infimum S_N could be 0.

The value of δ is not known. The proof uses the fact "if the action is small, the configuration is close to the gauge orbit of the vacuum" through **compactness**, not through an explicit estimate of distance. So we can say only that δ exists; no number comes out. Reading Numerically, $S_{\mathrm{V}}(2,m)$ is about 7.0 to 6.67 for $m=3$ to 8 (the numerical record of this book's study), and δ may be much smaller than that.

Cited This theorem is a formalisation, on the lattice, of the known picture that the action of a vortex (a fractional instanton) in a twisted box is positive ('t Hooft's twisted boundary conditions, 25-02) (Lean: `VortexPos.SN_pos`). The only tools are compactness and rigidity.

03

The route to positivity — labels and a telescoping product

We argue by contradiction. Take a configuration of very small action, and show that it is incompatible with the corner reversal.

Step 1 Split into parts. For each point $s=(i,j)$ ($0\leq i,j\leq m$) of the block's (3,4) plane, call the transverse torus T^2_N above it (the direction-1 and 2 links) the **fibre**. Split the action (1) into three kinds of parts.

- Fibre action a_s: the sum of the terms of the (1,2) plaquettes of the fibre.
- Link defect b_λ: how badly the direction-3 or 4 links joining two neighbouring points (N^2 of them) connect the two fibres. For unit quaternions $\|X-Y\|=\sqrt{2(1-\operatorname{Re}XY^*)}$, so $b_\lambda\leq 2N^2\sqrt{2S}$ (S is the action of the block) Proved here . This estimate is also used inside the Lean proof (`VortexPos.linkH_le`, `linkV_le`).
- Square part p_P: the sum of the terms of the N^2 plaquettes over the square P of the (3,4) plane.

The fibre actions and square parts are partial sums of (1), so they are at most S.

Step 2 Fibres are close to the vacuum. On the twisted T^2_N, configurations of action 0 lie in the gauge orbit of the twisted vacuum (Lean: `TwistBlocks.rigidT_all`). By compactness, if the fibre action is small, the fibre is close to the gauge orbit of the

vacuum. The only gauge transformations fixing the vacuum are the constants ± 1, and this too is extended by compactness to "if it nearly fixes it, it is close to ± 1".

Step 3 Attach labels. At each point s, take a gauge g_s that brings the fibre to the vacuum. On the frame the fibre is exactly the vacuum, so take $g_s = 1$ there. For each link λ, $g_s^{-1} U_\lambda g_{s'}$ is close to a constant $+1$ or -1; let its sign be the label $c(\lambda) \in \{\pm 1\}$. Links on the frame lines are 1 in the vacuum, so $c = +1$ there.

Step 4 Compatibility square by square. The product around a plaquette over square P, sandwiched by gauges, has the form $h_1 h_2 h_3^{*} h_4^{*}$, whose value is close to $\prod_{\lambda \in \partial P} c(\lambda)$. If $\eta(P) \;/= \prod_{\partial P} c$, the plaquette term is at least $2 - 4\varepsilon$ ($3/2$ for $\varepsilon = 1/8$). If all parts are small, then $\eta(P) = \prod_{\partial P} c$ for every square.

Step 5 Telescoping product. Multiplying $\prod_{\partial P} c$ over all squares, each interior link appears twice and the frame links are 1, so the product is 1 Proved here . But η is -1 on exactly one square, the corner, so $\prod_P \eta(P) = -1$. Contradiction.

Step 6 Conclusion. So some part is at least a threshold δ_0. For a fibre or a square, $S \geq \delta_0$; for a link, $2N^2\sqrt{2S} \geq \delta_0$. Together,

$$S_{\mathrm{v}}(N, m) \;\geq\; \delta := \min\!\left(\delta_0,\; \frac{\delta_0^2}{32N^4 + 1}\right) \quad (4)$$

holds for all $m \geq 1$. δ_0 comes from the compactness of step 2, so it is not explicit. The Lean core is a lemma of the form `TwistBlocks.grid_vortex_floor`, and all its part fields are discharged inside `VortexPos.Svortex_ge_unif`.

Detour: where the background twist matters

You may skip this. In step 3 the labels fall into $\{\pm 1\}$ because the only gauge transformations fixing the twisted vacuum are the constants ± 1 (Lean: `TwistBlocks.stab_vacT`). i and j appear on the links of the twisted vacuum, and the only unit quaternions commuting with both i and j are ± 1 Proved here . So the "offset" of each link is close to an element of the centre $\{\pm 1\}$, and the argument rides on $\mathbb{Z}_2$ bookkeeping: compatibility square by square and a telescoping product. The core of the proof is that the corner reversal cannot be cancelled in $\mathbb{Z}_2$ bookkeeping.

04

Upper bounds for N = 2, 3, 4 — statement

The upper side can be stated by exhibiting a single concrete configuration. Since $S_{\mathrm{v}}(N, m)$ is an infimum, the action of any configuration is an upper bound for it. The upper bounds machine-checked in Lean are the following three Computer-assisted .

N	Block side m	Upper bound	Corollary	Lean declarations
2	8	$S_{\mathrm{v}}(2,8) \le 667401/100000$	$S_2 < 8 = 2 \cdot 2^2$	`VortexCert.Svortex_two_eight_le`, `SN_two_le`, `SN_two_lt`
3	8	$S_{\mathrm{v}}(3,8) \le 909321/100000$	$S_3 < 18 = 2 \cdot 3^2$	`VortexCert.Svortex_three_eight_le`, `SN_three_le`, `SN_three_lt`
4	4	$S_{\mathrm{v}}(4,4) \le 1202057/100000$	$S_4 < 32 = 2 \cdot 4^2$	`VortexCert.Svortex_four_four_le`, `SN_four_le`, `SN_four_lt`

None has assumption fields. $S_N \le S_{\mathrm{v}}(N,m)$ follows from the definition of the infimum (Lean: `VortexCert.SN_le_Svortex`).

Put into the thick-vortex floor, the exponent becomes an explicit number. Lean: `TwistBlocks.thick_vortex_floor_two` (no assumption fields) says that, for each sufficiently large β, for sufficiently large M,

$$|\theta_{M+1}(\beta)| \;\le\; \exp\!\left(-(M+1)^2\, e^{-6.675\,\beta}\right) \quad (5)$$

`TwistBlocks.thick_vortex_floor_three` has exponent number 9.094, and `TwistBlocks.thick_vortex_floor_four` has 12.021. The proof just takes the ε of the thick-vortex floor (21-03, `TwistBlocks.thick_vortex_floor`) to be "exponent number $-$ SN". The certificate for the upper bound gives $\varepsilon > 0$.

One caution in reading. What we could say is **only an upper bound**. One cannot write "$S_2 = 6.674$". Since it is a single configuration at $m = 8$ ($m = 4$ for $N = 4$), it is not the value of the infimum over m either. Reading The configuration for $N = 4$ comes from a numerical minimisation at $m = 4$ that has not fully converged, and the true value of S_4 may be smaller.

05

How the certificate is made — rational unit quaternions and rounding up

The certificate is computer-assisted Computer-assisted . It is not a pillar of the analytic proof; it is placed as support for "exhibiting one configuration that gives an upper bound". It is made in four stages.

(a) Make the configuration rational. The interior link values $q = (q_0, \underline{q})$ obtained by numerical minimisation (cooling) are converted to $\tau = \underline{q}/(1+q_0)$ and then rounded to $t/2^{16}$ ($t \in \mathbb{Z}^3$). From this, quaternions are built in the Cayley form

$$q = \frac{\left(s^2 - |t|^2,\; 2s\,t\right)}{s^2 + |t|^2}, \qquad s = 2^{16} \quad (6)$$

Proved here Since $(s^2 - |t|^2)^2 + 4s^2|t|^2 = (s^2 + |t|^2)^2$, the norm is **exactly** 1. No square roots are needed (Lean: `VortexCert.normSq_cay`).

(b) Round up plaquette by plaquette. The plaquette term $1 - z_p\eta_p \operatorname{Re}(ABC^*D^*)$ is rational because the four links are rational. Round it **up** in units of 2^{-32} and add the resulting integers. Because the rounding is upward, the integer sum $/2^{32}$ is an upper bound on the action of the block.

(c) Count in the kernel. The integer inequality, for example for $N = 2$ "sum $\times 100000 \le 667401 \cdot 2^{32}$", is checked by a finite computation in the Lean kernel (`decide +kernel`) (Lean: `VortexCert.certN2`, `certN3`, `certN4`). The number of plaquettes is 1536 for $N = 2$, 3456 for $N = 3$, and 1536 for $N = 4$.

(d) Carry over to a real upper bound. The soundness theorem `VortexCert.Svortex_le_of_cert` derives $S_{\mathrm{v}}(N, m) \le b_n/b_d$ from the integer inequality. Its only assumptions are the integer inequality (and positivity of the denominator). Whatever the values in the table, each link is a unit quaternion because of the form (6), the frame values are fixed at V on the side of the definition, and it is shown inside Lean that the background and corner signs agree with the definition. So the configuration of the certificate belongs, **by construction**, to the set over which the infimum (2) is taken.

Reading The actions of the rational configurations are 6.6740076 for $N = 2$, 9.0932068 for $N = 3$ and 12.0205673 for $N = 4$, and the increase due to rounding is less than 4×10^{-7} (the numerical record of this book's study, computed exactly in fractions without rounding). The Lean upper bounds are, in each case, the smallest number on a 10^{-5} grid that passes.

What Lean checks is only the evaluation of integer polynomials. Where the configuration came from (whether the cooling is close to the minimum) has no bearing on the correctness of the upper bound.

