Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 23 The first road, first half — from the direction of time

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

> This PDF is a print of the chapters of the web edition. Interactive figures are frozen in the state in which they were printed. Links between chapters lead to the web edition, which holds the latest text.

Chapter		Page
23-00	Entrance — the staircase, and the random walk, the ruler and the clock	3
23-01	The line of construction and the diffusive-relaxation-time assumption	16
23-02	Roads tried on the four-dimensional lattice, and walls	42
23-03	The four steps of the staircase — why we start from 0+1 dimensions	66
23-04	The path measure of the double well, and kinks	77
23-05	The statement of the first step — the four double-well conditions and the upper-bound proposition	91
23-06	The in-phase local Poincaré condition	113
23-07	The map of the saddle-tube condition — seven routes and the seam of the ring	130
23-08	What computer-assisted means — the conditions on the chain inside the window	143

Chapter		Page
23-09	The sector framework and the three strands of the window log-Sobolev route	157
23-10	The list of detour plans — types of breakage	177
23-11	Box reduction and the gap inside the box — the constant of the core	186
23-12	The integrated table and the dependency graph	211
23-13	Lean statements and the limits of machine checking	228
23-14	Reconciliation — the numbers of the record and the known numbers	237
23-15	What remains and its cost — the distance to a theorem	251
23-16	What is not claimed	269
23-17	Insights and small questions	280

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time entrance, staircase, five quantities · Kiichi, Shiori and Rin

Entrance to Part 23 — the staircase, and the random walk, the ruler and the clock

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Part 23 is the part that examines a road to the mass gap with the diffusive-relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length) as its axis. The diffusive-relaxation-time assumption is one sentence that restates a quantity of the problem statement (the gap in Euclidean time) as a quantity that can be computed on the lattice and that machine checking can reach (the relaxation time of Langevin dynamics). This part divides that sentence into a staircase of four steps, and lays out how far the claim has been built on the first step (the path measure of the double well in 0+1 dimensions). In this chapter, after looking at the picture of the staircase and the map of the part, we set up quantities such as the correlation length ξ and the Poincaré constant C_P one by one, with the metaphors of the random walk, the ruler and the clock.

Prerequisite chapters — 2-10 (relaxation time and the dynamical exponent), 4-11 (definition of the mass gap and the correlation length), 2-07 (the path integral and imaginary time).

Said first — This chapter is an entrance that lines up quantities through metaphors, and proves nothing. The relaxation-time assumption is an unproved assumption. The precise definitions are placed in 23-01 (the side of the four-dimensional lattice) and 23-04 (the side of the double well).

The order of this chapter

1. A map of the part — the list of eighteen chapters, and how to read the labels
2. The four steps of the staircase — from the double well to the mass gap in the limit (Figure 1)
3. The ruler — the correlation length ξ — in how many steps correlations die out

4. The smearing time — the Poincaré constant C_P — L^2 steps to even out a length L (Table 3). Detour: why the random walk goes as a square
5. The scale and the clock — the remaining three quantities — the number of rescalings, the graduation of the clock, the clock that runs slow
6. The relaxation-time assumption in one sentence — smearing time $\asymp$ ruler^2. "Some f" and "every f"
7. How to walk through Part 23 — in which chapter each metaphor becomes a precise sentence

---

01

A map of the part — the list of eighteen chapters, and how to read the labels

This part is a study with the relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length) as its axis. The relaxation-time assumption is not the mass gap itself but a sentence that restates it as a quantity of a dynamics measurable on the lattice (23-01 § 05), and this part builds that sentence on the first step of a four-step staircase — the path measure of the double well in 0+1 dimensions. We split the claim of the first step into four conditions, and for each condition lay out in one table the rows where a proof is given, rows citing original sources, computer-assisted rows, rows of numerical reading, and assumption rows.

The eighteen chapters of Part 23

0. **23-00 Entrance — the staircase, and the random walk, the ruler and the clock** [Y2] (this chapter)
1. 23-01 The line of construction and the diffusive-relaxation-time assumption [Y2]–[Y3]
2. 23-02 Roads tried on the four-dimensional lattice, and walls [Y3]
3. 23-03 The four steps of the staircase — why we start from 0+1 dimensions [Y2]
4. 23-04 The path measure of the double well, and kinks [Y2]–[Y3]
5. 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition [Y3]
6. 23-06 The in-phase local Poincaré condition [Res]
7. 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring [Y3]

8. 23-08 What computer-assisted means — the conditions on the chain inside the window [Y2]–[Y3]
9. 23-09 The sector framework and the three strands of the window log-Sobolev route [Res]
10. 23-10 The list of detour plans — types of breakage [Y3]–[Res]
11. 23-11 Box reduction and the gap inside the box — the constant of the core [Res]
12. 23-12 The integrated table and the dependency graph [Y3]
13. 23-13 Lean statements and the limits of machine checking [Y2]–[Y3]
14. 23-14 Reconciliation — the numbers of the record and the known numbers [Y3]
15. 23-15 What remains and its cost — the distance to a theorem [Y3]
16. 23-16 What is not claimed [Y2]
17. 23-17 Insights and small questions [Y2]–[Y3]

The chapters run: why this assumption (23-00 to 23-02), the statement of the first step (23-03 to 23-05), the four conditions (23-06 to 23-11), the overall tables (23-12, 23-13), and reconciliation and what remains (23-14 to 23-17). The marks [Y2], [Y3] and [Res] are reader levels: [Y2] is second-year, [Y3] third-year and above (using measures, operators and Poincaré inequalities), and [Res] is research level. This chapter is [Y2]. Viewpoints other than the relaxation-time assumption, and new views that take the relaxation-time assumption as a premise, are placed in Part 24 The first road, second half.

The labels of this part

In this part the strength of a claim is divided by five labels (the labels of 0-01 § 03, refined for the material of this part). Each sentence gets one label; a sentence needing two is split.

Table 1 The five labels of this part (Lean is a column outside the labels)

Label	Meaning	What it does not guarantee
Proved here	Elementary. The proof is placed in the text or at the end of the chapter	That it contains no error (it is placed in a form the reader can check)
Cited	A known theorem cited from the original source (or a secondary source whose statement was checked)	That it is a result of this book
Computer-assisted	Computer-assisted. Checked as an inequality by interval arithmetic (premises such as rounding conventions are written on the same row)	That it holds outside the range computed

Label	Meaning	What it does not guarantee
Reading	A reading. Floating-point numbers, Monte Carlo, fits. The kind (a point value, a lower bound over a family, etc.) is added	That it is a proof of an upper or lower bound
Assumption	An assumption. A row put down without a proof. It is given a name	That it holds

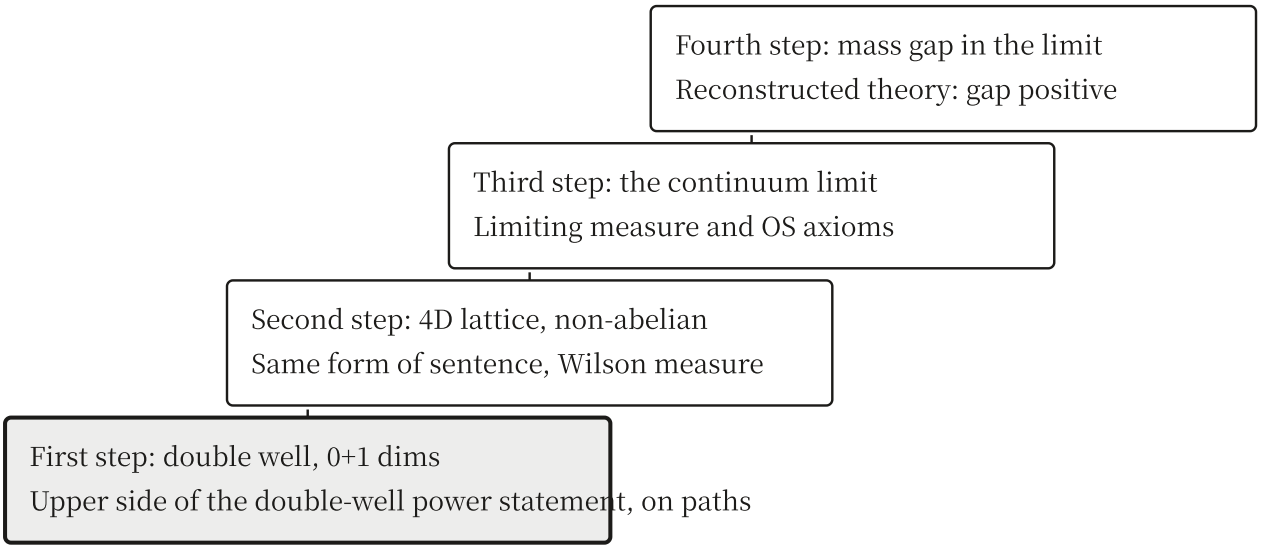
Lean is a column separate from the five labels. A Lean theorem is the "algebra from assumptions to conclusion" checked by machine, and the theorem name and the assumptions of that theorem are written on the same row. Whether the assumptions are true is outside the check (23-13).

02

The four steps of the staircase — from the double well to the mass gap in the limit

The relaxation-time assumption is the sentence "in some family of measures, the time over which fluctuations are smoothed out grows as a power of the square of the correlation length", and it can be stated independently of dimension. So we line up sentences of the same form in four steps, from easy places to hard ones.

Figure 1 The four steps of the staircase. This part deals with the inside of the first step; the arrows from the second step upwards stand for "a sentence of the same form", not for theorems of implication



Written out in words, Figure 1 is as follows. From the bottom: the first step = the double well in 0+1 dimensions (the path measure in imaginary time of a quantum-mechanical model); the second step = the non-abelian gauge theory on the four-dimensional lattice (the Wilson measure); the third step = the continuum limit with the lattice spacing taken to 0, and that limit satisfying the Osterwalder–Schrader axioms; the fourth step = the mass gap of the limiting theory.

Table 2 The objects of the four steps, and what can be called "closed" at each step

Step	Object	Form in which it can be called closed	State in this book

Step	Object	Form in which it can be called closed	State in this book
First step	The path measure of the double well in 0+1 dimensions (parameter $\hbar$)	Show the upper side of the double-well power statement (an upper bound on C P), with a table of finitely many assumption rows	Split into the four double-well conditions, and up to the algebra on top of them (the Double-Well Relaxation-Time Upper-Bound Proposition). Every condition leaves Assumption rows
Second step	The Wilson measure on the four-dimensional lattice (parameter β)	A sentence of the same form, as $\beta \to \infty$	The relaxation-time assumption remains an assumption. Only estimates from below (23-02)
Third step	The limiting measure as the lattice spacing a $\to 0$	The limit exists and satisfies the OS axioms	Not dealt with. That a theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory was proved in Lean by the construction published on 2026-11-10, according to the committee's assessment (Part 6). A probability measure for the continuum theory and uniqueness of the infinite-volume limit are not shown (6-04)
Fourth step	The reconstructed quantum field theory	The gap of the Hamiltonian is positive (4-11)	Not dealt with. The published construction does not show it either (6-04)

Even if the first step is closed, there is no guarantee that it lifts to the second step and beyond. What has a prospect of lifting is "the form of the sentence" and "the form of the tools"; constants and ranges of parameters do not lift. The reasons are given from both sides in 23-03 § 06.

03

The ruler — the correlation length ξ

Fluctuations are similar between nearby places and become unrelated when far apart. The length over which they become unrelated is the correlation length ξ. As a metaphor, it is the one ruler the measure carries.

As a formula, the correlation of two points (the expectation of the product with the mean subtracted) decreases exponentially with distance r, and the length of that decrease is ξ.

$$\langle O(0)\,O(r)\rangle - \langle O\rangle^2 \;\approx\; C\,e^{-r/\xi}, \qquad \xi = \frac{1}{m} \quad (1)$$

m is the gap of the transfer matrix (or transfer operator), that is, the difference between the lowest eigenvalue and the next, and r is the distance in the imaginary-time direction. That the decay of correlations and the gap are the same quantity is in 4-11 § 02, and the restatement as a correlation length is in 4-11 § 07 Cited .

Example 1 (free field). For the free field on a one-dimensional lattice (a Gaussian chain with neighbours joined by springs, mass m_0), correlations decrease exactly as $e^{-r/\xi}$, and ξ is determined by $m_0^2 = 2(\cosh(1/\xi) - 1)$. For $m_0 = 1$, $\xi \approx 1.04$; for $m_0 = 0.1$, $\xi \approx 10.0$ (18-03 § 05).

Example 2 (double well). In the quantum-mechanical double well (2-05 § 07), the gap is the tunnelling splitting Δ, determined by passing through between the two valleys, and it becomes exponentially small the higher the hill. Seen through paths in imaginary time, the correlation length is $\xi = \hbar/\Delta$, and as $\hbar$ is made smaller it grows at the rate $e^{S_0/\hbar}$ (S_0 is the action of the path that moves from valley to valley). This property, that the ruler grows exponentially, is one of the reasons for choosing the model of the first step (23-03 § 04, 23-04 § 03).

The ξ measurable on the lattice is a number in units of one graduation of the lattice (5-09 § 01). The physical length is ξ times the lattice spacing a, $a\xi$, and in the continuum limit $a\xi$ must remain finite while $a \to 0$. That is, ξ in lattice units must grow to infinity (5-09 § 05). The $\hbar \to 0$ of the double well is the smallest specimen of this regime, "the ruler becomes infinitely long compared with the graduations".

04

The smearing time — the Poincaré constant C_P

Drop a drop of ink on paper, and the ink gradually smears and spreads, and in the end becomes uniformly pale. What the smearing really is, is a random walk of the ink particles. Each particle moves one step left or right at random at each tick. After t steps the spread of the particles (the root-mean-square distance) is $\sqrt{t}$, and evening out a strip of length L takes on the order of L^2 steps.

The Poincaré constant C_P is this "longest time until evening out is finished", defined in general for fluctuations. As a metaphor, it is the reading of the one clock the measure carries: the smearing time.

We write it as a formula. Given a probability measure μ and the "size of the change from place to place" $\mathcal{E}(f)$ of a function f (the Dirichlet form),

$$\mathrm{Var}_\mu(f) \;\le\; C_P\,\mathcal{E}(f) \quad (\text{for every } f), \qquad C_P = \sup_f \frac{\mathrm{Var}_\mu(f)}{\mathcal{E}(f)} \tag{2}$$

the best constant is C_P (1-10 § 07). If there is a function with large variance but small change, that is, "a large, slowly varying fluctuation", C_P becomes large. For the diffusion dynamics (Langevin dynamics) with μ as its equilibrium, $1/C_P$ is the gap of that dynamics, and the slowest fluctuation decreases by a factor $1/e$ every time C_P Cited .

Example: the random walk on a single road

Consider a single road of L cells in a row (reflecting at both ends). μ is the uniform distribution, and $\mathcal{E}(f) = \frac{1}{L}\sum_i (f_{i+1} - f_i)^2$. Then the C_P of (2) is the reciprocal of the eigenvalue next after 0 of the Laplacian matrix of the road, and can be written in closed form.

$$C_P(L) = \frac{1}{2\,(1-\cos(\pi/L))} \;\approx\; \frac{L^2}{\pi^2} \qquad (L \to \infty) \tag{3}$$

Proved here Proof: $f_i = \cos(\pi(i-\frac{1}{2})/L)$ $(i = 1, \dots, L)$ is an eigenvector of the Laplacian matrix of the road reflecting at both ends, with eigenvalue $2(1-\cos(\pi/L))$ (use $\cos(A+B) + \cos(A-B) = 2\cos A \cos B$ on the sum of the two neighbouring terms, and check that at both ends the mirror conditions $f_0 = f_1$, $f_{L+1} = f_L$ hold). The other eigenvectors are $\cos(\pi k(i-\frac{1}{2})/L)$ $(k = 0, \dots, L-1)$, with eigenvalues increasing with k. Since μ is uniform, Var and $\mathcal{E}$ are divided by the same $1/L$, and the supremum of the ratio is the reciprocal of the eigenvalue next after 0. ∎

Proved here Table 3 C_P of the random walk on a single road (values of the closed formula (3)). C_P/L^2 approaches $1/\pi^2 \approx 0.10132$

L	**Gap** $2(1-\cos(\pi/L))$	C_P	C_P/L^2
16	0.0384294	26.02	0.10165
32	0.00963055	103.8	0.10140
64	0.00240909	415.1	0.10134
256	0.000150596	6640	0.10132

Doubling L makes the time until evening out is finished four times as long.

The C_P used in this part replaces this single road by "the space of configurations of whole paths". For the double well, the measure $e^{-S/\hbar}$ on the whole of the paths $x(\tau)$ in imaginary time is μ, $\mathcal{E}$ is the expected square of the change when a path is moved slightly, and Langevin dynamics diffuses whole paths along an "artificial time" s (23-04

§ 01). On the four-dimensional lattice, μ is the Wilson measure and what moves are the angles of the links (23-01, 7-09).

For the free field, the slowest fluctuation is in the direction of shifting the whole path uniformly, and its rate is the square of the mass, m_0^2. So $C_P = 1/m_0^2$, and from $m_0^2 = 2(\cosh(1/\xi) - 1)$ of Example 1, C_P/ξ^2 at $\xi = 1, 4, 16, 256$ is 0.921, 0.995, 0.9997, 0.999999, approaching 1 Proved here (values of the closed formula). The smearing time is the square of the ruler: that is the free field's answer.

Precisely, the C_P meant here is the Poincaré constant of "the distribution of configurations over the whole of spacetime (in 0+1 dimensions, imaginary time)" (the path measure). The Poincaré constant of the distribution at a single time (the $|\psi_0|^2$ of the ground state) is a different quantity, and it is tied to the gap of the Hamiltonian. The distinction among the three C_P is placed in 23-01. Known examples in which the power z (the dynamical exponent) with which C_P grows in ξ departs from 2 are in 2-10.

Detour: why the random walk goes as a square

A subsection you may skip. The position X_t after adding t independent steps ± 1 has mean 0, and its variance is t times the variance 1 of one step. This is because for sums of independent things, variances add (1-08).

$$\mathbb{E}[X_t^2] = \sum_{i=1}^{t} \mathbb{E}[\varepsilon_i^2] + \sum_{i \neq j} \mathbb{E}[\varepsilon i]\,\mathbb{E}[\varepsilon j] = t \quad (4)$$

The spread is $\sqrt{t}$, so reaching distance L takes $t \approx L^2$. A wave advances one step per tick, but a random walk goes back and forth, so reaching far is delayed by a square. The "square" of the relaxation-time assumption is this square of the random walk.

05

The scale and the clock — the remaining three quantities

Besides the ruler ξ and the smearing time C_P, there are three quantities that have been used in the record for the four-dimensional lattice. They are not the main players on the first step (the double well), but the reason for putting the relaxation-time assumption as a "power statement" lies here, so we only line up the metaphors.

The ruler	The smearing time	The number of rescalings	The graduation of the clock
The correlation length ξ	The Poincaré constant C_P		

the reciprocal of the gap m (§ 03)	L^2 steps to even out a length L (§ 04)	The bookkeeping length ξ_{RG} the number of steps that coarsen the map	The normalisation of $\mathcal{E}$ in what unit the same fluctuation is measured
`The clock that runs slow` Renormalisation of stochastic time the clock itself stretches with scale			

The number of rescalings. In four-dimensional lattice Yang–Mills there is no way to obtain ξ by formula as a function of the coupling β. So we coarsen the scale of the map one step at a time, and make separately a length that counts how many times it takes to reach a picture of fixed coarseness, the bookkeeping length ξ_{RG}. In the one-loop count, raising β just a little makes the bookkeeping length grow exponentially (4-07). That the bookkeeping length and ξ agree up to a constant factor is an assumption. The definitions and tables have been moved to 23-01 and 23-02.

The graduation of the clock. The C_P of (2) changes value depending on the unit in which $\mathcal{E}$ is measured. Multiplying $\mathcal{E}$ by c multiplies C_P by $1/c$. The same happens for the double well. Between the $\mathcal{E}$ that measures paths by the canonical field $x/\sqrt{\hbar}$ and the $\mathcal{E}_x = \mathcal{E}/\hbar$ that measures them by the position x as it is, $C_P^{(x)} = \hbar\, C_P$ Proved here Lean `ratioR_x_normalisation` (no assumptions; if $C_P^{(x)} = \hbar C_P$, the ratio is also multiplied by $\hbar$). The difference of graduation shows in "the width of a constant factor", but the factor of a power of $\hbar$ is small compared with $\ln \xi$, so it does not show in "the power".

The clock that runs slow. Langevin time itself stretches and shrinks when the scale is changed. In the perturbation theory of stochastic quantisation, the graduation of stochastic time also has to be renormalised (7-09) Cited (at the level of perturbation theory). On the four-dimensional lattice its effect is a factor of a fractional power of β, and this too does not show in the power. How to compare the four clocks is placed in a detour of 23-01.

Precisely, the three of them — the number of rescalings, the graduation of the clock and the clock that runs slow — move sentences stated in "the width of a constant

factor" (band statements), and do not move sentences stated in "a power". That is why the relaxation-time assumption in this part is a power statement.

06

The diffusive-relaxation-time assumption in one sentence — smearing time ≍ ruler²

With the quantities in place, we put the relaxation-time assumption into one sentence as a metaphor: "the smearing time grows with the same power as the square of the ruler". When a family of measures is lined up by one parameter, and moving the parameter makes the ruler grow without bound,

$$(\xi^2): \quad \frac{\ln C_P}{\ln \xi} \longrightarrow 2 \qquad (\xi \to \infty) \tag{5}$$

On the four-dimensional lattice the parameter is β with $\beta \to \infty$; for the double well the parameter is $\hbar$ with $\hbar \to 0$. For the free field it holds with $C_P/\xi^2 \to 1$, as in § 04. The question is whether it holds on the four-dimensional lattice with interaction, and the first step is to build the upper side on the double well, which is easier than that.

For the double well, the "width of a constant factor" sentence stronger than (5) ($C_P \le K\xi^2$) is read as being on the side that breaks. This is because the lower bound on the path C_P grows from 1 times to 6.5 times ξ^2 as $\hbar$ is lowered Reading (lower bound over a family, step $\varepsilon = 0.25$, $\hbar \in [0.07, 0.4]$, 23-04 § 04). In the count at the level of physics that counts kinks, the growth is first order in $S_0/\hbar \approx \ln \xi$ (the numbers in this range alone cannot distinguish it from exponential growth). If it is first order, the power statement (5) is kept. That is why the goal of the first step is not the band statement but the upper side of the power statement, $C_P \le K\xi^2 S_0/\hbar$ (23-03 § 05).

"Some f" and "every f"

The C_P of (2) is a supremum, so bounding it from below and from above differ in difficulty. To bound it from below (C_P is large), it is enough to show one test function with large $\mathrm{Var}(f)/\mathcal{E}(f)$. To bound it from above (C_P is not too large), the inequality must be shown for every f.

$$C_P \ge c\,\xi^2 \iff \exists f: \ \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \ge c\,\xi^2, \qquad C_P \le C\,\xi^2 \iff \forall f: \ \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \le C\,\xi^2 \tag{6}$$

As a metaphor, that smearing is slow can be said by showing one stain, but that it is not too slow must be said for every stain. What lies on the side of the mass gap is the upper side (C_P not too large), and test functions and numerics reach only the lower

side. The algebra that rereads the upper side as a lower bound on the mass, and the definitions of the upper and lower halves (the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption), are placed in 23-01.

Precisely, the lower bound $CP \geq \xi^2$ for the double well is a proposition of the form that a family of test functions local in time (functions of a single time summed over time) reaches up to equality (the One-Dimensional One-Sided Bridge Proposition in 23-04 § 04). On the upper side, even for the first step there are only conditional propositions.

07

How to walk through Part 23 — in which chapter each metaphor becomes a precise sentence

Each metaphor of this chapter is replaced once by a precise sentence somewhere in Part 23. Table 4 matches metaphors, symbols and the chapters that make them precise.

Table 4 Metaphors and the corresponding precise sentences

Metaphor	Symbol	Precise definition or sentence	State
The staircase	Four steps	23-03	The convention for how this part is built
The ruler	$\xi = 1/m$ (for the double well $\xi = \hbar/\Delta$)	4-11 § 07, 23-04 § 02	Cited Definition
The smearing time	CP (the path measure)	23-01, 23-04 § 01	Cited Definition. The value on the four-dimensional lattice is unknown
The number of rescalings	ξ_{RG}	23-01, 23-02	Convention (that it agrees with ξ is Assumption)
The graduation of the clock, the clock that runs slow	Normalisation of $\mathcal{E}$, stochastic time	23-01	Convention, level of perturbation theory
Smearing time $\asymp$ ruler^2	The relaxation-time assumption (power statement)	23-01, 23-03 § 05	Assumption
The upper side of the first step	$CP \leq K\xi^2 S_0/\hbar$ (the First-Step Upper-Bound Proposition)	23-05	Algebra on top of four conditions. The conditions leave Assumption rows

To follow the line first: 23-03 → 23-04 → 23-05 → 23-12 (the integrated table) → 23-15 (the distance to a theorem). To see the relation with the four-dimensional lattice first:

23-01 → 23-02. To see what is not claimed first, 23-16 is the shortcut. The Lean statements and assumptions are in 23-13, and how to read Lean is in A guide to Lean.

What this chapter can say

Said	Not said
Cited That the correlation length and the gap are the same quantity ((1)); the definition of the Poincaré constant, and that the gap of the diffusion dynamics is its reciprocal ((2))	That ξ diverges as $\beta \to \infty$ in four-dimensional lattice Yang–Mills; the value of C_P
Proved here The closed form (3) of C_P for the random walk on a single road; $C_P/\xi^2 \to 1$ for the one-dimensional free field; the mean-square distance $= t$ of the random walk ((4))	That the same ratio agrees in power in a model with interaction (the relaxation-time assumption)
Proved here That multiplying the normalisation by $\hbar$ also multiplies C_P by $\hbar$ (the graduation of the clock in § 05) Lean `ratioR_x_normalisation`	That the bookkeeping length ξ_{RG} of the four-dimensional lattice and the correlation length ξ agree up to a constant factor
Reading That the lower bound on the path C_P of the double well grows from 1 times to 6.5 times ξ^2 (lower bound over a family, $\varepsilon = 0.25$, $\hbar \in [0.07, 0.4]$)	An upper bound on C_P for the double well. The upper side of the first step is conditional on the four conditions (23-05)
The way of building it as a four-step staircase (a convention)	Implication from the first step to the second step and beyond. About the fourth step (the mass gap in the limit), this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
Correlation length and gap, correlations in imaginary time	Cited	The table of sources of 4-11.
Poincaré constant, gap of diffusion	Cited	The table of sources of 1-10.
C_P of the random walk on a single road ((3), Table 3)	Proved here	The proof in § 04 of the text. The values in the table are values of the closed formula (3).
Tunnelling splitting of the double well, and S_0	Cited	(9)(10) of 2-05 § 07 and its table of sources.
Reading of the lower bound on the path C_P (§ 06)	Reading	23-04 § 04 and its table of sources.

Item	Kind	Source / tools
Rereading the normalisation (the graduation of the clock in § 05)	Proved here, Lean	Theorem `ratioR_x_normalisation` (no assumptions). The statement is in 23-13.
Bookkeeping length, renormalisation of stochastic time	Convention, Cited (perturbation theory)	The tables of sources of 4-06, 4-07 and 7-09. The detailed formulas are in 23-01.
Figure 1 (the staircase)	Convention	A schematic drawing of how this part is built.

Words used in this chapter — correlation length, Poincaré constant, Dirichlet form, path measure, staircase (four steps), the relaxation-time assumption (power statement), the five labels (proved here, cited, computer-assisted, reading, assumption), tunnelling splitting. Their meanings are in A-02 Glossary.

Next chapter to read: 23-01 The line of construction and the diffusive-relaxation-time assumption — splits the problem into eight statements, and defines the diffusive-relaxation-time assumption and its upper and lower halves. To go straight to the first step, 23-03 The four steps of the staircase.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time line of construction, the relaxation-time assumption, the one-sided bridge · Kiichi, Shiori and Rin

The line of construction and the diffusive-relaxation-time assumption — eight statements, the direction of the lattice version of the mass gap, and the relaxation-time assumption in six stages

> This chapter is this book's own investigation. § 02 and § 08 are a map of open questions; § 03 is established knowledge.

What this chapter shows — We line up the Clay problem as a single line, "lattice → infinite volume → continuum limit → OS axioms → mass gap", and split it into the eight statements of construction. The mass gap on the lattice is not "the gap in lattice units is uniformly positive" but "the gap in physical units stays positive". The diffusive-relaxation-time assumption, the axis of this part (the time C_P over which fluctuations are smoothed out grows with the same power as the square of the correlation length), is explained in six stages, each with an example that can be checked by hand: what we want to measure; why we restate it as a quantity of a dynamics; the statement (band statement → power statement → upper and lower halves [the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption]); what the relaxation-time assumption gives; what it does not give; and analogous cases. On a lattice with continuous time the one-sided bridge $C_P \geq \kappa \xi^2$ can be shown, and together with the upper-bound assumption it gives a lower bound on the gap of the lattice in physical units. The converse does not hold in general; under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the lattice mass gap in physical units.

Prerequisite chapters — 0-03 (the problem statement), 5-09 (what to hold fixed when taking the limit), 4-11 (the gap and the correlation length), 1-10 (the Poincaré constant), 23-00 (the five quantities and the labels of the five grades).

Said first — This chapter does not solve the problem. Along the line of this chapter (for each β, first make the volume infinite, then β→∞), what is in hand in four dimensions is only the theorems of the reflection-positivity preservation statement and the OS reconstruction statement, and the infinite-volume uniqueness statement at small β (for the relation to the published construction, see the note in § 02). The relaxation-time assumption, the upper-bound assumption, the lower-bound assumption and the three chains are all assumptions. The one-sided bridge is a proposition on a finite spatial lattice with continuous time; towards the Wilson measure with discretised time and towards infinite volume, assumption rows remain. What was checked in Lean is only the algebra from assumptions to conclusions.

The order of this chapter

1. The conventions of this chapter — symbols, and the unit of the ruler
2. From the problem to the line of construction — eight statements (Table 1) and how they connect
3. The direction of the lattice version of the mass gap — a uniform lattice gap is on the opposite side (Table 2)

4. Restating with the Poincaré constant — three C_P (Table 3) and the free field (Table 4)
5. The relaxation-time assumption — (1) what we want to measure, (2) why we restate it, (3) the band statement, the power statement, the upper and lower halves (Table 5)
6. What the relaxation-time assumption gives and what it does not — (4) the one-sided bridge (Table 6), (5) a counterexample to the converse (the Free-Particle-on-a-Circle Counterexample) and the strong-coupling side (Table 7), (6) analogous cases, classification and the dependency graph (Table 8)
7. Detour: the number of rescalings and four clocks — logarithms that do not show in the power (Tables 9 and 10)
8. What of the line is in hand — on to 23-02 and 23-03

01

The conventions of this chapter — symbols, and the unit of the ruler

This chapter is about the side of the four-dimensional lattice. The group is $SU(2)$, the Wilson action is $S_W = \beta \sum_P (1 - \frac{1}{2} \operatorname{tr} U_P)$ (5-01), and $\beta = 4/g^2$ is the inverse of the coupling, not an inverse temperature. All quantities are counted in lattice units (lattice spacing 1).

$m_{\mathrm{lat}}(\beta)$ is the gap of the transfer matrix on gauge-invariant states (the mass in lattice units), $\xi = 1/m_{\mathrm{lat}}$ the correlation length, and $a(\beta)$ the lattice spacing measured in physical units. The mass in physical units is

$$\xi = \frac{1}{m_{\mathrm{lat}}}, \qquad \Delta(\beta) = \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)} \quad (1)$$

There is more than one way to fix $a(\beta)$. If the string tension σ is taken as the physical unit, $a\sqrt{\sigma}$ is the lattice spacing; in the four-dimensional record of this part, a is set to the reciprocal of a length fixed by counting the steps that coarsen the lattice, the bookkeeping length ξ_{RG} (§ 07). Which one is taken goes into the assumptions of the one-scale chain and the bookkeeping-length chain among the three chains of § 05.

Example: if $m_{\mathrm{lat}} = 0.514$ at $\beta = 2.60$ (a Monte Carlo value of Lucini–Teper), then $\xi = 1.95$ lattice spacings Reading . Δ is a quantity fixed only when this is divided by a; the numbers on the lattice alone do not fix it (5-09).

02

From the problem to the line of construction — eight statements

The problem statement (0-03) asks that a non-trivial quantum Yang–Mills theory exist on $\mathbb{R}^4$, and that the theory have a mass gap $\Delta > 0$. On the road entering from the lattice, we line this up as a single line.

Lattice	Infinite volume	Continuum limit	OS axioms	Mass gap
The Wilson measure on a finite box	Make the box infinite the infinite-volume uniqueness statement, the	$\beta \to \infty$ $(a \to 0)$ the Wilson-loop limit statement, the rotation-recovery statement	Hilbert space and Hamiltonian	The gap in physical units is positive

a probability measure for each β and each volume	reflection-positivity preservation statement		the OS reconstruction statement, the non-triviality statement	the physical-unit gap statement

The order is to make the volume infinite first for the measure at each β, and then take $\beta \to \infty$ (the order of Chatterjee 2018; there is also the order of taking $a \to 0$ first in a finite box, 16-07). Table 1 makes each stage of the line a statement, and the demand of the problem statement becomes the following overall picture.

$$\text{the demand of the problem statement} \impliedby (\mathrm{S1}) \wedge (\mathrm{S2}) \wedge (\mathrm{S3}) \wedge (\mathrm{S4}) \wedge (\mathrm{S6}) \wedge (\mathrm{S7}) \wedge (\mathrm{S8}) \quad (2)$$

(2) is an overall picture, not a theorem. The relaxation-time band at the bookkeeping length is not in (2); it is a restatement of the physical-unit gap statement (§ 04).

Table 1 The eight statements of construction and their current status in four dimensions (in formulas they are written with the labels (S1)–(S8))

Statement	Content	Current status in four dimensions	Chapter
Infinite-volume uniqueness statement ((S1) in formulas)	For each β, the infinite-volume limit measure is uniquely determined	Cited Limit points exist by compactness. At small β, existence of the limit (Osterwalder–Seiler 1978) and uniqueness (Shen–Zhu–Zhu 2023). At large β, 'in general unknown' (Chatterjee § 3)	5-04
Reflection-positivity preservation statement ((S2) in formulas)	A positive self-adjoint transfer operator. Preserved under weak limits	Cited Osterwalder–Seiler 1978, Lüscher 1977	5-03
Wilson-loop limit statement ((S3) in formulas)	As $a \to 0$, renormalised Wilson loops have a non-trivial limit	Unsolved (the continuum string-tension problem (Chatterjee's Problem 5.2))	5-09, 7-07
Physical-unit gap statement ((S4) in formulas)	$\Delta(\beta) \to \Delta_\infty \in (0, \infty)$	Unsolved (the half on the lattice side is the lattice correlation-length problem (Chatterjee's Problem 5.1))	4-11, 5-09
Relaxation-time band at the bookkeeping length (slowness of fluctuations) ((S5) in formulas)	$a^2 C_P \asymp 1$	Assumption The formulation of this part. There is no proof	§ 04, § 05
Rotation-recovery statement ((S6) in formulas)	In the limit, invariance under rotations and translations of $\mathbb{R}^4$ returns	Unsolved along this line (shown in two dimensions; for the side of the published construction, see the note below)	16-07
OS reconstruction statement ((S7) in formulas)	Build the Hilbert space and the Hamiltonian from the axioms	Cited A theorem (Osterwalder–Schrader). Along this line it remains that the limit satisfies the axioms (for the side of the published construction, see the note below)	4-10
Non-triviality statement ((S8) in formulas)	The limit is not Gaussian, and at short distances agrees with the predictions of asymptotic freedom	Unsolved. For four-dimensional ϕ^4 cut off by a lattice, the limit is Gaussian Cited	0-03, 7-08

Note (relation to the published construction). The "current status in four dimensions" of Table 1 is the status along the line of this chapter (for each β, first make the volume infinite, then β→∞). The construction published on 2026-11-10, with a different assembly (stacking steps of renormalisation to take the ultraviolet limit, and taking infinite volume along a subsequence), shows that a quantum field theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory, and the committee's assessment is that it is proved in Lean (Part 6). In that construction,

Euclidean invariance (recovery of rotations) and the OS axioms and reconstruction are shown for the theory of each phase (6-03). What is not shown is the convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap, Wilson loops and the area law, and one-particle states (6-04). The physical-unit gap statement and the Wilson-loop limit statement are open under either assembly.

The eight are not independent. Written with arrows meaning "the source is a prerequisite of the target", they are as follows (this book's arrangement; the arrow labelled with the band form of the diffusive-relaxation-time assumption holds only under the band statement of § 05).

$$(\mathrm{S1}) \to (\mathrm{S3}) \to (\mathrm{S6}),\ (\mathrm{S8}); \qquad (\mathrm{S1}),(\mathrm{S2}) \to (\mathrm{S4}) \xleftarrow{(\xi')\text{band}} (\mathrm{S5}); \qquad (\mathrm{S2}),(\mathrm{S6}) \to (\mathrm{S7}) \quad (3)$$

Along the line of this chapter, what is in hand in four dimensions is the reflection-positivity preservation statement and the OS reconstruction statement (the part that is the reconstruction theorem). One is tempted to say "once the limit can be taken, the rest is automatic", but the footnote of the problem statement does not accept weak existence by compactness alone as a solution unless the properties of the limit ('such as the existence of a mass gap and the axioms') are shown separately (Jaffe–Witten)Cited .

$$\underbrace{\mu_{\beta_j} \to \mu \ \ (\text{subsequence})}_{\text{compactness only}} \implies (\mathrm{S2})\ \text{survives};\ \ (\mathrm{S4})\cdot(\mathrm{S6})\cdot(\mathrm{S8})\ \text{nothing can be said} \quad (4)$$

Example. Three arrows enter the physical-unit gap statement, and only the one from the relaxation-time band at the bookkeeping length is an arrow under the band statement. Only the infinite-volume uniqueness statement enters the Wilson-loop limit statement because the order is to make the volume infinite first.

03

The direction of the lattice version of the mass gap — a uniform lattice gap is on the opposite side

If the "mass gap" on the lattice is read as "the gap in lattice units $m_{\mathrm{lat}}(\beta)$ is at least a fixed value for every β", in the continuum limit it means the opposite. As in (1), the mass in physical units is the mass in lattice units divided by the lattice spacing.

$$\inf_{\beta} m_{\mathrm{lat}}(\beta) \geq c > 0 \implies \Delta(\beta) \geq \frac{c}{a(\beta)} \to \infty \qquad (a \to 0) \quad (5)$$

Proved here Just divide (1) by $a(\beta)$. As a metaphor, if the graduations of a ruler are made finer and finer but the thing measured is always a few graduations long, the actual length of the thing shrinks to 0. Precisely, under (5), ξ is bounded, and the correlation length in physical units $a\xi$ goes to 0. What the physical-unit gap statement asks for is the opposite: that m_{lat} goes to 0 at the same rate as a.

$$(\mathrm{S4}): \quad m_{\mathrm{lat}}(\beta) \sim \Delta_{\infty}\, a(\beta), \qquad \text{in particular}\ \ \xi(\beta) \to \infty \quad (\beta \to \infty) \quad (6)$$

This direction is exactly the lattice correlation-length problem of Chatterjee (2018): 'Show that for any β > 0, there exists some ξ(β) ∈ (0, ∞) such that ... Moreover, prove that $\lim_{\beta\to\infty} \xi(\beta) = \infty$.' Cited How to fix the scale is in 5-09.

Example: Table 2 takes the string tension σ as the physical unit and regards $a\sqrt{\sigma}$ as the lattice spacing. Over $\beta = 2.25 \to 2.60$, a becomes 1/3.16 and m_{lat} 1/2.70, while $\Delta/\sqrt{\sigma}$ stays at 3.3–3.8. The last column is the case where m_{lat} stops at its value at $\beta = 2.25$; it increases in inverse proportion to a.

Reading Table 2 Monte Carlo values from Table 1 of Lucini–Teper (2001) ($SU(2)$, lattices 8^4 to 24^4), and their ratios (errors propagated as independent)

β	$a\sqrt{\sigma}$	m_{lat}	$\Delta/\sqrt{\sigma}$	Case with m_{lat} held
2.25	0.4231(25)	1.390(30)	3.285(74)	3.29
2.30	0.3108(17)	1.090(33)	3.507(108)	4.47
2.40	0.2634(14)	0.953(19)	3.618(75)	5.28
2.475	0.2016(13)	0.754(10)	3.740(55)	6.89
2.55	0.15896(63)	0.586(10)	3.686(65)	8.74
2.60	0.13395(62)	0.514(8)	3.837(62)	10.38

$\Delta/\sqrt{\sigma}$ mostly increases and approaches the extrapolated value 3.844(61) of the original from below (it drops once, within errors, over $\beta = 2.475 \to 2.55$) Reading . The window is $\xi \lesssim 2$ lattice spacings, and it is not evidence that the limit exists (16-08).

04

Restating with the Poincaré constant — the statement of the slowness of fluctuations, and three C_P

We restate the physical-unit gap statement with the Poincaré constant C_P of the lattice measure μ_β. It is the supremum of the ratio of the variance of an observable to its sensitivity to the motion of the links (the Dirichlet form $\mathcal{E}$), and it is the time over which the slowest fluctuation is smoothed out in Langevin dynamics, which shakes the links while moving them towards lower action (1-10, 23-00 § 04).

$$C_P = \sup_f \frac{\mathrm{Var}_{\mu_\beta}(f)}{\mathcal{E}(f)}, \quad \mathcal{E}(f) = \mathbb{E}_{\mu_\beta}\Big[\sum_\ell |\nabla_\ell f|^2\Big], \qquad (\mathrm{S5}):\ a(\beta)^2\, C_P(\beta) \asymp 1 \ (7)$$

∇_ℓ is the derivative in the direction of link ℓ (measured in the angle of the link), and $A \asymp B$ means that the ratio is bounded above and below by positive constants independent of β. Langevin time has the dimension of length squared (diffusion), so $a^2 C_P$ is the slowest relaxation measured in physical units, and the relaxation-time band at the bookkeeping length says it goes neither to 0 nor to infinity.

Three quantities are called by the same symbol C_P, so we distinguish them (Table 3). The C_P of this part is the path C_P, the Poincaré constant of the path measure (the distribution of all links in spacetime).

Table 3 Three C_P (the free-field column is for mass m with continuous time)

	Measure	Quantity it is tied to	Free field
Vacuum C_P	The vacuum distribution at a single time $\lvert\psi_0\rvert^2$	Gap of the Hamiltonian $\Delta = 1/(2C_P^{(\alpha)})$ Cited (ground-state transformation, 2-05, 1-10)	$1/(2m)$
Path C_P	The distribution of all fields in spacetime (paths)	The Langevin gap $\lambda_1 = 1/C_P$. Tied to m only through the relaxation chain	$1/m^2$
Path C_P measured in a graduation	The same measure as the path C_P	The graduation of $\mathcal{E}$. Multiplying $\mathcal{E}$ by c multiplies C_P by $1/c$	In angles $C_P^{(\vartheta)} = \frac{q^2}{4}\xi^2$, in fields $C_P^{(A)} = \xi^2$

For the free field $C_P^{(\beta)} = 4\,(C_P^{(\alpha)})^2$ Proved here (just square the free-field column of Table 3; on a time lattice of spacing 1 the right-hand side is multiplied by $1 + m^2/4$). The relaxation-time assumption is a sentence about the path C_P, so it can be reread as "the C_P of the paths agrees with the square of the C_P of the vacuum". In a gauge theory, however, the measure at a single time lives on the space of gauge orbits, and this rereading is an analogy from the form of the free field.

For the free field (with the graduation measured in field variables), C_P is the largest eigenvalue of the covariance $(-\Delta_{\text{lat}} + m_{\text{lat}}^2)^{-1}$ (1-10)Cited .

$$C_P = \frac{1}{m_{\text{lat}}^2} \quad \Longrightarrow \quad a^2 C_P = \frac{1}{\Delta^2} \ (8)$$

For the free field, the physical-unit gap statement and the relaxation-time band at the bookkeeping length say the same thing, and a uniform lattice gap $m_{\text{lat}} \geq c$ is $C_P \leq 1/c^2$, that is, the side of $a^2 C_P \to 0$. Table 4 is an example with $\Delta = 1$ held fixed and a made smaller.

Proved here Table 4 The free field on the four-dimensional lattice ($C_P = 1/m_{\text{lat}}^2$; ξ from $m_{\text{lat}}^2 = 2(\cosh(1/\xi) - 1)$. Values of the closed formulas)

a	m_{lat}	C_P	ξ	$a^2 C_P$	$a\xi$
1	1	1	1.0390	1	1.0390
1/4	0.25	16	4.0104	1	1.0026
1/16	0.0625	256	16.003	1	1.0002
2^{-k}	1 (held)	1	1.0390	4^{-k}	$1.039 \cdot 2^{-k}$

Each halving of a makes C_P four times and ξ twice as large, with $a^2 C_P = 1$ and $a\xi \to 1$. If the mass in lattice units is held at 1, $\Delta = 2^k$ diverges while $C_P = 1$ stays. Whether C_P and ξ^2 grow with the same power in a theory with interaction is the relaxation-time assumption of the next section.

Detour: two times

A subsection you may skip. For the free field the gap of the transfer matrix (Euclidean time) is $\approx m_{\text{lat}}$, and the gap $1/C_P$ of Langevin dynamics (the "fifth time") is m_{lat}^2. As a metaphor, it is the difference between the distance a wave reaches and the time ink takes to smear; precisely, Langevin dynamics is diffusion, and evening out a length ξ takes on the order of ξ^2 (18-03, the detour of 23-00).

05

The diffusive-relaxation-time assumption — what we want to measure, why we restate it, what it says

In one sentence, the relaxation-time assumption is the sentence "the time over which fluctuations are smoothed out grows with the same power as the square of the correlation length". In this section and the next we explain this sentence in six stages: (1) what we want to measure, (2) why we restate it as a quantity of a dynamics, (3) the statement, step by step, (4) what the relaxation-time assumption gives, (5) what the relaxation-time assumption does not give, (6) analogous cases. Each stage has a small example that can be checked by hand.

Its place in this part. The relaxation-time assumption is one restatement chosen in order to get a handle on the problem of the mass gap. It is neither the only road to the mass gap nor a road settled to be right. Viewpoints other than the relaxation-time assumption, and new ways of thinking that take the relaxation-time assumption as a premise, are dealt with in Part 24.

Why this part puts the relaxation-time assumption at the axis. The quantity the problem statement asks for is the gap in the direction of Euclidean time (the gap of the transfer matrix measured in physical units). The relaxation-time assumption replaces it with a quantity — the relaxation time C_P of Langevin dynamics and a power of the correlation length — that is fixed by the lattice measure alone, for which

tools exist to assemble inequalities "for every f" from local parts, and whose algebra of assembly can be checked by machine. As we see in § 06, this replacement can return to the side of the mass gap through the one-sided bridge; moreover, under the assumption of the one-sided bridge, the upper-bound assumption, the upper half of the relaxation-time assumption, is a stronger sentence than the gap in physical units on the lattice staying positive. Saying a stronger sentence in a language with more tools: this trade is the standpoint of this part, and all the chapters of Part 23 are built with the relaxation-time assumption at the axis. That is what the title of this part, a study taking the relaxation-time assumption as a handle, means.

(1) What we want to measure — the ruler and the smearing time

The part of the demand of the problem statement that can be stated on the lattice is (6): "the gap in physical units $\Delta = m_{\mathrm{lat}}/a$ stays positive as $\beta \to \infty$". Seen on the lattice, two things must happen at once: the ruler $\xi = 1/m_{\mathrm{lat}}$ grows without bound compared with the graduations of the lattice, and the way it grows matches the inverse of the lattice spacing ($a\xi$ goes neither to 0 nor to infinity). In terms of the five quantities of 23-00 § 05, the problem statement is a sentence asking about the relation between the ruler and the number of rescalings.

The ruler	The smearing time	The number of rescalings	The graduation of the clock	The clock that runs slow
The correlation length $\xi = 1/m$ lat the reciprocal of the gap in Euclidean time	The Poincaré constant C_P the reciprocal of the gap of Langevin dynamics	The bookkeeping length $\xi_{\mathrm{RG}} = 1/a$ the reciprocal of the lattice spacing (§ 07)	The normalisation of $\mathcal{E}$ the path C_P measured in a graduation, Table 3	Renormalisation of stochastic time the detour of § 07

The relaxation-time assumption brings the smearing time into this question. The time C_P over which the slowest fluctuation in the measure is smoothed out is a diffusion time, so evening out a length ξ should take on the order of ξ^2 (the detour of 23-00). So we make "the smearing time balances the square of the ruler" into one sentence, and state the relation between the ruler and the number of rescalings through the smearing time. That is the idea of the relaxation-time assumption.

Example 1 (counting the random walk on a single road by hand). On the single road of 23-00 § 04 (L cells, reflecting at both ends, uniform distribution, $\mathcal{E}(f) = \frac{1}{L}\sum_i (f_{i+1} - f_i)^2$), what corresponds to the ruler is the length of the road L. For $L = 2$, $f = (1, -1)$ is the slowest fluctuation, with $\mathrm{Var}(f) = 1$ and $\mathcal{E}(f) = \frac{1}{2} \cdot 2^2 = 2$, and the ratio is $\frac{1}{2}$. For $L = 3$, with $f = (1, 0, -1)$, $\mathrm{Var}(f) = \frac{2}{3}$ and $\mathcal{E}(f) = \frac{1}{3}(1 + 1) = \frac{2}{3}$, and the ratio is 1. Both agree with the values of $C_P(L) = 1/(2(1 - \cos(\pi/L)))$ from (3) of 23-00, and for $L = 4$ it is $1/(2 - \sqrt{2}) \approx 1.707$ Proved here . Doubling L makes C_P roughly four times as large, and for large L, $C_P \approx L^2/\pi^2$ (Table 3 of 23-00). This checks "the smearing time is the square of the ruler" on the smallest road.

Precisely, the L of the single road is a length given from outside, not a ruler the measure carries itself (a correlation length). The simplest example where the two agree is the free field: from $C_P = 1/m_0^2$ and $m_0^2 = 2(\cosh(1/\xi) - 1)$, $C_P/\xi^2 \to 1$ (23-00 § 04, Table 4 of this chapter). How the smearing time and the ruler balance in a measure with interaction is the question of the relaxation-time assumption.

(2) Why restate it as a quantity of a dynamics — measurable, with tools, within reach of checking

The quantity on the side of the ruler, m_{lat}, is the gap of the transfer matrix, a quantity of an operator on the Hilbert space of gauge-invariant states. Placing one trial state gives an upper bound on m_{lat} (a lower bound on ξ), but a lower bound needs all states. An upper bound on C_P also needs "every f", so the difficulty remains in the same direction. There are still three reasons to restate it.

(a) It can be measured from the lattice measure alone. As in (7), C_P is fixed by expectation values under μ_β alone: the variance and the Dirichlet form. It can be defined directly from the measure, without going through the Hilbert space or reconstruction; one test function gives a lower bound, and readings of lower bounds can also be taken from the autocorrelations of Langevin dynamics or Markov chains (23-02 § 04).

(b) There are tools to assemble "every f". For Poincaré inequalities, tools that assemble an upper bound on the whole from local parts are available in a form independent of dimension. With the conventions measure $\mu \propto e^{-S}$ and $\mathcal{E}(f) = \mathbb{E}|\nabla f|^2$,

$$\mathrm{Hess}\, S \geq \rho \;\Rightarrow\; C_P(\mu) \leq \frac{1}{\rho}; \qquad \mu' \propto e^{-W}\mu \;\Rightarrow\; C_P(\mu') \leq e^{\operatorname{osc} W}\, C_P(\mu); \qquad C_P(\mu_1 \otimes \mu_2) = \max\bigl(C_P(\mu_1),$$

The two on the left are Cited (the Bakry–Émery convexity criterion and the Holley–Stroock bounded perturbation; $\operatorname{osc} W = \sup W - \inf W$. Bibliographic data in the table of sources of 23-06), and the one on the right is Proved here (use the Poincaré inequality of each component on each term of the Efron–Stein inequality $\mathrm{Var}(f) \leq \mathbb{E}\,\mathrm{Var}_1(f) + \mathbb{E}\,\mathrm{Var}_2(f)$ to get $\leq \max$; equality for a function of one coordinate only). Adding to these the decomposition of the variance (the sum of the conditional variance and the variance of the conditional expectation, 23-05 § 01), these four are the parts from which the first step is assembled. The logarithmic Sobolev inequality (LSI) has tools of the same type, among them convexity criteria split by scale (23-02 § 05).

(c) Machine checking can reach it. The stage of combining the inequalities of the parts is algebra of real inequalities. The algebra can be checked in Lean, and numerical inequalities inside a finite window can be confirmed by interval arithmetic (computer-assisted). What is checked is the assembly from assumptions to conclusions, not the assumptions themselves (23-13).

Example 2 (where the tools work, for a Gaussian measure on two sites). Consider the measure $\mu \propto e^{-S}$ on $\mathbb{R}^2$, with $S(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2 + \frac{1}{2}m_0^2(x_1^2 + x_2^2)$. The Hessian is $\begin{pmatrix} 1+m_0^2 & -1 \\ -1 & 1+m_0^2 \end{pmatrix}$, with eigenvalues m_0^2 (direction $(1,1)$) and $m_0^2 + 2$ (direction $(1,-1)$). The criterion on the left of (9) gives $C_P \leq 1/m_0^2$. On the other hand, for $f = x_1 + x_2$, since the covariance is the inverse of the Hessian, $\mathrm{Var}(f) = 2/m_0^2$ and $\mathcal{E}(f) = |\nabla f|^2 = 2$, and the ratio is $1/m_0^2$. Upper and lower bounds agree, and $C_P = 1/m_0^2$ Proved here. For the single road (Example 1) we found C_P by diagonalising a matrix; in this example (9) gives the answer without diagonalisation. For the double well $V(x) = (x^2-1)^2/8$, at the top $V''(0) = -\frac{1}{2} < 0$, and the convexity criterion of (9) cannot be used on the whole. So one combines the decomposition of the variance, splitting valley by valley, with convexity and perturbation after the split (23-04 § 06, 23-05 § 04). That an upper bound can be assembled from parts for a measure with interaction that cannot be diagonalised — this is the first gain of the restatement.

(3) The statement of the relaxation-time assumption, step by step — the band statement, the power statement, the upper and lower halves

First step: the band statement. Written plainly, it becomes the sentence "C_P lies within the width of a constant factor of the square of the ruler". Comparing with the physical unit a and a width of constant

factors independent of β (a band),

$$\exists\,\beta_0,\ \exists\, 0<c\le C:\qquad c\le a(\beta)^2\,C_P(\beta)\le C\qquad(\beta\ge\beta_0)\ \text{(10)}$$

In numbers, it is the sentence that when $\xi_{\mathrm{RG}}=1/a=10$, C_P lies between $100c$ and $100C$, and if $\xi_{\mathrm{RG}}=1000$, between 10^6c and 10^6C. Langevin time has the dimension of length squared (diffusion), so a^2C_P is the slowest relaxation measured in physical units, and (10) says it goes neither to 0 nor to infinity.

Example 3 (the single road satisfies the band statement). Taking L as the ruler for the single road of Example 1 and setting $x=\pi/L$, $C_P(L)/L^2=\frac{1}{\pi^2}\cdot\frac{(x/2)^2}{\sin^2(x/2)}$. Since $t/\sin t$ increases on $(0,\pi/2]$, for $L\ge 2$

$$\frac{1}{\pi^2}\;<\;\frac{C_P(L)}{L^2}\;\le\;\frac{1}{8}\ \text{(11)}$$

(the equality on the right is $C_P=\frac12$ at $L=2$), and the band statement holds with $c=1/\pi^2\approx 0.101$, $C=1/8$ Proved here .

Inserting the correlation length, the band statement splits into three factors.

$$a^2C_P=\underbrace{(a\xi)^2}_{\text{chains (ii)(iii)}}\cdot\underbrace{\frac{C_P}{\xi^2}}_{\text{chain (i)}},\qquad (a\xi)^2=\frac{1}{m_{\mathrm{phys}}^2},\qquad \frac{C_P}{\xi^2}=\frac{m_{\mathrm{fat}}^2}{\lambda_1}\ \text{(12)}$$

The first factor is determined by the mass in physical units $m_{\mathrm{phys}}=m_{\mathrm{lat}}/a$, and the second is the factor of the relaxation chain, "the Langevin gap λ_1 agrees with the square of the mass". Further splitting the first factor into "the square of the mass agrees with the string tension" (the one-scale chain) and "the string tension agrees with the square of the lattice spacing" (the bookkeeping-length chain), the three ratios are the three chains (17-12). All three are assumptions Assumption . What the upper and lower halves of each chain need is seen in 23-02 § 05.

Second step: the power statement. The band statement is moved by the logarithmic factors brought in by the graduation of $\mathcal{E}$, the order of the bookkeeping and the renormalisation of stochastic time (the detour of § 07). Lowering it to a level that ignores not only constant factors but also factors of powers of $\ln\xi$ gives the relaxation-time assumption.

$$(\xi^2):\qquad \lim_{\beta\to\infty}\frac{\ln C_P(\beta)}{\ln\xi(\beta)}=2\ \text{(13)}$$

This sentence has the same form as (2) of the statement of the first step, 23-03 § 05, $\lim_{\hbar\to0}\ln C_P(\hbar)/\ln\xi(\hbar)=2$. Only the family of measures differs: here the Wilson measure μ_β with $\beta\to\infty$, on the first step the path measure of the double well with $\hbar\to0$. So in this part (13) is treated as a dimension-independent power statement, called "the diffusive-relaxation-time assumption" at the stage of the four-dimensional lattice and "the double-well relaxation-time power statement" on the first step. Being a power statement, it does not change when $\mathcal{E}$ is multiplied by a power of β ($\ln\beta/\ln\xi\to0$) Proved here Lean `exponent_invariant_of_slowly_varying` (assumptions: $X_n,\ell_n>0$, $\ln X_n/\ln a_n\to p$, $\ln\ell_n/\ln a_n\to0$. Conclusion: $\ln(X_n\ell_n)/\ln a_n\to p$. Algebra of sums of limits).

Example 4 (the difference between band and power, in numbers). Consider the family $C_P=\xi^2\ln\xi$. The ratio $C_P/\xi^2=\ln\xi$ keeps increasing, $2.30,\ 6.91,\ 13.8$ at $\xi=10,10^3,10^6$, so the band statement breaks. On the other hand $\ln C_P/\ln\xi=2+\ln\ln\xi/\ln\xi$ approaches 2 slowly, $2.362,\ 2.280,\ 2.190$, and the power statement holds Proved here (divide the values $0.834,\ 1.933,\ 2.626$ of $\ln\ln\xi$ by $\ln\xi=2.303,\ 6.908,\ 13.82$). Conversely the single road of Example 3 satisfies the band statement, but $\ln C_P/\ln L=2-(\ln\pi^2$

$+ o(1))/\ln L$ is $\ln 6640/\ln 256 \approx 1.587$ at $L = 256$, far from 2. The power statement says only "2 in the limit", and allows a slow approach by the amount of the constants. The band statement implies the power statement ($\ln c/\ln \xi \to 0$), and the converse fails, as $C_P = \xi^2 \ln \xi$ shows Proved here .

(10) is a sentence that combines the relaxation-time assumption with the one-scale chain and the bookkeeping-length chain, and in this book it is not called the relaxation-time assumption. The names are fixed as in Table 5.

Table 5 Readings and names of "C_P agrees with the square of a length" (eight detailed readings in A-06)

Level, length compared	Sentence	Name in this book	In the free approximation
Power, ξ	$\ln C_P/\ln \xi \to 2$	The diffusive-relaxation-time assumption	Holds
Power, bookkeeping length	$\ln C_P/\ln \xi_{\mathrm{RG}} \to 2$	The relaxation-time assumption with the one-scale chain and the bookkeeping-length chain (power)	Holds
Band, ξ	$C_P \asymp \xi^2$	$(\xi^2)_{\mathrm{band}}[\mathcal{E}_\vartheta]$, $(\xi^2)_{\mathrm{band}}[\mathcal{E}_A]$ (naming the graduation)	Equality in the field graduation; in the angle graduation the lower half breaks with ratio $g^2/4$
Band, bookkeeping length	$a^2 C_P \asymp 1$ ((10))	Band statement with the one-scale chain and the bookkeeping-length chain = the relaxation-time band at the bookkeeping length	Determined by the order of the bookkeeping (§ 07)

The two rows at the power level are moved neither by the choice of graduation nor by the logarithmic factors of § 07. That is why this part places the relaxation-time assumption as a power statement.

Third step: upper and lower halves. Separating the right and left of the band statement (10) gives the upper-bound assumption and the lower-bound assumption.

$$(\xi^2)\uparrow:\ a^2 C_P \le C\ \ (C_P \lesssim \xi_{\mathrm{RG}}^2), \qquad (\xi^2)\downarrow:\ a^2 C_P \ge c\ \ (C_P \gtrsim \xi_{\mathrm{RG}}^2) \quad (14)$$

C_P is the supremum of (7), and $\xi_{\mathrm{RG}} = 1/a$ is a length fixed separately from the measure, so the two have opposite quantifiers over the test function f.

$$(\xi^2)\downarrow \iff \forall \beta \ge \beta_0\ \exists f:\ \mathrm{Var}(f) \ge c\,\xi_{\mathrm{RG}}^2\,\mathcal{E}(f), \qquad (\xi^2)\uparrow \iff \forall \beta \ge \beta_0\ \forall f:\ \mathrm{Var}(f) \le C\,\xi_{\mathrm{RG}}^2\,\mathcal{E}(f) \quad (15)$$

Proved here This just writes out the definition of the supremum. The lower-bound assumption needs only one test function to be shown; the upper-bound assumption is a genuine Poincaré inequality over all functions. "Easy from above, hard from below" for eigenvalues (1-10) becomes, for their reciprocal C_P, "easy from below, hard from above". This asymmetry determines the whole of the roads and walls of 23-02.

Example 5 (families where only one holds). In the family $C_P = \xi_{\mathrm{RG}}^2 \ln \xi_{\mathrm{RG}}$, for $\xi_{\mathrm{RG}} \ge e$ we have $a^2 C_P \ge 1$, so the lower-bound assumption holds with $c = 1$, and since $a^2 C_P = \ln \xi_{\mathrm{RG}} \to \infty$, the upper-bound assumption breaks (the power statement holds, as in Example 4). Conversely, in a family where C_P is bounded independently of β (the form of the theorems on the strong-coupling side, Table 7 in § 06), the upper-bound assumption holds and the lower-bound assumption breaks Proved here . The band statement is the sentence that neither of these two ways of breaking happens.

Note — "the relaxation-time assumption" and "the four double-well conditions"

In this part "the relaxation-time assumption" refers only to the power statement (13). The upper-bound assumption and the lower-bound assumption are the halves of the band statement, with

names separate from the relaxation-time assumption. The four conditions into which the upper side of the double-well power statement is split on the first step are written "the four double-well conditions", always with the word "conditions" (23-03 § 05, 23-05). The in-phase local Poincaré condition is not "the first half of the relaxation-time assumption" but a different statement, "local Poincaré within a phase" for the model of the first step. The letters in Lean theorem names (H1 and so on) are not changed.

Note — topological sectors, and the order of making the volume infinite

The supremum of (7) runs over every f, so the (smoothed) topological charge Q, which can no longer move between sectors in the continuum limit, is also among the test functions. On four-dimensional lattices with periodic boundaries, the autocorrelation time of Q is reported to grow clearly faster than ξ^2 Reading (Del Debbio–Panagopoulos–Vicari 2002 with local heat bath and overrelaxation, Schaefer–Sommer–Virotta 2011 with HMC-type algorithms, Bonanno–Bonati–D'Elia 2021 with local algorithms; as a power the exponent is about 5, or else exponential in $1/a$, and all three write that power and exponential cannot be distinguished by the data). The groups of the three original sources are all $SU(3)$ or larger, and whether the exponent exceeds 2 for $SU(2)$, where this book's relaxation-time assumption is stated, is not settled in the primary sources found. There is one primary source found that measures the dependence of the autocorrelation of the topological charge on the lattice spacing for $SU(2)$ (Abe–Morikawa–Suzuki 2025; HMC, periodic boundaries, small boxes with the physical side held fixed, three lattice spacings). There the integrated autocorrelation time is reported to grow close to a^{-2} Reading . So the following reading is a reading on the original sources for $SU(3)$ and larger. For reversible dynamics, the integrated autocorrelation time of any observable does not exceed C_P ($\tau_{\mathrm{int}} \le C_P$ Cited , 23-02 § 05), so under the conjecture that HMC and Langevin are in the same universality class Assumption , the upper-bound assumption can be read as false in a periodic box. Open boundaries remove this obstacle only inside a finite box with the physical side held fixed (in Lüscher–Schaefer 2011, the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction Reading). In this book's order, which makes the volume infinite first, this is expected not to help Assumption . If the relaxation-time assumption is to be kept, candidate directions are to rewrite it as a statement sector by sector, or as a local statement with support limited to a few correlation lengths; that study is dealt with in Part 24 (24-03, 24-04).

06

What the diffusive-relaxation-time assumption gives and what it does not — the one-sided bridge, a counterexample, analogous cases

The remaining three of the six stages: (4) what the relaxation-time assumption gives, (5) what the relaxation-time assumption does not give, (6) analogous cases. (4) and (5) start from the fact that the quantity of the problem statement (the gap in Euclidean time) and the quantity of the relaxation-time assumption (the Langevin relaxation time) are different quantities, and fix precisely the relation between the relaxation-time assumption and the mass gap through the one-sided bridge joining the two and an example in which the converse of the bridge fails.

(4) What the relaxation-time assumption gives — rereading as a mass, and the one-sided bridge

The version with equality. If the relaxation chain is put as the equality $C_P = 1/m_{\mathrm{lat}}^2$, the two halves of (14) are reread as statements about the mass in physical units $m_{\mathrm{phys}} = m_{\mathrm{lat}}\,\xi_{\mathrm{RG}}$.

$$(\xi^2)\uparrow \implies m_{\mathrm{phys}} \;\geq\; C^{-1/2}, \qquad (\xi^2)\downarrow \implies m_{\mathrm{phys}} \;\leq\; c^{-1/2} \quad (16)$$

Proved here Proof: multiply both sides of $1/m_{\mathrm{lat}}^2 \leq C\xi_{\mathrm{RG}}^2$ in the upper-bound assumption by m_{lat}^2/C to get $1/C \leq (m_{\mathrm{lat}}\xi_{\mathrm{RG}})^2$. For the lower-bound assumption, the same with the inequality reversed. ∎ The level of proof is algebra, and two Lean theorems check this algebra.

```
theorem hup_is_gap_positivity {CP xi mlat K : ℝ}
    (hmlat : 0 < mlat) (hCP : CP = 1 / mlat ^ 2) (h : CP ≤ K * xi ^ 2) :
    1 ≤ K * (mlat * xi) ^ 2

theorem hdown_is_mass_finiteness {CP xi mlat c : ℝ}
    (hmlat : 0 < mlat) (hCP : CP = 1 / mlat ^ 2) (h : c * xi ^ 2 ≤ CP) :
    c * (mlat * xi) ^ 2 ≤ 1
```

Lean `LatticeGauge.hup_is_gap_positivity` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, $C_P \leq K\xi^2$. Conclusion: $1 \leq K(m_{\mathrm{lat}}\xi)^2$) and `LatticeGauge.hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, $c\,\xi^2 \leq C_P$. Conclusion: $c\,(m_{\mathrm{lat}}\xi)^2 \leq 1$). Inside Lean all four letters are named real numbers; that it is the Poincaré constant of the Wilson measure, and that C_P is a supremum (running over every f), are outside Lean (23-13). The upper-bound assumption is "the mass does not collapse to 0", that is, a lower bound on the side of the mass gap; the lower-bound assumption is the side "the mass does not escape to infinity".

The equality $C_P = 1/m_{\mathrm{lat}}^2$ is the value for the free field, and is too strong an assumption to put down for a theory with interaction. However, the proof that derives a lower bound on the mass from the upper-bound assumption uses only one side of the equality, $C_P \geq \kappa/m_{\mathrm{lat}}^2$. And on a lattice with continuous time, this one side can be shown as a proposition with its assumptions named.

The one-sided bridge (continuous time, finite spatial lattice, $T = \infty$)

Take the space to be a finite lattice, the space of configurations at a single time to be Q ($\mathbb{R}^n$, or a compact connected manifold), the metric of the kinetic term M, $H = -\frac{\hbar^2}{2}\Delta_M + V$, $A = (H - E_0)/\hbar$. Assume the following. (A1) The ground-state assumption: the spectrum of H is discrete, the ground state ψ_0 is non-degenerate, smooth and positive, and $\Delta = E_1 - E_0 > 0$. (A2) The stationary path-measure assumption: the path measure is the stationary measure at $T = \infty$ (the distribution at a single time is ψ_0^2, the transition is $e^{-\tau A}$). (A3) The metric comparison assumption: the metric of the Langevin Dirichlet form $\mathcal{E}_G(f) = \hbar \int \mathbb{E}\langle d_{q(\tau)}f, G\, d_{q(\tau)}f\rangle d\tau$ satisfies $G \leq \lambda_G M^{-1}$, and $\kappa := 1/\lambda_G$. Then, for $\xi = \hbar/\Delta$, $F = \psi 1/\psi 0$ and $f_T = \int_0^T F(q(\tau))\,d\tau$, for every $T > 0$

$$\frac{\mathrm{Var}(f_T)}{\mathcal{E}_G(f_T)} \;\geq\; \kappa\xi^2\Big[1 - \frac{\xi}{T}\big(1 - e^{-T/\xi}\big)\Big], \qquad \text{hence} \qquad C_P \;\geq\; \kappa\xi^2 = \frac{\kappa\hbar^2}{\Delta^2} \quad (17)$$

holds. With the canonical choice $G = M^{-1}$ ($\kappa = 1$), the supremum as $T \to \infty$ over the family of test functions local in time is exactly ξ^2.

Proved here Proof [Y3]. Set $m = \Delta/\hbar = 1/\xi$ and $\varphi = (F - \langle F\rangle)\psi_0$. Since $\langle F\rangle = \int \psi_0\psi_1 = 0$, $\varphi = \psi_1$ and $A\varphi = m\varphi$.

(i) The variance. Under the stationary path-measure assumption (A2), the correlations of the measure are $\mathrm{Cov}(F(q(s)), F(q(t))) = \langle \varphi, e^{-|s-t|A}\varphi\rangle = \|\varphi\|^2 e^{-m|s-t|}$ Cited (the formula writing correlations with the transfer operator, of the type of (12) in 2-07 § 08 and 23-04 § 04). Integrating over the square $[0,T]^2$ gives $\mathrm{Var}(f_T) = \|\varphi\|^2\big(2T/m - 2(1-e^{-mT})/m^2\big)$ (close $\iint e^{-m|s-t|} = 2\int_0^T (T-u)e^{-mu}du$ with one integration by parts).

(ii) The Dirichlet form. Differentiating f_T with respect to the path $q(\tau)$ gives $dF(q(\tau))$ for $\tau \in [0,T]$ and 0 otherwise. From stationarity and the metric comparison assumption (A3), $\mathcal{E}_G(f_T) = \hbar T \int \psi_0^2 \langle dF, G\, dF\rangle \le \hbar T \lambda_G \int \psi_0^2 |dF|_M^2 - 1$.

(iii) The ground-state transformation identity. For a smooth real function F,

$$\big\langle F\psi_0, (H - E_0)F\psi_0\big\rangle = \frac{\hbar^2}{2}\int \psi_0^2\, |dF|_M^2 - 1 \quad (18)$$

Proved here. Using $(V - E_0)\psi_0 = \frac{\hbar^2}{2}\Delta_M\psi_0$ gives $(H-E_0)(F\psi_0) = -\frac{\hbar^2}{2}\big(\psi_0\Delta_M F + 2\langle dF, d\psi_0\rangle_M - 1\big) = -\frac{\hbar^2}{2\psi_0}\,\mathrm{div}_M(\psi_0^2\,\nabla_M F)$, and multiplying by $F\psi_0$ and integrating by parts once gives the right-hand side (boundary terms vanish because Q is compact without boundary, or the functions decay at infinity). ∎ It has the same form as the one-dimensional calculation of 1-10 § 08 and 2-05 § 08. Since $F\psi_0 = \psi_1$, the left-hand side is $\Delta\|\varphi\|^2$, and $\int \psi_0^2|dF|_M^2 - 1 = (2m/\hbar)\|\varphi\|^2$; hence $\mathcal{E}_G(f_T) \le 2\lambda_G m T\|\varphi\|^2$.

(iv) The ratio. Dividing (i) by (ii)(iii) gives $\mathrm{Var}/\mathcal{E}_G \ge \kappa\big(1/m^2 - (1-e^{-mT})/(Tm^3)\big)$, which is the left of (17). C_P is a supremum, so it is at least the right-hand side, and as $T\to\infty$, $C_P \ge \kappa\xi^2$. For the supremum in the canonical case, any f local in time gives $\mathrm{Var}/T \to 2\langle\varphi, A^{-1}\varphi\rangle$ and $\mathcal{E}/T = 2\langle\varphi, A\varphi\rangle$, and using $1/a \le a/m^2$ for the eigenvalues $a \ge m$ of A it is at most ξ^2, with equality when φ lies in the eigenspace of eigenvalue m (the same as (13) of 23-04 § 04). ∎

Lean `LatticeGauge.HupBridge.cont_ratio_closed` (assumptions: $0 < m$, $0 < T$. Conclusion: the ratio of (i) divided by (ii)(iii) equals $1/m^2 - (1-e^{-mT})/(Tm^3)$) and `one_sided_bridge` (assumptions: a lower bound on the variance, an upper bound on the Dirichlet form, $\mathrm{Var}/\mathcal{E} \le C_P$, positive numbers. Conclusion: $\kappa/m^2 \le C_P$) check only the algebra of numbers in (i)–(iv). The path measure, the formula for correlations, (18), and that C_P is a supremum are outside Lean.

Table 6 The assumptions of the one-sided bridge, and where they are used

Symbol	Assumption	Where it is used
The ground-state assumption (A1)	Discrete spectrum, $\Delta > 0$, $\psi_0 > 0$ smooth and non-degenerate, ψ_1 exists	$F = \psi_1/\psi_0$ can be defined and is smooth
The stationary path-measure assumption (A2)	The stationary path measure at $T = \infty$ (the distribution at a single time is ψ_0^2, the transition is $e^{-\tau A}$)	The formula for correlations in (i)
The metric comparison assumption (A3)	$G \le \lambda_G M^{-1}$ ($\kappa = 1/\lambda_G$)	The comparison in (ii)
Continuous time	The kinetic term of the paths has the form $\frac{1}{2}\lvert\dot q\rvert_M^2$	(18) is used with the same metric as the $\mathcal{E}$ of the path measure
Finite spatial lattice	Q is finite-dimensional	The ground-state assumption and the integration by parts in (18)

κ does not depend on the length of time T. T enters only the factor $1-(\xi/T)(1-e^{-T/\xi})\in(0,1)$ of (17), which approaches 1 as $T/\xi\to\infty$. Setting $G=M^{-1}=1$ for a one-dimensional well gives the One-Dimensional One-Sided Bridge Proposition (23-04 § 04), and the one-sided bridge is its multidimensional version with a metric. In the language of the chains, the one-sided bridge shows one side of the relaxation chain, $\lambda_1\le m_{\mathrm{lat}}^2/\kappa$ (the half on the side of the lower-bound assumption), on a lattice with continuous time. Whether a claim of the same form exists in the literature has not been checked (it may be known).

Why the power is two. The variance is the time integral of correlations, $T\cdot(2/m)\|\varphi\|^2$, that is, it grows to first order in $1/\Delta$. The Dirichlet form, for the function $F=\psi_1/\psi_0$ of the slowest excitation, whose gradient is small, is $T\cdot 2m\|\varphi\|^2$ by (18), that is, it becomes small to first order in Δ. The ratio is $1/\Delta^2$. If one overlooks the smallness of the gradient and estimates the Dirichlet form as a quantity independent of Δ, one drops to a lower bound of first order, c/Δ, and the next (19) does not follow.

Note (two meanings of the symbol Δ). In this chapter Δ is used with two meanings. The $\Delta(\beta)=m_{\mathrm{lat}}/a$ of (1) is the mass in physical units. The $\Delta=E_1-E_0$ of the one-sided bridge and the examples of this section is the gap of the Hamiltonian with space made a finite lattice, tied to the correlation length in lattice units by $\xi=\hbar/\Delta$ (with $\hbar=1$ it corresponds to m_{lat}, and the $\Delta(\beta)$ of (1) is that divided by the lattice spacing a). In Part 24, Δ is restricted to the meaning of the one-sided bridge, E_1-E_0, and the mass in physical units is written m_{phys} (24-09 § 02).

Example 6 (checking the bridge by hand on a two-site lattice). Take the same two sites as in Example 2 as the spatial lattice, with continuous time ($\hbar=1$, $M=G=1$, $V=\frac12 x^{\mathsf T}Kx$, where K is the Hessian of Example 2). The normal modes are $u=(x_1+x_2)/\sqrt2$ (angular frequency m_0) and $v=(x_1-x_2)/\sqrt2$ (angular frequency $\sqrt{m_0^2+2}$); the first excitation raises the oscillation of u by one, so $\Delta=m_0$, $\xi=1/m_0$ and $F=\psi_1/\psi_0\propto u$. On the paths, $u(\tau)$ is a stationary Gaussian process with $\mathbb{E}[u(s)u(t)]=e^{-m_0|s-t|}/(2m_0)$. For $f_T=\int_0^T u\,d\tau$, $\mathrm{Var}(f_T)=\frac{1}{2m_0}\big(\frac{2T}{m_0}-\frac{2(1-e^{-m_0T})}{m_0^2}\big)$; since $|\nabla u|^2=\frac12+\frac12=1$, $\mathcal{E}(f_T)=T$, and the ratio is $1/m_0^2-(1-e^{-m_0T})/(Tm_0^3)$, exactly the right-hand side of (17). In numbers, with $m_0=\frac12$ ($\xi=2$) and $T=20$ the ratio is $4-0.99995/2.5\approx 3.600$, and with $T=200$ it is $4-1/25=3.96$, approaching $\xi^2=4$ from below. (18) can also be checked: the left-hand side is $\Delta\,\mathbb{E}[u^2]=m_0\cdot\frac{1}{2m_0}=\frac12$ and the right-hand side is $\frac12\,\mathbb{E}|\nabla u|^2=\frac12$ Proved here . In this example the path measure is Gaussian and the largest eigenvalue of the covariance $(-\partial_\tau^2+K)^{-1}$ is $1/m_0^2$, so $C_P=\xi^2$, and the bridge holds with equality.

Combining the upper-bound assumption with the bridge. Setting the $C_P\le C\xi_{\mathrm{RG}}^2$ of the upper-bound assumption beside the $C_P\ge\kappa/m_{\mathrm{lat}}^2$ of the one-sided bridge gives

$$\frac{\kappa}{m_{\mathrm{lat}}^2}\;\le\;C_P\;\le\;C\,\xi_{\mathrm{RG}}^2\qquad\Longrightarrow\qquad m_{\mathrm{phys}}^2=(m_{\mathrm{lat}}\,\xi_{\mathrm{RG}})^2\;\ge\;\frac{\kappa}{C}\quad(19)$$

Proved here . For example, with $\kappa=1$ and $C=4$, the gap in physical units stays at least $\frac12$ for every $\beta\ge\beta_0$. It is the form with the equality assumption of (16) weakened to one side, and in Lean

```
theorem hup_is_gap_positivity_of_lower {CP xi mlat K κ : ℝ}
    (hmlat : 0 < mlat) (hCP : κ / mlat ^ 2 ≤ CP) (h : CP ≤ K * xi ^ 2) :
    κ ≤ K * (mlat * xi) ^ 2
```

Lean `LatticeGauge.HupBridge.hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le C_P$, $C_P \le K\xi^2$. Conclusion: $\kappa \le K(m_{\mathrm{lat}}\xi)^2$). Setting $\kappa = 1$ recovers the theorem on the left of (16) (`hup_is_gap_positivity_recovered`). The right of (16), on the side of the lower-bound assumption, uses $C_P \le 1/m_{\mathrm{lat}}^2$ (the hard side), so it cannot be weakened with the one-sided bridge.

We list the conditions for (19) to become a sentence on the side of the mass gap. First, κ must be uniform in β. In the field graduation $\mathcal{E}_A$, κ is a number independent of the coupling (its value depends on the convention for normalising the metric of the group), and the condition is met. In the link-angle graduation $\mathcal{E}_\vartheta$, $\kappa \propto g^2 \to 0$, and (19) gives $m_{\mathrm{phys}}^2 \gtrsim g^2/C$, that is, it loses by a logarithm (the path C_P measured in a graduation, Table 3). Second, the treatment of time. The one-sided bridge is a sentence on a lattice with continuous time (a Hamiltonian lattice with only space made a lattice); for the Wilson measure, which also discretises time, there is no counterpart of the ground-state transformation identity (18). On a time lattice of step 1, with $q = e^{-\Delta}$ the second eigenvalue of the transfer operator $\hat{T}$ (divided by its largest eigenvalue), putting down an assumption comparing the increment between neighbouring times with the gradient,

$$(\mathrm{D}): \quad \mathbb{E}_{\psi_0^2}|dF|_G^2 \le c_D \,\langle \varphi, (1-\hat{T})\varphi\rangle \qquad \Longrightarrow \qquad C_P \;\ge\; \frac{1+q}{c_D(1-q)^2} \;\ge\; \frac{2}{c_D\,\Delta^2} \tag{20}$$

gives a lower bound of the same type (the Discrete-Time One-Sided Bridge Proposition) Proved here (on top of (D). The variance over a window of N sites is $\|\varphi\|^2 \sum_{s,t\le N} q^{|s-t|}$, the Dirichlet form is at most $N\,c_D(1-q)\|\varphi\|^2$, and the ratio as $N \to \infty$ is the middle expression. The right inequality, with $y = \Delta/2$, is the same as $\sinh^2 y \le y^2 \cosh y\, e^y$, which follows from $\tanh y \le y$ and $\cosh y \le e^y$) Lean `pairSum_closed` (closed form of the window sum), `disc_limit_ge` (a weak version with coefficient 1; assumptions: $0 < \Delta$, $0 < c_D$). When $\hat{T} = e^{-(H-E_0)}$ holds exactly, (18) gives $c_D = 2\Delta/(1-e^{-\Delta}) \to 2$, returning to the one-sided bridge with continuous time. The transfer operator of the Wilson measure is not of this form, so (D) remains an assumption, together with a c_D uniform in β Assumption. Third, the volume. The one-sided bridge is a sentence about a finite spatial box, and the stage connecting the C_P of the infinite-volume measure with the limit of the gaps of the boxes has not been examined Assumption.

