Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 22 Firm theorems — the rarity of large plaquettes

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Entrance — the rarity of large plaquettes

> This chapter is a guide.

What this chapter shows — The list of theorems of Part 22, the scope that applies to the whole part, and the idea of the tool, the Chessboard Estimate. We also set side by side how the published construction uses the same Chessboard Estimate (as a seminorm measuring the size of observables) and how this part uses it (as an estimate of the probability of rare events).

Prerequisite chapters — 19-01 the Wilson measure, 20-01 Reflection positivity, 26-05 the Chessboard Estimate and simultaneous flips.

Said first — The theorems of this part are statements at one scale, about the Wilson measure itself on a fine lattice at a fixed lattice spacing. They are uniform in the volume (the size of the torus), but say nothing about the measure at a coarse scale after renormalisation. The threshold $\beta \geq 81$ is a value chosen for the convenience of the proof, far from the band of physical interest. Each is a formalisation of a known type, the chessboard Peierls argument from reflection positivity Cited .

The order of this chapter

1. List of the theorems of this part — Table 1
2. The Chessboard Estimate as a tool — uniform patterns and the geometric mean
3. Two uses — measuring size, or measuring rarity — the difference from the published construction
4. The scope that applies to the whole part
5. Order of reading

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book.

01

List of the theorems of this part

Measure the "deviation" of a plaquette p by $1 - \operatorname{Re} U_p \in [0, 2]$, and call a plaquette exceeding a threshold y a **large plaquette** (for $y = 1$, $\operatorname{Re} U_p < 0$). The statement of this part is that the probability of configurations in which many large plaquettes appear at once is exponentially small in the number of plaquettes, with a rate that does not depend on the size of the torus.

Theorem	Scope	Gist of the statement	Lean
The Chessboard Estimate	Side $N = 2m$ ($m \geq 1$), all real β	The expectation of a product of functions placed block by block is at most the geometric mean of the expectations of the uniform patterns	`FoldWilson.wilson_chessboard_even`
Rarity uniform in the volume	$\beta \geq 81$, $m \geq 2$	The probability that $\operatorname{Re} U_p < 0$ on all of a set X of plaquettes is $\leq q_1^{\lfloor \lvert X \rvert / 6 \rfloor}$, with $q_1 < 1$ independent of N and X	`FoldCap.rarity_explicit`
The form with rate $1/4$	$0 < y \leq 2$, at or above some β_0, $m \geq 1$	Probability $\leq q_4^{\lfloor \lvert X \rvert / 24 \rfloor}$	`FoldQuarter.rarity_uniform4`

Table 1. The theorems of Part 22. Statements, proofs and numbers are in 22-01; the scope and what is not said are in 22-02.

02

The Chessboard Estimate as a tool

The introduction is in 26-05 and 20-01. For a reflection-positive measure, the expectation of the product of a function with its mirror image is at least 0, and from this follows an inequality of Cauchy–Schwarz form, the "folding inequality". Repeating

it for every mirror, one divides the torus into blocks, and for the expectation of a product with a function f_w placed in each block,

$$\left|\left\langle \prod_{w} f_w \right\rangle\right| \;\le\; \prod_{w} \left\langle \prod_{w'} f_w^{(w')} \right\rangle^{1/|\text{blocks}|} \tag{1}$$

holds (the Chessboard Estimate of Fröhlich–Israel–Lieb–Simon Cited). The $\prod_{w'} f_w^{(w')}$ on the right is the "uniform pattern" in which the same function f_w is carried by reflections to every block. The point is that any placement on the left is bounded by the geometric mean of the expectations of uniform patterns.

Each factor on the right is a quantity determined by a single function f,

$$\|f\|_{\mathrm{cb}} := \left\langle \prod_{w'} f^{(w')} \right\rangle^{1/|\text{blocks}|} \tag{2}$$

and we call it the **chessboard root**. Since it is a quantity per block, it takes a form independent of the volume even as the volume grows.

03

Two uses — measuring size, or measuring rarity

How the published construction uses it. From the reflection positivity of the Wilson measure, the published construction derives the chessboard inequality (`Cells.Tiling.tiles` , `YM/Cells/Chess.lean:80`) and the triangle inequality for the chessboard root, $\|F+P\|_{\mathrm{cb}} \le \|F\|_{\mathrm{cb}} + \|P\|_{\mathrm{cb}}$ (`Cells.Tiling.adds` , same file `:108`) Source . The root `rootTest` (`YM/Cells/Tiling.lean:226`) is a quantity of the form (2) Source . Because of the triangle inequality, the root can be used like a seminorm on the space of observables Inferred .

With this seminorm, one of the estimates placed at the interface between the ultraviolet side and the infrared side is written. Measured by the chessboard root, renormalised local observables are at most $A \cdot a_m^{-d-\varepsilon}$, uniformly in all volumes (`Extraction.Wcmain` , `YM/Extraction/Center.lean:38`) Source . This becomes the input of local boundedness and equicontinuity for taking an infinite-volume subsequence (12-02, 12-03) Source . This is not the only input received from the ultraviolet side Source .