06

Comparison with the thin-vortex floor — comparing rates, and its limits

The thin-vortex floor is $|\theta_M| \le \tanh(N^2|\beta|)^{M^2} = \exp(-M^2\sigma_0)$, $\sigma_0 = -\ln\tanh(N^2|\beta|)$. Proved here As $\beta \to \infty$, $\sigma_0 = 2e^{-2N^2\beta}(1 + o(1))$, so the rate at which the thin coefficient decreases is $2N^2$. The rate of the thick-vortex floor (5) is $S_N + \varepsilon$. For $N = 2$, $6.675 < 8$, and as a rate for $\beta \to \infty$, the thick coefficient becomes exponentially larger.

This comparison has three limits.

- **It is not a Lean theorem.** What Lean has is the numerical inequality $6675/1000 < 2 \cdot 2^2$ (`TwistBlocks.thick_vortex_floor_two_rate_lt`) and each of the two floors. "The thick-vortex floor is stronger than the thin-vortex floor" itself is a comparison of numbers set side by side by hand.
- **For small β it is the other way round.** Reading $e^{-6.675\beta}$ exceeds $-\ln\tanh 4\beta$ from $\beta > 0.523$. At $\beta = 1$ it is 1.26×10^{-3} versus 6.71×10^{-4}; at $\beta = 2$, 1.59×10^{-6} versus 2.25×10^{-7}.
- **The thresholds are not explicit.** β_0 and $M_0(\beta)$ of the thick-vortex floor are not explicit, so one cannot say "the thick bound is smaller at a concrete (β, M) with $\beta > 0.523$". What can be said is a comparison of rates as $\beta \to \infty$.

N	Thin-vortex rate $2N^2$	Upper bound on S_N (Lean)	Exponent number of the thick-vortex floor (Lean)
1	2	— (Reading $S_1 = 2$)	—
2	8	6.67401	6.675
3	18	9.09321	9.094
4	32	12.02057	12.021

$S_N < 2N^2$ for general N is not in Lean. For $N = 1$ there is no gain.

07

What is not said

- The value of S_N. What we could say is only $0 < S_N$ (with a non-explicit δ) and the upper bounds for $N = 2, 3, 4$.
- The number δ. It would need a quantitative estimate of rigidity (an explicit relation between action and distance from the vacuum).
- Lower or upper bounds uniform in N. Changing N changes the configuration space of the fibre, and this argument is for each N separately.
- The reverse direction of the thick-vortex floor (a lower bound on θ, an upper bound on the rate). There is no theorem in the reverse direction.
- The band (the statement that the vortex ratio θ becomes small exponentially in the area of the box, that is, that $-\ln\theta$ (corresponding to the slab tension times the area) grows in proportion to the area, stated with a single constant while moving the transverse and longitudinal sizes and the coupling together; 19-07). The quantities of this chapter are in a form with the lattice spacing and the width N held fixed. The distance is examined in 21-08.
- The lattice spacing is fixed; none of the continuum limit, the four-dimensional area law or the mass gap is said.

What this chapter can say

Can say	Cannot say
Lean `TwistBlocks.Svortex_le`, `TwistBlocks.SN_bounds`: $0 \le S_{\mathrm{v}}(N,m) \le 2N^2$, $0 \le S_N \le 2N^2$ (the configuration with the reversal put into the vacuum)	The value of S_N
Lean `VortexPos.Svortex_ge_unif`, `SN_pos`: for every $N \ge 1$ there is $\delta > 0$ uniform in m with $S_N \ge \delta > 0$	The number δ (it comes from compactness). A lower bound uniform in N
Computer-assisted $S_{\mathrm{v}}(2,8) \le 6.67401$, $S_{\mathrm{v}}(3,8) \le 9.09321$, $S_{\mathrm{v}}(4,4) \le 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	The value of the infimum over m. $S_N < 2N^2$ for general N
Lean `TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`: for each sufficiently large β, for sufficiently large M, floors with exponent numbers 6.675, 9.094, 12.021	The values of β_0 and $M_0(\beta)$
Reading For $N = 2$ the coefficient of the thick-vortex floor exceeds the thin one from $\beta > 0.523$	"The thick-vortex floor is stronger" itself is not a Lean theorem (a comparison of rates)

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (bibliography as in 25-01)
The action of fractional instantons and the dilute-gas picture	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (to the extent checked in 25-02)
The route to positivity (steps 1–6)	Proved here	§ 03 of the text. Lean: `VortexPos.Svortex_ge_unif`, `VortexPos.SN_pos`; core lemma `TwistBlocks.grid_vortex_floor`; rigidity of the fibre `TwistBlocks.rigidT_all`
The certificates for the upper bounds	Computer-assisted	Rational configurations in Cayley form ($s = 2^{16}$), integer sums rounded up plaquette by plaquette in units of 2^{-32}, evaluation in the Lean kernel (`decide +kernel`). Soundness `VortexCert.Svortex_le_of_cert`. `VortexCert.certN2`, `certN3`, `certN4`
Values of the action, $S_1 = 2$, $\beta > 0.523$	Reading	The numerical record of this book's study (a script written out from the Lean definitions, computing the action exactly in fractions without rounding; also checked that lowering the last digit of an upper bound by one makes it fail. The crossing β by floating-point root finding)

Item	Kind	Source / tool
Corollaries of the thick-vortex floor	—	Lean: `TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`; the numerical inequality `TwistBlocks.thick_vortex_floor_two_rate_lt`. The main theorem is in 21-03

Words used in this chapter — minimal action of a vortex $S_{\mathrm{v}}(N, m)$, S_N, twisted vacuum, fibre, link defect, label, telescoping product, Cayley form, rounding up, certificate, comparison of rates. Their meanings are in A-02 Glossary.

Next chapter to read: 21-06 The sign of θ and pairs of Polyakov loops — that the vortex ratio θ is never negative, and that θ bounds correlations of Polyakov loops.

The sign of θ and pairs of Polyakov loops

> This chapter is this book's own investigation.

What this chapter shows — that on a torus of width N and length M, if M is even then $Z_1 \le Z_0$, that is, $\theta_M \ge 0$, for every real β. That with the same reflection tools, the correlation of two Polyakov loops winding around direction 3 is bounded by $2\sqrt{\theta}/(1+\theta)$ at distance $M/2$, and by $2\tanh(N^2|\beta|)^{M\,\min(T,M-T)}$ at a general time separation T. Each is a formalisation, machine-checked in Lean, of an estimate of known type.

Prerequisite chapters — 19-02 the twist, the vortex ratio, Polyakov loops; 19-03 reflection; 21-02 The thin-vortex floor; 21-03 The thick-vortex floor

Said first — the expectation $\langle\cdot\rangle_0$ is with respect to the Wilson measure on a finite lattice with the background twist $n_{12} = 1$ and without the (3,4) twist. What can be said is a linear **lower bound** on the energy inside a transverse $N \times N$ tube, not uniform in N. The lattice spacing is fixed. Nothing is said about four-dimensional confinement, the area law or the mass gap. The case of odd M is not included.

The order of this chapter

1. The expectation $\langle\cdot\rangle_0$ and Polyakov loops
2. Splitting the lattice into two halves by a reflection in the time direction
3. θ ≥ 0 — statement and proof
4. Pairs of Polyakov loops at distance M/2, and θ
5. General time separation — repeating the reflection
6. Detour: the range in the parities of M, N and the sign of β
7. How to read it — a linear lower bound on the energy inside a tube
8. What is not said

01

The expectation $\langle\cdot\rangle_0$ and Polyakov loops

The lattice is the torus with transverse part $(\mathbb{Z}/N)^2$ (directions 1 and 2) and longitudinal part $(\mathbb{Z}/M)^2$ (directions 3 and 4). The definitions are in 19-02. Under the background twist $n_{12}=1$, let Z_0 and Z_1 be the partition functions without and with the (3,4) twist, and write the vortex ratio as $\theta_M=(Z_0-Z_1)/(Z_0+Z_1)$ (19-02).

The expectation in this chapter is

$$\langle F\rangle_0=\frac{1}{Z_0}\int F(U)\,w_0(U)\,dU,\qquad w_0(U)=\exp\Big(\beta\sum_p z_p^{(0)}\,\mathrm{Re}\,U_p\Big)\quad (1)$$

dU is the product of Haar measures, and $z_p^{(0)}$ is the sign of the background twist $n_{12}=1$ only. **The (3,4) twist is not included.** In Lean the weight is `wE N M β (fun _ => 1)` and the divisor is `ZW N M β true false`.

A Polyakov loop is the real part of the product of the links going once around direction 3, at transverse position $b=(x_1,x_2)$ and direction-4 position t (19-02).

$$P_b(t)=\mathrm{Re}\prod_{k=0}^{M-1}U_3\big((b),(k,t)\big)\quad (2)$$

The real part of a quaternion is $\frac{1}{2}\,\mathrm{tr}$ in the fundamental representation, and $|P_b(t)|\le 1$. In Lean it is `HalfSplit.polyLoop`, and the expectation of a pair is `PolyakovT.polyExp`.

Proved here The expectation of a single loop $\langle P_b(t)\rangle_0$ is 0. Multiplying all direction-3 links at $x_3=0$ by -1, every plaquette contains 0 or 2 of the reversed links, so w_0 is unchanged, while $P_b(t)$ alone changes sign. The meaningful quantity is the pair correlation $\langle P_b(0)P_{b'}(T)\rangle_0$.

02

Splitting the lattice into two halves by a reflection in the time direction

When M is even, the reflection $\Theta:\ x_4\mapsto -x_4$ fixes two planes of lattice points, $x_4=0$ and $x_4=M/2$. Links in directions 1, 2 and 3 only have their positions mapped; links in direction 4 have their orientation reversed and their values inverted (the general form of reflection is in 19-03).