Why the question becomes one-sided on the first step. Using the correlation length fixed by the gap, $\xi = 1/m_{\mathrm{lat}}$ ($\hbar = 1$), the bridge $\kappa\xi^2 \le C_P$ gives the lower half of the power statement automatically.

$$\frac{\ln C_P}{\ln \xi} \;\ge\; 2 + \frac{\ln \kappa}{\ln \xi} \qquad (\xi > 1), \qquad \text{so} \quad \liminf \frac{\ln C_P}{\ln \xi} \ge 2 \tag{21}$$

Proved here (take ln and divide by $\ln \xi > 0$; if κ is uniform the second term goes to 0) Lean `power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa\xi^2 \le C_P$. Conclusion: the left of (21)). Under the assumption of the bridge, the content of the relaxation-time assumption is only the upper half $\limsup \ln C_P/\ln \xi \le 2$. For the double well, the One-Dimensional One-Sided Bridge Proposition is exactly this bridge ($\kappa = 1$, one-dimensional), so the power statement on the first step becomes the one-sided question "can the upper side be shown", and its target is the Double-Well Relaxation-Time Upper-Bound Proposition of 23-05, $C_P \le K\xi^2 S_0/\hbar$.

Algebra on the side of the chains. Assuming the six band inequalities of the three chains puts $a^2 C_P = a^2/\lambda_1$ in a positive band.

$$c_1 m^2 \le \lambda_1 \le C_1 m^2, \;\; c_2\sigma \le m^2 \le C_2\sigma, \;\; c_3 a^2 \le \sigma \le C_3 a^2 \;\Longrightarrow\; c_1c_2c_3\,a^2 \le \lambda_1 \le C_1C_2C_3\,a^2 \tag{22}$$

Proved here Lean `chain_band`, `sq_times_poincare_band` (assumptions: the six band inequalities and positivity of the coefficients. Conclusion: (22), and that a^2/λ_1 lies in $[1/C, 1/c]$). The chains themselves are assumptions, and the converse (chains from the band statement) does not follow. The one-sided bridge gives the right inequality of the relaxation chain, $\lambda_1 \le C_1 m^2$ ($C_1 = 1/\kappa$), on a lattice with continuous time.

(5) What the relaxation-time assumption does not give — a different quantity, the converse, the limit

That it is a different quantity. C_P is the Poincaré constant of the path measure, that is, the reciprocal of the gap of Langevin dynamics running in an artificial time, while the mass of the problem statement is the gap of the transfer matrix (the Hamiltonian) (the path C_P and the vacuum C_P of Table 3). They are different quantities, and the bridge of (4) only puts a one-directional inequality between them. Assuming the relaxation-time assumption does not make the two the same.

The converse does not hold in general. (19) was "the upper-bound assumption ⇒ the gap in physical units is positive". Its converse, "the gap in physical units is positive ⇒ the upper-bound assumption", fails in the following example.

> The Free-Particle-on-a-Circle Counterexample
>
> Let $Q = S^1$ (a circle of circumference 2π), $M = 1$, $V = 0$, $H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2}$. The gap is $\Delta = \hbar^2/2 > 0$ and $\xi = \hbar/\Delta = 2/\hbar$, but the canonical C_P of the path measure at $T = \infty$ is $+\infty$.

Proved here Proof [Y3]. The eigenfunctions are e^{ikx} with eigenvalues $\hbar^2 k^2/2$, so $\Delta = \hbar^2/2$. The continuous lift $\tilde{x}(\tau)$ of a path $x(\tau)$ to the real line is determined up to a constant $2\pi k$. Take w with support $[0, L]$ vanishing at the ends, and set

$$f = \int w(\tau)\, d\tilde{x}(\tau) = -\int w'(\tau)\, \tilde{x}(\tau)\, d\tau, \qquad \mathrm{Var}(f) = \hbar \int w^2\, d\tau, \qquad \mathcal{E}(f) = \hbar \int w'^2\, d\tau \quad (23)$$

Since $\int w' = 0$, the indeterminacy of the constant of the lift disappears, and f is a function of the path. The variance follows from $\tilde{x}$ being a Brownian motion of strength $\hbar$ and the Itô isometry Cited, and the Dirichlet form from $\delta f/\delta x(\tau) = -w'(\tau)$. Stretching as $w(\tau) = w_0(\tau/L)$, the ratio is $L^2 \int w_0^2 / \int w_0'^2$, which grows without bound as $L \to \infty$. ∎ Lean `rotor_ratio_unbounded`, `rotor_gap_pos` (assumptions: $0 < r$ / $0 < \hbar$. Conclusions: for every B there is $L > 0$ with $B < L^2 r$ / $0 < \hbar^2/2$. Only the arithmetic of the skeleton of the Free-Particle-on-a-Circle Counterexample).

Example 7 (the random walk of the winding number). For $w_0(\tau) = \sin(\pi\tau)$ ($0 \le \tau \le 1$), $\int w_0^2 = \frac{1}{2}$ and $\int w_0'^2 = \pi^2/2$, so the ratio is L^2/π^2. It is the same number as the $C_P \approx L^2/\pi^2$ of the single road of Example 1, and not by chance. The winding number counts how many times the particle on the circle has gone round, and it performs a random walk on an endless single road. With $\hbar = 0.1$, $\xi = 20$ and $\xi^2 = 400$, but at $L = 100$ the ratio is about 1013, and at $L = 1000$ about 1.01×10^5, exceeding ξ^2 without limit. In the family $\hbar \to 0$, setting $a := 1/\xi$, $m_{\mathrm{phys}} = m\,\xi = 1$ is positive for every $\hbar$, yet since $C_P = \infty$, the upper-bound assumption breaks for every $\hbar$.

To name the type: relaxation becomes slower than the gap when a quantity not local in time and not reflected in the gap (a winding number, a topological charge, a number of kinks) is carried by diffusion. It is the same mechanism by which conserving dynamics has $C_P = \infty$ (the detour of 2-10); in gauge theory the freezing of the topological charge in a periodic box is a candidate of this type (23-02 § 05). Note the scope. For the four-dimensional Wilson measure on a finite periodic box the configuration

space is a compact connected finite-dimensional space, so $C_P < \infty$, and the Free-Particle-on-a-Circle Counterexample is not a counterexample in that class. What the Free-Particle-on-a-Circle Counterexample shows is a logical gap (a positive gap does not even imply finiteness of C_P); whether the converse is false within the class of Wilson measures is not settled.

In summary, under the assumption of the bridge, the upper-bound assumption implies a lower bound on the gap of the lattice in physical units, and the converse does not hold in general. **Under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the lattice mass gap in physical units.** The road of the relaxation-time assumption is a road that restates a stronger sentence in a language where the tools of (2) can be used.

The relaxation-time assumption is not non-triviality

For a free massive field, as in Table 4, $a^2 C_P = 1$, and in the field graduation the relaxation-time assumption, the band statement and the lower-bound assumption all hold with equality Proved here . So the lower-bound assumption is a condition separate from the "non-trivial" of the problem statement (that there is interaction, 0-03), and the relaxation-time assumption as a whole does not include non-triviality either. What the relaxation-time assumption says is only the form "for a family of measures heading to a limit with a mass gap, how the time of fluctuations grows".

The strong-coupling side is the opposite side

Many theorems of lattice gauge theory are on the side of small β (strong coupling), and what they show is a lattice gap uniform in the volume, or Poincaré and logarithmic Sobolev inequalities, that is, the form $C_P = O(1)$. Shen–Zhu–Zhu (2023) show both inequalities with Langevin dynamics, and write 'the infinite volume measure has a strictly positive mass gap' Cited . Converting the condition to Wilson's β gives $\beta < N^2/(16(d-1)) = 1/12$ $(N = 2, d = 4)$.

Table 7 The β where theorems exist and the β on the side of the limit ($SU(2)$, $d = 4$. ξ is a rough guide from the leading term of the strong-coupling expansion $m_0 = -4 \ln u$, $u = I_2(\beta)/I_1(\beta)$)

Region	β	ξ	What is said
Condition of Shen–Zhu–Zhu	`< 0.0833`	`0.065`	Cited Uniqueness, both inequalities, exponential decay
Curvature after one-link integration	`< 0.2722`	`0.093`	The lower bound on curvature $-27\beta^2$ is Lean `StaplePerturbation.hess_gauge` (assumptions: the period L is even with $L \geq 3$, links are unit quaternions, perturbations are pure imaginary). The criterion is Cited (Bakry–Émery). The stage proceeding to the gap is Assumption (5-11)
Numerical window (Table 2)	`2.25–2.60`	`0.72–1.95`	Reading $\Delta/\sqrt{\sigma} \approx 3.3$ to 3.8
Two-loop extrapolation	`3.24, 4.14, 5.02`	`10, 100, 1000`	Reading A rough guide assuming asymptotic scaling

The ξ column of Table 7 is Reading (a rough guide from the leading term). The strong-coupling theorems are important results giving the infinite-volume uniqueness statement at small β, but if the range of β is bounded, improving the constants does not bring the physical-unit gap statement or the relaxation-time band at the bookkeeping length any closer. If they extended as they are uniformly in β, the upper-bound assumption would be satisfied and the lower-bound assumption broken, falling into the trivial limit $m_{\mathrm{phys}} \to \infty$ (the direction is the same as (5); the second half of Example 5).

The limit and lifting. Even if the relaxation-time assumption closed on the four-dimensional lattice and the assumptions of the bridge were in place, what would be obtained goes only as far as a lower bound

on the lattice gap in physical units at each $\beta \geq \beta_0$. That $\Delta(\beta)$ has a limit (the rest of the physical-unit gap statement), the existence of the limiting measure (the infinite-volume uniqueness statement, the Wilson-loop limit statement), the recovery of rotations with the OS axioms and reconstruction (the rotation-recovery statement, the OS reconstruction statement), and the non-triviality statement lie outside the relaxation-time assumption. And even if the relaxation-time assumption closed on the first step (the double well), there is no theorem of implication to the relaxation-time assumption on the four-dimensional lattice; what has a prospect of lifting is only the form of the sentence and the form of the tools (23-03 § 06).

(6) Analogous cases — models where a question of the same type became a theorem

"Does the slowness of relaxation of a dynamics grow in the same form as a power of the correlation length" is the subject of theorems outside gauge theory.

Example 8 (the theorem of Bauerschmidt–Bodineau in one sentence). For the hierarchical version of the four-dimensional n-component $|\varphi|^4$ model (a model in which the lattice is replaced by nested boxes so that one step of renormalisation can be written down), in the family approaching the critical point from the high-temperature side, the spectral gap of Glauber-type dynamics goes to 0 with the same power as the gap of the free-field dynamics (with logarithmic corrections for $|\varphi|^4$) Cited (abstract and § 1– § 2 of Bauerschmidt–Bodineau, arXiv:1809.02075, checked). The gap of the free-field dynamics is the square of the mass, that is, it vanishes as a power of ξ^{-2}, so this is a sentence of the same type as the diffusive-relaxation-time assumption: "CP grows as a power of ξ^2, up to logarithmic factors" (the correspondence between mass and correlation length depends on definitions inside the model, so here we keep to a comparison of types).

Three things can be read from this example. First, the question of the type of the relaxation-time assumption stands as mathematics, and there is an example where it was answered for a four-dimensional model with interaction. Second, the answer comes with logarithmic corrections, and holds in the form of a power statement, not a band statement. This is the same situation as the reason for placing the relaxation-time assumption at the power level (the detour of § 07). Third, the method is an induction of Brascamp–Lieb type over discrete scales: the covariance is split scale by scale, and the spectral gap of the measure at each scale is handed on from the spectral gap at the next scale. The conclusion is in the form of a spectral gap (a Poincaré inequality); it uses as an assumption that the potential is uniformly close to convex at each scale, and takes results of equilibrium renormalisation as input. It is a result for scalar fields taking values in a flat space, not for gauge fields. How to transfer it to fields taking values in a group is dealt with, as a question outside this part, in 24-06 § 05 of Part 24.

Classification of the rows of the diffusive-relaxation-time assumption, and the dependency graph

We put the statements of (3)–(6) into one table in the format of the five grades (23-00 § 01). The numbers of the rows are the numbers of the nodes of the dependency graph below. The scope column says what the row is about (the range of β, the volume of space, the treatment of time).

Table 8 The relaxation-time assumption and the rows around it ("—" is an assumption row not shown in any range)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[01]	Upper-bound assumption: $a^2 C_P \leq C$	Assumption	(14)(15)	—	—
[02]	Lower-bound assumption: $a^2 C_P \geq c$	Assumption	(14)(15)	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[03]	The relaxation-time assumption (power statement): $\ln C_P / \ln \xi \to 2$	Assumption	(13)	—	—
[04]	Bands of the three chains (the relaxation chain, the one-scale chain, the bookkeeping-length chain)	Assumption	(12), 17-12	—	—
[05]	Equality in the relaxation chain, $C_P = 1/m_{\mathrm{lat}}^2$	Assumption	The assumption of (16) (the free-field value)	—	—
[06]	The ground-state assumption (A1): discreteness of the spectrum, simplicity and positivity of the ground state	Assumption	Assumption of the one-sided bridge (the statement in the original sources has not been checked in this part)	Finite spatial lattice	—
[07]	The stationary path-measure assumption (A2): the formula for correlations	Cited	(i), 2-07 § 08	$T = \infty$	—
[08]	The κ of the metric comparison assumption is uniform in β	Assumption	The first condition after (19) (a number independent of the coupling for $\mathcal{E}_A$, $\propto g^2$ for $\mathcal{E}_\vartheta$)	—	—
[09]	(D) Comparison of increments in discrete time with gradients, c_D uniform in β	Assumption	(20)	—	—
[10]	From the bridge in a box to the bridge in infinite volume	Assumption	The third condition after (19) (not examined)	—	—
[11]	The ground-state transformation identity (18)	Proved here	Proof of (iii)	Finite spatial lattice	—
[12]	The one-sided bridge: $C_P \geq \kappa \xi^2$	Proved here	(17), (i)–(iv) (conditional: (A1)–(A3), continuous time)	Finite spatial lattice, continuous time, $T = \infty$	`cont_ratio_closed`, `one_sided_bridge` (assumptions: positive numbers, a lower bound on the variance, an upper bound on the Dirichlet form)
[13]	The Discrete-Time One-Sided Bridge Proposition: $C_P \geq 2/(c_D \Delta^2)$	Proved here	(20) (conditional: the time-step comparison assumption (D))	Finite spatial lattice, time with step 1	`pairSum_closed`, `disc_limit_ge` (assumptions: $0 < \Delta$, $0 < c_D$)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
[14]	The Free-Particle-on-a-Circle Counterexample: $C_P = \infty$ with $\Delta > 0$ (negation of the converse)	Proved here	(23), Example 7	Circle, every $\hbar$	`rotor_ratio_unbounded`, `rotor_gap_pos` (arithmetic of the skeleton only)
[15]	(16) Rereading in the version with equality	Proved here	Proof of (16)	$\beta \geq \beta_0$	`hup_is_gap_positivity`, `hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m_{\mathrm{lat}}^2$, one side of (14))
[16]	(19) $m_{\mathrm{phys}}^2 \geq \kappa / C$ from the upper-bound assumption and the bridge	Proved here	(19)	$\beta \geq \beta_0$	`hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa / m_{\mathrm{lat}}^2 \leq C_P$, $C_P \leq K \xi^2$)
[17]	(21) The lower half of the power statement is automatic	Proved here	(21)	$\xi > 1$	`power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa \xi^2 \leq C_P$)
[18]	(22) Algebra of the bands of the chains	Proved here	(22)	$\beta \geq \beta_0$	`chain_band`, `sq_times_poincare_band` (assumptions: the six band inequalities and positivity of the coefficients)
—	The gap of the dynamics of hierarchical $\lvert\varphi\rvert_4^4$ has the same power as the free field (with logarithms)	Cited	Example 8 (an analogous case; not used in the rows of the relaxation-time assumption)	Hierarchical model, criticality and its vicinity	—

The "Proved here" rows are all either algebra taking assumption rows as input, or propositions with their assumptions named. The one-sided bridge is a proposition whose proof is given in this section, but it is not a theorem that applies as it is to the four-dimensional Wilson measure (for discretised time the time-step comparison assumption remains, for infinite volume the stage from the bridge in a box to the bridge in infinite volume, and for the graduation the uniformity of κ).

```
[01] ASM  relaxation-time upper-bound assumption: a²C_P ≤ C              —
[02] ASM  relaxation-time lower-bound assumption: a²C_P ≥ c              —
[03] ASM  diffusive-relaxation-time assumption: ln C_P / ln ξ → 2        —
[04] ASM  bands of the three chains                                      —
[05] ASM  equality in the relaxation chain, C_P = 1/m_lat²               —
[06] ASM  ground-state assumption: discrete spectrum,
          simple positive ground state                                   finite spa
[07] CIT  correlations of the stationary path measure                    T=∞
[08] ASM  κ uniform in β                                                 —
[09] ASM  time-step comparison assumption (c_D uniform)                  —
[10] ASM  from the box bridge to the infinite-volume bridge              —
[11] PRV  ground-state transformation identity (18) ⟸ [06]                finite sp
[12] PRV  one-sided bridge: C_P ≥ κξ² ⟸ [06][07][11]                      continuou
          Lean cont_ratio_closed, one_sided_bridge (algebra of numbers only)
[13] PRV  Discrete-Time One-Sided Bridge Proposition:
          C_P ≥ 2/(c_D Δ²) ⟸ [06][07][09]                                time with
          Lean pairSum_closed, disc_limit_ge
[14] PRV  Free-Particle-on-a-Circle Counterexample: Δ > 0 and C_P = ∞  every ħ
          Lean rotor_ratio_unbounded, rotor_gap_pos (arithmetic of the skeleton onl
[15] PRV  (16) upper-bound assumption ⇒ m_phys ≥ C^(-1/2),
          lower-bound assumption ⇒ m_phys ≤ c^(-1/2) ⟸ [01][02][05]
          Lean hup_is_gap_positivity, hdown_is_mass_finiteness
[16] PRV  (19) m_phys² ≥ κ/C ⟸ [01][08][10][12] (with discretised time, [13] inste
          Lean hup_is_gap_positivity_of_lower
[17] PRV  (21) liminf ln C_P / ln ξ ≥ 2 ⟸ [08][12] (or [13])
          Lean power_lower_half_of_bridge
[18] PRV  (22) a²C_P lies in a positive band ⟸ [04]
          Lean chain_band, sq_times_poincare_band
```

How to check. In every row with "⟸", check by eye that the numbers on the right are smaller than the number on the left ([11] from [06], [12] from [06][07][11], [13] from [06][07][09], [15] from [01][02][05], [16] from [01][08][10][12] (or [13]), [17] from [08][12] (or [13]), [18] from [04]). Every row keeps the rule, so there are no cycles. The ASM leaves are the nine [01]–[06] and [08]–[10]; the CIT leaf is [07]. The relaxation-time assumption [03] is a premise of no row. With the bridge, the lower half of the relaxation-time assumption follows automatically by [17], and the content of [03] is only the upper half: that is the content of (4). The Free-Particle-on-a-Circle Counterexample [14] is a counterexample with no premises, and shows that the converse of [16] does not hold in general.

Precisely, this graph is the dependency of the assembly of the relation between the relaxation-time assumption and the mass; it does not include the rows supporting the relaxation-time assumption itself (for the first step, the graph of 23-05 § 07). The one that joins everything together is the integrated graph of 23-12.

07

Detour: the number of rescalings and four clocks — logarithms that do not show in the power

A section you may skip. We look, through formulas for the three metaphors of 23-00 § 05 (the number of rescalings, the graduation of the clock, the clock that runs slow), at why the relaxation-time assumption is placed as a power statement rather than a band statement.

The number of rescalings — the bookkeeping length ξ_{RG}

In four-dimensional lattice Yang–Mills there is no way to obtain ξ by formula as a function of β. So we coarsen the scale of the map one step at a time, and make separately a length that counts how many times it takes to reach a picture of fixed coarseness. In one step that coarsens the lattice by a factor b (one step of the renormalisation group, 7-05), we count the coupling as decreasing as follows.

$$\beta \longmapsto \beta - \kappa\,\big(\ln b - \delta(b)\big), \qquad \kappa = \frac{11}{3\pi^2} \approx 0.3715 \quad (24)$$

κ is the one-loop coefficient (asymptotic freedom, 4-06) rewritten in the lattice β Cited , and $\delta(b)$ is what one step of bookkeeping misses (23-02 § 02). With $k(\beta)$ the number of steps it takes to come down to a fixed value β_*, set

$$\xi_{\mathrm{RG}} = b^{\,k(\beta)}, \qquad a := \frac{1}{\xi_{\mathrm{RG}}}, \qquad \xi_{\mathrm{RG}} = e^{(\beta-\beta_*)/\kappa} \quad (\delta = 0) \quad (25)$$

If $\delta = 0$, the number of steps is $(\beta - \beta_*)/(\kappa \ln b)$, and the bookkeeping length does not depend on b. It has the same form as the one-loop formula of dimensional transmutation (4-07).

Proved here Table 9 The number of rescalings (one loop, $\delta = 0$, $b = 2$, with $\beta_* = 2.0$ as an example. Values of the closed formula (25))

β	Number of steps k (number of halvings)	$\log_{10} \xi_{\mathrm{RG}}$
2.5	1.94	0.58
4	7.77	2.34
6	15.53	4.68
10	31.07	9.35
20	69.90	21.04

Each increase of β by 1 adds about 3.9 halvings. The choice of β_* affects only a constant factor. The bookkeeping length is not a length fixed directly by the measure but a length fixed by the convention of bookkeeping, and that it agrees with ξ up to a constant factor is the assumption of the one-scale chain and the bookkeeping-length chain Assumption . If the bookkeeping is truncated at one loop, the difference from the two-loop running (for pure $SU(N)$, $b_1/(2b_0^2) = 51/121$) remains as a factor $\beta^{51/121}$ Cited (at the level of perturbation theory, 4-06).

The graduation of the clock and the clock that runs slow

Writing an $SU(2)$ link as $U = e^{i\vartheta\cdot\sigma}$, the relation to the field of the continuum theory is $A = \sqrt{\beta}\,\vartheta$. $\mathcal{E}$ is a square of derivatives, so

$$\mathcal{E}_A = \frac{\mathcal{E}_\vartheta}{\beta}, \qquad C_P^{(A)} = \beta\, C_P^{(\vartheta)}, \qquad \text{in the free approximation } C_P^{(A)} = \xi^2,\ C_P^{(\vartheta)} = \frac{g^2}{4}\,\xi^2 \quad (26)$$

Proved here . Further, Langevin time itself stretches and shrinks with scale. When stochastic quantisation (7-09) is renormalised in perturbation theory, besides the field and the coupling, the graduation of stochastic time is also renormalised multiplicatively Cited (at the level of perturbation theory). Transferring its one-loop coefficient into the language of graduations, it can be read as

$$C_P^{(A)} \;\propto\; \xi^2\,\beta^{13/22}, \qquad C_P^{(\vartheta)} \;\propto\; \xi^2\,\beta^{-9/22} \quad (27)$$

can be read Assumption (a reading that presupposes that it is the leading one-loop logarithm, that the leading logarithm of the lattice agrees with that of dimensional regularisation, the multiplicative renormalisability of the five-dimensional theory, and the relaxation-time assumption itself). The power $\beta^{-9/22}$ in the angle graduation is the same power as the factor $g_0^{9/11}$ that Lüscher–Schaefer (2011) write for the autocorrelation time. It comes from the same one-loop coefficient, so it is not an independent check.

Table 10 lines up C_P/ξ^2 for four clocks: two graduations, with and without the slowing. ξ is replaced by the bookkeeping length ξ_{RG} of (25) ($\beta_* = 0$), and the constants are set to 1 at $\beta = 2.5$. The power is $d\ln C_P/d\ln\xi = 2 + \varepsilon\kappa/\beta$.

Proved here Table 10 Four clocks (schematic. Values of the closed formulas $\xi = e^{\beta/\kappa}$ and (26)(27))

β	$\log_{10}\xi$	Field A ($\varepsilon = 0$)	Angle ϑ ($\varepsilon = -1$)	Field A, slowed ($\varepsilon = 13/22$)	Angle ϑ, slowed ($\varepsilon = -9/22$)
2.5	2.92	1	1	1	1
4	4.68	1	0.625	1.32	0.8251
10	11.69	1	0.25	2.269	0.5672
100	116.90	1	0.025	8.844	0.2211
1000	1168.99	1	0.0025	34.48	0.0862
Power ($\beta = 2.5$)	—	2.0000	1.8514	2.0878	1.9392
Power ($\beta = 100$)	—	2.0000	1.9963	2.0022	1.9985

For every clock the power approaches 2, and the constant factor does not settle. Precisely, Table 10 is a schematic representing the interaction only through one-loop logarithmic factors; it is not a computation of the C_P of four-dimensional lattice Yang–Mills. The three logarithms of graduation, order of bookkeeping and stochastic time move the band statement, and do not move the power statement (13).

08

What of the line is in hand — on to 23-02 and 23-03

We line up the statements of this chapter as a single arrow, in the order of the line.

$$\underbrace{\text{one step of bookkeeping}}_{\text{23-02}} \xrightarrow{(\xi^{\prime})\wedge\sum\delta<\infty} (\text{S5}) \xrightarrow{(\xi^{\prime})\text{band}} (\text{S4}) \xrightarrow{(\text{S3})(\text{S6})(\text{S7})(\text{S8})} \text{the demand of the}$$

The first arrow is the sentence at the power level with the finiteness of the sum of the shortfalls of bookkeeping $\sum\delta$ added (23-02 § 02); the next arrow needs the band statement. Of (28), what is in hand is only part of the first box (the algebra of one step of bookkeeping), and none of the arrows is proved. The ultraviolet side has been crossed by a different road (a construction stacking steps of renormalisation) in the construction published on 2026-11-10. That a theory satisfying the Wightman axioms can be built as a limit of four-dimensional Wilson lattice gauge theory is proved in Lean, according to the committee's assessment (Part 6). Even so, a uniform mass gap is not shown (6-04). Where the setters name the need for new ideas is also the infrared: 'New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time.' (Jaffe–Witten § 6.5)Cited .

What follows if the relaxation-time assumption closes. As in § 06, if the upper-bound assumption closes on the four-dimensional lattice and the assumptions of the one-sided bridge are in place (a lattice with continuous time, or, on a lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge"); κ uniform in β; the stage from the box to infinite volume), it follows that at each $\beta \geq \beta_0$ the lattice gap in physical units stays at least $(\kappa/C)^{1/2}$ ((19)). From the relaxation-time assumption alone (a power statement, not the band statement), no gap in physical units follows. C_P is the reciprocal of the gap of Langevin dynamics running in an artificial time, a different quantity from the gap of the Hamiltonian of the transfer matrix (the mass) (the vacuum C_P and the path C_P of Table 3). The bridge is a one-directional inequality between them, and the converse does not hold in general (the Free-Particle-on-a-Circle Counterexample). Even with the relaxation-time assumption and the bridge in place, it does not follow that $\Delta(\beta)$ has a limit, nor the existence of the limit (the infinite-volume uniqueness statement, the Wilson-loop limit statement), the OS axioms and reconstruction (the

rotation-recovery statement, the OS reconstruction statement), or the non-triviality statement. And even if the relaxation-time assumption closed on the first step (the double well), there is no theorem of implication to the relaxation-time assumption on the four-dimensional lattice; what has a prospect of lifting is only the form of the sentence and the form of the tools (23-03 § 06).

The roads that pushed the arrows of (28) on the four-dimensional lattice as it is, and where each stopped, are gathered in 23-02. As seen there, on the side of the lower-bound assumption (some f) there are several roads that reach, and on the side of the upper-bound assumption (every f) no tool that reaches has been found. The relaxation-time assumption is a sentence independent of dimension, so the first step of the staircase is to move the same sentence to the smallest model where the relaxation-time assumption is expected to hold, and build the assembly of the upper side there first (23-03).

What this chapter can say

Said	Not said
Cited The split into eight statements and their current status in four dimensions (Table 1), the footnote of the problem statement and § 6.5, the direction of the lattice correlation-length problem, the strong-coupling theorems (Table 7), the relation of the vacuum C_P among the three C_P to the gap, the convexity and perturbation criteria of (9), the gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$ (Example 8, as an analogous case)	The infinite-volume uniqueness statement at large β; the Wilson-loop limit statement, the physical-unit gap statement, the rotation-recovery statement and the non-triviality statement (along the line of this chapter; for the relation to the published construction, see the note in § 02). This chapter shows nothing about them
Proved here That a uniform lattice gap means a divergent physical mass ((5)), that the physical-unit gap statement and the relaxation-time band at the bookkeeping length say the same thing for the free field ((8), Table 4), the relation between band and power ((11), Example 4), the asymmetry of quantifiers ((15)), the hand calculations of Examples 1–7, the conversion of graduations (26), the values of the closed formulas in Tables 9 and 10	Joining the physical-unit gap statement and the relaxation-time band at the bookkeeping length in a theory with interaction (the relaxation-time assumption, the band statement)
Proved here The one-sided bridge ($C_P \geq \kappa\xi^2$; conditions: (A1)–(A3), continuous time, finite spatial lattice, $T = \infty$) and the Discrete-Time One-Sided Bridge Proposition (condition: (D)), the ground-state transformation identity (18), the Free-Particle-on-a-Circle Counterexample (on the circle $\Delta > 0$ and $C_P = \infty$)	Applying the bridge to the four-dimensional Wilson measure ((D) for discretised time, uniformity of κ, the stage to infinite volume). Whether "positive gap ⇒ the upper-bound assumption" is false within the class of Wilson measures
Proved here The algebra rereading the upper-bound assumption and the lower-bound assumption as lower and upper bounds on the mass ((16)) Lean `hup_is_gap_positivity`, `hdown_is_mass_finiteness` (assumptions: $m_{\mathrm{lat}} > 0$, $C_P = 1/m^2_{\mathrm{lat}}$, one side of (14)). The version weakened to one side, (19), `hup_is_gap_positivity_of_lower` (assumptions: $0 < m_{\mathrm{lat}}$, $\kappa/m^2_{\mathrm{lat}} \leq C_P$, $C_P \leq K\xi^2$). The lower half of the power statement, (21), `power_lower_half_of_bridge` (assumptions: $0 < \kappa$, $1 < \xi$, $\kappa\xi^2 \leq C_P$). The algebra of the bands of the chains, (22), `chain_band` (assumptions: the six band inequalities)	The relaxation-time assumption, the upper-bound assumption, the lower-bound assumption and the three chains themselves. That $C_P = 1/m^2_{\mathrm{lat}}$ holds in a theory with interaction
Reading That $\Delta/\sqrt{\sigma}$ is roughly constant in the Monte Carlo values of the original source (Table 2); the rough guide to ξ in Table 7	That the extrapolated value is the value of the constructed theory
Assumption The exponents of stochastic time $13/22$ and $-9/22$ (a one-loop reading with premises), that the bookkeeping length and ξ agree up to a constant factor, the assumption rows of Table 8	That those exponents apply to the C_P of four-dimensional lattice Yang–Mills. About the mass gap and the continuum limit, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
Problem statement, footnote, § 6.5	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). Text checked (the table of sources of 0-03).
The lattice correlation-length problem and the continuum string-tension problem (Chatterjee's Problems 5.1 and 5.2), current status of the infinite-volume uniqueness statement	Cited	S. Chatterjee, arXiv:1803.01950 § 3, § 5 (Springer Proc. Math. Stat. 283, 2019). Text checked.
Uniform estimates at strong coupling and conversion of the condition	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737 (abstract, § 1 and formula (1.2) checked). K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only).
Transfer operator, OS reconstruction, triviality of ϕ^4_4	Cited	M. Lüscher, Commun. Math. Phys. 54 (1977) 283. K. Osterwalder, R. Schrader, Commun. Math. Phys. 31 (1973) 83, 42 (1975) 281. M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163. Bibliographic data only (tables of sources of each chapter).
The convexity criterion and bounded perturbation of (9)	Cited	D. Bakry, M. Émery, LNM 1123 (1985). The type of the Holley–Stroock perturbation lemma checked in M. Ledoux's restatement. Bibliographic data in the table of sources of 23-06.
Values in Table 2	Reading	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027, Tables 1 and 5 (text checked). Ratios by division and propagation of errors.
Tables 4, 9 and 10, Examples 1–7, (8)(11)(26)	Proved here	The closed formulas $C_P = 1/m^2$, $m^2 = 2(\cosh(1/\xi) - 1)$, $C_P(L) = 1/(2(1 - \cos(\pi/L)))$, the values of (25)–(27), and the hand calculations of the text.
The one-sided bridge, the Discrete-Time One-Sided Bridge Proposition, the Free-Particle-on-a-Circle Counterexample, and (18)	Proved here	The proofs in § 06 of the text. The formula for correlations in the one-sided bridge is 2-07 § 08; the Itô isometry is cited as a known fact. Whether a claim of the same form exists in the literature has not been checked.
The gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$	Cited	R. Bauerschmidt, T. Bodineau, Commun. Math. Phys. 373 (2020) 1167–1206, arXiv:1809.02075 (abstract and § 1– § 2 checked. The method is an induction of Brascamp–Lieb type over discrete scales; the conclusion is a spectral gap).
One-loop and two-loop coefficients	Cited	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343. H. D. Politzer, ibid. 1346 (bibliographic data only; the table of sources of 4-06).
Renormalisation of stochastic time, and (27)	Cited, Assumption	L. Baulieu, D. Zwanziger, Nucl. Phys. B 581 (2000) 604, arXiv:hep-th/9909006. M. Lüscher, S. Schaefer, JHEP 04 (2011) 104, arXiv:1103.1810. J. Zinn-Justin, D. Zwanziger, Nucl. Phys. B 295 (1988) 297 (abstract only). The premises of the reading of (27) are in the text of § 07.
The algebra of (15)(16)(19)(21)(22) and the Lean theorems	Proved here, Lean	Theorems `hup_is_gap_positivity`, `hdown_is_mass_finiteness`, `chain_band`, `sq_times_poincare_band` (bundle `LatticeGaugeOneScale`); `exponent_invariant_of_slowly_varying` (bundle `LatticeGaugePoincareNormalisation`); `hup_is_gap_positivity_of_lower`, `hup_is_gap_positivity_recovered`, `power_lower_half_of_bridge`, `cont_ratio_closed`, `one_sided_bridge`, `pairSum_closed`, `disc_limit_ge`, `rotor_ratio_unbounded`, `rotor_gap_pos` (bundle `LatticeGaugeHupBridge`). Statements and assumptions are in 23-13.

Item	Kind	Source / tools
The note of § 05 (topological sectors and the order of volume)	Reading, Cited, Assumption	L. Del Debbio, H. Panagopoulos, E. Vicari, arXiv:hep-th/0204125 § 2. S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228 § 4.1, § 5. C. Bonanno, C. Bonati, M. D'Elia, arXiv:2012.14000v2 § 1, Fig. 4. M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 (2025) 063B03, arXiv:2501.00286 § 3.2, Fig. 6 ($SU(2)$, HMC). M. Lüscher, S. Schaefer, JHEP 07 (2011) 036, arXiv:1105.4749 § 3, § 5.3. $\tau_{\mathrm{int}} \leq C_P$ is the Lean theorem of 23-02 § 05.
The eight statements of construction, the relaxation-time assumption, the band statement, the three chains, (A1), (D)	Assumption, convention	The formulation of this part. The origin of the three chains is 17-12. The list of readings is in A-06.

Words used in this chapter — the eight statements of construction, the lattice version of the mass gap, the Poincaré constant (three C_P), the diffusive-relaxation-time assumption (power statement), the band statement, the upper-bound assumption and the lower-bound assumption, the three chains, the one-sided bridge (the version with continuous time and the version with discretised time), the ground-state transformation identity, the Free-Particle-on-a-Circle Counterexample (winding number), the bookkeeping length ξ_{RG}, graduation (normalisation of $\mathcal{E}$), renormalisation of stochastic time, the four double-well conditions (name only). Their meanings are in A-02 Glossary.

To the next chapter — 23-02 Roads tried on the four-dimensional lattice, and walls: puts on one sheet the seventeen roads and twenty-one walls that pushed the arrows of (28) on the four-dimensional lattice as it is, and sees that they reach from below but not from above, and that this is why we go down to the first step. To go straight to the first step, 23-03 The four steps of the staircase.

Next chapter to read: 23-02 Roads tried on the four-dimensional lattice, and walls.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time seventeen roads, twenty-one walls (five types) · Kiichi, Shiori and Rin

Roads tried on the four-dimensional lattice, and walls — they reach from below, and not from above

> This chapter is this book's own investigation. § 01 is guide material; § 03, § 06 and § 08 are a map of open questions.

What this chapter shows — We put on one sheet the seventeen roads tried on the four-dimensional lattice towards the diffusive-relaxation-time assumption and the arrows of the line of construction, and the twenty-one places where they stopped (the walls). The roads through test functions, gauge fixing and spectroscopy all end in lower bounds on C_P or upper bounds on the mass; tools that reach the upper side (inequalities for every f) existed only on the strong-coupling side, where curvature works, and in a form moved into a single assumption. What was closed in Lean is the algebra from assumptions to conclusions. At the end of this chapter we say, through the correspondence between tools and walls, why we step down to the first step (the double well in 0+1 dimensions).

Prerequisite chapters — 23-01 (the eight statements, the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption, the three chains), 1-10 (the variational principle, the Poincaré constant), 4-11 (the gap and finite windows), 5-02 (the strong-coupling expansion), 7-05 (steps of the renormalisation group). Notation as in 23-01 ($SU(2)$, $d = 4$, $\beta = 4/g^2$, lattice units).

Said first — This chapter does not solve the problem. "Stopped" for a road means "we checked that this tool, in this family, does not reach this arrow"; it does not say that the proposition not reached is false. The seventeen roads and twenty-one walls of this chapter are the series on the side of the four-dimensional lattice, separate from the series of the first step (the four double-well conditions, routes, strands, detour plans). Many of the numbers are readings inside models, not computations for four-dimensional $SU(2)$.

The order of this chapter

1. How to read the map — the formula of arrows, and the names of the roads and walls
2. One step of bookkeeping — complete families, one-link integration, the deficit of bookkeeping, the shortfall and the form of the answer (Table 1)
3. The seventeen roads tried — one row each (Table 2)
4. Reaching from below — test functions, gauge fixing and spectroscopy end in upper bounds

5. Not reaching from above — curvature, tubes, one-dimensional chains of scales, boundaries
6. Twenty-one walls — five types (Table 3)
7. What remained — algebra closed in Lean, and numerical readings (Table 4)
8. Why we step down to the first step — which tool stopped at which wall (Table 5)

01

How to read the map — the formula of arrows, and the names of the roads and walls

The roads and walls of this chapter are attempts to push one of the arrows of the line of construction in 23-01 § 08. Rewritten with the band statement split into its upper and lower halves, it has the following form.

$$\underbrace{\text{one step of bookkeeping}}_{\S 02} \xrightarrow{\sum_k \delta(bk)<\infty} \text{(S5)} \xrightarrow{(\xi^{\angle})\downarrow \wedge (\xi^{\angle})\uparrow} \text{(S4)} \xrightarrow{\text{properties of the limit}} \text{t}$$

$\delta(b)$ is the shortfall of one step (§ 02); in the formula, (S5) (the relaxation-time band at the bookkeeping length) is $a^2 C_P \asymp 1$, (S4) (the physical-unit gap statement) is that the gap in physical units stays positive, and the properties of the limit are the infinite-volume uniqueness statement, the reflection-positivity preservation statement, the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement (23-01 Table 1). The lower-bound assumption is $C_P \gtrsim \xi_{\mathrm{RG}}^2$ (one test function suffices), and the upper-bound assumption is $C_P \lesssim \xi_{\mathrm{RG}}^2$ (an inequality over all test functions). None of the arrows of (1) is proved.

A road is the pair of one question and one set of tools applied to it, and is named after the content of the question. The road of comparison with the defect of a submission and the road of a one-dimensional chain of scales take up the original question again. A wall is a one-sentence statement of why a certain road, with certain tools, stopped short of a certain arrow; the walls are divided into five types (principle, structure, tools, assumptions, measurement and literature) (§ 06).

Example: the road of lower-bound test functions (test functions giving the lower-bound assumption) pushed the lower-bound assumption of (1), and stopped short of the upper-bound assumption at the one-sided test-function wall (test functions bound C_P only from below). The road of strong-coupling curvature is not on (1); it sits outside to the left. Uniform estimates at strong coupling are $C_P = O(1)$, the opposite side of the relaxation-time band at the bookkeeping length (23-01 § 06).

02

One step of bookkeeping — from complete families to the sum of shortfalls

One step is the map that coarsens the lattice spacing from a to ba, integrates out some of the fine variables, and makes the action of the coarse lattice (7-05). The bookkeeping here takes three moves: (i) choose links that can be integrated in closed form, (ii) write the remaining action with characters, (iii) read the coefficient of the fundamental representation of the coarse plaquette as β'. It is of the "decimation" type, keeping some of the original links (Migdal–Kadanoff is an approximation of the same type). The area law, dimension counting and the one-loop coefficient are known rewritings, not counted as results.

Complete families. If the links to be integrated never lie on the same plaquette, then with the rest held fixed the weight of each link separates independently and can be integrated. A family that removes exactly one link from every plaquette is called a complete family.

$$F \subset E \text{ is complete} \iff \text{for every plaquette } P,\ |P \cap F| = 1; \qquad f_x \le 2,\ \frac{1}{|V|}\sum_x f_x = \frac{d}{2} \implies d \le 4$$

Proved here An edge lies on $2(d-1)$ plaquettes, so counting the pairs $(P, \ell \in P \cap F)$ in two ways gives the density of the family $|F|/|E| = 1/4$. Two perpendicular edges at the same vertex are two adjacent edges of one plaquette, so the number f_x of edges of the family at a vertex x is at most the two along one axis, and comparing with the average $d/2$ gives $d \le 4$. The classification including existence is Table 1; the statements are Lean `PerfectFamily.perfectK_one_iff` (assumption: $2 \le d$. Conclusion: the box $(\mathbb{Z}/L)^d$ of period L has a complete family of edges $\iff$ $d \le 4$ and L is even) and `no_perfectZ_of_five` (assumption: $d \ge 5$. There is none on $\mathbb{Z}^d$). No prior record was found in what we searched, but it may be known. This 4 is a combinatorial number of the lattice; its agreement with the four dimensions of spacetime is a coincidence with no relation.

Proved here Table 1 Existence of complete families of edges on $(\mathbb{Z}/L)^d$ (the statement of `perfectK_one_iff` written out. In brackets, the number of edges in one family = 1/4 of the number of edges)

Period	$d=2$	$d=3$	$d=4$	$d=5$
$L=2$	Yes (2)	Yes (6)	Yes (16)	No
$L=3$	No	No	No	No
$L=4$	Yes (8)	Yes (48)	Yes (256)	No

One-link integration. The 6 plaquettes on which an edge ℓ of the family lies contain no other edges of the family, so U_ℓ pairs only with the sum M of staples (the products of the three edges of a plaquette other than ℓ), and the integral is determined by $|M|$ alone.

$$\int_{SU(2)} e^{\frac{\beta}{2}\operatorname{tr}(UM)}\,dU = \frac{2I_1(\beta|M|)}{\beta|M|} =: F(\beta|M|), \qquad \frac{\beta^2 t}{8} - \frac{\beta^4 t^2}{64} \le \ln F(\beta\sqrt{t}) \le \frac{\beta^2 t}{8} \quad (3)$$

The left is the standard formula of the $SU(2)$ heat-bath method Cited (5-07); the two-sided estimate on the right is Lean `log_F0_le`, `log_F0_ge` (assumption: only that the argument is non-negative). The conditional density of the remaining edges is a product of six sources and

does not close into the one-link type, so this trick works only once Assumption (there is a paper argument, but this book gives no proof).

The deficit of bookkeeping. After integrating a complete family, not a single "intact" plaquette with all 4 edges remaining is left, and the coupling can only be reborn from broken plaquettes. For the fraction e of remaining edges and the fraction q of intact plaquettes, counting pairs gives

$$e - q = \frac{1}{4|\mathcal{P}|} \sum_{P \text{ broken}} |P \cap E_K| \;\geq 0 \quad (4)$$

Proved here (double counting) Lean `Incidence.intact_frac_le` (assumption: the incidence is regular. Conclusion: $q \leq e$). Stacking steps does not reach it: in a sequence of complete steps the remaining edges are $(2^k+1)/2^{k+1} > 1/2$, `no_blocking` (assumption: a sequence of complete steps), and in any sequence removing at most $1/4$ at each step, reducing the edges to $1/16$ leaves at most $1/65536$ of the 4-loops, `blocking_shortfall` (assumption: fraction $\leq 1/4$). The reading of this subsection is that the wall of bookkeeping was neither curvature nor packing, but the way of bookkeeping that counts only intact plaquettes.

Discs and a weak-coupling step. Integrating the interior of a disc-shaped set of A plaquettes raises the coefficients of every channel of the character expansion to the power A: $u(\beta') = u(\beta)^A$, and for a 2×2 disc, $\beta' = \beta^4/64 + O(\beta^6)$ Cited (a rewriting of the strong-coupling area law and two-dimensional decimation, 5-02, 7-02). It is a semigroup in the area, with $\beta' < \beta$ everywhere Lean `discStep_semigroup`, `step_lowers_coupling` (assumption: surfaces of disc shape). In the Gaussian approximation at weak coupling, the b^2 plaquettes of a disc act in series and the b^{d-2} transverse layers in parallel, giving

$$\beta'_{\text{disc}} = \frac{\beta}{b^2}, \qquad \beta'_{\text{tree}} = \beta\, b^{d-4}, \qquad \kappa_{\text{disj}}(b) = \frac{1}{b^2} + \frac{2}{b^2+4b} < 1 \quad (5)$$

βb^{d-4} is the dimension count of the coupling, marginal at $d = 4$ Cited (4-04). The contribution of surfaces sharing no edges does not reach 1 Lean `kappaDisj_lt_one` (assumption: $b \geq 2$), and the bookkeeping of surfaces has the multiplicative form $\beta' = \kappa(b)\beta$, not the subtractive form of one loop $\beta' = \beta - \kappa \ln b$.

The one-loop slope and the shortfall. On top of the tree-level $\beta' = \beta$ sits the one-loop correction from the fast modes (momenta between $\pi/(ba)$ and π/a). Measured by the integral over a lattice window, it takes the form

$$\beta' = \beta - \kappa L_{\text{eff}}(b), \qquad L_{\text{eff}}(b) = \ln b + C - \delta(b), \qquad \delta(b) = \frac{c_2}{b^2} + O(b^{-4}) \quad (6)$$

$\kappa = 4Nb_0 = 11/(3\pi^2) = 0.3715110$ is the known one-loop coefficient Cited (4-06). The slope read from the difference of windows is 1.000028 at $b = 32 \to 64$; multiplied by κ it is 0.371521, agreeing with $11/(3\pi^2)$ to a relative 2.8×10^{-5} Reading (an agreement inside the model of the one-loop kernel; not a non-perturbative check). The coefficients of the shortfall are $c_2 = 0.0197(5)$ (weight w) and $0.4200(5)$ (weight w^2), and at $b = 2$, $\delta(2) = 0.1335$ and 0.2284 Reading

(fits in the same model). On the other side, the effective slope $\Delta\beta/\ln(a_1/a_2)$ read from the actual lattice $a\sqrt{\sigma}$ (Table 2 of 23-01) is 0.304 over $\beta = 2.25 \to 2.60$, reaching neither the one-loop 0.3715 nor the two-loop 0.397 Reading . Assuming the step-by-step form $\xi(\beta - \kappa \ln b) = \xi(\beta)/b$ for every $b \geq 1$ gives $\xi(\beta) = \xi(0)e^{\beta/\kappa}$ Proved here Lean `xi_eq_exp` (assumption: that functional equation; neither continuity nor monotonicity is assumed). The physical content lies on the side of the assumed functional equation (the one-loop running put down as an equation).

The form of the answer. Stacking steps with b fixed, the shortfall accumulates by the same amount each step, and if $\delta(b) \neq 0$ the sum diverges. For a sequence with growing step scales $b_k = k + 2$, $\sum_k c_2/(k+2)^2 < \infty$ and it stays bounded Proved here Lean `fixed_step_bounded_iff` (assumption: a constant error $\varepsilon_j = d \geq 0$. Conclusion: the partial sums are bounded $\iff d = 0$). For example, with the provisional value $c_2 = 1$, fixing $b = 2$ gives a total of 250 after 1000 steps, while with $b_j = j + 2$ the total stays within $\pi^2/6 - 1 = 0.645$ (values of closed formulas). What ties this shortfall to the first arrow of (1) is the algebra of two powers.

$$C_P \asymp \xi^{2-\eta}, \qquad a\,\xi \propto a^{\theta} \implies a^2 C_P \propto a^{p}, \qquad p = \eta + (2-\eta)\,\theta \ \text{(7)}$$

θ is the exponent of accumulation of the shortfall (with b fixed, $\delta(b)/\ln b$), and η is the dynamical power η_{dyn}. If $0 \leq \eta \leq 2$ and $\theta \geq 0$, then $p = 0 \iff \eta = 0 \wedge \theta = 0$ Proved here Lean `drain_exponent`, `drainExp_eq_zero_iff` (assumptions: the power form of (7) and the signs). In words, "under the relaxation-time assumption ($\eta = 0$), the physical-unit gap statement $\iff \sum_k \delta(b_k) < \infty$", and this is "the form of the answer". Besides the relaxation-time assumption, this statement stands on the power form of (7), a model in which the errors of the steps are additive and bounded, and δ being the value of the model of the one-loop kernel Assumption . The algebra that if the partial sums of the errors of the steps are bounded then $a_k^2 C_P^{(k)}$ stays in a positive band is also closed Lean `S4_two_sided` (assumptions: partial sums $|\sum_{j<k} \varepsilon_j| \leq M$, positivity of coefficients. Reading this as $a_k^2 C_P^{(k)}$ is on the side of the text).

A road that changes the kernel to make $\delta(2) = 0$ exists in the one-parameter family $c_1(q) = \alpha + (1-\alpha)\cos q$ at $\alpha_0 = 1.4400$, but the neighbouring weight is negative Reading . For kernels with positive weights, $c_2 > 0$ Lean `c2_ge_of_nonneg_weights` (assumption: c_2 is an increasing affine function $A + \kappa S$ of the kernel quantity $S \geq 0$, with $A, \kappa > 0$, a measured dependence. Not an unconditional claim). Negative weights can collide with reflection positivity, so within the families examined, removing the shortfall and the condition for returning to a quantum theory are not compatible. Outside the families it is undecided.

03

The seventeen roads tried — one row each

Table 2 lines up the seventeen roads one row each. The names in the "Where it stopped" column are the walls of § 06, and the "What remained" column is algebra closed in Lean (assumptions in brackets) or labelled statements.

Table 2 The seventeen roads tried on the four-dimensional lattice

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of strong-coupling curvature	Widen the β where the curvature method works (outside (1), strong-coupling side)	Lower bound on the Hessian form after one-link integration of a complete family, and the Bakry–Émery criterion	Up to $\beta < \sqrt{2/27} = 0.2722$. The correlation length is about 1/10 of a lattice spacing, on the opposite side from the limit (upper-bound tool wall)	Lean `hess_gauge` (assumptions: the period L is even with $L \geq 3$, unit quaternions, pure imaginary perturbations. Conclusion: $\partial_\tau^2 \Phi \geq -27\beta^2 \sum \lvert X_e \rvert^2$). The stage proceeding to the gap is Assumption
Road of one-step bookkeeping	Bookkeep the continuum limit as a sequence of steps (one-step bookkeeping)	Complete families, character expansion, disc steps, one-loop windows	One-link integration works only once (single-step integral wall). The bookkeeping of surfaces is multiplicative and does not produce the one-loop slope	`discStep_semigroup`, `step_lowers_coupling`, `kappaDisj_lt_one`, `intact_frac_le`, `no_blocking`, `blocking_shortfall`
Road of the sum of shortfalls	Is the sum of the shortfalls $\delta(b)$ finite (the summability-of-shortfalls assumption)	Window integrals of the model of the one-loop kernel	With b fixed the sum diverges. The kernel with $\delta(2) = 0$ has a negative neighbouring weight (positive-weight kernel wall). The difference of c_2 with the choice of gauge went only as far as reading it positive in two gauges (gauge dependence of the shortfall wall)	`fixed_step_bounded_iff`, `c2_ge_of_nonneg_weights` (assumption: the measured dependence). Reading c_2

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of the strong-coupling series	Can the dynamical power η be read from the strong-coupling series (the η of the relaxation-time assumption)	High-order strong-coupling expansion, Dlog-Padé	Near the pole the series takes the renewal form $G = A/(1 - K(p))$, which forces $\eta = 0$. At an essential singularity under asymptotic freedom the exponent cannot be read (renewal-series wall)	Lean `renewal_chi_mul_massSq`, `renewal_residue_pos` (assumption: coefficients of the renewal form $A, c_0 > 0$. Conclusion: $\chi m_0^2 = A/c_0 > 0$). That it takes the renewal form is Assumption
Road of the dimension of complete families	In which dimensions do complete families of edges and faces exist (one-step bookkeeping)	Double counting, exhaustive search for exact covers	For edges, only for $d \le 4$ with an even period. For k-cells there is an upper limit $d \le 3k + 1$, and $d = 9$ for $k = 3$ is undecided	`perfectK_one_iff`, `no_perfectZ_of_five`, `no_perfect_k4_d9`, `no_perfect_k5_d8` (assumption: period 2 only)
Road of the dimensional threshold	Why four dimensions is the borderline (the summability-of-shortfalls assumption)	The recursion for the drift of the steps $x_{k+1} = b^{d-4} x_k + \varepsilon_k$	Contracts for $d < 4$, accumulates for $d = 4$. The same as the standard dimension count; not a reproduction of the constructions in lower dimensions	Lean `dev_le_of_contracting` (assumptions: $0 \le \lambda < 1$, $\lvert\varepsilon_j\rvert \le E$. Conclusion: drift $\le E/(1 - \lambda)$), `dev_eq_of_marginal` (with $\lambda = 1$ and constant E, drift $= nE$)
Road of reflection positivity	Use reflection positivity for an estimate from below (the reflection-positivity preservation statement, the OS reconstruction statement)	Representation by the transfer matrix	For the overlap $\tilde{Z}$ with the lightest state, an upper bound $\tilde{Z} \le \chi m$ came out where a lower bound was wanted. The question moved to the road of lower-bound test functions	Lean `overlap_le_susc_mul_gap` (assumptions: $m > 0$, $\tilde{Z} \ge 0$, $\tilde{Z} \coth(m/2) \le \chi$)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of an area law in the bookkeeping	Does the form of the area law appear inside the bookkeeping (the Wilson-loop limit statement)	Coefficient $4(\beta/4)^{A}$	A rewriting of the strong-coupling area law, which holds even for abelian groups. Not counted as evidence of confinement. The content went to the road of the lower half at one scale	Cited (Wilson 1974, Osterwalder–Seiler 1978)
Road of multi-link integrals	Is the closed form of k-link integration analytic even in the continuum limit (one-step bookkeeping)	Closed form of a single sum	The two-link correction is of size β^{3} and not an obstacle. Analyticity has not been shown	Assumption (a calculation inside the strong-coupling expansion)
Road of comparison with the defect of a submission	Is the "small defect" of a submission that claims a construction the same quantity as δ (the summability-of-shortfalls assumption)	Comparison of types	The sum of defects is a quantity of the contracting sector; δ is the shortfall of the coefficient of the marginal sector. The words sit in the same place, but the quantities differ	We do not judge whether the submission is right or wrong (the policy of 16-01)
Road of multiscale curvature	Stack the curvature method scale by scale to make a tool for the upper-bound assumption (the upper-bound assumption)	Multiscale Bakry–Émery, curvature on the Gribov region, one step of gradient flow	Cannot be written beyond the cross terms of two scales (upper-bound tool wall). Details in § 05	Lean `comp_class_eq_of_equivariant_bijection` (assumption: the flow is a bijection commuting with the gauge action), `autocov_sum_le_on_invariant` (assumption: Poincaré on an invariant subspace)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of volume and boundary	Is the estimate of C_P uniform in volume and boundary condition (the infinite-volume uniqueness statement)	Bare link angles, comparison of boundary conditions	With bare link angles the ratio is 1 and the worry about zero modes disappeared. Instead the boundary condition matters, and the upper-bound assumption can break in a periodic box (periodic-boundary conjecture wall)	Assumption (a reading under the conjecture that HMC and Langevin are in the same universality class)
Road of lower-bound test functions	Is there a test function giving the lower-bound assumption (the lower-bound assumption)	Three families: plaquettes, bare link angles, gauge-fixed long-wavelength modes	Local operators have dimension 4, so $\eta \geq 2$ (local-operator dimension wall); link angles give ratio 1; gauge fixing runs into single-valuedness (gauge-fixing single-valuedness wall)	`profileQuotient_isGreatest`, `fisherEta_eq_zero_iff_canonical`, `eta_ge_two_of_bounded_susc` (assumption: susceptibility uniformly bounded), `prop_at_gap_scale` (assumption: pure-power propagator), `jump_energy_ge`
Road of gauge-fixed long wavelengths	Can gauge-fixed long-wavelength modes be made single-valued (the lower-bound assumption)	Weighted averages over orbits (the schemes of Parrinello–Jona-Lasinio and Zwanziger)	Sharp means multivalued, blunt means the long wavelengths vanish (gauge-fixing single-valuedness wall). The numbers are tests on compact $U(1)$	`orbitAvg_gauge_invariant`, `orbitAvg_continuous` (assumptions: orbits of a finite group, positive continuous weights)

Road	What (arrow pushed)	With which tools	Where it stopped	What remained
Road of spectroscopy	Bound the gap with smoothed operators and spectroscopy (the lower-bound assumption and upper bounds)	Wilson flow, APE, variational method, effective mass	Ends in upper bounds (spectroscopic upper-bound wall). A lower bound needs a lower bound on the overlap. The cost of measuring the numerator is $\geq cL^{d+2}$ (measurement-cost wall)	`meff_antitone`, `meff_ge_of_corr`, `gap_ge_of_overlap` (assumption: a lower bound q on the overlap), `light_admixture_invisible`, `cost_ge`, `sqQuotient_between`
Road of the lower half at one scale	Build the lower half at one scale, $m^2 \gtrsim \sigma$ (the one-scale chain)	Entropy of tubes, deficit $D = m - 4\sigma$, pillar model, Wulff shape	Moved into the tube transfer-operator inequality (the tube inequality), but there is no type of proof for it (tube-inequality wall)	`mass_sq_ge_of_tube` (assumption: a norm estimate that is a consequence of the tube inequality), `mass_bound_localized`, `linear_regime_breaks_onescale`, `casimir_only_no_bound`, `linear_deficit_breaks_inv`
Road of a one-dimensional chain of scales	Sandwich the upper-bound assumption with a one-dimensional chain of scales (the upper-bound assumption)	The radial type of holography, two-weight Hardy inequality, sandwiching autocorrelations	Sandwiched only under the assumption of projection onto a one-dimensional chain of scales. The projection assumption is not weaker than the relaxation-time assumption, and in a periodic box it breaks through the topological charge (projection-assumption wall)	`conditional_scale_bound` (assumptions: the projection assumption, shortfalls non-negative with bounded partial sums, geometric-series weights), `const_defect_scale_unbounded`, `tau_int_le`, `rayleigh_le_tau`

Seen arrow by arrow, there is a bias. Stopped roads gather at the lower-bound assumption and outside (1). Only two roads aimed at the upper-bound assumption, the road of multiscale curvature and the road of a one-dimensional chain of scales; the road of multiscale curvature stopped at the cross terms, and the road of a one-dimensional chain of scales sandwiched only under the projection assumption. No road of this chapter reached the rotation-recovery statement or the non-triviality statement (on the side of the published construction, the committee's assessment is that the recovery of rotations is shown for the theory phase by

phase; 6-03). The Wilson-loop limit statement and the rotation-recovery statement are claims of a kind that do not come out however many times the map of one step is composed (limit-property wall).

04

Reaching from below — test functions, gauge fixing and spectroscopy end in upper bounds

The road of reflection positivity, the road of lower-bound test functions, the road of gauge-fixed long wavelengths and the road of spectroscopy all aimed at estimates from below, and stopped at a place of the same form. C_P is a supremum over test functions, and the gap is an infimum over states, so what one test function or one operator gives is in either case only one side.

$$C_P = \sup_f \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} \;\geq\; \frac{\mathrm{Var}(f_0)}{\mathcal{E}(f_0)}, \qquad E_{\mathrm{low}} = \inf_{\psi \perp \Omega} \frac{\langle \psi, H\psi\rangle}{\langle \psi,\psi\rangle} \;\leq\; m_{\mathrm{eff}}(k) \quad (8)$$

Cited The variational principle (1-10). Ω is the vacuum, H the Hamiltonian of the transfer matrix, and $m_{\mathrm{eff}}(k) = \ln(C(k)/C(k+1))$ the effective mass at time k. The lower-bound assumption is a statement about "some f", the side test functions can reach; the upper-bound assumption is about "every f", the side they cannot.

Local operators (the road of lower-bound test functions). In the family made by translating one operator O_x and summing with weights, the freedom of the weights collapses to a single best momentum Lean `profileQuotient_isGreatest` (assumptions: finitely many momenta, the Fourier transform of the denominator is positive). The numerator is a susceptibility, and estimating it by the integral up to radius ξ,

$$\frac{\mathrm{Var}(f_O)}{\mathcal{E}(f_O)} \asymp \xi^{\,2-\eta_s}, \qquad \eta_s = \min\bigl(2\Delta_O - (d-2),\, 2\bigr), \qquad \Delta_O \geq 4 \Rightarrow \eta_s = 2 \quad (9)$$

(assuming a power form of the correlations). η_s is the static power of each operator, a different quantity from the dynamical η_{dyn} (18-05). The lowest dimension of gauge-invariant local operators is the 4 of $\mathrm{tr}\, F^2$ Cited , so the ratio does not grow with ξ Lean `eta_ge_two_of_bounded_susc` (assumption: the susceptibility is uniformly bounded; the boundedness comes from the physical assumptions of the form of correlations and $\Delta = 4$). For bare link angles, under the measure without gauge fixing each link angle is uniform and independent, and the ratio is 1 independently of β and the volume Proved here (a direct calculation from the invariance of the Haar measure; the same mechanism as Elitzur's theorem).

Gauge fixing (the road of gauge-fixed long wavelengths). Gauge-fixed long-wavelength modes are non-local and avoid the wall of dimension, but the choice of representative must be single-valued. Taking a weighted average at temperature T over the orbit makes it single-valued and continuous Lean `orbitAvg_gauge_invariant`, `orbitAvg_continuous`. In the smallest example, the $U(1)$ ring (L edges), the amplitude of the averaged mode vanishes exponentially

with L as $(I_1/I_0)^{L-1}$ for blunt fixing, and jumps at the total angle $\Phi = \pi$ for sharp fixing Proved here (closed formulas from Fourier series). The same functional serves both as "a tool for choosing among copies" and "a tool for evening out long wavelengths", so the two demands sit at the two ends of a single knob. The numbers measured for four-dimensional compact $U(1)$ (the scatter between copies being 3.25 standard deviations, and so on) are Reading ; they were not measured for $SU(2)$.

Spectroscopy (the road of reflection positivity, the road of spectroscopy). Smoothing in the three spatial directions only stays on one time slice, and correlations keep the form $C(k) = \sum_n w_n e^{-E_n k}$ $(w_n \geq 0)$ Cited (5-03). With this form the effective mass decreases monotonically Lean `meff_antitone` (assumption: a representation with non-negative weights), and at every k it is an upper bound on the lowest energy `meff_ge_of_corr`. A lower bound needs a different input.

$$E_{\text{low}} \;\geq\; \frac{1}{k}\Big[\ln q + \ln \frac{C(0)}{C(k)}\Big] \qquad (w_{\text{low}} \geq q\, C(0) \text{ assumed}) \quad (10)$$

Proved here Lean `gap_ge_of_overlap` (assumptions: a representation with non-negative weights and a lower bound q on the overlap). The price of the lower bound is exactly one lower bound on the overlap; without q, a light state of weight ε changes the correlation by only ε at every k `light_admixture_invisible` (a concrete two-level example). That no lower bound on the gap follows from finitely many measurements is a restatement of the known framework of the equivalence of a gap and exponential clustering Cited (4-11). On the numerical side, the effective mass of a smoothed operator at $\beta = 2.6$, 12^4 was $\leq 0.501 \pm 0.071$, touching the $am(0^{++}) = 0.514(8)$ of the original source from below Reading (Monte Carlo, reading of an upper bound). A four-dimensional flow that also smooths in the time direction does not stay on one slice, and breaks the upper-bound property with an effective mass that increases monotonically Proved here (closed formulas for the free field). The cost of measuring is $\geq cL^{d+2}$, from the relaxation time $\propto L^2$ of the softest mode in a box of side L `cost_ge` (assumptions: a lower bound on the relaxation time, and the number of sweeps needed for relative precision).

The three roads reached the same wall. Each needs "the overlap of gauge-invariant operators with the lightest fluctuation is bounded from below", which includes "no other light state is hidden", a statement about all states. It has the same form as the upper-bound assumption.

05

Not reaching from above — curvature, tubes, one-dimensional chains of scales, boundaries

The upper-bound assumption is an upper bound on CP. The types of tools that give upper bounds are limited; there are two representative ones.

$$\nabla^2 U \geq K > 0 \implies CP \leq \frac{1}{K} \quad \text{(Bakry–Émery)}, \qquad B \leq CP \leq 4B \quad \text{(one dimension, Muckenhoupt}$$

Cited The left comes from a lower bound on the curvature of the measure e^{-U} (1-10); the right sandwiches C_P within a factor of 4 by the maximum B of "the mass before a point $\times$ the resistance beyond it", which can be computed only in one dimension.

Curvature scale by scale (the road of strong-coupling curvature, the road of multiscale curvature). The known places where inequalities of the type of the upper-bound assumption stand as theorems are only curvature, diameter, convexity, strong coupling, and the multiscale Bakry–Émery uniform in the lattice regularisation for the high-temperature phase of continuum scalar fields in two and three dimensions Cited (Theorem 1.1 of Bauerschmidt–Dagallier; the criterion is Theorem 2.5 of Bauerschmidt–Bodineau). For four-dimensional gauge fields, the curvature after one-link integration is positive only at strong coupling (the road of strong-coupling curvature). Examining the curvature on the Gribov region in a finite-dimensional model, it was positive only on the strong-coupling side, and the non-convexity came from the zero-momentum mode Reading (on samples with $L = 2, 3, 4$). The ultraviolet fluctuations remaining after removing the zero modes cannot be overcome by one step of gradient flow: the flow is a bijection commuting with the gauge action, so it does not shrink the space of functions Lean `comp_class_eq_of_equivariant_bijection`; the only gain is in changing the metric, and the curvature of the pushed-forward measure became more deeply negative with flow time Reading (within the range measured: open box $\ell = 3$, typical points at $\beta = 2.3, 4$, $t \le 0.5$).

The deficit of tubes (the road of the lower half at one scale). We moved the lower half of the one-scale chain, $m^2 \gtrsim \sigma$, into an inequality between the transfer operator $\mathcal{T}$ of closed tubes and the perimeter operator $\hat{\ell}$.

$$T(c,1): \quad \mathcal{T} \;\le\; \exp\Big(-\big(\sigma\hat{\ell} + c/\hat{\ell}\big)\Big) \qquad \Longrightarrow \qquad m \;\ge\; \min_{\ell>0}\Big(\sigma\ell + \frac{c}{\ell}\Big) = 2\sqrt{c\sigma} \quad (12)$$

Proved here (the right is the AM–GM inequality) Lean `mass_sq_ge_of_tube` (assumption: the norm of the transfer operator is at most $e^{-2\sqrt{c\sigma}}$. Conclusion: $m^2 \ge 4c\sigma$). The power of $1/\ell$ is the only power that makes the coefficient c dimensionless. The content is on the side of the tube inequality, which itself has no type of proof. In the region where the strong-coupling surface expansion converges, the lower half at one scale is itself false `linear_regime_breaks_onescale` (assumption: a sandwich in the region of convergence); in counting with the pillar model the deficit can have either sign, the Wulff shape is a rhombus, and the tube inequality stands only after the anisotropy is subtracted `linear_deficit_breaks_inv` (negative control). The final form is three assumptions, "localised T on short loops, a Casimir floor on large loops, and a seam between them" `mass_bound_localized`; the floor alone gives no lower bound `casimir_only_no_bound`.

A one-dimensional chain of scales (the road of a one-dimensional chain of scales). In holographic models the gap is an eigenvalue problem in a single radial variable, and on that variable the band statement $C_P \asymp \xi^2$ becomes an identity needing no proof Proved here (ground-state transformation $H = P^\dagger P$ and the variational principle). To bring this onto the lattice, we put a one-dimensional chain on the scale $s = \ln \ell$ and sandwiched C_P with the discrete version of the right of (11).

$$(\mathrm{P}) \ \wedge \ \sum_k \delta(b_k) \le S_\delta \quad \Longrightarrow \quad e^{-2T} C_P \text{ is bounded independently of } T = \ln(1/a) \tag{13}$$

Lean `conditional_scale_bound` (assumptions: the projection assumption, shortfalls non-negative with bounded partial sums, geometric-series weights; the Hardy sandwich for finite chains is a reproof of a known result). As a negative control, with a constant shortfall the boundedness breaks `const_defect_scale_unbounded`. The content is on the side of the projection assumption, and the projection assumption splits into four, "the variance can be split scale by scale", "the conditional Poincaré constants of each stage are uniform", "the conductance across stages is bounded", "the state space is a single line of scales", of which the last three have not been shown Assumption . The projection assumption is not weaker than the relaxation-time assumption. The one-dimensional chain did not show the upper-bound assumption; it restated the upper-bound assumption as the projection assumption.

Boundaries and the topological charge (the road of volume and boundary, the road of a one-dimensional chain of scales). In a periodic box the state space splits into sheets by topological charge Q, connected only at the ultraviolet end. For reversible dynamics, for every observable

$$R(f) = \frac{\operatorname{Var} f}{\mathcal{E}(f)} \ \le \ \tau_{\mathrm{int}}(f) \ \le \ C_P \tag{14}$$

holds Cited Lean `tau_int_le`, `rayleigh_le_tau` (finite reversible chains; a reproof of known facts), and every measured τ_{int} is a lower bound on C_P. The original source reports for HMC-type dynamics an effective dynamical exponent of the square of the topological charge of "about 5" Reading (Schaefer–Sommer–Virotta 2011); combining this with the conjecture that HMC and Langevin are in the same universality class Assumption and (14), in a periodic box $C_P \gtrsim a^{-5}$ and the upper-bound assumption breaks. It is a conditional reading on top of a conjecture. "About 5" is a value for $SU(3)$; the same measurement for $SU(2)$ was found in one source (HMC, periodic boundaries, small boxes, three points), reported to be close to a^{-2} Reading (Abe–Morikawa–Suzuki 2025; the note in 23-01 § 05). Open boundaries remove this barrier only inside a finite box with the physical side held fixed (a count using the charge of a subregion as test function Assumption ; in the original source too, the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction Reading). In this book's order, which makes the volume infinite first, this is expected not to help, and the candidate direction is to rewrite the statement of the relaxation-time assumption itself as a sentence sector by sector of topology, or as a local sentence with support limited to a few correlation lengths (the note in 23-01 § 05; how to rewrite it is dealt with in Part 24).

Thus every road on the upper side stopped by naming the place to which the difficulty had moved.

$$m^2 \gtrsim \sigma \ \Longleftarrow \ T_{\mathrm{loc}}(c) \wedge \text{floor} \wedge \text{seam}, \qquad C_P \lesssim \xi^2 \ \Longleftarrow \ (\mathrm{P}) \wedge \textstyle\sum \delta < \infty \tag{15}$$

The arrows go only one way, from right to left, and every right-hand side is an assumption.

06

Twenty-one walls — five types

A wall is a one-sentence statement of why a certain road, with certain tools, stopped short of a certain arrow of (1). There are five types: walls of principle come from the direction of an inequality and do not depend on the details of the tool; walls of structure are where demands are not compatible within the chosen family, bookkeeping or series (outside the family it is undecided); walls of tools are where the type of inequality needed is known but there is no tool to show it; walls of assumptions are the footing of conditional statements; walls of measurement and literature are what was not settled by numbers, boxes or primary sources. No wall is a proof that it "cannot be crossed".

Table 3 Twenty-one walls ("To move" is what follows from the definition)

No.	Wall	One line	To move	Arrow
1	One-sided test-function wall	Test functions bound C_P only from below Cited	An inequality of a type other than test functions	Upper-bound assumption
2	Spectroscopic upper-bound wall	Spectroscopy ends in upper bounds on the gap. A light state with small weight is invisible in a finite window Cited	A lower bound on the overlap with the lightest state	Physical-unit gap statement
3	Local-operator dimension wall	Local gauge-invariant operators have dimension 4 or more, so $\eta_s = 2$ (algebra under the assumption that the susceptibility is bounded)	Smooth out up to radius ξ ($\to$ measurement-cost wall)	Lower-bound assumption
4	Gauge-fixing single-valuedness wall	Gauge fixing: sharp means multivalued, blunt means the long wavelengths vanish Reading ($U(1)$)	A construction separating selection from smoothing (no proposal)	Lower-bound assumption
5	Measurement-cost wall	The numerator of smoothed operators is the zero-momentum susceptibility, and the cost of measuring it is $\geq cL^{d+2}$	Bound the numerator by estimates rather than measurement	Lower-bound assumption
6	Positive-weight kernel wall	The kernel with $\delta(2) = 0$ has a negative neighbouring weight. For kernels with positive weights $c_2 > 0$ (under the measured dependence)	Kernels outside the family	Summability of shortfalls
7	Renewal-series wall	The strong-coupling series has the renewal form, forces $\eta = 0$, and does not let the exponent be read	Information from outside the series	The η of the relaxation-time assumption
8	Single-step integral wall	One-link integration of a complete family works for one step only	A different bookkeeping that stacks steps	One-step bookkeeping
9	Limit-property wall	The Wilson-loop limit statement and the rotation-recovery statement do not come out of compositions of the map of one step	Another tool to show the properties of the limit	Limit
10	Upper-bound tool wall	There is no tool of the type of the upper-bound assumption. The proposal of stacking curvature scale by scale stops at the cross terms	Estimates of the cross terms	Upper-bound assumption

No.	Wall	One line	To move	Arrow
11	Tube-inequality wall	$m^2 \gtrsim \sigma$ moved to the tube inequality, but the tube inequality has no type of proof	A gauge-invariant definition of the transfer operator on loop space	One-scale chain
12	Projection-assumption wall	The projection assumption is not weaker than the relaxation-time assumption. The lower-bound assumption does not drop to one dimension	Derive the projection assumption from a weaker statement	Upper-bound assumption
13	Checking-tool wall	The Riemannian geometry of the one-link argument, and the generalised eigenvalue problem of the matrix variation of CP, lack tools for machine checking	Building up the library	One-step bookkeeping
14	Wall of the assumption itself	The relaxation-time assumption and the three chains (the relaxation chain, the one-scale chain, the bookkeeping-length chain) are not proved Assumption	The upper-bound assumption itself, the projection assumption, or one of the lower bounds of the three chains	The η of the relaxation-time assumption, chains
15	Equivalence-scaffold wall	The scaffold of the equivalence for the physical-unit gap statement: the power form, $\eta \geq 0$, additive errors of the steps, the δ of the one-loop kernel Assumption	Remove each assumption	Summability of shortfalls
16	Periodic-boundary conjecture wall	The breaking of the upper-bound assumption with periodic boundaries depends on the conjecture that HMC and Langevin are in the same universality class Assumption	Dynamical exponent of local dynamics for $SU(2)$	Upper-bound assumption
17	Fitting wall	That the breaking of $m/\sqrt{\sigma}$ is $O(a^2)$ is a fit to tables of numbers Reading	Estimates of the distortion of the steps	One-scale chain
18	Box-size wall	The boxes are small (8^4 is in the small-box phase for $\beta \gtrsim 2.45$, and 12^3 has $L\sqrt{\sigma} < 3$)	Large boxes with matched physical volume	Lower-bound assumption
19	Gauge dependence of the shortfall wall	The difference of c_2 with the choice of gauge went only as far as reading it positive in both of two gauges inside the model of the one-loop kernel Reading	A gauge-independent definition of δ	Summability of shortfalls
20	Tube-model sign wall	In the tube model, the sign of the $1/P$ coefficient at finite u is undecided	An effective model of the zero modes of shape	One-scale chain
21	Primary-source wall	Gaps in primary sources (the dynamical exponent of local dynamics for $SU(2)$; the texts of Bałaban, Magnen–Rivasseau–Sénéor and Osterwalder–Seiler)	Obtain them	Physical-unit gap statement

Walls of principle. The one-sided test-function wall and the spectroscopic upper-bound wall are the two one-sided halves of (8), and they do not move however many measurements are stacked. For example, for the standard normal distribution ($C_P = 1$, with equality at $f = x$), the maximum ratio over a family of odd polynomials not containing x approaches 1 as the family is enlarged, but the numbers inside the family alone do not tell whether 1 is the final value Reading (lower bound over a family). One knows it is 1 because of an input from outside the family ($C_P \le 1$ from the convexity $V'' = 1$), and an input of this kind is what the upper-bound assumption needs.

Walls of structure. Incompatibilities inside the chosen family; they say nothing about outside it. The local-operator dimension wall, the gauge-fixing single-valuedness wall and the measurement-cost wall are the ways the three roads of § 04 stopped; the positive-weight kernel wall is the tension between reflection positivity and removing the shortfall, the same tension, in the opposite direction, as what the problem statement says about continuum regularisations, "once reflection positivity is lost there are few ways to recover it". The renewal-series wall, the single-step integral wall and the limit-property wall are limits of the form of bookkeeping and series.

Walls of tools. They gather in the column of the upper-bound assumption. The upper-bound tool wall, the tube-inequality wall and the projection-assumption wall are the three places of § 05 to which the difficulty moved; the checking-tool wall alone is not a mathematical obstacle but a lack of tools for checking. The $\beta < 0.2722$ where the curvature theorem exists and the $\beta = 2.60$ used by spectroscopy (correlation length 1.95 lattice spacings) are about 9.6 times apart, and an estimate crossing that interval uniformly is the content of the upper-bound tool wall.

Walls of assumptions. The band statement $a^2 C_P \asymp 1$ splits into the product of three ratios $(a^2/\sigma)(\sigma/m^2)(m^2/\lambda_1)$, and the central equivalence also stands on footing other than the relaxation-time assumption. Each assumption corresponds to a concept of Parts 1–5 and Parts 7, 17 and 18: the relaxation-time assumption, the one-scale chain and the bookkeeping-length chain are candidates for restating "a single scale" (4-07) in terms of the Poincaré constant; the relaxation chain is the dynamical exponent $z = 2$ (18-03); the one-scale chain is a single scale (17-12); the bookkeeping-length chain is the specialisation of asymptotic scaling (5-05) to the bookkeeping length. These are analogies, not mathematical equivalences.

Walls of measurement and literature. They move by computation or by obtaining sources, but even if they move, the logic of (1) does not change. The representative is the size of the box: $12 \times 0.15896 = 1.91$ ($\beta = 2.55$) and $12 \times 0.13395 = 1.61$ ($\beta = 2.60$), both small boxes that do not contain 3 physical lengths $1/\sqrt{\sigma}$ Reading .

The walls in the column of the upper-bound assumption are the one-sided test-function wall, the upper-bound tool wall, the projection-assumption wall and the periodic-boundary conjecture wall, with no wall of measurement and literature. That is, short of the upper-bound assumption there is no wall that moves by measuring. Where the setters wrote that "new ideas are needed" is also the side of the gap uniform in the volume (Jaffe–Witten § 6.5) Cited , the same place where the walls that measurement does not move gather.

07

What remained — algebra closed in Lean, and numerical readings

Most of what remained from the seventeen roads is finite inequalities and algebra with the assumptions made explicit as arguments. Table 4 lines up the theorems cited in the text by what their statements are about. Inside Lean every theorem is a sentence about real numbers, sequences, or sums over finite sets; that they are quantities of the Wilson measure is a reading attached by the text. There are no theorems about expectation values, correlations or gaps of the Wilson measure (23-13).