How this part uses it. This part uses the same Chessboard Estimate to estimate **the probability of rare events**. If the function placed is the indicator of a plaquette, "this plaquette is large", the uniform pattern becomes the event "that plaquette is large at every location". Its weight is exponentially small compared with the partition function, because the action loses heavily (energy), and the partition function is bounded below

by the weight of configurations with every link near the identity (entropy). The $1/|\text{blocks}|$-th power of the ratio becomes the rate per plaquette. This is the chessboard Peierls argument Cited .

	Published construction	This part
Function placed	Renormalised local observables	Indicator that a plaquette is large
What the root measures	The size of observables (of order $a_m^{-d-\varepsilon}$)	The rarity of events (rate per plaquette < 1)
Where it is used	Local boundedness and equicontinuity → subsequential limits	That configurations with many large plaquettes are rare
Mirrors	For the original Wilson measure, planes through lattice sites and planes through midpoints of edges, four axes, $\beta \geq 0$ Source	The same two kinds (the middle plane for gauge-invariant functions, $\beta \geq 0$, $m \geq 2$)
Triangle inequality	Present Source	Not used

Table 2. Two uses of the Chessboard Estimate.

In the language of the root, the theorems of this part can presumably be restated in the form "the root of the indicator that a plaquette of one orientation is large is at most $q < 1$" Inferred (steps 2–5 of the proof in 22-01 bound this quantity). The published construction does not use the chessboard to estimate the probability of sets of large plaquettes Inferred (within the range of a search by name). Large fields are handled by another mechanism within each step of renormalisation (22-02).

04

The scope that applies to the whole part

- **One scale.** These are statements about the Wilson measure itself on a fine lattice at a fixed lattice spacing. Nothing is said about the measure at a coarse scale pushed forward by the block map (19-04).
- **Uniform in the volume, not uniform in the lattice spacing.** The rate does not depend on N, but β is the coupling of a single lattice and has nothing to do with sequences sending the lattice spacing to 0.
- **The weak-coupling side, with an explicit threshold.** $\beta \geq 81$ ($m \geq 2$). For the form with rate $1/4$, only the existence of the threshold is given. Nothing is said about the band of physical interest (β around 2 to 3).
- **Even sides, no twist.** Only even sides, and only the measure without a background twist.

- **A formalisation of a known type.** A formalisation of the chessboard Peierls argument (of Fröhlich–Israel–Lieb–Simon type) for the SU(2) Wilson measure. No novelty is claimed.

05

Order of reading

22-01 (statements, routes of proof, numbers) → 22-02 (what is said and what is not, and the relation to how the published construction handles large fields). How this rarity was planned to be used on the side of the premises is in 28-04.

The scope of what this chapter says

Said	Not said
Lean The three theorems of Table 1 (details in 22-01)	Coarse scales, $\beta < 81$, odd sides, measures with a twist
Source The published construction has the chessboard inequality and the triangle inequality for the root, and uses an estimate of the size of observables measured by the root as an input to the infinite-volume limit	Where that estimate enters the dependency chain of the published theorems (read in 12-03)
Inferred The published construction does not use the chessboard to estimate the probability of large plaquettes. The theorems of this part can presumably be restated in the language of the root	A Lean form of the restatement

Sources and reproduction

Item	Kind	Source / tool
The Chessboard Estimate and the chessboard Peierls argument	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Checked against the summary in M. Biskup, arXiv:math-ph/0610025 (the text of the original paper not obtained). Same as 22-01
The chessboard and the root in the published construction	Source	`yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Cells/Chess.lean:80` `tiles`, `:108` `adds`; `YM/Cells/Tiling.lean:226` `rootTest`; `YM/Extraction/Center.lean:38` `Wcmain`
The theorems of Table 1	Lean	Lean declarations of this book's study (names in Table 1; details of the sources in 22-01)

Terms used in this chapter — large plaquette, Chessboard Estimate, uniform pattern, chessboard root, seminorm, Peierls argument, reflection positivity, uniform in the volume. Their meanings are in A-02 Glossary.

Next chapter to read: 22-01 Rarity of large plaquettes — the chessboard estimate — the statements and routes of proof of the three theorems of this part.

Rarity of large plaquettes — the chessboard estimate

This chapter is this book's own investigation. § 01 is established knowledge.

Scope — The theorems of this chapter are statements about the SU(2) Wilson measure itself at one scale (the fine lattice with the lattice spacing fixed). They say nothing about the blocked measure at a coarse scale, about fields with a background (such as twists), or about estimates uniform across steps. They are uniform in the volume (the size of the torus), but not in the lattice spacing. A summary of the scope of the whole part and of what is not said is in 22-02.

What this chapter shows — For the Wilson measure on the torus $(\mathbb{Z}/N)^4$ with even side $N = 2m$, we read three things: (1) a chessboard estimate for the expectation of products of functions of the links of a unit cube placed block by block (all β, all $m \geq 1$); (2) for $\beta \geq 81$ and $m \geq 2$, the probability that every plaquette of a set X of plaquettes has $\operatorname{Re} U_p < 0$ is $\leq q_1^{\lfloor |X|/6 \rfloor}$, with $q_1 < 1$ independent of N and X; (3) a form with rate $1/4$ that includes $m = 1$.