Since the planes pass through lattice points, no square crosses them. The squares on the two planes (those not containing direction 4) are shared half and half between the upper and lower halves. The **half weight** built in this way,

$$a_\eta(U) = \exp\Big(\beta \sum_p c_p\, z_p\, \eta_p\, \mathrm{Re}\, U_p\Big), \qquad c_p \in \{1, \tfrac{1}{2}, 0\} \tag{3}$$

(1 in the upper half, $\frac{1}{2}$ on the planes, 0 below), depends only on the links of the closed upper half. The whole weight then splits as

$$w_\eta(U) = a_{\eta 1}(U)\, a_{\eta 2}(\Theta U) \tag{4}$$

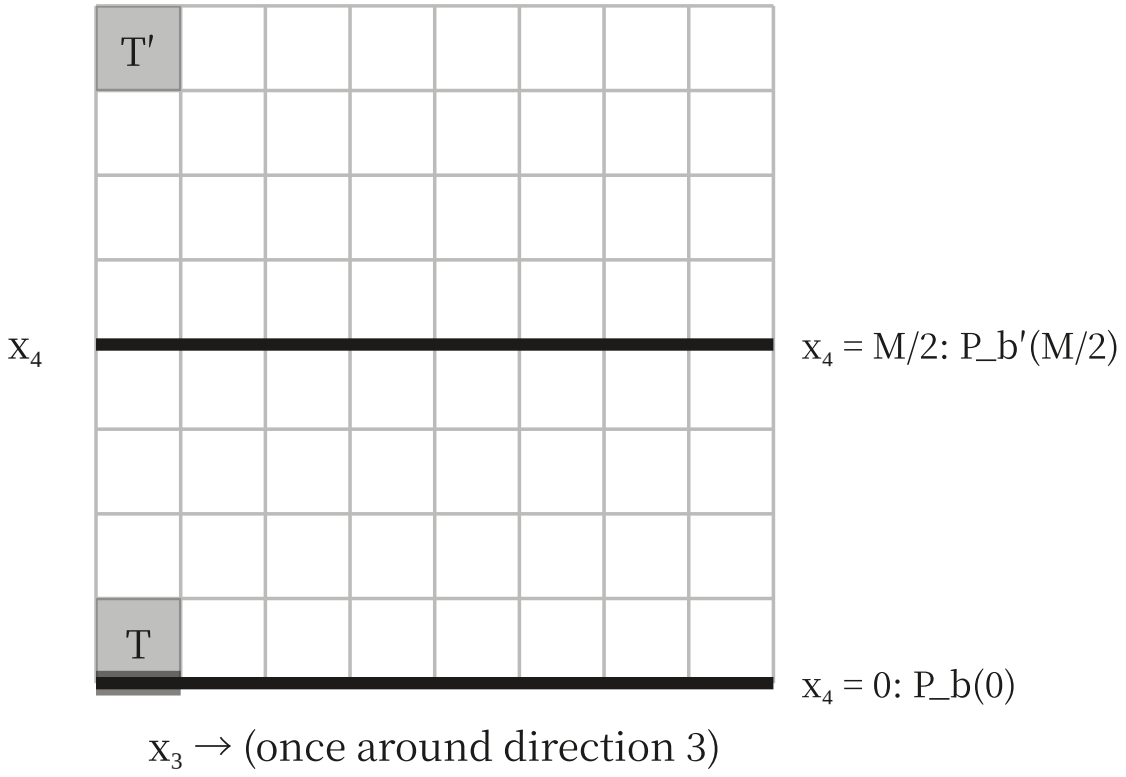
Here η gives the signs of the (3,4) plaquettes as $\eta 1$ in the upper half and as the mirror image of $\eta 2$ in the lower half (Lean: `HalfSplit.half_split`).

The product of Haar measures is invariant under the reflection, and for bounded measurable g depending only on the closed upper half,

$$\int g(\Theta U)\, g(U)\, dU \;\geq\; 0 \tag{5}$$

holds (reflection positivity; Lean: `HalfSplit.half_pairing_nonneg`). By (4) the weight is absorbed into g, so no condition on the sign of β is needed. Cited This form is of the type of Osterwalder–Seiler reflection positivity, and `HalfSplit.half_pairing_nonneg` is its formalisation on the twisted lattice (19-03).

Figure 1 The longitudinal (3,4) plane (example $M = 8$; the transverse T^2_N omitted). The thick horizontal lines are the reflection planes $x_4 = 0$ and $x_4 = M/2$. Two Polyakov loops lie on them. The shaded squares are the reversal T ($(0,0)$) and its mirror image T' ($(0,-1) = (0, M-1)$). The marked link is the sign reversal c of § 04



The figure only shows the two planes and the positions of the reversals used in § 03 and § 04; it carries no numbers.

03

θ ≥ 0 — statement and proof

The sign of the vortex ratio. If M is even, then for every real β and all $N \geq 1$,

$$Z_1 \leq Z_0, \qquad \text{equivalently} \qquad 0 \leq \theta_M(\beta). \tag{6}$$

Lean: `HalfSplit.Z1_le_Z0` (M even, every real β), `TwistBlocks.theta_nonneg` ($0 \leq \theta_M$ in the same range).

Proof. Let T be the reversal of the sign of the corner (3,4) plaquette $(x_3, x_4) = (0, 0)$ at all transverse positions, and put $g = a_1 - a_T$ into (5). Expanding with (4),

$$0 \leq \int g(\Theta U) g(U)\, dU = Z[1] - Z[T] - Z[T'] + Z[TT'] \tag{7}$$

T' is the reversal at $(0, -1)$, the mirror position of T, and TT' is the two reversals together. Proved here The partition function does not depend on the positions of the reversals; it is determined only by the product of the reversals $\prod \eta$ (positions of reversals can be moved by a change of variables that changes the signs of the links on

a cut; Lean: `TwistedWilson.ZEta_flip_any`, `TwistedWilson.ZEta_parity`). Hence $Z[T] = Z[T'] = Z_1$ and $Z[TT'] = Z_0$, and (7) is $0 \le 2Z_0 - 2Z_1$. □

This estimate removes the absolute value from the thick-vortex floor. Lean: `TwistBlocks.thick_vortex_floor_signed` (all $N \ge 1$) says that for any $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, if $M+1$ is even,

$$0 \le \theta_{M+1}(\beta) \le \exp\bigl(-(M+1)^2 e^{-\beta(SN+\varepsilon)}\bigr) \quad (8)$$

The thresholds in β and M are not explicit.

Odd M is not included, because the second plane $x_4 = M/2$ does not fall on lattice points. Moreover, for odd M and $\beta < 0$, $\theta \ge 0$ can fail. Reading At $N = 1$, $M = 1$, $\beta = -1.5$ we found $\theta = -0.055 \pm 0.005$ (SU(2) Monte Carlo, the numerical record of this book's study). For $\beta \ge 0$ it is expected to hold regardless of parity, by positivity of the transfer matrix, but this book has not checked it. Proved here Note that $\theta_M < 1$ follows for all M from $Z_1 > 0$ (Lean: `TwistBlocks.theta_lt_one`).

04

Pairs of Polyakov loops at distance M/2, and θ

A bound on pairs of Polyakov loops by θ. If M is even, then for every real β and all transverse positions b, b',

$$\bigl|\langle P_b(0)\, P_{b'}(M/2)\rangle_0\bigr| \;\le\; \frac{2\sqrt{\theta_M}}{1+\theta_M}. \quad (9)$$

Lean: `HalfSplit.polyakov_pair_theta` (M even, every real β). The left side is the expectation (1), with divisor Z_0.

θ is the quantity determined from the flux free energy $\Delta F = \ln(Z_0/Z_1)$ by $\theta = \tanh(\Delta F/2)$ (19-02; 26-03 of the present book also calls ΔF the "vortex free energy"), and (9) says that it is **a quantity that bounds the correlation of loops at long distance from above**. Cited Estimates that bound long loops by ratios of twisted partition functions belong to the line of Tomboulis–Yaffe's argument on the flux free energy, and `HalfSplit.polyakov_pair_theta` is a formalisation, in that line, of the form for pairs of Polyakov loops at $T = M/2$ (the inequality in the original paper is about Wilson loops; the text not checked).

The route.

1. **The central sign reversal** c**.** Multiply the direction-3 links at $(x_3, x_4) = (0,0)$ (N^2 transverse links) by -1. The product of Haar measures is unchanged. The (1,3) and

(2,3) plaquettes contain two reversed links and are unchanged; among the (3,4) plaquettes only the two, T and T', change sign (the marked link in Figure 1). Hence $w_0(cU) = wTT'(U)$.

2. **Parity of the loops.** In $P_b(0)$ only the first link of the loop is reversed, so it changes sign; $P_{b'}(M/2)$ is unchanged. The product $\psi = P_b(0)P_{b'}(M/2)$ satisfies $\psi(cU) = -\psi(U)$, so $\int \psi w_0 = \frac{1}{2}\int \psi\,(w_0 - wTT')$.
3. **Write as sum and difference.** With $a_\pm = a_1 \pm a_T$, (4) gives $w_0 - wTT' = \frac{1}{2}\big[(a_+ \circ \Theta)a_- + (a_- \circ \Theta)a_+\big]$.
4. **Cauchy–Schwarz.** ψ depends only on the links on the two planes (Figure 1), is unchanged by Θ, and $|\psi| \le 1$. Using Cauchy–Schwarz for the reflection pairing with ψ inserted, and putting in $\langle a_\pm, a_\pm\rangle = 2(Z_0 \pm Z_1)$, gives $|\int \psi w_0| \le \sqrt{Z_0^2 - Z_1^2}$ (Lean: `HalfSplit.odd_fix_bound`).
5. **Divide.** $\sqrt{Z_0^2 - Z_1^2}/Z_0 = 2\sqrt{\theta}/(1+\theta)$ Proved here . □

Reading A guide to its strength: in the sum over all configurations for the $\mathbb{Z}_2$ subgroup, $N = 1$, $M = 2$, (9) is an equality. In SU(2) samples the right side was 3–6 times the left side (the numerical record of this book's study).