Table 4 The main theorems closed in Lean on the side of the four-dimensional lattice (character: reproof = a known fact proved again / arrangement = algebra of definitions and assumptions / conditional = only the arrow from unproved assumptions to conclusions / negative control = an example that breaks when an assumption is removed / lattice theorem = a theorem about families of edges or effective actions, assuming only the shape of the lattice)

Theorem	Assumptions	Statement	Character
`log_F0_le`, `log_F0_ge`	Argument non-negative	Two-sided estimate (3) of the logarithm of one-link integration	Reproof
`Incidence.intact_frac_le`	Incidence regular	Fraction of intact faces $\leq$ fraction of remaining edges (4)	Arrangement
`no_blocking`, `blocking_shortfall`	Sequence of complete steps / fraction removed at each step $\leq 1/4$	Stacking steps does not bring intact faces back	Arrangement
`discStep_semigroup`, `step_lowers_coupling`	Surfaces of disc shape	Disc steps form a semigroup in the area, with $\beta' < \beta$	Reproof
`kappaDisj_lt_one`	$b \geq 2$	The contribution of surfaces sharing no edges is less than 1 (5)	Arrangement
`xi_eq_exp`	$\xi(\beta - t) = \xi(\beta)e^{-t/\kappa}$ for all steps	$\xi(\beta) = \xi(0)e^{\beta/\kappa}$	Arrangement
`fixed_step_bounded_iff`, `S4_two_sided`	Constant error / partial sums bounded	Partial sums bounded $\iff d = 0$ / stays in a positive band	Arrangement, conditional
`drain_exponent`, `drainExp_eq_zero_iff`	Power form of (7), $0 \leq \eta \leq 2, \theta \geq 0$	$p = 0 \iff \eta = \theta = 0$	Arrangement
`c2_ge_of_nonneg_weights`	$c2 = A + \kappa S$ (measured dependence)	$c2 > 0$	Conditional
`PerfectFamily.perfectK_one_iff`, `no_perfectZ_of_five`	$2 \leq d$ / $d \geq 5$	Classification of complete families of edges (Table 1)	Lattice theorem

Theorem	Assumptions	Statement	Character
`StaplePerturbation.hess_gauge`	Period even and ≥ 3, unit quaternions, pure imaginary perturbations	Hessian form of the effective action $\geq -27\beta^2$	Lattice theorem
`renewal_chi_mul_massSq`	Coefficients of the renewal form positive	$\chi m_0^2 = A/c_0$	Arrangement
`dev_le_of_contracting`, `dev_eq_of_marginal`	$0 \leq \lambda < 1$ / $\lambda = 1$	Drift bounded / proportional to the number of steps	Arrangement
`overlap_le_susc_mul_gap`	$m > 0$, $\tilde{Z} \coth(m/2) \leq \chi$	$\tilde{Z} \leq \chi m$	Arrangement
`profileQuotient_isGreatest`	Finitely many momenta, denominator positive	The supremum of the ratio over a rank-one family is the maximum of the componentwise ratios	Reproof
`eta_ge_two_of_bounded_susc`	Susceptibility uniformly bounded	$\eta_{\boldsymbol{s}} \geq 2$	Conditional
`orbitAvg_gauge_invariant`	Action of a finite group	Weighted averages over orbits are gauge-invariant	Reproof
`meff_antitone`, `meff_ge_of_corr`	Representation with non-negative weights	The effective mass decreases monotonically and is an upper bound on the lowest energy	Reproof
`gap_ge_of_overlap`, `light_admixture_invisible`	Lower bound q on the overlap / two-level example	(10) / a light state is invisible in the window	Conditional, negative control
`cost_ge`	Lower bound on the relaxation time, number of sweeps	Cost $\geq cL^{d+2}$	Conditional
`mass_sq_ge_of_tube`, `mass_bound_localized`	Consequence of the tube inequality / localised T, floor, seam	$m^2 \geq 4c\sigma$ (12)	Conditional
`linear_regime_breaks_onescale`, `casimir_only_no_bound`, `linear_deficit_breaks_inv`	Region of convergence / floor only / raw perimeter	Under each assumption the lower half of a single scale is false	Negative control
`conditional_scale_bound`, `const_defect_scale_unbounded`	The projection assumption, bounded shortfall / constant shortfall	(13) / boundedness breaks	Conditional, negative control
`tau_int_le`, `rayleigh_le_tau`	Finite reversible chains	(14)	Reproof

Theorem	Assumptions	Statement	Character
`comp_class_eq_of_equivariant_bijection`	Bijection, commuting with the action	The class of functions of the flowed field is the same as the class of invariant functions	Arrangement

The content of the conditional rows lies on the side of the assumptions, and the theorems do not take a single step towards showing those assumptions. Only the two rows of lattice theorems (the classification of complete families and the lower bound on the Hessian form of the effective action) are theorems on the lattice that assume only the shape of the lattice. The former is a combinatorial fact about the lattice; the latter is an estimate that works on the strong-coupling side, where the correlation length is about 1/10 of a lattice spacing. Both concern places far from the continuum limit.

What remained on the numerical side is readings Reading : the agreement of the slope in the model of the one-loop kernel with the known coefficient (relative 2.8×10^{-5}, inside the model), the values of c_2 (two weights and two gauges), the effective slope 0.304 read from the lattice $a\sqrt{\sigma}$ of the original source, the upper bound 0.501 ± 0.071 on the effective mass of smoothed operators ($\beta = 2.6$, 12^4), the scatter of gauge-fixed copies for compact $U(1)$, the strong-coupling range where the curvature of the Gribov region is positive (samples of a finite-dimensional model), and the curvature of the pushed-forward measure of the gradient flow (at the typical points measured). None of them is used in a proof of an upper or lower bound.

08

Why we step down to the first step — which tool stopped at which wall

Summing up § 04– § 06: there is no tool to attack the upper side of the relaxation-time assumption (the upper-bound assumption, every f) while staying on the four-dimensional lattice. The tools of the lower side (test functions, spectroscopy) stopped at walls of principle, and the tools of the upper side (curvature, tubes, one-dimensional chains) all stopped where they had moved the difficulty into a single assumption. Where each stopped is as in the following table, tool by tool. The right column is the prospect of what changes when the same tool is brought to the model of the first step (the path measure of the double well in 0+1 dimensions, 23-04).

Table 5 Correspondence of tools and walls, and what changes on the first step (the right column is a prospect; the content on the first step is in each chapter)

Tool	Wall where it stopped on the four-dimensional lattice	What changes on the first step
Test functions, spectroscopy (lower side)	One-sided test-function wall, spectroscopic upper-bound wall, local-operator dimension wall, gauge-fixing single-valuedness wall, measurement-cost wall	On the lower side, a family of test functions local in time reaches equality in $C_P \geq \xi^2$ (the One-Dimensional One-Sided Bridge Proposition, 23-04 § 04). That the upper side needs different tools does not change on the first step either

Tool	Wall where it stopped on the four-dimensional lattice	What changes on the first step
Curvature (Bakry–Émery)	Upper-bound tool wall (positive only at strong coupling; cross terms)	The measure as a whole has two valleys and is not convex, but inside one phase without kinks local Poincaré can be asked through an assembly of sectors (the in-phase local Poincaré condition, 23-06)
One-dimensional chain of scales (the projection assumption)	Projection-assumption wall (the projection assumption is not weaker than the relaxation-time assumption)	The slow degree of freedom is the position of the kinks, and the form of the projection to one dimension is visible from the model. The projection assumption splits into, and is named as, transport along the family of saddles (the saddle-tube condition) and reduction to the box dynamics (the box-reduction condition) (23-07, 23-11)
Transfer operator of tubes (the tube inequality)	Tube-inequality wall (there is no type of proof for the lower bound on the mass)	The side of the mass is not needed: the correlation length $\xi = \hbar/\Delta$ is obtained accurately and independently from the tunnelling splitting, and what it is compared with is clear (23-04)
One-step bookkeeping (the summability-of-shortfalls assumption)	Positive-weight kernel wall, single-step integral wall, equivalence-scaffold wall	No need to make a bookkeeping length: the correlation length grows in $\hbar$ at the rate $e^{S_0/\hbar}$, and the length to compare with is fixed directly by the measure
Boundaries and the topological charge	Periodic-boundary conjecture wall (the breaking in a periodic box rests on a conjecture)	Stated in a regime where the length of the paths T is made infinite first, so the question is set up in a form where the topological sheets of a periodic box do not appear
Boxes and primary sources	Walls of measurement and literature (box size, gauge dependence of the shortfall, tube-model sign, primary sources)	With numerics of the transfer operator, semiclassical formulas and interval arithmetic, many of the quantities compared are obtained to high precision or as inequalities (23-08)

Some things do not change. That the upper-bound assumption is an inequality over "every f", and that tools of the lower side cannot reach it in principle (the one-sided test-function wall), are the same on the first step. So what the first step does is to build the assembly of the upper side first, in the form of propositions with their assumption rows named. The relaxation-time assumption is a sentence independent of dimension (23-01 § 05), so we build the same sentence in the smallest model where the ruler grows exponentially in the parameter, the slow motion is the motion of "changing phase", and the quantities compared are known exactly.

Even if the first step is closed, there is no guarantee that it lifts to the four-dimensional lattice. What has a prospect of lifting is the form of the sentence and the form of the tools; constants and ranges of parameters do not lift (23-03 § 06). The table of walls in this chapter is kept as a table for reading which walls the tools of the first step might hit again on the second step.

The relaxation-time assumption is one restatement chosen as a handle, and the roads and walls of this chapter too are within the scope of a study with the relaxation-time assumption at its axis. Viewpoints other than the relaxation-time assumption are dealt with in Part 24.

What this chapter can say

Said	Not said
The arrangement of seventeen roads and twenty-one walls (Tables 2 and 3), the correspondence of tools and walls (Table 5)	None of the arrows of (1) is proved. "Stopped" does not mean that anything was shown about the mass gap or the continuum limit, nor that the propositions not reached are false
Cited The variational principle and finite windows, the lowest dimension of local operators, reflection positivity, the strong-coupling area law, the one-loop coefficient, dimension counting, the Bakry–Émery and Muckenhoupt inequalities, $R \le \tau_{\mathrm{int}} \le C_P$	Counting these as results
Proved here The count (2) giving $d \le 4$ for complete families, the identity (4) of the deficit of bookkeeping, the algebra of powers in (7), the algebra of (10) and (12), the ratio 1 for bare link angles, the closed formulas for gauge fixing on a ring. Lean The theorems of Table 4 (assumptions on the same row)	Proofs of the relaxation-time assumption, the three chains, the tube inequality, the projection assumption, boundedness of the susceptibility, lower bounds on the overlap, the measured dependence of c_2, the model of errors of the steps
Reading The agreement of κ in the model of the one-loop kernel and c_2, the effective slope of the lattice, upper bounds from spectroscopy, gauge fixing for $U(1)$, the curvature of the Gribov region and of the gradient flow, the effective exponent of the topological charge	That the numbers of the models are non-perturbative quantities of four-dimensional $SU(2)$. Using readings as proofs of upper or lower bounds
Assumption That one-link integration works only once, the renewal form, the breaking of the upper-bound assumption in a periodic box (on top of a conjecture), what changes on the first step (the prospects in the right column of Table 5)	That results of the first step lift to the second step and beyond. About the mass gap and the continuum limit, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
One-link integration, the heat-bath method, the strong-coupling area law, two-dimensional decimation	Cited	Source tables of 5-07, 5-02 and 7-02 (K. G. Wilson, Phys. Rev. D 10 (1974) 2445; K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440; A. A. Migdal, Zh. Eksp. Teor. Fiz. 69 (1975) 810. Bibliographic data only).
One-loop coefficient b_0	Cited	D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343. H. D. Politzer, ibid. 1346 (bibliographic data only).
Lattice $a\sqrt{\sigma}$ and $am(0^{++})$	Reading	B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027, Table 1 (text checked).

Item	Kind	Source / tools
Variational principle, finite windows, Bakry–Émery	Cited	The tables of sources of 1-10 and 4-11. D. Bakry, M. Émery, Lecture Notes in Math. 1123 (1985) 177–206 (bibliographic data only).
What multiscale Bakry–Émery has reached	Cited	Theorem 1.1 of R. Bauerschmidt, B. Dagallier, arXiv:2202.02295 (text checked). Theorem 2.5 of R. Bauerschmidt, T. Bodineau, Comm. Pure Appl. Math. 74 (2021).
Two-weight Hardy inequality	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38 (bibliographic data only).
Local gauge symmetry, Gribov, stochastic gauge fixing, averages over orbits	Cited	S. Elitzur, Phys. Rev. D 12 (1975) 3978. I. M. Singer, Commun. Math. Phys. 60 (1978) 7. C. Parrinello, G. Jona-Lasinio, Phys. Lett. B 251 (1990) 175. D. Zwanziger, Nucl. Phys. B 345 (1990) 461. D. Mehta, M. Schröck, Phys. Rev. D 89 (2014) 094512 (abstract checked; the others bibliographic data only).
Wilson flow, APE	Cited	M. Lüscher, JHEP 08 (2010) 071, arXiv:1006.4518 (abstract checked). M. Albanese et al., Phys. Lett. B 192 (1987) 163 (bibliographic data only).
Effective exponent of the topological charge, HMC and Langevin	Reading, Assumption	S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93, arXiv:1009.5228. M. Lüscher, S. Schaefer, JHEP 07 (2011) 036, arXiv:1105.4749 (abstracts of both checked). M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 (2025) 063B03, arXiv:2501.00286 § 3.2, Figure 6 (HMC for $SU(2)$; text checked).
Holographic models	Cited (inside the model)	U. Gürsoy, E. Kiritsis, F. Nitti, JHEP 0802 (2008) 019, arXiv:0707.1349. C. Csáki, H. Ooguri, Y. Oz, J. Terning, JHEP 9901 (1999) 017, hep-th/9806021 (abstracts checked). 17-09.
Problem statement § 6.5	Cited	A. Jaffe, E. Witten, the Clay problem statement (the table of sources of 0-03).
Theorems and assumptions of Table 4	Lean	Bundles `LatticeGaugeCoarseGraining`, `LatticeGaugeRunning`, `LatticeGaugeMultiStep`, `LatticeGaugeSpectralHinge`, `PerfectFamilyEdges`, `LatticeGaugeOneLink`, `LatticeGaugeOverlapExponent`, `LatticeGaugeTubeEntropy`, `LatticeGaugeWulffCorrection`, `LatticeGaugeHardyLadder`, `LatticeGaugeAutocorrelationBound` and others. The `#print axioms` at the end of each bundle is a subset of the three standard axioms. Statements and limits in 23-13.
Numbers of the models (c_2, slopes of windows, gauge fixing, curvature, flow)	Reading	Computations of this part's record for the four-dimensional lattice (the model of the one-loop kernel, finite-dimensional models, Monte Carlo in small boxes). Values and conditions in each row of the text.

Words used in this chapter — one-step bookkeeping, complete family, one-link integration, shortfall $\delta(b)$, the form of the answer, the seventeen roads tried on the four-dimensional lattice, the twenty-one walls (five types), renewal form, lower bound on the overlap, the tube inequality, the projection assumption, topological charge, negative control. Their meanings are in A-02 Glossary.

To the next chapter — 23-03 The four steps of the staircase — why we start from 0+1 dimensions: divides the relaxation-time assumption into four steps, and fixes the conditions for choosing the double well for the first step and the form of the statement of the first step

(the double-well relaxation-time power statement and the Double-Well Relaxation-Time Upper-Bound Proposition).

Next chapter to read: 23-03 The four steps of the staircase — why we start from 0+1 dimensions.

The four steps of the staircase — why we start from 0+1 dimensions

> This chapter is this book's own investigation. § 03 and § 06 are a map of open questions; § 07 is guide material.

What this chapter shows — the diffusive-relaxation-time assumption is a sentence of a dimension-independent form: "the relaxation time of a path measure grows as a power of the square of the correlation length of that measure". So we divide the diffusive-relaxation-time assumption into four steps (the double well in 0+1 dimensions / the four-dimensional lattice, non-abelian / the continuum limit and the OS axioms / the mass gap in the limit), and give the reasons for choosing the double well for the first step. The statement of the first step is "the double-well relaxation-time power statement" (below, the double-well power statement), $\ln C_P / \ln \xi \to 2$, and putting its upper side in the form $C_P \le K \xi^2 S_0/\hbar$ (the First-Step Upper-Bound Proposition) is the centre of the assembly of the first step. The First-Step Upper-Bound Proposition is a conditional proposition on top of four conditions, and is not proved. We give both the prospects of lifting from the first step to the second and beyond, and the reasons it does not lift.

Prerequisite chapters — 23-00 (the entrance, the five labels), 2-07 (the path integral and imaginary time), 2-05 § 07 (the tunnelling splitting of the double well). Definitions on the side of the four-dimensional lattice are in 23-01.

Said first — The relaxation-time assumption is an assumption both on the four-dimensional lattice and for the double well. The First-Step Upper-Bound Proposition is algebra on top of the four double-well conditions, and every one of the four conditions leaves assumption rows. This chapter fixes the form of the statement and proves nothing new.

The order of this chapter

1. The relaxation-time assumption is a sentence of a dimension-independent form — a family of measures, the correlation length, the relaxation time
2. Definition of the four steps — the form in which each step can be called "closed" (Table 1)

3. Why we do not attack it on the four-dimensional lattice as it is — there are no tools for the upper side
4. Conditions on the model chosen for the first step — a ruler that grows exponentially, two phases, quantities that can be compared (Table 2)
5. The power statement and the form of the First-Step Upper-Bound Proposition — the lower bound ξ^2 and the target for the upper bound
6. What lifts and what does not (Table 3)
7. A map inside the first step — a guide to 23-04 to 23-11

01

The diffusive-relaxation-time assumption is a sentence of a dimension-independent form — a family of measures, the correlation length, the relaxation time

Three things appear in the relaxation-time assumption: a family of probability measures μ_s lined up by one parameter, the correlation length $\xi(s)$ of each μ_s, and the Poincaré constant $C_P(s)$ of each μ_s (the relaxation time of the Langevin dynamics having that measure as equilibrium). When moving the parameter gives $\xi(s) \to \infty$, the relaxation-time assumption is the one sentence

$$(\xi^2): \quad \lim \frac{\ln C_P(s)}{\ln \xi(s)} = 2 \quad (1)$$

This sentence does not say on what space the measure lives. On the four-dimensional lattice μ_s is the Wilson measure with $s = \beta \to \infty$ (23-01); for the double well in 0+1 dimensions μ_s is the measure on paths in imaginary time $e^{-S/\hbar}$ with $s = 1/\hbar \to \infty$ (23-04). Sentences of the same form can be lined up with the object changed.

Example: for the free field (a Gaussian chain of mass m_0), $C_P = 1/m_0^2$ and $\xi \approx 1/m_0$, and (1) holds as $m_0 \to 0$ (23-00 § 04) Proved here . With interaction, new mechanisms that slow relaxation enter (crossing barriers, moving between configurations in different phases), and (1) is no longer trivial.

Precisely, "the relaxation-time assumption" in this part refers only to the sentence (1) at the power level. A sentence stated within the width of a constant factor (a band statement) is called by another name, not the relaxation-time assumption (23-01). Also, the four conditions of the first step are written "the four double-well conditions", always with the word "conditions" so as not to be confused with the relaxation-time assumption. The letters in Lean theorem names (H1 and so on) are not changed.

02

Definition of the four steps — the form in which each step can be called "closed"

We place four steps, from easy places to hard ones.

- **The first step**: the path measure of the double well in 0+1 dimensions. Parameter $\hbar \to 0$.
- **The second step**: non-abelian gauge theory on the four-dimensional lattice (the Wilson measure of SU(2) or SU(3)). Parameter $\beta \to \infty$.
- **The third step**: the continuum limit as the lattice spacing $a \to 0$. That the limiting measure exists and satisfies the Osterwalder–Schrader axioms (4-10).
- **The fourth step**: the mass gap of the limiting theory (4-11).

Table 1 Object, target, tools and state in this book, step by step

Step	Object	Form in which it can be called "closed"	Main tools	State in this book
First step	The path measure of the double well	State the upper side of the double-well power statement with a table of finitely many named assumption rows, and fill each row with a proof, an original source, or computer assistance	Splitting of the variance, local Poincaré, reduction to the box, interval arithmetic	Up to the First-Step Upper-Bound Proposition (algebra on top of the four double-well conditions). Every one of the four conditions leaves Assumption rows (23-05, 23-12)
Second step	The Wilson measure on the four-dimensional lattice	The sentence (1) of the same form, as $\beta \to \infty$	Prospect of transferring the types made on the first step (§ 06)	The relaxation-time assumption is Assumption. Only estimates from below have been reached (23-02)

Step	Object	Form in which it can be called "closed"	Main tools	State in this book
Third step	The measure of the continuum limit	Existence of the limit and the OS axioms	The sequence of renormalisation group steps, constructive field theory (Part 7)	Not dealt with. That a theory satisfying the Wightman axioms can be made as the limit of four-dimensional Wilson lattice gauge theory is, in the committee's assessment, proved in Lean by the construction published on 2026-11-10 (Part 6). The uniqueness of the infinite-volume limit and the continuum probability measure have not been shown (6-04)
Fourth step	The reconstructed quantum field theory	The gap of the Hamiltonian is positive	The OS reconstruction (4-10)	Not dealt with. The published construction does not show it either (6-04)

"Closed" for the first step means putting the proposition of the upper side into a form where "which rows are proofs and which are assumptions" can be read on one table. Only when every row is filled with a proof, an original source or computer assistance, and the stage of the First-Step Upper-Bound Proposition goes through as a single proof, does the first step become a theorem. At present it is not so (23-12).

To go up from the second step, one more bridge is needed between each pair of steps. Even if the sentence (1) of the second step holds, that alone does not give the existence of the continuum limit (the third step), and the gap in the limit (the fourth step) is a form that can be reread only after adding to the upper side of (1) the assumptions joining the correlation length to physical length (the chains of 23-01). This part has no theorem of implication for any bridge between steps.

```
03
```

Why we do not attack it on the four-dimensional lattice as it is — there are no tools for the upper side

In the record for the four-dimensional lattice (23-02), every road through test functions, gauge fixing and spectroscopy ended in lower bounds on C_P or upper bounds on the mass, and no tool reaching the upper side (C_P not too large, over every f) has been found. There are estimates on the strong-coupling side (C_P bounded) and the Bakry–Émery type that works for convex measures, but neither reaches the regime $\xi \to \infty$. In an example applying a known volume-uniform criterion to the path lattice of the double well, the upper bound obtained is of the type $e^{c\xi}$ (doubly exponential in $\hbar$), read as far from a power statement Cited (criterion: Theorem 2.1 of Bauerschmidt–Dagallier) Reading ($c \approx 1.4$ to 1.8; numbers for $\varepsilon = 0.25$, $\hbar \in [0.1, 0.4]$). So we first make an "assembly from above" on the smallest test bed where the relaxation-time assumption is expected to hold. The detailed roads and walls are in 23-02.

04

Conditions on the model chosen for the first step — a ruler that grows exponentially, two phases, quantities that can be compared

There are three conditions we ask of the model of the first step.

1. **The ruler grows exponentially in the parameter.** The correlation length of the four-dimensional lattice grows exponentially in β in the one-loop count (4-07). To make a test bed that does not confuse powers with exponentials, a ruler growing at the same rate is needed.
2. **The slow motion is the motion of "changing phase".** What slows relaxation is a rare event of moving between two (or more) phases. There are two scales: fluctuations inside the two phases are fast, and only the motion of changing phase is slow.
3. **The quantities compared are known exactly.** The correlation length can be found to high precision by an independent method. If what it is compared with is blurred, the growth of C_P/ξ^2 will be misread.

The path measure of the double well in 0+1 dimensions, $V(x) = (x^2-1)^2/8$, satisfies all three. The ruler grows exponentially as $\xi = \hbar/\Delta \propto \hbar^{1/2} e^{S_0/\hbar}$ (Δ is the tunnelling splitting, $S_0 = 2/3$ the action of the path moving from valley to valley); the two phases are "the intervals where the path is in the right valley" and "those where it is in the left valley", and the event of changing phase is a kink (a short transition with action S_0). Δ is found by two routes, the semiclassical formula ((10) of 2-05 § 07) and numerics of the transfer operator (23-04).

The action S_0 of one kink, and the correlation length $\xi \propto e^{S_0/\hbar}$. This pair has, in the smallest dimension, the structure in which "the action of an instanton" and "a correlation length growing exponentially" sit side by side on the four-dimensional lattice. This is an analogy, however, not a theorem of correspondence.

Table 2 How the double well and the four-dimensional lattice line up (the right column is an analogy, not a theorem of correspondence)

Item	Double well (first step)	Four-dimensional lattice (second step)
Measure	The path measure $e^{-S/\hbar}$ (the line of imaginary time)	The Wilson measure (four-dimensional lattice)
Parameter and limit	$\hbar \to 0$	$\beta \to \infty$
Ruler	$\xi \propto \hbar^{1/2} e^{S_0/\hbar}$ Cited (leading semiclassical term)	$\propto e^{3\pi^2\beta/11}$ from one-loop bookkeeping (SU(2); that it agrees with ξ is Assumption)
Slow degrees of freedom	The positions of kinks (action $S_0 = 2/3$)	The positions, sizes and orientations of instantons, the density of topological charge (analogy)
Lower side	$CP \geq \xi^2$ (One-Dimensional One-Sided Bridge Proposition, 23-04 § 04)	Lower bounds from test functions (23-02)
Upper side	Conditional on four conditions (the First-Step Upper-Bound Proposition)	None

The first step is an easy model, but not too easy. No known theorem giving the upper side of the double-well power statement was found in what we searched. There are theorems of Eyring–Kramers type that hold the length of the interval fixed and take $\hbar \to 0$, but the regime that makes the length of the paths T infinite first ($T \gg \xi$) is outside the scope of those theorems Cited (Brooks–Di Gesù 2026 state explicitly in the text that the regime of infinite length and the family of translations are outside their scope).

05

The power statement and the form of the First-Step Upper-Bound Proposition — the lower bound ξ² and the target for the upper bound

The statement of the first step, for the path measure of the double well (taking $T \to \infty$ first), is

$$\text{power form of Assumption } (\xi^2): \quad \lim_{\hbar \to 0} \frac{\ln C_P(\hbar)}{\ln \xi(\hbar)} = 2 \quad (2)$$

Setting $R := CP/\xi^2$, $\ln CP / \ln \xi = 2 + \ln R / \ln \xi$, so (2) is the same as "$\ln R$ is small compared with $\ln \xi$".

The label "power assumption (ξ^2)" on the left of (2) is what the text calls the double-well power statement (the double-well relaxation-time power statement).

The lower side. $R \geq 1$, that is $C_P \geq \xi^2$, holds for every well in 0+1 dimensions (the One-Dimensional One-Sided Bridge Proposition; the proof is in 23-04 § 04).

The target for the upper side. The position of a kink moves with diffusion coefficient $D = \hbar / S_0$, and the time to cross a length ξ by diffusion is $\xi^2 / D = \xi^2 S_0 / \hbar$ (23-04 § 03). So we set the target for the upper side as

$$C_P \;\leq\; K\,\xi^2\,\frac{S_0}{\hbar} \qquad (K \text{ does not depend on } \hbar\,) \quad (3)$$

In the count at the level of physics (counting kinks as a dilute gas) it is read as $C_P \approx \xi^2 S_0/\hbar$, and (3) is that with a margin of a constant factor Assumption (the count is heuristic, not a proof).

From (3) to (2). From (3) and $R \geq 1$, $1 \leq R \leq K S_0/\hbar$. When $\hbar$ is small, $\ln \xi \geq s/\hbar$ (for some $s > 0$), so

$$0 \;\leq\; \frac{\ln R}{\ln \xi} \;\leq\; \frac{\hbar\,\big(\ln(K S_0) + \ln(1/\hbar)\big)}{s} \;\longrightarrow\; 0 \qquad (\hbar \to 0) \quad (4)$$

and (2) follows Proved here Lean `power_H_of_kink_ratio` (assumptions: $0 < K$, $0 < s$, $0 < \delta$, and for $0 < \hbar < \delta$, $1 \leq R \leq K/\hbar$ and $s/\hbar \leq \ln \xi$. The K of Lean is the $K S_0$ of the text). $\ln \xi \geq s/\hbar$ is a row coming from the leading semiclassical term $\ln \xi \approx S_0/\hbar$, and in the lattice model it is Reading. The values are in 23-04 § 07.

The form of the First-Step Upper-Bound Proposition

(3) is not proved. The assembly of the first step splits (3) into the following four conditions, and puts it in the form of a conditional proposition: on top of them, (3) follows.

The First-Step Upper-Bound Proposition (conditional; statement only)

For the path measure of the double well in 0+1 dimensions, assuming the four double-well conditions, taking $T \to \infty$ first, as $\hbar \to 0$, $C_P \leq K \xi^2 S_0/\hbar$ (K depends only on the constants of the conditions). Hence the double-well power statement (2) follows.

- The in-phase local Poincaré condition (local Poincaré within a phase without kinks)
- The saddle-tube condition (transport along the family of flat saddles)
- The box-reduction condition (reduction to the box dynamics)

- The gap-inside-the-box condition

The content of the four conditions, the algebra of the splitting of the variance for the First-Step Upper-Bound Proposition, and how what has been shown is separated from the assumption rows are in 23-05. The current status of each condition is in 23-06 to 23-11, and the table putting all rows on one sheet is 23-12.

Precisely, every one of the four conditions leaves assumption rows at present. And there is not yet a single $\hbar$ at which all rows of all conditions hold at once (23-12). The First-Step Upper-Bound Proposition is a proposition naming "what must be shown for the first step to close", not a proposition saying that the first step has closed.

06

What lifts and what does not

Even if (3) closes on the first step, the sentence of the second step (the four-dimensional lattice) does not follow. This section states, with reasons, what has a prospect of lifting and what does not. Both are descriptions of prospects, not theorems of implication.

Table 3 From the first step to the second (prospects)

	Prospect of lifting	Reason it does not lift
Form	The sentences (1) and (2) do not depend on dimension. The two-scale split "fast inside a phase / slow when changing phase" can also be written in the same form for any measure with slow degrees of freedom	The slow degrees of freedom of the four-dimensional lattice (collective coordinates of instantons, the density of topological charge) increase in number in proportion to the volume. Even on the first step, when the path is much longer than ξ, a split taking only one slow variable cannot be used (the band of 23-04 § 05). In four dimensions this situation is expected to be stronger still
Tools	The splitting of the variance, the lemma of two projections, the Efron–Stein inequality and the stage of sums over sectors are algebraic stages that do not use the dimension of the space of the measure, and can be written as they are	Many stages of the first step depend on properties peculiar to one-dimensional nearest-neighbour measures (the transfer operator, the boundary between phases being a point, kinks being countable as a dilute gas). Gauge symmetry and the geometry of the orbit space (3-04) are absent from the first step
Constants	—	The constants of the first step were fixed by computer assistance and numerical readings at particular nodes and windows of $\hbar$, and give no information about the β of the four-dimensional lattice. The correspondence between $\hbar$ and β is also an analogy

We still start from the first step because the procedure itself of assembling the proposition of the upper side in a form where "which rows are assumptions" is visible is also needed on the second step. The procedure of naming the assumption rows, checking the direction of dependence (23-12), and separating what machine checking guarantees (23-13) does not depend on the model. Taking the procedure through on the first step means that when a proposition of the same form is set up on the second step, we can name which rows are newly needed.

Precisely, setting up "a sentence of the same form" on the second step is not evidence that the sentence of the second step holds. Even if the relaxation-time assumption is read as holding for the model of the first step (counts at the level of physics and readings of lower bounds), whether it holds on the second step is a separate question (23-15).

07

A map inside the first step — a guide to 23-04 to 23-11

Chapters of the first step

4. 23-04 The path measure of the double well, and kinks — definition of the setting, kinks, the lower bound $CP \geq \xi^2$ [Y2]–[Y3]
5. 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition — the splitting of the variance, and the assembly of the upper-bound proposition [Y3]
6. 23-06 The in-phase local Poincaré condition — local Poincaré in a phase without kinks [Res]
7. 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring [Y3]
8. 23-08 What computer-assisted means — the conditions on the chain inside the window — checking inequalities by interval arithmetic [Y2]–[Y3]
9. 23-09 The sector framework and the three strands of the window log-Sobolev route — the common wall of the plateau at the top [Res]
10. 23-10 The list of detour plans — types of breakage [Y3]–[Res]
11. 23-11 Box reduction and the gap inside the box — the constant of the core — the box-reduction condition, the gap-inside-the-box condition, and the constant of the core [Res]

The [Res] chapters place at the start a summary section that can be read at the [Y3] level, so one can read only the summaries and go on to 23-12.

What this chapter can say

Said	Not said
The way of building it by writing the relaxation-time assumption in the dimension-independent form (1) and dividing it into four steps (a convention)	Any implication between steps. The relaxation-time assumption on the second step, the continuum limit, the mass gap in the limit
Proved here The algebra (4) by which the double-well power statement (2) follows from (3), the lower bound $R \geq 1$ and $\ln \xi \geq s/\hbar$ Lean `power_H_of_kink_ratio` (assumptions are the rows of § 05)	(3) itself. The First-Step Upper-Bound Proposition is conditional on the four double-well conditions, and every condition leaves assumption rows
Cited That Eyring–Kramers-type theorems with the interval held fixed place the regime taking $T \to \infty$ first outside their scope	That there is no known theorem giving the upper side in this regime (only as far as: none was found in what we searched)
Prospects of lifting and reasons it does not lift (Table 3, prospects)	That a proposition of the same form can be set up, or holds, on the second step

Sources and reproduction

Item	Kind	Source / tools
Definitions of the Poincaré constant and the correlation length	Cited	The table of sources of 23-00.
Tunnelling splitting of the double well, and S_0	Cited	(9)(10) of 2-05 § 07 and its table of sources.
Eyring–Kramers-type theorems, and their scope	Cited	G. Di Gesù, D. Le Peutrec, arXiv:1506.04434 (Theorem 1.1); B. Brooks, G. Di Gesù, arXiv:2606.16083 (the statement of scope on p.4 of the text). Both checked in the text.
A volume-uniform criterion, and its values on the path lattice of the double well	Cited, Reading	R. Bauerschmidt, B. Dagallier, arXiv:2202.02295 (Theorem 2.1, checked in the text). Values from numerics of the transfer operator (step $\varepsilon = 0.25$, $T = \infty$).
The algebra (4) of the double-well power statement	Proved here, Lean	§ 05 of the text. Theorem `power_H_of_kink_ratio` (assumptions in § 05). The statement is in 23-13.
The count $C_P \approx \xi^2 S_0/\hbar$	Assumption	A count at the level of physics using a dilute gas of kinks and the diffusion coefficient of the projection (23-04 § 03).
Statement of the First-Step Upper-Bound Proposition	With a table of assumptions	The content is in 23-05.

Words used in this chapter — staircase (four steps), the double-well power statement, the four double-well conditions, the First-Step Upper-Bound Proposition, kink, tunnelling splitting, phase (phase without kinks), local Poincaré, box dynamics (box reduction). Their meanings are in A-02 Glossary.

Next chapter to read: 23-04 The path measure of the double well, and kinks — defines the setting of the first step, and derives the lower bound $C_P \geq \xi^2$ and the form of the target for the upper bound.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time double well, kinks, lower bound · Kiichi, Shiori and Rin

The path measure of the double well, and kinks

> In this chapter, § 01– § 03 are established knowledge and § 04 onward are this book's own investigation.

What this chapter shows — The setting of the first step is the measure $e^{-S/\hbar}$ on paths in imaginary time for the double well $V(x) = (x^2-1)^2/8$. The correlation length $\xi = \hbar/\Delta$ (Δ is the tunnelling splitting) grows at the rate $\hbar^{1/2}e^{S_0/\hbar}$, where $S_0 = 2/3$ is the action of the path moving from valley to valley (the kink). The Poincaré constant C_P of whole paths is at least ξ^2 (the One-Dimensional One-Sided Bridge Proposition), and the target for the upper side is a constant times the diffusion time of kinks, $\xi^2 S_0/\hbar$. At three values of $\hbar$, $\ln \xi$ is 218 / 661 / 2213 (readings).

Prerequisite chapters — 2-07 (the path integral, imaginary time, the transfer matrix), 2-05 § 07 (the double well), 1-10 § 07– § 09 (Poincaré inequalities), 23-03 (the staircase and the double-well relaxation-time power statement).

Said first — § 01– § 03 and § 07 are written at [Y2], § 04– § 06 at [Y3]. No upper bound on C_P is shown here. The One-Dimensional One-Sided Bridge Proposition is a lower bound, and the target for the upper side (15) is the conclusion of the conditional Double-Well Relaxation-Time Upper-Bound Proposition (23-05). All the numbers in this chapter are numbers of a 0+1-dimensional model, not of four-dimensional gauge theory.

The order of this chapter

1. Setting — the action, the path measure, the lattice ε=½, the Dirichlet form, C_P
2. Two times — the correlation length ξ=ħ/Δ and the Langevin gap 1/C_P
3. Kinks — action S_0=2/3, the position degree of freedom, diffusion coefficient D=ħ/S_0, how ξ grows
4. The lower bound C_P ≥ ξ² — the One-Dimensional One-Sided Bridge Proposition, and readings beyond it
5. The target for the upper side, and the band of long paths — why the naive reduction cannot be used
6. The plateau at the top — the curvature falls ¼ short of convexity

7. Numerical examples — $\ln \xi$ at three values of $\hbar$ (Table 2). Detour: the tunnelling splitting by Herring's formula

01

Setting — the action, the path measure, the lattice, the Dirichlet form

A particle of mass 1 sits in a potential with two valleys.

$$V(x) = \frac{(x^2-1)^2}{8}, \qquad V'(x) = \frac{x(x^2-1)}{2}, \qquad V''(x) = \frac{3x^2-1}{2} \quad (1)$$

The valleys are at $x = \pm 1$, the curvature at the bottom of a valley is $V''(\pm 1) = 1$ (the angular frequency of oscillation inside a valley is 1), and the top of the hill is at $x = 0$ with $V''(0) = -\frac{1}{2}$. This is (9) of 2-05 § 07 with $a = 1$.

To a path $x(\tau)$ in imaginary time τ ($0 \le \tau \le T$) we attach the Euclidean action

$$S[x] = \int_0^T \left[\frac{1}{2}\dot{x}(\tau)^2 + V\big(x(\tau)\big)\right] d\tau \quad (2)$$

and make a probability measure on paths with weight $e^{-S[x]/\hbar}$. This is the measure of (7) of 2-07 § 05 (mass $m = 1$); closing the path into a loop gives the partition function of (9) of 2-07 § 06, that is, a classical chain in which the length T of the loop plays the role of inverse temperature. In this part we take $T \to \infty$ (temperature 0, the side of the ground state) **first**, and then $\hbar \to 0$. The quantity with the order reversed (holding T fixed and taking $\hbar \to 0$) is a different quantity. For short paths ($T < 2\pi\sqrt{2}$), from the known theorem that the mean time to move between valleys grows at the rate $e^{T/(8\hbar)}$ Cited (Theorem 2.6 of Berglund–Gentz, of Eyring–Kramers type with the interval held fixed), C_P/ξ^2 is read as falling exponentially to 0 Reading (the rereading from mean times to eigenvalues is by reduction to two states).

The lattice. The numbers in the later chapters of this part are numbers on a lattice that discretises imaginary time with step $\varepsilon = \frac{1}{2}$. With x_i the value at site $i \in \mathbb{Z}$,

$$S_{\mathrm{lat}} = \sum_i \frac{(x_{i+1} - x_i)^2}{2\varepsilon} + \varepsilon \sum_i V(x_i), \qquad \varepsilon = \frac{1}{2} \quad (3)$$

and the measure at $T = \infty$ is a one-dimensional nearest-neighbour Gibbs measure. The integral kernel of the transfer operator ((13) of 2-07 § 08) is positive, so the measure is expected to be uniquely determined Assumption (using the integral-operator version of the Perron–Frobenius theorem of 2-07 § 08 on $\mathbb{R}$. A statement of this form in

an original source has not been checked in this part, so it is put as an assumption row). Some chapters also cite numbers with step $\varepsilon = 0.25$, and say so each time.

The Dirichlet form and C_P. For a function f of the path, we set the "size of the change when it is moved" to be

$$\mathcal{E}(f) = \hbar \int \mathbb{E}\Big|\frac{\delta f}{\delta x(\tau)}\Big|^2 d\tau, \qquad \text{on the lattice} \quad \mathcal{E}(f) = \frac{\hbar}{\varepsilon}\sum_i \mathbb{E}\,|\partial_i f|^2 \tag{4}$$

(the canonical Dirichlet form), and set the Poincaré constant as $C_P = \sup_f \mathrm{Var}(f)/\mathcal{E}(f)$ ((2) of 23-00 § 04). The $\hbar$ attached to $\mathcal{E}$ is a normalisation matching the form differentiated with respect to the canonical field $x/\sqrt{\hbar}$ rather than the position x (the scale of the clock in 23-00 § 05). The Langevin dynamics with this $\mathcal{E}$ is, on the lattice,

$$dx_i = -\frac{1}{\varepsilon}\frac{\partial S_{\mathrm{lat}}}{\partial x_i}\,dt + \sqrt{\frac{2\hbar}{\varepsilon}}\,dW_i \tag{5}$$

(W_i are independent Brownian motions); it has $e^{-S_{\mathrm{lat}}/\hbar}$ as equilibrium, and its gap is $1/C_P$ Cited (1-10 § 07). Along an artificial time t the whole path sways, and the slowest fluctuation decreases by a factor $1/e$ every time C_P.

Example (free field). With $V = \frac{1}{2}x^2$ the measure is Gaussian, $C_P = 1$, the correlation length is also $\xi = 1$, and the ratio C_P/ξ^2 is exactly 1 Proved here Lean `ratioR_free_eq_one` (assumptions: $C_P = 1/m^2$, $m \neq 0$). On the lattice it is the same in the zero-momentum direction.

Precisely, the measures of (2)–(5) are path measures of a 0+1-dimensional quantum-mechanical model, and no gauge field enters anywhere. The relation to four-dimensional gauge theory is only the analogy of 23-03 § 04.

02

Two times — the correlation length ξ=ħ/Δ and the Langevin gap 1/C_P

This model has two "times" of different character.

The first is a length in imaginary time, the correlation length ξ. With $\Delta = E_1 - E_0$ (the tunnelling splitting) the difference between the lowest two levels of the quantum-mechanical Hamiltonian $H = -\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V$, the correlation of two points on the path decreases as

$$\langle x(0)\,x(\tau)\rangle - \langle x\rangle^2 \;\approx\; C\,e^{-\tau\Delta/\hbar}, \qquad m := \frac{\Delta}{\hbar}, \qquad \xi := \frac{1}{m} = \frac{\hbar}{\Delta} \tag{6}$$

(the ratio of eigenvalues of the transfer operator; the same line of argument as 2-07 § 07) Cited . ξ is a length in the imaginary-time direction; counted in lattice sites it is $\xi/\varepsilon = 2\xi$.

The second is an artificial time, the smearing time C_P. It is the time for the whole path to approach equilibrium under the dynamics (5), a time in a direction different from imaginary time.

The quantity comparing the two is

$$R := \frac{C_P}{\xi^2} = C_P\, m^2 \tag{7}$$

For the free field $R = 1$, and the double-well relaxation-time power statement (the power statement) is the same as "$\ln R$ is small compared with $\ln \xi$" (23-03 § 05).

Precisely, the ξ of (6) is the definition for the model with continuous time; for the lattice model it is fixed by $\xi = \varepsilon/\ln(\lambda_0/\lambda_1)$ from the top two eigenvalues $\lambda_0 > \lambda_1$ of the transfer operator. As $\hbar \to 0$ the difference between the lattice value and the continuum value enters the exponent as the difference of actions $S_0^{\text{lat}} \neq S_0$, so lattice values are used when writing numbers (§ 03, § 07).

03

Kinks — action S₀=2/3, the position degree of freedom, diffusion coefficient D=ħ/S₀

When $\hbar$ is small, the weight $e^{-S/\hbar}$ concentrates on paths with small action. What makes the action 0 is the path staying in the right valley all along, $x \equiv 1$, and the path staying in the left valley all along, $x \equiv -1$. The path with the next-smallest action, "moving from valley to valley", is the **kink**.

The stationarity equation of (2) is $\ddot{x} = V'(x)$ (motion in the potential $-V$, 2-07 § 05), and solutions that settle into valleys as $\tau \to \pm\infty$ satisfy conservation of energy $\frac{1}{2}\dot{x}^2 = V(x)$. From (1), $\dot{x} = \sqrt{2V(x)} = (1 - x^2)/2$, and solving this gives

$$x_{\text{cl}}(\tau) = \tanh\frac{\tau - \tau_0}{2}, \qquad S_0 = \int \dot{x}_{\text{cl}}^2\, d\tau = \int_{-1}^{1} \sqrt{2V(x)}\, dx = \int_{-1}^{1} \frac{1 - x^2}{2}\, dx = \frac{2}{3} \tag{8}$$

Proved here (putting $\frac{1}{2}\dot{x}^2 = V$ into (2) makes the integrand $\dot{x}^2$, and $dx = \dot{x}\, d\tau$ turns it into an integral over x). The width of the kink is about 2 in τ, and its action is the number $2/3$, independent of $\hbar$. This is the $S_0 = 2a^2/3$ ($a = 1$) of 2-05 § 07, in the language of physics the action of an instanton (3-05).

The position degree of freedom. The τ_0 of (8) is arbitrary, and shifting the kink in the imaginary-time direction does not change the action. The second derivative of the action (the fluctuation operator) $-\partial_\tau^2 + V''(x_{\mathrm{cl}})$ has $\dot{x}_{\mathrm{cl}}$ as an eigenfunction with eigenvalue 0 Proved here (differentiating the stationarity equation $\ddot{x}_{\mathrm{cl}} = V'(x_{\mathrm{cl}})$ with respect to τ gives $\dddot{x}_{\mathrm{cl}} = V''(x_{\mathrm{cl}})\dot{x}_{\mathrm{cl}}$). This flat direction (the zero mode) is one slow degree of freedom, the position of the kink. The next eigenvalue is $3/4$, and above it is the continuous part $[1, \infty)$, as is known Assumption (an operator of Pöschl–Teller type; the statement in the original source has not been checked in this part).

The diffusion coefficient. Projecting the noise of (5) onto the direction of the position of the kink, the position X diffuses. The direction that shifts the position by one is $\dot{x}_{\mathrm{cl}}$, and the square of its length is $\int \dot{x}_{\mathrm{cl}}^2 d\tau = S_0$, so the noise strength $\hbar$ is divided by S_0, giving

$$\mathbb{E}\big[(X_t - X_0)^2\big] \approx 2Dt, \qquad D = \frac{\hbar}{S_0} \quad (9)$$

follows. Cited The same projection count reproduces the coefficient of the theorem that a single interface in the stochastic Allen–Cahn equation on an interval converges to Brownian motion with diffusion coefficient $3/4$ (the Single-Interface Diffusion Theorem (Theorem 2.1 of Bertini–Brassesco–Buttà)). That is, at the level of "the motion of a single kink", (9) is a restatement of a known theorem. Reading In Langevin numerics following one kink on a ring with step $\varepsilon = 0.25$, $D/(\hbar/S_0)$ was 1.00 ± 0.03 ($\hbar = 0.02$), 1.04 (0.07) and 1.14 ± 0.05 (0.1).

Precisely, (9) is a sentence about the process of the position of one kink, not about the gap $1/C_P$ of whole paths. Whether, on a long path with many kinks, the gap is determined by the diffusion of kinks is exactly the question of the upper side (the First-Step Upper-Bound Proposition).

How the ruler grows

The tunnelling splitting Δ is determined, in terms of paths in imaginary time, by "the weight of inserting one kink", $e^{-S_0/\hbar}$. The leading semiclassical term is ((10) of 2-05 § 07 written with $\hbar$)

$$\frac{1}{\xi} = \frac{\Delta}{\hbar} \approx \frac{4}{\sqrt{\pi\hbar}}\, e^{-S_0/\hbar}, \qquad \xi^2 \approx \frac{\pi\hbar}{16}\, e^{2S_0/\hbar}, \qquad \ln \xi = \frac{S_0}{\hbar} + \frac{1}{2}\ln\hbar + O(1) \quad (10)$$

Cited (the formula of the leading term is a copy from the source of 2-05. When Δ is found by two independent numerical methods, the ratio to the leading term approaches 1, from 0.47 at $\hbar = 0.5$ to 0.92 at $\hbar = 0.05$ Reading), and the correlation length grows exponentially as $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. In the lattice model, instead of S_0 the

action of the lattice kink $S_0^{\mathrm{lat}} = 0.665266$ ($\varepsilon = \frac{1}{2}$; it differs from $2/3$ by 0.2%) enters the exponent Reading . At $\hbar = 10^{-3}$ this difference moves $\ln \xi$ by 1.4, so lattice values are used when writing numbers.

04

The lower bound C_P ≥ ξ² — the one-dimensional one-sided bridge, and readings beyond it

C_P is a supremum, so one test function bounds it from below (23-00 § 06). The plainest test function is a function $h(x)$ of a single time, summed over time,

$$f = \int_0^T h\big(x(\tau)\big)\, d\tau \quad (11)$$

(a test function local in time). In this family the lower bound reaches exactly the square of the ruler.

The One-Dimensional One-Sided Bridge Proposition (continuous time, $T \to \infty$, any one-dimensional well)

The supremum of the limit as $T \to \infty$ of the Rayleigh quotient $\mathrm{Var}(f)/\mathcal{E}(f)$ of f of the form (11) is exactly $\xi^2 = 1/m^2$, attained at $h = \psi_1/\psi_0$ (ψ_0, ψ_1 are the lowest two eigenfunctions of H). Hence $C_P \geq \xi^2$, that is, $R \geq 1$.

Proved here Proof [Y3]. Set $\varphi := (h - \langle h \rangle)\psi_0$ and $A := (H - E_0)/\hbar$. A is the generator of imaginary time, and $A \geq m$ on functions orthogonal to ψ_0. Writing the correlations with the transfer operator (2-07 § 08),

$$\frac{\mathrm{Var}(f)}{T} \longrightarrow \int_{-\infty}^{\infty} \langle \varphi, e^{-|\tau|A}\varphi \rangle\, d\tau = 2\,\langle \varphi, A^{-1}\varphi \rangle \quad (12)$$

On the other hand $\delta f/\delta x(\tau) = h'(x(\tau))$, so $\mathcal{E}(f)/T = \hbar \int \psi_0^2 h'^2\, dx$, and from the ground-state transformation identity $\langle \varphi, (H - E_0)\varphi \rangle = \frac{\hbar^2}{2} \int \psi_0^2 h'^2\, dx$ (2-05 § 08, 1-10 § 08), $\mathcal{E}(f)/T = 2\langle \varphi, A\varphi \rangle$. With the eigendecomposition of A, $\varphi = \sum_k c_k e_k$ (eigenvalues $a_k \geq m$), $1/a \leq a/m^2$ gives

$$\frac{\sum_k |c_k|^2/a_k}{\sum_k |c_k|^2 a_k} \;\leq\; \frac{1}{m^2} = \xi^2 \quad (13)$$

with equality when φ lies in the eigenspace of eigenvalue m, that is, when $h - \langle h \rangle \propto \psi_1/\psi_0$. ∎ The transfer-operator formula (12) and the ground-state transformation identity

are cited as known facts Cited . Lean `local_rayleigh_ratio_le_one` (assumptions: $0 < m$, $a_k \geq m$, $c_k \geq 0$, $\sum c_k a_k > 0$) checks only the skeleton of finite sums of (13); the eigendecomposition, (12) and the ground-state transformation are outside Lean.

The One-Dimensional One-Sided Bridge Proposition holds not only for the double well but for every one-dimensional well. Whether a claim of the same form exists in the literature has not been checked (it may be known).

Readings beyond it. The One-Dimensional One-Sided Bridge Proposition also says that "for test functions local in time, $R = 1$ is the ceiling". Seeing $R > 1$ needs test functions that span times (for example, blurring the path in time and then looking at the sign, or counting the collective position of kinks). Evaluating the Rayleigh quotients of such test functions with samples drawn directly from the transfer operator (not samples approached by running a Markov chain), the lower bound on R grew from 1.06 at $\hbar = 0.4$ to 6.5 at $\hbar = 0.07$ Reading (lower bound over a family, step $\varepsilon = 0.25$, a window of the measure at $T = \infty$, variation between seeds ± 10–15%). It also agreed, for $\hbar = 0.3$ to 0.15, with values from measuring the decay of the slowest observable in Langevin dynamics Reading (both are lower bounds, not upper bounds).

The type of growth, in the count at the level of physics that counts kinks (a dilute gas of kinks, the diffusion coefficient of (9)), is $R \approx S_0/\hbar \approx \ln \xi$, that is,

$$C_P \;\approx\; \xi^2 \,\frac{S_0}{\hbar} \;\approx\; \frac{\pi}{24}\, e^{4/(3\hbar)} \qquad (14)$$

Assumption (the count is heuristic, not a proof). The numbers in this range alone cannot distinguish whether R grows as $1/\hbar$ or as $e^{c/\hbar}$. In the former case the double-well power statement holds; in the latter it breaks. This is why, even for the double well, the sentence stated within the width of a constant factor, $C_P \leq K\xi^2$, is read as being on the side that breaks.

05

The target for the upper side, and the band of long paths

Matching the form of (14), we put the target for the upper side as

$$C_P \;\leq\; K\,\frac{\xi^2}{D} = K\,\xi^2\,\frac{S_0}{\hbar} \qquad (K \text{ does not depend on } \hbar\,) \qquad (15)$$

Together with the One-Dimensional One-Sided Bridge Proposition this gives $1 \leq R \leq KS_0/\hbar$, and the double-well power statement follows ((4) of 23-03 § 05). (15) is the conclusion of the First-Step Upper-Bound Proposition on top of the four double-well conditions, and is not proved (23-05).

The naive road to showing (15) is to choose one slow variable (for example the mean position of the path, the magnetisation) and show that "functions orthogonal to it relax fast". For short paths ($T \ll \xi$, at most one pair of kinks) this form gives the right count. For long paths ($T \gg \xi$), however, this road cannot be used.

The reason can be seen in the Glauber dynamics of the one-dimensional Ising chain. The space spanned by functions of a single spin is closed under the generator, and the Fourier component of wave number k, $S_k = \sum_j e^{ikj}\sigma_j$, is an eigenfunction with eigenvalue $\alpha(1-\gamma\cos k)$ (α is the rate, $\gamma = \tanh 2K$). The magnetisation is $k=0$, and its ratio to the slowest component orthogonal to the magnetisation, $k = 2\pi/N$, is

$$\frac{1-\gamma\cos(2\pi/N)}{1-\gamma} = 1 + \frac{\gamma\,(1-\cos(2\pi/N))}{1-\gamma} \;\approx\; 1 + \Big(\frac{2\pi\xi}{N}\Big)^2 \qquad (16)$$

which approaches 1 when the chain is much longer than the correlation length ($N/\xi \to \infty$) Proved here (rewriting the ratio) Lean `glauber_band_ratio` (assumption: $\gamma \neq 1$). For the formula of the eigenvalues itself, we saw to 8 digits that the two eigenvalues appear as they are in the diagonalisation of all states of chains with $N = 8, 10$ Reading (the statement in Glauber's original paper has not been checked in this part). The slowest fluctuation is not an isolated single mode but the lower edge of a continuous band. On long paths of the double well too, long-wave fluctuations of the density of kinks make the same band, and the Poincaré constant of the conditional measure with the magnetisation fixed stays of the same size as ξ^2/D.

So on long paths one has to take a number of slow variables proportional to the volume, or cut the problem into boxes of length $C\xi$. The latter is the box-reduction condition (reduction to the box dynamics) and the gap-inside-the-box condition (23-11).

06

The plateau at the top — the curvature falls ¼ short of convexity

If the measure $e^{-S/\hbar}$ had a convex action (a log-concave density), the Bakry–Émery criterion would bound C_P by the reciprocal of the curvature of the action, independently of length and of $\hbar$ Cited (1-10). The path measure of the double well is not convex. We look on the lattice (3) at where convexity breaks.

The second derivative of the action (3) (the Hessian) is a matrix with $2/\varepsilon + \varepsilon V''(x_i)$ on the diagonal and $-1/\varepsilon$ between neighbours. For $\varepsilon = \frac{1}{2}$ the diagonal is $4 + \frac{1}{2}V''(x_i)$ and the neighbours -2. For uniform convexity over a long interval it suffices that the diagonal be at least the size 4 of the sum of the two neighbours (diagonal dominance).

Inside a valley ($|x_i|$ close to 1), $V'' \approx 1$ and the diagonal is 4.5, which is enough, but at the top of the hill $x_i = 0$, $V''(0) = -\frac{1}{2}$, so

$$\underbrace{4 + \tfrac{1}{2} \cdot \left(-\tfrac{1}{2}\right)}_{\text{curvature at one site at the top}} = \frac{15}{4} = 4 - \frac{1}{4} \quad (17)$$

which falls short by $\frac{1}{4}$ Proved here . For the path with every site at the top, $x \equiv 0$, the smallest eigenvalue of the Hessian (in the uniform direction) is exactly $\varepsilon V''(0) = -\frac{1}{4}$.

Between a pair of kinks, the interval where the path stays near the top makes a flat direction (a plateau) along which the action hardly changes. Conversely, confining the path to the convex core deep in a valley, $\{x_i \geq a\}$ ($a > 1/\sqrt{3}$), Bakry–Émery gives $C_P \leq 2/(3a^2 - 1)$ uniformly in length and $\hbar$ Proved here (lower bound on curvature) Cited (of Bakry–Émery type) Lean `convex_core_curvature` (assumptions: $1/3 < a^2$, $0 \leq a \leq x$). But the action of an excursion that leaves the core and comes back, $2/3 - a + a^3/3$, is smaller than the action of a kink $2/3$ (0.1535 at $a = 1/\sqrt{3}$) Lean `excursion_action_lt_kink` (assumption: $0 < a < 1$), and an interval of length ξ contains exponentially many excursions. The convex core alone does not give local Poincaré within a phase (the in-phase local Poincaré condition).

This shortfall of $\frac{1}{4}$ comes back, in changed form, to the same place on every road that builds the upper side. In later chapters it is called the common wall (23-09).

07

Numerical examples — ln ξ at three values of ħ

The later chapters of this part use three values of $\hbar$ as representatives. The order is always from large to small, $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$. The reason for the choice is that $\ln \xi$ changes widely, from about 218 to about 2213, making them representatives of the regime where the correlation length grows exponentially in $\hbar$. Table 2 gives values on the lattice $\varepsilon = \frac{1}{2}$.

Reading Table 2 $\ln \xi$ and $\ln(\xi^2 S_0/\hbar)$ at three values of $\hbar$ (lattice $\varepsilon = \frac{1}{2}$, $T = \infty$. ξ is a length in imaginary time. Floating-point values)

Quantity	$\hbar = 3 \cdot 10^{-3}$	$\hbar = 10^{-3}$	$\hbar = 3 \cdot 10^{-4}$
$\ln \xi$	218.08	661.04	2212.72
$S_0^{\mathrm{lat}}/\hbar$	221.76	665.27	2217.55
$\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2} \ln \hbar$	−0.774	−0.777	−0.778
$\ln(\xi^2 S_0/\hbar)$ (scale of the target for the upper side)	441.6	1328.6	4433.1

$\ln \xi$ is the value obtained for the difference of the top two eigenvalues of the transfer operator by Herring's formula, which involves no subtraction (the next detour). At intermediate $\hbar$ (0.2, 0.05, 0.04), the road of finding the eigenvalues of the matrix directly and the road of Herring's formula agreed to six digits Reading . That the remainder in the third row approaches a number independent of $\hbar$ is consistent with the form $\ln \xi = S_0/\hbar + \frac{1}{2}\ln \hbar + O(1)$ of (10), and the difference from the continuum semiclassical value $\frac{1}{2}\ln(\pi/16) \approx -0.814$ is read as a lattice correction.

The fourth row is the scale of the right-hand side of (15), and its difference from $2\ln \xi$ is $\ln(S_0/\hbar)$. The ratio of the two columns is close to 2, and approaches 2 as $\hbar$ is lowered. The "2" of the double-well power statement is a restatement of how small this difference is.

Precisely, Table 2 gives floating-point values, not values enclosed by interval arithmetic. ξ is of size e^{218} to e^{2213}, not a length that can be measured directly by Monte Carlo but a quantity read from the eigenvalues of the transfer operator. The three values of $\hbar$ are representative points, with a role different from the "nodes" and "windows" of later chapters (the distinction is in 23-12).

Detour: the tunnelling splitting by Herring's formula

A subsection you may skip. At $\hbar = 10^{-3}$, $\Delta/\hbar$ is of size e^{-661}, and if the two eigenvalues λ_0, λ_1 are found separately and subtracted, every digit of floating point is lost. So we use a formula that writes the difference directly.

The integral kernel $K(x,y)$ of the transfer operator is symmetric and unchanged under $x \to -x$, $y \to -y$. Making two kernels on the half-line $x, y > 0$, $A(x,y) = K(x,y)$ and $C(x,y) = K(x,-y)$, the even eigenfunctions are eigenfunctions of $A + C$ and the odd eigenfunctions of $A - C$ (eigenvalues λ_e and λ_o). Since A and C are both symmetric, for an even eigenfunction u and an odd eigenfunction v,

$$v \cdot (A+C)u - u \cdot (A-C)v = (\lambda_e - \lambda_o)\, u \cdot v \quad \Longrightarrow \quad \lambda_e - \lambda_o = \frac{2\, u \cdot Cv}{u \cdot v} \tag{18}$$

Proved here (on the left $v \cdot Au = u \cdot Av$ cancels, and two copies of $v \cdot Cu = u \cdot Cv$ remain). The right-hand side involves no subtraction. The kernel of $A - C$ is positive for $x, y > 0$ (since $(x+y)^2 > (x-y)^2$, $K(x,y) > K(x,-y)$), so the odd eigenfunction can also be taken positive on the half-line, and can be computed in the logarithmic domain keeping relative precision out to the tails. This is the transfer-operator version of Herring's formula of quantum mechanics (the formula writing the splitting through overlaps of wave functions).

Classification of the statements of this chapter

We put the statements of this chapter into one table in the format of the five grades of this part (Table 1 of 23-00 § 01). The columns are the same as in the tables of propositions in later chapters.

Table 3 The statements of this chapter (one grade per row; the machine-checking column is outside the grades)

Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Shape of the kink $\tanh(\tau/2)$ and action $S_0 = 2/3$	Proved here	(8) of § 03	Independent of $\hbar$	—
The fluctuation operator has $\dot{x}_{\mathrm{cl}}$ with eigenvalue 0 (the position degree of freedom)	Proved here	§ 03	Independent of $\hbar$	—
Next eigenvalue of the fluctuation operator $3/4$, continuous part $[1, \infty)$	Assumption	§ 03 (Pöschl–Teller type; original source unchecked)	Independent of $\hbar$	—
Diffusion coefficient of the position of one kink $D = \hbar/S_0$ (at the level of the process)	Cited	(9) of § 03. A copy of the coefficient of the Single-Interface Diffusion Theorem of Bertini–Brassesco–Buttà	$\hbar \to 0$	—
$D/(\hbar/S_0) = 1.00$ to 1.14	Reading	§ 03 (Langevin point values, $\varepsilon = 0.25$)	0.02, 0.07, 0.1	—
Leading semiclassical term (10) of ξ	Cited	§ 03, 2-05 § 07	$\hbar \to 0$	—
Lattice $\ln \xi$ (Table 2) and $S_0^{\mathrm{lat}} = 0.665266$	Reading	§ 07 (transfer operator, Herring's formula, floating point)	Three points	—

Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
The One-Dimensional One-Sided Bridge Proposition: the ceiling of the family local in time is ξ^2, hence $C_P \geq \xi^2$	Proved here	Proof in § 04 ((12) and the ground-state transformation cited). Continuous time, $T \to \infty$	Every $\hbar$	`local_rayleigh_ratio_le_one` (assumptions: $0 < m$, $a_k \geq m$, $c_k \geq 0$, $\sum c_k a_k > 0$. Skeleton of finite sums only)
The lower bound on R grows from 1.06 to 6.5	Reading	§ 04 (lower bound over a family, $\varepsilon = 0.25$)	$[0.07, 0.4]$	—
$C_P \approx \xi^2 S_0/\hbar$ ((14))	Assumption	§ 04 (count with a dilute gas; heuristic)	$\hbar \to 0$	—
The ratio (16) to the component orthogonal to the magnetisation approaches 1 (the band)	Proved here	§ 05 (rewriting the ratio)	Independent of $\hbar$	`glauber_band_ratio` (assumption: $\gamma \neq 1$)
The Glauber eigenvalues $\alpha(1 - \gamma \cos k)$ appear in the spectrum of all states	Reading	§ 05 (diagonalisation for $N = 8, 10, 8$ digits)	—	—
Curvature at one site at the top $15/4 = 4 - \frac{1}{4}$	Proved here	(17) of § 06	Independent of $\hbar$	—
$C_P \leq 2/(3a^2 - 1)$ on the convex core $\{x_i \geq a\}$	Proved here	§ 06 (the lower bound on curvature is proved here; the Bakry–Émery type is cited)	Independent of $\hbar$	`convex_core_curvature` (assumptions: $1/3 < a^2$, $0 \leq a \leq x$)
Target for the upper side (15), $C_P \leq K\xi^2 S_0/\hbar$	Assumption	Conclusion of the First-Step Upper-Bound Proposition (on top of the four double-well conditions). 23-05	$\hbar \to 0$	From (15) to the double-well power statement by `power_H_of_kink_ratio` (assumptions in 23-03 § 05)

What this chapter can say

Said	Not said
Proved here The kink action $S_0 = 2/3$ and the zero mode, the shortfall $\frac{1}{4}$ of curvature at the top, the transfer-operator version (18) of Herring's formula	Checking the rest of the spectrum of the fluctuation operator (the next eigenvalue) in the original source
Proved here The One-Dimensional One-Sided Bridge Proposition: $C_P \geq \xi^2$ (continuous time, $T \to \infty$; (12) and the ground-state transformation cited)	An upper bound on C_P. Whether R grows as $1/\hbar$ or exponentially
Cited The diffusion coefficient of one kink $D = \hbar/S_0$ (at the level of the process), the leading semiclassical term of ξ	That the gap of whole paths is determined by the diffusion of kinks (the question of the upper side)
Reading The growth of the lower bound on R (1.06→6.5), $\ln \xi$ at three values of $\hbar$ (Table 2), the numbers for D	Enclosing the values of Table 2 by interval arithmetic. Conversion between readings at $\varepsilon = 0.25$ and readings at $\varepsilon = \frac{1}{2}$
Assumption The target for the upper side (15) was put in the form of the conclusion of the First-Step Upper-Bound Proposition	(15) itself. The double-well power statement. About four-dimensional gauge theory, this chapter shows nothing

Sources and reproduction

Item	Kind	Source / tools
The path integral, the measure in imaginary time, the transfer matrix	Cited	The table of sources of 2-07.
Leading term of the tunnelling splitting of the double well, S_0	Cited	The table of sources of 2-05 (formula (2.8) of Zinn-Justin–Jentschura 2004). The transcription with $\hbar$ and the numbers for the ratio to the leading term are numbers of the record (two independent methods, $\hbar = 0.5$ to 0.05).
Poincaré constant, Langevin gap, the Bakry–Émery type	Cited	The table of sources of 1-10.
Diffusion coefficient of a single interface	Cited	L. Bertini, S. Brassesco, P. Buttà, arXiv:math-ph/0611076 (the Single-Interface Diffusion Theorem, checked in the text).
Transition times in the regime with the interval held fixed	Cited	N. Berglund, B. Gentz, arXiv:1202.0990 (Theorem 2.6, checked in the text). The rereading as eigenvalues is a reading.
Single-spin eigenvalues of Glauber dynamics	Assumption, Reading	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic data from memory, not checked in this part). Eigenvalues checked by diagonalisation for $N = 8, 10$.

Item	Kind	Source / tools
The One-Dimensional One-Sided Bridge Proposition, the finite sums of (13)	Proved here, Lean	§ 04 of the text. Theorem `local_rayleigh_ratio_le_one` (assumptions in the row of Table 3 for the One-Dimensional One-Sided Bridge Proposition).
$R = 1$ for the free field, the band ratio, the convex core, the action of excursions	Proved here, Lean	Theorems `ratioR_free_eq_one`, `glauber_band_ratio`, `convex_core_curvature`, `excursion_action_lt_kink` (assumptions in each row of the text). The list of statements is in 23-13.
Lower bound on R, numbers for D, Table 2	Reading	Floating-point computations of the record (Rayleigh quotients with samples drawn directly from the transfer operator, Langevin, the transfer operator and Herring's formula). Values not enclosed by interval arithmetic.

Words used in this chapter — double well, path measure, Dirichlet form, kink, zero mode, diffusion coefficient of kinks, tunnelling splitting, correlation length (of the double well), Herring's formula, plateau, convex core, phase (phase without kinks). Their meanings are in A-02 Glossary.

Next chapter to read: 23-05 The statement of the first step — the four double-well conditions and the upper-bound proposition — splits the upper side into four conditions by the algebra of the splitting of the variance, and states the First-Step Upper-Bound Proposition divided into the parts shown and the assumption rows.

The statement of the first step — the four double-well conditions and the upper-bound proposition

> This chapter is this book's own investigation. § 05 is established knowledge.

What this chapter shows — We split the upper side $CP \le K\xi^2 S_0/\hbar$ for the path measure of the double well into the four double-well conditions (the in-phase local Poincaré condition, the saddle-tube condition, the box-reduction condition and the gap-inside-the-box condition). The box-reduction condition reduces infinitely long paths to boxes of length $C\xi$, the gap-inside-the-box condition controls the gap inside a box, and the in-phase local Poincaré condition and the saddle-tube condition enter the gap-inside-the-box condition as the fast part inside the box. On top of this we state, as one line of argument, how $R = CP/\xi^2$ is squeezed into $1 \le R \le K'/\hbar$ and the double-well relaxation-time power statement follows, adding for each step "what it uses and which grade it has". At the end of the chapter the steps of the First-Step Upper-Bound Proposition are collected in a table of the five grades and a dependency graph.

Prerequisite chapters — 23-03 § 05 (the double-well power statement and the form of the First-Step Upper-Bound Proposition), 23-04 (the path measure, kinks, the One-Dimensional One-Sided Bridge Proposition), 1-10 § 07– § 09 (Poincaré inequalities), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. The First-Step Upper-Bound Proposition is algebra on top of four conditions, and every one of the four conditions leaves assumption rows. This chapter does not say that the first step is closed. It names what would have to be shown to close it. This part is a study that chose the diffusive-relaxation-time assumption as its foothold, and the split into the four conditions is one way of building within it. Other views are treated in Part 24.

The order of this chapter

1. The splitting of the variance — the algebra of splitting into two valleys, and why that alone is not enough for long paths
2. The parts shown — the algebra of the splitting, the convex core, the reduced plateau, the diffusion coefficient of kinks

3. The statements of the four conditions — for which measure, which constant, uniform in what
4. The First-Step Upper-Bound Proposition as one line — box reduction → inside the box → $C_P \le K\xi^2 S_0/\hbar \to R \le K'/\hbar$ → the double-well power statement
5. The distance from known theorems — where they reach and where they do not (Table 1)
6. The table of the five grades — the rows of the steps of the First-Step Upper-Bound Proposition (Table 2)
7. The dependency graph — checking by the numbers that there is no circle
8. A $\hbar$ that fits all, and where the four conditions stand — a guide to 23-06–23-12

01

The splitting of the variance — the algebra of splitting into two valleys

The setting is as in 23-04 § 01. The lattice is $\varepsilon = \frac{1}{2}$, the action S_{lat}, the measure $\mu \propto e^{-S_{\mathrm{lat}}/\hbar}$, the Dirichlet form $\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$, $C_P = \sup_f \mathrm{Var}(f)/\mathcal{E}(f)$, the diffusion coefficient of kinks $D = \hbar/S_0$, and $S_0 = 2/3$. We take $T \to \infty$ first and then $\hbar \to 0$.

The first tool for building the upper side is the algebra of splitting the measure into two parts. Split the space of configurations in two (for example by the sign of the smoothed value of the path) and write $\mu = Z_+\mu_+ + Z_-\mu_-$ ($Z_\pm \ge 0$, $Z_+ + Z_- = 1$, and $\mu_\pm$ the measures restricted to each part and normalised). Then for every f

$$\mathrm{Var}_\mu f = Z_+\,\mathrm{Var}_{\mu_+} f + Z_-\,\mathrm{Var}_{\mu_-} f + Z_+Z_-\big(\mathbb{E}_{\mu_+} f - \mathbb{E}_{\mu_-} f\big)^2 \quad (1)$$

holds Proved here (the identity for the variance of a mixture. Put $m_\pm = \mathbb{E}_{\mu_\pm} f$ and $m = Z_+m_+ + Z_-m_-$, split $\mathbb{E}_\mu(f-m)^2$ on each part into $f - m_\pm$ and $m_\pm - m$, and use $Z_+(m_+ - m)^2 + Z_-(m_- - m)^2 = Z_+Z_-(m_+ - m_-)^2$).

Suppose that each of the three terms on the right-hand side has the following inequality.

$$\mathrm{Var}_{\mu_\pm} f \le C_{\mathrm{loc}}\,\mathcal{E}_\pm(f), \qquad \big(\mathbb{E}_{\mu_+} f - \mathbb{E}_{\mu_-} f\big)^2 \le C_{\mathrm{md}}\,\mathcal{E}(f), \qquad Z_+\mathcal{E}_+(f) + Z_-\mathcal{E}_-(f) \le$$

The first is local Poincaré inside a valley, the second is the "transport of the mean difference", which controls the difference of the means of the two valleys by the Dirichlet form, and the third is the additivity of the form. Since $Z_+Z_- \le \frac{1}{4}$,

$$C_P \;\le\; C_{\mathrm{loc}} + \frac{C_{\mathrm{md}}}{4} \quad (3)$$

holds Proved here Lean `split_poincare_le` (assumptions: $Z_+ + Z_- = 1$, $Z_\pm \ge 0$, the splitting formula (1), the two local Poincaré inequalities, the mean difference, additivity of the form, non-negative constants). The splitting formula (1) has the same form as the Variance-Splitting Lemma (Lemma 2.4 of Menz–Schlichting) Cited (the skeleton of the Poincaré version of finite-dimensional Eyring–Kramers).

Of the two constants in (3), C_{loc} later becomes the in-phase local Poincaré condition and C_{md} becomes the saddle-tube condition. But (3) cannot be applied to long paths as it stands.

For long paths, one coarse variable is not enough. (1) is a split of the form "take one slow variable (which valley the path is in) and treat the rest as fast". When the path is much longer than the correlation length ($T \gg \xi$), long-wave fluctuations of the density of kinks form a continuous band, and the Poincaré constant of the conditional measure with one valley fixed stays of the same size as ξ^2/D (the Glauber band of 23-04 § 05). That is, C_{loc} does not become $O(1)$. So instead of taking a number of slow variables proportional to the volume, we cut the problem into boxes of length $C\xi$. This is the box-reduction condition and the gap-inside-the-box condition, and the in-phase local Poincaré condition and the saddle-tube condition are used inside the box (§ 04).

02

The parts shown — the algebra of the splitting, the convex core, the reduced plateau, the diffusion coefficient

First we list the parts of the construction of the upper side that can be said without assumption rows.

(a) The algebra of the splitting. (1)–(3) are as in § 01 Proved here .

(b) The convex core. For the measure restricted to the convex region deep in a valley, $\{x_i \ge a\ \forall i\}$ ($a > 1/\sqrt{3}$), the Hessian of the action is at least $\varepsilon\kappa_a$ ($\kappa_a = (3a^2-1)/2$), so Bakry–Émery on a convex region gives $C_P \le 2/(3a^2-1)$ uniformly in the length, $\hbar$ and the boundary Proved here (the lower bound on the curvature) Cited (Bakry–Émery on a convex region, Theorem 1.2 of Kolesnikov–Milman). The proof and the Lean statement are in 23-04 § 06. But excursions out of the core have smaller action than a kink, and exponentially many of them fit in an interval of length ξ, so the convex core alone does not give the in-phase local Poincaré condition.

(c) The two-state value of the reduced plateau. On a short ring ($T \ll \xi$) the path that moves between valleys is one pair of kinks, and the separation ℓ of the pair becomes

the one slow coordinate. In the one-dimensional model that keeps only ℓ (ℓ moves with diffusion coefficient $2D$, with an atom of a valley at each end and the plateau weight $\pi(\ell)$ in between), we control the mean difference by the Cauchy–Schwarz inequality, write the difference of the values at the two ends of ℓ as $\int f'(\ell)\,d\ell$, and

$$\big(f(\text{right valley}) - f(\text{left valley})\big)^2 \;\le\; \int \frac{d\ell}{\pi(\ell)} \cdot \int \pi(\ell)\, f'(\ell)^2\, d\ell \quad (4)$$

gives a value of the form $C_{\mathrm{md}}/4 = J/(4D)$, with equality nearly attained by the sign test function Proved here ((4) is the Cauchy–Schwarz inequality) Lean `reduced_two_state_value` (assumptions: $0 < T$, $0 < D$, $J = 4\xi^2 I/T$, with I the integral of the plateau weight). The reduced model itself (that everything other than ℓ may be thrown away) is an assumption of the model, and this value is "a value on the reduced model" Assumption .

(d) The diffusion coefficient of kinks. The position of a single kink moves with diffusion coefficient $D = \hbar/S_0$. At the level of the process of a single interface, this is a copy of the coefficient of a known theorem Cited ((9) of 23-04 § 03). It is not a statement about the gap of the whole path.

These four are all. The remaining parts, namely that local Poincaré inside a valley is uniform in the length and in $\hbar$, that transport along the family of flat saddles is uniform in the dimension, the reduction of long paths to boxes, and the gap inside a box, lie outside the assumptions of known theorems (§ 05). We name them as four conditions.

03

The statements of the four conditions — for which measure, which constant, uniform in what

We write the four conditions making three things explicit: the measure, the constant and the uniformity. All are statements about the 0+1-dimensional double well $V = (x^2 - 1)^2/8$, the lattice $\varepsilon = \frac{1}{2}$ and the Dirichlet form $\mathcal{E}$. "Uniform" means that for some $\hbar_0 > 0$ with $\hbar \in (0, \hbar_0]$, the constant does not depend on $\hbar$ (does not grow as $\hbar \to 0$).

The in-phase local Poincaré condition (local Poincaré within a phase)

Measure: the measure of the phase without kinks. The measure $\mu^{(+)} = \mu(\cdot \mid x_i \ge 0\ \forall i)$ restricted to configurations in which the path stays on the side of one valley, for example to configurations with $x_i \ge 0$ at every site (reflected at the wall $x_i = 0$).
Constant: the Poincaré constant C_{loc} of $\mu^{(+)}$, that is, $\mathrm{Var}_{\mu^{(+)}} f \le C_{\mathrm{loc}}\, \mathcal{E}(f)$.
Uniformity: $C_{\mathrm{loc}} = O(1)$ uniformly in $\hbar$ and in the length of the path. For use

inside the box, the version for segments with end values (length $\leq C\xi$) is needed (the segment local Poincaré assumption), and it must also be uniform in the end values.

The saddle-tube condition (transport along the family of flat saddles)

Measure: the conditional measure of configurations in the tube around the family of kink–antikink pairs (the family of saddles, parametrised by the centre X and separation ℓ of the pair, along which the action is almost flat). **Constant**: the constant of transport of the mean difference along that family from valley to valley (of the type of C_{md} in (2)), and the Poincaré constant inside the tube. **Uniformity**: uniform in the number of sites of the path (the dimension) and in $\hbar$. For use inside the box it becomes two rows: Poincaré in the tube uniform in tube length $\leq C\xi$ (the saddle-tube Poincaré assumption), and Poincaré of the core inside the tube (the interval that stays near the top) uniform in the core length k (the core Poincaré uniformity assumption).

The box-reduction condition (reduction to the box dynamics)

Measure: the measure μ on the lattice line $\mathbb{Z}$ with $T = \infty$, and the conditional measure $\mu_B^{a,b}$ with the values $(a, b) \in \mathbb{R}^2$ on both neighbours of a box B of length L sites ($L\varepsilon = C\xi$) fixed. **Constants**: $C_{\mathrm{box}}(L) := \sup_{a,b\in\mathbb{R}} C_P(\mu_B^{a,b})$ and the constant c_M of mixing between boxes. **Statement**: $C_P(\mu) \leq C_{\mathrm{box}}(L)/c_M$. **Uniformity**: c_M depends only on lengths in units of ξ (C and the fraction of overlap of the boxes), not on $\hbar$ or the boundary values.

The gap-inside-the-box condition

Measure: the conditional measure $\mu_B^{a,b}$ of a box of length $C\xi$. The number of kinks in the box is $O(1)$ on average (the density of kinks is $1/(2\xi)$, so the expectation is $C/2$). **Constant**: the gap $1/C_P(\mu_B^{a,b})$. **Statement**: $1/C_P(\mu_B^{a,b}) \geq c\,D/(C\xi)^2$, that is, $C_{\mathrm{box}} \leq K_4(C\xi)^2/D$ ($K_4 = 1/c$). **Uniformity**: c is uniform in $\hbar$ and in all boundary values (a, b) (it may depend on C).

The four are not conditions of the same weight. The box-reduction condition is the step that "moves the problem of long paths to a problem of boxes", the gap-inside-the-box condition is the step that "splits the inside of a box into slow variables (the arrangement of kinks) and fast variables", and the in-phase local Poincaré condition

and the saddle-tube condition are statements on the side of those fast variables. The next section puts into one line the place where the in-phase local Poincaré condition and the saddle-tube condition enter the gap-inside-the-box condition.

To be precise, the four double-well conditions are different from the relaxation-time assumption (the power statement, 23-01 § 05). The relaxation-time assumption is a statement on the side of the conclusion, and the four double-well conditions are conditions put in place to derive it in the first step (23-03 § 05).

04

The First-Step Upper-Bound Proposition as one line

The First-Step Upper-Bound Proposition (conditional)

For the path measure of the 0+1-dimensional double well (lattice $\varepsilon = \frac{1}{2}$, taking $T \to \infty$ first), assume the four double-well conditions and the premise rows of Step 0 below. Then as $\hbar \to 0$

$$C_P \;\le\; K\,\xi^2\,\frac{S_0}{\hbar} = K\,\frac{\xi^2}{D} \qquad (K \text{ depends only on the constants of the conditions, not on}$$

holds, hence $1 \le R = C_P/\xi^2 \le KS_0/\hbar$, and the double-well power statement $\ln C_P / \ln \xi \to 2$ follows.

(5) is the same formula as (15) of 23-04 § 05 and (3) of 23-03 § 05. Below, each step is given with "what it uses" and its grade.