Prerequisite chapters — 22-00 Entrance (the list of theorems of this part, and the scope that applies to the whole part); 19-01 The Wilson measure, plaquettes and holonomy, the product of Haar measures; 20-01 Reflection positivity; 26-05 § The Chessboard Estimate.

Said first — The scope is as written directly under the title. There are three further points. (1) The q_1 of (4) is written with the true volume $h(1/20)$ of the cap, but what Lean pins down as a number goes only as far as $h(1/20) \geq 83^{-4}$, and the upper bound $e^{-0.88\beta}83^{16}$ on q_1 is about $2.5 \cdot 10^{12}$ times the true q_1 Reading . The threshold 81 is determined by this upper bound; with the true h it would be about 48 Reading . (2) The threshold $\beta \geq 81$ is a value of convenience for the proof, far from the β of physical interest (the band 2–3). (3) The probability is a ratio to the partition function Z, not a ratio to any other quantity (such as a twisted partition function or a partition function restricted to small fields). Every theorem is a formalisation of a known type, the chessboard Peierls argument from reflection positivity Cited .

The order of this chapter

1. The chessboard and the Peierls argument — spreading an event into a uniform pattern by reflections, and comparing energy with entropy
2. Setting — large plaquettes and caps — the event $\mathrm{Re}\, U_p < 1 - y$, the volume $h(r)$ of a cap
3. The chessboard estimate — every even side, every β
4. Rarity uniform in the volume — $\beta \geq 81 - q_1^{\lfloor |X|/6 \rfloor}$, $q_1 < 1$
5. The route of the proof of rarity — reflection positivity → uniform pattern → an upper bound on the action and a lower bound on the partition function → pigeonhole
6. Detour: the bookkeeping behind $48r^2$ and $h(r)^4$ — the numbers of plaquettes and links per lattice site
7. The form with rate $1/4$ — including $m = 1$ — the threshold exists, nothing more
8. Strength — in numbers — the lower bound on the volume of the cap and its true value, the values of the thresholds
9. What is not said — coarse scales, backgrounds, $\beta < 81$, other ratios

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01

The chessboard and the Peierls argument

The introduction is 26-05 § The chessboard estimate and 20-01. For a reflection-positive measure, the expectation of a product of functions placed block by block is bounded by the geometric mean of the expectations of the "uniform patterns" in which each function is carried by reflections to all the blocks (the chessboard estimate, Fröhlich–Israel–Lieb–Simon)Cited .

Using this to estimate rare events is the chessboard Peierls argumentCited . When the event to be estimated (a plaquette at some place is large) is carried by reflections to all the blocks, it becomes the event of a "uniform pattern" in which the plaquettes are large at every place. The weight of the uniform-pattern event loses a lot in the action, so it is exponentially small compared with the partition function Z. The $1/|\text{number of blocks}|$-th power of that ratio becomes the rate per plaquette. The key point of this argument is that the rate does not depend on the volume.

The theorems of this chapter formalise this argument for the SU(2) Wilson measure. They are not results of a new type.

02

Setting — large plaquettes and caps

The lattice is $(\mathbb{Z}/N)^4$, $N = 2m$. The measure is the Wilson measure of 19-01 (not normalised),

$$d\mu_W(U) = \prod_{\ell} dh(U_\ell)\ \exp\Big(\beta \sum_{p} \operatorname{Re} U_p\Big), \qquad Z = \mu_W(\text{whole space}), \tag{1}$$

where h is the Haar probability measure of $SU(2)$, and the sum over plaquettes is over the $6N^4$ unoriented plaquettes $(x; \mu < \nu)$. Lean: `TorusRP.wilsonMeasure`; Z is `FoldPeierls.Zw` (the total mass). This chapter numbers the directions 1 to 4 (the Lean indices 0 to 3 are directions 1 to 4; 19-01 § Numbering of directions).

Large plaquettes. For a threshold $y \in (0, 2]$, a plaquette p is called "large" when $1 - \operatorname{Re} U_p > y$. For $y = 1$ this is $\operatorname{Re} U_p < 0$. Lean: `FoldEntropy.bigP N y ω p` (an indicator function, strict inequality). $\operatorname{Re} U$ is $\frac{1}{2}\operatorname{tr} U$, and $1 - \operatorname{Re} U = |U - 1|^2/2$ (regarding U as a unit quaternion).

Caps. We write $h(r)$ for the Haar volume of the open cap $\{U \in SU(2) : |U - 1| < r\}$ around the identity ($|\cdot|$ is the quaternion norm). Lean: `FoldEntropy.ballQ r`; $h(r)$ is `haarBall.real (ballQ r)`.

For a set X of plaquettes, $\int \prod_{p \in X} \mathbf{1}[p \text{ is large}]\, d\mu_W / Z$ is the probability of the event "all plaquettes of X are large". No condition is put on plaquettes outside X.