Putting a floor into (9) gives a floor for loop correlations. From $2\sqrt{\theta}/(1+\theta) \le 2\sqrt{\theta}$ and the thin-vortex floor $\theta \le \tanh(N^2|\beta|)^{M^2}$, $|\langle P_b(0)P_{b'}(M/2)\rangle_0| \le 2\tanh(N^2|\beta|)^{M^2/2}$ (M even, every real β; Lean: `PolyakovT.half_pair_tanh`). From the signed floor (8), Lean: `HalfSplit.polyakov_pair_floor` (all $N \ge 1$) gives, for any $\varepsilon > 0$, for each sufficiently large β, for sufficiently large M, if $M+1$ is even,

$$\left|\langle P_b(0)\,P_{b'}\left(\tfrac{M+1}{2}\right)\rangle_0\right| \;\le\; 2\exp\!\Big(-\tfrac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)}\Big) \quad (10)$$

(the lattice has length $M+1$). The thresholds are not explicit, and the coefficient $e^{-\beta(S_N+\varepsilon)}$ becomes exponentially small as $\beta \to \infty$.

05 General time separation — repeating the reflection

Pairs of Polyakov loops at general time separation. If $M \equiv 0 \pmod 4$, then for every real β, all $0 \le T \le M$ and all b, b',

$$\left|\langle P_b(0)\,P_{b'}(T)\rangle_0\right| \;\le\; 2\,\tanh(N^2|\beta|)^{\,M\cdot\min(T,\,M-T)}. \quad (11)$$

Lean: `PolyakovT.polyakov_pair_all` ($M \equiv 0 \bmod 4$, every real β).

Cited Decay of correlations inside a tube of finite cross-section is an estimate of known type along the lines of Mack–Petkova and Tomboulis–Yaffe, and `PolyakovT.polyakov_pair_all` is its formalisation.

The case $T = M/2$ follows from § 04. For general T, we move the reflection plane and repeat Cauchy–Schwarz, reducing to convexity of the logarithm of the correlation. Write $G_{bb'}(T) = \int P_b(0) P_{b'}(T)\, w_0\, dU$ (unnormalised).

Step 1 Translation. The direction-4 translation $x_4 \mapsto x_4 + 1$ preserves the product of Haar measures and w_0, so $\int P_b(s) P_{b'}(t)\, w_0$ depends only on $t - s$ (Lean: `PolyakovT.corr_shiftZ`). So the reflection plane can be placed at any time. Also $G_{bb'}(T) = G_{b'b}(M - T)$ (`PolyakovT.Gc_swap`).

Step 2 The weighted reflection pairing. For bounded F, G depending only on the closed upper half, $\langle F, G\rangle_w = \int F(\Theta U) G(U)\, w_0\, dU$ is positive semidefinite because $w_0 = a_1 \cdot (a_1 \circ \Theta)$ by (4), and Cauchy–Schwarz holds (every real β; Lean: `PolyakovT.wpair_nonneg`, `PolyakovT.wpair_cs`).

Step 3 Midpoint convexity for even separations. Taking $F = P_b(u)$, $G = P_{b'}(v)$ and using $P_b(u) \circ \Theta = P_b(-u)$ and translation, for $2u, 2v \le M$,

$$G_{bb'}(u+v)^2 \;\le\; G_{bb}(2u)\, G_{b'b'}(2v), \qquad G_{bb}(2u) \ge 0 \quad (12)$$

(Lean: `PolyakovT.cs_even`; the non-negative side is `PolyakovT.Gc_even_nonneg`).

Step 4 The chord inequality. Proved here If $g \ge 0$ satisfies $g(k)^2 \le g(k-1) g(k+1)$ $(1 \le k \le m-1)$, then $g(k)^m \le g(0)^{m-k} g(m)^k$ (Lean: `LogChord.chord`). Applying this with $g(k) = G_{bb}(2k)$ and $m = M/4$, for $T = 2k \le M/2$, $G(T)$ is bounded by a weighted geometric mean of $G(0) \le Z_0$ and $G(M/2)$.

Step 5 Put in the ends. By § 04 and the thin-vortex floor, $G(M/2)$ is at most $2\, t^{M \cdot M/2}$ Z_0 $(t = \tanh(N^2|\beta|))$ (`PolyakovT.half_pair_tanh`). For even $T \le M/2$ with $b = b'$, this gives (11). For $b \neq b'$, use (12) with $u = v = T/2$ to return to $G_{bb'}(T)^2 \le G_{bb}(T) G_{b'b'}(T)$. For odd $T < M/2$, since $M/2$ is even, $T + 1 \le M/2$, and we use (12) with $u = (T-1)/2$, $v = (T+1)/2$. $T > M/2$ is brought back by the symmetry of step 1.

The constant 2 does not depend on β, N, M or T. Using the trace of the transfer matrix and Hölder's inequality, the θ form $(\theta^{T/M} + \theta^{1-T/M})/(1+\theta)$ is to be expected, but it is not in Lean. (11) is weaker than that in only two respects: the constant 2, and the replacement of θ by t^{M^2} from the thin-vortex floor.

Detour: the range in the parities of M, N and the sign of β

You may skip this. With reflections in planes of lattice points alone, odd separations are not covered when $M \equiv 2 \pmod 4$. Adding reflections in planes through the middle of links covers them, but such a plane crosses squares, so $\beta \geq 0$ is needed (Lean: `PolyakovT.link_rp`, M even ≥ 4, $\beta \geq 0$). If N and M are both even, a change of variables by staggered signs sends $\beta \mapsto -\beta$ without changing the expectation of a pair (`PolyakovT.polyExp_neg`).

Lean declaration	M	N	β
`PolyakovT.polyakov_pair_all`	$M \equiv 0 \bmod 4$	Any	All real numbers
`PolyakovT.polyakov_pair_evenM`	Even, $M \geq 4$	Any	$\beta \geq 0$
`PolyakovT.polyakov_pair_evenNM`	Even, $M \geq 4$	Even	All real numbers

All conclusions have the same form as (11). Not covered are the case of odd M, and the case $M \equiv 2 \pmod 4$ with N odd and $\beta < 0$.

06

How to read it — a linear lower bound on the energy inside a tube

Setting $V(T) = -\frac{1}{M} \ln |\langle P(0)P(T)\rangle_0|$, (11) becomes

$$V(T) \;\geq\; \sigma_0 \cdot \min(T, M-T) - \frac{\ln 2}{M}, \qquad \sigma_0 = -\ln \tanh(N^2|\beta|) \quad (13)$$

A pair of Polyakov loops is a pair of static charges when direction 3 is viewed as time, and $V(T)$ corresponds to the energy of separating the pair to distance T. (13) is **a linear lower bound on the energy inside a transverse $N \times N$ tube**.

- **Only a lower bound.** It says neither that V is linear nor anything about an upper bound.
- **Not uniform in N.** $\sigma_0 \approx 2e^{-2N^2|\beta|}$ goes to 0 as $N \to \infty$. The thicker the tube, the weaker the lower bound.
- **The lattice spacing is fixed.** With N held fixed the system is effectively two-dimensional, and Cited in two-dimensional lattice gauge theory an area law at every β is the usual outcome. This is not a claim about four-dimensional confinement, the area law or the mass gap.

A guide to the non-trivial range. $|\langle P(0)P(T)\rangle_0| \leq \langle P^2\rangle_0$ is trivial. Reading $\langle P^2\rangle_0$ is about 0.25–0.3 (1/4 at $\beta = 0$). The smallest exponent $X = M \min(T, M-T)$ needed for the right side of (11) to fall below 1/4 is as follows Reading .

N	β = 0.1	0.3	0.5	1	2
1	1	2	3	8	57
2	3	12	57	3100	$9.2\cdot10^{6}$
3	7	231	8425	$6.8\cdot10^{7}$	$4.48\cdot10^{15}$

Reading Since $X \le M^2/2$ (with equality at $T = M/2$), for example at $N = 2$, $\beta = 1$, $X \ge 3100$ requires $M \ge 78.74$; with $M \equiv 0 \pmod 4$ in fact $M \ge 80$, and even then the right side falls below 1/4 only for T near $M/2$. At large β and large N, (11) is loose, and that side is covered by forms of the thick-vortex floor such as (10).

07

What is not said

- $\theta \ge 0$ for odd M; pairs of Polyakov loops for odd M.
- The Hölder form itself ((11) is weaker by the constant 2 and by replacing θ with t^{M^2}).
- An upper bound on $V(T)$, the linearity of V, the value of the tension.
- Estimates uniform in the width N. The band (the statement that the vortex ratio θ becomes small exponentially in the area of the box, that is, that $-\ln\theta$ (corresponding to the slab tension times the area) grows in proportion to the area, stated with a single constant while moving the transverse and longitudinal sizes and the coupling together; 19-07) is not reached. The estimates of this chapter are in a form with the lattice spacing and the width held fixed (21-08).
- The values of the thresholds in β and M in (8) and (10).
- The continuum limit; four-dimensional confinement, the area law, the mass gap.