Step 0 (the step aligning the assumptions) The premise rows

That the measure μ with $T = \infty$ is uniquely determined (simplicity of the principal eigenvalue, from the positivity of the integral kernel of the transfer operator) Assumption (the original source for the version with a positive kernel on $\mathbb{R}$ has not been checked; the same row as in 23-04 § 01). The relation between $C_P(T)$ of the finite ring and C_P of the measure with $T = \infty$ (the step of the limit), and the regularity of the functions used in the steps below (conditional measures, smoothness of the kink coordinates) Assumption . These three are premises placed outside the First-Step Upper-Bound Proposition, and they are counted as assumption rows in the integrated table (23-12) too.

Step 1 (the box-reduction step) Box reduction — the box-reduction condition in the form of the Box-Reduction Proposition

Cover the line $\mathbb{Z}$ by boxes of length L sites. Neighbouring boxes overlap in l sites ($l \le L/3$), and the boxes are divided alternately into an even family E and an odd family O. Let $\mathcal{G}_E$ be the σ-algebra generated by the values at the sites outside the even family, define $\mathcal{G}_O$ in the same way, and let the maximal correlation of the two σ-algebras be

$$c_{\mathrm{alt}} := \sup\{\operatorname{corr}_\mu(F,G) : \ F \in L^2(\mathcal{G}_E),\ G \in L^2(\mathcal{G}_O)\} \quad (6)$$

.

The Two-Projection Lemma. If $c = c(\mathcal{G}_1,\mathcal{G}_2) < 1$, then for all $f \in L^2(\mu)$, $\operatorname{Var} f \le (1-c)^{-1}\big(\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_1) + \mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_2)\big)$.

Proved here Proof. Put $\tilde f = f - \mathbb{E}f$ and $a_i = \mathbb{E}[f \mid \mathcal{G}_i] - \mathbb{E}f$. Then $s := \|a_1\|^2 + \|a_2\|^2 = \langle \tilde f, a_1 + a_2\rangle \le \|\tilde f\|\,\|a_1+a_2\|$, while $\|a_1+a_2\|^2 \le s + 2c\|a_1\|\|a_2\| \le (1+c)s$. Hence $s \le (1+c)\|\tilde f\|^2$, and adding $\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_i) = \|\tilde f\|^2 - \|a_i\|^2$ gives at least $(1-c)\|\tilde f\|^2$. ∎ Lean `two_proj_var` (assumption: $A + B \le (1+c)V$. The geometry of the projections is outside Lean).

The product over boxes. Even boxes are $L - 2l \ge 1$ sites apart, so by the Markov property of the nearest-neighbour interaction they are mutually independent under $\mathcal{G}_E$. The variance of a function of independent components is at most the sum of the conditional variances component by component (of the Efron–Stein type), so

$$\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_E) \ \le \ C_{\mathrm{box}}(L) \sum_{B \in E} \mathcal{E}_B(f) \quad (7)$$

holds Proved here (the Markov property and the variance inequality for independent products; the proof is placed in 23-11). The same holds for the odd family. Each site lies in the even family at most once and in the odd family at most once, and only the overlaps lie in both.

Averaging over shifts. Shift the arrangement of boxes by l at a time, in $n = \lfloor (L-l)/l \rfloor$ ways. Every shift has the same c_{alt}, and the overlap sites are disjoint between shifts. Applying the Two-Projection Lemma and (7) to each shift and averaging gives

$$C_P(\mu) \ \le \ \Big(1 + \frac{1}{n}\Big)\frac{C_{\mathrm{box}}(L)}{1 - c_{\mathrm{alt}}} \quad (8)$$

Proved here Lean `box_two_family`, `box_shift_average` (assumptions: the two-family inequality for each shift, $C_{\mathrm{box}} \ge 0$, the sum of the forms on the overlaps at most $\mathcal{E}$). This is the Box-Reduction Proposition, and c_M of the box-reduction condition

corresponds to $(1-c_{\mathrm{alt}})/(1+1/n)$. This step uses only Poincaré inequalities, not the logarithmic Sobolev inequality.

One assumption row remains in (8): that the alternating maximal correlation $c_{\mathrm{alt}}<1$ is uniform in $\hbar$ (the alternating maximal-correlation assumption for boxes). Its value is read as $c_{\mathrm{alt}}\approx 0.73$ at $C=2$ and overlap $\theta=l/L=\frac{1}{4}$ Reading (a family value copied from the closed formula for the two-state telegraph chain; not a proof of an upper bound). The type of the box reduction has the same form as the Box-Dynamics Proposition for Glauber dynamics of discrete spins (Propositions 3.4 and 3.5 of Martinelli's lectures) Cited (discrete spins, in the form of a total-variation coupling; here rewritten in one dimension, in L^2, in the form of maximal correlation).

Step 2 (the step inside the box) Inside the box — the gap-inside-the-box condition in the form of the Gap-Inside-the-Box Proposition, and where the in-phase local Poincaré condition and the saddle-tube condition enter

Fix a box B ($L\varepsilon=C\xi$) and boundary values (a,b). Let the slow variable inside the box be Y = the arrangement of kinks (the number of kinks, and the position of each kink defined by projection onto the manifold of kinks). Splitting the variance by Y gives

$$\operatorname{Var} f=\mathbb{E}\operatorname{Var}(f\mid Y)+\operatorname{Var}\big(\mathbb{E}[f\mid Y]\big) \quad (9)$$

Proved here (the identity for conditional expectations).

First term = the fast side. The in-phase local Poincaré condition and the saddle-tube condition enter here. By the nearest-neighbour Markov property, the conditional measure with Y fixed splits into a product of the following parts. (i) Segments of the phase without kinks between kinks (length $\le C\xi$, with values at both ends). (ii) The tube around each kink or kink pair, and the core inside it (the interval that stays near the top). The Poincaré constant of a product measure is the maximum of the constants of the components (the variance inequality for independent products), so

$$\mathbb{E}\operatorname{Var}(f\mid Y)\ \le\ C_{\mathrm{loc}}\,\mathcal{E}(f),\qquad C_{\mathrm{loc}}=\max\big(C_{(\mathrm{H1}^{\mathrm{seg}})},\ C_{(\mathrm{H2}^{\mathrm{tube}})},\ C_{(\mathrm{T0}'_{\mathrm{P}})}\big) \quad (10)$$

Proved here (the Poincaré constant of a product measure is the maximum of the constants of the components. That it splits into a product depends on how Y is defined, and its regularity is included in the row of Step 0). The constant of (i) is the segment version of the in-phase local Poincaré condition, and the constants of (ii) are the tube and core rows of the saddle-tube condition Assumption (all three are assumption rows).

Second term = the slow side. Longitudinal transport along the family of saddles enters here. $\mathbb{E}[f\mid Y]$ is a function of Y alone, and its fluctuations relax by the diffusion

of kink positions and by the creation and annihilation of kink pairs. The Dirichlet form $\mathcal{E}_Y$ on Y has mobility per kink of the form $D = \hbar/S_0$ (the kink diffusion-coefficient row: the projection formula, 23-04 § 03). Here we put two rows.

- The kink-gas gap assumption: the Poincaré constant of the box for the reduced kink gas, $C_{\mathrm{kg}} \le (C\xi)^2/\big(c_{\mathrm{kg}}(C)(1-\eta_{\mathrm{KG}})D\big)$, with $\eta_{\mathrm{KG}} < 1$ uniform in $\hbar$ and the boundary values Assumption (the assumption that the kink gas falls into the same type as the Glauber dynamics of the one-dimensional Ising model).
- The cross-term uniformity assumption: the cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1+\chi)\,\mathcal{E}(f)$, with χ uniform in $\hbar$ and the boundary values Assumption (of the type of the cross constant of a two-scale inequality; a statement of this form is not taken from a known theorem).

c_{kg} of the kink-gas gap assumption comes from reading in the continuum limit the closed formula for the gap of the Glauber box with fixed ends, $(1-\gamma)+\gamma(1-\cos(\pi/(L+1)))$,

$$\frac{D}{\xi^2} + \frac{\pi^2 D}{L_\tau^2} = \big(C^2+\pi^2\big)\frac{D}{(C\xi)^2} \qquad (L_\tau = C\xi) \quad (11)$$

which gives $c_{\mathrm{kg}} = C^2+\pi^2$ Proved here (the algebra of (11)) Lean `kg_constant` (assumption: $C\xi \neq 0$). The first term D/ξ^2 is the creation and annihilation of pairs, and the second is the diffusion of kink positions. When a boundary value is near the top, half a kink can sit at the end of the box (corresponding to a free end), and in the sup including that case $c_{\mathrm{kg}} = c_{\mathrm{ff}}(C)$, read as 7.17 at $C = 2$ Reading (a family value obtained from lattice values of the gap of the free-end box of the one-dimensional Ising model).

Combining (9)(10) with the kink-gas gap assumption and the cross-term uniformity assumption gives

$$C_{\mathrm{box}} \;\le\; C_{\mathrm{loc}} + (1+\chi)\,C_{\mathrm{kg}} \;\le\; K_4\,\frac{(C\xi)^2}{D}, \qquad K_4 = \frac{1+\chi}{c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})} + C_{\mathrm{loc}}\,\frac{D}{(C\xi)^2} \quad (12)$$

Proved here Lean `H4_two_scale` (assumptions: the inequality of decomposition (9), first term $\le C_{\mathrm{loc}}\mathcal{E}$, the kink-gas gap assumption $\le C_{\mathrm{kg}}\mathcal{E}_Y$, the cross-term uniformity assumption, $C_{\mathrm{kg}} \ge 0$), `H4_gap_form` (assumptions: $D>0$, $C\xi \neq 0$, $c_{\mathrm{kg}}>0$, $\eta_{\mathrm{KG}}<1$, $1+\chi \ge 0$, the two inequalities above). This is the Gap-Inside-the-Box Proposition.

The second term of K_4 is C_{loc} multiplied by $D/(C\xi)^2$, and $D/\xi^2 = \hbar/(S_0\xi^2)$ is of size $e^{-441.6}/e^{-1328.6}/e^{-4433.1}$ at the three values of $\hbar$ ($\hbar = 3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$) Reading (Table 2 of 23-04 § 07). Unless C_{loc} grows exponentially in $\hbar$, this term vanishes as $\hbar \to$

0. In this sense the demand of the in-phase local Poincaré condition for $C_{\mathrm{loc}} = O(1)$ is a demand with room to spare.

A note on where the saddle-tube condition goes. In the two-valley split of § 01, the saddle-tube condition was the mean-difference constant C_{md} itself. In the box split, the direction along the family of saddles (the kink positions) was taken as the slow variable Y, so transport in that direction is handled by the kink-gas gap assumption and the cross-term uniformity assumption in the second term, and what remains in the saddle-tube condition is the Poincaré of the tube and the core with Y fixed. In this book the tube and core rows are counted as rows of the saddle-tube condition, and the kink-gas gap assumption and the cross-term uniformity assumption as rows of the gap-inside-the-box condition (the same counting as the integrated table, 23-12).

Step 3 (the composition step) Composition — $C_P \le K\xi^2 S_0/\hbar$

Putting (12) into C_{box} of (8) and using $L\varepsilon = C\xi$ and $D = \hbar/S_0$ gives

$$C_P(\mu) \;\le\; K\,\xi^2\,\frac{S_0}{\hbar}, \qquad K = \Big(1+\frac{1}{n}\Big)\frac{C^2 K_4}{1-c_{\mathrm{alt}}} \quad (13)$$

Proved here Lean `H3_H4_compose` (assumptions: the conclusion (8) of the Box-Reduction Proposition, the conclusion $C_{\mathrm{box}} \le K_4 (C\xi)^2/D$ of the Gap-Inside-the-Box Proposition, $D = \hbar/S_0$, $n > 0$, $\hbar > 0$, $S_0 > 0$, $c_{\mathrm{alt}} < 1$). That K does not depend on $\hbar$ is the uniformity of c_{alt}, χ, η_{KG}, c_{kg} and C_{loc}, that is, the very content of the assumption rows.

Putting in the values of the reduced model ($\chi = \eta_{\mathrm{KG}} = 0$, $n = 3$, $\theta = \frac{1}{4}$) gives $K \approx 2.76$ at $C = 2$ (the sup including boundary values at the top), 1.43 (ends on the well side only), and $K \approx 1.78$ at $C = 4$ Reading (family values, not upper bounds on K) Lean `K_ff_C2`, `K_ff_C4` (assumptions: the measured values of c_{ff} and c_{alt}; arithmetic only). The value is the same at the three values of $\hbar$ because $\hbar$ does not enter the reduced model.

Step 4 (the step bracketing the ratio) Bracketing the ratio R — $1 \le R \le K'/\hbar$

Dividing (13) by ξ^2 gives $R = C_P/\xi^2 \le KS_0/\hbar =: K'/\hbar$ Proved here (division). From below, the One-Dimensional One-Sided Bridge Proposition (23-04 § 04) gives $R \ge 1$ Proved here (continuous time, $T \to \infty$; the proof is placed in 23-04). Together,

$$1 \;\le\; R \;\le\; \frac{K'}{\hbar}, \qquad K' = KS_0 \quad (14)$$

The One-Dimensional One-Sided Bridge Proposition is a lower bound that uses no conditions, and the left inequality of (14) does not depend on the four double-well

conditions.

Step 5 (the step reading the power) The double-well power statement — $\ln C_P / \ln \xi \to 2$

$\ln C_P / \ln \xi = 2 + \ln R / \ln \xi$, and (14) gives $0 \le \ln R \le \ln K' + \ln(1/\hbar)$. The remaining input is how the ruler grows, $\ln \xi \ge s/\hbar$ (for some $s > 0$ and small $\hbar$), which comes from $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$. The semiclassical leading term of the continuum model is a copy of a known formula Cited , and for the lattice model it is a reading that $\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2}\ln \hbar$ settles to a constant, $-0.774/-0.777/-0.778$, at the three values of $\hbar$ Reading (23-04 § 07). On this input

$$0 \;\le\; \frac{\ln R}{\ln \xi} \;\le\; \frac{\hbar\big(\ln K' + \ln(1/\hbar)\big)}{s} \;\longrightarrow\; 0 \qquad (\hbar \to 0) \quad (15)$$

and the double-well power statement follows Proved here Lean `power_H_of_kink_ratio` (assumptions: $0 < K$, $0 < s$, $0 < \delta$, and $1 \le R \le K/\hbar$ and $s/\hbar \le \ln \xi$ for $0 < \hbar < \delta$. K in Lean is K' in the text).

Checking the single line

Shrinking Steps 1–5 to one line gives

$$\underbrace{(\mathrm{H1}^{\mathrm{seg}}),\ (\mathrm{H2}^{\mathrm{tube}}),\ (\mathrm{T0}'_{\mathrm{P}})}_{C_{\mathrm{loc}}} \xrightarrow{(9)(10),\ (\mathrm{KG}),\ (\mathrm{X})} \underbrace{C_{\mathrm{box}} \le K_4 \frac{(C\xi)^2}{D}}_{\text{Proposition H4}} \xrightarrow{\text{Proposition H3, (M}}$$

Of the inputs written above the arrows, the assumption rows are the segment local Poincaré assumption, the saddle-tube Poincaré assumption, the core Poincaré uniformity assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes, and the premises of Step 0; the rest are elementary algebra, known theorems or readings.

Here we also check that the statements of the assumption rows do not contain the quantities of the conclusion. The quantities of the conclusion are $C_P(\mu)$ of the whole measure with $T = \infty$ and the ratio R. The segment local Poincaré assumption, the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption are constants of the measures of segments, tubes and cores, the kink-gas gap assumption is a constant of the reduced gas, the cross-term uniformity assumption is a comparison of forms, and the alternating maximal-correlation assumption for boxes is a quantity of correlation. None of them contains $C_P(\mu)$ or R. A form that puts the conclusion among the premises (a circle of dependence) does not appear in the way these steps are built (for how to check this, see 23-13).

To be precise, the conclusion of the First-Step Upper-Bound Proposition is the first-step statement that "the relaxation time C_P of Langevin dynamics grows as a power of ξ^2". C_P is a different quantity from the gap of the transfer operator (the tunnelling splitting), and even if every row of the First-Step Upper-Bound Proposition were in place, the statements from the second step on would not follow (23-01 § 08, 23-03 § 06).

05

The distance from known theorems

Within the range searched, no known theorem gives the upper bound (5) of the First-Step Upper-Bound Proposition. Table 1 summarises how far nearby theorems reach and where they stop.

Table 1 Where known theorems reach and where they do not (the grade is the grade of the row that cites the theorem)

Known theorem	Where it reaches	Where it does not	Grade
The Poincaré version of Eyring–Kramers (Bovier–Gayrard–Klein; the Variance-Splitting Lemma of Menz–Schlichting, Theorems 2.9 and 2.12)	For a Morse function at fixed dimension, from the splitting of the variance, local Poincaré and transport of the mean difference up to the asymptotics of the gap and the prefactor	The constants are not uniform in the dimension. Degenerate families of saddles such as translation of a kink are outside the assumptions	Cited
Eyring–Kramers for the stochastic Allen–Cahn equation on a fixed interval (Di Gesù–Le Peutrec; Brooks–Di Gesù)	Asymptotics of the gap with the length of the interval fixed below the bifurcation threshold	The regime with infinite interval length and the translation family are stated by the authors themselves in the text to be out of scope	Cited
Transition times of short paths (Berglund–Gentz Theorem 2.6)	The mean time to move between valleys for $T < 2\pi\sqrt{2}$ (uniform saddle)	Long paths ($T \gg \xi$)	Cited
Diffusion of a single interface (Bertini–Brassesco–Buttà Theorem 2.1)	That the position of a single kink approaches Brownian motion with diffusion coefficient D (at the level of the process)	Not a statement about the spectral gap of the whole path	Cited
A criterion uniform in the volume (Bauerschmidt–Dagallier Theorem 2.1)	Applies directly to the lattice of paths and gives an upper bound uniform in the volume	The value is $C_P \le e^{c\xi}$ with $c \approx 1.4 \to 1.8$ ($\varepsilon = 0.25$, $\hbar = 0.4 \to 0.1$), doubly exponential in $\hbar$. It does not reach the double-well power statement	Cited (theorem) Reading (value)

Known theorem	Where it reaches	Where it does not	Grade
The Box-Dynamics Proposition (Propositions 3.4 and 3.5 of Martinelli's lectures)	For discrete spins, the type that controls the whole gap by the box dynamics and the gap inside the box	Unbounded spins, sup over boundary values. Step 1 borrows only the type, rewritten in one dimension in the L^2 form	Cited (type)

As the last three rows of Table 1 show, the known tools sit at the two ends, "a single kink" and "an upper bound uniform in the volume but coarse", and what lies between them (an upper bound on long paths sharp up to the diffusion time ξ^2/D of the kink gas) is the place of the First-Step Upper-Bound Proposition. The Bauerschmidt–Dagallier bound becoming $e^{c\xi}$ is read as follows: the criterion looks at a single quantity, the susceptibility, so it cannot distinguish slow but local relaxation from slow and global relaxation, and ξ enters the exponent Reading .

06

The table of the five grades — the rows of the steps of the First-Step Upper-Bound Proposition

We put the steps of § 04 into one table in the format of the five grades (23-00 § 01). Each row has one grade, and the column of machine checking is outside the grades. The row numbers are the numbers of the nodes of the dependency graph in § 07 (the labels in the formulas, (H1^{seg}), (H2^{tube}), (T0′$_{\mathrm{P}}$), (KG) and (X), are the assumptions [02]–[06] respectively).

Table 2 The rows of the steps of the First-Step Upper-Bound Proposition ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Uniqueness of the measure with $T = \infty$, the step of the limit, regularity	Assumption	§ 04 Step 0 (the original source for the principal eigenvalue of a positive kernel on $\mathbb{R}$ is unchecked)	—	—
[02]	The segment local Poincaré assumption: Poincaré of segments without kinks uniform in the length, the end values and $\hbar$	Assumption	§ 03, (10) of § 04 Step 2. The node propositions for the ring version are in 23-06	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[03]	The saddle-tube Poincaré assumption: Poincaré of the tube uniform in length $\leq C\xi$	Assumption	§ 03, (10)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[04]	The core Poincaré uniformity assumption: Poincaré of the core uniform in the core length	Assumption	§ 03, (10). The proposition with an assumption at the lower end of the window is in 23-09 and 23-11. For the core on the sector side (both ends in the well), it is supported for every $\hbar \in (0,1]$ by the Dobrushin criterion of the coarse chain: conditional on a proof in the paper (the Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$, 239 intervals, interval arithmetic, $\rho_{\mathrm{col}} \leq 0.400$) + a paper proof and Computer-assisted (the Small-$\hbar$ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \leq 10^{-5}$, $\rho_{\mathrm{col}} \leq 0.874$) + Cited (the Poincaré Version of Helffer's Criterion in Bobkov–	Sector side: $(0,1]$	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
			Ledoux 2000 (Proposition 3.1 of Ledoux)), and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper). There are no Reading rows. The core on the box side is not covered (the paper "Convex blocks and a coarse Dobrushin chain")		
[05]	The kink-gas gap assumption: the box gap of the reduced kink gas $\geq c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})D/(C\xi)^2$	Assumption	§ 04 Step 2 (the assumption that it falls into the Glauber type)	—	—
[06]	The cross-term uniformity assumption: cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \leq (1+\chi)\mathcal{E}(f)$	Assumption	§ 04 Step 2	—	—
[07]	The alternating maximal-correlation assumption for boxes: $c_{\mathrm{alt}} < 1$ uniformly in $\hbar$	Assumption	§ 04 Step 1	—	—
[08]	Mobility of a single kink $D = \hbar/S_0$	Cited	§ 02 (d), 23-04 § 03 (a copy of the coefficient of a theorem at the level of the process)	$\hbar \to 0$	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[10]	The Two-Projection Lemma	Proved here	Proof in § 04 Step 1	Independent of $\hbar$	`two_proj_var` (assumption: $A + B \le (1 + c)V$)
[11]	Conditional independence of even boxes and the product variance inequality (7)	Proved here	§ 04 Step 1 (proof in 23-11)	Independent of $\hbar$	—
[14]	The Box-Reduction Proposition: (8) $C_P \le (1 + 1/n)C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$	Proved here	§ 04 Step 1 (the Two-Projection Lemma, the product variance of even boxes, averaging over shifts)	Independent of $\hbar$	`box_two_family`, `box_shift_average` (assumptions: two-family inequality, $C_{\mathrm{box}} \ge 0$, sum over overlaps $\le \mathcal{E}$)
[12]	Decomposition of the variance (9) and into a product (10)	Proved here	§ 04 Step 2 (the product decomposition depends on how Y is defined; regularity is in the row of common premises [01])	Independent of $\hbar$	—
[13]	$c_{\mathrm{kg}} = C^2 + \pi^2$ (the algebra of (11))	Proved here	§ 04 Step 2 (the closed Glauber formula is inside the kink-gas gap assumption)	Independent of $\hbar$	`kg_constant` (assumption: $C\xi \;/= 0$)
[15]	The Gap-Inside-the-Box Proposition: (12) $C_{\mathrm{box}} \le K_4 (C\xi)^2 / D$	Proved here	§ 04 Step 2	Independent of $\hbar$	`H4_two_scale`, `H4_gap_form` (assumptions are the rows of § 04 Step 2)
[16]	The First-Step Upper-Bound Proposition: (13) $C_P \le K \xi^2 S_0 / \hbar$	Proved here	§ 04 Step 3	$\hbar \to 0$	`H3_H4_compose` (assumptions: (8), (12), $D = \hbar / S_0$, $n, \hbar, S_0 > 0$, $c_{\mathrm{alt}} < 1$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
—	$K \approx 2.76/1.43$ ($C = 2$), 1.78 ($C = 4$)	Reading	§ 04 Step 3 (reduced model, $\chi = \eta$ $\mathrm{KG} = 0$, family values)	Independent of $\hbar$	`K_ff_C2`, `K_ff_C4` (assumptions: measured values; arithmetic only)
[17]	The One-Dimensional One-Sided Bridge Proposition: $R \geq 1$	Proved here	23-04 § 04	Every $\hbar$	`local_rayleigh_ratio_le_one` (the skeleton of the finite sum only; the assumptions are in Table 3 of 23-04)
[18]	(14) $1 \leq R \leq K'/\hbar$	Proved here	§ 04 Step 4	$\hbar \to 0$	—
[09]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{S0/\hbar}$)	Reading	§ 04 Step 5 (the continuum leading term is Cited, the values of the lattice model are Reading)	Read at three points; $\hbar \to 0$ from the leading term	—
[19]	The double-well power statement: $\ln C_P / \ln \xi \to 2$	Proved here	(15) of § 04 Step 5	$\hbar \to 0$	`power_H_of_kink_ratio` (assumptions: $0 < K, s, \delta$, $1 \leq R \leq K/\hbar$, $s/\hbar \leq \ln \xi$)

The "Proved here" rows of Table 2 are all algebra that takes assumption rows as input. That "Proved here" rows line up does not mean that the conclusion of the First-Step Upper-Bound Proposition has been shown. The conclusion becomes unconditional only when every "Assumption" row and the "Reading" row (the row of the growth of the correlation length, [09]) have been replaced.

07

The dependency graph — checking by the numbers that there is no circle

We turn the rows of Table 2 into a list of numbered nodes. The format is common to this part (the integrated graph of 23-12 is the same), and **the numbers of the premises to the right of "⟸" are always placed smaller than the number of the conclusion on the left**. The leaves (assumptions, known results, readings) have small numbers, and the conclusion has the largest number.

```
[01] ASM  row of common premises: uniqueness of the T=∞ measure, limi
[02] ASM  segment local Poincaré assumption: Poincaré of segments wit
[03] ASM  saddle-tube Poincaré assumption: Poincaré of the tube (leng
[04] ASM  core Poincaré uniformity assumption (uniform in core length
[05] ASM  kink-gas gap assumption: box gap of the reduced kink gas
[06] ASM  cross-term uniformity assumption: cross term ℰ_Y ≤ (1+χ)ℰ
[07] ASM  alternating maximal-correlation assumption for boxes: c_alt
[08] CIT  kink diffusion-coefficient row: mobility of a single kink D
[09] RDG  row of the growth of the correlation length: ln ξ ≥ s/ħ (ξ
[10] PRV  Two-Projection Lemma
          Lean two_proj_var (assumption: A+B ≤ (1+c)V)
[11] PRV  conditional independence of even boxes, product variance (7
[12] PRV  decomposition of variance (9), into a product (10) ⟸ [01]
[13] PRV  c_kg = C²+π² (algebra of (11))
          Lean kg_constant (assumption: Cξ ≠ 0)
[14] PRV  Box-Reduction Proposition (8) ⟸ [07][10][11]
          Lean box_two_family, box_shift_average (assumptions: two-fa
[15] PRV  Gap-Inside-the-Box Proposition (12) ⟸ [02][03][04][05][06]
          Lean H4_two_scale, H4_gap_form (assumptions: decomposition,
[16] PRV  First-Step Upper-Bound Proposition (13) C_P ≤ Kξ²S₀/ħ ⟸ [0
          Lean H3_H4_compose (assumptions: (8), (12), D = ħ/S₀, n, ħ,
[17] PRV  One-Dimensional One-Sided Bridge Proposition R ≥ 1 ⟸ [01]
          Lean local_rayleigh_ratio_le_one (skeleton of the finite su
[18] PRV  (14) 1 ≤ R ≤ K'/ħ ⟸ [16][17]
[19] PRV  double-well power statement ln C_P / ln ξ → 2 ⟸ [09][18]
          Lean power_H_of_kink_ratio (assumptions: 0 < K, s, δ; 1 ≤ R
```

How to check. In every row with "⟸", look to see that the numbers on the right are smaller than the number on the left. [11] and [12] use only [01], [14] uses [07][10][11], [15] uses [02]–[06][08][12][13], [16] uses [08][14][15], [17] uses [01], [18] uses [16][17], and [19] uses [09][18]. Every row keeps the rule. If the rule is kept, following the premises always lowers the number and eventually reaches a leaf, so there is no circle.

Among the leaves, "Assumption" covers the seven [01]–[07], "Cited" is [08], and "Reading" is [09]. The conclusion [16] of the First-Step Upper-Bound Proposition depends on all of [01]–[08] (on [01] also through [11][12]), and the double-well power statement [19] depends further on [09]. Only the One-Dimensional One-Sided Bridge Proposition [17] depends on nothing but [01] and does not use the four double-well conditions.

To be precise, this graph is the dependency of the construction of the steps of the First-Step Upper-Bound Proposition, and it does not include the content of each assumption

row (for example the Six-Node In-Phase Poincaré Proposition, which supports [02] at the nodes). Those are in the graphs of the chapters on each condition, and the graph that joins them all into one is the integrated graph of 23-12.

08

A $\hbar$ that fits all, and where the four conditions stand

There is no $\hbar$ at which all rows are in place. The node propositions of the in-phase local Poincaré condition are at six points with $\hbar \geq 0.005$, the conditional propositions that support the core row are at the lower end of the window, of order $\hbar \leq 10^{-3}$, and the two do not meet. The kink-gas gap assumption, the cross-term uniformity assumption and the saddle-tube Poincaré assumption are assumption rows at every $\hbar$. The band figure and the way of fitting them together are placed in 23-12.

We list where each of the four conditions stands, one line each. The details are in each chapter.

Where the four conditions stand

1. The in-phase local Poincaré condition — the Six-Node In-Phase Poincaré Proposition for the ring version: margin ≥ 0.067 at the six nodes $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, with neighbours and ends in $[0, 2.5]$ (computer-assisted). What remains is between the nodes, $\hbar < 0.005$, ends > 2.5, and the segment version (23-06)
2. The saddle-tube condition — a map split into seven routes. The condition on the chain outside the window is computer-assisted on the whole window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, five steps of the sector framework are propositions, and for the one remaining proposition three strands stop at the curvature shortfall $\frac{1}{4}$ at one site at the top (23-07–23-10)
3. The box-reduction condition — the Box-Reduction Proposition (§ 04 Step 1). The alternating maximal-correlation assumption for boxes and the $\sup$ over boundary values remain (23-11)
4. The gap-inside-the-box condition — the Gap-Inside-the-Box Proposition (§ 04 Step 2). The cross-term uniformity assumption, the kink-gas gap assumption, the segment local Poincaré assumption and the core row remain, and the core constant is controlled in three steps (23-11)

This part is a study that chose the relaxation-time assumption as its foothold, and the First-Step Upper-Bound Proposition is one way of splitting to build the upper side of

the first step. Other views are treated in Part 24.

What this chapter can say

What is said	What is not said
Proved here The algebra of the splitting of the variance (3), the Two-Projection Lemma, the Box-Reduction Proposition (8), the Gap-Inside-the-Box Proposition (12), the composition (13), the bracketing of the ratio (14), the algebra to the double-well power statement (15). All are algebra on named assumption rows Lean (the assumptions are the rows of Table 2)	None of the assumption rows [01]–[07]. Hence neither (5) itself nor the double-well power statement for the first step
Proved here The One-Dimensional One-Sided Bridge Proposition: $R \geq 1$ (uses no conditions)	Whether R grows like $1/\hbar$ or exponentially
Cited The diffusion coefficient of a single kink; how far known theorems reach (Table 1)	A known theorem giving an upper bound on long paths sharp up to the diffusion time of the kink gas (only as far as "not found within the range searched")
Reading $K \approx 2.76$ and others (reduced model), $c_{\mathrm{alt}} \approx 0.73$, how $\ln \xi$ grows	Using these in proofs of upper or lower bounds
That the dependency graph has no circle (the rule of the numbers)	A $\hbar$ at which all rows are in place. Implications from the first step to the second and later steps

Sources and reproduction

Item	Kind	Source / tools
The splitting of the variance, the skeleton of the Poincaré version of Eyring–Kramers	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014), arXiv:1202.1510 (the Variance-Splitting Lemma, Theorems 2.9 and 2.12; checked in the text). A. Bovier, V. Gayrard, M. Klein, J. Eur. Math. Soc. 7 (2005) 69–99 (bibliographic data).
Bakry–Émery on a convex region	Cited	D. Bakry, M. Émery, LNM 1123 (1985); A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526 (Theorem 1.2; text checked).
Eyring–Kramers on a fixed interval	Cited	G. Di Gesù, D. Le Peutrec, arXiv:1506.04434 (Theorem 1.1); M. Brooks, G. Di Gesù, arXiv:2606.16083 (the explicit statement of what is out of scope; text checked).
Transition times of short paths, diffusion of a single interface	Cited	The table of sources of 23-04 (Berglund–Gentz, Bertini–Brassesco–Buttà).

Item	Kind	Source / tools
A criterion uniform in the volume	Cited, Reading	R. Bauerschmidt, B. Dagallier, Comm. Pure Appl. Math. 77, arXiv:2202.02295 (Theorem 2.1; text checked). The application to the lattice of paths and the value $e^{c\xi}$ are floating-point computations of the record (transfer matrix, $\varepsilon = 0.25$, 1.0049 on the harmonic positive control).
The type of box dynamics	Cited	F. Martinelli, LNM 1717 (1999) 93–191 (checked in § 3.6 and Propositions 3.4 and 3.5 of the preprint).
The lemma of two projections, the Efron–Stein type	Proved here	§ 04 Step 1 of the text (the original sources for the Friedrichs angle and Efron–Stein are bibliographic only).
The closed formula for the box gap of Glauber dynamics, values with free ends	Assumption, Reading	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic only). The fixed-end formula agrees to 6 digits with diagonalisation over all states of small chains; free ends are lattice values (floating point).
Values of the alternating maximal correlation	Reading	Family values copied from the closed formula for the two-state telegraph chain (floating point).
The algebra of the steps	Proved here, Lean	Theorems `split_poincare_le`, `reduced_two_state_value`, `two_proj_var`, `box_two_family`, `box_shift_average`, `kg_constant`, `H4_two_scale`, `H4_gap_form`, `H3_H4_compose`, `K_ff_C2`, `K_ff_C4`, `power_H_of_kink_ratio` (the assumptions are the rows of Table 2 and of the text). The list of statements is in 23-13.

Words used in this chapter — splitting of the variance, local Poincaré, transport of the mean difference, the four double-well conditions, the First-Step Upper-Bound Proposition, phase (phase without kinks), segment, tube, core, box dynamics (box reduction), maximal correlation, alternating maximal correlation, the Two-Projection Lemma, Efron–Stein inequality, reduced kink gas, cross term, dependency graph. Their meanings are in A-02 Glossary.

Next chapter to read: 23-06 The in-phase local Poincaré condition — turns the Poincaré constant of the measure of the phase without kinks into node propositions at six values of $\hbar$, by a two-scale construction of kept sites and bridges, and describes the method and cost of filling in between the nodes.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time the in-phase local Poincaré condition, the Six-Node In-Phase Poincaré Proposition, six nodes · Kiichi, Shiori and Rin

The in-phase local Poincaré condition

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — The in-phase local Poincaré condition is the statement that the Poincaré constant of the measure restricted to the phase without kinks (configurations in which the path stays on the side of one valley) is $O(1)$ uniformly in $\hbar$ and in the length of the path. In this chapter we turn the version on the ring into the Six-Node In-Phase Poincaré Proposition, by a two-scale construction that splits the path into kept sites and bridges of 11 sites between them. The Six-Node In-Phase Poincaré Proposition holds at six nodes of $\hbar$, $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, and its key inequality (margin on the coarse side ≥ 0.067) is computer-assisted. At the end we list the method and cost of filling in the $\hbar$ between the nodes, and what remains that needs other methods, such as $\hbar < 0.005$ and large end values.

Prerequisite chapters — 23-05 (the four conditions and the Double-Well Relaxation-Time Upper-Bound Proposition, the format of the table of the five grades and the dependency graph), 23-04 § 06 (the plateau at the top and the convex core), 1-10 § 07– § 09 (Poincaré inequalities). For how to read computer-assisted results, see 23-08.

Said first — § 01 is a summary readable at [Y3], and § 02 onwards is at the research level [Res]. You can read only the summary and go on to 23-07. The Six-Node In-Phase Poincaré Proposition is not a proof of the in-phase local Poincaré condition. The $\hbar$ outside the six nodes, end values > 2.5, and the segment version used inside the box remain as assumption rows.

The order of this chapter

1. Summary [Y3] — statement, construction, the Six-Node In-Phase Poincaré Proposition, what remains
2. The statement and the measure of the phase — what is hard
3. The two-scale construction — kept sites, bridges, nesting, the criterion on the coarse side, the algebra of the rate
4. The Six-Node In-Phase Poincaré Proposition — the margins at the six nodes (Table 1) and the constant

5. The table of the five grades — the rows of the Six-Node In-Phase Poincaré Proposition (Table 2)
6. The $\hbar$ between the nodes — method and cost (Tables 3 and 4)
7. The remaining rows — what needs other methods (Table 5)
8. The dependency graph

01

Summary [Y3]

Statement. On the lattice of paths of the double well ($\varepsilon = \frac{1}{2}$), consider the measure $\mu^{(+)}$ restricted to $x_i \geq 0$ at every site. The in-phase local Poincaré condition says "$C_P(\mu^{(+)})$ is at most a constant independent of $\hbar$ and of the length". Bakry–Émery on the convex core works only deep in a valley ($x_i \geq a > 1/\sqrt{3}$), and exponentially many excursions within the valley fit in an interval of length ξ, so this measure is not convex (23-04 § 06).

Construction. Keep one site in every 12 (the coarse side), and treat the 11 sites between them as bridges with both ends fixed (the fine side). Split the variance into "the conditional variance with the kept sites fixed" and "the variance of its expectation". The fine side is controlled by splitting each bridge further into three small bridges and two sites, and the coarse side is judged on the chain of kept sites by the gap at one site and the strength of the coupling between neighbours. The key to the criterion on the coarse side is the margin inequality

$$\underbrace{\hbar\rho}_{\text{gap at one site}} - 2 \underbrace{\hbar\kappa}_{\text{coupling to neighbours}} > 0 \quad (1)$$

, and interval arithmetic is used to check it for every combination of end values.

The Six-Node In-Phase Poincaré Proposition. At each of the six nodes of $\hbar$, the margin ≥ 0.067 is checked computer-assisted for all neighbour and end values in $[0, 2.5]$, and on top of the remaining rows (assumptions and known theorems) $C_P(\mu^{(+)}) \leq C$ with $C \leq 1.6 \times 10^{10}$ (from paper and interval-arithmetic inputs only). With numerical readings as input, $C \leq 658$. Neither depends on the length of the ring.

What remains. For the $\hbar$ between the nodes, there is a scheme that fills in subinterval by subinterval by making $\beta = 1/\hbar$ a third direction of tilting, and its cost is estimated at tens of hours on one core. $\hbar < 0.005$ needs infinitely many subintervals and so needs another method (asymptotics as $\hbar \to 0$). End values > 2.5, the sharpness of the constant, the scrap version used inside the box, and checking the bibliography of the cited theorems also remain.

02

The statement and the measure of the phase — what is hard

The measure of the phase. From the lattice action on a ring of $N = 12n$ sites ($n \geq 3$),

$$S = \sum_i \frac{(x_{i+1} - x_i)^2}{2\varepsilon} + \varepsilon \sum_i V(x_i), \qquad V(x) = \frac{(x^2-1)^2}{8}, \qquad \varepsilon = \frac{1}{2} \quad (2)$$

we form the measure $\mu^{(+)} \propto e^{-S/\hbar} \mathbf{1}\{x_i \geq 0 \ \forall i\}$ restricted to configurations with $x_i \geq 0$ at every site. The orthant is convex, so this measure is the equilibrium of Langevin dynamics reflected at the wall $x_i = 0$ (Neumann condition). The Dirichlet form is $\mathcal{E}(f) = (\hbar/\varepsilon) \sum_i \mathbb{E}|\partial_i f|^2$. The path stays on the side of the valley $x = 1$ and has no kinks. This is "the phase without kinks".

The in-phase local Poincaré condition (ring version)

There exist $\hbar_0 > 0$ and $C < \infty$ such that $\mathrm{Var}_{\mu^{(+)}} f \leq C\, \mathcal{E}(f)$ for all $\hbar \in (0, \hbar_0]$ and all n.

The version used inside the box (the segment local Poincaré assumption used in the step inside the box, 23-05 § 04) uses, instead of the ring, segments of length $\leq C\xi$ with values at both ends, and asks for uniformity in the end values as well.

What is hard. Three naive roads all lose a factor exponential in $\hbar$.

- The convex core: restricting to $x_i \geq a > 1/\sqrt{3}$ gives $C_P \leq 2/(3a^2 - 1)$ by Bakry–Émery, but the action of an excursion out of the core (0.1535 at $a = 1/\sqrt{3}$) is smaller than the kink action $2/3$, and $e^{(S_0 - c_a)/\hbar}$ excursions fit in an interval of length ξ Proved here (23-04 § 06).
- Global perturbation (of the Holley–Stroock type): the oscillation of the difference from a convex action is proportional to the length of the path, and the constant is of the type $e^{T\delta/\hbar}$ Cited (the Holley–Stroock perturbation lemma; type checked in the restatement as Proposition 6.1 of Ledoux).
- Perturbation site by site: the non-convexity per site is $\varepsilon/(8\hbar)$, but the coupling between neighbours $(x_{i+1} - x_i)^2/(2\varepsilon)$ is strong, so sites cannot be detached and treated one at a time Reading .

The numbers do not contradict the in-phase local Poincaré condition. Measuring the integrated autocorrelation time of the magnetisation for Langevin dynamics on the ring with a wall gave $0.86/1.40/1.50/1.39$ at $\hbar = 0.3/0.15/0.1/0.07$, sizes that do not depend on $\hbar$ Reading (the autocorrelation of a single observable gives only a lower bound on the Poincaré constant, and is not evidence for an upper bound).

03

The two-scale construction

Kept sites and bridges. Keep the sites $y_j := x_{12j}$, and treat the 11 sites $x_{12j+1}, \dots, x_{12j+11}$ between y_j and y_{j+1} as a bridge with both ends fixed at y_j, y_{j+1}. Splitting the variance by the kept sites y gives

$$\mathrm{Var}_\mu(+)\ f = \mathbb{E}\big[\mathrm{Var}(f \mid y)\big] + \mathrm{Var}\big(\mathbb{E}[f \mid y]\big) \quad (3)$$

(the variance-decomposition row: the decomposition of the variance is elementary; the regularity and density of the functions used are assumptions). Because the interaction is nearest-neighbour, the bridges are independent once y is fixed, and the marginal measure of y is a nearest-neighbour chain of the kept sites (the nearest-neighbour chain row: the Markov property).

The fine side: nesting inside the bridge. A bridge of 11 sites is not convex (the smallest eigenvalue of the Hessian of the bridge is $(1/\varepsilon)\lambda_{\min}(L_{11}) - \varepsilon/2 = -0.11$). So we keep sites 4 and 8 of the bridge as one more level and split it into three small bridges (3 sites) and two sites (x_4, x_8).

- A small bridge of 3 sites is log-concave: Hessian $\geq (1/\varepsilon)\lambda_{\min}(L_3) - \varepsilon/2 = 2(2-\sqrt{2}) - \frac{1}{4} \approx 0.92$. Bakry–Émery gives the gap $\rho_f = 0.922$ Proved here (lower bound on the curvature) Cited (Bakry–Émery on a convex region).
- The cross term between the small bridges and the two sites is $\kappa_f = 1.98$ by the Brascamp–Lieb inequality Cited .
- The gap of the marginal measure of the two sites (x_4, x_8) is $\lambda_c \geq 1.3 \times 10^{-4}$ (end values $[0,4]^2$, $\hbar \in (0, 0.2]$) Computer-assisted (the two-site gap row).

Combining the three by the nested two-scale algebra, the gap of the bridge is at least the smaller root of $t^2 - (\rho_f + \lambda_c + \kappa_f^2/\rho_f)t + \rho_f \lambda_c = 0$, and $\lambda_S \geq 2.3 \times 10^{-5}$ Proved here (arithmetic of the root).

The coarse side: the gap at one site and the coupling. The one-dimensional conditional measure of y_j with the neighbour values c, d fixed is $\propto e^{-U(t)/\hbar}$, with

$$U'(t) = \varepsilon V'(t) + \frac{2t - e(c,t) - e(d,t)}{\varepsilon}, \qquad e(a,b) := \mathbb{E}_{a,b}\, x_{11} \quad (4)$$

($\mathbb{E}_{a,b}$ is the expectation over the bridge with ends a, b). Write $\hbar\rho(c,d)$ for its Poincaré gap multiplied by $\hbar$, and let the strength of the coupling between the two ends of the bridge be

$$\hbar\kappa(a,b) := \frac{\mathrm{Cov}_{a,b}(x_1, x_{11})}{\varepsilon^2 \hbar} = \frac{1}{\varepsilon}\,\partial_a e(a,b) \ \geq\ 0 \quad (5)$$

(the second equality is the identity obtained by differentiating $\partial_a \log Z(a,b)$ in b Proved here ; non-negativity follows from e being increasing in a Cited (the FKG correlation inequality)). Using the Poincaré Version of Helffer's Criterion in the Gershgorin form on the ring, the gap of the chain on the coarse side is at least the margin

$$\lambda := \min_{c,d} \hbar\rho(c,d) - 2\max_{a,b} \hbar\kappa(a,b) \quad (6)$$

Cited (the Poincaré Version of Helffer's Criterion, Proposition 3.1 of Ledoux) Proved here (the Gershgorin form) Lean `ring_gershgorin` (the inequality for quadratic forms on the ring; assumption: $\kappa \geq 0$). (1) means $\lambda > 0$.

The cross term and the algebra of the rate. The strength of the cross term linking the fine side and the coarse side is $\kappa^2 = 2(v/\hbar)\lambda_S/\varepsilon^2$, with $v/\hbar \leq 2\big(644/1455 + (60/97)^2/\lambda_c\big) \leq 5888$ (the crossing row) Proved here (arithmetic) Lean `T4_paper_numeric` (assumption: $\lambda_c \geq 1.3 \times 10^{-4}$) Cited (Brascamp–Lieb). Both λ and λ_S are in units of the gap for the flat Dirichlet form $\int |\nabla f|^2$ multiplied by $\hbar$. Then if $0 \leq t < \lambda$ satisfies

$$t\,\kappa^2 \;\leq\; \lambda_S(\lambda_S - t)(\lambda - t) \quad (7)$$

, we have $C_P(\mu^{(+)}) \leq \varepsilon/t$ Proved here Lean `H1_assembly` (algebra with the assumptions named: decomposition (3) [`hT5`], the fine side [`hFine`; ρ in Lean is λ_S in the text], the conclusion of the criterion on the coarse side [`hLedoux`], the margin $\lambda \leq \hbar\rho - 2\hbar\kappa$ [`hT1`], $0 < \lambda$, the cross term [`hT4`], the rate condition (7)).

The whole construction, as a tree of decompositions of the variance, is as follows.

```
Var f = E[Var(f|y)] + Var(E[f|y])                                  (varia
 ├ fine side: bridges of 11 sites with y fixed
 │    └ nesting {4,8}: small bridges BE ρ_f=0.922, cross term BL κ_f=1.9
 ├ coarse side: ring of kept sites (nearest-neighbour chain row) ── Poin
 │    └ margin row: gap at one site ħρ (Muckenhoupt, FKG band, tail) − 1
 └ crossing: κ² = 2(v/ħ)λ_S/ε² (crossing row ← two-site gap row)
 → rate condition tκ² ≤ λ_S(λ_S−t)(λ−t) (H1_assembly) → C_P ≤ ε/t
```

04

The Six-Node In-Phase Poincaré Proposition — the margins and the constant

The Six-Node In-Phase Poincaré Proposition (0+1 dimensions, ring version, with a table of assumptions)

Let $V = (x^2-1)^2/8$, $\varepsilon = \frac{1}{2}$, the ring $N = 12n$ ($n \geq 3$), the measure of the phase $\mu^{(+)}$ with the wall $x_i \geq 0$, the kept sites $y_j = x_{12j}$, and the nested sites inside the bridge $\{4,$

8}. Under the "Assumption" and "Cited" rows of Table 2, at each point $\hbar \in \{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$,

$$\mathrm{Var}_{\mu}(+)\, f \;\le\; C\,\mathcal{E}(f), \qquad C \le 1.6 \times 10^{10} \quad (8)$$

holds independently of n. With numerical readings as input, $C \le 658$.

The key row is the margin (6), and the values of Table 1 are obtained at the six nodes for every combination of neighbour and end values in $[0, 2.5]$.

Computer-assisted Table 1 The margin on the coarse side at each node (neighbour and end values $[0, 2.5]^2$; $\hbar\rho$ is a lower bound and $\hbar\kappa$ an upper bound, values rounded outwards)

$\hbar$	$\hbar\rho \ge$	$\hbar\kappa \le$	**Margin** $\hbar\rho - 2\hbar\kappa \ge$
0.2	0.7755	0.0358	0.7039
0.1	0.7140	0.0664	0.5812
0.05	0.8121	0.1222	0.5677
0.02	0.8846	0.2636	0.3574
0.01	0.8670	0.39996	0.0670
0.005	0.92182	0.41868	0.0844

The interval arithmetic is implemented by adding outward widths to float64 operations (rounding of non-negative sums is widened outwards by a relative 10^{-10}, interval arithmetic operations by a relative 10^{-12}, weights below e^{-700} are replaced from above, and the region $\varepsilon V/\hbar > 690$ is covered by a tail upper bound). It is not rounding with a direction specified for each operation, but an added width that presupposes correct IEEE rounding. This premise is written in the same row, row [01] of Table 2. The arithmetic that gives the margin ≥ 0.067 from the six pairs of values is Lean `T1_margin_six_nodes` (no assumptions, numbers), and the step from the interval output to the margin at one node is `T1_node_margin` (assumptions: the two inequalities of the interval output and $0.067 \le r - 2k$).

Tools that extend the intervals to the continuum between nodes. Table 1 gives values "for every combination of neighbour and end values", and to pass from computations on finitely many boxes to the continuum we use three lemmas.

- The $\hbar\rho$ side: Muckenhoupt's estimate, which squeezes the Poincaré constant of a one-dimensional measure by a Hardy-type quantity B as $B \le C_P \le 4B$ Cited ; a lemma that builds a band from the values at the nodes to the continuum of neighbours by the monotonicity (FKG) of $e(c,t)$ in the neighbour value Cited (FKG); and a lemma for the tail on the side of large t (the continuous version of Holley's comparison and a tilted Gaussian bridge) Assumption (the theorem number of the continuous version has not been checked in the original source).

- The $\hbar\kappa$ side: the Tilted Second-Order Taylor Lemma, which tilts the measure at the centre of a box of end values and controls the whole box by the second-order Taylor expansion of the cumulant generating function and a Hölder remainder Proved here (the Taylor remainder and Hölder's inequality). This formula is exact whatever constant is chosen for the centring.
- The smaller of two upper bounds is also an upper bound, and the reflection symmetry $\hbar\kappa(a,b) = \hbar\kappa(b,a)$ covers half Proved here Lean `min_of_two_upper`, `symm_upper` (assumptions: two upper bounds / symmetry and a one-sided upper bound).

The moves that made the margin positive at $\hbar = 0.01$ and 0.005. When $\hbar$ is small the upper bound on $\hbar\kappa$ swells, and in a coarse computation the margin was negative at $\hbar = 0.01$, with 0.634. What limits it differs from place to place, and three moves bring it down. (i) At the corner of the ends $(0,0)$, the centring constant of the Tilted Second-Order Taylor Lemma is retaken for each subrectangle as the median of that range (the formula is exact for any centre, so only the upper bound goes down). (ii) The second-order term acts as the square of the box width δ, so δ is halved. (iii) At the large end $(2.5, 2.5)$, the bridge jumps far into the well in one step, so the quadrature error is the limit, and the quadrature step is halved. At $\hbar = 0.005$, refining the quadrature on the $\hbar\rho$ side raises the lower bound from 0.6945 to 0.92182. The last two rows of Table 1 are the results of these moves Computer-assisted .

The constant. $C \le 1.6 \times 10^{10}$ in (8) is the ε/t that follows from $t = 3.2 \times 10^{-11}$ satisfying the rate condition (7) with the paper and interval-arithmetic inputs ($\lambda \ge 0.067$, $\lambda_S \ge 2.3 \times 10^{-5}$, $v/\hbar \le 5888$) Proved here (arithmetic) Lean `H1_rate_paper_six_nodes` (no assumptions, numbers). Taking the gap of the two sites to be the numerical $\lambda_C = 0.57$ gives $\lambda_S \ge 0.093$, $v/\hbar \le 1.0015$, then $t = 7.6 \times 10^{-4}$ and $C \le 658$ Reading (the input is a floating-point reading) Lean `H1_rate_numeric_six_nodes` (no assumptions, numbers). Most of the difference between the two values comes from the weakness of the paper value of λ_C.

05

The table of the five grades — the rows of the Six-Node In-Phase Poincaré Proposition

We put the inputs of the Six-Node In-Phase Poincaré Proposition into one table in the format of the five grades. Items whose paper proof is placed in this book are "Proved here", those whose proof is not are "Assumption", interval arithmetic is "Computer-assisted" with the premises of its implementation written in the same row, and cited theorems are split into "Cited" or "Assumption" according to the level to which their bibliography has been checked. The numbers in the row column are the numbers of the nodes of the dependency graph in § 08, and "—" marks a row not in the graph.

Table 2 The rows of the Six-Node In-Phase Poincaré Proposition ("six nodes" in the range of $\hbar$ means $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbours and ends $[0, 2.5]^2$)	Computer-assisted	§ 04 Table 1. float64 + outward widths (relative 10^{-10}, 10^{-12}), presupposing correct IEEE rounding; tails covered by upper bounds	Six nodes	`T1_margin_six_nodes` (numbers), `T1_node_margin` (assumption: the interval output)
[02]	The Tilted Second-Order Taylor Lemma (second-order Taylor of the tilt and Hölder remainder, any centre)	Proved here	§ 04 (Taylor remainder, Hölder's inequality)	Independent of $\hbar$	—
[03]	Squeezing the one-dimensional Poincaré constant, $B \leq C_P \leq 4B$ (Muckenhoupt)	Cited	Statement checked in the restatement as Theorems 1 and 2 of Barthe–Roberto	Independent of $\hbar$	—
[04]	Monotonicity of $e(c, t)$ and $\hbar\kappa \geq 0$ (the MTP_2 correlation inequality)	Cited	Proposition 2.1 of Karlin–Rinott (checked in the restatement as Proposition 3.5 of Fallat et al.)	Independent of $\hbar$	—
[05]	The continuous version of Holley's comparison used for the tail lemma	Assumption	§ 04 (the theorem number in the original source is unchecked)	Six nodes	—
[06]	The Poincaré Version of Helffer's Criterion (gap of the chain on the coarse side $\geq \lambda$)	Cited	The Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux; checked in the original text)	Independent of $\hbar$	—
[14]	The Gershgorin form on the ring	Proved here	§ 03 (6)	Independent of $\hbar$	`ring_gershgorin` (assumption: $\kappa \geq 0$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[11]	Nearest-neighbour chain row: the marginal of the kept sites is a nearest-neighbour chain	Proved here	§ 03 (the Markov property)	Independent of $\hbar$	—
[07]	Two-site gap row: $\lambda_c \geq 1.3 \times 10^{-4}$ (ends $[0,4]^2$)	Computer-assisted	§ 03. Interval arithmetic (multiprecision intervals and float64 + outward widths)	$(0, 0.2]$	—
[08]	The Poincaré–Miranda theorem used to cover the boxes of the two-site gap row	Assumption	The statement in the original source is unchecked (bibliographic only)	$(0, 0.2]$	—
[15]	Nesting: small bridge $\rho_f = 0.922$ (Bakry–Émery)	Proved here	§ 03 (lower bound on the curvature; Bakry–Émery is Cited via Theorem 1.2 of Kolesnikov–Milman)	Independent of $\hbar$	—
[09]	Nesting: cross term $\kappa_f = 1.98$ (also used in the crossing row; Brascamp–Lieb)	Cited	Theorem 4.1 of Brascamp–Lieb (checked in a restatement); convex regions by Kolesnikov–Milman	Independent of $\hbar$	—
[16]	Crossing row: arithmetic of $v/\hbar \leq 5888$ and $\lambda_S \geq 2.3 \times 10^{-5}$	Proved here	§ 03	$(0, 0.2]$	`T4_paper_numeric` (assumption: $\lambda_c \geq 1.3 \times 10^{-4}$)
[12]	Variance-decomposition row: decomposition of the variance (3)	Proved here	§ 03	Independent of $\hbar$	—
[10]	Variance-decomposition row: regularity and density of the functions used	Assumption	§ 03 (not checked in the original source)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[17]	The algebra of the rate (7) $\Longrightarrow C_P \leq \varepsilon/t$	Proved here	§ 03	Independent of $\hbar$	`H1_assembly` (the assumptions are the seven named ones of § 03)
[18]	$C \leq 1.6 \times 10^{10}$	Proved here	§ 04 (arithmetic from the values of [01], [07] and [16])	Six nodes	`H1_rate_paper_six_nodes` (numbers)
—	$C \leq 658$	Reading	§ 04 ($\lambda_c = 0.57$ is floating point)	Six nodes	`H1_rate_numeric_six_nodes` (numbers)
—	Margin row (large-end side): neighbours and ends > 2.5	Assumption	§ 07	—	—
—	Margin row (between nodes): the $\hbar$ between the nodes	Assumption	§ 06 (method and cost)	Outside the nodes of $(0.005, 0.2)$	—
—	Margin row (small $\hbar$): $\hbar < 0.005$	Assumption	§ 07	—	—
—	Two-site gap row (large-end side): ends of the two sites > 4	Assumption	§ 07	—	—
—	The segment local Poincaré assumption: from the ring to segments (with end values)	Assumption	§ 07 (the transfer to the chain version of the criterion is a reading)	—	—

The "Computer-assisted" rows [01] and [07] of Table 2 are inequalities at the computed nodes and within the computed ranges. They say nothing about $\hbar$ or end values outside those ranges, and that is what the five "Assumption" rows at the end of Table 2 (with "—" in the number column) take on.

06

The $\hbar$ between the nodes — method and cost

The Six-Node In-Phase Poincaré Proposition is a statement at six points. The in-phase local Poincaré condition is a statement on the continuum of $\hbar$, so the gaps between the points must be filled. This section describes the method for that, the result of trying the method on one subinterval, and an estimate of the cost for the whole range.

(a) β as a third direction of tilting. $\beta = 1/\hbar$ enters the measure $e^{-\beta S}$ only as an exponent of the weight, so we regard an interval $[\hbar_1, \hbar_2]$ of $\hbar$ as an interval $[\beta_{\mathrm{lo}}, \beta_{\mathrm{hi}}]$ of β, and add a third direction (the observable $-S$) to the two directions of tilting in the end values in the Tilted Second-Order Taylor Lemma. β enters the margin (6) in six places (the weight, the width of the tilt, the prefactor, the quadrature step, the tail, the monotonicity band), and for each it is determined at which end of the interval the worst case lies Proved here . A guide for the ratio of one step, $r = \beta_{\mathrm{hi}}/\beta_{\mathrm{lo}}$, is $r \approx 1.17$, from the fluctuation of the action $\mathrm{Var}(\beta S)$ (about half the degrees of freedom of the bridge, near $11/2$ Reading), and $\hbar \in [0.005, 0.2]$ can be covered by 24 subintervals Proved here $(0.2/1.17^{24} \le 0.005)$ Lean `nodes_reach_117` (no assumptions, numbers).

(b) The chord form breaks. With the end values (a, b) fixed, $f(\beta) = \hbar\kappa(a, b; 1/\beta)$ is smooth in β, and a lower bound on the second derivative $f'' \ge -C$ gives the chord upper bound

$$f(\beta) \;\le\; \max\big(K2(\beta_{\mathrm{lo}}), K2(\beta_{\mathrm{hi}})\big) + \frac{(\beta_{\mathrm{hi}} - \beta_{\mathrm{lo}})^2}{8} \sup|f''| \quad (9)$$

Proved here Lean `chord_beta_upper` (assumptions: $f \in C^2$, $f'' \ge -C$, $0 \le C$, $\beta_{\mathrm{lo}} \le \beta_{\mathrm{hi}}$, upper bounds at the ends). But f'' is a connected cumulant, and controlling it by a product of absolute values (Hölder) gives 5×10^3–1.8×10^6 times the true value Reading , so at $r = 1.17$ the remainder is 0.25–0.64, far beyond what is allowed Lean `holder_r117_exceeds` (numbers).

(c) A step-by-step scheme (a scheme that carries $\hbar$ as an interval). Compute the unnormalised integral $J_X(\beta) = \int X e^{-\beta H}\, dx$ ($X \ge 0$ is the weight of the product of end observables) at three points of β, and interpolate by a quadratic. The interpolation error is $|\omega(\tau)|/6 \cdot \sup|J_X'''|$, with $\max|\omega| = 2\Delta^3/(3\sqrt{3})$ Proved here (ω is the cubic that vanishes at the three nodes. Choosing K so that $g(t) = J_X(t) - p_X(t) - K\omega(t)$ vanishes at the three interpolation points and at τ, four points in all, Rolle's theorem used three times gives a zero of g''', and $K = J_X'''(\eta)/6$). The third derivative is controlled by $|y|^3 \le c_3(\lambda_+)e^{\lambda+y} + c_3(\lambda_-)e^{-\lambda-y}$ and the log-convexity of J_X, through the maximum at the ends of the integrals at four shifted values of β Proved here . Splitting the interval of β further into small pieces, on each piece the expectation is a linear-fractional function, and its value lies between the values at the two ends of the piece, so it can be enclosed by the interval hull Proved here Lean `mobius_between` (assumption: the denominator is positive at both ends of the piece), `lebesgue_le` (assumption: $|u| \le 1$), `cube_le_exp_split` (assumption: $\lambda_\pm > 0$). This scheme never uses monotonicity. The remainder enters as a third-order quantity of raw integrals that needs no cancellation, and the Hölder loss of (b) does not appear.

(d) A trial on one subinterval. Table 3 is the result of running (c) in interval arithmetic on $\hbar \in [0.008547, 0.01]$ ($r = 1.17$) and the rectangle of end values $[0, 0.18]^2$ Computer-assisted (the same implementation premises as Table 1).

Computer-assisted Table 3 Trial on one subinterval ($\hbar \in [0.008547, 0.01]$, rectangle $[0, 0.18]^2$, values of the largest box)

Quantity	Value	How to read it
Upper bound at the point $\hbar = 0.01$ (cross-check)	0.18996	Agrees with the computation of Table 1 to five digits
Upper bound on the subinterval (including the remainder)	0.23638	Swelling from the point upper bound $+0.0464$
Share of the remainder	0.00968	Within the uniform allowance 0.0335
Breakdown of the swelling	–	Growth of the node upper bound with β 48%, interpolation and hull of the pieces 31%, remainder 21%

The allowance. What is needed on the subinterval at $\hbar = 0.01$ is "point upper bound + swelling $\leq \hbar\rho/2 = 0.4335$" in every box. If the same amount is allowed to every box, the uniform allowance set by the worst box (upper bound 0.39996) is $0.4335 - 0.39996 \approx 0.0335$, and the swelling 0.0464 of Table 3 exceeds it. On the other hand, the allowance for the boxes in the rectangle of Table 3 is $0.4335 - 0.19 \approx 0.24$, and the swelling falls within it.

What limits it. The limit is not the remainder but the growth of the node upper bound itself with β (the quadrature width widens with β). The upper bound 0.39996 of the worst box is mostly quadrature width, and the floating-point value for the same box is 0.0076 Reading . Recomputing the band containing this box with finer quadrature brings the upper bound of the box down to 0.0751 Computer-assisted , but the maximum over the whole range only comes down to 0.39971. This is because about 32,000 boxes with upper bounds above 0.19 remain at the coarse setting in a frame along the four sides of the square of end values.

Cost. Table 4 is the estimate for the whole range $\hbar \in [0.005, 0.2]$ (computation time on one core; all are estimates, none has been run yet). The cost of one subinterval is about 3.4 times that of a point computation (measured in the trial of Table 3), and the smaller $\hbar$ is, the larger the cost of one point.

Table 4 Estimated cost of filling in between the nodes (the $\hbar\kappa$ side; the $\hbar\rho$ side is needed separately)

Road	Premise	r	Subintervals	Computation time
After recomputing the frame with finer quadrature	That the point upper bounds in the frame come down to 0.318 or below (not yet computed)	1.17	24	About 23–39 hours (+ 30–50 minutes to recompute the frame)
Point upper bounds as they are (a model in which the swelling acts additively)	The swelling of Table 3 does not depend on the box	1.12	33	About 22 hours

Road	Premise	r	Subintervals	Computation time
Point upper bounds as they are (a model in which the swelling acts multiplicatively)	The ratio 1.363 of Table 3 does not depend on the box	1.04	94	About 63 hours

Which of the two models is right cannot be decided from a trial on one rectangle. The road that recomputes the frame first passes at $r = 1.17$ regardless of the model if the upper bound in the frame falls below 0.318, so it also narrows the range of the cost estimate. The $\hbar\rho$ side (a one-dimensional measure, 23 degrees of freedom) is read by the same guide as $r \approx 1.12$ and 33 subintervals Reading .

To be precise, Table 3 is the result for one rectangle of one subinterval, and does not close that subinterval. To close a subinterval, every rectangle in it must be computed by the scheme of (c).

07

The remaining rows — what needs other methods

Table 5 lists the rows that remain in going from the Six-Node In-Phase Poincaré Proposition to the in-phase local Poincaré condition, and to the segment local Poincaré assumption used inside the box. The cost column is filled only where there is an estimate.

Table 5 The remaining rows of the in-phase local Poincaré condition

Row	Content	Current state	What is needed
Remaining item between nodes	The $\hbar \in (0.005, 0.2)$ between the nodes	The scheme of the method of § 06 exists, and has been tried on one rectangle of one subinterval	Recomputing the frame, and computing all subintervals and all rectangles (Table 4). The $\hbar\rho$ side by the same scheme too
Small-ħ remaining item	$\hbar < 0.005$	The computer-assisted scheme cannot be used: with the ratio r fixed, the number of subintervals $\ln(0.005/\hbar)/\ln r$ grows without bound as $\hbar \to 0$, and the cost of each one grows too	An argument uniform in the asymptotics $\hbar \to 0$ (another method)
Large-end-value remaining item	Neighbour and end values > 2.5	The road of covering by monotonicity is closed: FKG gives an order of likelihood ratios, and the gap is not monotone (the Gap Non-Monotonicity Proposition). Also $e(c, t)$ is not bounded in c (it grows at the rate $c^{3^{-11}}$)	A paper argument that the one-dimensional U becomes uniformly convex for large neighbours. The finite range $[2.5, 4]$ is expected to be covered by the same computation if $\hbar \geq 0.05$

Row	Content	Current state	What is needed
Sharp-constant remaining item	Sharpness of the constant	1.6×10^{10} with paper inputs, 658 with numerical inputs. The difference is carried by the weakness of the paper value of the gap of the two sites λ_c	Rebuilding the two-site gap row. The statement of the in-phase local Poincaré condition does not need the size of the constant (only that it does not depend on $\hbar$)
Segment-version remaining item	The segment version (the segment local Poincaré assumption)	The Six-Node In-Phase Poincaré Proposition is a statement on the ring. Inside the box, segments with values at both ends are needed	Transfer to the chain version of the criterion (fewer neighbours than the ring). At present at the level of a reading
Bibliographic remaining item	Bibliography of the cited theorems	Muckenhoupt and FKG have their statements checked in restatements in secondary sources, Ledoux in the original. The theorem number of Holley's continuous version and the original source of the Poincaré–Miranda theorem are unchecked	Obtaining the original sources. If they cannot be obtained, the rows stay "Assumption" ([05] and [08] of Table 2)

The Gap Non-Monotonicity Proposition is a small proposition of a negative type. The variance $p(1-p)$ of the measure $(1-p, p)$ on two points $\{0, 1\}$ is not monotone when p is raised in the order of likelihood ratios $(3/16 < 1/4 > 3/16)$ Proved here Lean `tilt_variance_not_monotone` (no assumptions, numbers). The same holds for the family obtained by tilting the symmetric double well linearly, and FKG alone does not give "the gap is monotone in the neighbour".

08

The dependency graph — the Six-Node In-Phase Poincaré Proposition

We turn the rows of Table 2 into a list of nodes in the same format as 23-05 § 07. The numbers of the premises are always placed smaller than the number of the conclusion.

```
[01] CMP  margin row: margin ≥ 0.067 (neighbours and ends [0,2.5]²)
[02] PRV  Tilted Second-Order Taylor Lemma (any centre)
[03] CIT  Muckenhoupt squeeze B ≤ C_P ≤ 4B
[04] CIT  MTP₂ correlation inequality (monotone e, ħκ ≥ 0)
[05] ASM  continuous version of Holley's comparison (tail lemma)
[06] CIT  Poincaré version of Helffer's criterion (Ledoux Prop. 3.1)
[07] CMP  two-site gap row: λ_c ≥ 1.3×10⁻⁴ (ends [0,4]²)
[08] ASM  Poincaré–Miranda theorem
[09] CIT  Brascamp–Lieb (cross term κ_f, crossing row)
[10] ASM  variance-decomposition row: regularity, density
[11] PRV  nearest-neighbour chain row: kept sites form a nearest-neighb
[12] PRV  variance-decomposition row: decomposition of the variance (3)
[13] PRV  continuum statement of the margin λ ⟸ [01][02][03][04][05]
          Lean T1_margin_six_nodes (numbers), T1_node_margin (assumptio
[14] PRV  coarse-side gap ≥ λ (Gershgorin) ⟸ [06][11][13]
          Lean ring_gershgorin (assumption: κ ≥ 0)
[15] PRV  nesting λ_S ≥ 2.3×10⁻⁵ ⟸ [07][08][09]
[16] PRV  crossing row: v/ħ ≤ 5888 ⟸ [07][09]
          Lean T4_paper_numeric (assumption: λ_c ≥ 1.3×10⁻⁴)
[17] PRV  algebra of the rate C_P ≤ ε/t ⟸ [12][14][15][16]
          Lean H1_assembly (assumptions: hT5, hFine, hLedoux, hT1, 0 < 
[18] PRV  Six-Node In-Phase Poincaré Proposition: C ≤ 1.6×10¹⁰ ⟸ [17]
          Lean H1_rate_paper_six_nodes (numbers)
```

In every row with "⟸" the numbers on the right are smaller than the one on the left, so there is no circle. Among the leaves, "Assumption" covers [05][08][10] and "Computer-assisted" covers [01][07]. $C \leq 658$ from numerical inputs (the row of Table 2 with numerical inputs) puts reading values into the algebra of [17], and does not enter the conclusion [18] of this graph. The five rows at the end of Table 2 are not premises of the Six-Node In-Phase Poincaré Proposition but rows for going from the Six-Node In-Phase Poincaré Proposition to the in-phase local Poincaré condition and the segment local Poincaré assumption, and in the integrated graph (23-12) they come between [18] and the node of the in-phase local Poincaré condition.

The construction of this chapter is one way of handling one condition within the first step, with the diffusive-relaxation-time assumption as the foothold. Other views are treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted At the six nodes, margin ≥ 0.067 for all neighbours and ends in $[0, 2.5]^2$ (Table 1, with the premises of the implementation). Gap of the two sites $\lambda_c \geq 1.3 \times 10^{-4}$	The $\hbar$ between the nodes, $\hbar < 0.005$, $\hbar > 0.2$. Neighbours and ends > 2.5
Proved here The algebra of the two-scale construction and the rate condition (7), the constant 1.6×10^{10} of the Six-Node In-Phase Poincaré Proposition (on the assumption rows of Table 2) Lean (the assumptions are in Table 2)	The in-phase local Poincaré condition itself (a statement uniform on the continuum of $\hbar$). The segment version (the segment local Poincaré assumption)
Cited The Muckenhoupt squeeze, the MTP_2 correlation inequality, the Poincaré Version of Helffer's Criterion, Bakry–Émery, Brascamp–Lieb	Checking the continuous version of Holley's comparison and the Poincaré–Miranda theorem in the original sources ([05] and [08] of Table 2 are Assumption)
Reading $C \leq 658$ with numerical inputs, the autocorrelation time of the phase with a wall, the guide for the step ratio between nodes	Using these as upper bounds
The scheme of the method for filling in between the nodes, the trial on one rectangle of one subinterval (Table 3), and the cost estimate (Table 4)	That even one subinterval has been closed. The times in Table 4 are estimates, not times actually run

Sources and reproduction

Item	Kind	Source / tools
Squeezing the one-dimensional Poincaré constant	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38 (bibliographic data). The statement checked in Theorems 1 and 2 of F. Barthe, C. Roberto, Potential Anal. 29 (2008) 167–193, arXiv:math/0612026.
The MTP_2 correlation inequality, Holley's comparison	Cited, Assumption	Proposition 2.1 of S. Karlin, Y. Rinott, J. Multivariate Anal. 10 (1980) 467–498 (checked in the restatement as Proposition 3.5 of S. Fallat et al., Ann. Statist. 45 (2017), arXiv:1510.01290). R. Holley, CMP 36 (1974) 227–231 (theorem number of the continuous version unchecked).
The Poincaré Version of Helffer's Criterion	Cited	The Poincaré Version of Helffer's Criterion in M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (checked in the text). Original: B. Helffer, Ann. IHP Probab. Statist. 35 (1999) 483–508.
Bakry–Émery, Brascamp–Lieb	Cited	D. Bakry, M. Émery, LNM 1123 (1985); H. J. Brascamp, E. H. Lieb, J. Funct. Anal. 22 (1976) 366–389; convex regions: Theorem 1.2 of A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526.
The Poincaré–Miranda theorem	Assumption	W. Kulpa, Amer. Math. Monthly 104 (1997) 545–550 (bibliographic only; statement unchecked).

Item	Kind	Source / tools
Interval values of Tables 1 and 3	Computer-assisted	Interval arithmetic of the record (float64 + outward widths, the premise of row [01] of Table 2). Cross-checked that floating-point values from an independent route (trapezoidal rule, different steps) all lie below the upper bounds at random boxes and the largest box of each rectangle.
Costs of Table 4	Estimate	Estimated from the measurement in the trial of Table 3 (one subinterval is about 3.4 times a point) and a fit of the $\hbar$-dependence of the computation time per node.
The algebra and numbers of the construction	Proved here, Lean	Theorems `H1_assembly`, `ring_gershgorin`, `T1_margin_six_nodes`, `T1_node_margin`, `T4_paper_numeric`, `H1_rate_paper_six_nodes`, `H1_rate_numeric_six_nodes`, `min_of_two_upper`, `symm_upper`, `tilt_variance_not_monotone`, `chord_beta_upper`, `holder_r117_exceeds`, `nodes_reach_117`, `mobius_between`, `lebesgue_le`, `cube_le_exp_split` (the assumptions are in the text and the rows of Table 2). The list of statements is in 23-13.

Words used in this chapter — phase (phase without kinks), kept site, bridge, nesting, Helffer's criterion, margin, node, subinterval, allowance, swelling, frame, segment version (the segment local Poincaré assumption), computer-assisted, interval arithmetic. Their meanings are in A-02 Glossary.

Next chapter to read: 23-07 The map of the saddle-tube condition — seven routes and the seam of the ring — puts on one sheet the map that splits transport along the family of saddles (the saddle-tube condition) into seven routes, and the seam of the ring (the outer chain closes on the whole window, the inner chain has finite coefficients).

The map of the saddle-tube condition — seven routes and the seam of the ring

> This chapter is this book's own investigation. § 02 is a map of open questions; § 07 is guide material.

What this chapter shows — We put the saddle-tube condition (transport along the family of flat saddles) on one sheet, as a map split into seven routes. For each route we give, in one line, what it tries to show, its current grade (proved here, cited, computer-assisted, reading, assumption), what remains, and the chapter that treats it in detail. The second half deals with the place where a kink and an antikink meet on a periodic path (the seam of the ring). The seam is the plateau at the top itself. The outer chain becomes a computer-assisted inequality on the whole computed window of $\hbar$, and the inner chain remains with a finite coefficient above 1.

Prerequisite chapters — 23-05 § 03 (the statements of the four conditions), 23-05 § 04 The step inside the box (where the saddle-tube condition enters the box), 23-04 § 06 (the plateau), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. This part places the diffusive-relaxation-time assumption (the time over which fluctuations are smoothed out grows as a power of the square of the correlation length, 23-01) at the axis of the study. The diffusive-relaxation-time assumption rephrases the quantities of the problem statement as quantities that can be computed on the lattice and pursued with machine checking, and the saddle-tube condition is one of the four pillars that support its first step. This chapter does not say that the saddle-tube condition is closed. It puts on one sheet which routes have come how far and what remains. Another view is treated in Part 24.

The order of this chapter

1. The statement of the saddle-tube condition and two places — the rows left in the saddle-tube condition by the box split, the tube of the bridge and the seam of the ring
2. The map of the seven routes — grade and what remains, one line each (Table 1)
3. The seam of the ring — the plateau at the top, the outer chain and the inner chain (Table 2)
4. The remaining numbers of the route of the window-constant numerics and their cost — estimates of the number of runs and the time
5. The table of the five grades — the rows of the saddle-tube condition (Table 4)
6. The dependency graph — checking by the numbers that there is no circle
7. A guide to 23-08–23-11

01

The statement of the saddle-tube condition and two places

The statement of the saddle-tube condition is as in 23-05 § 03. We copy it here.

The saddle-tube condition — transport along the family of flat saddles (copied from 23-05)

Measure: the conditional measure of configurations in the tube around the family of kink–antikink pairs (the family of saddles, parametrised by the centre X and separation ℓ of the pair, along which the action is almost flat). **Constant**: the constant of transport of the mean difference along that family from valley to valley, and the Poincaré constant inside the tube. **Uniformity**: uniform in the number of sites of the path (the dimension) and in $\hbar$. For use inside the box it becomes two rows: Poincaré in the tube uniform in tube length $\leq C\xi$ (the saddle-tube Poincaré assumption), and Poincaré of the core inside the tube (the interval that stays near the top) uniform in the core length k (the core Poincaré uniformity assumption).

What the box split leaves in the saddle-tube condition. In the box split of 23-05, the direction along the family of saddles (the kink positions) was taken as the slow variable, so longitudinal transport in that direction is handled by the rows of the gap-inside-the-box condition (the kink-gas gap assumption and the cross-term uniformity assumption) (the note in the step inside the box, 23-05 § 04). What remains in the saddle-tube condition are the **tube and core rows** after the arrangement of kinks is fixed, namely the saddle-tube Poincaré assumption and the core Poincaré uniformity assumption. The map of this chapter is a map of the tools that support these two rows.

Why it is hard. On the core $Q_k = (-x_c, 0)^k$ ($x_c = 0.89$; k sites of boxes, each running from the side of the valley -1 to just short of the top 0), the action is almost flat at the top and the curvature at one site is negative (the summit-curvature shortfall wall of Table 4 in § 05; for details see 23-09 § 06). Convexity tools (Bakry–Émery) do not apply to the whole core, and the core length k grows up to the correlation length, so that the constant is uniform in k must be shown by another means.

Two places — the tube of the bridge and the seam of the ring

The rows of the saddle-tube condition appear in two places where the shape of the path differs.

- **The tube of the bridge.** The tube around a kink, for configurations with one kink on an interval with both ends fixed in the two valleys (a bridge). The core is the interval inside the tube that stays near the top. The saddle-tube Poincaré assumption and the core Poincaré uniformity assumption inside the box have this shape.
- **The seam of the ring.** Configurations on a short periodic path (a ring, length $T \ll \xi$) with one kink–antikink pair, at the place where the separation ℓ of the pair shrinks and the two meet. The configuration just before the pair disappears is the plateau at the top itself (§ 03). The argument for the tube of the bridge uses the fact that the kinks are apart, so the seam needs a different treatment.

In both places, the tool that controls the core constant is of the same type. We attach a coarse mark "valley side or top side" to the sites in the core, and combine the logarithmic Sobolev constant A_W of the marginal of the marks on a range of 17 sites around one site (the site window W) with the contraction ratio q of the chain extending outside the window. The contraction condition $q < 1$ of

this "chain outside the window" is the centre of the route of the chain outside the window and of the outer chain in the route of the seam of the ring, and the size of A_W is the centre of the route of the window-constant numerics, the route of the sector framework and the route of the window log-Sobolev inequality. The site window W is different from the range of $\hbar$ treated computer-assisted (the window of $\hbar$, $[3\cdot10^{-4},3\cdot10^{-3}]$). The window in "the chain outside the window" and "the stages inside the window" is the site window, while the window in "the whole window" and "the lower end of the window" is the window of $\hbar$. Where they could be confused we add "site".

02

The map of the seven routes

Seven routes were set up to close the core rows of the saddle-tube condition. The routes are called by the names of their content. The order of the rows of the table indicates neither order nor weight. "Grade" in Table 1 uses the symbols of the five grades of 23-00 § 01, and when two or more grades are mixed in one route they are listed from the main one.