03

The chessboard estimate

The blocks are unit cubes, one for each point w of $(\mathbb{Z}/N)^4$, N^4 in all. A bounded measurable function that depends only on the links inside the cube at the origin (links whose starting point has all coordinates 0 or 1 and whose coordinate in their own direction is 0) is called "a function of the block at the origin". Ψ_w is a map of configurations which, for each coordinate direction, composes a translation if the coordinate of w is even and a reflection if it is odd. For a function f of the block at the origin, $f \circ \Psi_w$ is the function carried to block w.

The chessboard estimate Lean: `FoldWilson.wilson_chessboard_even`. For $N = 2m$ ($m \geq 1$), every real β, a family (f_i) of functions of the block at the origin, and an assignment $F : (\mathbb{Z}/N)^4 \to I$,

$$V(F) = \int \prod_{w} f_{F(w)}\big(\Psi_w U\big)\, d\mu_W(U), \qquad |V(F)|^{N^4} \;\le\; \prod_{w} V\big(\text{the constant assignment } F(w)\big)$$

Meaning. Whatever function is placed on whatever block, the expectation of the product is bounded by the geometric mean of the expectations of the uniform patterns in which each function is placed on all the blocks. The side need not be a power of 2. For $f \equiv 1$ both sides are Z^{N^4} and equality holds.

The route of the proof has three stages. (i) On a ring $\mathbb{Z}/2m$ in one direction, derive $|V(F)|^{2m} \le \prod_t V(\text{constant } F(t))$ from the "folding inequality" $V(F)^2 \le V(F^+)V(F^-)$ for reflections in all planes and from $V(\text{constant}) \ge 0$ (`FoldChess.chess_one`). To avoid being restricted to powers of 2, we use the argument of Fröhlich–Israel–Lieb–Simon, which takes the maximum of a ratio over a finite family of ranges and uses that folding preserves the maximum. In a model with a one-dimensional transfer kernel, the conclusion is not violated for reflection-positive kernels, and is violated for kernels that are not positive Reading . (ii) Apply it once in each of the four directions (`FoldChess.chess_four`). (iii) Derive the folding inequality from reflection positivity for planes through lattice sites in Cauchy–Schwarz form (20-01, every real β), and from the invariance of the measure under reflections and translations (`FoldChess.chess_data`; the concrete discharge is `FoldWilson.data`).

04

Rarity uniform in the volume — $\beta \ge 81$

Theorem of rarity uniform in the volume Lean: `FoldCap.rarity_explicit`. Set

$$q_1 = \frac{e^{\beta(48\cdot(1/20)^4 - 1)}}{h(1/20)^4} = \frac{e^{-0.88\,\beta}}{h(1/20)^4} \quad (3)$$

Then $q_1 < 1$ for every $\beta \ge 81$. Moreover, for every $m \ge 2$ ($N = 2m$) and every set X of unoriented plaquettes of $(\mathbb{Z}/N)^4$,

$$\int \prod_{p \in X} \mathbf{1}\big[\operatorname{Re} U_p < 0\big]\, d\mu_W \;\le\; Z \cdot q_1^{\lfloor |X|/6 \rfloor}. \quad (4)$$

The order of quantifiers is "for each β, there is a q_1 determined by β alone, and for all N and all X". q_1 depends neither on N nor on X.

Meaning. When the coupling is weak (β large), configurations in which many plaquettes are strongly twisted at once are exponentially unlikely in the number of twisted plaquettes, at a rate that does not depend on the size of the lattice. There is content from $|X| \ge 6$ onward (for $|X| < 6$ the right-hand side is Z). The event is not

empty. There is a configuration with $\mathrm{Re}\, U_p = -1$ on every plaquette (a fully frustrated $\mathbb{Z}_2$ configuration), and positive weight sits on an open set near it, so the left-hand side is positive for every X Proved here .

An intermediate form (also in Lean): for a general threshold $0 < y \le 2$, there exist $r > 0$ and β_0 (in the proof $r = y/20$) such that for $\beta \ge \beta_0$, $q1 = e^{\beta(48r^2 - y)}/h(r)^4 < 1$ and the form (4) holds (`FoldMix.rarity_uniform_all_mix`, $m \ge 2$). For plaquettes of a single orientation only, the rate is 1 per plaquette (`FoldMix.rarity_orient`).