What this chapter can say

Can say	Cannot say
Lean `HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`: $0 \le \theta_M$ for even M and every real β	Odd M (Reading there is an example where it fails at $\beta < 0$)
Lean `TwistBlocks.thick_vortex_floor_signed`: at large β and large M ($M+1$ even), $0 \le \theta_{M+1} \le \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds
Lean `HalfSplit.polyakov_pair_theta`: for even M and every real β, $\lvert\langle P_b(0) P_b'(M/2)\rangle_0\rvert \le 2\sqrt{\theta}/(1+\theta)$	Distances other than $M/2$ (those are the next row)

Can say	Cannot say
Lean `PolyakovT.polyakov_pair_all`: for $M \equiv 0 \bmod 4$, every real β, $0 \le T \le M$, $\lvert\langle P_b(0)P_{b'}(T)\rangle_0\rvert \le 2\tanh(N^2\lvert\beta\rvert)^{M\,\min(T,M-T)}$. `_evenM` ($M$ even ≥ 4, $\beta \ge 0$), `_evenNM` (N, M even, every β)	The Hölder form (weaker by the constant 2 and by $\theta \to t^{M^2}$). Odd M. A tension uniform in N
Lean `HalfSplit.polyakov_pair_floor`: a floor at distance $(M+1)/2$, combined with the thick-vortex floor	The values of the thresholds. Four-dimensional confinement
Lean `PolyakovT.half_pair_tanh`: for even M (including $M = 2$) and every real β, $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2t^{M\cdot M/2}$	Other T for $M \equiv 2 \bmod 4$ (`_evenM` if $M \ge 4$)
Reading The non-trivial range (at $N = 2$, $\beta = 1$: $M \ge 80$, with T near $M/2$)	—

Sources and reproduction

Item	Kind	Source / tool
Reflection positivity (planes of lattice points)	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 (bibliography in 19-03). Lean: `HalfSplit.half_split`, `HalfSplit.half_pairing_nonneg`
The line of argument bounding long loops by the flux free energy	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 (the text not checked; the same bibliography as 25-04)
The line of argument on vortices and confinement at finite cross-section	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (the text not checked; the same bibliography as 25-01)
The area law in two-dimensional lattice gauge theory	Cited	A. A. Migdal, Sov. Phys. JETP 42 (1975) 413 (bibliography only)
Twisted boundary conditions and the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 (25-01)
$\theta \ge 0$, pairs at distance M/2 (the routes of § 03 and § 04)	Proved here	The text. Lean: `PolyakovT.half_pair_tanh`, `TwistBlocks.theta_lt_one`, `HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`, `HalfSplit.odd_fix_bound`, `HalfSplit.polyakov_pair_theta`, `TwistBlocks.thick_vortex_floor_signed`, `HalfSplit.polyakov_pair_floor`
General time separation (steps 1–5 of § 05)	Proved here	The text. Lean: `PolyakovT.cs_even`, `PolyakovT.Gc_even_nonneg`, `LogChord.chord`, `PolyakovT.polyakov_pair_all`, `PolyakovT.link_rp`, `PolyakovT.polyakov_pair_evenM`, `PolyakovT.polyExp_neg`, `PolyakovT.polyakov_pair_evenNM`

Item	Kind	Source / tool
The example with $\theta < 0$ for odd M, the strength ratios, the table of the non-trivial range	Reading	The numerical record of this book's study (SU(2) Monte Carlo and weighted averages over uniform samples, sums over all $\mathbb{Z}_2$ configurations, floating-point computation of $2t^X < 1/4$)

Words used in this chapter — expectation $\langle\cdot\rangle_0$, Polyakov loop, vortex ratio θ_M, reflection positivity, half weight, central sign reversal, midpoint convexity for even separations, chord inequality, staggered signs, linear lower bound on the energy inside a tube. Their meanings are in A-02 Glossary.

Next chapter to read: 21-07 The vortex ratio — continuity, limit, matching — for the vortex ratio with the lattice held fixed, its continuity and limits in the coupling, and matching it to a chosen value.

The vortex ratio — continuity, limit, matching

This chapter is this book's study.

What this chapter shows — For the vortex ratio $\theta_{N,M}(\beta) = (Z_0 - Z_1)/(Z_0 + Z_1)$ of the SU(2) Wilson measure on a torus of transverse size N and longitudinal size M, we read five things that are machine-checked in Lean with the lattice held fixed. (1) $\beta \mapsto \theta$ is continuous and $\theta(0) = 0$. (2) A box with two twists has no configuration in which the holonomy of every plaquette equals the sign of the twist (no flat configuration). (3) The deficit (the loss of action) of that box has a positive lower bound, and in a symmetric box it is at least $1/2$, independent of N. (4) As $\beta \to \infty$, $\theta \to 1$, and the slope of the flux free energy converges to the minimal deficit. (5) For every κ_0 with $0 \le \kappa_0 < 1$ there is a $\beta \ge 0$ with $\theta(\beta) = \kappa_0$, and among the solutions there is a largest one (matching).

Prerequisite chapters — 19-02 the torus of transverse size N and longitudinal size M, the sign of the twist, the vortex ratio, the twisted vacuum; 21-02 The thin-vortex floor. The introduction is in 25-01.

Said first — The lattice spacing is fixed at 1, and nothing is said about the continuum limit. Every theorem is a statement **with the lattice sizes N, M held fixed**. The speed of $\theta \to 1$, the constants in the estimates coming from the lower bound on the deficit, and the matching solution β^* depend on N, M, and nothing is said about sequences in which N and β move together (matched sequences). Nor do we say anything about the monotonicity of θ, the uniqueness of the matching solution, the continuity of β^* in κ_0, or that the largest solution for $\kappa_0 = 0$ is 0. Nor do we compare the true value of the minimal deficit with the value in the continuum theory. Each is a formalisation of the lattice form of the known picture of 't Hooft's twisted box Cited .

The order of this chapter

1. Setting — the vortex ratio, the flux free energy, the deficit — the background with two twists, and its minimal deficit
2. Statement — continuity and the value at β = 0 — θ(0) = 0
3. Statement — a twisted box has no flat configuration — with only one twist there is one

4. Statement — a lower bound on the deficit — $2N^2M^2/(N^2+M^2)^2$, and $1/2$ in a symmetric box
5. Statement — $\theta \to 1$ as $\beta \to \infty$ — the form of the lower bound, and the slope of the free energy
6. Statement — the largest matching solution — every $\kappa_0 \in [0, 1)$, the lower envelope from the right
7. Route of the proof — discrete Stokes, anticommutation of quaternions, positivity of Haar measure on open sets, intermediate values
8. As questions — how things map when the lattice is doubled — what is not a theorem
9. What is said, and what is not

Labels — Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book (Reading is a numerical reading, Lean a Lean theorem of this book).

01

Setting — the vortex ratio, the flux free energy, the deficit

The lattice is the torus $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($N, M \geq 1$) of transverse size N (directions 1, 2) and longitudinal size M (directions 3, 4); the weight is $e^{\beta \sum_p z_p \operatorname{Re} U_p}$, and the reference measure is the product of normalised Haar measures (19-02). With the background twist $n_{12} = 1$, write the partition functions without and with the longitudinal twist n_{34} as

$$Z_0 = Z_{N,M}(\beta; 1, 0), \qquad Z_1 = Z_{N,M}(\beta; 1, 1) \quad (1)$$

and write the vortex ratio and the flux free energy as

$$\theta = \theta_{N,M}(\beta) = \frac{Z_0 - Z_1}{Z_0 + Z_1}, \qquad \Delta F = \ln \frac{Z_0}{Z_1} = \ln \frac{1+\theta}{1-\theta} \quad (2)$$

(19-02; the equality on the right is Lean `ThetaLimit.phi_theta_eq`). Lean: `TwistBlocks.thetaM N M β`; Z_0 and Z_1 are `ZW N M β true false` and `ZW N M β true true`. This chapter deals mainly with $\beta \geq 0$.

The deficit. In the background with two twists $(n_{12}, n_{34}) = (1, 1)$, define the deficit of a configuration ω as

$$D(\omega) := \sum_p \big(1 - z_p \operatorname{Re} U_p(\omega)\big) \ \geq 0 \quad (3)$$

(z_p is the sign of the twist of this background). It is the loss from the maximum of the action (the number of plaquettes) to the actual action. $D(\omega) = 0$ means $U_p = z_p$ on every plaquette, that is, a flat configuration matching the twist. Write the minimal deficit as

$$D_{\min}(N, M) := \inf_{\omega} D(\omega) \quad (4)$$

Lean: `TwistFlat.defect11`, `ThetaLimit.Smin`.

Caution. $D_{\min}$ is the minimal deficit over the whole twisted box. It is a different quantity from the minimal action of a vortex S_N (19-02, 21-05; the quantity in which the boundary of a block of side m is held at the twisted vacuum).

02

Statement — continuity and the value at β = 0

Statement (continuity of the vortex ratio)

Let $N, M \geq 1$. $\beta \mapsto Z_{N,M}(\beta; n_{12}, n_{34})$ is continuous on the whole real line, and $\beta \mapsto \theta_{N,M}(\beta)$ is continuous too. At $\beta = 0$, $Z = 1$ and $\theta = 0$.

Lean Continuity of a bounded exponential integral. Lean: `ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero` (the only assumption is $N, M \geq 1$).

The proof is by dominated convergence. In the range $|\beta| \leq |\beta_0| + 1$, the integrand is bounded by $\exp((|\beta_0| + 1) \cdot \text{number of plaquettes})$. The denominator $Z_0 + Z_1$ is positive, so θ is continuous too. At $\beta = 0$, the reference measure is a probability measure, so $Z = 1$.

The range of values is $-1 < \theta \leq 1$ (`ThetaLimit.theta_gt_neg_one`, `theta_le_one`), and if the longitudinal size M is even, $\theta \geq 0$ (21-06, `TwistBlocks.theta_nonneg`). That $\theta > 0$ for $\beta > 0$ is not in Lean.

03

Statement — a twisted box has no flat configuration

Statement (a box with two twists has no flat configuration)

Let $N, M \geq 1$ (including the degenerate lattices $N = 1$, $M = 1$). In the background $(n_{12}, n_{34}) = (1, 1)$, there is no SU(2) configuration with $z_p \operatorname{Re} U_p = 1$ on every plaquette.

Lean The lattice form of the known type ('t Hooft): "a box with one twist in directions 1, 2 and one in directions 3, 4 has no flat connection". Lean: `TwistFlat.no_flat_twist11` (no assumptions).