Table 1 The seven routes (window = $\hbar\in[3\cdot10^{-4},3\cdot10^{-3}]$, three points = $\hbar=3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$)

Route	What it shows	Current grade	Range of $\hbar$	What remains	Chapter
The route of the chain outside the window (contraction of the chain outside the window)	In the core of the bridge, the Stein-kernel ratio $q_{\mathrm{free}}<1$ of the unmarked chain extending outside the site window	Computer-assisted $q_{\mathrm{free}}\le0.998854904$ (float64 + outward widths, presupposing correct IEEE rounding, $\hbar$ as an interval, 25 subintervals)	The whole window	Says nothing outside the window or as $\hbar\to0$	23-08
The route of the window-constant numerics (remaining numbers)	Constants of the site window: minimal weight π_*, rarity Λ, one-site Poincaré C_{LD}^{W}, gap of the mark dynamics gap_W, ratio of the inner chain of the window of the bridge	Reading (floating point at three points) Assumption (the paper reduction to a number of computations is not placed in the book)	Three points	The number of interval-arithmetic runs and the time (§ 04). Decided not to run them because of the floor of the product form	This chapter, § 04
The route of the sector framework	The five steps of the sector sum (sector sum, Gibbs decomposition, componentwise localisation, volume locality, sector composition) and the closing condition $b(A_W c_H+D_W+\ln2)<\kappa\ell_{\min}$	Proved here (the proof of the five steps is placed in 23-09). Inputs are Assumption , Reading	Steps independent of $\hbar$ / inputs at three points	Inputs (the marginal log-Sobolev input for the window, the bridge volume-locality input, E_m, the defect rows, regularity)	23-09 § 02

Route	What it shows	Current grade	Range of $\hbar$	What remains	Chapter
The route of the sector input (small assumptions)	The small assumptions of the sector input (eleven rows: the bridge volume-locality input, the core log-Sobolev uniformity assumption, the longitudinal term and regularity, plus seven rows whose content has not yet been written in one line)	Assumption (list only; not started)	—	Each row	This chapter, § 05
The route of the window log-Sobolev inequality (the one remaining proposition)	Window Log-Sobolev Threshold Proposition: the logarithmic Sobolev constant of the marginal on the site window lies within the needed value $A_{\max}(\hbar)$	Assumption . None of the three strands (componentwise, scalar, Feynman–Kac) is closed	Window (the Scalar-Strand Proposition is on $[3\cdot 10^{-4}, 10^{-3}]$)	The remaining step of each of the three strands	23-09 § 05
The route of the seam of the ring	The core on the ring: contraction of the outer chain and the coefficient of the inner chain	Outer: Computer-assisted (pin ≤ 0; the seam limit assumption and the random-end-value assumption are Assumption) / inner: Reading $q_{\text{in}} \approx 1.24$–$1.28$	Window	The seam limit assumption, the random-end-value assumption, the ratio conditioned on the plateau, the boundary stage, putting the inner chain into intervals	This chapter, § 03
The route of the range of $\hbar$	Outside the window and $\hbar \to 0$	Outside the scope of this part	—	Paper asymptotics	23-12, 23-15

Three notes on how to read Table 1.

(a) Only two rows are computer-assisted. They are the route of the chain outside the window and the outer chain in the route of the seam of the ring. Both are closed in the sense that "within the computed window of $\hbar$, the inequality holds given the table of outputs of interval arithmetic", and say nothing outside the window or as $\hbar \to 0$ (23-08).

(b) Even if the route of the chain outside the window closes, the saddle-tube condition does not close. The ratio q_{free} of the route of the chain outside the window enters only one factor $S_W = 1 + 2\big(16 + 1/(1-q_{\text{free}})\big)$ of the upper bound on the logarithmic Sobolev constant A_W of the site window ([12] of the dependency graph in § 06). In the current construction of the upper bound, the upper bound on A_W lies outside the needed value (by 4–5 orders of magnitude in the product form of the factors for the 17-site window, and by 29–74 times even in the form for the 3-site window with the needed side raised, 23-09 § 03), and the one remaining proposition (the route of the window log-Sobolev inequality) is being attacked by other strands.

(c) The route of the range of ħ was decided to be out of scope. The double-well relaxation-time power statement is a statement as $\hbar \to 0$, so in the end the four double-well conditions must be stated in the asymptotics $\hbar \to 0$, and computer assistance on a fixed window alone does not reach that. Outside the window, paper asymptotics are needed. In this part we first check the grades of the rows inside the window, and how to fit them together is described in 23-12 and 23-15.

03

The seam of the ring — the plateau at the top, the outer chain and the inner chain

A ring is a periodic path of length $T \ll \xi$ with one kink–antikink pair, and we write ℓ for the length of the arc on the negative side. The reduced plateau model of 23-05 § 02 (c) kept only this ℓ. The seam is where ℓ becomes small and the pair disappears.

The seam is the plateau at the top itself

The continuum minimal action $S(\ell)$ with the length of the negative arc fixed at ℓ tends, as $\ell \to 0^+$, not to 0 but to S_0 (the path touches the top and returns). For $\ell \le \pi\sqrt{2}$ the minimising configuration has no kinks: it is a plateau lying flat on the top 0, with action $S_0 + \ell/8$. For $\ell > \pi\sqrt{2}$ a dip appears and the action tends to $2S_0$ ($2S_0 - S \approx 8e^{-\ell}$) Assumption (a continuum variational argument and a cross-check with floating-point minima on the lattice; the proof is not placed in the book). On the lattice ($\varepsilon = \frac{1}{2}$) too, the minimum with m sites on the negative side lies on $S_{\mathrm{lat}}(1) + (m-1)/16$ for $m \le 9$ Reading (floating-point minima, difference $\le 5 \times 10^{-11}$). **The configuration at the seam is the core itself (the interval that stays near the top)**, and the same difficulty as for the core of the tube of the bridge appears here.

The ℓ direction is easy. One-dimensional transport in ℓ alone can be controlled, seam included. On each of the band, the collar and the plateau of the seam, the Muckenhoupt quantity, which squeezes the one-dimensional Poincaré constant within a factor 4, is at most $64S_0\hbar$, $L_0^2/(8D)$ and T^2/D, and the algebra passes machine checking Assumption (paper estimates on the one-dimensional model; the reduction to the model is itself an assumption). The hard part is the side of the coordinates: at the seam, the number of sign changes does not become a slow variable. So we fold by sectors that do not depend on the sign, namely runs of sites with $|z| < \theta$ ($\theta = \frac{1}{2}$) (**lumps**). With the rest fixed, the action of a lump of k sites is convex for $k \le 7$, from $V'' \ge -\frac{1}{2}$ Proved here (the smallest eigenvalue $8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4}$ of the tridiagonal matrix $\mathrm{tridiag}(-2, 4+w, -2)$, $w \ge -\frac{1}{4}$, is positive for $k \le 7$).

The outer chain and the inner chain

On a site window W of the ring (17 sites) we put marks: one lump (length $k \le 7$, possibly absent), the other sites of the window on the valley side, and outside the window the side of fixed sign. The construction of the logarithmic Sobolev constant of the window involves contraction ratios of two kinds of chain.

- **The outer chain**: the chain extending outside the site window. It moves in the region of fixed sign $[0, \infty)$, and mapped to $(-\infty, 0]$ by the reflection $x \mapsto -x$ it becomes a chain of the same type as the outer chain of the core of the bridge. The action is invariant under the reflection Proved here (V is even) Lean `chain_action_reflect` (no assumptions; the image of the measure and the reversal of the Stein kernel are outside Lean), but it is not the same chain as the chain of the bridge. The box of the bridge has a lower wall $-x_c$, and the outer chain of the ring has none.
- **The inner chain**: the stages inside the site window. From stage to stage the box alternates between "valley side" and "lump", and the lump box $(-\frac{1}{2}, \frac{1}{2})$ straddles 0.

The difference between the two chains can be read from the corner at the wall. The classical corner ratio at a wall at position b is $r(b) = \Delta/(\Delta - \varepsilon^2 V'(b))$. It is exactly 1 at a wall with $V'(b) = 0$ (the wall 0 of the outer chain), and exceeds 1 at a wall where the force pushes out of the box Proved here (the algebra of the sign of the ratio) Lean `corner_eq_one` (assumption: $\Delta \neq 0$), `corner_gt_one` (assumption: $0 < c < \Delta$). The formula for the ratio itself comes from a computation of the envelope, and its regularity is Assumption. At the upper wall $+\frac{1}{2}$ of the lump, $\varepsilon^2 V'(\frac{1}{2}) = -3/64$, so the force points towards the wall and the corner exceeds 1 (the classical value for an isolated wall is 1.2857).

Table 2 The outside and inside of the seam of the ring (triples in the order $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$)

Quantity	Value	Grade	Conditions
Outer chain (unconditioned form) q_{out}	≤ 0.998888530 at every $\hbar$ of the window (the maximum is at the lower end of the window; on the upper half $[10^{-3}, 3 \cdot 10^{-3}]$ it is ≤ 0.996606914)	Computer-assisted	All pins $p \le 0$, evaluation points $s \le 0$, end $z_0 = -1$. The seam limit assumption, the random-end-value assumption and regularity are Assumption. float64 + outward widths, presupposing correct IEEE rounding
Deep side of the outer chain (pin $p < -3$)	≤ 0.592898559	Computer-assisted	As above (uses the upper bound on the tail variance ≤ 0.357608022)
Boundary stage (pin $\in (0, \frac{1}{2})$)	≈ 1.004–1.045 (at the four points $\hbar = 3 \cdot 10^{-3}/10^{-3}/(3/9100)/3 \cdot 10^{-4}$: $1.0043/1.0282/1.0436/1.0446$)	Reading	A finite stage that appears only in configurations where the lump touches the end of the window. The outer-chain rows above do not cover this pin
Inner chain q_{in}	$1.2707/1.2765/1.2421$ (in the order of the triple; 1.2449 at the fourth point $\hbar = 3/9100 \approx 3.3 \cdot 10^{-4}$)	Reading	Floating-point transfer; it moves upwards as the grid is refined. The limit is set by the stage inside the lump with the pin at the edge of the far wall of the lump
Finite coefficient $\bar{q} = \max(q_{\text{in}}, 1)$	$\bar{q} \approx 1.3$, $\bar{q}^{16} \approx 27$–67 (applied to each of S_W and T_W; $\bar{q}^{32}$ for the product A_W)	Reading	$\bar{q} = 1$ does not hold
$S_W = 1 + 2\bar{q}^{16}(16 + 1/(1 - q_{\text{out}}))$	≤ 122745 (≤ 1833 if $\bar{q} = 1$)	Proved here (arithmetic)	Input $q_{\text{in}} \le 13/10$ is Reading, q_{out} is Computer-assisted. Lean `sw_inner_le` (assumptions: $q_{\text{in}} \le 13/10$, $q_{\text{out}} \le 0.998888530$)

In one line, Table 2 says: **the outside is closed (inside the window, conditional on pin ≤ 0), and the inside has a finite coefficient**. "Closed" here means that within the computed window of $\hbar$ the contraction of the outer chain became a computer-assisted inequality, not that the saddle-tube condition is closed on the ring. The result for the outer chain was first closed for the chain conditioned on the box (upper bound 0.998888525) and then extended to the unconditioned chain (the tail term put into the remainder of the interval arithmetic, deep pins covered by a closed-form upper bound). The extension removed the need for the four assumption rows that returned the conditioned chain to the original measure.

Why the inner chain exceeds 1, and the cost of putting it into intervals. What limits the inner chain is the stage with the pin at the edge of the far wall of the lump box, which is exactly the corner pushing outwards described above (close to the classical value 1.2857). The interval

machinery of the outer chain is built on the premises that the right wall is at 0 with $V'(0) = 0$, that there is one box, and that the pins and the grid coincide, so it cannot be used for the inner chain just by swapping the inputs. The estimated cost of putting the inner chain into interval arithmetic is 1–2 paper steps + 1–2 scripts + about 50 runs, about 42 hours (one core), and what it gives is only an upper bound on the finite coefficient $\bar{q} \approx 1.3$. This coefficient enters A_W as a factor $\bar{q}^{32} \approx 10^{3.6}$, so it is not a road to meeting the closing condition of the sector framework (23-09 § 02).

The rows that remain at the seam of the ring. The seam limit assumption (that the limit of stacking stages without end falls inside the enclosure of the fixed point at each $\hbar$), the random-end-value assumption (the case where the value at the end of the chain is given not by a point but by a distribution), the ratio conditioned on the plateau (the ratio conditioned on the length of the plateau), the boundary stage, and putting the inner chain into intervals. All remain Assumption or Reading .

04

The route of the window-constant numerics — the remaining numbers and their cost

In the current construction (the Product Bound for the Window Log-Sobolev Constant), the upper bound on the logarithmic Sobolev constant of the site window has the form of a product of factors $A = c_W \cdot C_{\mathrm{LD}}^{W} \cdot S_W \cdot T_W$. c_W is a constant of the marginal measure of the marks, built from the minimal weight π_* and the gap of the mark dynamics gap_W, and S_W and T_W are built from the contraction ratios of the chains. The route of the window-constant numerics is the paper reduction and estimate of the number of runs for moving the values of the factors from floating point at three points to interval arithmetic.

Table 3 The numbers of the route of the window-constant numerics, and the number of runs to move them to interval arithmetic (the numbers of runs and the times are estimates; not a single run has been made yet)

Quantity	Paper reduction (not placed in the book)	Estimated runs	Current value
Minimal weight π_*	The marginal measure of the marks is MTP_2, and the sequence giving the minimal weight is pushed to the extreme sequence (all on the valley side) (the preservation theorem of Karlin–Rinott Cited , statement checked in a restatement in a secondary source)	2–3 runs	Reading all on the valley side is the minimum at the three points
Rarity Λ	The sup over boundary values of a monotone function is attained by a single chain at the extreme (MTP_2 association)	1–2 runs	Reading
One-site Poincaré C_{LD}^{W}	The Muckenhoupt criterion quantity Cited (checked in a restatement in a secondary source), cell by cell of boundary values	About 20 runs	Reading of order ≤ 0.66
Gap gap_W	Since one mark screens the interaction of the interfaces, the congestion of the flow is controlled by the chain of the extreme sequence	4 runs	Reading 0.959–1.000
Ratio of the inner chain of the window of the bridge	The Stein-kernel ratio is not monotone in the sequence of marks (it fails for around 300 of the 665 pairs enumerated), so it is split by type of sequence and a band on the far side	About 13 runs	Reading < 1 at the three points

The estimate totals about 42 runs and about 21 hours (one core). **We decided not to run these 42.** The reason is the floor of the product form itself. From $S_W \geq 1 + 2/(1 - q_{\mathrm{free}})$, $T_W \geq |W|$ and c_W

$\geq \ln\bigl(1/\nu(\sigma_i = \text{valley side})\bigr) \approx E_U/\hbar$, even with the true values of the factors, $A \gtrsim 1.4 \times 10^4$–$1.4 \times 10^6$ Reading (estimate), which lies outside the needed value of the sector framework (23-09 § 03, tens to hundreds) by tens to tens of thousands of times. Even if the factors are confirmed in interval arithmetic, this form as it stands does not reach the closing condition. The paper reduction of the route of the window-constant numerics (screening, the extreme sequence) is kept as a tool that enters unchanged into upper bounds not of product form (the strands of the route of the window log-Sobolev inequality).

05

The table of the five grades — the rows of the saddle-tube condition

We put the rows of Tables 1–3 into one table in the format of the five grades. Each row has one grade, and the column of machine checking is outside the grades. The numbers of the rows are the numbers of the nodes of the dependency graph in § 06, and "—" marks a row not in the graph.

Table 4 The core rows of the saddle-tube condition ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Premises of interval arithmetic: correct rounding of IEEE 754 arithmetic, upper bounds on summation errors, upper bounds on the errors of our own exponential function	Assumption	23-08 § 03	—	—
[02]	The seam limit assumption, the random-end-value assumption, regularity	Assumption	§ 03	—	—
[03]	Window Log-Sobolev Threshold Proposition: $A_W < A_{\max}(\hbar)$	Assumption	The route of the window log-Sobolev inequality, 23-09 § 03	—	—
[04]	All the small assumptions of the route of the sector input	Assumption	Table 1	—	—
[05]	Product Bound for the Window Log-Sobolev Constant: $A_W \leq c_W C_{\mathrm{LD}}^{W} S_W \tau_W$ (product form)	Assumption	§ 04 (the paper construction is not placed in the book)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[06]	The constants of the route of the window-constant numerics, π_*, Λ, C_{ID}^{W}, gap_W, and the ratio of the inner chain of the window of the bridge	Reading	Table 3 (floating point at three points)	Three points	—
[07]	Inner chain of the ring $q_{\mathrm{in}} \approx 1.24$–$1.28$	Reading	Table 2 (floating-point transfer; not an upper bound)	Four points	—
[08]	Boundary stage $q \approx 1.004$–1.045	Reading	Table 2	Four points	—
[09]	Row of the contraction of the chain outside the window: $q_{\mathrm{free}} \le 0.998854904$	Computer-assisted	23-08 § 05 (the premise is the row of interval-arithmetic premises)	Window	`win_qfree_window_le_of_interval_table` (assumption: the table of outputs of interval arithmetic on 25 subintervals)
[10]	Outer chain of the ring $q_{\mathrm{out}} \le 0.998888530$ (pin ≤ 0)	Computer-assisted	23-08 § 06 (the premises are the row of interval-arithmetic premises and the row of the seam limit assumption)	Window	`unc_window_lt_one_of_table` (assumption: the table of outputs of interval arithmetic on 25 subintervals)
[11]	The five steps of the sector framework	Proved here	23-09 § 02 (the inputs are the row of small assumptions and the row of the seam limit assumption)	Independent of $\hbar$	`cell_sum_sup`, `comp_close` (the assumptions are in the table of 23-09)
[12]	$S_W \le 1 + 2(16 + 1/(1-q)) \approx 1779.6$ of the bridge	Proved here	Sum of a geometric series (23-08 § 02)	Window	—
[13]	$S_W \le 122745$ of the ring	Proved here	Table 2 (the input q_{in} is a reading)	Window	`sw_inner_le` (assumptions: $q_{\mathrm{in}} \le 13/10$, $q_{\mathrm{out}} \le 0.998888530$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[14]	Values of A_W in product form $\approx 2.5 \times 10^6 / 5.4 \times 10^6 / 4.7 \times 10^7$	Reading	§ 04 (one point of c_W is an estimate including extrapolation)	Three points	—
[15]	A positive lower bound on the gap of the core, uniform in k (in the form of sector composition)	Proved here	23-09 § 02 (on the window log-Sobolev threshold row and the sector-framework row)	—	`comp_close` (assumptions: the three inequalities of Barta, componentwise localisation and Bochner)
—	The curvature at one site at the top falls $\frac{1}{4}$ short of 4 (the common wall)	Proved here	23-09 § 06	Independent of $\hbar$	—

The two "Computer-assisted" rows of Table 4 ([09] and [10]) are both computer-assisted on the premise, the row of interval-arithmetic premises, and what Lean checked is "the construction with the table of outputs of interval arithmetic placed as an assumption" (23-08 § 08). The "Proved here" core lower-bound row is algebra on the assumption, the window log-Sobolev threshold row, and does not mean that a lower bound on the gap of the core has been shown.

06

The dependency graph — checking by the numbers that there is no circle

We turn the rows of Table 4 into numbered nodes. **The numbers of the premises to the right of "⟸" are always placed smaller than the number of the conclusion on the left** (the format common to this part, 23-05 § 07).

```
[01] Assumption        row of interval-arithmetic premises (correct IEEE roundi
[02] Assumption        seam limit assumption, random-end-value assumption, regu
[03] Assumption        Window Log-Sobolev Threshold Proposition: A_W < A_max(ħ)
[04] Assumption        small assumptions of the sector input
[05] Assumption        Product Bound for the Window Log-Sobolev Constant (A_W ≤
[06] Reading           constants of the route of the window-constant numerics
[07] Reading           inner chain of the ring q_in ≈ 1.24–1.28
[08] Reading           boundary stage q ≈ 1.004–1.045
[09] Computer-assisted row of the contraction of the chain outside the window:
                        Lean win_qfree_window_le_of_interval_table (assumption:
[10] Computer-assisted outer chain of the ring q_out ≤ 0.998888530 (pin ≤ 0) ⟸
                        Lean unc_window_lt_one_of_table (assumption: table of i
[11] Proved here       the five steps of the sector framework ⟸ [02][04]
                        Lean cell_sum_sup, comp_close (assumptions in the table
[12] Proved here       S_W of the bridge ≤ 1779.6 ⟸ [09]
[13] Proved here       S_W of the ring ≤ 122745 ⟸ [07][08][10]
                        Lean sw_inner_le (assumptions: q_in ≤ 13/10, q_out ≤ 0.
[14] Reading           A_W in product form ≈ 2.5·10⁶–4.7·10⁷ ⟸ [05][06][12]
[15] Proved here       positive lower bound on the core gap uniform in k ⟸ [0
                        Lean comp_close (assumptions: the three inequalities of
[16] Proved here       the form of the core Poincaré uniformity assumption of 2
```

How to check. [09] has only [01] on the right, [10] only [01][02], [11] only [02][04], [12] only [09], [13] only [07][08][10], [14] only [05][06][12], [15] only [03][11], and [16] only [02][15]. In every row the numbers on the right are smaller than the one on the left, so there is no circle.

What to read from this graph. (1) The road to the core row [16] passes through [15], and [15] depends on the assumption row [03] (the one remaining proposition of the route of the window log-Sobolev inequality). (2) The computer-assisted [09][10] enter only the product-form value [14] through [12][13], and [14] does not satisfy [03] (it lies outside the needed value). That is, **in the current graph the two computer-assisted rows are not on the road to [16]**. When an upper bound not of product form satisfies [03], [09][10] become part of the grounds of [03] if the ratio q enters that bound. (3) None of the reading rows [06][07][08] is used in the proof of an upper bound.

We also check that the statements of the assumption rows do not contain the quantities of the conclusion. The quantities of the conclusion are the gap of the measure of the core ([15]) and the constant of the core Poincaré uniformity assumption ([16]). [03] is the logarithmic Sobolev constant of the marginal of the marks on the site window, [04] constants of the sector input, [05] the construction of the product-form upper bound, and [02] limits and regularity. None of them contains the gap of the core itself.

07

A guide to 23-08–23-11

```
Chapters to read after this one
```

1. 23-08 What computer-assisted means — using the route of the chain outside the window and the outer chain in the route of the seam of the ring as examples, how interval arithmetic turns floating-point values into inequalities, its premises, and what it does not say
2. 23-09 The sector framework and the three strands of the window log-Sobolev route — the proof of the five steps of the route of the sector framework, the derivation of the needed value $A_{\max}$, the three strands and the common wall
3. 23-10 The list of detour plans — the detour plans tried in the route of the window log-Sobolev inequality, and the types of breakage that appeared repeatedly
4. 23-11 Box reduction and the gap inside the box — the constant of the core — the construction that controls the core constant in three steps from the side of the box

This part places the relaxation-time assumption at the axis of the study and treats the saddle-tube condition as one of the pillars of its first step. Another view is treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted $q_{\text{free}} \le 0.998854904$ at every $\hbar$ of the window $[3\cdot 10^{-4}, 3\cdot 10^{-3}]$, and $q_{\text{out}} \le 0.998888530$ for the outer chain of the ring (pin ≤ 0). On the premises of interval arithmetic	Contraction outside the window and as $\hbar \to 0$. The boundary stage (pin $\in (0, \frac{1}{2})$)
Proved here The five steps of the sector framework, and the lower bound on the core gap on the assumption row (the Window Log-Sobolev Threshold Proposition) (23-09). The arithmetic of S_W	The Window Log-Sobolev Threshold Proposition itself. Hence the core Poincaré uniformity assumption, the saddle-tube Poincaré assumption and the saddle-tube condition
Reading The inner chain of the ring $q_{\text{in}} \approx 1.24$–$1.28$, the constants of the route of the window-constant numerics	Upper bounds on these. Interval machinery for the inner chain
That the dependency graph has no circle (the rule of the numbers)	A road by which the computer-assisted rows reach the core rows of the saddle-tube condition (none in the current graph)

Sources and reproduction

Item	Kind	Source / tools
Contraction of the chain outside the window, the outer chain of the ring	Computer-assisted, Lean	Interval arithmetic in Python float64 (outward widths for each operation, our own exponential function). Procedure and cross-checks in the table of sources of 23-08. Lean: `win_qfree_window_le_of_interval_table`, `unc_window_lt_one_of_table`, `vt_window_of_table` (assumption: the table of outputs of interval arithmetic).
Preservation of MTP_2	Cited	S. Karlin, Y. Rinott, J. Multivariate Anal. 10 (1980) 467–498; statement in S. Fallat et al., Ann. Statist. 45 (2017) 1152–1184, arXiv:1510.01290 (checked in Proposition 3.5).
The Muckenhoupt criterion quantity	Cited	B. Muckenhoupt, Studia Math. 44 (1972) 31–38; statement in F. Barthe, C. Roberto, Potential Anal. 29 (2008) 167–193 (checked in Theorems 1 and 2).

Item	Kind	Source / tools
The plateau of the seam of the ring, convexity of lumps, the corner at the wall	Assumption, Proved here, Reading	Continuum variational argument and floating-point minima (Assumption, Reading), smallest eigenvalue of the tridiagonal matrix (Proved here), sign of the corner ratio (Proved here, `corner_eq_one`, `corner_gt_one`).
The inner chain of the ring, the boundary stage	Reading	Floating-point transfer (grid $h = 2^{-10}$, limiting pairs rechecked at 2^{-11}). The regular Gaussian chain reproduces $2/3$ to 3 digits (positive control), and a chain with the wall moved in the direction that pushes outwards gives > 1.2 (negative control).
Reflection	Proved here, Lean	`chain_action_reflect` (symmetry of each term of the action).

Words used in this chapter — route, core, site window, window (of $\hbar$), chain outside the window, Stein-kernel ratio, ring, seam, lump, outer chain, inner chain, boundary stage, corner at the wall, computer-assisted, floor of the product form. Their meanings are in A-02 Glossary.

Next chapter to read: 23-08 What computer-assisted means — the conditions on the chain inside the window — explains the mechanism and premises by which interval arithmetic turns floating-point computations into proofs of inequalities, using the contraction condition of the chain outside the window as an example.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time computer-assisted, interval arithmetic, window · Kiichi, Shiori and Rin

What computer-assisted means — the conditions on the chain inside the window

> This chapter is this book's own investigation. § 01 and § 03 are established knowledge.

What this chapter shows — We make it possible to read what results labelled "computer-assisted" (Computer-assisted) in this part guarantee and what they do not. The example is the contraction condition $q < 1$ for the chain extending outside the core. Instead of writing floating-point values as they are, we hold every number as an interval that contains the true value, make $\hbar$ itself an interval, and check that the inequality holds at every point of the computed window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ of $\hbar$. We describe this mechanism and its premises (the rounding conventions). At the end we look at an example where computer assistance returns "no", and confirm that it says nothing about what lies outside the window.

Prerequisite chapters — 23-07 § 02 (the route of the chain outside the window and the route of the seam of the ring), 23-00 § 01 (the labels of the five grades), 23-04 § 01 (the path measure).

Said first — The level is [Y2]–[Y3]. § 01– § 03 are written to be readable at [Y2], and § 04 onwards lean towards [Y3]. The inequalities treated in this chapter are one input to the construction of the first step with the diffusive-relaxation-time assumption at its axis, and by themselves they do not close the saddle-tube condition (23-07 § 06).

The order of this chapter

1. Floating-point values, and inequalities written with intervals — a small example that can be checked by hand
2. What the chain condition $q < 1$ guarantees — the Stein-kernel ratio, and the geometric series of influence
3. The premises of interval arithmetic — the rounding conventions and our own exponential function
4. Making $\hbar$ an interval — why it breaks when done naively, and three lemmas
5. Example 1: the outer chain of the core of the bridge — 25 subintervals (Table 1)

6. Example 2: the outer chain of the ring — from the conditioned chain to the unconditioned chain
7. Cross-checks, controls, pre-registration — reproduction by another implementation, and positive and negative controls
8. What Lean checked — the construction with the table of outputs of interval arithmetic placed as an assumption
9. Examples where computer assistance returns "no" — the inner chain, and a subinterval with first-order tilting
10. What computer assistance does not say, and the cost — how to read the label (box)

01

Floating-point values, and inequalities written with intervals

This section and § 03 summarise existing knowledge (floating point and interval arithmetic in general). The line between theorem and observation is introduced in 5-07 § 09.

Real numbers on a computer are floating-point numbers, and the result of most operations is the true value rounded to the nearest representable number. For example, $0.1 + 0.2$ becomes 0.30000000000000004 in double precision. Each rounding is small, so in many computations it can be ignored. But when claiming an inequality "$q < 1$" about a value close to 1, such as $q \approx 0.9989$, one must show that accumulated rounding does not flip the direction of the inequality.

Interval arithmetic is computation in which every number is held as "an interval that is sure to contain the true value". Intervals are built from intervals: addition is $[a, b] + [c, d] = [a + c, b + d]$, multiplication takes the minimum and maximum of the four products of the ends, and so on. When the ends are computed, rounding is taken **outwards** (the lower end downwards, the upper end upwards). Then the final interval $[q^-, q^+]$ is sure to contain the true q, and if $q^+ < 1$, then $q < 1$ follows as an inequality. In this part, inequalities obtained in this way carry the label Computer-assisted . Floating-point values written as they are carry Reading .

An example that can be checked by hand. The true range of $g(x) = x - x^2$ on $x \in [0, 1]$ is $[0, \frac{1}{4}]$. A naive evaluation in interval arithmetic gives, from $x \in [0, 1]$ and $x^2 \in [0, 1]$, $x - x^2 \in [0 - 1,\ 1 - 0] = [-1, 1]$. This contains the true value, but the width has grown eightfold. The cause is that the intervals forget the **dependence** that x and x^2 come from the same x. There are two ways to fix this.

- **Split finely.** Splitting into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ gives $[0 - \frac{1}{4}, \frac{1}{2} - 0] = [-\frac{1}{4}, \frac{1}{2}]$ and $[\frac{1}{2} - 1, 1 - \frac{1}{4}] = [-\frac{1}{2}, \frac{3}{4}]$, together $[-\frac{1}{2}, \frac{3}{4}]$, and the width shrinks to 1.25.
- **Expand about the centre.** With $g(x) = g(c) + g'(\xi)(x - c)$, $c = \frac{1}{2}$ and $g'(\xi) = 1 - 2\xi \in [-1, 1]$, we get $\frac{1}{4} + [-1, 1] \cdot [-\frac{1}{2}, \frac{1}{2}] = [-\frac{1}{4}, \frac{3}{4}]$, and the width is 1. This takes the first-order part of the dependence outside the interval.

What we meet in § 04 when making $\hbar$ an interval is a large-scale version of these two fixes. We split the window of $\hbar$ into 25 subintervals (split finely), and within each subinterval expand the expectations in $\hbar$ to take the first- and second-order movement outside the interval (expand about the centre).

02

What the chain condition $q < 1$ guarantees

Seen from a site in the core, consider the chain $x_{i-1}, x_{i-2}, \dots$ extending to one side. The term of the action between neighbours is $(x_{i-1} - x_i)^2/(2\varepsilon)$, and the measure is $e^{-S/\hbar}$. Fixing $x_i = y$ and integrating out the far side, the conditional law of x_{i-1} takes the form

$$\rho_y(x) \;\propto\; \exp\Big(-\frac{1}{\hbar}\Big[\frac{(x-y)^2}{2\varepsilon} + U(x)\Big]\Big) \qquad (x \text{ in the box } [L, R]\ ) \quad (1)$$

(U is the effective potential with the far side integrated out, and does not depend on y). Let $m(y)$ be the mean of ρ_y, and let the Stein kernel be

$$\tau_y(x) = \frac{1}{\rho_y(x)} \int_L^x \big(m(y) - t\big)\rho_y(t)\,dt \quad \geq 0 \quad (2)$$

. Integration by parts gives $\mathrm{Cov}_y(\varphi(X), X) = \mathbb{E}_y[\varphi'(X)\tau_y(X)]$ (the boundary term vanishes since $\int_L^R (m - t)\rho_y\,dt = 0$). On the other hand, the logarithmic derivative of (1) in y is $(x - m(y))/(\varepsilon\hbar)$, so

$$\frac{d}{dy}\,\mathbb{E}_y[\varphi(X)] = \frac{\mathrm{Cov}_y(\varphi(X), X)}{\varepsilon\hbar} = \frac{\mathbb{E}_y[\varphi'(X)\tau_y(X)]}{\varepsilon\hbar}, \qquad \Big|\frac{d}{dy}\,\mathbb{E}_y[\varphi(X)]\Big| \;\leq\; q\,\sup|\varphi'|, \quad q :=$$

Proved here (differentiation under the integral sign and integration by parts; the box is bounded and the density is smooth). (3) reads: "each step back, the slope of a function (its Lipschitz constant) becomes at most q times as large". Going back d steps gives at most q^d times. **If $q < 1$, the influence of fluctuations at distant sites on nearby sites decreases as a geometric series.** This is the content of "the chain condition", and we call q the Stein-kernel ratio.

A check. If ρ_y is a normal distribution with variance σ^2, computing (2) gives $\tau_y \equiv \sigma^2$, and $q = \sigma^2/(\varepsilon\hbar)$. For the harmonic chain (U a quadratic), one can check by hand that this ratio converges stage by stage to a fixed value.

What it is used for. In the construction of the logarithmic Sobolev constant of the site window (the product form of 23-07 § 04), the quantity SW that adds up the influence of all stages outside the window enters as a factor. Adding the contribution of the stages inside the window and the contribution of the stages outside the window $\sum_{d\ge 0} q^d = 1/(1-q)$ on the two sides, left and right, gives

$$SW = 1 + 2\Big(16 + \frac{1}{1-q}\Big) \quad (4)$$

(the form with the ratio of the stages inside the window set to 1. The paper construction is not placed in the book, so the step that uses (4) sits on an Assumption row). If $q < 1$, then SW is finite, and if $q \le 0.998854904$, then $1/(1-q) \le 873.3$ and $SW \le 1779.6$ Proved here (arithmetic).

Why q is close to 1. The right wall of the box of the outer chain of the core is the top 0, and there $V'(0) = 0$. At the corner of this wall the classical ratio is exactly 1 (the corner at the wall in 23-07 § 03), and the departure from 1 is set by the width $\eta \approx 0.5\hbar$ of the thin layer near the wall, of size $1 - q \approx \eta \ln(1/\eta)$ Reading (an expansion of the classical map, cross-checked with transfer-matrix values). The smaller $\hbar$ is, the closer q comes to 1. **This also shows that checking $q < 1$ inside a fixed window of $\hbar$ leaves the behaviour as $\hbar \to 0$ as a separate question.**

03 The premises of interval arithmetic

The computer assistance of this part is interval arithmetic built on Python's double-precision floating point (float64). We collect the premises in one place.

- **Rounding conventions.** We assume that each arithmetic operation returns, as the IEEE 754 standard specifies, the floating-point number nearest the true value (correct rounding). The result of each operation is made into an interval by moving it outwards by one to the neighbouring floating-point number (the lower end one down, the upper end one up). The error of correct rounding is at most half a unit, so moving one unit outwards contains the true value. It is not a scheme that switches the rounding direction for each operation. Fused multiply–add (FMA) is not used.
- **Summation error.** Long sums are widened outwards by a known upper bound on summation error.

- **The exponential function.** The error of the standard mathematical functions is not guaranteed, so we use our own implementation of the exponential function and add an upper bound on its error to the interval. The arithmetic of the constants of the upper bound is checked in Lean.
- **Integrals.** For each grid cell we build upper and lower envelopes of the integrand (with a concavity correction) and squeeze the integral outwards. Truncation of the integral (cells with weight below e^{-40} of the maximum) enters the interval as an upper bound on the truncated mass.

In this part this premise row is Assumption (the row of interval-arithmetic premises in 23-07 § 05). That the computer and the language implementation keep correct IEEE 754 rounding is placed as a premise, not something we checked. When citing a computer-assisted result, we write this premise in the same row.

04

Making $\hbar$ an interval — why it breaks when done naively, and three lemmas

To get $q^{+} < 1$ at a point value of $\hbar$ (for example $\hbar = 3 \cdot 10^{-3}$), the interval arithmetic of § 03 is enough. To say it for **every** $\hbar$ in the window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, $\hbar$ itself must be put into an interval. Even if a downward trend is seen at three points, that the upper bound is monotone in $\hbar$ cannot be derived on paper (the formula for the upper bound contains two terms that are largest at opposite ends of $\hbar$).

We fix notation. $\beta = 1/(\varepsilon\hbar) = 2/\hbar$; for a subinterval $\hbar \in [\hbar_1, \hbar_2]$, $\beta \in [\beta_2, \beta_1]$; midpoint $\beta_0 = (\beta_1 + \beta_2)/2$, half-width $\Delta = (\beta_1 - \beta_2)/2$, and $\tau = \beta - \beta_0 \in [-\Delta, \Delta]$. $\hbar$ enters only through β.

Naive intervals break. Multiplying the exponent of the density $\beta(sx - A(x))$ by the interval of β as it is allows the weight of each grid cell to move independently by factors $e^{\pm\Delta|sx - A|}$, and the width of the exponent is of order 10^2. It is the same loss of dependence as $x - x^2$ in § 01. Indeed, for the chain of the double well, at the second stage the interval of the mean spread over the whole box (width 0.89), and the map of the stages lost its meaning.

Linear-Fractional Lemma (making the expectation a linear-fractional function of τ). Putting $h = g - g^*$ (g^* a constant) and $\varphi_0 = \beta_0 h$, with $\beta = \beta_0 + \tau$,

$$\mathbb{E}_\beta[f] = \frac{\int f e^{\beta g}}{\int e^{\beta g}} = \frac{S_0 + \tau S_1 + R_S(\tau)}{Z_0 + \tau Z_1 + R_Z(\tau)}, \qquad 0 \le e^{\tau h} - 1 - \tau h \le \frac{(\Delta|h|)^2 e^{\Delta|h|}}{2} \quad (5)$$

($S_0 = \int f e^{\varphi_0}$, $S_1 = \int f h e^{\varphi_0}$, $Z_0 = \int e^{\varphi_0}$, $Z_1 = \int h e^{\varphi_0}$; R are the integrals of the remainders). With the values of the remainders fixed, $t \mapsto (S_0 + R_S + tS_1)/(Z_0 + R_Z$

$+ tZ_1$) is linear-fractional, and if the denominator is positive at $t = \pm\Delta$ it is monotone on $[-\Delta, \Delta]$. Hence for all $|\tau| \le \Delta$ the expectation lies between its two values at the ends $t = \pm\Delta$, and evaluating those two in intervals encloses it Proved here Lean `mobius_between` (assumptions: the denominator is positive at $t = \pm\Delta$, $-\Delta \le t \le \Delta$). The first-order term $\tau S_1/Z_0$ represents the real change with $\hbar$, and only Δ times the quadrature error and the remainder (of order Δ^2) are left in the width of the interval.

Tilted-Enclosure Lemma (first and second forms; holding the enclosure tilted in τ). The fixed points of the map of the stages (the conditional means m_k of each stage of the chain) really move with $\hbar$. When the width of a subinterval is 3%, that movement alone is 1.5×10^{-4}, and putting it into the width of the enclosure ruins the downstream estimates. So we hold the enclosure in the form "$m_k - \tau\mu_k - \tau^2\nu_k$ lies in an interval" (μ_k and ν_k are floating-point sequences). On the output side the identity

$$\frac{A + tB}{C + tD} = \frac{A}{C} + t\,\frac{B - (A/C)D}{C + tD} \tag{6}$$

extracts the first-order movement Proved here Lean `shear_identity` (assumptions: $C \neq 0$, $C + tD \neq 0$). Tilting to second order, the remaining width is of order Δ^3. This is the form "expand about the centre" of § 01.

Moving-Centre Lemma (moving the centre with τ). The upper bound on the variance has the form of a sum of two one-sided parts $\beta\,\mathbb{E}[(X - c)_+^2] + \beta\,\mathbb{E}[(c - X)_+^2]$, and when the mean moves by $\tau\mu$ one increases and the other decreases. Maximising the two separately at the ends of τ adds movements that in truth cancel. Moving the centre as $c_\tau = c + \tau\kappa$ and bounding by $(y - t)_+^2 \le y_+^2 - 2ty_+ + t^2$ brought this loss down from 3.6×10^{-3} to 2.2×10^{-4} Proved here Lean `dvtau_pos` (no assumptions).

Invariant sets. The chain is infinitely long, so we cannot run the map of the stages a finite number of times and stop. The map $\Phi_{[\hbar_1,\hbar_2]}$ on a subinterval is built to contain the results of the maps $\Phi_\hbar$ for all $\hbar \in [\hbar_1, \hbar_2]$, so if interval arithmetic confirms $\Phi_{[\hbar_1,\hbar_2]}(E^*) \subset E^*$ for some enclosure E^*, then $\Phi_\hbar(E^*) \subset E^*$ for every $\hbar$ Proved here (transitivity of inclusion). All later stages lie inside E^*, and it is enough to evaluate the upper bound on q once on E^*.

Table 1 The balance on one subinterval (outer chain of the core of the bridge; the upper bound at the point $\hbar = 3 \cdot 10^{-3}$ is 0.995611069)

Version	$\hbar \in [3/1030, 3/1000]$ **(width 3%)**	$\hbar \in [3/1100, 3/1000]$ **(width 10%)**
Naive intervals	Breaks at the second stage	Breaks at the second stage
Linear-Fractional Lemma only	Computer-assisted $q^+ =$ 0.998757877	—

Version	$\hbar \in [3/1030, 3/1000]$ **(width 3%)**	$\hbar \in [3/1100, 3/1000]$ **(width 10%)**
Linear-Fractional Lemma + first-order tilt + Moving-Centre Lemma	Computer-assisted $q^{+} = 0.996449644$	The upper bound is 1.00114, above 1, and no certificate is produced (§ 09)
Linear-Fractional Lemma + second-order tilt + Moving-Centre Lemma	—	Computer-assisted $q^{+} = 0.997808483$

Second-order tilting made subintervals of width 10% usable, and the window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ was covered by 25 subintervals with ends $\hbar = 3/N$ ($N = 1000, 1100, 1210, \dots, 9100, 10000$, ratio between neighbours ≤ 1.1). The grid width is 2^{-11}, and the number of stages is 20 on the upper half and 25 on the lower half.

05

Example 1: the outer chain of the core of the bridge — $q_{\mathrm{free}} \le 0.998854904$ on the whole window

Write q_{free} for the ratio of the unmarked chain extending outside the site window of the core of the bridge (box $(-x_c, 0)$, $x_c = 0.89$) (the route of the chain outside the window in 23-07 § 02). On each of the 25 subintervals we built the enclosures of the map of the stages and an invariant set in interval arithmetic, and obtained an upper bound q_i^{+} on q for all chain lengths $n \ge 2$ and all pins of the box (the values at the end of the chain). The results are the left column of Table 2.

Result (computer-assisted)

For the path measure of the 0+1-dimensional double well ($V = (x^2 - 1)^2/8$, $\varepsilon = \frac{1}{2}$, $x_c = 0.89$, mark threshold $\theta = \frac{1}{2}$), under the premises of § 03, for every $\hbar$ in the window $\hbar \in [3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$

$$q_{\mathrm{free}}(\hbar) \;\le\; \max_i q_i^{+} \;=\; 0.998854904 \;<\; 1 \quad (7)$$

On the upper half $\hbar \in [10^{-3}, 3 \cdot 10^{-3}]$, ≤ 0.998098180. The maximum is on the subinterval at the lower end of the window. Regularity and the inputs of the sector framework are Assumption .

The values in the table (rounded up at 9 digits) increase as $\hbar$ decreases, with a bulge only at the two at the upper end. The limiting place is the cell where the value at the end of the chain lies just below the wall 0. It is exactly the place in § 02 where "the corner at the wall is exactly 1".

Table 2 Upper bounds on the 25 subintervals (from the large-$\hbar$ side. Left = outer chain of the core of the bridge q_i^{+}, right = outer chain of the ring, unconditioned form Q_i^{unc}. Both rounded up at 9 digits; computer-assisted)

#	Subinterval of $\hbar$	Bridge q_i^{+}	Ring Q_i^{unc}
01	$[3/1100,\ 3/1000]$	0.997808483	0.993773312
02	$[3/1210,\ 3/1100]$	0.998098180	0.993531447
03	$[3/1330,\ 3/1210]$	0.996740073	0.993402112
04	$[3/1460,\ 3/1330]$	0.995689889	0.993847141
05	$[3/1600,\ 3/1460]$	0.994979345	0.994267993
06	$[3/1760,\ 3/1600]$	0.995310186	0.994690320
07	$[3/1930,\ 3/1760]$	0.995450880	0.995072576
08	$[3/2120,\ 3/1930]$	0.995739245	0.995442678
09	$[3/2330,\ 3/2120]$	0.996022449	0.995791525
10	$[3/2560,\ 3/2330]$	0.996294103	0.996116234
11	$[3/2810,\ 3/2560]$	0.996551317	0.996416208
12	$[3/3000,\ 3/2810]$	0.996695721	0.996606914
13	$[3/3300,\ 3/3000]$	0.996969923	0.996887371
14	$[3/3630,\ 3/3300]$	0.997198625	0.997139630
15	$[3/3990,\ 3/3630]$	0.997410737	0.997371392
16	$[3/4380,\ 3/3990]$	0.997607042	0.997583254
17	$[3/4810,\ 3/4380]$	0.997791698	0.997780790
18	$[3/5290,\ 3/4810]$	0.997966990	0.997966387
19	$[3/5810,\ 3/5290]$	0.998126215	0.998135077
20	$[3/6390,\ 3/5810]$	0.998276990	0.998293614
21	$[3/7020,\ 3/6390]$	0.998414660	0.998437926
22	$[3/7720,\ 3/7020]$	0.998559421	0.998572746
23	$[3/8490,\ 3/7720]$	0.998691543	0.998697045
24	$[3/9100,\ 3/8490]$	0.998741829	0.998780022
25	$[3/10000,\ 3/9100]$	**0.998854904**	**0.998888530**

```
06
```

Example 2: the outer chain of the ring — from the conditioned chain to the unconditioned chain

The outer chain of the seam of the ring (23-07 § 03) is a chain moving in a region of fixed sign, and mapped to $(-\infty, 0]$ by reflection it becomes the same type as Example 1. But there is no lower wall $-x_c$ as in the box of the bridge, so the certificate of Example 1 cannot be used as it is. We swapped the grid of the same machinery and closed it in two stages.

1. **The chain conditioned on a box.** For a chain conditioned on a finite box for each subinterval ($[-7/5, 0]$ on the upper half, $[-6/5, 0]$ on the lower half), the upper bound on the whole window is 0.998888525 Computer-assisted . As it stands, four assumption rows are needed to pay separately for configurations that leave the box.
2. **The unconditioned chain.** Widen the box to $[-3, 0]$, and put the contribution of the tail $(-\infty, -3]$ into the remainder of the interval arithmetic by a tangent upper bound from the convexity of the tail. The upper bound on the tail variance, in a form that pushes the law onto a single row at the end of the tail, is ≤ 0.357608022 at every $\hbar$ (within < 5.25 needed for convexity and ≤ 1 needed for the correlation inequality). Pins deeper than -3 are covered by a closed-form upper bound, ≤ 0.592898559. The results are the right column of Table 2, with maximum 0.998888530 Computer-assisted (all pins $p \le 0$, evaluation points $s \le 0$, end value $z_0 = -1$; the seam limit assumption, the random-end-value assumption and regularity are Assumption).

Comparing the two columns, the upper bound for the bridge bulges at the two at the upper end, while that for the ring is smallest on the third subinterval and then increases monotonically. At the lower end the ring is slightly closer to 1 (difference 3.4×10^{-5}), and the order of the floating-point point values (the ring closer to 1) appears in the upper bounds as well. Each row of the unconditioned chain is higher than the corresponding row of the conditioned chain by only 2.4×10^{-9}–6.7×10^{-7}, with the same limiting place and shape.

The boundary stage is not covered. The claim of the right column is a claim for pins $p \le 0$. In configurations where the lump touches the end of the window, the pin of the first outer stage falls in $(0, \frac{1}{2})$, and there the floating-point reading is $q \approx 1.004$–1.045 Reading . It is a single finite stage, so it does not break the finiteness of the construction, but when citing the right column of Table 2 the condition "pin ≤ 0" cannot be dropped.

07

Cross-checks, controls, pre-registration

The correctness of the interval-arithmetic scripts themselves has not been formally verified. Instead they pass the following checks. All are checks to see "whether what we think we checked has drifted from what we wanted to check".

- **Cross-check with another implementation.** On every subinterval of Example 1, we ran the map of the stages by a floating-point recursion separate from the interval arithmetic, at three values of $\hbar$ (the two ends and the midpoint of the subinterval), and counted whether its values fell inside the interval enclosures at every stage and every grid point (75 runs, 0 misses). For the unconditioned chain of the ring, the same check at one $\hbar$ per subinterval (25 runs, 0 misses).
- **Cross-check with point values.** The floating-point point values obtained from the transfer matrix (26 ends) all lay below the upper bound of the subinterval containing that end (the smallest difference, at the lower end of the window, was 2.85×10^{-4}).
- **Positive control.** Whether the machinery reproduces the answer where it is known in closed form. For the pure Gaussian chain (variance per stage $v_k = (k-1)/k$, mean $m_k(s) = (k-1)s/k$, neither depending on $\hbar$), every point fell inside the enclosures with $\hbar$ kept as an interval. With the width of the interval of $\hbar$ set to 0, it gave the same values as the machinery at a point $\hbar$ to order 10^{-12}. The unconditioned machinery for the ring reproduced the limiting place and shape of the version conditioned on a box on the same subintervals, with values within an upward difference of order 10^{-8} (they are not the same chain, so this is a check of reproduction, not agreement).
- **Negative control.** Whether the machinery rejects claims that should not hold. We confirmed that lowering the target upper bound at the limiting cell by 10^{-7} makes the Lean check fail to close, that replacing one entry of the table by 1 makes < 1 on the whole window fail to close, and that setting the tail variance above the convexity line stops the tail stage.
- **Pre-registration.** Before running the computations, we wrote down the expected range of values, the limiting place, and the pass lines of the positive and negative controls, and recorded after the results whether they were right. In Example 1 we expected 60–120 subintervals and 25 sufficed (wrong), and the limiting place coming at the lower end of the window was right. That a prediction was wrong does not weaken the result, but it is a record that the pass lines were not moved after seeing the results.

```
08
```

What Lean checked — the construction with the table of outputs of interval arithmetic placed as an assumption

The Lean column attached to computer-assisted results splits into the following three layers.

1. **The algebraic core.** The monotonicity of the linear-fractional function in the Linear-Fractional Lemma (`mobius_between`), the tilting identity (`shear_identity`), the inequality of the Moving-Centre Lemma (`dvtau_pos`), and the algebra of passing from the cover to the whole window (`cover_le`, assumption: $f \le c$ on each subinterval).
2. **Recomputation of cells.** From the input values output by the interval arithmetic (exact values of the floating-point numbers) and the rational ends of the subintervals, for the limiting cells and randomly chosen cells (46 in Example 1, 23 in the conditioned version for the ring), the formula for the upper bound at the last stage was recomputed inside Lean, closing "at most the interval-arithmetic value + 2×10^{-9}". This is not verification of every cell (a few dozen out of more than several million cells in one subinterval).
3. **Assembly of the table.** `win_qfree_window_le_of_interval_table` (Example 1), `unc_window_lt_one_of_table` (Example 2) and `vt_window_of_table` (tail variance) place as an **assumption** "$q(\hbar) \le q_i^+$ for every $\hbar$ in each subinterval" (the table of outputs of interval arithmetic; q is an abstract function), and give as conclusion "$q \le \max_i q_i^+ < 1$ for every $\hbar$ in the window".

Outside Lean lie the map of the stages (the interval evaluation of the Linear-Fractional Lemma, the Tilted-Enclosure Lemma and the Moving-Centre Lemma), quadrature, the invariant sets, the layers of the formula for the upper bound on q, and the choice of evaluation points. So the machine-checking column guarantees "if the outputs of the interval arithmetic are correct, the conclusion on the whole window is correct as algebra", and "the outputs of the interval arithmetic are correct" rests on the premises of § 03 and the checks of § 07.

09

Examples where computer assistance returns "no"

When the upper bound from interval arithmetic exceeds 1, all one can say is "this construction produces no certificate", not "the true value exceeds 1". We distinguish two cases.

(a) An example where it fails on the side of the tool. On the subinterval of width 10% in Table 1, the version with first-order tilting only gave the upper bound 1.00114. On

the same subinterval the version with second-order tilting shows the true value is ≤ 0.997808483, so this "no" comes from the width of the tool (having put the curvature of the true m in $\hbar$ into the interval). It is the kind of "no" that disappears with finer subintervals or a higher order of expansion.

(b) An example where it fails on the side of the object — the inner chain of the ring. For the stages inside the site window of the ring (the inner chain), measuring q_{in} by floating-point transfer gave $1.2707/1.2765/1.2421$ at $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$ Reading , moving upwards as the grid was refined. The limit is set by the stage with the pin at the edge of the far wall of the lump box, the corner where the wall force points out of the box (classical value 1.2857) (23-07 § 03). Here, even building interval machinery is not expected to give $q^{+} < 1$. What it would give is only a finite upper bound $\bar{q} = \max(q_{\mathrm{in}}, 1) \approx 1.3$. This "no" is of a kind that does not disappear however fine the tool is made, and it can only be absorbed on the side of the construction (making a form that does not need $q_{\mathrm{in}} < 1$, or paying a coefficient $\bar{q}^{16}$ on each of S_W and T_W, and $\bar{q}^{32}$ on A_W).

When reading a computer-assisted result and seeing a report that the upper bound exceeded 1, check whether (a) or (b) is what is written.

10

What computer assistance does not say, and the cost

Box: how to read the "Computer-assisted" label

An inequality carries Computer-assisted only when all of the following are in place. (1) It is the output of a computation that holds every number as an outward interval, not a floating-point value written as it is. (2) The premises (correct IEEE 754 rounding, upper bounds on summation error, upper bounds on the error of our own exponential function, regularity) are written in the same row. (3) The scope of the claim (the window of $\hbar$, the chain lengths, the range of pins) is written in the same row. It says nothing outside the window of $\hbar$, as $\hbar \to 0$, or for pins outside the range.

What it does not say. (i) $\hbar$ outside the window $[3\cdot 10^{-4}, 3\cdot 10^{-3}]$. As seen in § 02, q approaches 1 as $\hbar \to 0$, so the further the window is extended downwards, the finer the subintervals and grid must be, and no finite computation can cover the whole of $\hbar \to 0$. The double-well relaxation-time power statement is a statement as $\hbar \to 0$, so that is work for paper asymptotics. (ii) Anything outside the model on the lattice $\varepsilon = \frac{1}{2}$ (the difference from the continuum path measure). (iii) Regularity (smoothness of eigenfunctions and conditional expectations) and the inputs of the sector framework.

(iv) The boundary stage of Example 2 (pin $\in (0, \frac{1}{2})$). (v) The saddle-tube condition itself (23-07 § 06).

Table 3 Cost (one core; "estimate" means not run)

Computation	Runs	Time	Size of one run
Example 1, outer chain of the core of the bridge (whole window)	27 runs	About 4 hours 10 minutes	6–16 minutes, 107 MB or less
Example 2, outer chain of the ring, form conditioned on a box	25 runs	About 3 hours 19 minutes	5–12 minutes, 114 MB or less
Example 2, outer chain of the ring, unconditioned form	25 runs	About 6 hours 44 minutes	13–22 minutes, 123 MB or less
Putting the inner chain of the ring into intervals (estimate)	About 50 runs	About 42 hours	About 50 minutes
Remaining numbers of the route of the window-constant numerics (estimate; decided not to run)	About 42 runs	About 21 hours	Within 30 minutes

This part is a study with the diffusive-relaxation-time assumption at its axis, and within it computer assistance is used as a tool that raises individual inequalities inside the window from floating-point readings to inequalities. Another view is treated in Part 24.

What this chapter can say

What is said	What is not said
Computer-assisted $q_{\mathrm{free}} \le 0.998854904$ at every $\hbar$ of the window, and ≤ 0.998888530 for the outer chain of the ring (pin ≤ 0). On the premises of § 03	Outside the window, $\hbar \to 0$, the boundary stage
Proved here The derivation of the chain condition (3), the algebra of the Linear-Fractional Lemma, the Tilted-Enclosure Lemma and the Moving-Centre Lemma, the inclusion of invariant sets Lean (the algebraic core and the assembly of the table)	The formal correctness of the interval-arithmetic scripts (only as far as checked by cross-checks and controls)
Reading The inner chain of the ring $q_{\mathrm{in}} \approx 1.24$–$1.28$	An interval upper bound on it

Sources and reproduction

Item	Kind	Source / tools
The ideas of interval arithmetic, loss of dependence and the centred form	—	Standard methods (interval arithmetic, Taylor-form enclosures). This chapter cites no bibliography and places the example of § 01 and the lemmas of § 04 in the text.

Item	Kind	Source / tools
The Stein kernel and the chain condition	Proved here	(2)(3) of § 02 (integration by parts and differentiation under the integral sign).
Implementation of interval arithmetic	Computer-assisted	Python float64, one floating-point step outwards per operation, our own exponential function, grid 2^{-11}, stages 20 / 25, 25 subintervals.
Cross-checks	Reading	A separate floating-point recursion (three values of $\hbar$), transfer-matrix point values (26 points).
Lean	Lean	`mobius_between`, `shear_identity`, `dvtau_pos`, `cover_le`, `win_qfree_window_le_of_interval_table`, `unc_window_lt_one_of_table`, `vt_window_of_table` (the assumptions are the rows of the text). The list of statements is in 23-13.

Words used in this chapter — computer-assisted, interval arithmetic, outward rounding, loss of dependence, expanding about the centre, Stein-kernel ratio, chain condition, subinterval, tilting, invariant set, cross-check, positive and negative controls, pre-registration, boundary stage. Their meanings are in A-02 Glossary.

Next chapter to read: 23-09 The sector framework and the three strands of the window log-Sobolev route — states the five steps of the sector sum and the closing condition, the derivation of the needed value, how far the three strands that attacked the one remaining proposition got, and the common wall, the shortfall of the curvature at one site at the top.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time sector framework, closing condition, the common wall · Kiichi, Shiori and Rin

The sector framework and the three strands of the window log-Sobolev route

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — We prove, split into five steps, the construction that controls the gap of the core uniformly in the core length k (the sector framework), and derive the condition under which it closes, $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$, and the needed value $A_{\max}$ determined by it. We sort the constants of the condition into those that come from the structure and those that come from the choice of tools, and list the directions that raise the needed value and that lower the insufficient side. We put in one table how far the three strands (componentwise, scalar, Feynman–Kac) that attacked the one remaining proposition (that the logarithmic Sobolev constant of the site window lies within the needed value) got, and state in one line of diagonal dominance why all three stop at the same place: the curvature at one site at the top falls $\frac{1}{4}$ short of convexity.

Prerequisite chapters — 23-07 (the positions of the route of the sector framework and the route of the window log-Sobolev inequality), 23-08 § 02 (the chain condition), 23-05 § 03 (the saddle-tube condition and the core Poincaré uniformity assumption), 1-10 § 07– § 09 (Poincaré inequalities).

Said first — The level is [Res], and only § 01 is written to be readable at [Y3]. § 02 onwards are sections at the research level, and you can skip ahead to Table 4 in § 05 and the common wall in § 06 and still read the conclusions of this chapter. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter says how far the core row of the saddle-tube condition of its first step could be built. The sector framework has been shown, but the one remaining proposition is not closed by any of the three strands.

The order of this chapter

1. Summary — a section readable at [Y3]
2. The five steps of the sector framework — sum over sectors, Gibbs in marginal form, componentwise localisation, locality of volume, composition

3. The closing condition and the needed value $A_{\max}$ — where the constants come from, and two directions (Tables 1 and 2)
4. The threshold mismatch — an example where the condition, taken literally, gave nothing
5. The three strands of the window log-Sobolev route — componentwise, scalar (the Scalar-Strand Proposition, Table 3), Feynman–Kac, and how far they got (Table 4)
6. The common wall — the curvature at one site at the top, $15/4$ against 4
7. The table of the five grades (Table 5)
8. The dependency graph

01

Summary

The core $Q_k = (-x_c, 0)^k$ is an interval of k sites where the path stays near the top, and here the Hessian of the action has negative directions. To control the gap of the measure of the core (the core Poincaré uniformity assumption) uniformly in k, one must pay using the fact that places with negative curvature are **rare**. The sector framework splits that payment into five steps.

1. **Sum over sectors.** Even when the space of configurations is split into a direct product of boxes, the Dirichlet form, the variance and the entropy split into sums over sectors, and the number of sectors does not enter the constants.
2. **Gibbs in marginal form.** The weight on a rare set B (measure at most $e^{-\ell}$) can be paid with the entropy and $\ln 2$: $\ell \int_B g^2 \le \mathrm{Ent} + \ln 2 \int g^2$.
3. **Componentwise localisation.** Applying the Gibbs step in marginal form to each component of the gradient with the logarithmic Sobolev constant A_W of the marginal of the marks on the 17 sites around that site (the site window), k does not enter the sum.
4. **Locality of volume.** The rarity exponent $\ell_{\min}$ does not depend on k.
5. **Composition.** Combining the above with the two inequalities of Barta and Bochner gives, for the gap of the core λ, $\lambda(1 + bA_W/\ell_{\min}) \ge \kappa - b(A_W c_H + D_W + \ln 2)/\ell_{\min}$. The condition for the right-hand side to be positive is the **closing condition** $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$.

The closing condition rewritten for A_W is the needed value $A_{\max} = (\kappa \ell_{\min}/b - D_W - \ln 2)/c_H$. At the three values $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$, the original form on the road that defines defects by three sites is negative at $3 \cdot 10^{-3}$ (it closes for no A_W), 0.20 and 7.13, and correcting two choices of tools raises it to $25.1/90.3/312.3$ Reading (arithmetic on the measured rarity exponents). On the other hand, in the current construction the upper bound on A_W is of order $1857/2794/9106$ Reading (including estimates), 29–74 times short.

The one remaining proposition "$A_W < A_{\max}$" was attacked by three strands. The componentwise strand stopped at 1.4–5.5 times the needed value, the scalar strand became a proposition with a table of assumptions (the Scalar-Strand Proposition) for $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, and the Feynman–Kac strand lies within 0.54 times the threshold by a Monte Carlo reading, but its upper bound is 10^3 times the threshold. All three stop at the same place: at the top, the diagonal at one site, $4 - \frac{1}{4} = 15/4$, does not reach the sum of the couplings to the neighbours, 4 (§ 06).

02

The five steps of the sector framework

Setting. Take the core $Q_k = (-x_c, 0)^k$ ($x_c = 0.89$), fix the end values $z_0, z_{k+1} \le -x_c$, and let $\mu \propto e^{-S/\hbar} \mathbf{1}_{Q_k}$ and $\mathcal{E}(f) = \hbar \int |\nabla f|^2 d\mu$ (Neumann on the faces of the box). Let λ be the first non-zero eigenvalue of $-L$, f the eigenfunction, $u = \nabla f$, and put

$$X = \int |u|^2 d\mu, \qquad Z = \hbar \int \|\nabla u\|^2 d\mu, \qquad Y = \sum_{j=1}^{k} \int_{N_j} u_j^2 \, d\mu \quad (1)$$

. N_j is the defect set of site j (configurations where j or one of its neighbours lies on the top side of the threshold $-\theta_B$, $\theta_B = 0.57$). Attach a mark $\sigma_j = \mathbf{1}[z_j > -\theta]$ to each site, and write σ_{W_j} for the set of marks on the window W_j around site j. A sector is a direct product of boxes for each type of marks, $\Omega_\sigma = \prod_j I_{\sigma_j}$.

Sector sum

For any partition $\mathcal{P}$ into direct products of boxes, (a) $\sum_P \mu(P) \mathcal{E}_P(f) = \mathcal{E}(f)$ (faces have measure 0, and $|\nabla f|^2$ is the same function on each sector), (b) $\mathrm{Var}_\mu f = \sum_P \mu(P) \, \mathrm{Var}_{\mu_P} f + \mathrm{Var}_\nu(\mathbb{E}[f \mid \mathcal{P}])$ (the law of total variance; the entropy has a chain rule of the same form), (c) if $\mathrm{Var}_{\mu_P} f \le C_P \mathcal{E}_P(f)$ on each sector, the first term is at most $(\sup_P C_P) \, \mathcal{E}(f)$ Proved here . The number of sectors appears only in the weights $\mu(P)$, which

sum to 1. Lean `var_split_cells` (no assumptions), `cell_sum_sup` (assumptions: $w_i \geq 0$, $E_i \geq 0$, $C_i \leq C_{\max}$).

Gibbs decomposition — the Gibbs step in marginal form

The Gibbs variational inequality. For a probability measure ν, $G \geq 0$ and a function F,

$$\int FG\,d\nu \;\leq\; \mathrm{Ent}_\nu(G) + \Big(\int G\,d\nu\Big)\ln \int e^F\,d\nu \tag{2}$$

Proved here Proof. Put $\tilde{G} = G/\int G$. Jensen's inequality gives $\ln \int e^F\,d\nu \geq \ln \int_{\tilde{G}>0} e^{F-\ln\tilde{G}}\tilde{G}\,d\nu \geq \int (F-\ln\tilde{G})\tilde{G}\,d\nu$. Multiplying both sides by $\int G$ gives (2). ∎

Taking $F = t\mathbf{1}_B$ and $p = \nu(B) \leq e^{-\ell}$ in (2) gives $\ln \int e^F = \ln(1-p+pe^t)$, and with $t = \ell$ this is $\leq \ln 2$ Lean `gibbs_ln2_cell` (assumptions: $p \geq 0$, $\ell \geq 0$, $pe^{\ell} \leq 1$). If B is a set determined by a coarse variable $\mathcal{F}$, taking $G = \mathbb{E}[g^2 \mid \mathcal{F}]$ gives

$$\ell \int_B g^2\,d\mu \;\leq\; \mathrm{Ent}_\nu\big(\mathbb{E}[g^2 \mid \mathcal{F}]\big) + \ln 2 \int g^2\,d\mu \tag{3}$$

. This is the form that pays for the weight on a rare set with the entropy of the marginal of the coarse variable.

Componentwise localisation

As input we assume, for all j and g, the logarithmic Sobolev inequality of the marginal on the window

$$(\mathrm{LS}_W) \qquad \mathrm{Ent}_{\nu W_j}\big(\mathbb{E}[g^2 \mid \sigma W_j]\big) \;\leq\; A_W\,\mathcal{E}(g) + D_W \int g^2 \tag{4}$$

Assumption , and let $N_j \subset B_j$ (B_j a set determined by σW_j, $\ell_{\min} = \min_j \ln(1/\mu(B_j))$). Applying (3) to $g = u_j$ and summing over j, from $\sum_j \mathcal{E}(u_j) = Z$ and $\sum_j \int u_j^2 = X$,

$$\ell_{\min}\,Y \;\leq\; A_W\,Z + (D_W + \ln 2)\,X \tag{5}$$

holds Proved here Lean `comp_sum_le` (assumptions: $0 < \ell \leq \ell_j$, the inequality for each component). k **does not enter.** For contrast, if (3) is applied to a single union $\bigcup_j B_j$ with a scalar g, without splitting the gradient into components, just putting in the constant function $g \equiv 1$ makes the defect term at least $\sum_j \mu(B_j)$, and it is not uniform in k Proved here Lean `scalar_not_uniform` (assumption: $p > 0$; the conclusion is "there is no d with $kp \leq d$ for all k"). **Uniformity in k and having A_W enter the condition are a trade-off.**

Volume locality

$\ell_{\min}$ is the rarity exponent of the defect set, and does not depend on k. On the road that defines defects by three sites, the measured values $\hbar \ln \mu(N_j)$ give $\ell_{\min} = 38.3/121.9/406.8$ ($\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$) Reading (floating-point transfer matrix; $3 \cdot 10^{-4}$ is an extrapolation, and the remeasured value is 420.0). On the road that defines defects by lumps of length 8 or more, $\ell_{\min} \geq E_8/\hbar - \ln V_{\mathrm{loc}}$ ($E_8 = 0.402$, $\ln V_{\mathrm{loc}} = 2.3$) Reading . To say the same on the bridge, one needs that the remainder of the transfer operator of a scrap is small uniformly in the end values (the bridge volume-locality input) Assumption .

Sector composition

We combine the following three inequalities.

- **Barta**: $\lambda X \geq \kappa X - bY$. It comes from the row inequality $(\mathcal{M}\psi)_j \geq (\kappa - b\mathbf{1}_{N_j})\psi_j$ obtained by applying a weight ψ to the componentwise operator $\mathcal{M} = -L + \operatorname{Hess} S$ ($u = \nabla f$ satisfies $\mathcal{M}u = \lambda u$). $\kappa = 3/1000$ is the lower bound on the rows outside the defects, and $b = 493/1000$ is the depth of the defects, determined by the worst row over the whole box Assumption (the paper estimate of the rows is not placed in the book).
- **(5)**: componentwise localisation itself.
- **Bochner**: $Z = \lambda X - \int u \cdot \operatorname{Hess} S\, u\, d\mu \leq (\lambda + c_H)X$, $\operatorname{Hess} S \geq -c_H$, $c_H = \frac{1}{4}$. The identity is the integrated form of Bochner's formula Cited (Proposition 1.3 of Ledoux's lectures, checked in the original text), and the vanishing of the boundary terms on the flat faces of the box and the smoothness of the eigenfunction up to the corners are Assumption .

Putting Bochner into (5), and that into Barta, gives

$$\lambda\Big(1 + \frac{bA_W}{\ell_{\min}}\Big) \;\geq\; \kappa - \frac{b\,(A_W c_H + D_W + \ln 2)}{\ell_{\min}} \quad (6)$$

Proved here Lean `comp_close` (assumptions: $X > 0$, $b, A \geq 0$, $\ell > 0$, and the three inequalities Barta, (5), Bochner). If the right-hand side is positive, that is, if

$$b\,(A_W c_H + D_W + \ln 2) \;<\; \kappa \ell_{\min} \quad (7)$$

, the gap of the core λ is bounded below by a positive number that does not depend on k. (7) is the **closing condition**. Of the five steps, the constructions of the sector sum, the Gibbs decomposition, componentwise localisation and sector composition have elementary proofs placed in the text, and what remains as assumptions is the marginal log-Sobolev input for the window, the bridge volume-locality input, the Barta rows, the

regularity for Bochner, and whether A_W is small enough to satisfy (7). **What remains as an assumption is not structure but the size of a constant.**

03

The closing condition and the needed value $A_{\max}$

Solving (7) for A_W gives the needed value

$$A_W \;<\; A_{\max}(\hbar) := \frac{\kappa \ell_{\min}/b - D_W - \ln 2}{c_H} \tag{8}$$

follows Proved here (rearranging (7)). That A_W is smaller than this is the one remaining proposition (the Window Log-Sobolev Threshold Proposition), the route of the window log-Sobolev inequality in 23-07 § 02.

Where the constants come from. We sort the six constants of (7) into those that come from the structure and those that come from the choice of tools.

Table 1 Where the constants of the closing condition (7) come from

Constant	Value	Origin	Side
κ	$3/1000$	Lower bound on the Barta rows outside the defects (small because the threshold $\theta_B = 0.57$ is placed just above the one-site threshold 0.5653)	Structure
b	$493/1000$	The worst value of the same rows over the whole box (-0.49 on the top side)	Structure
$\ell_{\min}$	Table 2	The rarity exponent of the defect set (not a length)	Structure
c_H	$\frac{1}{4}$	In the Bochner step, the minimum $w(0) = -\frac{1}{4}$ of the negative curvature $w = \varepsilon V''$ paid **uniformly over the whole** core	Choice of tools
$\ln 2$	0.693	The value with the tilt t in the Gibbs step fixed at $t = \ell$	Choice of tools
D_W	0	A construction that puts no defect term in the marginal log-Sobolev input for the window	—

The reason A_W enters the condition multiplied by c_H is this: when $A_W Z$ on the right of (5) is replaced by $(\lambda + c_H)X$ using Bochner, the λ part can be moved to the left side, but $c_H X$ remains without a factor λ and can only be paid by κ. The amplification $b/\kappa \approx 164$ multiplies $A_W c_H$ here.

Two directions — raising the needed side, lowering the insufficient side

Raising the needed side (correcting the choice of tools).

- **Localising Bochner.** The negative curvature $-w(z) = (1 - 3z^2)/4$ exists only for $z > -1/\sqrt{3}$, and that region is already inside the defect set N_j. From the one-line interval bound $-w(z) \le \delta + \gamma\,\mathbf{1}[z > -\theta_R]$ ($\delta = \max(0, (1 - 3\theta_R^2)/4)$, $\gamma = c_H - \delta$) Proved here Lean `negw_le_localized` (assumptions: $\theta \ge 0$, $z \le 0$, $(1 - 3\theta^2)/4 \le \delta \le \frac{1}{4}$), and using that the kinetic term $2T_k$ is positive semi-definite, we get

$$Z \;\le\; (\lambda + \delta)X + \gamma Y \quad (9)$$

  Proved here Lean `bochner_localized` (assumptions: the identity $Z = \lambda X - H$, $-H \le \delta X + \gamma Y$). Y is a term already paid by (5), so the coefficient of A_W goes down from $bcH = 0.123$ to $b\delta + \kappa\gamma = 0.0038$ ($\theta_R = 0.57$; about 1/32).
- **Optimising the Gibbs tilt.** t in (2) may also satisfy $t < \ell$, and at $t^* = \ell - \ln((b - \kappa)/\kappa)$ $= \ell - 5.10$, which maximises $\kappa t - b\ln(1 + e^{t-\ell})$, the cost $b\ln 2 = 0.342$ is replaced by 0.0183 Proved here (maximum of a concave function of one variable). Lean's `tilt_cost_le` (assumptions: $b \ge 0$, $L \le 1/178$) shows only an upper bound of 0.0184 on the cost at t = ℓ − 5.2, not at the optimal point Lean (the optimisation itself is not in Lean).

The closing condition with both combined is $b(A_W\delta + D_W + L(t^*)) < \kappa(t^* - A_W\gamma)$, and the needed value is $A_{\max} = (\kappa\ell_{\min} - 0.0183 - bD_W)/(b\delta + \kappa\gamma)$. Raising the threshold θ further to near $1/\sqrt{3}$ gives $\delta \le 0$, and the condition becomes in effect $A_W \lesssim 4\ell_{\min}$.

Table 2 The needed value $A_{\max}$ ($D_W = 0$; triples are $\hbar = 3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$; values are arithmetic on the measured $\ell_{\min}$, Reading)

Form	$A_{\max}$	Remarks
Original form (road defining defects by lumps of length 8 or more)	0.43/6.96/29.8	$\ell_{\min} \ge E_8/\hbar - \ln V_{\text{loc}}$
Original form (road defining defects by three sites)	Negative / 0.20 / 7.13	At $3\cdot10^{-3}$ it does not close even with $A_W = 0$
Gibbs tilt only	0.78/2.82/9.75	Three-site road
Localised Bochner only	Negative / 6.23 / 228	As above

Form	$A_{\max}$	Remarks
Both combined	25.1/90.3/312.3	As above. The value is Reading . What Lean's `judge_i_inside` shows goes only as far as this: for the two values of ħ on the small side (the cases where the lower bound on ℓ is 406 and 121), the condition for closing holds if $0 \le A \le 22$ (that is, the needed value, namely the upper limit of the A for which it closes, is 22 or more) Lean . The third ħ, and the values themselves, are not in Lean (assumptions: $\ell \ge 406$, $\ell \ge 121$, $0 \le A \le 22$, $0 \le L \le 1/178$)
Raising the threshold to $1/\sqrt{3}$	128/452/1604	$\kappa(\theta)$ and $\ell(\theta)$ are floating-point lattice values

Lowering the insufficient side (rebuilding the upper bound on A_W). The upper bound on A_W for the three-site window $|W| = 3$ is 1857/2794/9106 in the current construction Reading (including estimates), which is **74 / 31 / 29 times** outside the needed value with both combined, and 14.5 / 6.2 / 5.7 times outside the needed value with the threshold raised (the latter is an estimate that assumes the same upper-bound technique gives the same value when the threshold is raised). On the other hand, the lower bound on A_W measured by test functions is 1.71/1.74/2.59 and does not exclude the needed value. **That the true A_W lies within the needed value is not excluded; what falls short is the way the upper bound is built.** This is the starting point of the three strands of the route of the window log-Sobolev inequality.

04

The threshold mismatch — an example where the condition, taken literally, gave nothing

Componentwise localisation uses $N_j \subset B_j$, with B_j a set determined only by the marks of the window σ_{W_j}. In Table 1, κ, b and $\ell_{\min}$ were values measured with the defect threshold $\theta_B = 0.57$, while A_W was measured with the mark threshold $\theta_\sigma = \frac{1}{2}$. If $\theta_\sigma < \theta_B$, the atom of marks "all marks in the window on the valley side" contains points with $z_i \in (-\theta_B, -\theta_\sigma]$, that is, points of N_j. The only set determined by the marks that contains N_j is the whole space, so $\mu(B_j) = 1$ and $\ell = 0$ Proved here (one line of measurability). **With the two thresholds mismatched, (7), taken literally, gives nothing.** The fix is not to lower θ_B to $\frac{1}{2}$ (at $\theta = \frac{1}{2}$ the lower bound κ on the Barta rows becomes negative), but to measure A_W with marks at $\theta_\sigma = \theta_B$, which is an order to the insufficient side. It is an example showing that, when numerical constructions are stacked, a step is needed that checks literally whether each of the conditions being composed holds on the same definitions.

05

The three strands of the window log-Sobolev inequality

The one remaining proposition was attacked in three forms that differ in the functional form of the condition. None of the strands has shown sector composition, the core Poincaré uniformity assumption or the saddle-tube condition. All numbers are values on the 0+1-dimensional double well on the lattice $\varepsilon = \frac{1}{2}$. The full list of detour plans and the types of breakage are sent to 23-10.

The componentwise strand — the componentwise form

On top of the localised Bochner and the Gibbs tilt, adding a form that pays the negative curvature at each point by the rarity of that point (paying the curvature with a function of height $h = (-w(z_j))_+$) raises the needed value to $537/1754/6052$ Reading . The upper bound is built in the product form of "a coarse step" and "a propagation step".

- **The coarse step is exhausted.** With a mark chain that is multi-valued only in the middle, the logarithmic Sobolev constant of the coarse step is $\leq 3.584/3.544/3.613$, a ratio of $1.016/1.005/1.003$ to the lower bound from the indicator of a single point Reading (a rational decomposition on double-precision data). Nothing more can be gained from this step.
- **The propagation step has no upper bound.** Moving the rest into a weighted conditional Poincaré constant C_w, the needed value is $C_w \leq 39.3/138.2/476.3$, and the lower bound from a family of test functions is 15.4–15.6 (almost independent of $\hbar$) Reading . With the Lyapunov-function criterion (of the Bakry–Barthe–Cattiaux–Guillin type Cited , checked in a restatement in a secondary source), the family of Lyapunov functions that do not depend on the far end was $16.6/14.7/14.7$ times outside the needed value on the top face $x \equiv 0$. Measuring by duality the best of a split family extended to the far end goes beyond this leading term, but on the finest grid it stops at $215/355/654 = 5.5/2.6/1.4$ times the needed value Reading (grid values; the lower $\hbar$ goes, the more sensitive to the grid). The floor that holds for all Lyapunov functions is 31 (uniform in $\hbar$) Reading Lean `tilt_g1` (assumption: $\hat{\theta}\,\mathrm{Var} \leq \hbar b$), and the form of the criterion itself can get below the needed value.

The remaining step: a Lyapunov function with a coupling term between neighbours that lowers the finite-$\hbar$ correction. Grades: Reading and Assumption .

The scalar strand — the scalar form, and the Scalar-Strand Proposition

This form does not split the gradient into components: it applies Bakry–Émery to the convex bottom (the box on the valley side) and pays for the band at the top by the Gibbs step at each site. The condition is only $\ln k < \ell_{\mathrm{band}}$, and A_W, κ, b and c_H are not

needed, but k enters as $\ln k$, so it works only over a finite range of core lengths. The Poincaré constant of the core becomes the sum of $C_P^{(1)} \approx 50$–64 and a remainder (Rem_2), and, restricted to the first eigenfunction, the remainder only needs to be small, $R_2(f)/\int g^2 < \frac{1}{4}$. The Scalar-Strand Proposition turns this into a proposition with a table of assumptions.

Scalar-Strand Proposition (with a table of assumptions)

0+1-dimensional double well, $\varepsilon = \frac{1}{2}$, $x_c = 0.89$, core length $k = 20$, central lump $\{9, 10\}$, f the first non-zero eigenfunction. Under the rows of Table 3, at each node $\{3, 4, \dots, 10\} \cdot 10^{-4}$ of $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, the remainder quantity R satisfies $R \le 2A$, with $A = 2kP_v e^{-Ev} \int f^2$ and $2kP_v e^{-Ev} \le \frac{1}{4} e^{-3}$.

Lean `path5_conditional` (assumptions, named: the remainder inequality, the proxy ratio, the rarity, the arithmetic margin $c_0 + q_g + l_\mu \le -0.7$), `p5_assembly` (assumptions: the proxy ratio, the rarity, the threshold, arithmetic, monotonicity).

Table 3 The rows of the Scalar-Strand Proposition (five grades)

Row	Statement (one line)	Grade	Notes
Spectral regularity	Discrete spectrum, existence and regularity of eigenfunctions, simplicity of λ_1	Assumption	Regularity of the Neumann problem on a box with corners does not follow from standard theorems
Logarithmic Sobolev at one site of the band	Logarithmic Sobolev with a defect at one site of the band	Assumption	Not yet
Convex bottom and propagation in the band	Convex bottom and propagation in the band ($C_P^{(1)} \approx 50$–64)	Assumption	The paper argument is not placed in the book
Upper bound on λ_1	Upper bound on λ_1 uniform in k (Rayleigh quotient)	Reading	Interval-arithmetic scheme (1 run per node), not yet run
Localisation in the direction of exit	Estimate of localisation in the direction of exit	Assumption	A paper estimate. Interval-arithmetic scheme with 1 closed formula per node, not yet run. The IMS localisation formula is Cited (checked in a secondary source)
Rarity of the target	Rarity of the target $\mu(\mathrm{Rem}_v) \le e^{l_\mu(\hbar)}$, $l_\mu \le -4.97$	Reading	Floating-point transfer. Interval-arithmetic scheme with 7 runs per node, not yet run

Row	Statement (one line)	Grade	Notes
Rarity on non-convex fibres	Rarity on a non-convex fibre (the proportion of f^2 is at most e^{-7})	Assumption	Markov's inequality and transfer. Interval-arithmetic scheme with 2 runs per node, not yet run. The lower bound on the action difference ΔS_{nc} is not in interval-arithmetic form
Lifting the proxy ratio	Proxy ratio $c_1 \le c_*(\hbar)$ ($c_* \ge 5.39$, $\hbar \le 10^{-3}$)	Assumption	**The only assumption that lifts beyond the measured range**: a linear lift from the measured values for two-site lumps, 0.002–0.033, by +0.048 per step (tolerance ≥ 0.67)
Comparison of level sets	Comparison of level sets $\ln \Phi \le -4.209$	Reading	Interval-arithmetic scheme with 3 runs per node, not yet run
Arithmetic of the margin	Arithmetic of the margin $c_1 + \ln \Phi \le -3.7$	Proved here	Arithmetic
Enclosure between the nodes	Enclosure of $\hbar$ between the nodes	Assumption	Not yet. The range $[3 \cdot 10^{-4}, 10^{-3}]$ is point values at 8 nodes

The number of runs to move to interval arithmetic is 14 per node — upper bound on λ_1 1, localisation in the direction of exit 1, rarity of the target 7, rarity on non-convex fibres 2, comparison of level sets 3 — making 112 over the 8 nodes, plus the enclosures between the nodes, and not a single one has been run yet. Slightly beyond 10^{-3} the two conditions no longer hold together on the same window (they do not hold together at $1.35 \cdot 10^{-3}$ Reading), and it does not extend towards $\hbar \to 0$ either, because the estimate in the direction of exit is not uniform, by $\hbar^{-1/2}$. **The Scalar-Strand Proposition is a proposition with a table of assumptions on a finite range at the lower end of the window.**

The Feynman–Kac strand — the Feynman–Kac form

In the form that keeps the Feynman–Kac weight on the diagonal of the comparison, what is needed is that the logarithmic Sobolev constant of the whole core is $A_{\mathrm{core}} \lesssim a/\hbar$ with $a < 1.29/1.47/1.55$ (the question extends to the coefficient of $1/\hbar$). The lower bound from a family of test functions is $\hbar A_{\mathrm{core}} \approx 0.78$–$0.83$ Reading , a margin of 1.6–1.9 times. Splitting into radial chains, the chain constant is 0.54–0.56 times the needed value (on the model chain) Reading , and what remains is the quantity $K^2 = \sup |\nabla \psi|^2$ in the Helffer–Sjöstrand representation Cited (bibliographic only; the statement is unchecked, so this book treats it as Assumption), with threshold $1 + K^2 \le W$, that is, $K^2 \le W - 1 = 0.83$–0.86 (W is the ratio of the needed value to the chain constant).