05

The route of the proof of rarity

1. **Reflection positivity in four directions.** In directions 1 and 2 we use reflections in planes through lattice sites (every real β); in directions 3 and 4, reflections in planes through link midpoints (gauge-invariant functions, $\beta \ge 0$, $m \ge 2$) (20-01). Mixing in the midpoint planes makes the blocks lattice sites, and the carried plaquettes are distinct across all N^4 blocks. The indicator of a plaquette is gauge invariant ($\mathrm{Re}\, U_p$ is invariant). This builds the vessel of § 03 (`FoldMix.dataMix`). $m \ge 2$ is the condition of this stage.
2. **To a uniform pattern.** Place the block function "the plaquette of orientation $(1,2)$ at the origin is large" on the blocks of a set S of lattice sites, place 1 elsewhere, and use (2): the probability is bounded by the $|S|/N^4$-th power of the weight of the event "all N^4 plaquettes of orientation $(1,2)$ are large".
3. **The weight of the uniform pattern (energy).** On that event, $\mathrm{Re}\, U_p < 1 - y$ on N^4 plaquettes and $\mathrm{Re}\, U_p \le 1$ on the others, so the action is $\le 6N^4 - yN^4$ and the weight is $\le e^{\beta(6N^4 - yN^4)}$ (in the form of `FoldMix.action_le_mix`).
4. **A lower bound on the partition function (entropy).** If every link is placed in the cap $|U - 1| < r$, each plaquette is a product of four, so $|U_p - 1| \le 4r$ and hence $\mathrm{Re}\, U_p \ge 1 - 8r^2$. The Haar weight of this set of configurations is $h(r)^{4N^4}$, so $Z \ge e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$ (`FoldEntropy.Zw_ge`).
5. **Take the ratio.** Dividing 3 by 4 and taking the $1/N^4$-th power gives $q = e^{\beta(48r^2 - y)}/h(r)^4$ per plaquette.
6. **Align the orientations.** The measure is unchanged by involutions that map the six orientations to $(1,2)$. A general X has at least $\lfloor |X|/6 \rfloor$ plaquettes of one of the six orientations (pigeonhole), and the indicator functions take values in $[0,1]$, so we can drop to that subset (`FoldMix.rarity_all_mix`).

7. **An explicit threshold.** With $y = 1$ and $r = 1/20$, $48r^2 - 1 = -0.88$. The volume of the cap is bounded by $h(1/20) \geq 83^{-4}$ using a covering by the lattice $(1/41)\mathbb{Z}^4$ (`FoldCap.ballQ_ge`), and from $e^{71} > 2.718^{71} > 83^{16}$, $q_1 < 1$ for $\beta \geq 81$ (`FoldCap.q_lt_one_explicit`).

Detour: the bookkeeping behind $48r^2$ and $h(r)^4$

You may skip this. Per lattice site there are 6 plaquettes (the number of pairs of directions $\mu < \nu$) and 4 links. In the lower bound on the partition function, each plaquette loses at most $1 - \mathrm{Re}\, U_p \leq 8r^2$, so the loss of action per lattice site is $6 \cdot 8r^2 = 48r^2$, and the Haar weight is $h(r)^4$. On the side of the uniform-pattern event, one plaquette per lattice site loses at least y. Taking the ratio leaves $e^{\beta(48r^2 - y)}/h(r)^4$ per lattice site Proved here . Making r smaller makes $48r^2$ smaller, but $h(r)$ also shrinks at the rate r^3, so to get $q < 1$, β must be taken to be several times $\log(1/h(r))$. This is why the threshold is large.

06

The form with rate $1/4$ — including $m = 1$

Theorem of rarity with rate $1/4$ Lean: `FoldQuarter.rarity_uniform4`. For $0 < y \leq 2$, there exist $r > 0$ and β_0 such that for every $\beta \geq \beta_0$, $q_4 = e^{\beta(48r^2 - y/4)}/h(r)^4 < 1$, and for every $N = 2m$ ($m \geq 1$, in particular including $N = 2$) and every set X of plaquettes,

$$\int \prod_{p \in X} \mathbf{1}\left[1 - \mathrm{Re}\, U_p > y\right] d\mu_W \;\leq\; Z \cdot q_4^{\lfloor |X|/24 \rfloor}. \tag{5}$$

There are two differences. Since it is built only with reflections in planes through lattice sites, without those through link midpoints, the condition $m \geq 2$ is not needed; in exchange, the distinct plaquettes in a uniform pattern number $N^4/4$, and the rate becomes $1/4$. A general X is split into 24 classes (6 orientations $\times$ 4 parities of the outer directions) and pigeonhole is used.

The Lean statement asserts only the existence of r and β_0. In the proof, $r = y/1000$, and β_0 is taken as an expression containing $h(r)$; its numerical value is not in Lean. Putting in numbers with the formula for the volume of the cap gives $\beta_0 \approx 357$ at $y = 1$ Reading . The only explicit threshold held as a number inside Lean is the $\beta \geq 81$ of § 04 ($m \geq 2$).

07

Strength — in numbers

Quantity	Value	Label
True value of the volume $h(1/20)$ of the cap (formula $(\vartheta - \sin\vartheta\cos\vartheta)/\pi$, $\vartheta = 2\arcsin(1/40)$)	$2.652 \cdot 10^{-5}$	Reading
Lean's lower bound 83^{-4}	$2.107 \cdot 10^{-8}$	Reading
Ratio (true value / lower bound)	1259	Reading
Loss in the upper bound on q_1 (1259^4)	$\approx 2.5 \cdot 10^{12}$	Reading
q_1 at $\beta = 81$: Lean's upper bound / with the true h	0.56 / $2.2 \cdot 10^{-13}$	Reading
Threshold for $q_1 < 1$: with Lean's lower bound / with the true h	80.34 / 47.9	Reading
Threshold of the form with rate $1/4$ ($y = 1$, $r = y/1000$ as in the proof)	357	Reading

The upper bound should be far from the true probability, but there is no record comparing it numerically with the true probability at β above the threshold. What can be said goes only as far as the form of the rate and the threshold. A direct estimate without the chessboard diverges as $N \to \infty$ with X fixed, so it is not uniform in the volume. At small volumes the direct estimate can be the stronger one. The value of this chapter's estimate is that it is uniform in the volume, and in exchange the constants are weak.