In the background $(1, 0)$ with only one twist, the statement of the same form is false. The twisted vacuum (19-02: the unit quaternion $\mathbf{i}$ on the links of direction 1 at $x_1 = 0$, $\mathbf{j}$ on the links of direction 2 at $x_2 = 0$, and 1 elsewhere) is a configuration with deficit 0 (`ThetaLimit.flat_exists_10`). Because of this contrast, the statement does not hold trivially merely because of how the signs of the twist are placed.

Meaning. In a box with two twists, every configuration contributing to the twisted partition function Z_1 pays a positive loss of action. On the other hand, Z_0 (one twist) has configurations with zero loss. This difference is the source of $Z_1/Z_0 \to 0$, that is $\theta \to 1$, as $\beta \to \infty$ (§ 05).

04

Statement — a lower bound on the deficit

Statement (the minimal deficit is positive and is attained)

Let $N, M \geq 1$. There is a configuration $\omega^\star$ with $0 < D(\omega^\star) \leq D(\omega)$ (for all ω). Hence $D_{\min}(N, M) > 0$.

Lean From compactness and § 03. Lean: `TwistFlat.defect11_floor`, `minDefect11_pos` (no assumptions).

Statement (a uniform lower bound on the deficit)

For all $N, M \geq 1$ and all configurations ω,

$$D(\omega) \;\geq\; \frac{2N^2M^2}{(N^2+M^2)^2}. \quad (5)$$

In particular, in a symmetric box $N = M$, $D(\omega) \geq 1/2$, and $D_{\min}(N, N) \geq 1/2$ holds independently of N.

Lean A coarse uniform version, on the lattice, of the known type of lower bound on the action of a twisted box. Lean: `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half` (no assumptions).

In the background with one twist, there is no positive lower bound of the same form (`TwistFloor.no_floor_10`).

The core of (5) is an inequality for quaternions. For unit quaternions a, b, c, d,

$$\|ab + ba\| + \|cd + dc\| + \|ac - ca\| + \|bc - cb\| \;\geq\; 2 \quad (6)$$

(Lean `QuadFloor.quad_floor_two`). The left-hand side is the sum of the failures of four relations, "a, b anticommute, c, d anticommute, and c commutes with both a and b", and in SU(2) these four are incompatible. The constant 2 is attained, for example at $(a, b, c, d) = (\mathbf{i}, \mathbf{j}, 1, \mathbf{k})$, and cannot be raised.

In numbers Reading . The minimal deficit for $N = M = 1$ is numerically 2.000; for $N = M = 2$ and 4 the deficits of configurations found by cooling are 7.84 and 9.80, and these are upper bounds on the minimum. There is still a gap of a factor of 16 to 20 from the lower bound $1/2$. The lower limit of the action of the corresponding sector in the continuum theory is $\pi^2 \approx 9.87$ in units of $\beta = 4/g^2$, from the self-dual estimate and the fractional topological charge $1/2$ Cited . How $D_{\min}(N, N)$ changes with N is not known.

05

Statement — θ → 1 as β → ∞

Statement (the limit of the vortex ratio)

With $N, M \geq 1$ held fixed, $\theta_{N,M}(\beta) \to 1$ as $\beta \to \infty$. Moreover, for all $\beta \geq 0$,

$$\theta_{N,M}(\beta) \;\geq\; 1 - \frac{2}{v}\, e^{-D_{\min}\beta/2} \quad (7)$$

where $v > 0$ is the Haar volume of a certain open set containing the twisted vacuum in the background with one twist (it does not depend on β, depends on N, M, and has no explicit value).

Lean From the lower bound on the deficit and the positivity of Haar measure on open sets. Lean: `ThetaLimit.tendsto_theta_one`, `theta_lower`, `Z11_le`, `Z10_ge`,

`volV_pos`, `ratio_tendsto_zero`.

Statement (the slope of the flux free energy)

With $N, M \geq 1$ held fixed,

$$\frac{\Delta F}{\beta} = \frac{1}{\beta} \ln \frac{Z_0}{Z_1} \longrightarrow D_{\min}(N, M) \qquad (\beta \to \infty). \tag{8}$$

In a symmetric box, this limit is at least $1/2$, independent of N.

Lean Estimates from both sides using the same parts. Lean: `ThetaLimit.tendsto_logratio_div`, `TwistFloor.logratio_limit_half`.

Meaning. With the lattice held fixed and the coupling made weaker, the sector with two twists loses weight exponentially by the amount of the deficit, and the vortex ratio approaches 1. The slope of that exponent is exactly the minimal deficit. This is $\beta \to \infty$ with the lattice held fixed, not the continuum limit (the direction in which κ_0 is held fixed and the lattice is made finer).

06

Statement — the largest matching solution

Statement (matching the vortex ratio)

Fix $N, M \geq 1$. For every κ_0 with $0 \leq \kappa_0 < 1$ there is a $\beta \geq 0$ with $\theta_{N,M}(\beta) = \kappa_0$, and the set of these solutions has a largest element $\beta^*(\kappa_0; N, M)$ within $\beta \geq 0$. If $\kappa_0 > 0$, then $\beta^* > 0$.

Lean Continuity, the intermediate value theorem and compactness. Lean: `ThetaLimit.tuning_all`, `tuning_all_pos` (the only assumption is $0 \leq \kappa_0 < 1$). An example where the assumptions can be met: for $N = M = 2$, a positive solution for $\kappa_0 = 1/2$ (`ThetaLimit.sanity_half`).

As a weak form there is also the statement that if $0 \leq \kappa_0 \leq \theta(\beta_1)$, there are a solution and a largest solution in $[0, \beta_1]$ (`ThetaTuning.tuning_exists`, `tuning_max_exists`). By § 05 such a β_1 can be taken for every $\kappa_0 < 1$, which gives the full form above (`ThetaLimit.exists_theta_gt`).

The lower envelope from the right. Whether θ is monotone in β is not known. We therefore use $\theta(\beta) := \inf_{\beta' \geq \beta} \theta(\beta')$. θ is increasing; at the largest solution β^*, $\theta(\beta^*) = \kappa_0$; for $\beta > \beta^*$, $\theta(\beta) > \kappa_0$; and β^* is also the largest among the solutions of $\theta = \kappa_0$ (`ThetaLimit.thetaLow`, `thetaLow_mono`, `envelope_max_sol`). Even for a θ that is not monotone, the largest solution is characterised by this monotone function. We do not claim continuity of θ.

What the matching is for. The matching in the statements to be shown (19-07) is a statement about how to fix the bare coupling along a sequence of ever finer lattices. The matching in this chapter is the part, lattice by lattice, of one candidate for it, "on each lattice, match the vortex ratio to a chosen value κ_0": the existence on each lattice. What holds as a sequence (how β^* depends on N) is not said.

07

Route of the proof

No flat configuration (§ 03). In two steps.

1. **Discrete Stokes.** Suppose quaternion fields A, B on an additive group G satisfy at each point the exchange relation $A(x)B(x+e) = s(x)\, B(x)A(x+f)$ ($s(x)$ a sign), and $Le = 0$, $L'f = 0$. Then the product along a line going once around in direction e and the product along a line going once around in direction f commute up to the product of all the s on the cut (Lean `FlatStokes.torus_commutator`). This is shown by a two-stage induction, first column by column, then sheet by sheet. For a configuration with deficit 0, $U_p = z_p$ on each plaquette is precisely this exchange relation.
2. **A contradiction in the quaternions.** For the four loops W_1, W_2 (transverse) and W_3, W_4 (longitudinal) through the origin, the product of the signs on the cut is -1 for the cuts in directions 1, 2 and in directions 3, 4, and $+1$ for the cuts in directions 1, 3 and 2, 3. Hence $W_1W_2 = -W_2W_1$, $W_3W_4 = -W_4W_3$, and W_3 commutes with both W_1 and W_2. Only real numbers commute with an anticommuting pair (W_1, W_2) of unit quaternions, so $W_3 = \pm 1$, which contradicts $W_3W_4 = -W_4W_3$ (`FlatStokes.no_twisted_quadruple`). Only four pairs of directions are used.

The lower bound on the deficit (§ 04). Extend (6) to a quantitative form. If the fields are unit-valued and the failure of the exchange relation is at most $d(x)$ at each point, then the failure of the loops to commute is at most the sum of d over the cut (`FlatStokes.approx_torus_commutator`; each sheet peeled off adds one error). The failure on a plaquette is $\|U_p - z_p\| = \sqrt{2(1 - z_p \operatorname{Re} U_p)}$. Applying (6) to the four loops at each base point, summing over the base points, recounting by translation and using

Cauchy–Schwarz twice gives (5). When $N \neq M$, the weights of the cuts in directions 1, 2 (N^2 sheets) and in directions 3, 4 (M^2 sheets) do not balance, and the bound becomes $2/(\rho + 1/\rho)^2$ in the ratio $\rho = N/M$. We take its approach to 0 as $\rho \to 0$ or $\rho \to \infty$ to be a weakness of the method Assumption . The only grounds are small examples: on the degenerate lattice $N = 1$, the minimal deficit for $M = 1, 2, 3$ stays numerically at 2.000 and does not drop Reading .

$\theta \to 1$ (§ 05). For $\beta \geq 0$, action = number of plaquettes $-D$, so $Z_1 \leq e^{\beta(|P| - D_{\min})}$ ($|P|$ is the number of plaquettes). On the other hand, the action in the background with one twist is continuous and takes its maximum $|P|$ at the twisted vacuum, so the set of configurations whose action exceeds $|P| - D_{\min}/2$ is an open set containing the twisted vacuum. The product of Haar measures puts positive weight on non-empty open sets, so its volume v is positive, and $Z_0 \geq v\, e^{\beta(|P| - D_{\min}/2)}$. Taking the ratio, $Z_1/Z_0 \leq v^{-1} e^{-D_{\min}\beta/2}$, and $\theta = 1 - 2Z_1/(Z_0 + Z_1) \geq 1 - 2Z_1/Z_0$ gives (7). (8) uses the same parts from above and below, with an open set around the minimiser. No precise Laplace-type estimate is needed.