- **Monte Carlo reading**: values from below after searching for starting points, $L = 0.4333 \pm 0.0067$ (10^{-3}) and 0.4163 ± 0.0079 ($3 \cdot 10^{-4}$); adding 3σ gives 0.453 = **0.54 times** the threshold 0.835 Reading (the maximum over starting points, not an upper

bound over the whole fibre = the assumption that the maximum over searched starting points suffices).

- **Paper upper bound**: a pathwise Grönwall-type upper bound counted at the same points along the same paths gives $U^2 = 798$ = about 10^3 times the threshold Reading . Most of the width comes from the step where the scalar Grönwall discards the direction of the gradient and reduces every direction at the softest rate. The version that controls the contraction by the smallest eigenvalue of the time-averaged matrix is false Lean `mbar_bound_false` (a two-dimensional counterexample, no assumptions).

The remaining step: an upper bound that keeps the direction (a matrix Lyapunov function along the path, or an estimate split into soft and hard subspaces).

The three side by side

Table 4 How far the three strands got (triples are $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$)

Strand	How far it got	Grade	Range of $\hbar$	The remaining step
2 Scalar	The Scalar-Strand Proposition (with a table of assumptions, one lifting assumption, 112 interval-arithmetic runs not yet made)	Assumption + Reading	8 nodes of $[3 \cdot 10^{-4}, 10^{-3}]$	Interval arithmetic at each node, enclosures between the nodes, making the lifting assumption explicit
3 Feynman–Kac	0.54 times the threshold by the Monte Carlo reading. The paper upper bound is 10^3 times the threshold	Reading	$10^{-3}, 3 \cdot 10^{-4}$	An upper bound that keeps the direction
1 Componentwise	Went beyond the leading term of the Lyapunov criterion, but 5.5 / 2.6 / 1.4 times the needed value. The floor for all Lyapunov functions is 31	Reading	Three points	A Lyapunov function with coupling terms

Table 4 is not a ranking of which strand is "close". The scalar strand has a form only at the lower end of the window, and only the componentwise strand and the Feynman–Kac strand have forms that cover the whole window. Read progress by the grade of each row.

06

The common wall — the curvature at one site at the top, $15/4$ against 4

The Hessian of the action of the core is, on the lattice $\varepsilon = \frac{1}{2}$, the tridiagonal matrix

$$\operatorname{Hess} S = \begin{pmatrix} 4+w_1 & -2 & & \\ -2 & 4+w_2 & -2 & \\ & \ddots & \ddots & \ddots \\ & & -2 & 4+w_k \end{pmatrix}, \qquad w_j = \varepsilon V''(z_j) = \frac{3z_j^2 - 1}{4} \quad (10)$$

(the neighbour term $(z_{j+1} - z_j)^2/(2\varepsilon)$ gives 4 on the diagonal and -2 next to it). Looking at one row, the diagonal is $4 + w_j$, and the sum of the absolute values of the couplings to the neighbours is $2 + 2 = 4$. At the top $z_j = 0$, $w_j = -\frac{1}{4}$, so

$$\text{diagonal} \;=\; 4 - \tfrac{1}{4} \;=\; \tfrac{15}{4} \;<\; 4 \;=\; \text{sum of couplings} \quad (11)$$

, and **diagonal dominance fails by $\frac{1}{4}$** Proved here . Row-by-row estimates (Gershgorin discs) only go as far as "smallest eigenvalue $\geq \min_j w_j = -\frac{1}{4}$", and in fact on the face $x \equiv 0$ where every site is at the top, the smallest eigenvalue of a lump of k sites with both ends fixed is $8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4}$, negative for $k \geq 8$ Proved here (the closed formula for the eigenvalues of $\operatorname{tridiag}(-2, 4, -2)$). To check by hand: at $k = 7$, $8\sin^2(\pi/16) \approx 0.304 > \frac{1}{4}$; at $k = 8$, $8\sin^2(\pi/18) \approx 0.241 < \frac{1}{4}$.

The three strands came back to this one-line shortfall in changed forms.

- However the Barta weight is chosen, the floor of the row of the one-site weight at the top is $F(0,0) = w(0) = -\frac{1}{4}$.
- The criterion of the uniform road of the componentwise strand was $A - 2B = -\frac{1}{4}$ with diagonal $A = 4 - \frac{1}{4}$ and coupling $B = 2$.
- The top face where the Lyapunov criterion of the componentwise strand stopped is exactly the face $x \equiv 0$ where the gradient of the action vanishes.
- In the Feynman–Kac strand, the coefficient of $1/\hbar$ lived in the radius from the corner at the top (the non-convex direction of the plateau).

Why the wall is universal (one line). Tools whose entrance is convexity (a positive lower bound on the Hessian) — Bakry–Émery, Brascamp–Lieb, the Barta rows, diagonal dominance — must, since the Hessian really has a negative direction on the plateau at the top, pay this $\frac{1}{4}$ somewhere with the rarity (entropy) of the plateau, and $\hbar$ enters that payment. A detailed study of this view, and other viewpoints that do not rely on convexity, are treated in Part 24 (24-06).

07

The table of the five grades — the rows of the sector framework and the route of the window log-Sobolev inequality

Table 5 The rows of this chapter ("—" in the range of $\hbar$ marks an assumption row not shown at any $\hbar$; the numbers in the row column are the numbers of the dependency graph in § 08)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[01]	Marginal log-Sobolev input for the window (the form of logarithmic Sobolev for the marginal on the window)	Assumption	§ 02 (4)	—	—
[02]	Bridge volume-locality input (volume locality on the bridge)	Assumption	§ 02 Volume locality	—	—
[03]	Barta rows $\kappa = 3/1000$, $b = 493/1000$	Assumption	§ 02 Sector composition (the paper argument is not placed in the book)	Independent of $\hbar$	—
[04]	Vanishing of the Bochner boundary terms and regularity of eigenfunctions	Assumption	§ 02 Sector composition	—	—
[05]	The lifting assumption of the Scalar-Strand Proposition and the other Assumption rows of Table 3	Assumption	§ 05 Table 3	—	—
[07]	Integrated Bochner Formula	Cited	Integrated Bochner Formula (Proposition 1.3 of Ledoux's lectures; checked in the original text)	Independent of $\hbar$	—
[08]	$\ell_{\min} = 38.3/121.9/406.8$	Reading	§ 02 Volume locality (transfer matrix, including extrapolation)	Three points	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[09]	Upper bound $1857/2794/9106$ and lower bound $1.71/1.74/2.59$ on A W	Reading	§ 03 (including estimates / family values)	Three points	—
[10]	Sector sum	Proved here	§ 02	Independent of $\hbar$	`var_split_cells` (no assumptions), `cell_sum_sup` (assumptions: $w_i, E_i \geq 0$, $C_i \leq C_{\max}$)
[11]	Gibbs decomposition: the Gibbs variational inequality (2) and (3)	Proved here	Proof in § 02	Independent of $\hbar$	`gibbs_ln2_cell` (assumptions: $p \geq 0$, $\ell \geq 0$, $pe^{\ell} \leq 1$)
[12]	Componentwise localisation (5) $\Longleftarrow$ [01]	Proved here	§ 02	Independent of $\hbar$	`comp_sum_le` (assumptions: $0 < \ell \leq \ell_j$, the inequality for each component)
[13]	The scalar form is not uniform in k	Proved here	§ 02 Componentwise localisation	Independent of $\hbar$	`scalar_not_uniform` (assumption: $p > 0$)
[17]	Sector composition (6) and the closing condition (7)	Proved here	§ 02	Independent of $\hbar$	`comp_close` (assumptions: $X > 0$, $b, A \geq 0$, $\ell > 0$, Barta, (5), Bochner)
[18]	The needed value (8)	Proved here	§ 03 (the values are arithmetic on the measured ℓ_min, hence Reading)	Three points	—
[15]	Localised Bochner (9)	Proved here	§ 03	Independent of $\hbar$	`negw_le_localized`, `bochner_localized` (the assumptions are in the text of § 03)
[16]	Optimising the Gibbs tilt	Proved here	§ 03	Independent of $\hbar$	`tilt_cost_le` (up to the upper bound on the cost at t = ℓ − 5.2; assumptions: $b \geq 0$, $L \leq 1/178$)

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
[19]	The needed value with both combined $25.1/90.3/312.3$	Reading	§ 03 Table 2	Three points	`judge_i_inside` (for two values of $\hbar$, closes if $A \le 22$; assumptions: lower bounds on the measured ℓ, $A \le 22$)
[20]	With mismatched thresholds $\ell = 0$	Proved here	§ 04	Independent of $\hbar$	—
[21]	Scalar-Strand Proposition	Assumption	§ 05 the scalar strand (a construction on the assumption lifting the proxy ratio and the row of spectral regularity)	8 nodes of $[3 \cdot 10^{-4}, 10^{-3}]$	`path5_conditional`, `p5_assembly` (the assumptions are in § 05)
[22]	How far the componentwise strand got (1.4–5.5 times the needed value, floor 31)	Reading	§ 05 the componentwise strand	Three points	`tilt_g1` (assumption: $\hat{\theta}\,\mathrm{Var} \le \hbar b$)
[23]	How far the Feynman–Kac strand got (reading 0.54 times, paper upper bound 10^3 times)	Reading	§ 05 the Feynman–Kac strand	Two points	`mbar_bound_false` (counterexample, no assumptions)
[24]	Diagonal dominance fails by $\frac{1}{4}$ at one site at the top	Proved here	§ 06 (11)	Independent of $\hbar$	—
[06]	Window Log-Sobolev Threshold Proposition $A_W < A_{\max}$	Assumption	The route of the window log-Sobolev inequality	—	—

The "Proved here" rows of Table 5 are all either algebra on assumption rows or general inequalities whose proofs are placed in the text. They do not mean that a lower bound on the gap of the core has been shown unconditionally.

08

The dependency graph

The format is the same as 23-05 § 07, and **the numbers to the right of "⟸" are always placed smaller than the number on the left**.

```
[01] Assumption        marginal log-Sobolev input for the window
[02] Assumption        bridge volume-locality input
[03] Assumption        Barta rows κ = 3/1000, b = 493/1000
[04] Assumption        Bochner boundary terms and regularity
[05] Assumption        Assumption rows of the Scalar-Strand Propositi
[06] Assumption        Window Log-Sobolev Threshold Proposition: A_W
[07] Cited             Integrated Bochner Formula
[08] Reading           ℓ_min = 38.3 / 121.9 / 406.8
[09] Reading           upper and lower bounds on A_W
[10] Proved here       sector sum
                        Lean var_split_cells, cell_sum_sup
[11] Proved here       Gibbs decomposition (Gibbs variational inequal
                        Lean gibbs_ln2_cell (assumptions: p ≥ 0, ℓ ≥
[12] Proved here       componentwise localisation (5) ⟸ [01][10][11]
                        Lean comp_sum_le
[13] Proved here       the scalar form is not uniform in k
                        Lean scalar_not_uniform (assumption: p > 0)
[14] Reading           volume locality: ℓ_min does not depend on k ⟸
[15] Proved here       localised Bochner (9) ⟸ [04][07]
                        Lean negw_le_localized, bochner_localized
[16] Proved here       optimising the Gibbs tilt
                        Lean tilt_cost_le (up to the upper bound)
[17] Proved here       sector composition (6), closing condition (7)
                        Lean comp_close (assumptions: Barta, (5), Boc
[18] Proved here       needed value (8) ⟸ [17]
[19] Reading           needed value 25.1 / 90.3 / 312.3 ⟸ [08][15][1
                        Lean judge_i_inside (for two values of ħ, up
[20] Proved here       with mismatched thresholds ℓ = 0
[21] Assumption        Scalar-Strand Proposition ⟸ [05]
                        Lean path5_conditional, p5_assembly
[22] Reading           how far the componentwise strand got ⟸ [19]
[23] Reading           how far the Feynman–Kac strand got
[24] Proved here       diagonal dominance fails by ¼ at the top
[25] Proved here       positive lower bound on the core gap uniform i
```

How to check. The rows with ⟸ are [12] ([01][10][11]), [14] ([02][08]), [15] ([04][07]), [17] ([03][04][07][12][14]), [18] ([17]), [19] ([08][15][16][18]), [21] ([05]), [22] ([19]) and [25] ([06] [17]). In each the numbers on the right are smaller than the one on the left, so there is no circle.

What to read. The conclusion [25] depends, through [17], on the structural steps ([10]–[14]) and the input assumptions ([01]–[04]), and further on the assumption row [06] (the one remaining proposition). The candidates to support [06] are the three strands [21] [22][23], and none of them shows [06] ([21] is a proposition with a table of assumptions at the lower end of the window, [22][23] are readings). [20] is a checking row: unless [01] and [03] are measured at the same threshold, [12] is empty. [24] explains where the three strands stopped, and is a premise of no conclusion.

We also check that the statements of the assumption rows do not contain the quantity of the conclusion. The quantity of the conclusion is the gap λ of the measure of the core. [01] and [06] are logarithmic Sobolev constants of the marginal of the marks on the window, [02] the remainder of the transfer operator of a scrap, [03] the lower bound on the weighted rows, [04] the boundary terms and regularity, [05] the remainder quantity and the proxy ratio. None of them contains λ.

09

To the next chapters

The full list of detour plans tried in the route of the window log-Sobolev inequality (twenty-two, plus two born along the way, and their derivatives) and the types of breakage that appeared repeatedly are treated in 23-10. The construction that controls the core constant in three steps from the side of the box is 23-11. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter has said how far the core row of the saddle-tube condition of its first step could be built. A viewpoint that handles the negative curvature at the top without relying on convexity is treated in Part 24 (24-06).

What this chapter can say

What is said	What is not said
Proved here The constructions of the sector sum, the Gibbs decomposition, componentwise localisation and sector composition in the sector framework, the closing condition (7), the needed value (8), the localised Bochner and the Gibbs tilt, the consequence of the threshold mismatch, the failure of diagonal dominance at the top Lean (algebra)	The inputs (the marginal log-Sobolev input for the window, the bridge volume-locality input, the Barta rows, regularity)
Reading The needed value $25.1/90.3/312.3$, the gap of 29–74 times to the upper bound on A_W, how far the three strands got	The one remaining proposition (the Window Log-Sobolev Threshold Proposition). Hence the core row (the core Poincaré uniformity assumption) and the saddle-tube condition

What is said	What is not said
Assumption The Scalar-Strand Proposition (8 nodes of $\hbar \in [3 \cdot 10^{-4}, 10^{-3}]$, with a table of assumptions)	Interval arithmetic for the Scalar-Strand Proposition, between the nodes, the upper half of the window and $\hbar \to 0$

Sources and reproduction

Item	Kind	Source / tools
Integrated Bochner Formula	Cited	M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (the integrated form of Bochner's formula; text checked).
Lyapunov functions and criteria for local Poincaré	Cited	D. Bakry, F. Barthe, P. Cattiaux, A. Guillin, Electron. Commun. Probab. 13 (2008); D. Bakry, P. Cattiaux, A. Guillin, J. Funct. Anal. 254 (2008) 727–759 (Theorem 3.6 checked in a restatement in a secondary source).
The IMS localisation formula	Cited	H. L. Cycon, R. G. Froese, W. Kirsch, B. Simon, *Schrödinger Operators*, Springer 1987 (checked in a restatement of § 3).
The Helffer–Sjöstrand representation	Assumption	B. Helffer, J. Sjöstrand, J. Stat. Phys. 74 (1994) 349–409 (bibliographic only; statement unchecked).
The Gibbs variational inequality, sum over sectors, componentwise localisation, composition	Proved here, Lean	Proofs in § 02 of the text. `var_split_cells`, `cell_sum_sup`, `gibbs_ln2_cell`, `comp_sum_le`, `comp_close`, `scalar_not_uniform`.
The two corrections on the needed side	Proved here, Reading, Lean	§ 03. `negw_le_localized`, `bochner_localized`, `tilt_cost_le`, `judge_i_inside`. The rarity exponents from the transfer matrix (midpoint rule, the smaller of grids 600 and 1200).
Numbers of the three strands	Reading, Lean	The componentwise strand: duality and the primal problem on floating-point grids (grids 40–160), `tilt_g1`. The scalar strand: floating-point transfer, `path5_conditional`, `p5_assembly`. The Feynman–Kac strand: Monte Carlo of the adjoint estimator (32 paths, search for starting points), `mbar_bound_false`.

Words used in this chapter — sector framework, sector, mark, site window, defect set, rarity exponent, Gibbs variational inequality, componentwise localisation, closing condition, needed value, choice of tools and structure, threshold mismatch, the three strands, Scalar-Strand Proposition, lifting assumption, the common wall, diagonal dominance. Their meanings are in A-02 Glossary.

Next chapter to read: 23-10 The list of detour plans — types of breakage — puts the detour plans into one table with "the line that killed it", and reads the three sides (the needed side, the form, the floor) and the types of breakage that appeared repeatedly.

The list of detour plans — types of breakage

This chapter is a map of open questions. § 04 is this book's own investigation; § 07 is guide material.

What this chapter shows — We put into one table the twenty-four detour plans tried (the list of detour plans), arranged by which factor of the closing condition of the route of the window log-Sobolev inequality, $b(A_W c_H + D_W + \ln 2) < \kappa \ell_{\min}$, they move. Each row gives "the line that killed it" (for a dead plan) or "the remaining assumption" (for a live plan), and whether the factor moved comes from the structure or from the choice of tools. From the table we extract seven types of breakage that appeared repeatedly, and for each type give the plans it caught and a question with which readers can test their own plans.

Prerequisite chapters — 23-09 § 03 (the closing condition, the needed value, Table 1 of where the constants come from), 23-09 § 06 (the common wall), 23-07 § 02 (the position of the route of the window log-Sobolev inequality).

Said first — The level is [Y3]–[Res], and § 01, § 02 and the names of the types and the questions in § 04 can be read at [Y3]. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter tidies up after the plans tried towards the core row of the saddle-tube condition of its first step. None of the plans has shown the saddle-tube condition. All numbers are values on the 0+1-dimensional double well on the lattice $\varepsilon = \frac{1}{2}$, and triples are in the order $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$.

The order of this chapter

1. Summary — a section readable at [Y3]
2. The starting point — the closing condition, and three sides
3. The list of detour plans (Table 1, the count)
4. Types of breakage — seven types, and questions to test your own plan (Table 2)
5. Where the survivors went — into the three strands (Table 3)
6. The table of the five grades — the main lemmas for each type (Table 4)

01

Summary

The one remaining proposition of the route of the window log-Sobolev inequality was that the logarithmic Sobolev constant A_W of the site window lies within the needed value $A_{\max}$ (23-09 § 03). A detour plan moves one side of this inequality, or the form of the inequality itself.

- **The results of the twenty-four.** The twenty-two listed first split into 13 dead, 2 on hold, 3 conditional and 4 alive. Of the 4 alive, 2 (detour plan "localising negative curvature" and detour plan "optimising the Gibbs tilt") enter every later form as parts, and the other 2 (detour plan "scalar payment" and detour plan "replacing the functional inequality"), together with detour plan "paying with a height function", which was born along the way, became the three strands of 23-09.
- **The side of tools and the side of structure.** What moved the needed value a lot were the plans that corrected the choice of tools (localising negative curvature and optimising the Gibbs tilt: $0.20 \to 90.3$ at 10^{-3}, about 460 times), while the plans that moved factors coming from the structure (the depth of defects b, the rarity of the defect set ℓ) all stopped under a ceiling of 1.15 times.
- **Seven types of breakage.** One source (the curvature at one site at the top); a ceiling for structural factors that enter as ratios; paying for the same place twice; mixing up currencies; the address of $1/\hbar$; a mismatch between the direction of fluctuation and the soft direction; transfer. Of the 13 dead plans, 12 fall under one of these.

02

The starting point — the closing condition, and three sides

Copied from 23-09. The sector framework, which controls the gap λ of the core Q_k uniformly in the core length k, combines the three inequalities of Barta, componentwise localisation and Bochner, and closes under the following condition (23-09 (7)).

$$b\,(A_W c_H + D_W + \ln 2) \;<\; \kappa\,\ell_{\min}, \qquad A_W \;<\; A_{\max} = \frac{\kappa\ell_{\min}/b - D_W - \ln 2}{c_H} \quad (1)$$

Of the six constants, $\kappa = 3/1000$, $b = 493/1000$ and $\ell_{\min}$ are quantities of the **structure**, determined by the model and by how the defect set is defined, while $c_H = \frac{1}{4}$ (the negative curvature paid uniformly over the whole core) and $\ln 2$ (the Gibbs tilt fixed at $t = \ell$) are quantities of the **choice of tools** (23-09 Table 1). In the original form the needed value was negative / 0.20 / 7.13, while the upper bound on A_W was $1857/2794/9106$ Reading (including estimates) and the lower bound $1.71/1.74/2.59$ Reading (values from a family of test functions).

What a detour plan is. To close (1), one raises the needed value, lowers the upper bound, or changes the form of the condition. In this chapter we split the plans into three sides by that direction.

Figure 1 The three sides (three columns of text)

The needed side	Form	The insufficient side (floor)
Make $A_{\max}$ larger. Make κ and $\ell_{\min}$ on the right of (1) larger, and b, c_H and $\ln 2$ on the left smaller	Change the functional form of (1). A construction in which A_W does not enter the condition; a construction with Poincaré instead of logarithmic Sobolev	Lower the upper bound on A_W. Reduce the structural loss (floor) of the construction of the upper bound
Localising negative curvature, optimising the Gibbs tilt, raising the threshold, layering defects by depth, narrowing the defect set, defects as long clusters, non-linear Barta weight, extension to the classical limit, Hölder and hypercontractive steps, paying with a height function	Scalar payment, detour plan "Agmon-type payment", detour plan "re-examining downstream needs", replacing the functional inequality	Paying for the summit band, coarse and propagation steps at once, log-Sobolev with clusters fixed, perturbing only the summit band, known uniform log-Sobolev, finite-volume criteria, Glauber log-Sobolev, two-step cluster expansion, one- or two-site windows, localising at the cluster core

The other axis is the character of the factor moved. Factors of the **structure** are quantities determined by the model, and moving them requires changing the definitions of the defect set or the weights. Factors of the **tools** are choices of steps in the proof, and can be changed with the same sets. The "Structure / tools" column of Table 1 is this axis, and "tools → structural floor" marks plans that changed tools but stopped on hitting a structural floor.

03

The list of detour plans

The words of the verdicts in Table 1 mean the following. **Alive** = in that form it came within the needed value (still not a proof of (1); remaining assumptions exist). **Conditional** = the remaining assumptions can be named. **On hold** = not decided on paper. **Dead** = the killing test put it outside the needed value, or no form came out. The line that killed it shrinks to one line the fact that stopped the plan, and all numbers are readings from floating-point grids or transfer matrices Reading (lower bounds are values within a family of test functions).

Table 1 List of detour plans (triples are $\hbar = 3\cdot10^{-3}/10^{-3}/3\cdot10^{-4}$; "Type" is the type of breakage in § 04)

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Localising negative curvature	Localise the negative curvature of Bochner to the defect set	c_H (needed side)	Tools	Alive (absorbed as a part)	Together with optimising the Gibbs tilt, needed value $25.1/90.3/312.3$. Remaining: regularity of the Bochner step; measuring A_W with marks at the same threshold as the defects	Double-payment type, 23-09 § 03
Optimising the Gibbs tilt	Set the Gibbs tilt to $t^* = \ell - 5.10$	$\ln 2$ (needed side)	Tools	Alive (absorbed)	Cost $b\ln 2 = 0.342 \to 0.0183$. No remaining assumption (a one-line inequality)	Double-payment type, 23-09 § 03
Raising the threshold	Raise the threshold to $1/\sqrt{3}$, and the remainder of localisation δ to 0	δ, κ (needed side)	Tools	Conditional	The needed value is in effect $A_W \lesssim 4\ell$: $128/452/1604$ against the upper bound $1629/2444/8153$ (12.8 / 5.41 / 5.08 times outside). Remaining: the tail outside the window	Type of the ceiling on structural factors
Layering defects by depth	Split the depth of defects b into layers	b (needed side)	Structure	Dead	Factor $1.05/1.01/1.004$. Plans that move only b and κ without changing the set have $A_{\max} < 4\ell$ (ceiling $1.15/1.04/1.01$ times the current value)	Type of the ceiling on structural factors
Narrowing the defect set	Narrow the defect set to the set where the rows fall below κ	$\ell_{\min}$ (needed side)	Structure	Dead	The set became 1.40 times rarer, but the rarity on the Barta side acts only multiplied by $\kappa/(\beta_0 - \kappa) \approx 1/47$ (factor $1.11/1.05/1.02$)	Type of the ceiling on structural factors

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Defects as long clusters	Move to the road that defines defects by lumps of length 8 or more	$\ell_{\min}$ (needed side)	Structure	Dead	For short lumps of length ≥ 2, the best lower bound on the Barta rows is negative (-0.0096 at length 2)	Summit-curvature type
Paying for the summit band	Pay for the band at the top as a defect	Floor	Structure	Dead	The rarity of the band and the tail through the band are both $\propto 1/\hbar$ and cancel, and the tail of paths that touch the band and leave equals the whole tail	Type of the address of $1/\hbar$
Scalar payment	Pay for the defects of the sectors in scalar form	Form	Tools	Alive (conditional)	The condition is only $\ln k < \ell_{\text{band}}$; AW, κ, b and c_H are not needed. Remaining: the rows of the table of the Scalar-Strand Proposition	23-09 the scalar strand
Non-linear Barta weight	Make the Barta weight non-linear	b (needed side)	Tools → structural floor	Dead	The worst row rises $-0.4865 \to -0.25$, but the floor at the top $F(0,0) = w(0) = -\frac{1}{4}$ is there for every one-site weight (factor $1.02/1.006/1.002$)	Summit-curvature type
Agmon-type payment	Pay the weight of the defects with the gradient of the true eigenfunction alone (Agmon type)	Form	Tools → structural floor	Dead	Agmon's gain is only where the curvature has a positive margin, and the defect set is the one place where the curvature is negative = the most permissive side	Summit-curvature type
Re-examining downstream needs	Re-examine what is needed downstream	Form	Tools	On hold → absorbed into scalar payment	Downstream, "uniform in k, polynomial in $\hbar$" suffices, but the only tools that give the input uniformly in k were of logarithmic Sobolev type	23-09 the scalar strand
Coarse and propagation steps at once	Do the coarse step and the propagation step at once rather than as a product	Floor	Tools	Conditional	The coarse step is exhausted, with ratio $1.016/1.005/1.003$ to the lower bound. Remaining: the weighted conditional Poincaré constant C_w of the propagation step	23-09 the componentwise strand
Replacing the functional inequality	Poincaré instead of logarithmic Sobolev / modified LSI / Φ-entropy	Form	Tools	Poincaré alive (conditional), the other two dead	Modified LSI only moves $\ln(1/\pi_*)$ to the energy side, and Φ-entropy cannot beat the Poincaré endpoint	Transfer type, the scalar strand
Log-Sobolev with clusters fixed	Logarithmic Sobolev for the conditional measure with the state of the lumps fixed	Floor	Tools	Dead	After conditioning, the measure of the defect set is 0 or 1, and $\ell = 0$	Currency-mix-up type
Perturbing only the summit band	Bakry–Émery outside the window, Holley–Stroock perturbation only for the band at the top	Floor	Tools	Dead	The one-site oscillation $\mathrm{osc}/\hbar = 3.7/11/37$ enters the exponent and is multiplied by the length along the chain; paid in the defect term it is 2000–4900 times the allowance	Type of the address of $1/\hbar$
Known uniform log-Sobolev	Known theorems on uniform logarithmic Sobolev in one dimension	Floor	Tools	Dead	The uniform constant is a finite-volume constant for a box covering the correlation length (length $\gtrsim 230$–810), and is not below the window constant	Transfer type
Finite-volume criteria	Finite-volume criteria (of the Martinelli–Olivieri and Lu–Yau types)	Floor	Tools	Dead	Same as detour plan "known uniform log-Sobolev": the box of the criterion must cover the correlation length	Transfer type
Glauber log-Sobolev	Known logarithmic Sobolev for Glauber dynamics	Floor	Tools	Dead	The coarse step is not the limit, and the Glauber constant ≥ 39–400 is if anything worse	—
Two-step cluster expansion	Two-level cluster expansion, block spins	Floor	Tools	Dead	The pin of the largest path stays at the top, and the tail gain ≤ 1.8 (10^{-3}), ≤ 3 ($3 \cdot 10^{-3}$) falls short of the needed 5–13 times	Transfer type

Detour plan	Plan (one line)	What it moves	Structure / tools	Result	The line that killed it (dead) / remaining assumption	Type, chapter
Extension to the classical limit	Take $\hbar$ below the window (the classical limit)	Needed side (range)	Structure	On hold	Removing the band, the contraction of the boundary map is $0.976/0.952/0.929 < 1$ (band width $0.1/0.2/0.3$), and the tail is bounded independently of $\hbar$. Remaining: the self-loop lemma for the boundary and the band lemma (detour plan "paying for the summit band" is dead)	—
One- or two-site windows	Make the window one or two sites (half rows)	Floor	Tools	Conditional (small gain)	The effect is about 2/3, a factor gap of $8.45/3.59/3.38$. The propagation tail remains	Transfer type
Localising at the cluster core	Localise at the core of the lump instead of the window	Floor	Tools	Dead	Even in the limit with the window tail set to 1, $4.2/1.8/1.7$ times outside (the origin of the floor does not depend on how the window is chosen)	Transfer type
Hölder and hypercontractive steps	Replace the Gibbs step by Hölder and hypercontractivity	The needed side	Tools	On its own, dominated by detour plan "raising the threshold"	$\sqrt{k}$ does not appear, but A enters $\ln C^2$ linearly and the ceiling is the same, $\approx 4\ell$ (needed value 0.80–0.91 times that of raising the threshold). Its merit is that measurability of the set of marks is not needed	Type of the ceiling on structural factors
Paying with a height function	Pay the negative curvature by a function of height $h = (-w(z_j))_+$ (the curvature at each point by the rarity of that point)	The needed side	Tools	Conditional	Needed value $537/1754/6052$ (3.8–4.2 times that of raising the threshold). Dead if Rothaus's lemma is used as it is. Remaining: the propagation tail	The componentwise strand; together with detour plan "Hölder and hypercontractive steps", the Feynman–Kac strand

Counting. The first twenty-two (from localising negative curvature to localising at the cluster core in Table 1) are 4 alive (localising negative curvature and optimising the Gibbs tilt absorbed as parts; scalar payment and the Poincaré form of replacing the functional inequality conditional), 3 conditional (raising the threshold, detour plan "coarse and propagation steps at once", detour plan "one- or two-site windows"), 2 on hold (re-examining downstream needs, absorbed into scalar payment; detour plan "extension to the classical limit") and 13 dead. The 13 dead split into 8 on the side of the floor (paying for the summit band, log-Sobolev with clusters fixed, perturbing only the summit band, known uniform log-Sobolev, finite-volume criteria, Glauber log-Sobolev, two-step cluster expansion, localising at the cluster core), 4 on the needed side (detour plan "layering defects by depth", detour plan "narrowing the defect set", detour plan "defects as long clusters", detour plan "non-linear Barta weight") and 1 on form (Agmon-type payment). Of the two born along the way, Hölder and hypercontractive steps is, on its own, dominated by raising the threshold, and paying with a height function is conditional and entered the componentwise strand; the form combining it with Hölder and hypercontractive steps became the Feynman–Kac strand.

Contrasting the side of tools with the side of structure. Putting the "Structure / tools" column of Table 1 next to the results column, every row that moved the needed value a lot is "tools". Localising negative curvature and optimising the Gibbs tilt kept the same defect set and only corrected how the negative curvature is paid and how the Gibbs tilt is chosen, and moved the needed value from negative / 0.20 / 7.13 to $25.1/90.3/312.3$ (about 460 times at 10^{-3}, about 44 times at $3 \cdot 10^{-4}$, from negative to positive at $3 \cdot 10^{-3}$) Reading . The "structure" rows (layering defects by depth, narrowing the defect set, defects as long clusters, paying for the summit band) and the rows that hit a structural floor (non-linear Barta weight, Agmon-type payment), on the other hand, did not exceed a factor 1.15, or no form came out. Most of the loss on the needed side lay not in the structure of the model but in the choice of tools. Conversely, the gap that remains after the tools have been fully corrected (5–13 times in the form of raising the threshold) goes back to the structural floor of the summit-curvature type in § 04.

▶ List of derivatives (forms tried within the strands, collapsed)

04

Types of breakage

Lining up the lines that killed the plans in Table 1 and Table 1′, the same reasons appear again and again in changed forms. We group them into seven types. For each type we give the plans it caught, a paragraph on why plans break in that way, and a question to test your own plan. The types are not mutually exclusive, and one plan can fall under two types.

The summit-curvature type — one source (the curvature at one site at the top)

Plans caught: defects as long clusters, non-linear Barta weight, Agmon-type payment, and the uniform road and the Lyapunov criterion without sectors in Table 1′.

In one row of the Hessian of the action of the core, at the top the diagonal $4 - \frac{1}{4} = \frac{15}{4}$ falls $\frac{1}{4}$ short of the sum of the couplings to the neighbours 4 (23-09 (11)). This shortfall appears with the same value in the Barta weight rows, in the contraction criterion for chains and in the Lyapunov-function criterion, on the top face $x \equiv 0$ (every site at the top). For example, the Barta row of a one-site weight $\psi_j = \phi(z_j)$ has the form $F(z, n) = w(z) + (\text{log-derivative of the weight}) \times (\text{drift})$, and at the top $z = 0$ with neighbours $n = 0$ the drift $\varepsilon V'(0) + 8(0 - 0)$ is 0, so however the weight is chosen Proved here

$$F(0, 0) = w(0) = -\tfrac{1}{4} \ (2)$$

(a one-line computation on the same assumption row as the Barta rows of 23-09) Lean `floor_top` (assumption: weight value $/= 0$). The Agmon type in Agmon-type payment is a tool that gains where the curvature has a positive margin, and the defect set is exactly where the curvature is negative, so the coefficient of gain there is ≤ 0 Lean `agmon_no_gain_on_negative_curvature` (assumptions: curvature $w \leq 0$, $\lambda \geq 0$ and others; the framework of Agmon estimates is not placed in the book). In defects as long clusters, defining defects by long clusters made the rows of short clusters of length 2 or more negative, and the same shortfall came out on the side of short clusters.

Question to test: what does your plan pay on the top face $x \equiv 0$? If the quantity paid there is a constant independent of $\hbar$, the shortfall of $\frac{1}{4}$ remains as it is. Only the rarity (entropy) of the plateau at the top can pay for this shortfall, and $\hbar$ enters that payment.

The type of the ceiling on structural factors — structural factors that enter as ratios have a ceiling

Plans caught: layering defects by depth, narrowing the defect set, Hölder and hypercontractive steps, and the ceiling of raising the threshold.

Raising the threshold, with remainder of localisation $\delta \geq 0$, remaining coefficient γ, Gibbs tilt t and $L(t) = \ln(1 + e^{t-\ell})$, the closing condition is $b(A\delta + D_W + L(t)) < \kappa(t - A\gamma)$ (the form of localising negative curvature, optimising the Gibbs tilt and raising the threshold).

Ceiling lemma. If $0 < \kappa < b$ and $\delta, D_W \geq 0$, any $A \geq 0$ satisfying this condition satisfies $A\gamma < \ell$.

Proved here Proof. The left-hand side is at least $bL(t) > 0$, so $t - A\gamma > 0$. From $b > \kappa$, $b(t - A\gamma) > \kappa(t - A\gamma) > bL(t) \geq b(t - \ell)$ ($\ln(1 + e^x) \geq x$). Dividing by b gives $A\gamma < \ell$. ∎ Lean `ceiling_same_set` (assumptions: $0 < \kappa < b$, $t - A\gamma > 0$, $t - \ell \leq L$, the closing condition)

Plans that move only b and κ without changing the defect set (hence ℓ) cannot push the needed value above ℓ/γ (4ℓ in the form of raising the threshold). The needed value in the form of raising the threshold is already at $1/1.15/1/1.04/1/1.01$ of the ceiling Reading , and the $1.05/1.01/1.004$ times of layering defects by depth lies within it. In narrowing the defect set, which changes the set, the rarity on the Barta side (ℓ_B of the set where the rows fall below κ) enters the condition only multiplied by $\kappa/(\beta_0 - \kappa) \approx 1/47$ Assumption (the paper argument is not placed in the book) Lean `ceiling_split`, and making the set 1.40 times rarer stopped at a factor of 1.11. Hölder and hypercontractive steps changed the tool of the Gibbs step, but A enters in the same form, so the ceiling is the same.

Question to test: does your plan change the defect set? If not, the needed value does not exceed ℓ/γ. If it does, first write down which coefficient of the condition the rarity of the new set is multiplied by when it enters.

The double-payment type — the same place was paid for twice

Plans caught (on the side that fixed it): localising negative curvature, optimising the Gibbs tilt, and the construction of the window in Table 1′.

The Bochner step of the original construction paid the negative curvature $-\frac{1}{4}$ over the whole core X. But negative curvature exists only for $z > -1/\sqrt{3}$, and that region is inside the defect set already paid by the Gibbs step as Y. Paying the same place again on the X side made the coefficient of A_W equal to $bc_H = 0.123$, through the amplification $b/\kappa \approx 164$. Localisation (localising negative curvature) brings the coefficient down to 0.0038 (23-09 (9)). Fixing the Gibbs tilt at $t = \ell$ (optimising the Gibbs tilt), and the $(1 + \sqrt{C_W})^2$ in the construction of the propagation step (counting the same quantity twice), are of the same type. This is a type that has a large effect when fixed.

Question to test: is the quantity paid in that step on a set already paid for by an earlier step? Drawing the places where the two steps pay in the same figure makes it visible.

The currency-mix-up type — speed of exit and rarity

Plans caught: detour plan "log-Sobolev with clusters fixed", the form that pays on the band by rarity and the double threshold in Table 1′, and the naive decomposition of the variance in 23-11 § 04.

What a Poincaré inequality measures is the speed of exit (the ratio of the variance to the Dirichlet form), and what can pay for rarity $\mu(B) \le e^{-\ell}$ is the currency of entropy (23-09 (3): $\ell \int_B g^2 \le \mathrm{Ent} + \ln 2 \int g^2$). The ratio of the variance to the Dirichlet form does not change when the function is multiplied by a constant, so putting in a function supported on a rare set makes the rarity cancel between numerator and denominator. "The fluctuations are small because the path sticks to the band" is an intuition about the variance, and what the step that pays for rarity needs is an estimate of entropy. Log-Sobolev with clusters fixed is more direct: if the defect set is determined by the σ-algebra of the conditioning, the measure of that set under the conditional measure is 0 or 1 Proved here (the definition of conditional probability), and no rarity ℓ is left to pay with.

Question to test: does the step that pays for the rarity $\mu(B)$ pass through entropy (or an inequality of the same type)? If it uses only the variance and the Dirichlet form, rarity has no effect. Also check whether rarity is still left after conditioning.

The type of the address of 1/ħ — the address of $1/\hbar$

Plans caught: paying for the summit band, detour plan "perturbing only the summit band", and in Table 1′ the linear collective coordinate, the sector-split chain, the multi-stage cut at the wall, the chain of wall marks, and the chain with the mean height added.

What the Feynman–Kac strand needs is the coefficient of $1/\hbar$ in the logarithmic Sobolev constant of the core. If one mistakes which quantity that $1/\hbar$ lives in, the plan only moves $1/\hbar$ to another place. Three facts came up again and again.

- **$1/\hbar$ lives not in $\sum z$ but in the radius.** On the plateau at the top, the action changes not in the direction of the linear collective coordinate $\sum_j z_j/\sqrt{k}$ but along the radius from the corner at the top (the direction of $\sum_j z_j^2$), and its length is the crossing length $\ell_\times \approx 0.34/\hbar$ Reading . In the two-scale split into the collective coordinate and the transverse directions, the smallest curvature in the directions orthogonal to the collective coordinate on the top face, $4 - 4\cos(2\pi/(k+1)) - \frac{1}{4}$, becomes negative at $k = 17$ Proved here (the closed formula for tridiagonal eigenvalues) Lean `transverse_fails_k17` (no assumptions), so convexity holds only for $k \le 16$.
- **Cutting into sectors only allocates $1/\hbar$.** The chain rule of entropy $\mathrm{Ent} = \mathrm{Ent}(\text{marginal of the marks}) + \mathbb{E}\,\mathrm{Ent}(\text{inside the sector})$ is an identity Proved here Lean `chain_rule_cells` (assumptions: the chain rule, upper bounds on the two terms), and cutting the plateau at the position of the wall allocates $\approx 0.8/\hbar$ between the side of the marks and the side of the sectors Reading . The sum does not decrease.
- **A mark represents only the one step that crosses the face.** In the chain of wall marks the constant of the chain itself becomes small, but a mark does not distinguish positions on the plateau and represents only the step across the face, so $1/\hbar$ moved into the constant $\ge 13/26/66$ that compares the mark chain with the original measure Reading . Whether a variable added to the chain is good or bad is decided by whether its rate function is flat at the extremum (if flat, the cost of adding it is $O(1)$) Lean `S_ext_01_iff` (algebra); the mean height is not flat and pays $\propto 1/\hbar$, while the radius passed.

Paying for the summit band (the rarity of the band and the tail through the band are both $\propto 1/\hbar$ and cancel) and perturbing only the summit band (the cost of the Holley–Stroock perturbation $e^{\mathrm{osc}/\hbar}$) are also examples that only moved the place where $1/\hbar$ is paid.

Question to test: in your plan, which quantity is $1/\hbar$ attached to? If it has only moved to a sector, a mark, a comparison constant or the cost of a perturbation, it has not disappeared. Draw the rate function of the variable you add near its extremum.

The type of the misaligned fluctuation direction — mismatch between the direction of fluctuation and the soft direction

Plans caught: in Table 1′, radius and angles on the sphere, the three steps, the variance version of two-scale, and the interior radius; and the paper upper bound of the Feynman–Kac strand.

In a two-scale composition, taking the gradient outside the fibre with $\sup$ (the direction of fluctuation) and paying for it with the constant of the softest direction of the fibre (of the type $\kappa \times A_{\mathrm{fib}}$) makes the upper bound too large by orders of magnitude wherever the two directions are not aligned. Even in the version replaced by the variance, the lower bound on the loss, 0.66–1.8, was outside the needed 0.18–0.50 Reading . What must be paid is the gradient of the solution of the Poisson equation on the fibre ($|M^{-1}\nabla_\perp G|$ in the Helffer–Sjöstrand representation), and it must be estimated while keeping the direction. The gap in the Feynman–Kac strand, where the Monte Carlo reading is 0.54 times the threshold and the paper upper bound 10^3 times the threshold, also came from the step where the scalar Grönwall discards the direction of the gradient and reduces every direction at the softest rate (23-09 the Feynman–Kac strand).

Question to test: somewhere in your upper bound, have you crushed a vector quantity into a scalar? If there is such a step, it pays at the rate of the softest direction there. Draw the direction of large fluctuation and the soft direction in the same figure.

The transfer type — the weight only moves to another column

Plans caught: the modified LSI of replacing the functional inequality, known uniform log-Sobolev, detour plan "finite-volume criteria", detour plan "two-step cluster expansion", one- or two-site windows, detour plan "localising at the cluster core", and the proxy series in Table 1′.

Known uniformity theorems (known uniform log-Sobolev, finite-volume criteria) do not make the constant; they move the problem to the finite-volume constant of a box covering the correlation length (length $\gtrsim$ **230–810**). The window is finite, so existence was trivial from the start, and the reason they do not help is size. Plans that change how the window is taken (one- or two-site windows, localising at the cluster core) reduced the loss of the window, but the floor of the propagation step remained. In the cluster expansion (two-step cluster expansion) the pin of the largest path stayed at the top, and in the modified LSI (replacing the functional inequality) $\ln(1/\pi_*)$ moved to the energy side. In the proxy series of the Scalar-Strand Proposition, when the assumption column became lighter by changing proxy or direction, the computed column became heavier by the same amount (an identity in which the sum of the two columns does not depend on the direction) Assumption (the definitions are not placed in the book) Lean `threshold_gap_invariant` (assumption: the identity of the two sums).

Question to test: in your plan, has another column (the size of a box, the propagation step, a computed quantity) become heavier by as much as one column became lighter? If the sum is invariant, the margin has not moved.

Table 2 Types of breakage (one line each)

Type	One line	Plans caught	Question to test
Summit-curvature type	At one site at the top, the diagonal $15/4$ falls $\frac{1}{4}$ short of the coupling 4	Defects as long clusters, non-linear Barta weight, Agmon-type payment, the uniform road, Lyapunov without sectors	What is paid on the top face $x \equiv 0$
Type of the ceiling on structural factors	Moving b and κ without changing the set still gives $A\gamma < \ell$	Raising the threshold, layering defects by depth, narrowing the defect set, Hölder and hypercontractive steps	Does it change the defect set
Double-payment type	A later step pays again for a place an earlier step paid for	Localising negative curvature, optimising the Gibbs tilt (on the side that fixed it), the construction of the window	Have you drawn the places paid in the same figure
Currency-mix-up type	Poincaré is the speed of exit; rarity is the currency of entropy	Log-Sobolev with clusters fixed, the band by rarity, the double threshold, the naive decomposition of the variance	Does the step that pays for rarity pass through entropy
Type of the address of 1/ħ (the address of $1/\hbar$)	$1/\hbar$ lives in the radius, and sectors and marks only allocate it	Paying for the summit band, perturbing only the summit band, collective coordinate, sector-split chain, multi-stage, mark chain, mean height	Which quantity is $1/\hbar$ attached to
Type of the misaligned fluctuation direction	Paying a $\sup$ gradient in the softest direction is too large by orders of magnitude	Angles and radius, three steps, variance version, interior radius, Grönwall	Have you crushed a vector into a scalar
Transfer type	Another column becomes heavier by as much as one becomes lighter	Replacing the functional inequality, known uniform log-Sobolev, finite-volume criteria, two-step cluster expansion, one- or two-site windows, localising at the cluster core, the proxy series	Is the sum an invariant

Among the dead and on-hold plans, those whose type column in Table 1 is "—" are detour plan "Glauber log-Sobolev", which fixed a step that was not the limit, and extension to the classical limit, which is on hold (for the alive and conditional plans that column gives the chapter they went to). This table format, which keeps the line that killed each dead plan, can be used as it is to tidy up after sorting out detour plans for other problems too.

05

Where the survivors went — into the three strands

The alive and conditional plans were gathered into the table of needed values and the three strands. How far the three strands got and their remaining steps are in 23-09 Table 4, and here we list only the correspondence from the plans.

Table 3 Where the surviving plans went

Plan	Destination	What remains there
Localising negative curvature, optimising the Gibbs tilt (, raising the threshold)	The table of needed values (23-09 Table 2). Parts of every later form	Regularity of the Bochner step; measuring A_W with marks at the same threshold as the defects (23-09 § 04)
Paying with a height function, coarse and propagation steps at once (and the derivatives of the componentwise strand in Table 1′)	The componentwise strand — the componentwise form	An upper bound on C_w of the propagation step (a Lyapunov function with coupling terms)
Scalar payment, the Poincaré form of replacing the functional inequality, re-examining downstream needs	The scalar strand — the scalar form (the Scalar-Strand Proposition)	The interval-arithmetic rows, between the nodes, the one lifting assumption

Plan	Destination	What remains there
The form combining Hölder and hypercontractive steps with paying with a height function	The Feynman–Kac strand — the Feynman–Kac form	An upper bound that keeps the direction (the type of the misaligned fluctuation direction)

The place where all three strands stopped is the source of the summit-curvature type, the same as the common wall of 23-09 § 06. Another construction that controls the core constant from the side of the box rather than by the sector framework is treated in 23-11.

06

The table of the five grades — the main lemmas for each type

This chapter is a list, so its rows are limited to the lemmas supporting the types and the values of Table 1. None of the rows is a premise of the conclusion of the route of the window log-Sobolev inequality, and the dependence of the conclusion is in the graph of 23-09 § 08. So this chapter has no dependency graph.

Table 4 The rows of this chapter

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Floor at the top	The floor of the Barta row of a one-site weight $F(0,0) = w(0) = -\frac{1}{4}$	Proved here	§ 04 (2) (the form of the row is on the Assumption row of 23-09 (the assumption of the Barta row))	Independent of $\hbar$	`floor_top` (assumption: weight value $/= 0$)
Agmon's gain	Where the curvature is negative, the coefficient of Agmon's gain is ≤ 0	Assumption	§ 04 summit-curvature type (the framework of Agmon estimates is not placed in the book)	—	`agmon_no_gain_on_negative_curvature` (algebra)
Ceiling lemma	The ceiling for plans that do not change the set, $A\gamma < \ell$	Proved here	Proof of the type of the ceiling on structural factors in § 04	Independent of $\hbar$	`ceiling_same_set` (assumptions: $0 < \kappa < b$, $t - A\gamma > 0$, $t - \ell \leq L$, the closing condition)
Rarity on the Barta side	The rarity on the Barta side enters only multiplied by $\kappa/(\beta_0 - \kappa)$	Assumption	§ 04 type of the ceiling on structural factors (the paper argument is not placed in the book)	—	`ceiling_split` (algebra)
Fixing the double payment	Localised Bochner and the Gibbs tilt (fixing the double payment)	Proved here	23-09 § 03	Independent of $\hbar$	`bochner_localized`, `tilt_cost_le` (the assumptions and scope are in 23-09)
Rarity after conditioning	Under the conditional measure, a set determined by the σ-algebra of the conditioning has measure 0 or 1	Proved here	§ 04 currency-mix-up type	Independent of $\hbar$	—
Transverse convexity	On the top face, the smallest curvature orthogonal to the collective coordinate is negative at $k = 17$	Proved here	§ 04 type of the address of $1/\hbar$ (closed formula)	Independent of $\hbar$	`transverse_fails_k17` (no assumptions)
Chain rule of entropy	The chain rule of entropy (cutting into sectors does not decrease the sum)	Proved here	§ 04 type of the address of $1/\hbar$ (identity)	Independent of $\hbar$	`chain_rule_cells` (assumptions: the chain rule, upper bounds on the two terms)
Values of the address of $1/\hbar$	The address of $1/\hbar$ (crossing length $0.34/\hbar$, allocation $\approx 0.8/\hbar$, comparison constant $\geq 13/26/66$)	Reading	§ 04 type of the address of $1/\hbar$	Three points	—
Loss from the mismatch of directions	The loss from the mismatch of directions (lower bound 0.66–1.8 against needed 0.18–0.50)	Reading	§ 04 type of the misaligned fluctuation direction	Three points	—
Invariance of the proxy series	Invariance of the sum of the two columns of the proxy series	Assumption	§ 04 transfer type (the definitions are not placed in the book)	—	`threshold_gap_invariant` (assumption: the identity of the two sums)
Reading of known uniformity	Known theorems on uniform logarithmic Sobolev in one dimension give the constant of a box	Assumption	§ 04 transfer type (the statements in the original sources have not been checked)	—	—

Row	Statement (one line)	Grade	Grounds	Range of $\hbar$	Machine checking
Values of Table 1	Factors, ceilings and needed values of Tables 1 and 1′	Reading	§ 03 (floating-point grids and transfer matrices; lower bounds within families)	Three points	—

The "Proved here" rows are general facts that identify the types, and do not mean that any detour plan succeeded. None of the rows of Table 4 shows the closing condition (1).

07

To the next chapter

The construction that controls the core constant in three steps from the side of the box-reduction condition and the gap-inside-the-box condition rather than the sector framework, and the drawing of the boundary of the end sets, are in 23-11. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter tidied up after the plans aimed at the core row of the saddle-tube condition of its first step. Another viewpoint that handles the negative curvature at the top without relying on convexity is treated in Part 24 (24-06).

What this chapter can say

What is said	What is not said
Proved here The floor at the top (2), the ceiling for plans that do not change the set, the disappearance of rarity after conditioning, the threshold of transverse convexity, the chain rule of entropy Lean (algebra)	The closing condition (1). Hence the one remaining proposition of the route of the window log-Sobolev inequality and the saddle-tube condition
Reading The factors and needed values of Table 1; that plans on the side of tools moved the needed value by 44–460 times while plans on the side of structure stopped under a ceiling of 1.15 times	That the types exhaust all plans. There are plans that fit no type (Glauber log-Sobolev, extension to the classical limit)
Assumption Agmon's gain, how the rarity on the Barta side enters, the invariance in the proxy series, the reading of known uniformity theorems	That the propositions of the dead plans are false (death is a property of the tool, not the truth of the proposition)

Sources and reproduction

Item	Kind	Source / tools
The integrated form of Bochner's formula	Cited	M. Ledoux, Sém. Probab. XXXV, LNM 1755 (2001) 167–194 (Proposition 1.3; text checked).
The Holley–Stroock perturbation	Cited	R. Holley, D. Stroock, J. Stat. Phys. 46 (1987) 1159–1194 (type checked in the restatement as the Holley–Stroock Perturbation Lemma (Proposition 6.1 of Ledoux 2001)).
Rothaus's lemma	Cited	O. S. Rothaus, J. Funct. Anal. 64 (1985) 296–313 (statement checked in a restatement).
Uniform logarithmic Sobolev for unbounded spins, finite-volume criteria	Assumption	Th. Bodineau, B. Helffer, J. Funct. Anal. 166 (1999) 168–178; F. Martinelli, E. Olivieri, Comm. Math. Phys. 161 (1994) 447–486, 487–514 (bibliographic only; statements unchecked).
Glauber dynamics of the one-dimensional Ising model	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294–307 (bibliographic only; values are readings from diagonalisation on the lattice).
The Helffer–Sjöstrand representation	Assumption	B. Helffer, J. Sjöstrand, J. Stat. Phys. 74 (1994) 349–409 (bibliographic only; statement unchecked).
Lemmas of the types	Proved here, Lean	§ 04 of the text. `floor_top`, `ceiling_same_set`, `ceiling_split`, `agmon_no_gain_on_negative_curvature`, `transverse_fails_k17`, `chain_rule_cells`, `S_ext_01_iff`, `threshold_gap_invariant`.
Numbers of Tables 1 and 1′	Reading	Floating-point grids, transfer matrices (midpoint rule), lower bounds from families of test functions, rational decompositions. Interval arithmetic is not used.

Words used in this chapter — detour plan, the line that killed it, needed side, form, insufficient side (floor), structure and choice of tools, ceiling, types of breakage, paying twice, currency (speed of exit and entropy), the address of $1/\hbar$, crossing length, direction of fluctuation and soft direction, transfer. Their meanings are in A-02 Glossary.

Next chapter to read: 23-11 Box reduction and the gap inside the box — the constant of the core — turns box reduction and the gap inside the box into propositions, and controls the core constant in three steps.

Box reduction and the gap inside the box — the constant of the core

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-05 and 2-07.

What this chapter shows — We state Step 1 (box reduction) and Step 2 (inside the box) of the Double-Well Relaxation-Time Upper-Bound Proposition of the first step (23-05 § 04) as the Box-Reduction Proposition (box reduction, using only the Poincaré inequality) and the Gap-Inside-the-Box Proposition (the gap inside the box, with a table of assumptions), and we give the proof of the product inequality for boxes, which 23-05 graded "Proved here". Next, we bound the constant of the core that enters the fast side of the Gap-Inside-the-Box Proposition in three stages by the length of the core (convexity, two blocks, and the reduction to the box applied to the core). Finally, we state how far the set of values at the ends of the core is covered, and how the bistability that appears at ends beyond the top is paid for by the margin of the action barrier.

Prerequisite chapters — 23-05 § 03, § 04 (the statements of the box-reduction condition and the gap-inside-the-box condition, and the single line of argument), 23-04 (the path measure, the correlation length ξ, kinks), 23-09 § 06 (the curvature of one site at the top).

Said first — The level is [Res], and only § 01 is written so that it can be read at [Y3]. § 02 onwards are sections at the research level; you can skip them and go to the table of remaining rows in § 07 and still read the conclusion of this chapter. This part places the diffusive-relaxation-time assumption at the axis of the study (23-01), and this chapter states how far the box-reduction condition and the gap-inside-the-box condition of its first step could be put in the form of propositions. The Box-Reduction Proposition and the Gap-Inside-the-Box Proposition are propositions that leave assumption rows in place; they do not show the box-reduction condition or the gap-inside-the-box condition. All numbers are values on the 0+1-dimensional double well $V = (x^2 - 1)^2/8$ on the lattice $\varepsilon = \frac{1}{2}$, and triples are in the order $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$.

The order of this chapter

1. Summary — a section readable at [Y3]
2. Box-Reduction Proposition — box reduction (Figure 1, Table 1)
3. Proof of the product inequality for boxes — the Markov property and Efron–Stein
4. Gap-Inside-the-Box Proposition — the gap inside the box (Table 2)
5. Three stages for the constant of the core — the Three-Range Core-Constant Proposition (Tables 3 and 4)
6. The set of ends — what is covered, and how to pay for bistability
7. The list of remaining rows (Table 5)
8. The table of the five grades (Table 6)
9. The dependency graph

01

Summary

The box-reduction condition bounds the Poincaré constant of a long path by the Poincaré constant of a box of length $C\xi$, and the gap-inside-the-box condition bounds the gap inside that box at the order of $D/(C\xi)^2$.

1. **Box-Reduction Proposition.** Cover the line with overlapping boxes and split them into even boxes and odd boxes; then $C_P \le (1+1/n)\, C_{\mathrm{box}}/(1-c_{\mathrm{alt}})$. It uses only the Two-Projection Lemma, the product inequality for boxes and averaging over shifts, and does not use the logarithmic Sobolev inequality. The remaining assumption rows are that the maximal correlation between even and odd, $c_{\mathrm{alt}} < 1$, is uniform, and the $\sup$ over the boundary values. The correlation length is read as $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and a box does not cut the plateau at the top.
2. **Gap-Inside-the-Box Proposition.** Splitting the inside of the box by the configuration of kinks Y gives $C_{\mathrm{box}} \le C_{\mathrm{loc}} + (1+\chi) C_{\mathrm{kg}}$. If the reduced kink gas is put in the same type as the Glauber dynamics of the one-dimensional Ising model, the gap is the sum of "creation and annihilation of pairs" and "diffusion of kinks", $(C^2+\pi^2)D/(C\xi)^2$. The remaining rows are the cross term (the cross-term uniformity assumption), the type of the gas (the kink-gas gap assumption), the version for segments (the segment local Poincaré assumption), and the constant of the core.
3. **Three stages for the constant of the core.** Core length $k \le 7$ is bounded by convexity, ≤ 9.18 (independent of $\hbar$ and of the end values); $8 \le k \le 14$ by the lemma

of two blocks; and $k \geq 15$ by applying the reduction to the box to the core. Taking the end values as samples in $[0, x_c]$ gives $\sup_k C_P(\text{core}) \leq 22.1/30.5/66.4$ Reading .

4. **The set of ends.** In the core of the sector framework, the ends are only on the side of the well, and there is no hole. In the core on the box side, the ends can lie beyond the top, and there the wall of conditioning creates bistability; but its barrier $\Delta S_* \approx 10^{-2}$ is about 95 times smaller than $2S_0$, and the local term of the Gap-Inside-the-Box Proposition vanishes as $\hbar \to 0$ (on the reading at $\hbar = 0$).

02

Box-Reduction Proposition

Setting. Take the action $S = \sum_i (x_{i+1} - x_i)^2/(2\varepsilon) + \varepsilon \sum_i V(x_i)$ on the lattice line $\mathbb{Z}$ (spacing $\varepsilon = \frac{1}{2}$) at $T = \infty$, the infinite-volume measure μ of $e^{-S/\hbar}$, and the Dirichlet form $\mathcal{E}(f) = (\hbar/\varepsilon) \sum_i \mathbb{E}_\mu |\partial_i f|^2$ (the uniqueness of μ and the nearest-neighbour Markov property are in the row of common premises (Step 0 in 23-05 § 04)). A box B is an interval of L sites, the boundary values (a, b) are the values at the neighbouring sites to the left and right of B, $\mu_B^{a,b}$ is the conditional measure with these fixed, and $\mathcal{E}_B$ is the sum over the sites of B only. We put

$$C_{\text{box}}(L) := \sup_{a,b \in \mathbb{R}} C_P\left(\mu_B^{a,b}; \mathcal{E}_B\right)$$

The maximal correlation of two σ-algebras is $c(\mathcal{G}_1, \mathcal{G}_2) := \sup\{\text{corr}_\mu(F, G) : F \in L^2(\mathcal{G}_1), G \in L^2(\mathcal{G}_2)\}$.

Two-Projection Lemma. If $c = c(\mathcal{G}_1, \mathcal{G}_2) < 1$, then for every $f \in L^2(\mu)$

$$\operatorname{Var} f \;\leq\; \frac{1}{1-c}\Big(\mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_1) + \mathbb{E}\operatorname{Var}(f \mid \mathcal{G}_2)\Big) \quad (1)$$

Proved here (proof in Step 1 in 23-05 § 04) Lean `two_proj_sum_le` (assumptions: Cauchy–Schwarz and the inequality of maximal correlation), `two_proj_var` (assumption: $A + B \leq (1 + c)V$).

The family of boxes. Line up boxes B_m of L sites so that neighbours overlap in l sites ($l \leq L/3$), and let E be the family of even-numbered boxes and O the family of odd-numbered ones. $\mathcal{G}_E$ is the σ-algebra generated by the values at the sites outside E, $\mathcal{G}_O$ likewise, and $c_{\text{alt}}(L, l) := c(\mathcal{G}_E, \mathcal{G}_O)$ (independent of translation).

```
Figure 1  Even and odd boxes, and the shift (example with L = 8, l =
sites       ....|....|....|....|....|....|....|....
even E      [ B0   ]       [ B2   ]       [ B4   ]
odd O            [ B1   ]       [ B3   ]
overlap          ^^     ^^      ^^     ^^          ← l sites each
shift s=1   move the whole row of boxes l sites right (the overlaps m
with n = ⌊(L−l)/l⌋ shifts, the overlap sites are mutually disjoint
```

The product over boxes. Even boxes are $L - 2l \geq 1$ sites apart from each other, and the following inequality holds (proof in § 03).

$$\mathbb{E}\,\mathrm{Var}(f \mid \mathcal{G}_E) \;\leq\; C_{\mathrm{box}}(L) \sum_{B \in E} \mathcal{E}_B(f) \quad (2)$$

Box-Reduction Proposition. $n := \lfloor (L-l)/l \rfloor$. If $c_{\mathrm{alt}}(L,l) < 1$, then

$$C_P(\mu) \;\leq\; \Big(1 + \frac{1}{n}\Big) \frac{C_{\mathrm{box}}(L)}{1 - c_{\mathrm{alt}}(L,l)} \quad (3)$$

Proved here Proof. Fix one shift s and let its even and odd families be E^s, O^s. From (1) and (2) for the two families, $(1 - c_{\mathrm{alt}})\,\mathrm{Var}\, f \leq C_{\mathrm{box}} \big(\sum_{B \in E^s} \mathcal{E}_B + \sum_{B \in O^s} \mathcal{E}_B \big)$. The boxes cover the line, each site lies in the even family at most once and in the odd family at most once, and only the overlap sites lie in both, so the bracket is $\mathcal{E}(f) + \mathcal{E}_{\mathrm{ov}^s}(f)$ (the sum over the overlap sites only). Averaging over the shifts $s = 0, \ldots, n-1$, the overlap sites are mutually disjoint across shifts, so $\sum_s \mathcal{E}_{\mathrm{ov}^s} \leq \mathcal{E}$, hence $(1 - c_{\mathrm{alt}})\,\mathrm{Var}\, f \leq C_{\mathrm{box}}(1 + 1/n)\mathcal{E}(f)$. ∎ Lean `box_two_family` (assumptions: the Two-Projection Lemma, (2) for each family, $\mathcal{E}_E + \mathcal{E}_O \leq \mathcal{E} + \mathcal{E}_{\mathrm{ov}}$, $C_{\mathrm{box}} \geq 0$), `box_shift_average` (assumptions: the two-family inequality for each shift, the sum over overlaps is at most $\mathcal{E}$).

(3) is built only from Poincaré inequalities and does not use the logarithmic Sobolev inequality. The in-phase local Poincaré condition and the saddle-tube condition enter only through C_{box} (§ 04). Putting in $C_{\mathrm{box}} \leq K_4 (C\xi)^2 / D$ of the gap-inside-the-box condition together with $L\varepsilon = C\xi$ and $D = \hbar / S_0$ gives $C_P \leq (1 + 1/n) K_4 C^2 / (1 - c_{\mathrm{alt}}) \cdot \xi^2 S_0 / \hbar$, the form of the First-Step Upper-Bound Proposition Lean `H3_with_H4`, `H3_H4_compose` (assumptions: (3), the gap-inside-the-box condition, $D = \hbar / S_0$, $n > 0$, $c_{\mathrm{alt}} < 1$). It is the same type as the box dynamics of the Glauber dynamics of discrete spins (Propositions 3.4 and 3.5 of Martinelli's lectures) Cited ; here we rewrite it in one dimension, in L^2, and in the form of maximal correlation.

The remaining assumption rows, and the numbers

- **Alternating maximal-correlation assumption for boxes: $c_{\mathrm{alt}} < 1$ uniformly in $\hbar$.** There is no paper proof Assumption . As values, on the reading that the slow part is a two-state chain of the sign (the telegraph chain), boxes with $\theta = l/L = \frac{1}{4}$ give $c_{\mathrm{alt}} = 0.731/0.410/0.138$ for $C = 2/4/8$ Reading (values of a family that copies the values of the closed formula of the two-state chain). These values agree with $\varphi(s,t)$ of the Closed Formula for the Alternating Maximal Correlation in § 05 with $s = e^{-C\theta}$ and $t = e^{-C(1-2\theta)}$ (at $C = 2$, $\varphi = 0.7308$).
- **The sup over boundary values.** C_{box} takes the sup over all real boundary values. We see in § 04 that boundary values near the top are the worst.
- **The size of ξ.** Computing the difference of the even and odd eigenvalues of the transfer operator in the logarithmic domain gives $\ln \xi = 218.08/661.04/2212.72$ and $\ln \xi - S_0^{\mathrm{lat}}/\hbar - \frac{1}{2}\ln\hbar \to -0.778$ Reading (two independent computational routes agree to six digits at intermediate $\hbar$). That is, $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and its ratio to the length of the plateau at the top (the crossing length $\ell_\times \approx 0.34/\hbar$, 23-10 type of the address of $1/\hbar$) is $\xi/\ell_\times \geq e^{213}$ Lean `xi_over_ellx` (assumption: the measured $\ln\xi - \ln\ell_\times \geq 213$). **A box of length $C\xi$ does not cut the plateau; it contains it whole.** In exchange, the constant of the core inside the box has to be uniform in the length of the core (§ 05).

Table 1 Positive controls for the inequality of the Box-Reduction Proposition (models in which both sides are known from a closed formula or by diagonalisation)

Model	What was compared	Result
Pure Gaussian ($V = x^2/2$)	The right-hand side $(1 + 1/n)C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$ against $CP = 1$	Right-hand side 1.10–1.33 ≥ 1 ($L = 4$–128) Reading . Approaches 1 as $l/L \to 0$, $l \to \infty$ Lean `gauss_box_le_one`
Heat-bath Glauber for the one-dimensional Ising model (rings of 12 and 16 sites, couplings $K = 0.3, 0.6, 1.0$)	Both sides of (3)	Holds in all 9 cases, with a loss of 1.17–1.35 times Reading (even at $c_{\mathrm{alt}} = 0.87$) Lean `ising_K1_holds` (assumption: measured values)
Negative control	$\mathcal{G}_1 = \mathcal{G}_2$ ($c = 1$)	The reduction says nothing Lean `no_bound_when_c_one`, `blowup_as_c_to_one`

In the Glauber dynamics examples, even when the gap of the ring (the lower edge of the band) was only about a sixth of the gap of the box, $1 - c_{\mathrm{alt}}$ carried exactly that difference. The band structure of the kink gas does not break the form of the box.

```
03
```

Proof of the product inequality for boxes

We give the proof of (7) of 23-05 ((2) of this chapter). The box-product inequality row was graded "Proved here" in the table of the five grades in 23-05 because of the proof in this section.

Proposition (product over boxes)

When the boxes of the even family E are $L - 2l \geq 1$ sites apart from each other, (2) holds for smooth functions f that depend on finitely many sites only.

Step (a) Conditional independence. The action S is the sum of terms between neighbours $(x_{i+1} - x_i)^2/(2\varepsilon)$ and one-site terms $\varepsilon V(x_i)$. If all values outside a box B are fixed, the conditional density of x_B is determined only by the terms of $e^{-S/\hbar}$ that contain x_B, which are the terms inside B and the two terms joining the two ends of B to the two neighbours a_B, b_B. So the conditional law is $\mu_B^{a_B, b_B}$, and it depends on the values outside B only through the two neighbours (the nearest-neighbour Markov property). Between even boxes there is at least one site fixed by $\mathcal{G}_E$, so given $\mathcal{G}_E$, the conditional density of the even boxes splits into a product of factors, one per box,

$$\prod_{B \in E} \frac{1}{Z_B} \exp\Big(- \frac{S_B(x_B; a_B, b_B)}{\hbar} \Big)$$

and the values of the even boxes $(x_B)_{B \in E}$ are mutually independent with the product law $\bigotimes_{B \in E} \mu_B^{a_B, b_B}$ Proved here (one line on the density splitting into a product in finite volume; that the measure at $T = \infty$ has the same conditional law is in the row of common premises).

Step (b) The Efron–Stein inequality for product measures. Let $\nu = \nu_1 \otimes \cdots \otimes \nu_m$ be a product measure on $X_1 \times \cdots \times X_m$, $F \in L^2(\nu)$, and $\mathrm{Var}_j F$ the variance obtained by moving only the j-th coordinate (the other coordinates fixed). Then

$$\mathrm{Var}_\nu F \;\leq\; \sum_{j=1}^{m} \mathbb{E}_\nu \big[\mathrm{Var}_j F \big] \qquad (4)$$

Proved here Proof. Put $F_j := \mathbb{E}[F \mid X_1, \ldots, X_j]$ ($F_0 = \mathbb{E}F$, $F_m = F$) and $\Delta_j := F_j - F_{j-1}$. The Δ_j are mutually orthogonal, so $\mathrm{Var}\, F = \sum_j \mathbb{E}\Delta_j^2$. Since the coordinates are independent, F_{j-1} is F_j integrated over the j-th coordinate with ν_j, and $\mathbb{E}\Delta_j^2 = \mathbb{E}[\mathrm{Var}_j F_j]$. Moreover F_j is F averaged over the later coordinates $X_{j+1}, \ldots, X_m$, and $G \mapsto \mathrm{Var}_j G = \| G - \int G \, d\nu_j \|^2_{L^2(\nu_j)}$ is convex (the square of a seminorm), so Jensen's

inequality gives $\mathrm{Var}_j F_j \le \int \mathrm{Var}_j F \, d\nu_{j+1} \cdots d\nu_m$. Taking expectations, $\mathbb{E}\Delta_j^2 \le \mathbb{E}\,\mathrm{Var}_j F$, and summing over j gives (4). ∎

Step (c) Poincaré for the box. Apply (4) to the conditional product of Step (a) (with the coordinate X_j the value x_B of the box B). The variance obtained by moving only the coordinates of a box B is the variance of the measure $\mu_B^{a_B,b_B}$, so

$$\mathrm{Var}_B f \;\le\; C_P(\mu_B^{a_B,b_B})\,\frac{\hbar}{\varepsilon}\sum_{i\in B}\int |\partial_i f|^2\, d\mu_B^{a_B,b_B} \;\le\; C_{\mathrm{box}}(L)\,\frac{\hbar}{\varepsilon}\sum_{i\in B}\int |\partial_i f|^2\, d\mu_B^{a_B,b_B}$$

by the sup in the definition of C_{box}.

Step (d) Putting it together. From (4) and Step (c), $\mathrm{Var}(f \mid \mathcal{G}_E) \le C_{\mathrm{box}} \sum_{B\in E} \mathbb{E}\big[(\hbar/\varepsilon) \sum_{i\in B} |\partial_i f|^2 \mid \mathcal{G}_E\big]$. Taking expectations, by the tower property of conditional expectation each term on the right becomes $\mathcal{E}_B(f)$, which is (2). Since f depends on finitely many sites only, finitely many boxes are involved, and the terms of boxes that do not touch the support of f are 0. ∎

The extension to general f in the domain of $\mathcal{E}$ (approximation by functions depending on finitely many sites) is included in the row of common premises. The odd family is the same. (4) of Step (b) is the Efron–Stein inequality (Efron–Stein 1981); since we give the proof here, the grade is "Proved here".

04

Gap-Inside-the-Box Proposition

Fix a box B ($L\varepsilon = C\xi$) and boundary values $(a,b) \in \mathbb{R}^2$, and let $\mu = \mu_B^{a,b}$ and $D = \hbar/S_0$. The expected number of kinks in the box is $C/2$ from the kink density $1/(2\xi)$, which is $O(1)$ Proved here Lean `pair_fugacity` (assumption: $\xi \;/= 0$).

The naive split goes round in a circle. Splitting by a rare set E (for example, configurations with a long core), with $p = \mu(E)$, gives $\mathrm{Var}\, f = p\,\mathrm{Var}_E f + (1-p)\,\mathrm{Var}_{E^c} f + p(1-p)(m_E - m_{E^c})^2$ Proved here (an identity). The second term can be paid with a constant on E^c, but the first term is as large as $\mathrm{Var}\, f$ if f is concentrated on E, and the smallness of p does not come out. What can be paid is up to $p \cdot \mathrm{osc}(f)^2/4$, which is not a gap but a weak form of the Poincaré inequality Lean `naive_split_weak` (algebra). It is an example of 23-10 currency-mix-up type: rarity cannot be paid with variance. So we take the split not by a rare set but by a slow variable.

Splitting by the slow variable. Let Y be the configuration of kinks in the box (the number of kinks, and the position of each kink defined by projection onto the manifold of kinks), and take the identity for conditional expectation Proved here

$$\operatorname{Var} f = \mathbb{E}\operatorname{Var}(f \mid Y) + \operatorname{Var}\big(\mathbb{E}[f \mid Y]\big)$$

as the starting point.

- **The first term (the fast side).** By the nearest-neighbour Markov property, the conditional measure with Y fixed splits into a product of scraps of the phase without kinks (length $\le C\xi$, with values at both ends), the tubes around the kinks, and the cores inside them. The Poincaré constant of a product measure is at most the maximum of the constants of the components ((4) and Poincaré for each component), so $\mathbb{E}\operatorname{Var}(f \mid Y) \le C_{\mathrm{loc}}\,\mathcal{E}(f)$ with $C_{\mathrm{loc}} = \max(C_{(\mathrm{H1}^{\mathrm{seg}})}, C_{(\mathrm{H2}^{\mathrm{tube}})}, C_{(\mathrm{T0}'_{\mathrm{P}})})$ Proved here (the algebra of products). The three constants are assumption rows Assumption, and the constant $C_{(\mathrm{T0}'_{\mathrm{P}})}$ of the core is bounded in § 05 and § 06.
- **The second term (the slow side).** $\mathbb{E}[f \mid Y]$ is a function of Y alone and relaxes by diffusion of the kink positions (mobility D, the kink diffusion-coefficient row Cited) and by the creation and annihilation of kink pairs. The kink-gas gap assumption: the Poincaré constant of the box of the reduced kink gas, $C_{\mathrm{kg}} \le (C\xi)^2/(c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})D)$ Assumption. The cross-term uniformity assumption: the cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1+\chi)\mathcal{E}(f)$ Assumption (of the type of the cross constant of a two-scale inequality; a statement of this form has not been drawn from a known theorem).

Gap-Inside-the-Box Proposition. Under the rows above, for every (a, b)

$$C_{\mathrm{box}} \;\le\; C_{\mathrm{loc}} + (1+\chi)\,C_{\mathrm{kg}} \;\le\; K_4\,\frac{(C\xi)^2}{D}, \qquad K_4 = \frac{1+\chi}{c_{\mathrm{kg}}(1-\eta_{\mathrm{KG}})} + C_{\mathrm{loc}}\,\frac{D}{(C\xi)^2} \quad (5)$$

Proved here (the sum of the two terms above) Lean `H4_two_scale` (assumptions: the decomposition inequality, first term $\le C_{\mathrm{loc}}\mathcal{E}$, the kink-gas gap assumption, the cross-term uniformity assumption, $C_{\mathrm{kg}} \ge 0$), `H4_gap_form` (assumptions: $D > 0$, $C\xi \neq 0$, $c_{\mathrm{kg}} > 0$, $\eta_{\mathrm{KG}} < 1$, $1+\chi \ge 0$). (5) is the same formula as (12) of 23-05.

Comparison with the Glauber dynamics of the one-dimensional Ising model

The constant c_{kg} of the kink-gas gap assumption is fixed by the assumption that puts the reduced kink gas in the same type as the heat-bath Glauber dynamics of the one-dimensional Ising model. The gap of a box of L sites with the spins at both ends fixed has the closed formula

$$\mathrm{gap}_{\mathrm{box}} = 1 - \gamma\cos\frac{\pi}{L+1} = (1-\gamma) + \gamma\Big(1-\cos\frac{\pi}{L+1}\Big), \qquad \gamma = \tanh 2K \quad (6)$$

Assumption (the original source is bibliographic only; diagonalisation on the lattice agrees to the 6 digits shown, and it is the same whether the boundary values are $(+,+)$

or $(+,-)$ Reading) Lean `glauber_split` (the right-hand equality, no assumptions). In the limit of long correlation length, the first term $1-\gamma \approx 1/(2\xi^2)$ is the infinite-volume gap = **creation and annihilation of pairs**, and the second term $\approx \pi^2/(2L^2)$ is the **diffusion of kinks** in an interval of length L. Written in the Glauber unit $D=\frac{1}{2}$, it is $D/\xi^2+\pi^2 D/L^2=(C^2+\pi^2)D/(C\xi)^2$, that is, $c_{\mathrm{kg}}=C^2+\pi^2$ Proved here (algebra) Lean `kg_constant` (assumption: $C\xi\ /=0$). The pair term is not a small perturbation; it enters additively as one of the main terms of the gap. The local term of the part without kinks does not appear at the lower edge of the gap.

When the boundary values are near the top, a half kink can sit at the end of the box. This corresponds to a free end of the Ising model, and there is no closed formula. Read by diagonalisation on the lattice, the gap does not fall below $1-\gamma$ for any type of boundary (ratio ≥ 1.17), but a free end is slower than a fixed end.

Table 2 The box constant of the reduced gas $c_{\mathrm{kg}}=\mathrm{gap}\cdot(C\xi)^2/D$ and the coefficient K of the First-Step Upper-Bound Proposition ($\chi=\eta\mathrm{KG}=0$, $n=3$, $\theta=\frac{1}{4}$; all are values of models independent of $\hbar$ Reading)

C	1	2	3	4	6
Fixed ends $C^2+\pi^2$	10.87	13.87	18.87	25.87	45.87
Free ends c_{ff} (lattice values)	2.86	7.17	12.94	20.32	39.07
c_{alt} (two-state chain)	0.915	0.731	0.550	0.410	0.234
K (fixed ends only)	1.44	1.43	1.41	1.40	1.37
K (the sup including boundary values at the top)	5.46	2.76	2.06	1.78	1.60

$K=(1+1/n)C^2/(c_{\mathrm{kg}}(1-c_{\mathrm{alt}}))$ Lean `K_fixed_C2`, `K_ff_C2`, `K_ff_C4` (assumptions: the measured c_{ff} and c_{alt}; arithmetic only). Since K_4 decreases with C, with fixed ends only K hardly depends on C, and including boundary values at the top, the larger C is taken, the better.

The rows remaining in the gap-inside-the-box condition. The cross-term uniformity assumption, the kink-gas gap assumption (the Glauber type and free ends), the segment local Poincaré assumption (the Six-Node In-Phase Poincaré Proposition of 23-06 is a statement on the ring, and there is no version for segments with end values), the saddle-tube Poincaré assumption, and the row of the core. The second term C_{loc} $D/(C\xi)^2$ of K_4 vanishes if C_{loc} is sufficiently smaller than $e^{2S0/\hbar}$, because D/ξ^2 is of size $e^{-441.6}/e^{-1328.6}/e^{-4433.1}$ Reading . We use this margin for the core in § 06.