08

What is not said

- **Coarse scales.** These are theorems at one scale (about the Wilson measure on the fine lattice itself). They do not reach estimates of large fields for the measure at a coarse scale pushed forward by the block map (19-04). Estimates of large fields across steps are on the side of the premises (28-04).
- **Lattice spacing.** They are uniform in the volume, but β is the coupling on a fixed lattice, and has nothing to do with the limit in which the lattice spacing is sent to 0.
- **Range of β.** § 04 covers only $\beta \geq 81$, and § 06 only couplings at or above a β_0 that is known only to exist. Nothing is said about the physical band (β between 2 and 3). 81 is a value of convenience for the proof.
- **Parity of the side.** Only even sides. § 04 needs $m \geq 2$; $m = 1$ ($N = 2$) is covered only by the form with rate $1/4$ of § 06.
- **What the ratio is taken against.** The probability is a ratio to Z. It is not a ratio to a partition function restricted to small-field events, nor to a twisted partition function.

- **Backgrounds.** Only the Wilson measure without twist (19-02).

What this chapter can say

Can say	Cannot say
Chessboard estimate: for $N = 2m$ ($m \geq 1$), every real β, and a family of functions of the links of a unit cube, $\lvert V(F)\rvert^{N^4} \leq \prod_w V(\text{constant})$. Lean: `FoldWilson.wilson_chessboard_even`	Odd sides. Measures with a twist
$\beta \geq 81$, $m \geq 2$: $P(X\text{: all its plaquettes have } \operatorname{Re} U_p < 0) \leq q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, with $q_1 < 1$ independent of N and X. Lean: `FoldCap.rarity_explicit`	$\beta < 81$, $m = 1$. The measure at a coarse scale. The threshold 81 is a value fixed by a crude lower bound on the volume of the cap (with the true h about 48 Reading)
$0 < y \leq 2$, at or above some existing β_0, all $m \geq 1$: the form with rate $1/4$, $\leq q_4^{\lfloor \lvert X\rvert/24 \rfloor}$. Lean: `FoldQuarter.rarity_uniform4`	The value of β_0 (about 357 at $y = 1$ is Reading)
$Z \geq e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$ ($\beta \geq 0$). Lean: `FoldEntropy.Zw_ge`	—
$h(1/20) \geq 83^{-4}$. Lean: `FoldCap.ballQ_ge`	A Lean form of the true value of the volume of the cap

Sources and reproduction

Item	Kind	Source / tools
The Chessboard Estimate and the chessboard Peierls argument	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Confirmed from the abstract of M. Biskup, Reflection positivity and phase transitions in lattice spin models, arXiv:math-ph/0610025 (the text of the original paper not obtained). Same as 26-05
Reflection positivity (planes through lattice sites, planes through link midpoints)	Cited	The sources of 20-01 (K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471)
The theorems of § 03– § 06	—	Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, confirmed by re-checking with the kernel). The list is in 22-02
The true value of the volume of the cap, the floating-point value of the lower bound, the ratio 1259, the values of q_1, the thresholds 80.34, 47.9 and 357	Reading	The numerical record of this book's study (the formula for the volume of the cap cross-checked by Monte Carlo with $2 \cdot 10^6$ points at $r = 0.3, 0.5, 1$; the thresholds from the formula)

Item	Kind	Source / tools
That the one-directional core of the chessboard fails for kernels that are not reflection positive (a control)	Reading	The numerical record of this book's study (a model with a one-dimensional transfer kernel, $n = 4, 6, 10$; 0 violations for positive-definite kernels, violations for kernels that are not positive definite)

Terms used in this chapter — large plaquette, cap, chessboard estimate, uniform pattern, reflection positivity, plane through lattice sites, plane through link midpoints, Peierls argument, partition function, pigeonhole. Their meanings are in A-02 Glossary.

Next chapter to read: 22-02 What this part says, and what it does not — the theorems of Part 22 laid out in one table, checking the scope and what is not said.

What this part says, and what it does not

> This chapter is a guide. § 03 and § 04 are a map of open questions.

What this chapter shows — We lay out the theorems of Part 22 in one table and check their scope and what they do not say. We then describe what it means that this rarity is "a statement at one scale", how it relates to the published construction's handling of large fields by recounting them within each step of renormalisation, and the distance between statements of this type and the questions that remain (as an inference).

Prerequisite chapters — 22-00 Entrance, 22-01 Rarity of large plaquettes — the chessboard estimate. The published construction's handling of large fields is in 11-03 and 11-04.