Matching (§ 06). Since $\theta(0) = 0$, θ is continuous and $\theta \to 1$, the intermediate value theorem gives a solution. The set of solutions is closed, and there are no solutions beyond some b_0 ($\theta > \kappa_0$), so as a bounded closed set it has a largest element.

08

As questions — how things map when the lattice is doubled

What follows is not a theorem; these are questions lying beyond the theorems of this chapter.

- **How doubling maps.** With the coupling matched to $\theta = \kappa_0$ in a symmetric box of side N (the largest solution $\beta^*(\kappa_0; N)$), measuring the vortex ratio of a box of side $2N$ determines a value $\Sigma(\kappa_0; N) := \theta_{2N}(\beta^*(\kappa_0; N))$. Whether $\Sigma(\cdot; N)$ approaches a single function as $N \to \infty$, and if so how that function behaves between 0 and 1, remains a question. For two-dimensional SU(2), the corresponding function can be written by a formula on the continuum side, and it behaves like κ_0^4 for small κ_0 Reading (a formula using Bessel coefficients, and numbers; no proof is given). The Monte Carlo numbers on small four-dimensional lattices have large errors and are not at a stage where anything can be said.
- **The limit of the fixed-lattice direction.** What § 05 gives is only that the slope of ΔF as $\beta \to \infty$ with the lattice held fixed is $D_{\min}(N, N)$, and that this is at least $1/2$. If $D_{\min}(N, N)$ and $D_{\min}(2N, 2N)$ differ, no statement bounding $\Delta F_N - \Delta F_{2N}$ uniformly can be made in the fixed-lattice form. Even if they are equal, a

comparison of the terms slower than linear in β remains. The cooling numbers are all upper bounds, so whether they differ is not known.

- **Relation to the band.** The band statement (19-07; the correct form is in 26-01) says that the vortex ratio becomes exponentially small in the longitudinal area, uniformly in the lattice spacing and the size of the box. The theorems of this chapter are all statements with the lattice held fixed, and do not reach the band statement.

09

What is said, and what is not

- **Statements with the lattice held fixed.** v, $D_{\min}$, β^* and the speed of convergence in (7) depend on N, M. What happens to θ along sequences in which N and β move together is not said.
- **The shape of θ.** Monotonicity, $\theta > 0$ for $\beta > 0$, uniqueness of the matching solution, continuity of β^* in κ_0, and that the largest solution for $\kappa_0 = 0$ is 0, are not said.
- **The value of the deficit.** The true value of $D_{\min}$, its comparison with π^2, and its dependence on N are not said. The lower bound $1/2$ is a coarse uniform version. For $N \neq M$, the lower bound approaches 0 when the aspect ratio becomes lopsided.
- **Reaches neither the band nor the mass gap.** The lattice spacing is fixed, and the continuum limit is not treated.
- **Not a new result.** That a twisted box has no flat connection, the lower bound on its action, and the behaviour of the vortex free energy at weak coupling are a known picture; this chapter is its formalisation on the lattice.

The scope of what this chapter says

Said	Not said
Lean Continuity and $\theta(0) = 0$, $-1 < \theta \leq 1$. Lean `ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero`; `ThetaLimit.theta_gt_neg_one`, `theta_le_one`	Monotonicity; $\theta > 0$ for $\beta > 0$
Lean The background $(1, 1)$ has no flat configuration (all $N, M \geq 1$, no assumptions). Lean `TwistFlat.no_flat_twist11`. Contrast: `ThetaLimit.flat_exists_10`	—

Said	Not said
Lean The lower bound $2N^2M^2/(N^2+M^2)^2$ on the deficit, and $1/2$ in a symmetric box. The minimum is positive and attained. Lean `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half`; `TwistFlat.defect11_floor`, `minDefect11_pos`; parts `QuadFloor.quad_floor_two`, `FlatStokes.approx_torus_commutator`	The true value of $D_{\min}$ and its dependence on N; comparison with π^2
Lean $\theta \to 1$, the lower bound (7), $\Delta F/\beta \to D_{\min}$ (limit $\geq 1/2$ in a symmetric box). Lean `ThetaLimit.tendsto_theta_one`, `theta_lower`, `tendsto_logratio_div`; `TwistFloor.logratio_limit_half`	That the speed is uniform in N; uniform estimates at finite β
Lean A largest solution β^* for every $\kappa_0 \in [0,1)$; $\beta^* > 0$ for $\kappa_0 > 0$; a characterisation by the lower envelope from the right. Lean `ThetaLimit.tuning_all`, `tuning_all_pos`, `envelope_max_sol`	Uniqueness; continuity in κ_0; dependence of β^* on N; properties of matched sequences
Reading Numerical values of the minimal deficit (2.000; upper bounds 7.84 and 9.80), and the behaviour of the doubling function in two dimensions	Not proofs of upper or lower bounds. The doubling function in four dimensions

Sources and reproduction

Item	Kind	Source / tool
Twisted boundary conditions; flat connections and the lower bound on the action in a twisted box	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliography only; the text not checked). The introduction is in 25-01
Continuity in § 02, and the weak form of matching	Lean	`ThetaTuning.ZW_continuous`, `thetaM_continuous`, `ZW_zero`, `thetaM_zero`, `tuning_exists`, `tuning_max_exists`
The statements and parts of § 03–§ 06	Lean	`TwistFlat.no_flat_twist11`, `defect11`, `defect11_floor`, `minDefect11_pos`; `FlatStokes.torus_commutator`, `no_twisted_quadruple`, `approx_torus_commutator`; `QuadFloor.quad_floor_two`; `TwistFloor.defect11_ge_two`, `defect11_sym_half`, `defect11_uniform_half`, `Smin_sym_half`, `logratio_limit_half`, `no_floor_10`; `ThetaLimit.Smin`, `Z11_le`, `Z10_ge`, `volV_pos`, `theta_lower`, `tendsto_theta_one`, `exists_theta_gt`, `tuning_all`, `tuning_all_pos`, `sanity_half`, `ratio_tendsto_zero`, `flat_exists_10`, `phi_theta_eq`, `tendsto_logratio_div`, `thetaLow`, `thetaLow_mono`, `envelope_max_sol`. sorry 0, the three standard axioms, confirmed by re-checking with the kernel, statements checked against the wording of the source
$\theta \geq 0$ (M even)	Lean	`TwistBlocks.theta_nonneg` (21-06)

Item	Kind	Source / tool
Numerical values of the minimal deficit, upper bounds from cooling, the doubling function in two dimensions	Reading	The numerical record of this book's study (quaternion minimisation, cooling, numerical values of the formula with Bessel coefficients; the cooling values are upper bounds)

Terms used in this chapter — vortex ratio, flux free energy, 't Hooft twist, deficit, flat configuration, twisted vacuum, discrete Stokes, anticommutation, positivity of Haar measure on open sets, matching, largest solution, lower envelope from the right. Their meanings are in A-02 Glossary.

Next chapter to read: 21-08 What this part says, and what it does not — the theorems of Part 21 in one table. The vortex floor in the strong-coupling window, with a rate independent of the transverse size (the prefactor N^2 remains), is in 20-06.

What this part says, and what it does not

> This chapter is this book's own investigation. The table of § 01 is a list of the theorems of Part 21.

What this chapter shows — the theorems of Part 21 in a single table, with their scope, what they do not say and their Lean declarations. After that, a short look at how far the floors of this part are from the band.

Prerequisite chapters — 21-00 Entrance and each chapter of Part 21. The definitions of the objects are in 19-02

Said first — apart from the first, abstract implication, every theorem of this part is an equality or inequality at a fixed lattice spacing on a finite lattice of transverse size N and longitudinal size M. Inside the strong-coupling window there is a form whose rate does not depend on the transverse size (the prefactor, the transverse area, remains; 20-06). Outside the window, none of the rates of the floors of this part (upper bounds on θ) is uniform in the transverse size N. The lower bound on the deficit in 21-07 (1/2 in a symmetric box, independent of the transverse size) is a statement in the opposite direction from a floor (on the side of lower bounds on θ). Nothing is said about the continuum limit, four-dimensional confinement or the mass gap.

The order of this chapter

1. The list of theorems — scope and what is not said
2. What depends on the width N
3. The distance from the floor to the band

01

The list of theorems — scope and what is not said

The object throughout is the SU(2) Wilson measure on the torus with transverse part $(\mathbb{Z}/N)^2$ and longitudinal part $(\mathbb{Z}/M)^2$, with background twist $n_{12} = 1$ (19-02). We write

$\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ and $t = \tanh(N^2|\beta|)$. Each is a formalisation of a result or picture of known type (the Cited rows of each chapter).

Only the first row is an abstract implication with assumption fields. The other rows are theorems about concrete objects, whose assumption fields are only the ranges written in the table.