05

The three-stage proposition for the constant of the core

Setting. A core is a maximal interval of sites at which the path stays between the top and the well, $(0, x_c)$ ($x_c = 0.89$). In this chapter we reverse the direction by symmetry and write the core as $(0, x_c)^k$ (the mirror image of $Q_k = (-x_c, 0)^k$ in 23-09). The measure of the core $\mu_k^{\alpha,\beta}$ is $e^{-S/\hbar}$ with the outside values at both ends α, β fixed, restricted to $(0, x_c)^k$ with Neumann boundary, and the Dirichlet form is $\mathcal{E} = (\hbar/\varepsilon) \sum_i \mathbb{E}|\partial_i f|^2$ (converted from the convention $\hbar \int |\nabla g|^2$ of 23-09 by $C^{(23\text{-}09)} = C/\varepsilon = 2C$). What the fast side of the gap-inside-the-box condition needs is that $C_P(\mu_k^{\alpha,\beta})$ is bounded uniformly in the length of the core k and in the end values (the row of the core Poincaré uniformity assumption).

The short-core step $k \le 7$: convexity (the Short-Core Convexity Lemma)

The Hessian of the action of the core is

$$\operatorname{Hess} S = \frac{1}{\varepsilon} T_k + \varepsilon \operatorname{diag} V''(x_i), \qquad T_k = \operatorname{tridiag}(-1, 2, -1)$$

, and the end values α, β enter S only in first-order terms, so they do not appear in the Hessian. From $\lambda_{\min}(T_k) = 4 \sin^2 \frac{\pi}{2(k+1)}$ Proved here (the closed formula for the eigenvalues of a tridiagonal matrix) and $\inf V'' = V''(0) = -\frac{1}{2}$, at $\varepsilon = \frac{1}{2}$

$$\operatorname{Hess} S \;\ge\; \kappa(k) := 8 \sin^2 \frac{\pi}{2(k+1)} - \frac{1}{4} \quad (7)$$

. This is the same formula that appeared in 23-09 § 06 as the smallest eigenvalue on the surface of the top; it is positive for $k \le 7$ and negative at $k = 8$ ($8 \sin^2(\pi/16) \approx 0.304 > \frac{1}{4}$, $8 \sin^2(\pi/18) \approx 0.241 < \frac{1}{4}$) Lean `kappa7_pos`, `kappa8_neg` (no assumptions).

Short-Core Convexity Lemma. If $\kappa(k) > 0$, then for all end values $\alpha, \beta \in \mathbb{R}$ and all $\hbar > 0$, $C_P(\mu_k^{\alpha,\beta}) \le \varepsilon/\kappa(k)$.

Proved here Proof. By (7), $S/\hbar$ is uniformly convex with $\kappa/\hbar$. A uniformly log-concave measure restricted with Neumann boundary to a convex domain (the box $(0, x_c)^k$) satisfies $\operatorname{Var} f \le (\hbar/\kappa) \int |\nabla f|^2$ Cited (Bakry–Émery on convex domains, checked in Theorem 1.2 (1) of Kolesnikov–Milman), and rewritten in $\mathcal{E} = (\hbar/\varepsilon) \int |\nabla f|^2$ this is $\le (\varepsilon/\kappa)\mathcal{E}$. ∎ Lean `lemma_BV` (assumptions: the lower bound of the Hessian is $\ge \kappa > 0$ independently of the end values, and the pointwise Bakry–Émery inequality).

The values are $\varepsilon/\kappa(k) = 0.133/0.286/0.543/0.973/1.749/3.42/9.18$ ($k = 1, \dots, 7$), independent of both $\hbar$ and the end values. The sup measured at $k = 1, 2$ (0.128 and 0.26)

lie within these Reading . **If the blocks are cut to 7 sites or fewer, the difficulty of the sup over end values disappears by convexity.** The price moves to the side of the correlation at the seams of the cut blocks.

The medium-core step $8 \le k \le 14$: two blocks (the Two-Block Lemma)

End-Reduction Lemma. If the chain $x_1, \dots, x_k$ is Markov, then for $a < b$, $c\big(\sigma(x_1, \dots, x_a), \sigma(x_b, \dots, x_k)\big) = c\big(\sigma(x_a), \sigma(x_b)\big)$.

Proved here Proof. Right-hand side $\le$ left-hand side follows from the inclusion of σ-algebras. Conversely, let F be a centred function of the front piece and G one of the back piece. By the Markov property $\mathbb{E}[G \mid x_1, \dots, x_a] = \mathbb{E}[G \mid x_a] = \mathbb{E}\big[\mathbb{E}[G \mid x_b] \mid x_a\big]$, so putting $\hat F = \mathbb{E}[F \mid x_a]$ and $\hat G = \mathbb{E}[G \mid x_b]$ gives $\mathbb{E}[FG] = \mathbb{E}[\hat F\, \mathbb{E}[\hat G \mid x_a]] = \mathbb{E}[\hat F \hat G] \le c(\sigma(x_a), \sigma(x_b))\|\hat F\|\|\hat G\| \le c(\sigma(x_a), \sigma(x_b))\|F\|\|G\|$. ∎

Two-Block Lemma. Let $1 \le a < b \le k$, $k - a \le 7$, $b - 1 \le 7$. Then

$$C_P(\mu_k^{\alpha,\beta}) \;\le\; \frac{M(a,b)}{1 - c(x_a, x_b)}, \qquad M(a,b) = \begin{cases} \varepsilon/\kappa(k-a) + \varepsilon/\kappa(b-1) & (b \ge a+2) \\ \max\big(\varepsilon/\kappa(k-a),\, \varepsilon/\kappa(b-1)\big) & (b = a+1) \end{cases} \tag{8}$$

Proved here Proof. Apply the Two-Projection Lemma to $\mathcal{G}_1 = \sigma(x_1, \dots, x_a)$ and $\mathcal{G}_2 = \sigma(x_b, \dots, x_k)$. By the End-Reduction Lemma, $c(\mathcal{G}_1, \mathcal{G}_2) = c(x_a, x_b)$. The conditional law given $\mathcal{G}_1$ is the chain $x_{a+1}, \dots, x_k$ of length $k - a \le 7$ (with ends x_a and β) restricted to the box, and the Short-Core Convexity Lemma gives $\mathrm{Var}(f \mid \mathcal{G}_1) \le (\varepsilon/\kappa(k-a))(\hbar/\varepsilon) \sum_{i>a} \mathbb{E}[|\partial_i f|^2 \mid \mathcal{G}_1]$. $\mathcal{G}_2$ likewise gives $\sum_{i<b}$ and $\varepsilon/\kappa(b-1)$. Taking expectations and putting them into (1), the coefficient of site i is $[i > a]\varepsilon/\kappa(k-a) + [i < b]\varepsilon/\kappa(b-1)$; both are switched on only when $a < i < b$, which happens only when $b \ge a + 2$. ∎ Lean `tb_far`, `tb_adjacent` (assumptions: the conclusion of the Two-Projection Lemma, the Short-Core Convexity Lemma for each block, $\mathcal{E}_i \le \mathcal{E}$, $c < 1$).

From the conditions $k - a \le 7$, $b - 1 \le 7$, $a < b$ we get $k \le 14$, and at $k = 15$ there is no (a, b) Lean `tb_range`, `tb_none_at_15` (arithmetic of natural numbers). The input is a single **two-point** maximal correlation $c(x_a, x_b)$, which is the second singular value of the normalised matrix of the joint distribution of two sites Cited (the singular-value representation of Witsenhausen and Rényi, checked in a restatement). For the measure on the lattice, this value can be computed from products of transfer matrices.

Values. Taking the ends as samples in $[0, x_c]$, the sup over $8 \le k \le 14$ of the upper bound (the Two-Block Lemma) is $22.1/30.5/66.4$; at 10^{-3} and $3 \cdot 10^{-4}$ it is attained at $k = 8$ with both ends 0, and at $3 \cdot 10^{-3}$ at $k = 13$ (at $3 \cdot 10^{-3}$, the value for $k = 8$ with both ends 0 is 15.5). The lower bound by polynomial test functions (Rayleigh–Ritz) is

$8.39/17.3/38.5$ Reading . $k = 8$ lies just outside the threshold of convexity; on the reading at $\hbar = 0$, the smallest eigenvalue of the Hessian at the constrained minimiser is 0.0175, heading towards $C_P \to \varepsilon/\lambda_{\min} = 28.55$. There is a peak at $\hbar \approx 10^{-4}$ (lower bound 57, upper bound 99) Reading , and it is not monotone in $\hbar$.

Why the form of the long-core step is not used here. In a core with $k = 8$ and both ends at the top, the alternating maximal correlation of subboxes of length 5 rises to $0.879/0.938/0.971$, and the coefficient of the long-core step becomes 76 or more Reading Lean `coeff_k8` (assumption: the measured $\bar{c} \geq 0.9713$). This is because the soft direction ($\lambda_{\min} = 0.0175$) pushes the correlation at the seams towards 1; the form of two blocks avoids the alternating correlation and can be paid with the two-point correlation alone.

The long-core step $k \geq 15$: applying box reduction to the core (the Box-Reduction Inequality for Long Cores)

The measure of the core is also one-dimensional and nearest-neighbour Markov (restriction to a product box preserves the Markov property), so the construction of § 02 and § 03 can be used literally inside the core. Taking subboxes of length $L_c = 5$ with overlap $l_c = 1$ ($n_c = 4$), the constant of the subboxes (all of 5 sites or fewer, including short ones cut at the ends) is $\leq \varepsilon/\kappa(5) = 1.749$ uniformly in the end values by the Short-Core Convexity Lemma, so for every k and all end values

$$C_P(\mu_k^{\alpha,\beta}) \;\leq\; \Big(1 + \frac{1}{4}\Big)\,\frac{\varepsilon/\kappa(5)}{1 - \bar{c}_{\text{core}}} = \frac{2.186}{1 - \bar{c}_{\text{core}}} \quad (9)$$

holds Proved here (the proof of the Box-Reduction Proposition transferred to the core) Lean `core_uniform_L5` (assumptions: the conclusion of the Box-Reduction Proposition, $C_{\text{box}} \leq 7/4$, the measured $\bar{c}_{\text{core}}$). $\bar{c}_{\text{core}}$ is the alternating maximal correlation between the even and odd subboxes inside the core; the values measured at $k \geq 15$ are ≤ 0.535 over end values (including negative ends beyond the top), and at $k = 20$–30 they are saturated to 7 digits Reading (a measurement of a **lower bound** of the maximal correlation by additive polynomial functions and two points at the seams). Putting this in gives $C_P \leq 4.70$.

The form of the alternating maximal correlation (the Closed Formula for the Alternating Maximal Correlation). In a stationary Gaussian chain $\ldots, L_j, R_j, L_{j+1}, R_{j+1}, \ldots$ with correlation t inside a piece and correlation s at the seams ($\operatorname{corr}(L_j, R_j) = t$, $\operatorname{corr}(R_j, L_{j+1}) = s$, $|s|, |t| < 1$; the chain is taken to continue infinitely on both sides), the maximal correlation between the even pieces and the odd pieces is $\varphi(|s|, |t|)$, where

$$\varphi(s,t) = \frac{s(1+t)}{1+s^2 t}, \qquad 1 - \varphi(s,t) = \frac{(1-s)(1-st)}{1+s^2 t} \quad (10)$$

. Derivation: for the uniform type $F = \sum_{j\ \mathrm{even}}(L_j + R_j)$, $G = \sum_{j\ \mathrm{odd}}(L_j + R_j)$, the covariance per period is $S_{AB} = 2s(1+t)^2/(1-s^2t^2)$, because by the Markov property a two-point correlation is the product of the correlations of the links in between; the variance is $S_{AA} = 2(1+t) + 2(1+t)^2 s^2 t/(1-s^2t^2)$, and the ratio is φ Proved here (sum of a geometric series) Lean `gauss_mode0`, `one_sub_phi`. This ratio falls within the framework of the convergence rate of two-block Gibbs sampling for a Gaussian target distribution Cited . The rate of Gibbs sampling that alternately redraws all the even pieces and all the odd pieces is the square of the largest eigenvalue of the block Jacobi matrix (the Two-Block Gibbs-Sampler Rate Theorem (Theorem 5 of Roberts–Sahu) of Roberts–Sahu 1997, and § 2.4), and that largest eigenvalue is the maximal correlation φ of the two pieces. The case of pieces of length 1 ($L_j = R_j$; the value $2s/(1+s^2)$ obtained by putting $t = 1$ in (10)) is in Theorem 1 of Pitt–Shephard 1999 and Corollary 2 of Mider–Jenkins–Pollock–Roberts 2022, its square being the rate in the limit of infinite chain length Cited . The value of (10) for general piece length is the same quantity as the convergence radius of line Jacobi iteration in numerical linear algebra (the bibliography on that side has not been cross-checked). That the maximal correlation of a Gaussian chain equals the linear canonical correlation is Cited (Kolmogorov–Rozanov, checked in a restatement); that the uniform type is the maximum within it has only been confirmed by computation Reading . Since $\varphi \geq s \geq st$, the naive product $c \leq s \cdot t$ is false Lean `naive_product_false` (assumption: $0 < s, t < 1$). The $c_{\mathrm{alt}} = 0.7308$ of the two-state chain in § 02 is the φ at $s = e^{-1/2}$, $t = e^{-1}$.

It does not reduce to two-point quantities. In the core, the Gaussian proxy φG built from the two-point maximal correlation agrees with the measured value at the order of 10^{-4} Reading . Even so, for a general Markov chain there is no inequality that bounds the alternating maximal correlation by two-point quantities alone. A chain of three variables X_1–X_2–X_3: $X_2 \sim \mathrm{Bern}(p)$; if $X_2 = 1$ then $X_1 = X_3 = 1$; if $X_2 = 0$ then X_1, X_3 are independently 1 with probability e; $p = 10^{-4}$, $e = 10^{-3}$. A function of a binary variable is affine, so the maximal correlation between binary variables is $|\,\mathrm{corr}\,|$, and $\mathrm{corr}(X_1, X_2)^2 = p(1-p)(1-e)^2/(m_1(1-m_1)) \leq 1/10$ ($m_1 = p + (1-p)e$). But $Z = X_1 X_3$ has $\mathrm{corr}(Z, X_2)^2 = p(1-p)(1-e^2)^2/(m_Z(1-m_Z)) \geq 98/100$ ($m_Z = p + (1-p)e^2$) Proved here (computation of covariances) Lean `rareR_pair`, `rareR_joint` (arithmetic of rational numbers). It is an example in which a non-linear function that holds a rare state from both sides is effective; the paper for the row of the alternating maximal-correlation assumption for the core needs a comparison theorem that uses the fact that the conditional law of the core is close to log-concave.

The Three-Range Core-Constant Proposition and the table of the three stages

Three-Range Core-Constant Proposition. For a set of end values $\mathfrak{E}$, let $B_2(\hbar;\mathfrak{E}) := \sup_{8\le k\le 14}\sup_{(\alpha,\beta)\in\mathfrak{E}^2}\min_{(a,b)} M(a,b)/(1-c_{k,\alpha,\beta}(x_a,x_b))$ and $B_3(\hbar;\mathfrak{E}) := 2.186/(1-\bar{c}_{\ge 15}(\mathfrak{E}))$. Then for every $k\ge 1$ and $(\alpha,\beta)\in\mathfrak{E}^2$

$$C_P(\mu_k^{\alpha,\beta}) \;\le\; \max\{\varepsilon/\kappa(7),\; B_2(\hbar;\mathfrak{E}),\; B_3(\hbar;\mathfrak{E})\} \quad (11)$$

Proved here (split into cases by k and apply the three steps) Lean `three_stage` (assumption: the upper bound of each stage).

Table 3 Core length and stage (ends sampled in $[0,x_c]$; triples are $\hbar = 3\cdot 10^{-3}/10^{-3}/3\cdot 10^{-4}$)

Stage	k	Tool	Grade	Upper bound	Lower bound
1	≤ 7	Short-Core Convexity Lemma (convexity)	Proved here + Cited (Bakry–Émery on convex domains)	≤ 9.18 (independent of $\hbar$ and the ends)	—
2	8–14	Two-Block Lemma + two-point maximal correlation	Proved here + Reading	$22.1/30.5/66.4$	$8.39/17.3/38.5$
3	≥ 15	Box-Reduction Inequality for Long Cores + alternating maximal correlation	Proved here + Reading + Assumption (the alternating maximal-correlation assumption for the core)	≤ 4.70 ($\bar{c}$ is a measurement of a lower bound)	—

From Table 3, $\sup_k C_P(\text{core}) \le 22.1/30.5/66.4$ Reading Lean `three_stage_3em4` (assumption: the measured values of the medium-core step and the long-core step). The ratios of upper to lower bound are $2.6/1.76/1.72$. The input of the medium-core step (the two-point maximal correlation) is a product of matrices and a second singular value, so it is of a type that can be put on interval arithmetic; but it needs a paper estimate of the difference between the lattice measure and the continuum measure, and the cost of running it is estimated at the order of 10^3 CPU hours (not yet). In the long-core step, what is measured is a lower bound, and there is no paper for the upper bound.

06

The set of ends — what is covered, and how to pay for bistability

Table 3 took the end values as samples in $[0,x_c]$. A core is a **maximal** interval of sites in $(0,x_c)$, so the neighbouring sites outside it actually take values in $(-\infty,0]\cup[x_c,\infty)$. We separate which ends are needed in which construction.

The core of the sector framework — ends only on the side of the well

The core of the sector framework (23-09) is defined as having both end values on the side of the well ($[x_c, \infty)$ in the direction of this chapter). So ends beyond the top do not enter there Proved here (by definition) Lean `p5_end_not_negative` (assumption: $x_c = 0.89$), and the set of ends has no hole. In the stationary chain, a fraction $1 - e^{-0.63/\hbar}$ of the mass of runs of length k is of this type, and the minimiser on the reading at $\hbar = 0$ is a single shape with every site pressed against the wall x_c Reading .

The core on the box side — ends beyond the top, and the wall of conditioning

In the core on the fast side of the Gap-Inside-the-Box Proposition (the core after splitting by the configuration of kinks), the outside neighbour can take negative values beyond the top. There, two local minimisers at $\hbar = 0$ appear in cores with $k = 9, 10$, the barrier is $\Delta S \in [1.1 \cdot 10^{-4}, 2.5 \cdot 10^{-3}]$ (depending on the end values), and the lower bound of CP for $k = 9$ with ends $(-0.035, -0.035)$ is $13.9/44.6/526$, growing in the manner of $e^{\Delta S/\hbar}$ Reading .

- **Both minimisers sit on the wall of conditioning.** One is a bump spanning the core interval as it is; the other has the zeros at both ends moved inwards by one site each. Because we conditioned the core interval to be "exactly this interval", a wall stands in the direction in which the zeros want to move, and the two types compete. The Peierls–Nabarro barrier of a discrete kink on this lattice is 10^{-12} or less, a different order from the bistability barrier $0.5\text{–}2.0 \cdot 10^{-3}$ Reading Lean `pn_vs_bistab` (assumption: measured values). **The bistability is not physical metastability; it is made by the choice of conditioning.**
- **Cutting by the binary of the bump brings the fibres back.** Splitting by the binary Y_3 of whether a site x_m near the top is at or above the threshold θ^* (whether there is a bump), the CP of each fibre returns, by the Two-Block Lemma (the same proof applies to products of intervals), to upper bound $25.8/46.2/101$ and lower bound $8.9/17.4/32.8$, the same order as the medium-core step of Table 3 Reading .
- **$e^{\Delta S/\hbar}$ only moves.** In exchange, the Poincaré constant of the marginal of the binary Y_3 takes on $e^{\Delta S/\hbar}$, with $\mathrm{Var}(\mathbb{E}[f \mid Y_3]) \geq 470\,\mathcal{E}(f)$ $(3 \cdot 10^{-4})$ Reading Lean `y_side_k9_3em4` (assumption: measured values). Cutting does not remove the barrier; it takes the form of transfer of 23-10 transfer type. Extending the constant of the core to the actual set of ends while keeping it $O(1)$ cannot be done by this split.

Paying with the margin of the action barrier — the row of the core action barrier

In the Gap-Inside-the-Box Proposition the constant of the core enters C_{loc}, and appears in K_4 only in the form $C_{\mathrm{loc}} \cdot D/(C\xi)^2$ ((5)). $D/(C\xi)^2$ is of the order of $e^{-2S_0/\hbar}$,

so the constant of the core need not be $O(1)$; it suffices that the action barrier is smaller than $2S_0$. We put this as an assumption row.

The row of the core action-barrier assumption. There are $c_*, p \geq 0$ and $\Delta S_* < 2S_0$ such that, for the conditional measures of the core on the box side and of the windows containing it (the core, its two neighbours and m sites outside, with arbitrary outer ends), $C_P \leq c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ (for small $\hbar$) Assumption .

Reading. On the reading at $\hbar = 0$ (numerical minimisers checked for stationarity and positive definiteness, and the height of the connection bracketed from above and below), the sup of the depth of non-global minimisers over the range $k = 8$–14, $m \leq 12$, outer ends $[-1.3, -0.01]$ is $\Delta S_*^0 \in [9.6 \cdot 10^{-3}, 1.4 \cdot 10^{-2}]$; its ratio to $2S_0 = 4/3$ is $\leq 1.05 \cdot 10^{-2}$, a margin of about 95 times Reading Lean `measured_ratio` (assumption: the measured upper bound $\Delta S^0 \leq 0.01401$). It is 5–8 times deeper than the value with the ends fixed ($1.77 \cdot 10^{-3}$), and the sup is attained where moving the outer ends makes the two types degenerate. The lower bound of $\hbar \ln C_P$ approaches ΔS^0 from below ($0.97 \Delta S^0$ by a fit), but the prefactor $\hbar^{-p}$ of the upper bound has not been measured.

Proposition (the local term vanishes). Under the core action-barrier assumption and the reading of ξ, $D/(C\xi)^2 \leq d\, e^{-2S_0/\hbar}$, for any $\delta > 0$

$$C_{\mathrm{loc}} \cdot \frac{D}{(C\xi)^2} \;\leq\; c_* d\, \frac{p!}{\delta^p}\, e^{-(2S_0 - \Delta S_* - \delta)/\hbar} \qquad (12)$$

Proved here Proof. From $e^x \geq x^p/p!$ ($x = \delta/\hbar > 0$), $\hbar^{-p} \leq (p!/\delta^p) e^{\delta/\hbar}$. Multiplying this by the two assumptions gives $e^{\Delta S_*/\hbar} e^{-2S_0/\hbar} e^{\delta/\hbar}$, which is (12). ∎ Lean `cloc_term_poly`, `cloc_term_vanishes` (assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$); negative control `no_vanish_if_no_gap` (if $\Delta S_* \geq 2S_0$, the estimate does not vanish).

If δ is taken smaller than $2S_0 - \Delta S_*$, the right-hand side of (12) goes to 0 as $\hbar \to 0$, and the measured values leave a margin of $2S_0 - \Delta S_*^0 \geq 1.319$. **The bistability of the core on the box side does not affect the coefficient K of the First-Step Upper-Bound Proposition, whether one conditions or not** (on the core action-barrier assumption and the reading at $\hbar = 0$). Cores with negative ends appear in the sector framework inside the sector of kink positions (the bridge), and their mass in the stationary chain has the rarity $e^{-1.13/\hbar}$ ($k = 9$) Reading .

07

The list of remaining rows

Table 5 The rows remaining in the box-reduction condition, the gap-inside-the-box condition and the core

Row	What remains	Present support	Grade
Row of common premises	Uniqueness of the measure at $T = \infty$, the nearest-neighbour Markov property, extension from local functions, regularity of the conditional measures and the coordinates of kinks	That the kernel of the transfer operator is positive (the original source for the version on $\mathbb{R}$ not yet checked)	Assumption
Alternating maximal-correlation assumption for boxes	The alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$ is uniform in $\hbar$ and the boundary values (a paper on the mixing of boxes)	The reading 0.731 of the two-state chain ($C = 2$), the form of the Closed Formula for the Alternating Maximal Correlation	Assumption
Uniformity in the boundary values	The $\sup$ over boundary values: near the top (the free-end type) and far away (steep descent)	The reading of c_{ff} for free ends; far away, only the prospect that the gap does not vanish	Assumption
Cross-term uniformity assumption	The cross term $\mathcal{E}_Y(\mathbb{E}[f \mid Y]) \le (1 + \chi)\mathcal{E}(f)$, with χ uniform	None (the type of a two-scale inequality)	Assumption
Kink-gas gap assumption	That the reduced kink gas falls in the Glauber type, and the loss at free ends	The closed formula of the Ising model and diagonalisation	Assumption
Segment local Poincaré assumption	Local Poincaré for segments with end values	The Six-Node In-Phase Poincaré Proposition in the ring version (23-06)	Assumption
Saddle-tube Poincaré assumption	Poincaré for the tube, uniform in the tube length $\le C\xi$	None	Assumption
Mechanisation of the medium-core step	Enclosing the two-point maximal correlation in interval arithmetic, and a paper on the difference between lattice and continuum	Floating-point transfer (cost estimated at the order of 10^3 CPU hours)	Reading
Alternating maximal-correlation assumption for the core	An upper bound for $\bar{c}_{\ge 15} < 1$ of the long-core step	Measurement of a lower bound ≤ 0.535. Does not reduce to two-point quantities (counterexample)	Assumption
Core action-barrier assumption	$C_P \le c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ for the core on the box side and its windows, in particular the prefactor $\hbar^{-p}$ and windows with $k \ge 15$, $m > 12$	The reading at $\hbar = 0$, $\Delta S_*^0 \le 1.4 \cdot 10^{-2}$, and the lower bound of $\hbar \ln C_P$	Assumption

The construction of the steps of the box-reduction condition and the gap-inside-the-box condition itself ((1)–(5), (7)–(12)) is algebra for which elementary proofs are given, and what remains as assumptions are the rows of Table 5. There is still no $\hbar$ at which all the rows of the four conditions are in place together (23-05 § 08).

08

The table of the five grades — the rows of box reduction, the gap inside the box and the core

Table 6 The rows of this chapter ("—" in the range of $\hbar$ marks an assumption row that has not been shown at any $\hbar$; the numbers in the row column are the numbers of the dependency graph in § 09, and "—" marks a row not in the graph)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[01]	Row of common premises (uniqueness, Markov property, extension, regularity)	Assumption	§ 03, Step 0 in 23-05 § 04	—	—
[16]	Two-Projection Lemma (1)	Proved here	Proof of Step 1 in 23-05 § 04	Independent of $\hbar$	`two_proj_var` (assumption: $A + B \le (1 + c)V$)
[17]	Efron–Stein for product measures (4)	Proved here	Proof in § 03 Step (b)	Independent of $\hbar$	—
[18]	Product over boxes (2) $\Longleftarrow$ [01], [17]	Proved here	§ 03 Steps (a)–(d)	Independent of $\hbar$	—
[19]	Box-Reduction Proposition (3)	Proved here	Proof in § 02	Independent of $\hbar$	`box_two_family`, `box_shift_average` (assumptions: the two-family inequality, $C_{\mathrm{box}} \ge 0$, the sum over overlaps $\le \mathcal{E}$)
[02]	Alternating maximal-correlation assumption for boxes	Assumption	§ 02	—	—
[12]	$c_{\mathrm{alt}} = 0.731/0.410/0.138$ ($C = 2/4/8$)	Reading	§ 02 (values of a family copying the values of the two-state chain)	Model independent of $\hbar$	—
[11]	$\ln \xi = 218.08/661.04/2212.72$, $\xi/\ell_\times \ge e^{213}$	Reading	§ 02	Three points	`xi_over_ellx` (assumption: the measured difference)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
—	Positive controls (Gaussian, Ising)	Reading	Table 1	—	`gauss_box_le_one`, `ising_K1_holds` (assumption: measured values)
[20]	Decomposition of the variance and the product on the fast side	Proved here	§ 04	Independent of $\hbar$	—
[03], [04]	Segment local Poincaré assumption, saddle-tube Poincaré assumption	Assumption	§ 04	—	—
[05], [06]	Kink-gas gap assumption, cross-term uniformity assumption	Assumption	§ 04	—	—
[09]	Mobility of a single kink $D = \hbar / S_0$	Cited	23-04 § 03, the step inside the box in 23-05	$\hbar \to 0$	—
—	The gap of the Glauber box with fixed ends (6)	Assumption	§ 04 (original source bibliographic only; a 6-digit reading by diagonalisation on the lattice)	—	`glauber_split` (equality)
[21]	$c_{\mathrm{kg}} = C^2 + \pi^2$	Proved here	§ 04 (algebra of the limit of (6))	Independent of $\hbar$	`kg_constant` (assumption: $C\xi \neq 0$)
—	c_{ff} for free ends and the values of K (Table 2)	Reading	§ 04	Model independent of $\hbar$	`K_ff_C2`, `K_ff_C4` (assumption: measured values)
[31]	Gap-Inside-the-Box Proposition (5)	Proved here	§ 04	Independent of $\hbar$	`H4_two_scale`, `H4_gap_form` (assumptions in § 04)
[10]	Bakry–Émery on convex domains	Cited	Kolesnikov–Milman Theorem 1.2 (1)	Independent of $\hbar$	—
[22]	Short-Core Convexity Lemma ($C_P \le \varepsilon / \kappa(k)$ for $k \le 7$)	Proved here	§ 05 (7) and proof	Independent of $\hbar$	`lemma_BV`, `kappa7_pos`, `kappa8_neg`
[23]	End-Reduction Lemma	Proved here	Proof in § 05	Independent of $\hbar$	—

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[24]	Two-Block Lemma (8), for $k \leq 14$ only	Proved here	Proof in § 05	Independent of $\hbar$	`tb_far`, `tb_adjacent`, `tb_range`, `tb_none_at_15`
—	Singular-value representation of maximal correlation	Cited	Witsenhausen 1975, Rényi 1959 (checked in a restatement)	—	—
[13]	The two-point maximal correlation of the medium-core step and B_2 $(22.1/30.5/66.4)$	Reading	§ 05 Table 3	Three points	—
[27]	Box-Reduction Inequality for Long Cores (9)	Proved here	§ 05 (a copy of the Box-Reduction Proposition)	Independent of $\hbar$	`core_uniform_L5` (assumption: the measured $\bar{c}$)
[07]	Alternating maximal-correlation assumption for the core: an upper bound for $\bar{c}_{\geq 15} < 1$	Assumption	§ 05	—	—
[14]	$\bar{c}_{\geq 15} \leq 0.535$ (measurement of a lower bound)	Reading	§ 05	Three points	—
[25]	The ratio of the Closed Formula for the Alternating Maximal Correlation (10)	Cited (framework) + Proved here (closed formula)	§ 05 (geometric series), the Two-Block Gibbs-Sampler Rate Theorem of Roberts–Sahu 1997	Independent of $\hbar$	`gauss_mode0`, `one_sub_phi`, `naive_product_false`
—	For Gaussians, maximal correlation = linear canonical correlation	Cited	Kolmogorov–Rozanov (checked in a restatement)	—	—
[26]	Two-point quantities cannot bound the alternating maximal correlation (example with three variables)	Proved here	§ 05	—	`rareR_pair`, `rareR_joint` (arithmetic of rational numbers)

Row	Statement (one line)	Grade	Basis	Range of $\hbar$	Machine checking
[28]	Three-Range Core-Constant Proposition (11)	Proved here	§ 05	Independent of $\hbar$	`three_stage` (assumption: the upper bound of each stage)
[29]	The ends of the core of the sector framework are only on the side of the well	Proved here	§ 06 (by definition)	Independent of $\hbar$	`p5_end_not_negative`
[15]	Bistability of the core on the box side, the fibres of the binary of the bump, the marginal of Y_3	Reading	§ 06	Three points	`pn_vs_bistab`, `y_side_k9_3em4` (assumption: measured values)
[08]	Core action-barrier assumption	Assumption	§ 06	—	—
—	$\Delta S_*^0 \le 1.4 \cdot 10^{-2}$, ratio to $2S_0 \le 1.05 \cdot 10^{-2}$	Reading	§ 06 (reading at $\hbar = 0$)	$\hbar = 0$	`measured_ratio` (assumption: the measured upper bound)
[30]	The local term vanishes (12)	Proved here	Proof in § 06	$\hbar \to 0$	`cloc_term_poly`, `cloc_term_vanishes` (assumptions: [08], [11], $\Delta S_* < 2S_0$)

The "Proved here" rows of Table 6 are either algebra on top of assumption rows or general inequalities whose proofs are given in the text. They do not mean that the box-reduction condition and the gap-inside-the-box condition have been shown, nor that the constant of the core has been bounded unconditionally.

09

The dependency graph

The format is the same as 23-05 § 07: **the numbers to the right of "⟸" are always smaller than the number on the left**.

[01]	Assumption	row of common premises: uniqueness at T=∞, Mar
[02]	Assumption	alternating maximal-correlation assumption for
[03]	Assumption	segment local Poincaré assumption
[04]	Assumption	saddle-tube Poincaré assumption: Poincaré for
[05]	Assumption	kink-gas gap assumption: box gap of the reduce
[06]	Assumption	cross-term uniformity assumption: cross term ∂
[07]	Assumption	alternating maximal-correlation assumption for
[08]	Assumption	core action-barrier assumption: C_P ≤ c ħ^(−p)
[09]	Cited	kink diffusion-coefficient row: mobility D = ħ
[10]	Cited	Bakry–Émery on convex domains
[11]	Reading	row of the growth of the correlation length: ξ
[12]	Reading	values of c_alt
[13]	Reading	two-point maximal correlation of the medium-co
[14]	Reading	c̄ ≤ 0.535 (measurement of a lower bound)
[15]	Reading	bistability, fibres of the binary of the bump
[16]	Proved here	Two-Projection Lemma (1) Lean two_proj_var
[17]	Proved here	Efron–Stein for products (4)
[18]	Proved here	product over boxes (2) ⟸ [01][17]
[19]	Proved here	Box-Reduction Proposition (3) ⟸ [02][16][18] Lean box_two_family, box_shift_average
[20]	Proved here	decomposition of the variance ⟸ [01]
[21]	Proved here	c_kg = C²+π² Lean kg_constant
[22]	Proved here	Short-Core Convexity Lemma (7) ⟸ [10] Lean lemma_BV (assumptions: lower bound of th
[23]	Proved here	End-Reduction Lemma ⟸ [01]
[24]	Proved here	Two-Block Lemma (8) ⟸ [16][22][23] Lean tb_far, tb_adjacent, tb_range
[25]	Proved here	ratio of the Closed Formula for the Alternatin
[26]	Proved here	example that does not reduce to two-point quan
[27]	Proved here	Box-Reduction Inequality for Long Cores (9) ⟸
[28]	Proved here	Three-Range Core-Constant Proposition (11) ⟸ Lean three_stage (assumption: upper bound of
[29]	Proved here	ends of the core of the sector framework on th
[30]	Proved here	the local term vanishes (12) ⟸ [08][11] Lean cloc_term_poly, cloc_term_vanishes
[31]	Proved here	Gap-Inside-the-Box Proposition (5) ⟸ [03][04] Lean H4_two_scale, H4_gap_form
[32]	Proved here	K_4 independent of ħ ⟸ [28][30][31]

```
[33] Proved here        composition C_P ≤ Kξ²S₀/ħ ⟸ [19][32]
                         Lean H3_H4_compose
```

How to check. The rows with ⟸ are [18] ([01][17]), [19] ([02][16][18]), [20] ([01]), [22] ([10]), [23] ([01]), [24] ([16][22][23]), [27] ([01][16][17][22]), [28] ([07][13][14][22][24][27]), [30] ([08][11]), [31] ([03]–[06][09][17][20][21]), [32] ([28][30][31]) and [33] ([19][32]); in every one the numbers on the right are smaller than the one on the left, and there is no circle.

What to read. The conclusion [33] depends on all the assumption rows [01]–[08] ([07] [08] through the constant of the core [28] and the local term [30]). [12][15][25][26][29] are premises of no conclusion; they are rows that explain the reading of values, the nature of the bistability, the form of the alternating correlation and the drawing of the line at the ends. The range of $\hbar$ of [28] is "three points" because the values of the medium-core step and the long-core step are readings at three values of $\hbar$; the form of (11) itself does not depend on $\hbar$.

We also check that the statements of the assumption rows do not contain the quantity of the conclusion. The quantity of the conclusion is $C_P(\mu)$ of the whole measure at $T = \infty$. [02][07] are quantities of correlation, [03][04][08] are constants of scraps, tubes, and cores and windows, [05] is a constant of the reduced gas, [06] is a comparison of forms, and [01] is a property of the measure; none of them contains $C_P(\mu)$.

Correspondence with the graph of 23-05. In the integrated graph (23-12), the leaves [02] (the segment local Poincaré assumption), [03] (the saddle-tube Poincaré assumption), [05] (the kink-gas gap assumption), [06] (the cross-term uniformity assumption) and [07] (the alternating maximal-correlation assumption for boxes) of 23-05 are the same nodes as [03][04][05][06][02] of this chapter; [04] (the core Poincaré uniformity assumption) of 23-05 is replaced by [28] (the Three-Range Core-Constant Proposition) and [30] (the local term) of this chapter, and the leaves move to [07][08]. [11] (the variance of the product of even boxes), [14] (the Box-Reduction Proposition), [15] (the Gap-Inside-the-Box Proposition) and [16] (the First-Step Upper-Bound Proposition) of 23-05 are [18], [19], [31] and [33] of this chapter.

10

To the next chapter

23-12 joins the dependency graphs of 23-05 to 23-11 into one, puts every row into one table of the five grades, and shows in a figure of bands that there is still no $\hbar$ at which all rows are in place. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter was the step that put the box-reduction condition and the gap-inside-the-box condition of its first step in the form of propositions. The

viewpoint of carrying the form of box reduction over to the four-dimensional lattice or to other quantities is taken up in Part 24 (24-03).

What this chapter can say

Can say	Cannot say
Proved here The product over boxes (2) and its proof, the Box-Reduction Proposition (3), the Gap-Inside-the-Box Proposition (5), the Short-Core Convexity Lemma, the End-Reduction Lemma and the Two-Block Lemma, the Box-Reduction Inequality for Long Cores, the ratio of the Closed Formula for the Alternating Maximal Correlation, the example that does not reduce to two-point quantities, the construction of the Three-Range Core-Constant Proposition, the local term vanishes (12) Lean (algebra)	The box-reduction condition and the gap-inside-the-box condition. The rows of the alternating maximal-correlation assumption for boxes, the cross-term uniformity assumption, the kink-gas gap assumption, the segment local Poincaré assumption, the saddle-tube Poincaré assumption, the alternating maximal-correlation assumption for the core and the core action-barrier assumption
Reading The size of ξ, the values of c_{alt}, c_{ff} and K, the constant of the core $22.1/30.5/66.4$, the bistability barrier and the 95-fold margin	Mechanisation of the upper bound for the constant of the core, uniformity as $\hbar \to 0$, the prefactor of the core action-barrier assumption
Cited Bakry–Émery on convex domains, the singular-value representation of maximal correlation, maximal correlation of Gaussians, the type of the box dynamics	A $\hbar$ at which all rows of the four conditions are in place

Sources and reproduction

Item	Kind	Source, tool
Bakry–Émery on convex domains	Cited	D. Bakry, M. Émery, Sém. Probab. XIX, LNM 1123 (1985) 177–206; A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017) 1680–1702 (Theorem 1.2 (1), checked in the text).
Box dynamics (discrete spins)	Cited	F. Martinelli, "Lectures on Glauber dynamics for discrete spin models", LNM 1717 (1999) 93–191 (Propositions 3.4 and 3.5, checked in the text of the preprint).
Singular-value representation of maximal correlation	Cited	H. S. Witsenhausen, SIAM J. Appl. Math. 28 (1975) 100–113; A. Rényi, Acta Math. Acad. Sci. Hungar. 10 (1959) 441–451 (statement checked in a restatement).
For Gaussians, maximal correlation = linear canonical correlation; maximal correlation of stationary chains	Cited	A. N. Kolmogorov, Yu. A. Rozanov, Theory Probab. Appl. 5 (1960) 204–208; R. C. Bradley, Probab. Surveys 2 (2005) 107–144 (checked in a restatement).

Item	Kind	Source, tool
The rate of two-block Gibbs sampling (the framework of the Closed Formula for the Alternating Maximal Correlation, and the case of pieces of length 1)	Cited	G. O. Roberts, S. K. Sahu, J. R. Stat. Soc. B 59 (1997) 291–317, doi:10.1111/1467-9868.00070 (the Two-Block Gibbs-Sampler Rate Theorem (Theorem 5), Corollary 3, § 2.4, checked in the text); M. K. Pitt, N. Shephard, J. Time Series Anal. 20 (1999) 63–85 (Theorem 1; checked in the text of the 1996 working-paper version, the published version not cross-checked); M. Mider, P. A. Jenkins, M. Pollock, G. O. Roberts, arXiv:2009.10440v2 (2022) (Theorem 2, Corollary 2, checked in the text). The bibliography on the line Jacobi side has not been cross-checked.
The Efron–Stein inequality	Proved here	B. Efron, C. Stein, Ann. Statist. 9 (1981) (bibliographic; the proof is given in § 03).
Glauber dynamics of the one-dimensional Ising model	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294–307 (bibliographic only; the values are readings by diagonalisation on the lattice).
Eyring–Kramers type (the barrier and $\hbar \ln CP$)	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014) (the statement in the case without boundary checked in the text; the version for minimisers on a wall has not been obtained, and the core action-barrier assumption is an assumption row).
Algebra of the construction	Proved here, Lean	Proofs in the text. `two_proj_var`, `box_two_family`, `box_shift_average`, `H4_two_scale`, `H4_gap_form`, `H3_H4_compose`, `lemma_BV`, `tb_far`, `tb_adjacent`, `three_stage`, `gauss_mode0`, `cloc_term_poly`.
Numbers	Reading	Floating point for the transfer operator (two routes: the power method in the logarithmic domain and dense eigenvalues), diagonalisation of the Ising model on the lattice, polynomial Rayleigh–Ritz, numerical minimisers and heights of connection at $\hbar = 0$. Interval arithmetic is not used.

Terms used in this chapter — box dynamics (reduction to the box), even and odd boxes, averaging over shifts, maximal correlation, alternating maximal correlation, the Two-Projection Lemma, the Efron–Stein inequality, the reduced kink gas, free end, cross term, core, the threshold of convexity, the Two-Block Lemma, reduction to the ends, the Three-Range Core-Constant Proposition, the wall of conditioning, the binary of the bump, the action barrier. Their meanings are in the A-02 glossary.

Next chapter: 23-12 The integrated table and the dependency graph — puts every row of the four conditions into one table of the five grades, and confirms that the dependency graph of the steps has no circle.

The integrated table and the dependency graph

> This chapter is this book's own investigation. § 01 and § 08 are guide material; § 06 and § 07 are a map of open questions.

What this chapter shows — For the four double-well conditions of the first step, we put every row of the tables of the five grades of 23-05, 23-06, 23-07, 23-09 and 23-11 into one table, and renumber the five dependency graphs and join them into one. We put it in a form in which readers can check by the rule of numbering (the number of a premise is smaller than the number of a conclusion) that there is no circle, and we see in figures where the computer-assisted rows lie on the road to the conclusion, and that there is still no $\hbar$ at which all rows are in place together.

Prerequisite chapters — 23-05 § 07 (the format of the dependency graph), 23-06 § 05, 23-07 § 05, 23-09 § 07, 23-11 § 08 (the table of each condition), 23-00 § 01 (the labels of the five grades).

Said first — The level is [Y3]. The integrated table is a map of "how far things are shown, and where assumption begins"; it is not a ruler for measuring progress. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter lays out on one sheet the whole construction of its first step.

The order of this chapter

1. How the integrated table is made — the rules for putting five tables on one sheet, how identical rows are merged, the meaning of the columns
2. The integrated table — every row of the first step (Table 1), and the number of rows per grade (Table 2)
3. The dependency graph — the trunk (Figure 1) and the whole, how to check that there is no circle, and that the assumption rows do not contain the quantity of the conclusion
4. The family of bridges — the one-dimensional and multi-dimensional one-sided bridges
5. The position of the computer-assisted rows — what is on the road to the conclusion and what is not
6. There is no $\hbar$ at which all rows are in place — bands of the range (Figure 2, Table 4), and what is needed to bring them into line
7. The rows needed to reach the power statement — two ways of supporting the row of the core (Table 5)
8. A note on how to read it — a map, not a fraction

01

How the integrated table is made — putting five tables on one sheet

The claim of the first step is the Double-Well Relaxation-Time Upper-Bound Proposition (23-05 § 04): on the four double-well conditions and the premise rows, $C_P \le K\xi^2 S_0/\hbar$, and hence the double-well relaxation-time power statement, $\ln C_P / \ln \xi \to 2$. Each of the four conditions is written in its own chapter as a table of rows in the five grades (Proved here, Cited, Computer-assisted, Reading, Assumption) and as a dependency graph of numbered nodes. In this chapter we put them on one sheet by the following rules.

1. **Count by rows.** The unit is each row of the tables, not the name of a condition (whether the in-phase local Poincaré condition is "closed"). This is because it is usual for rows with proofs and assumption rows to be mixed within one condition.
2. **Make identical rows one.** When the same statement appears in two chapters, we make it one row and write the nodes of both chapters in the source column. For example, [14] of 23-05 and [19] of 23-11 are both the Box-Reduction Proposition, and the window log-Sobolev threshold row of 23-07 and [06] of 23-09 are both the remaining single proposition (the Window Log-Sobolev Threshold Proposition). The core Poincaré uniformity assumption of 23-05 ([04]) is supported in 23-11 by the pair of the Three-Range Core-Constant Proposition and the local term, and in 23-07 and 23-09 by the lower bound of the core on the sector side, so we made it one node with an "or" of the two ways of support.
3. **Rows left out.** The rows of the list of detour plans (23-10) are not included, because none of them is a premise of any conclusion of the present construction. Of the rows of 23-04, the uniqueness of the measure is included in the row of common premises, and the remaining spectrum of the fluctuation operator (of Pöschl–Teller type) is included as a single explanatory row.
4. **Line up the family of bridges.** The One-Dimensional One-Sided Bridge Proposition (23-04 § 04) and the one-sided bridge (23-01 § 06) are the one-dimensional and multi-dimensional versions of the same bridge, "the relaxation time of the path measure is at least the square of the correlation length", so we included them as a family of adjacent nodes (§ 04). The other rows of Table 8 of 23-01 (the relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption and the rereading of mass, the chain, the version with discrete time, the infinite-volume step) do not enter the road to the conclusion of the first step, so they stay in the graph of 23-01.

The columns of Table 1 are as follows. **No.** is the number of the node in the dependency graph of § 03; **Grade** is the label of the five grades; **Range of $\hbar$** is what the row speaks about (nodes, window, three points, independent of $\hbar$, $\hbar \to 0$, or "—" for not shown at any $\hbar$); **Condition** is which condition the row belongs to; **Premise of which conclusion** gives the numbers of the nodes that take the row as a direct premise, and whether following the edges of implication reaches the final conclusion (the double-well power statement); **Source** gives the tables and graph nodes of each chapter. Machine checking (Lean theorem names and assumptions) is placed under each row of the graph in § 03.

02

The integrated table — every row of the first step

Table 1 is divided into five blocks by grade (within a block, in the order of the graph numbers). The smaller the number, the more it is a leaf (Assumption, Cited, Computer-assisted, Reading); the larger, the more it is a step of the construction; the last, [111], is the double-well power statement.

Table 1 The integrated table (every row of the first step. Numbers are the node numbers of the dependency graph in § 03, and the order of the table is the same as the graph. "To the double-well power statement" is whether following the edges of implication from the row reaches the conclusion [111])

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
Assumption	**Assumption rows (put without proof) (30 rows)**					
[001]	Uniqueness of the measure at T=∞, Markov property, extension from local functions, the limit step, regularity	Assumption	—	Common	[071][073][077][081][106]. To the double-well power statement: reaches	23-05 [01], 23-11 [01], 23-04 § 01
[002]	Saddle-tube Poincaré assumption: Poincaré for the tube, uniform in tube length ≤ Cξ	Assumption	—	Saddle-tube condition	[075]. To the double-well power statement: reaches	23-05 [03], 23-11 [04]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[003]	Kink-gas gap assumption: the box gap of the reduced kink gas is of the Glauber type	Assumption	—	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [05], 23-11 [05]
[004]	Closed formula for the gap of the Glauber box with fixed ends (original source bibliographic only)	Assumption	—	Gap-inside-the-box condition	Premise of no node. Part of the content of the kink-gas gap assumption (a reading not made an independent node in the graph)	23-11 Table 6 (the row of the Glauber box with fixed ends)
[005]	Cross-term uniformity assumption: cross term $\mathcal{E}_Y(E[f\|Y]) \le (1+\chi)\mathcal{E}(f)$, χ uniform	Assumption	—	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [06], 23-11 [06]
[006]	Alternating maximal-correlation assumption for boxes: the alternating maximal correlation between boxes c_alt < 1 is uniform	Assumption	—	Box-reduction condition	[072]. To the double-well power statement: reaches	23-05 [07], 23-11 [02]
[007]	Alternating maximal-correlation assumption for the core: upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	Assumption	—	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [07]
[008]	Core action-barrier assumption: C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	Assumption	—	Saddle-tube condition	[084]. To the double-well power statement: reaches	23-11 [08]
[009]	Continuous version of Holley's comparison (the tail lemma)	Assumption	Six nodes	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [05]
[010]	The Poincaré–Miranda theorem (covering by boxes)	Assumption	(0,0.2]	In-phase local Poincaré condition	[063]. To the double-well power statement: reaches	23-06 [08]
[011]	Variance-decomposition row: regularity and density of the functions used	Assumption	—	In-phase local Poincaré condition	[060]. To the double-well power statement: reaches	23-06 [10]
[012]	Margin row (large-end side): neighbour and end values > 2.5	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[013]	Margin row (between the nodes): ħ between the nodes (in uniform form)	Assumption	Outside the nodes	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[014]	Margin row (small ħ): ħ < 0.005 (in uniform form)	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[015]	Two-site gap row (large-end side): end values of the two sites > 4	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[016]	Transfer from the ring to segments (with end values)	Assumption	—	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 Table 2
[017]	Premises of interval arithmetic (IEEE correct rounding, errors of sums, our own exp)	Assumption	—	Saddle-tube condition	[042][043]. To the double-well power statement: **does not reach**	23-07 [01], 23-08 § 03
[018]	Seam limit assumption, random-end-value assumption, regularity at the seam of the ring	Assumption	—	Saddle-tube condition	[043][099][103]. To the double-well power statement: reaches	23-07 [02]
[019]	The small assumptions of the route of the sector input (inputs to the sectors)	Assumption	—	Saddle-tube condition	[099]. To the double-well power statement: reaches	23-07 [04]
[020]	Product Bound for the Window Log-Sobolev Constant (construction of the upper bound for A_W of the product form)	Assumption	—	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [05]
[021]	Marginal log-Sobolev input for the window (the form of LSI for the marginals of site windows)	Assumption	—	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [01]
[022]	Bridge volume-locality input (locality of volume on the bridge)	Assumption	—	Saddle-tube condition	[089]. To the double-well power statement: reaches	23-09 [02]
[023]	Barta's row $\kappa = 3/1000$, $b = 493/1000$	Assumption	Independent of ħ	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [03]
[024]	The boundary term of Bochner and regularity of eigenfunctions	Assumption	—	Saddle-tube condition	[090][092]. To the double-well power statement: reaches	23-09 [04]
[025]	The provisional rows of the Scalar-Strand Proposition (lifting assumption, regularity of the spectrum and others)	Assumption	—	Saddle-tube condition	[096]. To the double-well power statement: **does not reach**	23-09 [05]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[026]	Window Log-Sobolev Threshold Proposition: A_W < A_max(ħ); the remaining single proposition of the route of the window log-Sobolev inequality	Assumption	—	Saddle-tube condition	[099]. To the double-well power statement: reaches	23-09 [06], 23-07 [03]
[027]	The next eigenvalue 3/4 of the fluctuation operator (of Pöschl–Teller type)	Assumption	Independent of ħ	Explanation	Premise of no node. An explanatory row that does not enter the premises of the conclusion of the first step	23-04 Table 3
[028]	Ground-state assumption: discreteness of the spectrum, simplicity and positivity of the ground state (finite spatial lattice)	Assumption	—	Bridge	[107][108]. To the double-well power statement: **does not reach**	23-01 [06]
[029]	κ of the metric comparison assumption is uniform in β (the graduation of the multi-dimensional version)	Assumption	—	Bridge	[109]. To the double-well power statement: **does not reach**	23-01 [08]
[096]	Scalar-Strand Proposition (the lower end of the window, with a table of assumptions)	Assumption	8 nodes	Saddle-tube condition	Premise of no node. A candidate to support the remaining single proposition (the Window Log-Sobolev Threshold Proposition), with no edge of implication	23-09 [21]
Cited	**Rows citing an original source (10 rows)**					
[030]	Mobility of a single kink D = ħ/S_0 (at the level of the process)	Cited	ħ→0	Gap-inside-the-box condition	[075][105]. To the double-well power statement: reaches	23-05 [08], 23-11 [09]
[031]	Bakry–Émery on convex domains (Kolesnikov–Milman)	Cited	Independent of ħ	Saddle-tube condition	[076]. To the double-well power statement: reaches	23-11 [10]
[032]	Muckenhoupt's bracket B ≤ C_P ≤ 4B	Cited	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [03]
[033]	Correlation inequality for MTP_2 (monotonicity of e, ħκ ≥ 0)	Cited	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [04]
[034]	Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux)	Cited	Independent of ħ	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [06]
[035]	Brascamp–Lieb (the cross κ_f, the crossing row)	Cited	Independent of ħ	In-phase local Poincaré condition	[063][064]. To the double-well power statement: reaches	23-06 [09]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[036]	Integrated Bochner Formula (Proposition 1.3 of Ledoux)	Cited	Independent of ħ	Saddle-tube condition	[090][092]. To the double-well power statement: reaches	23-09 [07]
[037]	Singular-value representation of maximal correlation (Witsenhausen)	Cited	—	Saddle-tube condition	Premise of no node. A representation used for how the reading values of the medium-core step are obtained	23-11 Table 6
[038]	For Gaussians, maximal correlation = linear canonical correlation (Kolmogorov–Rozanov)	Cited	—	Saddle-tube condition	Premise of no node. Used for reading the Closed Formula for the Alternating Maximal Correlation	23-11 Table 6
[039]	Stationary path-measure assumption: formula for correlations of the stationary path measure	Cited	T=∞	Bridge	[108]. To the double-well power statement: **does not reach**	23-01 [07]
Computer-assisted	**Computer-assisted rows (4 rows)**					
[040]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbour and end in $[0,2.5]^2$)	Computer-assisted	Six nodes	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches (only by the road on which it joins the Assumptions [012]–[016] at [068])	23-06 [01]
[041]	Two-site gap row: $\lambda_c \geq 1.3\times10^{-4}$ (ends in $[0,4]^2$)	Computer-assisted	(0,0.2]	In-phase local Poincaré condition	[063][064]. To the double-well power statement: reaches (only by the road on which it joins the Assumptions [012]–[016] at [068])	23-06 [07]
[042]	$q_{free} \leq 0.998854904$ (the chain outside the window)	Computer-assisted	Window	Saddle-tube condition	[100]. To the double-well power statement: **does not reach**	23-07 [09]
[043]	$q_{out} \leq 0.998888530$ (the outer chain of the ring, pin ≤ 0)	Computer-assisted	Window	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [10]
Reading	**Reading rows (19 rows)**					
[044]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{(S_0/\hbar)}$), the size of $D/(C\xi)^2$	Reading	Three points, leading term	Conclusion	[084][111]. To the double-well power statement: reaches	23-05 [09], 23-11 [11]
[045]	c_{alt} = 0.731 / 0.410 / 0.138 (C = 2 / 4 / 8)	Reading	Model	Box-reduction condition	Premise of no node	23-11 [12]
[046]	Two-point maximal correlation of the medium-core step and B_2 (22.1 / 30.5 / 66.4)	Reading	Three points	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [13]
[047]	$\bar{c}_{(\geq 15)} \leq 0.535$ (measurement of a lower bound)	Reading	Three points	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [14]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[048]	Bistability of the core on the box side, the fibres of the binary of the bump	Reading	Three points	Saddle-tube condition	Premise of no node	23-11 [15]
[049]	Positive controls for the Box-Reduction Proposition (pure Gaussian, Ising)	Reading	—	Box-reduction condition	Premise of no node	23-11 Table 6
[050]	K ≈ 2.76 / 1.43 (C = 2), 1.78 (C = 4), and c_ff for free ends	Reading	Model	Gap-inside-the-box condition	Premise of no node	23-05 Table 2, 23-11 Table 6
[051]	$\Delta S^{0}_{*} \le 1.4\cdot10^{-2}$ (ratio to $2S_0 \le 1.05\cdot10^{-2}$)	Reading	ħ = 0	Saddle-tube condition	Premise of no node	23-11 Table 6
[052]	The constants of the route of the window-constant numerics π_*, Λ, C_1D, gap_W, the inner chain of the bridge	Reading	Three points	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [06]
[053]	The inner chain of the ring q_in ≈ 1.24–1.28	Reading	Four points	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [07]
[054]	The boundary stage q ≈ 1.004–1.045	Reading	Four points	Saddle-tube condition	[101]. To the double-well power statement: **does not reach**	23-07 [08]
[055]	ℓ_min = 38.3 / 121.9 / 406.8	Reading	Three points	Saddle-tube condition	[089][094]. To the double-well power statement: reaches	23-09 [08]
[056]	Upper bound of A_W 1857 / 2794 / 9106, lower bound 1.71 / 1.74 / 2.59	Reading	Three points	Saddle-tube condition	Premise of no node	23-09 [09]
[057]	How far the Feynman–Kac strand got (reading 0.54 times, paper upper bound 10^3 times)	Reading	Two points	Saddle-tube condition	Premise of no node	23-09 [23]
[067]	C ≤ 658 (numerical input)	Reading	Six nodes	In-phase local Poincaré condition	Premise of no node	23-06 Table 2
[089]	Volume locality: ℓ_min does not depend on k	Reading	Three points	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [14]
[094]	Needed value 25.1 / 90.3 / 312.3	Reading	Three points	Saddle-tube condition	[097]. To the double-well power statement: **does not reach**	23-09 [19]
[097]	How far the componentwise strand got (1.4–5.5 times the needed value)	Reading	Three points	Saddle-tube condition	Premise of no node	23-09 [22]
[102]	A_W of the product form ≈ $2.5\cdot10^{6}$–$4.7\cdot10^{7}$	Reading	Three points	Saddle-tube condition	Premise of no node	23-07 [14]
Proved here	**Rows with proofs (elementary proofs, algebra on named premises) (48 rows)**					

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[058]	Tilted Second-Order Taylor Lemma (second-order Taylor of the tilt and the Hölder remainder)	Proved here	Independent of ħ	In-phase local Poincaré condition	[061]. To the double-well power statement: reaches	23-06 [02]
[059]	Nearest-neighbour chain row: the marginal of the kept sites is a nearest-neighbour chain	Proved here	Independent of ħ	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [11]
[060]	Variance-decomposition row: decomposition of the variance	Proved here	Independent of ħ	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [12]
[061]	The continuum statement of the margin λ	Proved here	Six nodes	In-phase local Poincaré condition	[062]. To the double-well power statement: reaches	23-06 [13]
[062]	Gap on the coarse side $\geq \lambda$ (Gershgorin on the ring)	Proved here	Six nodes	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [14]
[063]	Nesting $\lambda_S \geq 2.3\times10^{-5}$	Proved here	(0,0.2]	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [15]
[064]	Crossing row: $v/\hbar \leq 5888$	Proved here	(0,0.2]	In-phase local Poincaré condition	[065]. To the double-well power statement: reaches	23-06 [16]
[065]	Algebra of rates $C_P \leq \varepsilon/t$	Proved here	Six nodes	In-phase local Poincaré condition	[066][067]. To the double-well power statement: reaches	23-06 [17]
[066]	Six-Node In-Phase Poincaré Proposition: $C \leq 1.6\times10^{10}$ (ring version)	Proved here	Six nodes	In-phase local Poincaré condition	[068]. To the double-well power statement: reaches	23-06 [18]
[068]	In-phase local Poincaré condition, segment local Poincaré assumption: assembly when the rows [012]–[016] hold in uniform form	Proved here	(0,0.2]	In-phase local Poincaré condition	[075]. To the double-well power statement: reaches	23-05 [02], 23-11 [03], 23-06 § 07
[069]	Two-Projection Lemma	Proved here	Independent of ħ	Box-reduction condition	[072][078][081]. To the double-well power statement: reaches	23-05 [10], 23-11 [16]
[070]	Efron–Stein for product measures (martingale differences and Jensen)	Proved here	Independent of ħ	Box-reduction condition	[071][075][081]. To the double-well power statement: reaches	23-11 [17]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[071]	Product inequality for boxes (conditional independence of even boxes)	Proved here	Independent of ħ	Box-reduction condition	[072]. To the double-well power statement: reaches	23-05 [11], 23-11 [18]
[072]	Box-Reduction Proposition: $C_P \le (1+1/n)C_{box}/(1-c_{alt})$	Proved here	Independent of ħ	Box-reduction condition	[105]. To the double-well power statement: reaches	23-05 [14], 23-11 [19]
[073]	Decomposition of the variance and splitting into a product on the fast side	Proved here	Independent of ħ	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [12], 23-11 [20]
[074]	$c_{kg} = C^2+\pi^2$ (algebra of the limit of the closed formula)	Proved here	Independent of ħ	Gap-inside-the-box condition	[075]. To the double-well power statement: reaches	23-05 [13], 23-11 [21]
[075]	Gap-Inside-the-Box Proposition: $C_{box} \le C_{loc} + (1+\chi)C_{kg}$	Proved here	Independent of ħ	Gap-inside-the-box condition	[104]. To the double-well power statement: reaches	23-05 [15], 23-11 [31]
[076]	Short-Core Convexity Lemma (convexity for $k \le 7$)	Proved here	Independent of ħ	Saddle-tube condition	[078][081][082]. To the double-well power statement: reaches	23-11 [22]
[077]	End-Reduction Lemma	Proved here	Independent of ħ	Saddle-tube condition	[078]. To the double-well power statement: reaches	23-11 [23]
[078]	Two-Block Lemma (two blocks for $8 \le k \le 14$)	Proved here	Independent of ħ	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [24]
[079]	The ratio in the Closed Formula for the Alternating Maximal Correlation (geometric series)	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [25]
[080]	Example in which two-point quantities cannot bound the alternating maximal correlation	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [26]
[081]	Box-Reduction Inequality for Long Cores ($k \ge 15$)	Proved here	Independent of ħ	Saddle-tube condition	[082]. To the double-well power statement: reaches	23-11 [27]
[082]	Three-Range Core-Constant Proposition	Proved here	Three points	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-11 [28]
[083]	The ends of the core of the sector framework are only on the side of the well	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-11 [29]
[084]	The local term vanishes as $\hbar \to 0$	Proved here	$\hbar \to 0$	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-11 [30]
[085]	Sector sum	Proved here	Independent of ħ	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [10]
[086]	Gibbs decomposition: the Gibbs variational inequality	Proved here	Independent of ħ	Saddle-tube condition	[087]. To the double-well power statement: reaches	23-09 [11]
[087]	Componentwise localisation	Proved here	Independent of ħ	Saddle-tube condition	[092]. To the double-well power statement: reaches	23-09 [12]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[088]	The scalar form is not uniform in k	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [13]
[090]	Localised Bochner	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [15]
[091]	Optimising the Gibbs tilt	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [16]
[092]	Sector composition and the closing condition	Proved here	Independent of ħ	Saddle-tube condition	[093][099]. To the double-well power statement: reaches	23-09 [17], 23-07 [11]
[093]	The formula for the needed value A_max	Proved here	Independent of ħ	Saddle-tube condition	[094]. To the double-well power statement: **does not reach**	23-09 [18]
[095]	If the thresholds disagree, $\ell = 0$	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [20]
[098]	At the single site at the top, diagonal dominance fails by ¼	Proved here	Independent of ħ	Saddle-tube condition	Premise of no node	23-09 [24], 23-07 Table 4
[099]	A positive lower bound for the gap of the core, uniform in k (sector side)	Proved here	—	Saddle-tube condition	[103]. To the double-well power statement: reaches	23-09 [25], 23-07 [15]
[100]	S_W of the bridge $\leq$ 1779.6	Proved here	Window	Saddle-tube condition	[102]. To the double-well power statement: **does not reach**	23-07 [12]
[101]	S_W of the ring $\leq$ 122745	Proved here	Window	Saddle-tube condition	Premise of no node	23-07 [13]

No.	Content (one line)	Grade	Range of ħ	Condition	Premise of which conclusion	Source (chapter nodes)
[103]	Core Poincaré uniformity assumption: Poincaré for the core, uniform in core length and end values	Proved here	—	Saddle-tube condition	[104]. To the double-well power statement: reaches. For the core on the sector side (both ends in the well), it is supported for every $\hbar \in (0,1]$ by the Dobrushin criterion of the coarse chain: conditional on the paper's proof (Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$, 239 intervals, interval arithmetic, $\rho_{col} \leq 0.400$) + a paper proof and Computer-assisted (Small-ħ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \leq 10^{-5}$, $\rho_{col} \leq 0.874$) + Cited (Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) in Bobkov–Ledoux 2000) and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper). No Reading rows. The core on the box side is not covered (the paper "The coarse-chain Poincaré inequality")	23-05 [04], 23-07 [16], 23-11 [28][30]
[104]	K_4 does not depend on ħ	Proved here	$\hbar \to 0$	Gap-inside-the-box condition	[105]. To the double-well power statement: reaches	23-11 [32]
[105]	First-Step Upper-Bound Proposition: $C_P \leq K\xi^2 S_0/\hbar$	Proved here	$\hbar \to 0$	Conclusion	[110]. To the double-well power statement: reaches	23-05 [16], 23-11 [33]
[106]	Family of bridges, one-dimensional version: One-Dimensional One-Sided Bridge Proposition ($R \geq 1$)	Proved here	Every ħ	Bridge	[110]. To the double-well power statement: reaches	23-05 [17], 23-04 § 04
[107]	Identity of the ground-state transformation	Proved here	Finite spatial lattice	Bridge	[108]. To the double-well power statement: **does not reach**	23-01 [11]
[108]	Family of bridges, multi-dimensional version: the one-sided bridge ($C_P \geq \kappa\xi^2$)	Proved here	Continuous time, $T=\infty$	Bridge	[109]. To the double-well power statement: **does not reach**	23-01 [12]
[109]	The lower half of the power for the multi-dimensional version	Proved here	$\xi > 1$	Bridge	Premise of no node	23-01 [17]
[110]	$1 \leq R \leq K'/\hbar$	Proved here	$\hbar \to 0$	Conclusion	[111]. To the double-well power statement: reaches	23-05 [18]
[111]	Double-well power statement: $\ln C_P / \ln \xi \to 2$	Proved here	$\hbar \to 0$	Conclusion	The final conclusion	23-05 [19]

Counting the rows by grade gives Table 2.

Table 2 The number of rows of Table 1 (by grade; the breakdown in the condition column is counted by rows, not by the names of conditions)

Grade	Rows	Main content
Assumption	30	The row of common premises, the saddle-tube Poincaré assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes, the alternating maximal-correlation assumption for the core, the core action-barrier assumption, [012]–[016] extending the in-phase local Poincaré condition beyond the nodes, the premises of interval arithmetic, the inputs to the sectors, the remaining single proposition (the Window Log-Sobolev Threshold Proposition), the Scalar-Strand Proposition (with a table of assumptions) and others
Cited	10	Mobility of kinks, Bakry–Émery on convex domains, Muckenhoupt, MTP_2, Helffer's criterion, Brascamp–Lieb, Bochner, two representations of maximal correlation, the formula for stationary correlations
Computer-assisted	4	The margin of the in-phase local Poincaré condition (six nodes), the gap of two sites, the chain outside the window, the outer chain of the ring
Reading	19	How the correlation length grows, maximal correlations and the values of the steps of the core, the constants of the route of the window-constant numerics, the inner chain, the rarity exponent, the needed value, the coefficient K of the reduced model and others
Proved here	48	The assembly of the Six-Node In-Phase Poincaré Proposition, the Two-Projection Lemma, the product over boxes, the Box-Reduction Proposition, the Gap-Inside-the-Box Proposition, the Three-Range Core-Constant Proposition, the five stages of the sectors, the First-Step Upper-Bound Proposition, the One-Dimensional One-Sided Bridge Proposition, the algebra to the double-well power statement and others
Total	111	Breakdown by condition: in-phase local Poincaré condition 25, saddle-tube condition 57, box-reduction condition 7, gap-inside-the-box condition 9, common 1, family of bridges 7, conclusion 4, explanation 1

There are many rows of the saddle-tube condition in Table 2 because the rows of the core of the saddle-tube condition are built from two sides (the three steps on the box side and the five stages on the sector side); it does not mean that the saddle-tube condition is further along than the others. Conversely, there are few rows of the box-reduction condition because the construction of the Box-Reduction Proposition is short and the remaining assumption rows are gathered into one, the alternating maximal-correlation assumption for boxes. The number of rows counts where there are proofs and where assumptions remain; it does not measure how many of the four conditions are done (§ 08).

03

The dependency graph — joining five graphs into one

We turn the 111 rows of Table 1 into a list of nodes in the same format as 23-05 § 07. **The numbers of the premises to the right of "⟸" are always smaller than the number of the conclusion on the left.** There is just one node, [103] (the core Poincaré uniformity assumption), for which either of two ways of support suffices, and there we write "(...) or (...)". The numbers in both brackets are smaller than [103], so the rule works in the same way.

The whole is long, so we first give Figure 1, which extracts only the trunk. The numbers in the trunk are the same as in the whole graph.

Figure 1 The trunk of the integrated graph (of all 111 nodes, only the nodes on the main roads to the conclusion [111]; numbers as in the whole)

```
[001] ASM  row of common premises (uniqueness, Markov property, extension, limit, regula
[002][003][005] ASM saddle-tube Poincaré assumption, kink-gas gap assumption, cross-term
[006] ASM alternating maximal-correlation assumption for boxes      [007][008] ASM alter
[012]–[016] ASM rows extending the in-phase local Poincaré condition beyond the nodes, b
[026] ASM Window Log-Sobolev Threshold Proposition (the remaining single proposition)
[040][041] CMP margin of the in-phase local Poincaré condition, gap of two sites (six no
[044] RDG ln ξ ≥ s/ħ
[066] PRV Six-Node In-Phase Poincaré Proposition (ring version, six nodes) ⟸ [065] (wi
[068] PRV in-phase local Poincaré condition, segment local Poincaré assumption ⟸ [066]
[072] PRV Box-Reduction Proposition ⟸ [006][069][071]
[075] PRV Gap-Inside-the-Box Proposition ⟸ [068][002][003][005][030][070][073][074]
[082] PRV Three-Range Core-Constant Proposition (core on the box side) ⟸ [007][046][04
[084] PRV the local term vanishes ⟸ [008][044]
[099] PRV lower bound of the core (sector side) ⟸ [026][092][019][018]
[103] PRV core Poincaré uniformity assumption ⟸ ([082][084]) or ([099][018])
[104] PRV K₄ independent of ħ ⟸ [103][075]
[105] PRV First-Step Upper-Bound Proposition ⟸ [030][072][104]
[106] PRV family of bridges, One-Dimensional One-Sided Bridge Proposition (one-dim. vers
[110] PRV 1 ≤ R ≤ K'/ħ ⟸ [105][106]
[111] PRV double-well power statement ⟸ [044][110]

(side road) [042][043] CMP chain outside the window, outer chain of the ring → [100][101
```

▶ Open the whole graph (111 nodes)

How to check. There are three steps by which readers can check for themselves. (1) Look one by one at the rows of the whole graph that have "⟸", and check that the numbers on the right are all smaller than the number on the left. At [103], look at both brackets. (2) If this rule is kept in every row, the number strictly decreases each time a premise is followed, so you eventually reach a leaf (a row without "⟸") and never return to the original row. That is all there is to "there is no circle". (3) If even one edge is found that does not keep the rule, it is a candidate for a circle of dependence, and the text has to name which step takes the conclusion of which step as a premise. In the graph of this chapter there was no such edge (all edges of the 111 nodes were checked by comparing numbers).

The assumption rows do not contain the quantity of the conclusion. The rule of numbering does not prevent the form in which the conclusion has slipped into the *content* of an assumption row (the form that puts the conclusion as a premise). So we read the statements of the assumption rows one by one. The quantities of the conclusion are three: the Poincaré constant $C_P(\mu)$ of the whole measure at $T=\infty$, the ratio $R = C_P/\xi^2$, and the power $\ln C_P / \ln \xi$. The 30 assumption rows fall into (i) properties of the measure ([001], [011], [018], [024]), (ii) constants of partial measures (scraps, tubes, cores, site windows: [002], [008], [012]–[016], [021], [022], [026]), (iii) quantities of correlation ([006], [007]), (iv) reduced models and comparisons of forms ([003], [004], [005]), (v) premises of computation and citation ([009], [010], [017], [019], [020], [023], [025], [096]), and (vi) the multi-dimensional version and explanatory rows ([027], [028], [029]); none of them contains $C_P(\mu)$, R or the power. The Poincaré constants contained in the rows of (ii) belong to the *partial* measures of scraps, tubes, cores and windows, not to the whole measure. This check is done by eye; machine checking does not replace it (23-13 § 05).

04

The family of bridges — the one-dimensional and multi-dimensional one-sided bridges

[106] the One-Dimensional One-Sided Bridge Proposition and [108] the one-sided bridge are the one-dimensional and multi-dimensional versions of a sentence of the same form. Both say "for the path measure,

the ceiling of the Rayleigh quotient of test functions local in time is the square of the correlation length", and from this come lower bounds for the relaxation time, $C_P \geq \xi^2$ (the One-Dimensional One-Sided Bridge Proposition) and $C_P \geq \kappa\xi^2$ (the one-sided bridge).

Table 3 The two versions of the family of bridges

Version	Statement	Premises	Role in this part
One-dimensional version [106]	One-Dimensional One-Sided Bridge Proposition: $R = C_P/\xi^2 \geq 1$ for the 0+1-dimensional double well	[001] (continuous time, $T \to \infty$; proof in 23-04 § 04)	The left inequality of (14). Gives the lower half of the double-well power statement without conditions
Multi-dimensional version [108]	One-sided bridge: $C_P \geq \kappa\xi^2$ on a finite spatial lattice	[028] assumption on the spectrum, [039] formula for correlations, [107] ground-state transformation (23-01 § 06)	[109] the lower half of the power for the multi-dimensional version (with the uniformity of the graduation [029]). Does not enter the road to the conclusion [111] of the first step

Putting the two side by side, one reads in the same form that the content of the diffusive-relaxation-time assumption is only "the upper half". The lower half is given automatically by the bridge (the one-dimensional version needs only the premise [001]; the multi-dimensional version needs [028][039] and the uniformity of the graduation [029]), and what the four double-well conditions take on is the upper half $C_P \leq K\xi^2 S_0/\hbar$. For the multi-dimensional version to apply to the four-dimensional Wilson measure, the rows of discrete time, infinite volume and graduation remain ([08][09][10] of the graph of 23-01).

05

The position of the computer-assisted rows — what is on the road to the conclusion and what is not

For the four "Computer-assisted" rows of Table 1, we look at the road to the conclusion [111].

- **[040] the margin, [041] the gap of two sites (in-phase local Poincaré condition).** There is a road [061] the continuum statement of the margin → [062], [063], [064] → [065] the algebra of rates → [066] the Six-Node In-Phase Poincaré Proposition → [068] the in-phase local Poincaré condition → [075] the Gap-Inside-the-Box Proposition → ... → [111]. But it is only the road on which they join the assumption rows [012]–[016] at [068]; there is no road that reaches the conclusion without passing through [068]. The computer assistance at the six nodes becomes a premise of the conclusion only when combined with the assumption rows extending it beyond the nodes, beyond the ends and to segments.
- **[042] the chain outside the window, [043] the outer chain of the ring (saddle-tube condition; the route of the chain outside the window and the route of the seam of the ring).** [042] → [100] S_W of the bridge → [102] A_W of the product form (Reading) stops there, and [043] → [101] S_W of the ring stops there. **Neither is on the road to the conclusion.** This is because the value of A_W of the product form lies outside the needed value that the remaining single proposition [026] asks for (23-09 § 03).

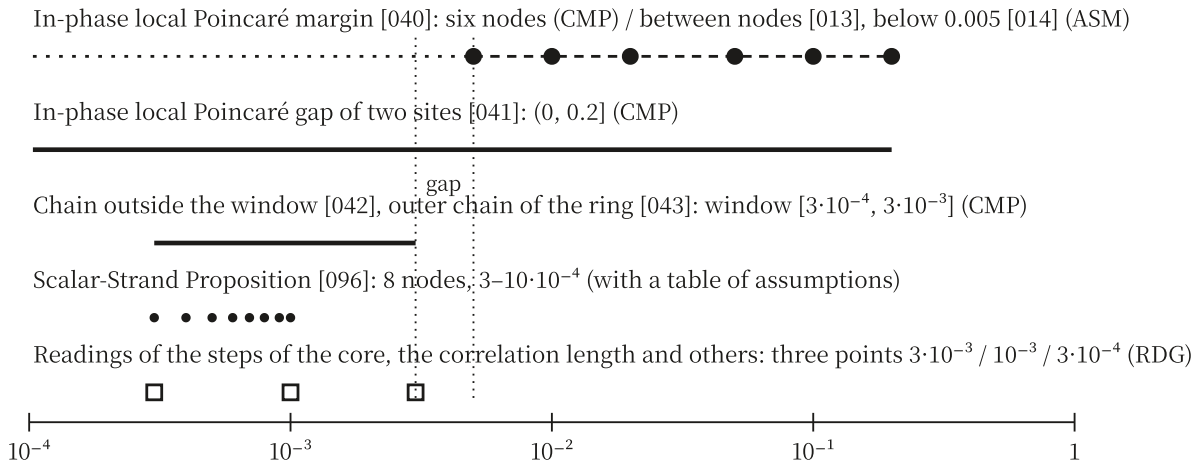
These two rows turn the contraction of the chains into inequalities over the whole window $\hbar \in [3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, and they stand as results of computer assistance within the window in their own right (23-08). But in the present construction they are not used as premises of the conclusion. They enter the road when an upper bound for A_W not of the product form satisfies [026] and the ratio of the chain q enters that upper bound (23-07 § 06). Put the other way round, the number of computer-assisted rows does not show how close one is to the conclusion.

06

There is no $\hbar$ at which all rows are in place

We overlay the range of $\hbar$ of each row on one number line (Figure 2). Assumption rows whose range is "—" ([002], [003], [005], [006] and others, most of the 30 assumption rows) are rows not shown at any $\hbar$, so they are not drawn; only the rows that have a range are drawn.

Figure 2 Bands of the range of $\hbar$ (the horizontal axis is $\log_{10}\hbar$, from 10^{-4} to 1). Solid lines and circles are computer-assisted, dashed lines are assumptions, squares are readings, and small dots are the node values of a proposition with a table of assumptions



There are three things to read from Figure 2. (a) The computer assistance for the margin of the in-phase local Poincaré condition is at six nodes with $\hbar \geq 0.005$, and the smallest node 0.005 is larger than the upper end of the window $3 \cdot 10^{-3}$ (the "gap" between the vertical dotted lines in the figure). (b) The Scalar-Strand Proposition is point values at 8 nodes at the lower end of the window, $[3 \cdot 10^{-4}, 10^{-3}]$, and a little beyond 10^{-3} the two conditions can no longer hold together in the same window (the reading that they do not hold together at $1.35 \cdot 10^{-3}$, 23-09 § 05). (c) The six nodes of the in-phase local Poincaré condition and the 8 nodes of the Scalar-Strand Proposition do not meet, and there is no node of the in-phase local Poincaré condition inside the window. Moreover, the assumption rows whose range is "—" (the saddle-tube Poincaré assumption, the kink-gas gap assumption, the cross-term uniformity assumption, the alternating maximal-correlation assumption for boxes and others) remain assumptions at every $\hbar$. Therefore **there is still not a single $\hbar$ at which all the rows of the four conditions are in place together.**

Table 4 Three kinds of range of $\hbar$ (do not mix them; four kinds together with the three points of 23-04 § 07)

Kind	Range	Which rows	Grade
The nodes of the in-phase local Poincaré condition	$\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$	[040], [061]–[066]	Computer-assisted , Proved here (between the nodes and below 0.005, Assumption of [013] [014])
The computer-assisted window	$[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ (25 subintervals)	[042], [043], [100], [101]	Computer-assisted , Proved here
The range of the Scalar-Strand Proposition	8 nodes in $[3 \cdot 10^{-4}, 10^{-3}]$ (no enclosure between the nodes)	[096]	Assumption (a proposition with a table of assumptions)
The three points of reading	$\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$	[044], [046], [047], [052]–[056] and others	Reading

What is needed to bring them into line. To bring them into line over the same range of $\hbar$, one of the following is needed.

1. **Extend the in-phase local Poincaré condition to $\hbar \leq 10^{-3}$.** That is, show row [014] ($\hbar < 0.005$). The type of computer assistance by subintervals cannot be used, because with a fixed ratio the number of subintervals, $\ln(0.005/\hbar)/\ln r$, grows without bound as $\hbar \to 0$; a paper argument uniform as $\hbar \to 0$ is needed (the small-ħ remaining item of 23-06 § 07).
2. **Extend the Scalar-Strand Proposition to $\hbar \geq 0.005$.** This corresponds to the route of the range of ħ (outside the window), a route decided to be outside the scope of this part (23-07 § 02). The form of the Scalar-Strand Proposition itself also stops at $1.35 \cdot 10^{-3}$.

Whichever is done, they only come into line over a fixed range. The double-well power statement is a statement about $\hbar \to 0$, so in the end every row has to be stated in a form uniform as $\hbar \to 0$, and computer

assistance in the window alone cannot reach it. Outside the window, paper asymptotics are needed. The assumption rows whose range is "—" remain at the stage before ranges are brought into line: first being shown at a single $\hbar$.