Said first — This chapter adds no new theorem. The comparisons and assessments of § 03 and § 04 come from reading the statements in the source of the published construction Source and from inferences drawn from them Inferred . We do not say that the rarity of this part yields uniqueness of the infinite-volume limit or the mass gap.

The order of this chapter

1. List of theorems — scope and what is not said — Table 1
2. A statement at one scale — nothing is said about scales after renormalisation
3. Large fields in the published construction — recounting step by step — an inequality on the side of the action, and rarity on the side of the measure
4. Distance to the questions that remain — an estimate of the type that controls large fields (an inference)
5. What the part as a whole does not say

Labels — Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment (details in "Labels" in 6-00). Proved here, Cited, Computer-assisted, Reading, Assumption and Lean are the labels of the whole book; Reading is a numerical reading of this book, and Lean a Lean theorem of this book. Source: `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0).

01

List of theorems — scope and what is not said

Theorem	Scope	What is not said	Lean
The Chessboard Estimate	$N = 2m$ ($m \geq 1$), all real β, families of functions of the links of a unit cube	Odd sides, measures with a twist, measures at a coarse scale	`FoldWilson.wilson_chessboard_even`
Rarity uniform in the volume	$\beta \geq 81$, $m \geq 2$, any set X of plaquettes: probability $\leq q_1^{\lfloor \lvert X\rvert/6\rfloor}$, with $q_1 < 1$ independent of N and X	$\beta < 81$, $m = 1$, coarse scales. 81 is a value fixed by a coarse lower bound on the volume of the cap (with the true volume it would be about 48 Reading)	`FoldCap.rarity_explicit`
The form with rate $1/4$	$0 < y \leq 2$, at or above some β_0, all $m \geq 1$: probability $\leq q_4^{\lfloor \lvert X\rvert/24\rfloor}$	The value of β_0 (about 357 for $y = 1$ is Reading)	`FoldQuarter.rarity_uniform4`
Lower bound on the partition function	$\beta \geq 0$: $Z \geq e^{6N^4\beta(1-8r^2)} h(r)^{4N^4}$	—	`FoldEntropy.Zw_ge`
Lower bound on the volume of the cap	For $K \geq 2$ and $2/K < r$, $h(r) \geq 1/(2K+1)^4$. With $K = 41$, $r = 1/20$, $h(1/20) \geq 83^{-4}$	A Lean form of the true value of the volume of the cap	`FoldCap.ballQ_ge`

Table 1. The theorems of Part 22. Each is a formalisation in this book's Lean of the type of the chessboard Peierls argument from reflection positivity Lean . Notation as in 22-01.

Probabilities are ratios to the partition function Z. They are not ratios to a partition function restricted to small-field events, nor to a twisted partition function. The upper bounds should be far from the true probabilities, but there is no record of a numerical

comparison with the true probabilities at β above the threshold. What can be said goes only as far as the form of the rate and the threshold (22-01 § 07).

02

A statement at one scale

The theorems of this part are statements about the Wilson measure itself on a fine lattice at a fixed lattice spacing. β is the coupling of that lattice, and no operation changing the lattice spacing is involved.

- **Nothing is said about scales after renormalisation.** Whether large coarse plaquettes are rare for the measure pushed forward to a coarse lattice by the block map (19-04) does not follow from the theorems of this part. The push-forward averages and composes the deviations of plaquettes, so there is no guarantee that the rarity of fine plaquettes becomes, as it stands, the rarity of coarse plaquettes Proved here .
- **Rarity uniform across steps is on the side of the premises.** Rarity of the same form at every step of renormalisation (with constants independent of the step and the lattice spacing) is placed as a premise in this book's study and is not shown (28-04, the large-field rarity condition).
- **Uniform in the volume, not uniform in the lattice spacing.** In the continuum limit β moves with the lattice spacing, so a statement at fixed β is not, as it stands, an estimate for any step of a continuum-limit sequence Proved here .

03

Large fields in the published construction — recounting step by step

The published construction too has to deal with large fields (places where the deviation of a plaquette exceeds a threshold). Its handling is of a different kind from the rarity of this part.

- **The form of the density.** The effective density at each step is carried in the form of a product of the exponential of an action truncated to be quadratic up to the threshold and linear beyond it, and a gas of large-field activities (forced activities) Source (11-01).
- **Recounting step by step.** The "seeds" of large fields are recounted at each step from the field of that step. It is not of the type in which a region that has once become a large field is fixed from then on Source . A persistent defect has its price paid again at each step Inferred .

- **Energy per seed.** The key of one step is a configuration-wise inequality: the difference of actions inside the halo is bounded below in proportion to the number of seeds (`Next.Fluctuations.grants`, `YM/Next/FLUCTUATIONS/Grants.lean:24`; with assumptions such as a gauge slice (`SliceSet`), regions (`Plot`) and `Funds`) Source . In the record of how declarations are used, this inequality is reached from one of the headline theorems (`SharpGram.habitat`) Source . How to read it is in 11-04. Where this inequality enters the dependency chain of the published theorems has not been checked.