Theorem	Scope	What is not said	Lean declarations
Three results on sign systems (the covering inequality, $\mathbb{Z}_2$ monotonicity, comparison with $\mathbb{Z}_2$)	Abstract groups and sign characters. Implications with a column of assumptions	On their own they are not estimates for a concrete measure (the application is inside the next row)	`CoveringInequality.covering_ineq`, `mon_ineq`, `dominated_covering_ineq`, `invariant_dominated`
The thin-vortex floor (ratio form)	All real β, $N, M \geq 1$, any background twist: `twisted_vortex_bound` is the signed form $(Z_0 - Z_1)/(Z_0 + Z_1) \leq t^{M^2}$, and `_abs` the absolute-value form $\lvert Z_0 - Z_1\rvert/(Z_0 + Z_1) \leq t^{M^2}$	The sign ($Z_1 \leq Z_0$ is a separate row). A rate uniform in N	`TwistedWilson.twisted_vortex_bound`, `twisted_vortex_bound_abs`
The thin-vortex floor (exponential form)	$\beta \neq 0$: $\sigma_0 = -\ln t > 0$, $\lvert\theta_M\rvert \leq e^{-\sigma_0 M^2}$	$\sigma_0 \to 0$ ($N \to \infty$)	`TwistedWilson.thin_vortex_floor`
The block estimate	$mk \leq M$, all real β: $\lvert\theta_M\rvert \leq \tanh(K_{\text{eff}})^{k^2}$	When blocks of different shapes are mixed, it is weaker than the product over blocks	`TwistBlocks.block_bound_abs`
The thick-vortex floor	All $N \geq 1$, any $\varepsilon > 0$, for every sufficiently large β and sufficiently large M: $\lvert\theta_{M+1}\rvert \leq \exp(-(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds in β and M. The value of S_N. The opposite direction (a lower bound on the rate)	`TwistBlocks.thick_vortex_floor`

Theorem	Scope	What is not said	Lean declarations
Positivity of the minimal action of a vortex	All $N \geq 1$: $\delta \leq S_{\mathrm{v}}(N, m)$ with a $\delta > 0$ uniform in m, and $S_N > 0$	The number δ (not explicit). A lower bound uniform in N	`VortexPos.Svortex_ge_unif`, `VortexPos.SN_pos`
Certificates for the upper bounds Computer-assisted	$S_{\mathrm{v}}(2,8) \leq 6.67401$, $S_{\mathrm{v}}(3,8) \leq 9.09321$, $S_{\mathrm{v}}(4,4) \leq 12.02057$, hence $S_N < 2N^2$ $(N = 2,3,4)$	The value of S_N (only upper bounds, from a single configuration). General N	`VortexCert.Svortex_two_eight_le`, `Svortex_three_eight_le`, `Svortex_four_four_le`, `SN_two_lt`, `SN_three_lt`, `SN_four_lt` (soundness: `VortexCert.Svortex_le_of_cert`)
The thick-vortex floor with explicit exponents	$N = 2,3,4$, for every sufficiently large β and sufficiently large M: exponent numbers 6.675, 9.094, 12.021	The values of the thresholds. "Stronger than the thin-vortex floor" is a comparison of rates and not a Lean theorem (Reading from $\beta > 0.523$ for $N = 2$)	`TwistBlocks.thick_vortex_floor_two`, `_three`, `_four`
The sign of the vortex ratio	M even, all real β: $Z_1 \leq Z_0$, $\theta_M \geq 0$	M odd (Reading examples where it fails for $\beta < 0$)	`HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg`
The signed thick-vortex floor	In the range of the thick-vortex floor, with $M+1$ even: $0 \leq \theta_{M+1} \leq \exp(-(M+1)2_e{-\beta(S_N+\varepsilon)})$	The values of the thresholds	`TwistBlocks.thick_vortex_floor_signed`
Continuity and matching of the vortex ratio	With the size of the lattice held fixed, for all real couplings: the vortex ratio is continuous in the coupling, is 0 at coupling 0, and approaches 1 as the coupling grows. For every target value at least 0 and less than 1 there is a coupling matched to it (the largest solution)	Monotonicity, uniqueness of the solution, sequences in which the transverse size and the coupling move together	`ThetaTuning.thetaM_continuous`, `ThetaLimit.tendsto_theta_one`, `ThetaLimit.tuning_all`

Theorem	Scope	What is not said	Lean declarations
A lower bound on the deficit of a twisted box	A box with twists in two orthogonal planes has no flat configuration. The deficit of a symmetric box is at least 1/2, independent of the transverse size	The true value of the minimal deficit	`TwistFlat.no_flat_twist11`, `TwistFloor.defect11_sym_half`
Estimate of pairs of Polyakov loops by θ	M even, all real β, all b, b': $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2\sqrt{\theta}/(1+\theta)$	Only the distance $M/2$	`HalfSplit.polyakov_pair_theta`
Pairs of Polyakov loops at a general time separation	$M \equiv 0 \bmod 4$, all real β, $0 \le T \le M$: $\lvert\langle P_b(0)P_{b'}(T)\rangle_0\rvert \le 2t^{\frac{M}{4}\min(T,M-T)}$	A Hölder form (weaker by the constant 2 and by replacing θ with t^{M^2}). M odd. Linearity; an upper bound	`PolyakovT.polyakov_pair_all`
The same (other parities)	M even ≥ 4 with $\beta \ge 0$ / N, M even ($M \ge 4$) with all β	M odd. $M = 2$ (only $T = M/2$, in the next row). $M \equiv 2 \bmod 4$ with N odd and $\beta < 0$	`PolyakovT.polyakov_pair_evenM`, `polyakov_pair_evenNM`
Pairs at distance $M/2$ and the thin-vortex floor	M even (including $M = 2$), all real β: $\lvert\langle P_b(0)P_{b'}(M/2)\rangle_0\rvert \le 2t^{M\cdot M/2}$	Other time separations (the two rows above)	`PolyakovT.half_pair_tanh`
Pairs of Polyakov loops combined with the thick-vortex floor	For every sufficiently large β and sufficiently large M, $M+1$ even: the pair at distance $(M+1)/2$ is $\le 2\exp(-\frac{1}{2}(M+1)^2 e^{-\beta(S_N+\varepsilon)})$	The values of the thresholds. Four-dimensional confinement	`HalfSplit.polyakov_pair_floor`

Common to all rows: the lattice spacing is fixed, and none of the continuum limit, the four-dimensional area law or the mass gap is said. The Polyakov-loop estimates are linear **lower bounds** on the energy inside a transverse $N \times N$ tube.

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02
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What depends on the width N

Every theorem of this part has the form of holding N fixed and moving M and β. We list how each depends on N.

Quantity	Dependence on N	What is known
The rate of the thin-vortex floor $\sigma_0 = -\ln\tanh(N^2\lvert\beta\rvert)$	Proved here $\approx 2e^{-2N^2\lvert\beta\rvert}$, 0 as $N \to \infty$	Explicit
The exponent of the thick-vortex floor S_N	$0 < S_N \leq 2N^2$ (`VortexPos.SN_pos`, `TwistBlocks.SN_bounds`)	Value unknown. Upper bounds only for $N = 2, 3, 4$
The thresholds of the thick-vortex floor $\beta_0(N)$, $M_0(\beta, N)$	Unknown	Not explicit
The positivity lower bound $\delta(N)$	Unknown	Not explicit
The rate of the vortex floor at strong coupling (20-06; a theorem of Part 20)	Independent of it (only inside the strong-coupling window; the transverse area in the prefactor remains)	Explicit. Nothing is said outside the window
The lower bound on the tension inside a tube ((13) of 21-06)	Same as σ_0; goes to 0 as $N \to \infty$	Explicit
The minimal deficit $D_{\min}(N, N)$ of a twisted box (21-07; not a floor, but on the side of lower bounds on θ)	The lower bound $1/2$ does not depend on N. v and β^* depend on N	The lower bound is explicit. The true value is unknown

Outside the strong-coupling window, the explicit quantities of the floors go to 0 with N. For the rate coming from S_N, the dependence on N is unknown, and the thresholds are not explicit either.

03

The distance from the floor to the band

The floors of this part are **upper bounds** on θ, for each width N. They bound the slab tension $\sigma_N(\beta) = \liminf_M -\frac{1}{M^2}\ln|\theta_M|$ (21-03) from below, for each N.

The statement of the band is in 19-07. Assumption For a sequence of bare couplings β_n fixed by matching, there is $c > 0$ such that for every L_0, for all sufficiently large n, $\theta_{2^n L_0, 2^n L_0}(\beta_n) \leq \exp(-c\,L_0^2)$. It is the form that refines the transverse and longitudinal sizes together, moves the coupling as well, and bounds the vortex ratio exponentially in the area with a single constant c.

The floors of this part concern holding the lattice spacing and the transverse size N fixed and moving the longitudinal size M, and do not reach this form. To state the

band, one needs estimates uniform in the transverse and longitudinal sizes and in the coupling. As the table of § 02 shows, every floor theorem of this part is for each N separately and contains no such uniformity (the lower bound on the deficit in 21-07 does not depend on the transverse size, but it is a statement on the side of lower bounds on θ, not a floor). The form in 20-06 inside the strong-coupling window has a rate independent of the transverse size, but since the coupling stays inside the window, it does not reach matched sequences on the side where the coupling grows large.

In this book, this uniform estimate is placed on the side of the premises. The list of premises and the form of the implications are in Part 28.

What this chapter can say

Can say	Cannot say
Each row of the table of § 01 (machine-checked in Lean; the first row is an abstract implication)	The "what is not said" column of each row
Proved here The rate of the thin-vortex floor and the lower bound on the tension inside a tube go to 0 as $N \to \infty$	Estimates uniform in N and in the coupling (the statement of the band, 19-07)

Sources and reproduction

Item	Kind	Source / tool
The theorems of each row	—	The "Sources and reproduction" table of each chapter (21-01–21-07). Lean declarations in the table of § 01
$\sigma_0 \approx 2e^{-2N^2\lvert\beta\rvert}$	Proved here	$\tanh x = 1 - 2e^{-2x} + O(e^{-4x})$ and $-\ln(1-y) = y + O(y^2)$

Words used in this chapter — vortex floor, vortex ratio θ_M, minimal action of a vortex S_N, Polyakov loop, band, estimate uniform in the width. Their meanings are in A-02 Glossary.

Next chapter to read: 22-00 Entrance — the rarity of large plaquettes — on to the theorem on rarity at one scale, on the weak-coupling side.