The difference from the rarity of this part is as shown in Table 2 Inferred .

	Large fields in the published construction	**Rarity in this part**
Kind of statement	A configuration-wise inequality for the action (for all configurations, with conditions)	A statement about the measure (probabilities of events)
Scale	Each step of renormalisation, recounted step by step	One scale of the fine lattice
Coupling	Inside the weak-coupling band at every step	$\beta \geq 81$ (the coupling of one lattice)
Volume	The ledger of one step is a pointwise sum and does not depend on the volume. Estimates at the level of the density contain an exponential in the volume Source	The rate is uniform in the volume
Tools	Decomposition of the action, regions, partitions of unity, polymer sums	Reflection positivity, the chessboard, Peierls

Table 2. Two ways of handling large fields.

Neither follows from the other Inferred . The inequality of the published construction is a statement about the action within a step and says nothing about the rarity of a measure. The rarity of this part is a statement about the measure at one scale and says nothing about estimates across steps. What the published construction uses as estimates uniform in the volume are such things as the chessboard-root estimate on the side of the Wilson measure (22-00 § 03), and there too the chessboard is not used for the probability of large plaquettes Inferred (within the range of a search by name).

04

Distance to the questions that remain

This section is an inference Inferred .

One of the open questions in the assessment is the convergence and uniqueness of the infinite-volume limit (16-02). The published construction takes the infinite-volume

limit as a subsequence by compactness, and includes neither estimates of differences when the volume is changed nor material for uniqueness Source .

Arguments that show uniqueness usually take as a component some "lack of disorder" uniform in the volume. For uniqueness in the strong-coupling window (20-03), this was the Dobrushin condition (the smallness of the influence that one link receives from its surroundings). The rarity of this part provides, as a statement about the measure, a lack of disorder of the form "configurations with many large plaquettes are rare, at a rate independent of the volume" on the weak-coupling side. It is an estimate of the type that controls large fields and, like the Dobrushin condition in the strong-coupling window, has the form of a lack of disorder uniform in the volume. Its object, however, is the measure at one lattice spacing, which is different from the object of the assessment's question (the infinite-volume limit of the continuum theory).

For the following three reasons, the rarity of this part does not yield uniqueness.

1. Rarity alone says nothing about long-range correlations on the side of small fields (plaquettes that are not large). Uniqueness needs decay on that side, or an argument that replaces it.
2. It is a statement about one lattice with $\beta \geq 81$, a different object from the one for which the published construction takes infinite volume (correlation functions with the continuum limit taken first).
3. There is no rarity at renormalised scales (§ 02), so it is not in a form usable along a continuum-limit sequence.

Whether a statement of the same type can become a component of an argument for the mass gap (uniform decay) likewise remains open.

05

What the part as a whole does not say

- **Coarse scales.** Rarity of the pushed-forward measure and rarity uniform across steps are not said.
- **The range of β.** $\beta \geq 81$ (explicit), and at or above a β_0 whose existence only is given. Nothing is said about the physical band (β around 2 to 3).
- **Sides and background.** Only even sides and measures without a twist.
- **What the ratio is taken against.** Probabilities are ratios to Z.
- **Uniqueness, the mass gap, the continuum limit.** None of these follows from the theorems of this part.
- **Not a new result.** It is a formalisation of the chessboard Peierls argument for the SU(2) Wilson measure.

The scope of what this chapter says

Said	Not said
Lean The five items of Table 1 (the Chessboard Estimate, rarity uniform in the volume, the form with rate 1/4, the lower bound on the partition function, the lower bound on the volume of the cap). Details in 22-01	Coarse scales, $\beta < 81$, odd sides, measures with a twist
Source The published construction recounts large fields at each step and handles them by a configuration-wise inequality on the energy per seed	The path by which that inequality enters the chain of the published theorems (not checked)
Inferred Neither handling yields the other. The rarity of this part is an estimate of the type that controls large fields (in the form of a lack of disorder uniform in the volume)	A road to uniqueness or the mass gap

Sources and reproduction

Item	Kind	Source / tool
The theorems of Table 1	Lean	Lean declarations of this book's study (names in Table 1). The Chessboard Estimate is the standard argument of Fröhlich–Israel–Lieb–Simon (1978). Details of the sources in 22-01
The threshold values 48 and 357	Reading	The numerical record of this book's study (from the formula for the volume of the cap; 22-01 § 07)
The published construction's handling of large fields	Source	`yang-mills-group/OpenAI-YM-20261028`: `YM/Next/FLUCTUATIONS/Grants.lean:24` `grants`. Readings of the form of the density and of recounting are in 11-01, 11-03, 11-04
Open question (convergence and uniqueness)	Assessment	The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8, item 1

Terms used in this chapter — large plaquette, rarity, one scale, push-forward, truncated action, forced activity, seed, recounting, uniform in the volume, uniqueness. Their meanings are in A-02 Glossary.

Next chapter to read: 23-00 Entrance — the staircase, and the random walk, the ruler and the clock — from here on is the record of the study heading toward the band. To see the parts of firm theorems together, go to Table 1 of 16-04.