07

The rows needed to reach the power statement — two ways of supporting the row of the core

Following the graph from the conclusion [111] in the direction of the premises picks up all the leaves the conclusion needs. The only fork is the "or" at [103] (the core Poincaré uniformity assumption), so there are two required sets of leaves (Table 5). This applies to the graph of the first step the way of thinking that counts the sets that reach the conclusion when the assumption rows are put as "provisionally true" (for the list of walls on the side of the four-dimensional lattice, see 23-02 § 06).

Table 5 The leaves needed for the conclusion [111] (for each way of supporting [103]. In either set, every row listed is needed)

Grade of leaf	Support A: the core on the box side ([082] and [084])	Support B: the core on the sector side ([099])
Assumption	15 rows: [001], [002], [003], [005], [006], [007], [008], [009], [010], [011], [012]–[016]	20 rows: [001], [002], [003], [005], [006], [009], [010], [011], [012]–[016], [018], [019], [021], [022], [023], [024], [026]
Cited	[030], [031], [032], [033], [034], [035]	[030], [032], [033], [034], [035], [036]
Computer-assisted	[040], [041]	[040], [041]
Reading	[044], [046], [047]	[044], [055] (and [089], which depends on it)

What to read from Table 5. (1) The assumption rows common to both ways of support are the common premise [001], the rows inside the box [002][003][005], [006] between boxes, and [009]–[016] extending the in-phase local Poincaré condition. These remain whichever road is taken. (2) Support A puts the alternating maximal correlation of the core [007] and the action barrier [008] as assumptions, and needs the readings of the medium-core step and the long-core step, [046][047], to be replaced by upper bounds. Support B puts the remaining single proposition [026] and the inputs to the sectors [019], [021], [022], [024] and Barta's row [023] as assumptions. (3) Both sets still contain "Reading" rows. For the conclusion to become unconditional, every "Assumption" row has to be replaced by a proof, and every "Reading" row by an upper bound (Proved here, Cited or Computer-assisted).

08

A note on how to read it — a map, not a fraction

The integrated table is a map that shows where in the claim of the first step there are proofs and where assumptions remain. A proportion such as "48 of 111 rows are Proved here" does not show progress. There are two reasons. One is that the rows do not have equal weight (the single row [001] holds everything from the uniqueness of the measure to regularity, while the single row [074] is the algebra of a sum of two terms). The other is that many of the "Proved here" rows are algebra that takes assumption rows as input, and as long as one assumption row remains, the conclusion stays conditional (the same as the note after Table 2 of 23-05).

When reading it as a map, the following order helps. (a) Follow the trunk of Figure 1 from the conclusion [111], (b) look at the required set of leaves in Table 5, and (c) trace the "Assumption" rows in it from the source column of Table 1 to the text of each chapter. What remains and its cost are laid out row by row of this table in 23-15.

This part places the relaxation-time assumption at the axis of the study, and has built the upper half of the first step by dividing it into four conditions. This map, which puts the whole on one sheet, is also a map within that way of dividing. Other viewpoints are taken up in Part 24.

What this chapter can say

Can say	Cannot say
The integrated table putting the rows of the five chapters of the first step on one sheet (111 rows: Assumption 30, Cited 10, Computer-assisted 4, Reading 19, Proved here 48), and the dependency graph that keeps the rule of numbering on every edge (no circle)	Any of the 30 assumption rows. Hence the conclusion of the First-Step Upper-Bound Proposition and the double-well power statement of the first step
That the statements of the assumption rows do not contain the quantities of the conclusion $CP(\mu)$, R or the power (a check by reading the statements)	That the assumption rows are compatible with each other, or that none of them is false
The position of the four computer-assisted rows: the two rows of the in-phase local Poincaré condition become premises of the conclusion only on the road where they join [012]–[016], and the two rows of the saddle-tube condition are not on the road to the conclusion	Carrying the computer-assisted results outside the window or to $\hbar \to 0$
That there is no $\hbar$ at which all rows are in place together (by comparing ranges), and the two things needed to bring them into line	Either of the two things to bring them into line. Rows in a form uniform as $\hbar \to 0$
That there are two required sets of leaves for the conclusion (Table 5)	Which of the two ways of support is easier

Sources and reproduction

Item	Kind	Source, tool
The statement, grade and range of each row of Table 1	This book	23-05 Table 2, 23-06 Table 2, 23-07 Table 4, 23-09 Table 5, 23-11 Table 6, 23-04 Table 3, 23-01 Table 8. The original sources of the cited theorems are in the source table of each chapter
Edges of the dependency graph	This book	The "⟸" of each chapter's dependency graph copied as it is, with identical rows merged (the rules of § 01). The "or" at [103] comes from the notes on correspondence in 23-05, 23-07 and 23-11
Checking the rule of numbering, counting the rows, the required set of leaves	Text processing	A small script that, from data copying out the rows and edges of the table, compares the numbers on every edge, counts rows by grade, and collects leaves from the conclusion in the direction of the premises (not a mathematical computation)
Lean theorem names and assumptions	Lean	Under each row of the graph. The list of statements and the limits of checking are in 23-13

Terms used in this chapter — integrated table, dependency graph, node, leaf, trunk, the rule of numbering, circle of dependence, the form that puts the conclusion as a premise, family of bridges, way of support, required set of leaves, bands of the range of $\hbar$ (nodes, window, the range of the Scalar-Strand Proposition, the three points of reading). Their meanings are in the A-02 glossary.

Next chapter: 23-13 Lean statements and the limits of machine checking — looks, through the list of bundles and examples, at what machine checking guarantees, "the conclusion follows from the propositions as written", and what it does not guarantee (the truth of the assumptions, the encoding of the objects, closing by putting the conclusion into a definition, circles of dependence).

Lean statements and the limits of machine checking

> This chapter is a guide. § 01, § 02 and § 05 are established knowledge.

What this chapter shows — We confirm, with the list of bundles of the first step, the convention that what machine checking (Lean) guarantees is only that "the conclusion follows from the propositions as written (the list of assumptions)", and that it guarantees neither that the assumptions are true nor that the real numbers inside Lean represent physical quantities. We look, through examples with their sources withheld, at two types that cannot be detected even when `#print axioms` stays within the three standard axioms (putting the conclusion into a definition, and a circle of dependence), and we give a procedure by which readers can inspect statements themselves.

Prerequisite chapters — A-04 Lean and machine checking (the meaning of the label "Lean"), 23-12 § 03 (the dependency graph and the rule of numbering), 23-08 § 08 (assemblies that put the output of interval arithmetic as an assumption). For how to read Lean itself, see the Lean guide.

Said first — The level is [Y2]–[Y3]. § 01– § 02 and § 06 can be read at [Y2]. This part places the diffusive-relaxation-time assumption at the axis of the study, and Lean is used as a tool to check the algebraic steps of its construction. This chapter shows neither the mass gap nor the continuum limit.

The order of this chapter

1. What machine checking guarantees — the arrow from a list of assumptions to a conclusion (example: the composition of the Double-Well Relaxation-Time Upper-Bound Proposition)
2. The three axioms and `#print axioms` — the criterion of cleanness, why `native_decide` is not used
3. The bundles of the first step — the number of theorems in each bundle, what algebra it is, how the assumptions are put (Table 1), the side of the four-dimensional lattice (Table 2)
4. Assumptions next to theorem names — the discipline of reading, and the fact that the layer of interval arithmetic is outside Lean
5. Four limits — the truth of the assumptions, the encoding of the objects, putting the conclusion into a definition, circles of dependence (Table 3)
6. A procedure for readers to inspect for themselves — six questions
7. Translating conventions — places where the same letter or the same name refers to different quantities (Table 4)
8. How the bundles are divided — by article, paper and subject

---

01

What machine checking guarantees — the arrow from a list of assumptions to a conclusion

Lean is a system in which a mathematical statement is written as a type and its proof as a term of that type, and a small core (the kernel) mechanically checks that the proof matches the statement (Lean guide 01). The rows in this book labelled "`Lean`" refer to theorems that have passed that check. The label guarantees only one thing.

The convention of this book

What is guaranteed in a row labelled "Lean" is that **if all the assumptions written in the statement of the theorem hold, then the conclusion written there holds**. It is not guaranteed that the assumptions hold, that the symbols in the statement represent the quantities of the text, or that the statement means the same as the claim in the text.

As an example, we copy the statement of the theorem for the composition of the First-Step Upper-Bound Proposition (23-05 § 04 Step 3 (composition)).

```
theorem H3_H4_compose (CP Cbox n ca K4 C ξ D hbar S0 : ℝ) (hn : 0 < n) (hca : ca < 1)
    (hh : 0 < hbar) (hS : 0 < S0) (hD : D = hbar / S0)
    (hH3 : CP ≤ (1 + 1 / n) * Cbox / (1 - ca)) (hH4 : Cbox ≤ K4 * (C * ξ) ^ 2 / D) :
    CP ≤ (1 + 1 / n) * K4 * C ^ 2 / (1 - ca) * (ξ ^ 2 * S0 / hbar)
```

How to read it. Everything in the brackets of the first line is a real number (`ℝ`); `CP` and `ξ` are just real numbers with names. `hH3` takes the conclusion of the Box-Reduction Proposition, $C_P \le (1+1/n)C_{\mathrm{box}}/(1-c_{\mathrm{alt}})$, and `hH4` the conclusion of the Gap-Inside-the-Box Proposition, $C_{\mathrm{box}} \le K_4(C\xi)^2/D$, as **assumptions**. What the theorem checked is the algebra by which (13) $C_P \le K\xi^2 S_0/\hbar$ follows from these two and $D = \hbar/S_0$. That `CP` is the Poincaré constant of the path measure at $T=\infty$, and that the conclusions of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition actually hold (that is, the four double-well conditions and the premise rows), are outside Lean.

This character of "only the arrow" is not a weakness but a premise of how it is used. Many of the "Proved here" nodes in the dependency graph (23-12 § 03) are algebra that takes assumption rows as input, and Lean checks that this algebra has no copying errors or sign errors. Because what enters as an assumption is named in the statement, it can be cross-checked against the list of assumption rows (the integrated table).

02

The three axioms and `#print axioms`

This section summarises existing knowledge (the three axioms of Lean and #print axioms). The introduction is in A-04 § 04.

On top of the type theory of Lean's kernel, the standard toolkit adds three axioms: propositional extensionality `propext` (equivalent propositions are equal), soundness of quotients `Quot.sound` (equivalent elements are equal in the quotient), and choice `Classical.choice` (extracting an element from a proof of existence). Which axioms a theorem depends on is listed by the command `#print axioms theorem_name`.

In the bundles of this book, we put `#print axioms` for the main theorems at the end of each file, and our criterion of cleanness is that **every line of its output is a subset of the three axioms**. Using `sorry`, which skips a proof, adds `sorryAx` to the list, and declaring an `axiom` oneself adds its name, so this criterion is a mechanical confirmation that "the proof has no hole and no arbitrary axiom has been added". In every bundle of Table 1 (the "copy of the axiom output" column of Table 1; including the copies retaken on a second machine on 2026-10-09), we reread that there is not a single axiom name other than the three. Some theorems do not use `Classical.choice` among the three (for example, `tb_range` uses only `propext` and `Quot.sound`).

`native_decide`: why it is not used
`decide` checks a proposition decided by finite case analysis by computation inside the kernel. `native_decide` translates the same computation into a Lean program, runs it as machine code, and trusts the answer. It is fast, but in exchange the range of trust widens from the kernel to the translator and the execution environment, and that trust appears in the list as an additional axiom (depending on the version, an axiom with a different name appears for each declaration). In the bundles of this book we do not use `native_decide`, and we put the criterion of cleanness as "every line of the list is a subset of the three axioms". Looking for the presence or absence of a particular axiom name would miss things when the naming changes between versions.

03

The bundles of the first step — what algebra was checked

The Lean theorems of the first step are divided into files (bundles) by subject. Table 1 lists the main bundles in the order of the chapters that cite them in the text. "Number of theorems" is the number of `theorem` / `lemma` declarations in the file, and the column "Representative theorem" gives the theorem name with that theorem's assumptions in the same column.

Table 1 The main bundles of the first step (the number of theorems is the number of declarations. Copy of the axiom output: ○ = the output of `#print axioms` at the end remains separately, and we reread that every line is a subset of the three axioms. Includes the copies retaken on a second machine on 2026-10-09)

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeOneScale`	12	The relaxation-time upper-bound assumption, the relaxation-time lower-bound assumption and the rereading of mass (the equality version)	`hup_is_gap_positivity` ($0 < m$, $CP = 1/m^2$, $CP \le K\xi^2$)	23-01	○
`LatticeGaugeHupBridge`	18	The one-sided bridge, the lower half of the power, negative controls	`hup_is_gap_positivity_of_lower` ($0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le CP$, $CP \le K\xi^2$), `first_power_does_not_bridge` (negative control)	23-01	○
`LatticeGaugeDoubleWellTestbed`	7	The skeleton of the finite sum of the One-Dimensional One-Sided Bridge Proposition	`local_rayleigh_ratio_le_one` ($0 < m$, $a_k \ge m$, $c_k \ge 0$, $\sum c_k a_k > 0$)	23-04	○
`LatticeGaugeTwoScalePoincare`	8	Splitting of the variance, curvature of the convex core, values of the reduced two-state model, the band ratio of Glauber	`split_poincare_le` (the splitting formula, two local Poincaré inequalities, the mean difference, additivity of forms, non-negative constants)	23-04, 23-05	○
`LatticeGaugeDoubleWellKramers`	10	From the bracketing of the ratio to the double-well relaxation-time power statement	`power_H_of_kink_ratio` ($0 < K, s, \delta$, $1 \le R \le K/\hbar$, $s/\hbar \le \ln \xi$)	23-04, 23-05	○
`LatticeGaugeH1Assembly`	17	The algebra of rates of the Six-Node In-Phase Poincaré Proposition, the arithmetic of the margin at six nodes, Gershgorin on the ring	`H1_assembly` (seven named ones: decomposition, the fine side, Helffer's criterion, the margin, $0 < \lambda$, the crossing row, the condition on rates)	23-06	○
`LatticeGaugeCoarseSideUniform`	20	Arithmetic of the crossing row, the two-sided form of Muckenhoupt	`T4_paper_numeric` ($\lambda_c \ge 1.3 \times 10^{-4}$)	23-06	○

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeIntervalQfreeWindow`	150	The linear-fractional map of the Linear-Fractional Lemma, the identity of the tilt, from covering to window, recomputation of the rate-limiting cells	`win_qfree_window_le_of_interval_table` (the table of interval-arithmetic output)	23-08	○
`LatticeGaugeRingUnconditionedWindow`	20	Assembly of the window for the outer chain of the ring	`unc_window_lt_one_of_table` (the table of interval-arithmetic output)	23-07, 23-08	○
`LatticeGaugeRingInnerChain`	22	The ratio at the corner of the wall, S_W of the ring	`sw_inner_le` ($q_{\mathrm{in}} \le 13/10$, $q_{\mathrm{out}} \le 0.998888530$)	23-07	○
`LatticeGaugePartitionFramework`	14	The sector sum, the Gibbs step, composition and the closing condition	`comp_close` ($X > 0$, b, $A \ge 0$, $\ell > 0$, and the three inequalities of Barta, componentwise localisation and Bochner)	23-09	○
`LatticeGaugeRemVProxy`	23	Assembly of the Scalar-Strand Proposition	`path5_conditional` (with the remainder inequality, the proxy ratio, rarity and the arithmetic margin named)	23-09	○
`LatticeGaugeDetourKills`, `…2`	17, 16	The arithmetic of "the line that killed it" for the detour plans	Lemmas for each type (23-10 Table 4)	23-10	○
`LatticeGaugeBoxReduction`	23	The Two-Projection Lemma, even and odd boxes, averaging over shifts, positive and negative controls	`box_shift_average` (the two-family inequality for each shift, $C_{\mathrm{box}} \ge 0$, the sum over overlaps $\le \mathcal{E}$)	23-05, 23-11	○
`LatticeGaugeBoxGap`	18	The Gap-Inside-the-Box Proposition, the algebra of c_{kg}, the composition of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition	`H3_H4_compose` (the statement of § 01)	23-05, 23-11	○

Bundle	Number of theorems	What algebra	Representative theorem (assumptions)	Chapter	Copy of the axiom output
`LatticeGaugeCoreMixing`, `CoreMaxCorr`, `CoreThreeStage`	15, 20, 18	The three stages of the core (the Short-Core Convexity Lemma, the Closed Formula for the Alternating Maximal Correlation, the Two-Block Lemma, combining the three stages)	`three_stage` (the upper bound of each stage)	23-11	○
`LatticeGaugeCoreActionGap`	15	The vanishing of the local term, drawing the line at the ends	`cloc_term_vanishes` (the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$)	23-11	○
`LatticeGaugeGlobalLyapunov`	21	The algebra of the Lyapunov criterion and the positive control of the one-dimensional Gaussian	`gauss1d_bbcg`, `gauss1d_equality` (no assumptions)	23-14	○

The theorems of Table 1 are all inequalities and equalities on real numbers, finite sums and sequences. There is no theorem that has measures, operators or spaces of paths as types. The bundle of interval arithmetic (most of the 150 declarations of `IntervalQfreeWindow`) recomputes the formula for the upper bound of the last step cell by cell in rational numbers; there are many declarations, but the content is arithmetic. The index of all bundles is shared with Appendix A-04.

The side of the four-dimensional lattice. The theorems of the roads tried on the four-dimensional lattice (23-02 § 07 Table 4), divided by the type of the object their statements speak about directly, are as in Table 2.

Table 2 The type of object and the character of the theorems on the side of the four-dimensional lattice (rows are theorems; the character symbols correspond to the classification of Table 4 of 23-02: R = reproof, A = arrangement, C = conditional, N = negative control, M = lattice theorem)

Type of object	Number of theorems	Examples
Real numbers (and natural numbers) only	23	`chain_band`, `drain_exponent`
Sequences, real functions	11	`S4_two_sided`, `conditional_scale_bound`
Sums over finite sets and finite groups	8	`intact_frac_le`, `meff_ge_of_corr`
Operators on an inner-product space	1	`tau_int_le`
Lattice gauge fields (quaternions, Haar measure)	3	`one_link`, `hess_gauge`
Total	46	Counted by character (41 rows, counting two paired theorems as one row): R 11, A 15, C 7, N 5, M 3

There are three theorems that have lattice gauge fields as a type, and there is no theorem about expectations, correlations or gaps of the Wilson measure. As for the correspondence of characters, "reproof, arrangement, conditional, negative control, lattice theorem" of Table 4 of 23-02 correspond one to one to R, A, C, N, M of this table.

04

Assumptions next to theorem names

In the text and tables of this part, when we write a theorem name, **we list the assumptions of that theorem in the same column** (example: `H1_assembly` (assumptions: seven named ones)). If only the theorem name is written, it is read at the stage of summary as "settled in Lean", and conditional algebra swells into an unconditional result. In the integrated table (23-12), too, we wrote the theorem names and assumptions under each node of the graph.

In the computer-assisted rows, the Lean column splits into three layers (23-08 § 08): (1) the algebraic core (monotonicity of the linear-fractional map, the identity of the tilt, the algebra from covering to window); (2) recomputation of the rate-limiting cells and of randomly chosen cells (not all cells); (3) the assembly that puts the table of interval-arithmetic output as an **assumption**

and derives the conclusion for the whole window. Interval arithmetic itself (interval evaluation of the map of a step, quadrature, invariant sets, the conventions of rounding) is outside Lean, and its correctness rests on cross-checks and control tests (23-08 § 07). The label of a theorem that puts the table as an assumption is read as "if the output of interval arithmetic is correct, the conclusion for the whole window is correct as algebra".

05

Four limits — what cannot be detected even within the three axioms

Even for a theorem that meets the criterion of cleanness (§ 02), the following four are not guaranteed. The first two restate the convention of § 01; the last two are types in which the way the statement is written itself empties the conclusion.

(1) The truth of the assumptions. A theorem does not prove a single step of its assumptions. If `hH4` of `H3_H4_compose` is false, the conclusion says nothing.

(2) The encoding of the objects. `CP` inside Lean is a real number with a name, and that it is the Poincaré constant of the path measure is a reading of the text. The check passes even if the same name refers to a different quantity in another bundle. For example, a theorem named `measured_ratio` exists in two bundles of the first step (`CoreActionGap` and `HierarchicalLink`), and they are statements about different quantities in different namespaces. A name is no guarantee of content.

(3) Putting the conclusion into a definition. If the desired property is written as it is inside a definition and that definition is taken as an assumption, the conclusion becomes a copy of the assumption, and the proof closes in one line. The following is pseudo-Lean showing only the type.

Putting the conclusion into a definition closes anything (pseudo-Lean, an example of the type)

```
def InSpectrum (E : ℝ) : Prop := E = 0 ∨ c ≤ E          -- write the desired property into the definition of "spectrum"
theorem gap (E : ℝ) (h : InSpectrum E) : E = 0 ∨ c ≤ E := h   -- just returns the assumption as it is
def massGap : ℝ := 1 / 2                                  -- put the value literally, and positivity closes too
```

All three lines pass the check, and the list from `#print axioms` is either empty or within the three axioms. Even so, "spectrum" is not tied to the spectrum of any operator, so there is no mathematical content. One document wrote one of the conjuncts of its main theorem in this type, and another as a number put literally. It is an example in which the theorem name contains "mass gap" and every check passes, yet the statement does not encode the object.

(4) A circle of dependence. Lean checks theorems one at a time. Even if theorem A takes the conclusion of B as an assumption and theorem B takes the conclusion of A as an assumption, both pass the check. Neither conclusion has been derived unconditionally, yet putting the two side by side makes it look "shown". The same holds when proofs are written in prose: one document put the final conclusion, that the gap is positive, as a premise of intermediate steps (clustering, uniqueness of the vacuum), "to be shown later". Drawn as a dependency graph, an edge goes back from a conclusion with a large number to a step with a small number, and the rule of numbering (23-12 § 03) is broken.

Table 3 The four limits, and how this book guards against them

Limit	Visible to the check?	How this book guards against it
(1) The truth of the assumptions	Not visible (assumptions are arguments)	Write the assumptions next to the theorem name (§ 04); count the assumption rows one by one in the integrated table (23-12 Table 1)
(2) The encoding of the objects	Not visible (it only gives names to real numbers)	State the types of the Lean objects in the text ("real numbers, finite sums" of § 03); translate conventions (§ 07); write, chapter by chapter, the correspondence between the quantities of the text and the symbols of the theorem statements
(3) Putting the conclusion into a definition	Not visible (stays within the three axioms)	Copy the theorem statement into the text and read the definitions back (questions 3 and 4 of § 06); give negative-control theorems (examples that collapse when an assumption is removed)
(4) A circle of dependence	Not visible (theorems are checked one at a time)	The rule of numbering of the dependency graph; checking that the assumption rows do not contain the quantity of the conclusion (23-12 § 03)

Negative-control theorems work as a guard against (3). For example, `no_bound_when_c_one` (if the two σ-algebras are the same, $c = 1$, the reduction to the box says nothing) and `first_power_does_not_bridge` (with the first power, the bridge cannot be built) show that the conclusion collapses when an assumption is removed, confirming that the assumption is not decoration but acts on the conclusion.

06

A procedure for readers to inspect for themselves — six questions

When reading a claim labelled Lean, ask the following six in order. They apply equally to the theorems of this book and to theorems of other documents.

1. **Cleanness.** Is every line of the output of `#print axioms` a subset of `propext`, `Classical.choice`, `Quot.sound`? If you do not have the output at hand, it is "unconfirmed".
2. **The list of assumptions.** Write out all the brackets before the colon in the theorem statement (the arguments whose types are propositions). Does a claim that the text calls "unconditional" still have propositional arguments?
3. **Trace back the definition of the conclusion.** Open the definitions of the names that appear in the conclusion (such as `InSpectrum`), and see whether they contain the same sentence as the conclusion. Are the assumption and the conclusion unfoldings of the same proposition?
4. **The type of the object.** Do the objects the claim should speak about (measures, operators, fields) appear as types in the theorem statement? If what appears is only real numbers, natural numbers and finite sums, the tie to the objects lies in the reading of the text.
5. **The direction of dependence.** If the claim joins several theorems, arrange as numbered nodes which theorem takes the conclusion of which theorem as an assumption, and see whether the numbers of premises can be placed smaller than the numbers of conclusions. If not, it is a candidate for a circle.
6. **Negative control.** Is an example given in which the conclusion collapses when one assumption is removed? If not, check for yourself whether that assumption acts on the conclusion.

Example. Applied to `H3_H4_compose` of § 01: (1) a subset of the three axioms; (2) the propositional arguments are the seven `hn`, `hca`, `hh`, `hS`, `hD`, `hH3`, `hH4`; (3) the conclusion is an inequality of real numbers and no tracing back of definitions is needed; (4) the objects are all real numbers; (5) `hH3` and `hH4` are the conclusions of the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition, and in the integrated graph [072], [075] and [104] are smaller than [105]; (6) `no_bound_when_c_one` shows that at $c_{\mathrm{alt}} = 1$ the reduction to the box says nothing (`hca` is not decoration). The answer is "if the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition hold, the inequality of the First-Step Upper-Bound Proposition follows as algebra", and no more.

07

Translating conventions — places where the same letter or the same name refers to different quantities

Lean theorems do not know the meaning of symbols, so the check passes even if the same letter refers to different quantities in different files. Table 4 translates the conventions inside the theorems into the notation table of this book (Appendix A-01) and writes discrepancies as discrepancies. The upper six rows are on the side of the four-dimensional lattice, the lower five on the side of the first step.

Table 4 Conventions inside Lean and the notation of this book

Quantity	Inside Lean	This book	Discrepancy
β	One-link integration $e^{\beta\langle u,M\rangle}$, $\langle u,M\rangle = \frac{1}{2}\mathrm{Re}\,\mathrm{tr}(UM\dagger)$	$\beta = 2N/g^2$	None ($\beta = 4/g^2$ at $N=2$). Against the convention of Shen–Zhu–Zhu, $\beta = 4\beta_{\mathrm{SZZ}}$
tr and angle	Link angle $U = e^{i\vartheta\cdot\sigma}$	$T^a = \sigma^a/2$, $U = e^{i\theta^a T^a}$	$\theta = 2\vartheta$, $C_P^{(\theta)} = 4C_P^{(\vartheta)}$. A constant factor, so it does not affect the power
a	`a` of `drain_exponent`	Lattice spacing	Lean's a is the reciprocal of the bookkeeping length, $1/\xi_{\mathrm{RG}}$
η	The static exponent, the exponent of susceptibility, the exponent of overlap	η_s and η_{dyn} separated by subscripts	Present (the same letter for three quantities)
κ	The one-loop slope, the coefficient of a model, the coefficient of the Casimir term, the contribution of faces sharing no edge	Only the one-loop slope $11/(3\pi^2)$	Present (the same letter for four quantities)
C_P (four dimensions)	A real number with a name	The Poincaré constant of Langevin. The normalisation is named	With the normalisations of the field A and the angle ϑ, $C_P^{(A)} = \beta C_P^{(\vartheta)}$
$\hbar$ and ε	`hbar` (a real number). The lattice spacing enters as a number	$\mathcal{E}(f) = (\hbar/\varepsilon)\sum_i \mathbb{E}\lvert\partial_i f\rvert^2$, $\varepsilon = \frac{1}{2}$	None. The conversion between the readings at $\varepsilon = 0.25$ and at $\varepsilon = \frac{1}{2}$ is not written (23-14 § 05)
The graduation of C_P for the core	C_P of the theorems of the stages of the core	23-11 uses the convention of $\mathcal{E}$, 23-09 the convention of $\hbar\int\lvert\nabla g\rvert^2$	$C^{(23\text{-}09)} = C/\varepsilon = 2C$ (23-11 § 05)

Quantity	Inside Lean	This book	Discrepancy
K	`K` of `power_H_of_kink_ratio`	K of the First-Step Upper-Bound Proposition and $K' = KS_0$	Lean's K is K' of the text (23-05 § 04 Step 5)
Theorem names	`measured_ratio`	—	Different theorems with the same name in two bundles ((2) of § 05)
Reading values	Numerical theorems such as `K_ff_C2`, `three_stage_3em4`	Reading	Arithmetic with measured values put as assumptions. The grade of the values stays "Reading" (it does not become "Proved here" through the Lean label)

08

How the bundles are divided — by article, paper and subject

The bundles (files) are divided by subject (one proposition and its lemmas). It is usual for the theorems cited by one chapter or article to span several bundles, and the "Chapter" column of Table 1 gives the correspondence. There are four disciplines for dividing them.

- **One subject per bundle.** Even mutually dependent subjects such as the Box-Reduction Proposition and the Gap-Inside-the-Box Proposition are put in separate bundles, and the composition goes into a bundle of its own. The boundaries between bundles become the boundaries between nodes of the dependency graph, and one can follow which bundle takes the conclusion of which bundle as an assumption.
- **`#print axioms` at the end.** At the end of each bundle we put `#print axioms` for the main theorems, and keep the output under the same name as the bundle (the "copy of the axiom output" of Table 1).
- **No internal names in names.** Bundle names and theorem names are given by mathematical content (`LatticeGauge…` followed by the name of the subject), and names visible to readers do not contain working names or names of people.
- **Decide the bundles handed to articles and papers.** When an article or paper cites theorems of some bundle, the list of the bundles containing the cited theorems is also placed on the article's side, together with the version of the bundles (the output of which check). If a bundle is later corrected, the list on the side of the citing article is corrected too.

This part places the diffusive-relaxation-time assumption at the axis of the study, and Lean was used for the algebraic steps of the construction of the first step. The viewpoint of doing proof checking in another form (for example, a formalisation that makes the objects themselves types) is taken up in Part 24.

What this chapter can say

Can say	Cannot say
`Lean` The theorems of the bundles of Table 1 pass machine checking in a form that makes the assumptions explicit as arguments. In every bundle of Table 1, every line of the remaining `#print axioms` output is a subset of the three axioms	The truth of the assumption rows. A check of reading Lean's real numbers as quantities of the path measure itself
The Lean theorems of the first step are all statements on real numbers, finite sums and sequences; there is no theorem that has measures or operators as types	A formalisation that makes measures and operators types
The four limits and how to guard against them (Table 3), the six questions for inspection	That a claim that has passed the six questions is true (the questions are necessary checks, not sufficient ones)
That the layer of interval arithmetic is outside Lean, and only the assembly that puts the table as an assumption is inside Lean	Formal verification of the implementation of interval arithmetic

Sources and reproduction

Item	Kind	Source, tool
The declarations and numbers of the theorems of the bundles	Lean	We read the declarations of the source of each bundle (Lean 4, Mathlib) and counted the `theorem` / `lemma` lines. Theorem names and assumptions are in each row of the text
Copy of the axiom output	Lean	We read the copy of the `#print axioms` output of each bundle and counted the absence of axiom names other than the three (`sorryAx` and others), after confirming with a control copy that contains such a name that it would be detected
The three axioms and `#print axioms`	Cited	J. Avigad, L. de Moura, S. Kong, S. Ullrich et al., "Theorem Proving in Lean 4", chapter Axioms and Computation

Item	Kind	Source, tool
The layers of type and the characters on the side of the four-dimensional lattice (Table 2)	This book	The count of theorems in 23-02 § 07 (the same 46 theorems)
Examples of the type that puts the conclusion into a definition and the type of a circle of dependence	This book	A reading of two external documents that claim a construction or a proof (the sources are withheld and only the types are written; no judgement is made on whether the documents are right or wrong)

Terms used in this chapter — machine checking, bundle, the criterion of cleanness, the three axioms, `#print axioms`, `native_decide`, encoding of the objects, putting the conclusion into a definition, circle of dependence, negative-control theorem, translating conventions. Their meanings are in the A-02 glossary.

Next chapter: 23-14 Reconciliation — the numbers of the record and the known numbers — lays out, row by row against the numbers of the record, the distance between each condition of the first step and the nearest known theorem, a summary of the bibliographic checks, and the known numbers used for positive controls.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time reconciliation, distance to known theorems, bibliography · Kiichi, Shiori and Rin

Reconciliation — the numbers of the record and the known numbers

> This chapter is this book's own investigation. § 01 and § 07 are guide material; § 02 is established knowledge; § 05 is a map of open questions.

What this chapter shows — For each of the four conditions of the first step, how far the nearest known theorem reaches and where it stops. A summary of the results of checking the statements of the known theorems cited in the first step against original sources and secondary sources, and the types of discrepancy that had occurred before the check. We lay out, row by row, the known numbers (closed formulae) used as positive controls against the numbers of the record, and divide them into agreement, conditional agreement, disagreement and not comparable. The nineteen-row cross-check on the side of the four-dimensional lattice is kept as it is.

Prerequisite chapters — 23-05 § 05 (where known theorems reach), 23-12 (the integrated table), 23-13 § 07 (translating conventions).

Said first — The level is [Y3]. Numbers agreeing shows neither that the assumptions are true nor that the mass gap exists. Many of the numbers that agreed are checks that recompute known values by another road. This part places the diffusive-relaxation-time assumption at the axis of the study, and this chapter holds the numbers and tools used in its construction up against outside numbers and tools.

The order of this chapter

1. The conventions of reconciliation — four verdicts, and three things to align
2. Distance to known theorems — for each condition, the nearest known result and what differs (Table 1)
3. Summary of the bibliographic checks — the support of the main citations, and the types of discrepancy (Table 2)
4. The numbers of the record and the known numbers — cross-check against the closed formulae used as positive controls (Table 3)
5. Rows that cannot be compared — numbers with no counterpart to check against
6. The nineteen rows on the side of the four-dimensional lattice — one table of cross-checks (Table 4)

7. A way of reading dependency graphs learned from outside documents — two types

---

01

The conventions of reconciliation — four verdicts, and three things to align

There are four verdicts.

- **Agreement**: the same quantity compared under the same conditions agrees within the error or the digits shown.
- **Conditional agreement**: it agrees, but how it agrees depends on one choice (the lattice spacing, the box, the convention, the form of extrapolation).
- **Disagreement**: compared as the same quantity, it differs beyond the error. Even when the reason is known, we write disagreement.
- **Not comparable**: the quantities or conditions differ, and no counterpart to check against is found.

Before comparing, we align three things: the **convention** (the same CP changes value with the normalisation of the Dirichlet form, Table 4 of 23-13), the **conditions** (the lattice spacing ε, the boundary, whether inside or outside a model), and the **kind of number** (closed formula, computer-assisted upper bound, floating-point reading, lower bound of a family). Especially in the first step, the counterpart is often "a model whose answer is known by a closed formula"; agreement there is a check that the tool works correctly (a positive control), not a check of the claim on the double well.

02

Distance to known theorems — condition by condition

This section summarises existing knowledge (known criteria and theorems for Poincaré inequalities). The introduction is in 1-10 § 08.

Table 1 of 23-05 listed where known theorems reach for the Double-Well Relaxation-Time Upper-Bound Proposition as a whole. Table 1 divides it by condition, and writes in one line the nearest known result for each condition and what differs.

Table 1 The nearest known result for each condition (the grade is the grade of the row citing the known theorem; statements checked in the text)

Condition	Nearest known result	Where it reaches	What differs	Grade

Condition	Nearest known result	Where it reaches	What differs	Grade
The in-phase local Poincaré condition	Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux), Bakry–Émery on convex domains (Kolesnikov–Milman Theorem 1.2)	The two-scale construction of the chain on the coarse side and the bridge on the fine side, the convex core at the bottom of the valley	The inputs to the criterion (the margin, the gap of two sites) are values at each node of $\hbar$, and it gives no input uniform in $\hbar$	Cited
The saddle-tube condition (transport along the family of saddles)	Poincaré version of Eyring–Kramers (Menz–Schlichting Lemma 2.4, Theorems 2.9 and 2.12)	Asymptotics and prefactor of the gap between valleys of a Morse function	**Fixes the dimension**: the constants are not uniform in dimension, and a degenerate family of saddles such as the translations of a kink is outside the assumptions	Cited
The saddle-tube condition (same as above)	Eyring–Kramers for the stochastic Allen–Cahn equation on a fixed interval (Di Gesù–Le Peutrec, Brooks–Di Gesù)	Asymptotics of the gap with the length of the interval fixed short of the bifurcation threshold	**Excludes the family**: the regime in which the interval becomes infinitely long, and the family of translations, are stated by the authors themselves to be outside the scope	Cited
The box-reduction condition	The box dynamics of Glauber dynamics (Propositions 3.4 and 3.5 of Martinelli's lectures)	The type that bounds the whole gap by the box dynamics and the gap inside the box	Discrete spins, in the form of total-variation coupling. The rewriting into one dimension, L^2 and the form of maximal correlation is on this book's side (23-11 § 02)	Cited (type)
The gap-inside-the-box condition	The closed formula for the Glauber dynamics of the one-dimensional Ising model, diffusion of a single interface (Bertini–Brassesco–Buttà Theorem 2.1)	The formula for the gap of a box with fixed ends, the diffusion coefficient D of the position of a single kink	The former is for discrete spins, and that the reduced kink gas falls in that type is an assumption (the kink-gas gap assumption). The latter is a statement at the level of the process, not a statement on the box gap with many kinks	Assumption (the original source of the closed formula has not been checked), Cited

Condition	Nearest known result	Where it reaches	What differs	Grade
The whole (uniform in volume)	A criterion uniform in volume (Bauerschmidt–Dagallier Theorem 2.1)	Applies directly to the lattice of paths, an upper bound uniform in length	**Uniform in volume but doubly exponential**: the value is $CP \le e^{c\xi}$ ($c \approx 1.4 \to 1.8$, read at $\varepsilon = 0.25$, $\hbar = 0.4 \to 0.1$), and since ξ grows exponentially in $\hbar$, it is doubly exponential in $\hbar$. It does not reach the double-well relaxation-time power statement	Cited, Reading (values)
The whole (an analogous case of a theorem of the same type)	The gap of the dynamics of hierarchical $\lvert\varphi\rvert^4_4$ (Bauerschmidt–Bodineau)	A theorem of the same type as the diffusive-relaxation-time assumption: the gap vanishes with the same power as the free field (with a logarithmic correction)	**Hierarchical**: a model of nested boxes, neither the lattice model itself nor the path measure of the double well. Cited as evidence that a question of this type stands as mathematics (23-01 § 06)	Cited

From Table 1, the known tools lie in four places, "asymptotics between valleys with the dimension fixed", "a single kink", "an upper bound uniform in volume but coarse" and "a theorem of the same type in another model", and within the range searched we found no theorem that gives a sharp upper bound on a long path up to the diffusion time of the kink gas, ξ^2/D. No row can replace any one of the four conditions wholesale by a known theorem; they take the form of citing a step within a condition (a criterion, a type, a coefficient).

03

Summary of the bibliographic checks — support, and the types of discrepancy

We picked up every row in the assumption tables of the four conditions of the first step marked "cites a known theorem", and checked it against the original source or a secondary source restating the statement. 55 items were picked up, of which 26 are main items entering the columns of the condition tables directly. Table 2 gives where the 26 main items went.

Table 2 Where the 26 main citations went (level of checking. "Changed into assumptions" means renamed as an assumption row rather than a citation row)

Destination	Items	Content

Destination	Items	Content
Statement checked in the text of the original source	4	Poincaré version of Helffer's criterion (Ledoux), Bakry–Émery on convex domains and Brascamp–Lieb (by a single theorem of Kolesnikov–Milman), Sturm's oscillation theorem
Statement checked in a restatement in a secondary source	8	Muckenhoupt's bracket (in the form of Barthe–Roberto), correlation inequality for MTP_2 (Karlin–Rinott), the type of Lyapunov and local Poincaré, the Poincaré constant of the exponential distribution, IMS localisation, the bathtub principle, the maximal correlation of n steps of a reversible chain, maximal correlation of Gaussians
Proof given in this book (elementary)	7	Decomposition of the variance and the steps around it, the direction of min-max, the ground-state transformation (integration by parts), the lemma of two projections, the Efron–Stein type
Original source needs to be obtained	4	The Poincaré–Miranda theorem, regularity of the weighted Neumann operator on a box, stochastic monotonicity (Herbst–Pitt), the principal eigenvalue of a positive integral kernel on $\mathbb{R}$
Changed into assumptions	3	The cross term (cross-term uniformity assumption), the type of the kink gas (kink-gas gap assumption), Eyring–Kramers for minimisers on a wall (core action-barrier assumption)
Total	26	19 with support (the top three rows)

The 4 items that need to be obtained are counted in the integrated table as "Assumption" rows ([010], [024], part of [001] and others in 23-12 Table 1). The rule of this part is that citations that cannot be checked in the text of the original source do not go into "Cited" (23-00 § 01).

The type of discrepancy — one word of the setting. Between the record before checking and the original sources there were 15 discrepancies. 12 were reattributions that do not change the mathematics (citing the version for convex domains through another theorem, correcting attributions, correcting pages), and 3 needed rewriting of assumption rows. The 3 rewrites were (a) a row that had put regularity of eigenfunctions on a box up to the corners as "standard" (for a box with corners and an asymmetric weight it does not follow from the standard results, so it was split off as an assumption row), (b) a row in which the theorem number of the continuous version of monotonicity in the end values was unconfirmed (left in the row as unconfirmed), and (c) a row that used a logarithmic Sobolev lemma in uncentred form (the restatement of the original source is in centred form).

The discrepancies all arose from **one word of the setting** — whether there is a boundary, whether it is finite, whether it is centred. And they arose not in rows where the record wrote the source as "memory" but in rows where it wrote "standard". Rows written as "standard" have a weak motive to be checked. This is why the five grades of this part have no grade "standard". Every cited theorem is always assigned to one of "statement checked in an original or secondary source (Cited)", "proof given in this book (Proved here)" and "not checked (Assumption)".

04

The numbers of the record and the known numbers — closed formulae used as positive controls

The tools of the first step (criteria, the machinery of interval arithmetic, reduction to the box) were first applied to models whose answers are known by closed formulae, and applied to the double well only after confirming that they work correctly. Table 3 is the list of those cross-checks; the ratio is "record side ÷ known side".

Table 3 Positive controls and calibration of the first step (grades on the record side by label; row numbers continue after Table 4 of § 06)

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
20	The best constant from applying the criterion of Lyapunov and local Poincaré (of BBCG type) to the one-dimensional Gaussian	Reading 5.828427 by scanning weights ($u = 0.41422$)	Closed formula $3 + 2\sqrt{2}$ ($u = \sqrt{2} - 1$)	1.000000	Agreement	6 digits. 8 digits in the dual form too. Lean `gauss1d_bbcg`, `gauss1d_equality` (no assumptions)
21	The true Poincaré supremum of the same model	Reading 4.4979 (400 finite elements), 4.4994 (800)	The value of the criterion $3 + 2\sqrt{2}$	0.77	Conditional agreement	The loss of the criterion is 1.30 times, and the direction (criterion $\geq$ true value) is right
22	CP of the pure Gaussian $V = x^2/2$ and the right-hand side of the Box-Reduction Proposition	Reading right-hand side 1.10–1.33 ($L = 4$–128)	$CP = 1$ (equality in Bakry–Émery)	1.10–1.33	Agreement	A check of the direction of the inequality. Approaches 1 as the proportion of overlap $\to 0$ Lean `gauss_box_le_one`
23	Both sides of the Box-Reduction Proposition for heat-bath Glauber of the one-dimensional Ising model	Reading holds in all 9 cases, loss 1.17–1.35 times	The gap by diagonalisation over all states	–	Agreement	Rings of 12 and 16 sites, couplings 0.3, 0.6, 1.0. A check of the direction of the inequality

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
24	The gap of the Glauber box with fixed ends	Reading diagonalisation over all states of small chains	Closed formula $1-\gamma\cos(\pi/(L+1))$	1.00000	Agreement	The 6 digits shown. The same for boundary values $(+,+)$ and $(+,-)$. The original source is bibliographic only
25	Alternating maximal correlation ($C=2$, $\theta=\frac{1}{4}$)	Proved here $\varphi(e^{-1/2},e^{-1})$ of the Closed Formula for the Alternating Maximal Correlation	The closed formula of the two-state telegraph chain, 0.7308	1	Agreement	Two expressions of the same quantity (23-11 § 02)
26	The action of the kink on the lattice	Reading $S_0^{\mathrm{lat}}=0.665266$ ($\varepsilon=\frac{1}{2}$)	The continuum $S_0=2/3$	0.998	Conditional agreement	The difference of 0.2% is due to the lattice spacing. There is no estimate of the difference between continuum and lattice
27	The logarithm of the correlation length $\ln\xi$	Reading two routes: the difference of eigenvalues, and the transfer-operator version of Herring's formula	The semiclassical leading term $\xi\propto\hbar^{1/2}e^{S_0/\hbar}$	–	Conditional agreement	The two routes agree to 6 digits at intermediate $\hbar$. $\ln\xi-S_0^{\mathrm{lat}}/\hbar-\frac{1}{2}\ln\hbar$ settles towards a constant, $-0.774/-0.777/-0.778$, which agrees with the form of the leading term. There is no counterpart for comparing the constant term
28	The diffusion coefficient of a single kink	Reading $D/(\hbar/S_0)=1.00$–1.14	$D=\hbar/S_0$ (the coefficient of the theorem at the level of the process)	1.00–1.14	Conditional agreement	Point values at $\varepsilon=0.25$, $\hbar=0.02$, 0.07, 0.1. Corrections for finite $\hbar$ and the spacing

#	Quantity	Record side	Known side	Ratio	Verdict	Conditions, notes
29	The machinery for fitting the criterion uniform in volume	Reading 1.0049 for the harmonic model	Closed formula 1	1.005	Agreement	A positive control for the transfer-matrix computation applying the criterion to the lattice of paths ($\varepsilon = 0.25$)
30	The machinery of interval arithmetic: the pure Gaussian chain	Computer-assisted every point is inside the enclosure with $\hbar$ kept as an interval	Variance per step $(k-1)/k$, mean $(k-1)s/k$	1	Agreement	With the interval width set to 0, the same as the point machinery to the order of 10^{-12} (23-08 § 07)
31	The machinery of interval arithmetic: another implementation and point values	Computer-assisted upper bound	Reading 75 floating-point recursions, 26 point values of the transfer matrix	—	Agreement	0 outliers. All point values below the upper bound (smallest difference 2.85×10^{-4})
32	The gap of the hierarchical Gaussian free field	—	L^{-2N} at depth N of the nested boxes (known)		Not comparable	A known number put as a counterpart for comparison. The number on the side of the free field of the Langevin dynamics of links has not yet been computed

The rows of agreement in Table 3 are all checks that "the tool worked correctly on a model whose answer is known"; they do not show that the rows on the double well are correct. The conditions of the rows of conditional agreement are the lattice spacing (rows 26 and 28), finite $\hbar$ (row 28), and the missing constant term of the counterpart (row 27). Row 21 measures, on a model whose answer is known, the loss by which the criterion comes out 1.30 times larger than the true value; it serves as a rough guide when comparing the upper bound of the criterion with the needed value on the double well (23-09 § 05).

```
05
```

Rows that cannot be compared — numbers with no counterpart to check against

We list the numbers of the first step for which no counterpart to check against is found. All of them are numbers at the places where the construction comes closest to the upper half of the diffusive-relaxation-time assumption, and only there is there no counterpart.

- **Conversion between the two lattice spacings.** The readings of the diffusion coefficient of kinks and of the lower bound of the ratio R are at spacing $\varepsilon = 0.25$, while the table of correlation lengths and the steps of the First-Step Upper-Bound Proposition are at $\varepsilon = \frac{1}{2}$, and the conversion between the two spacings is not written (the table of ranges in 23-04).
- **The coefficient of $1/\hbar$ of the logarithmic Sobolev constant of the core.** A quantity needed in the Feynman–Kac strand; the Monte Carlo reading and the pathwise upper bound are three orders of magnitude apart (23-09 § 05).
- **The reading of $\sup |\nabla \psi|^2$.** A reading that it is within 0.54 times the threshold; it is not an upper bound. No known value of the same quantity was found in the range searched.
- **The lifting assumption of the Scalar-Strand Proposition.** An assumption that lifts the measured values of the two-site lump linearly beyond the measured range; the counterpart for comparison is the lifted value itself (23-09 § 05).
- **Comparison with the hierarchical free field.** Row 32 of Table 3. The known number L^{-2N} exists, and the number on the record side does not yet.

06

The nineteen rows on the side of the four-dimensional lattice

We keep one table that holds the numbers computed by the record on the side of the four-dimensional lattice (the one-loop slope κ, the shortfall $\delta(b)$, the ratio $m/\sqrt{\sigma}$, upper bounds of mass, C_P and autocorrelation times, strong-coupling thresholds, the exponent of stochastic time) up against existing computations and the numbers of primary lattice sources. The ratio is "record side ÷ existing side", and it is left blank for rows that cannot be compared. The details of each row and the conversion of conventions are in the chapter on the side of the four-dimensional lattice (23-02) and the source tables of Part 5.

Table 4 Cross-checks on the side of the four-dimensional lattice (rows 1–19)

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
1	κ (window model)	0.371521	0.371511	1.00003	Agreement	Within the one-loop model

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
2	Two-loop exponent	51/121	$b_1/(2b_0^2)$	1	Agreement	Arithmetic of rational numbers
3	Lattice slope / one loop	0.304(2)	0.3715	0.819	Disagreement	$\beta \leq 2.60$, before asymptotic scaling
4	Lattice slope / two loops	0.304(2)	0.3972	0.766	Disagreement	Two loops widen the gap
5	c_2 of $\delta(b)$	0.0197, 0.4200	—		Not comparable	Different variables from Symanzik and MCRG
6	$m/\sqrt{\sigma}$ (p=2) / LT	3.844(43)	3.844(61)	1.000	Conditional agreement	The same data table; choosing p=2
7	$m/\sqrt{\sigma}$ (p=1) / LT	4.053(73)	3.844(61)	1.054	Conditional agreement	χ^2 cannot choose p
8	$m/\sqrt{\sigma}$ (p=2) / AT	3.844(43)	3.781(23)	1.017	Conditional agreement	1.3 times the combined error
9	Upper bound β=2.40 (APE)	1.025(140)	0.953(19)	1.076	Agreement	The same spatial side 12 ($L\sqrt{\sigma} \approx 3.2$)
10	Upper bound β=2.55 (APE)	0.531(50)	0.586(10)	0.906	Conditional agreement	-1.1σ. $12^3 \times 24$ ($L\sqrt{\sigma} \approx 1.9$) against 20^4
11	Upper bound β=2.60 (APE)	0.558(51)	0.514(8)	1.086	Conditional agreement	$12^3 \times 24$ ($L\sqrt{\sigma} \approx 1.6$) against 24^4
12	Upper bound β=2.60 (flow)	0.501(71)	0.514(8)	0.975	Conditional agreement	12^4 ($L\sqrt{\sigma} \approx 1.6$) against 24^4
13	Ratio m(2.40)/m(2.60)	1.837(302)	1.854	0.991	Conditional agreement	β=2.60 is a small box with $L\sqrt{\sigma} \approx 1.6$
14	Straight-line threshold / SZZ	1/12	1/48 (converted by $\beta = 4\beta_{SZZ}$)	1.000	Agreement	Rederivation of a known result
15	Threshold after one-link integration	$\sqrt{(2/27)} = 0.2722$	—		Not comparable	Threshold of the marginal measure. The constant moves by 3.27 times within the same criterion. Not a claim of a widened range

#	Quantity	Record side	Existing side	Ratio	Verdict	Conditions, notes
16	C_P of the free field	C_P^(A)=ξ², 4C_P(time 0)²	Gaussian measure, ground-state transformation	1	Agreement	Recomputation
17	τ_int/ξ² of Langevin	0.5–1.4	—		Not comparable	No primary source for the same quantity at zero temperature towards the continuum limit was found in the range searched
18	The pole in stochastic time	13/6, −11/3	Baulieu–Zwanziger	1	Agreement	Only one independent primary source
19	Exponent s	−9/22	g₀^{9/11} (Lüscher–Schaefer equation (1.1))	1	Conditional agreement	Conditional on the one-loop leading logarithm and the relaxation-time assumption. Not an independent check

On the side of the four-dimensional lattice there are 6 agreements, 8 conditional agreements, 2 disagreements and 3 not comparable (rows 3 and 4 are two readings of the same disagreement). On the side of the first step (rows 20–32 of Table 3) there are 8 agreements, 4 conditional agreements and 1 not comparable. Read together, the rows of agreement are recomputations of known values or checks of tools on models whose answers are known, and the rows with no counterpart gather, on both sides, at the numbers that come closest to the relaxation-time assumption ($\tau_{\mathrm{int}}/\xi^2$ on the side of the four-dimensional lattice; the coefficient of the core and the lifting assumption on the side of the first step).

07

A way of reading dependency graphs learned from outside documents

We read two outside documents that claim to solve the existence and mass gap of Yang–Mills, and drew from them, not their correctness, but two types useful as **ways of reading the construction of a claim**. We do not give the sources; we write only the types. This book does not judge whether those documents are right or wrong.

1. **The type that puts the conclusion as a premise of intermediate steps.** One document piled up steps in the form of theorems and proofs; intermediate steps (clustering, uniqueness of the vacuum, exponential decay) took as a premise the

positivity of the mass gap, "to be shown later", and the positivity of the mass gap itself was supported by physical relations and dimensional analysis. Read step by step, each step derives its conclusion from its premises, but when the steps are arranged as numbered nodes, an edge goes back from the final conclusion to the intermediate steps, and the rule of numbering is broken. It is a type that is **invisible to step-by-step checking, and visible once arranged in a dependency graph**.

2. **The type with an object for its name and numbers for its content.** The other document closed its main theorem in Lean. Tracing the conjuncts of the main theorem back to their definitions, "lies in the spectrum" was defined as the same proposition as the conclusion, and the value of the gap was a rational number put literally; operators, fields and measures appeared only as names. It is a type that is **invisible even when machine checking passes and the list of axioms is clean, and visible once the definitions are traced back** (23-13 § 05).

These two types should be applied in the same way to the construction of this book. In the integrated graph of the first step, we checked the rule of numbering on every edge, and checked by reading that the statements of the assumption rows do not contain the quantities of the conclusion (23-12 § 03). For the Lean theorems, that there is no name on the side of the conclusion whose definition needs tracing back (only inequalities of real numbers) can be checked by the six questions of 23-13.

This part places the relaxation-time assumption at the axis of the study, and has held the numbers and tools of its first step up against outside numbers and tools. Other viewpoints are taken up in Part 24.

What this chapter can say

Can say	Cannot say
Cited Where the nearest known result for each condition stops (Table 1). Within the range searched, no known theorem gives a sharp upper bound on a long path up to the diffusion time of the kink gas	That no such theorem exists (outside the range searched)
Of the 26 main citations, 19 are supported by an original source, a secondary source or a proof in this book, 4 need to be obtained, and 3 were changed into assumption rows (Table 2). The type of discrepancy is one word of the setting	The statements of the 4 items that need to be obtained
The cross-check of positive controls (rows 20–32 of Table 3): that the tools worked correctly on models whose answers are known	That this shows the rows on the double well are correct

Can say	Cannot say
The nineteen-row cross-check on the side of the four-dimensional lattice (Table 4)	Explaining the disagreement between the slope of the lattice data tables and the one-loop value beyond the reason that it is before asymptotic scaling
The two types drawn from outside documents, and the check of applying them to the construction of this book	Whether the outside documents are right or wrong

Sources and reproduction

Item	Kind	Source, tool
Statements of known theorems (Table 1)	Cited	G. Menz, A. Schlichting, Ann. Probab. 42 (2014), arXiv:1202.1510. G. Di Gesù, D. Le Peutrec, arXiv:1506.04434; M. Brooks, G. Di Gesù, arXiv:2606.16083. F. Martinelli, LNM 1717 (1999) 93–191. L. Bertini, S. Brassesco, P. Buttà, Arch. Ration. Mech. Anal. 190 (2008) 307–345. R. Bauerschmidt, B. Dagallier, Comm. Pure Appl. Math. 77, arXiv:2202.02295. R. Bauerschmidt, T. Bodineau, arXiv:1809.02075. M. Ledoux, LNM 1755 (2001) 167–194. A. V. Kolesnikov, E. Milman, J. Geom. Anal. 27 (2017), arXiv:1310.2526. The same as the source table of each chapter
Closed formula for Glauber dynamics	Assumption	R. J. Glauber, J. Math. Phys. 4 (1963) 294 (bibliographic only). The formula was checked independently by diagonalisation over all states of small chains
Summary of the bibliographic checks (Table 2)	This book	A summary of the table checking the 55 items cited in the assumption tables of the first step against original sources, secondary sources and bibliographic queries. The bibliography of each item is in the source table of each chapter
Numbers on the record side of Table 3	Reading, Computer-assisted, Proved here	Floating-point computation (transfer matrices, finite elements, diagonalisation over all states), interval arithmetic, closed formulae in the text. In the source table of each chapter
The nineteen rows of Table 4	This book	One table of cross-checks on the side of the four-dimensional lattice (the bibliography of the original materials is in the source tables of Part 5 and 23-02)
The two types	This book	A reading of two outside documents that claim a construction or a proof (sources withheld)

Terms used in this chapter — reconciliation, agreement, conditional agreement, disagreement, not comparable, positive control, calibration, distance to known theorems, one word of the setting, the type that puts the conclusion as a premise of intermediate steps, the type with an object for its name and numbers for its content. Their meanings are in the A-02 glossary.

Next chapter: 23-15 What remains and its cost — the distance to a theorem — lays out what remains and its cost for each assumption row of the integrated table, and

separates the distance within the first step from the distance from the second step on.

What remains and its cost — the distance to a theorem

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — That the rows needed to reach the conclusion of the first step (the upper side of the double-well relaxation-time power statement) split two ways at the single fork of the integrated table. For each of the two roads (the core on the box side, the core on the sector side), the assumption and reading rows needed, and the kind of work needed to close each row together with cost estimates. What is needed to produce a $\hbar$ at which all rows are in place. Outside the first step, what the distance from the second step on looks like when measured by three levels of the goal on the four-dimensional lattice, and the meaning of the one line "what is missing is uniformity".

Prerequisite chapters — 23-12 Table 1 (the integrated table) and 23-12 § 07 (the required sets of leaves). The content of each row is in the chapters 23-05 to 23-11. For the second step on, 23-01 and 23-03 § 06.

Said first — The level is [Y3]. Every cost is an estimate, and nothing has been run. We do not add up numbers of rows or hours and read them as "so many tenths to go". The rows do not have equal weight, and only rows expected to close by computation have time estimates; rows whose method is unknown have no estimate at all. This part places the diffusive-relaxation-time assumption at the axis of the study, and what this chapter measures is what remains of the first step on that axis.

The order of this chapter

1. The conventions of this chapter — distance is a list, cost is an estimate
2. The fork and the two roads — the core on the box side, the core on the sector side (Table 1)
3. Comparing the two roads — counting by kind of work (Table 2, Figure 1)
4. Remaining work not on the road to the conclusion — work worth having (Table 3)
5. What is needed to produce a ħ that fits all — why computer assistance in the window cannot reach it / The third branch of the fork (the road of blocks and mixing) (Tables C1 and C2)
6. Conditions common to the four conditions — five premises (Table 4)
7. The distance from the second step on — three levels of the goal, and the number of steps in β (Tables 5 and 6) / Detour: the lesson of ϕ^4_4
8. How to read the distance — a one-line conclusion

01

The conventions of this chapter — distance is a list, cost is an estimate

The "distance" to a theorem is a metaphor. In this chapter we call distance the **list** of those premises of the target proposition that have no proof.

$$\text{distance} \;=\; \{\text{the rows among the premises of the target proposition that have no proof}\} \qquad \text{(a set; we do not}$$

The goal of the first step is the upper side of the double-well power statement for the path measure of the double well, that is, the conclusion of the Double-Well Relaxation-Time Upper-Bound Proposition, $C_P \le K\xi^2 S_0/\hbar$, and $\ln C_P / \ln \xi \to 2$, which follows from it (23-05 § 04). The premises as a whole are the 111 rows of the integrated table (23-12 Table 1), and the rows without proof are the "Assumption" and "Reading" rows. The "Reading" rows are at hand as numbers, but they are not propositions of upper or lower bounds, so as premises of the conclusion they count as rows without proof.

To each row we attach the kind of work needed to close it. There are six: **interval arithmetic** (replace it by a computer-assisted inequality; 23-08), **paper** (write a proof), **bibliography** (check a known theorem in the original source and raise it to "Cited"), **measurement** (measure a quantity not yet measured), **another method** (it is known that the present tools cannot reach it, and another tool is needed), and **method undecided** (it is not yet decided with which tool to attack it). Time estimates are written only for rows with an estimate in the record, and rows with "no estimate" are left blank.

Example. Row [013] of the in-phase local Poincaré condition, between the nodes, has the estimate "24 subintervals, 23–39 hours in total". Row [002] for the tube, on the other hand, has no estimate. It is wrong to read the former as "closer" than the latter. The former is work expected to close if computed; the latter is work for which it is not yet known what would close it. A sum of computing hours does not show the size of the distance.

02

The fork and the two roads — the core on the box side, the core on the sector side

Following the dependency graph of the integrated table from the conclusion [111] in the direction of the premises, there is only one fork with "or": the core Poincaré uniformity assumption (the Poincaré constant of the core is uniform in core length and end values, [103]). This row is supported in two ways (23-12 § 07). For the road of the core on the sector side, the paper "The coarse-chain Poincaré inequality", which came out of this book, supports the core Poincaré uniformity assumption for every $\hbar \in (0,1]$ on the core whose two ends lie in the well: conditional on the paper's proof (Coarse-Chain Dobrushin Theorem (Theorem CC of the paper)) + Computer-assisted ($\hbar \in [10^{-5},1]$ by interval arithmetic over 239 intervals, $\rho_{\text{col}} \le 0.400$) + a paper proof and Computer-assisted (Small-$\hbar$ Coarse-Chain Proposition (Proposition PIN of the paper): $\hbar \le 10^{-5}$, $\rho_{\text{col}} \le 0.874$) + Cited (Poincaré Version of Helffer's Criterion (Proposition 3.1 of Ledoux) in Bobkov–Ledoux 2000) and the Coarse-Chain Brascamp–Lieb Proposition (Proposition BLc of the paper), with no Reading rows. The core on the box side (the core that enters the Gap-Inside-the-Box Proposition on the road of the core on the box side) is not covered.

- **The road of the core on the box side**: supported by the Three-Range Core-Constant Proposition (three stages for the constant of the core, [082]) and the vanishing of the local term as $\hbar \to 0$ ([084]) (23-11 § 05).
- **The road of the core on the sector side**: supported by a uniform lower bound for the gap of the core of the sector framework ([099]) and regularity at the seam (the seam limit assumption) (23-09 § 02).

Whichever road is taken, the rows needed for the conclusion are fixed as one set. Table 1 lists all the "Assumption", "Reading" and "Computer-assisted" rows in the union of the two roads, and writes the work and cost needed to close each row. The table, the graphs and the counts were made from the same single set of data (rows, edges) as the integrated table, with a column of costs added.

Table 1 The assumption, reading and computer-assisted rows needed for the conclusion [111], and the work needed to close them (numbers and contents as in the integrated table. A = the road of the core on the box side, B = the road of the core on the sector side. Every cost is an estimate; nothing has been run)

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
Assumption	**Assumption rows (22 rows)**						
[001]	Uniqueness of the measure at T=∞, Markov property, extension from local functions, the limit step, regularity	○	○	Cite from the original source or show on paper the uniqueness of the measure (the principal eigenvalue of a positive integral kernel on ℝ), the Markov property, extension, the limit step and regularity	Bibliography, paper	Obtaining 1 original source (positive kernel) and paper. No time estimate	23-05
[002]	Saddle-tube Poincaré assumption: Poincaré for the tube, uniform in tube length ≤ Cξ	○	○	The Poincaré constant of the tube is uniform in tube length ≤ Cξ	Method undecided	No estimate (not started)	23-07
[003]	Kink-gas gap assumption: the box gap of the reduced kink gas is of the Glauber type	○	○	Show the box gap of the reduced kink gas without copying the values of the one-dimensional Ising model	Another method	No estimate (a comparison theorem is needed)	23-11
[005]	Cross-term uniformity assumption: cross term $\mathcal{E}_Y(E[f\|Y]) \le (1+\chi)\mathcal{E}(f)$, χ uniform	○	○	χ of the cross term is uniform	Method undecided	No estimate (not started)	23-11
[006]	Alternating maximal-correlation assumption for boxes: the alternating maximal correlation between boxes c_alt < 1 is uniform	○	○	The alternating maximal correlation between boxes c_alt < 1, uniformly	Another method	No estimate (a comparison theorem avoiding the example that does not reduce to two-point quantities is needed)	23-11
[007]	Alternating maximal-correlation assumption for the core: upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	○	—	Upper bound for the alternating maximal correlation in the core $\bar{c} < 1$ ($k \ge 15$)	Another method	No estimate (the saturation is only a floating-point reading)	23-11
[008]	Core action-barrier assumption: C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	○	—	C_P ≤ c ħ^(−p) e^(ΔS/ħ) for core and window	Paper, measurement	The prefactor ħ^(−p) has not been measured. The next move is the capacity at the degenerate point (no estimate)	23-11
[009]	Continuous version of Holley's comparison (the tail lemma)	○	○	Continuous version of Holley's comparison	Bibliography	Confirming the version and the theorem number	23-06
[010]	The Poincaré–Miranda theorem (covering by boxes)	○	○	The Poincaré–Miranda theorem	Bibliography	Obtaining 1 original source	23-06

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[011]	Variance-decomposition row: regularity and density of the functions used	○	○	Regularity and density of the functions used	Bibliography, paper	No estimate	23-06
[012]	Margin row (large-end side): neighbour and end values > 2.5	○	○	Neighbour and end values > 2.5	Paper	No estimate (the road through monotonicity does not close, and the lemma of tangents to a convex tail is far from the margin)	23-06
[013]	Margin row (between the nodes): ħ between the nodes (in uniform form)	○	○	ħ between the six nodes (0.005–0.2)	Interval arithmetic	After one rerun of the frame (30–50 minutes), 24 subintervals, 23–39 hours in total (one core; cut into jobs of at most 25 minutes each, 60–170 jobs)	23-06
[014]	Margin row (small ħ): ħ < 0.005 (in uniform form)	○	○	ħ < 0.005	Paper + interval arithmetic	Interval arithmetic alone would need infinitely many subintervals. On paper it is reduced to the ħ→0 limit of the margin ħρ − 2ħκ ≥ 0.067 of the margin row alone, and the limiting margin is ≥ 0.758 (interval arithmetic + paper). The quantitative uniform Laplace method and a one-dimensional lemma on the ħρ side are needed (not done). With an explicit $\hbar_0$ the infinitely many become finitely many (§ 05). On the upper-end side, ħ∈[0.001949, 0.005] was filled with six subintervals by interval arithmetic (margin ≥ 0.067; the arithmetic on the subintervals in Lean: LatticeGaugeH1HbarInterval). Below that, the estimate of the corner tail does not close with this computing machinery, and the machinery has to be replaced (§ 05)	23-06
[015]	Two-site gap row (large-end side): end values of the two sites > 4	○	○	End values of the two sites > 4	Paper	No estimate	23-06
[016]	Transfer from the ring to scraps (with end values)	○	○	From the statement on the ring to the statement on scraps (with end values)	Paper	No estimate	23-06
[018]	Seam limit assumption, random-end-value assumption, regularity at the seam of the ring	—	○	Seam limit assumption, random-end-value assumption, regularity at the seam	Paper	No estimate	23-07

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[019]	The small assumptions of the route of the sector input (inputs to the sectors)	—	○	The small assumptions of the route of the sector input (the vertical term, the small assumptions of the sector input, regularity)	Paper	A list only, not started (no estimate)	23-07
[021]	Marginal log-Sobolev input for the window (the form of LSI for the marginals of site windows)	—	○	Marginal log-Sobolev input for the window	Paper	No estimate	23-09
[022]	Bridge volume-locality input (locality of volume on the bridge)	—	○	Bridge volume-locality input (locality of volume on the bridge)	Paper	No estimate	23-09
[023]	Barta's row $\kappa = 3/1000$, $b = 493/1000$	—	○	Barta's row $\kappa = 3/1000$, $b = 493/1000$ (at threshold 0.57)	Paper, interval arithmetic	No estimate (the paper evaluating the row is not given in this book)	23-09
[024]	The boundary term of Bochner and regularity of eigenfunctions	—	○	The boundary term of Bochner and regularity of eigenfunctions (C^3 up to the corners does not follow from the standard results)	Another method	No estimate	23-09
[026]	Window Log-Sobolev Threshold Proposition $A_W < A_{max}(\hbar)$: the remaining single proposition of the route of the window log-Sobolev inequality	—	○	The remaining single proposition of the route of the window log-Sobolev inequality, $A_W < A_{max}(\hbar)$	Method undecided	No estimate. All three strands stop at the shortfall of ¼ in the curvature of the single site at the top	23-09
Reading	**Reading rows (need to be replaced by propositions of upper or lower bounds) (5 rows)**						
[044]	$\ln \xi \geq s/\hbar$ ($\xi \propto \hbar^{1/2} e^{(S_0/\hbar)}$), the size of $D/(C\xi)^2$	○	○	$\ln \xi \geq s/\hbar$ as a lower-bound proposition	Bibliography, paper	Expected to rise to "Cited" if the original source of the semiclassical asymptotics of the tunnelling splitting is checked (no estimate)	23-04
[046]	Two-point maximal correlation of the medium-core step and B_2 (22.1 / 30.5 / 66.4)	○	—	The two-point maximal correlation of the medium-core step ($8 \leq k \leq 14$) as an upper bound	Interval arithmetic, paper	Interval arithmetic for each of the seven values of k and each point of ħ, and a paper on the difference between lattice and continuum (no time estimate)	23-11
[047]	$\bar{c}_{(\geq 15)} \leq 0.535$ (measurement of a lower bound)	○	—	$\bar{c}_{(\geq 15)} \leq 0.535$ as an upper bound	Another method	No estimate (at present only a measurement of a lower bound)	23-11
[055]	$\ell_{min} = 38.3$ / 121.9 / 406.8	—	○	$\ell_{min} = 38.3$ / 121.9 / 406.8 as a lower bound	Interval arithmetic	No estimate	23-09
[089]	Volume locality: ℓ_{min} does not depend on k	—	○	Volume locality: ℓ_{min} does not depend on k	Paper	No estimate	23-09
Computer-assisted	**Computer-assisted rows (done within the nodes and ranges) (2 rows)**						

No.	Content (integrated table)	A	B	Work needed to close it	Kind	Cost estimate	Text
[040]	Margin row: margin $\hbar\rho - 2\hbar\kappa \geq 0.067$ (neighbour and end in $[0,2.5]^2$)	○	○	(Done) computer-assisted at six nodes	—	Closed within the nodes. Outside the nodes, [013][014]	23-06
[041]	Two-site gap row: $\lambda_c \geq 1.3\times10^{-4}$ (ends in $[0,4]^2$)	○	○	(Done) computer-assisted row	—	The range of $\hbar$ as in the column of the integrated table	23-06

What to read from Table 1. (1) The rows common to the two roads are 13 assumption rows and 1 reading row: the common premise [001], the rows inside the box [002][003][005], [006] between boxes, [009]–[016] extending the in-phase local Poincaré condition, and the reading of how the correlation length grows [044]. These remain whichever road is taken. (2) Needed only for the road of the core on the box side are four rows: the alternating maximal correlation of the core [007], the action barrier [008], and the readings of the medium-core step and the long-core step [046][047]. (3) Needed only for the road of the core on the sector side are the remaining single proposition of the route of the window log-Sobolev inequality [026], 7 assumption rows including the inputs to the sectors [019][021][022][024], Barta's row [023] and regularity at the seam [018], and the two reading rows [055][089]. (4) The two computer-assisted rows [040][041] are done within the nodes and ranges, and common to both roads.

03

Comparing the two roads — counting by kind of work

We compare the costs of the two roads not by the number of rows but by the kind of work (Table 2).

Table 2 The assumption and reading rows of Table 1 counted by the kind of work needed to close them (a row with two kinds is counted in both; the two computer-assisted rows are done and excluded)

Kind of work	Common to both roads	A only	B only
Interval arithmetic	1 [013]	1 [046]	2 [023], [055]
Paper	6 [001], [011], [012], [015], [016], [044]	2 [008], [046]	6 [018], [019], [021], [022], [023], [089]
Bibliography	5 [001], [009], [010], [011], [044]	0	0
Measurement	0	1 [008]	0
Another method	3 [003], [006], [014]	2 [007], [047]	1 [024]
Method undecided	2 [002], [005]	0	1 [026]

There are three ways to read it.

First, **the only row with a time estimate is [013], common to both roads** (24 subintervals, 23–39 hours, not run). There are other rows expected to close by interval arithmetic ([046], [023], [055]), but there are as yet no estimates of the number of jobs or the time. And among the common rows is [014] ($\hbar < 0.005$, infinitely many subintervals needed), which is known to be out of reach of interval arithmetic. So neither of the two is a road that "closes given enough computer time".

Second, **the additions of the road of the core on the box side lean towards "another method", and the additions of the road of the core on the sector side lean towards "paper"**. [007][047] of the road of the core on the box side need a comparison theorem bounding the alternating maximal correlation of the core uniformly in k, and it must take a form that avoids the example that does not reduce to two-point quantities (23-11 § 05). Many of the additions of the road of the core on the sector side are papers on the inputs to the sectors, but behind them lies one "method undecided" row, [026]: the remaining single proposition of the route of the window log-Sobolev inequality, the place where all three strands stopped at the shortfall $\frac{1}{4}$ in the curvature of the single site at the top (23-09 § 06).

Third, **the road of the core on the sector side, with more rows, is not necessarily heavier**. The additional papers of the road of the core on the sector side are all rows for which what has to be shown is fixed as a statement. The two additional rows of the road of the core on the box side, [007][047], and the one row of the road of the core on the sector side, [026], are all rows for which "what to show" is fixed but "how to show it" is not. The difference between the two roads is whether these rows of unknown method lie on the side of the maximal correlation of the core or on the side of the curvature at the top.

Figure 1 is the dependency graph for each of the two roads, keeping only the ancestors of the conclusion. From the graph of the integrated table (111 nodes), only the nodes reached by choosing one side of the "or" at [103] and following the direction of the premises are listed, with the same numbers.

Figure 1 Dependency graph for each road (number, grade, symbol, content; the numbers to the right of $\Longleftarrow$ are premises. The rule of numbering is the same as in the integrated table: the numbers of premises are smaller than that of the conclusion)

- ▶ The road of the core on the box side: the core on the box side (55 nodes)
- ▶ The road of the core on the sector side: the core on the sector side (59 nodes)

Example. In the graph of the road of the core on the box side the premises of [103] are [082][084]; in the graph of the road of the core on the sector side they are [099][018]. The rest of the trunk (the gap of the box and the composition from [104] on) consists of the same nodes in both graphs, and the work before the fork acts in common on both roads.

04

Remaining work not on the road to the conclusion — work worth having

The integrated table contains work that is not on the road to the conclusion yet has been named as remaining. Closing any of it would not move the conclusion [111], but it is worth having for the certainty of the numbers or as material for other roads.

Table 3 Remaining work not on the road to the conclusion (costs copied from the estimates in the record; none has been run)

Remaining work	Row of the integrated table	Work and cost estimate	What could be said if it closed	Text
Putting the inner chain of the ring on interval arithmetic	[053], [054]	With 1–2 rounds of paper study and 1–2 rounds of scripting, build a machine separating the lattice of pins from the lattice of positions and a proposition for the layer of the outward-pushing wall; about 50 jobs, about 42 hours over the 25 subintervals of the window (about 85 hours without the paper on bands)	The coefficient of the inner chain $\bar{q} \approx 1.3$ becomes an upper bound. Not a road that meets the closing condition of the sectors ((7) of 23-09)	23-07 § 03
Extending the pin of the boundary stage to $(0, \frac{1}{2})$	[054]	Part of the machine above (extending the domain of the outer machine)	The condition "pin ≤ 0" on the outer chain is removed	23-07 § 03
The remaining numbers of the route of the window-constant numerics	[052]	About 42 jobs, about 21 hours, for the constants of the window, the minimal weight and the congestion. Since the floor of the product form is 4–5 orders of magnitude above the needed value, the decision is not to run them	The upper bound of the product form becomes Computer-assisted (it does not reach the conclusion)	23-07 § 04

Remaining work	Row of the integrated table	Work and cost estimate	What could be said if it closed	Text
Aligning the thresholds and remeasuring A_W	[056]	Measure A_W with the mark of the defect threshold $\theta_B = 0.57$ (no estimate)	The closing condition (7) of 23-09 has literal meaning	23-09 § 04
Making the rows of the Scalar-Strand Proposition computer-assisted	[025], [096]	12 jobs per node (3 for the bathtub column, 7 for rarity, 2 for non-convex fibres), estimated 60–250 for the nodes. The lower bound of the action difference of non-convex fibres is not yet in the form of interval arithmetic. The lifting assumption remains	The rows of the Scalar-Strand Proposition at the lower end of the window become Computer-assisted. There is no edge of implication to the remaining single proposition [026]	23-09 § 05
Sharpening the constant of the in-phase local Poincaré condition	[066]	Reworking the two-site gap row (no estimate)	$C \le 1.6 \times 10^{10}$ becomes smaller (the form of the conclusion does not change)	23-06 § 07
Conversion between the two spacings	—	Connect the readings at lattice spacing $\varepsilon = 0.25$ with the tables at $\varepsilon = \frac{1}{2}$ (no estimate)	One fewer "not comparable" row in the reconciliation	23-14 § 05
Obtaining bibliography	[010] and others	4 items to obtain (Poincaré–Miranda, elliptic regularity of the Neumann problem on a box, stochastic monotonicity, the principal eigenvalue of a positive integral kernel)	Rows that can be obtained rise from "Assumption" to "Cited"	23-14 § 03

The first three rows of Table 3 are, in amount of computation, larger work than [013] of Table 1. Even so, they are not on the road to the conclusion because closing them does not remove a single "Assumption" row of the integrated table (the same reason as "the two Computer-assisted rows of the saddle-tube condition are not on the road to the conclusion" in 23-12 § 05). The size of the computation and the closeness to the conclusion are different rulers.

05

What is needed to produce a ħ that fits all — why computer assistance in the window cannot reach it

Even if all the rows of Table 1 closed, if the ranges of $\hbar$ in which each row holds do not intersect, the conclusion cannot be stated at even a single $\hbar$. The present ranges are of four kinds (23-12 § 06): the nodes of the in-phase local Poincaré condition $\{0.2, 0.1, 0.05, 0.02, 0.01, 0.005\}$, the computer-assisted window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$, the nodes in the range of the Scalar-Strand Proposition $[3 \cdot 10^{-4}, 10^{-3}]$, and the three points of reading $\hbar = 3 \cdot 10^{-3}/10^{-3}/3 \cdot 10^{-4}$. The side of the in-phase local Poincaré condition has nothing below 0.005 and the side of the saddle-tube condition has nothing above $3 \cdot 10^{-3}$, so the intersection is empty.

There are only two roads to bring them into line: (a) extend the in-phase local Poincaré condition to $\hbar \le 10^{-3}$ (row [014]), or (b) extend the side of the saddle-tube condition to $\hbar \ge 0.005$. (b) was decided to be outside the scope of this part (the route of the range of ħ, 23-07 Table 1). (a) cannot be reached by interval arithmetic: the width of subintervals shrinks with $\hbar$, and covering down to $\hbar \to 0$ would need infinitely many subintervals. What is needed is another method, writing the asymptotics as $\hbar \to 0$ on paper. That line has come as far as the following form. Of the inputs of the Six-Node In-Phase Poincaré Proposition, the two-site gap row, the crossing row, λ_S and the nesting close for $\hbar \in (0, 0.2]$, and the

condition on the rate is monotone in the margin, so what remains is only to show the margin of the margin row, $\hbar\rho - 2\hbar\kappa \geq 0.067$, as $\hbar \to 0$ [Paper] . The limiting formula is $\kappa_0(a,b) = (H^{-1})_{1,11}/\varepsilon^2$ (the Hessian at the minimiser of the 11-site bridge), and for all ends $(a,b) \in [0,2.5]^2$ there is a unique global minimiser in the interior with $\lambda_{\min} H \geq 0.2$, $\kappa_0 \leq 0.1282$, and in the coarse landscape $U_0''(t^*) \geq 1.943$ [Computer-assisted] (interval arithmetic; the prerequisite paper lemmas — KKT, the truncation $x \leq \max(a,b,1)$, the leading principal minors of Jacobi — are not placed in the book; the action gap g is floating point [Computed]). From this the limit of the margin is $\geq 1.014 - 2 \times 0.1282 = 0.758$ (after paying for Muckenhoupt), larger than the 0.067 of the six nodes, and the true margin at finite $\hbar$ (1.67–1.79) is also nearly flat [Computed] . To bring the $\hbar \to 0$ limit down to finite $\hbar$, the quantitative uniform Laplace method and a one-dimensional lemma on the Muckenhoupt asymptotics on the $\hbar\rho$ side are needed [Assumption] (the constants of the uniform Laplace method are not made explicit in Theorem 2.12 of Katsevich arXiv:2406.12706 [Cited]). If an explicit $\hbar_0$ is obtained, the infinitely many subintervals of [014] become finitely many on $[\hbar_0, 0.005]$ [Reading] (the estimate of $\hbar_0$ is 4×10^{-4}–8×10^{-4}).

Subintervals from the upper end. On the upper-end side of the window $[\hbar_0, 0.005]$, we filled it by extending the same interval-arithmetic machinery as for the six nodes to intervals of $\hbar$. On the $\hbar\kappa$ side, the covariance of the bridge of 11 sites is bounded from above over an interval of $\hbar$ with interpolation in β at three points and a cubic remainder. On the $\hbar\rho$ side, the band of the expectation of the boundary values is extended to an interval of $\hbar$ with the same type, and the forward sums appearing in the Muckenhoupt exponent with band are compared cell by cell at the two ends of $\hbar$, giving a lower bound valid for every $\hbar$ in the interval [Paper] . As a result, on all six subintervals $[0.001949, 0.002281]$, $[0.002281, 0.002668]$, $[0.002668, 0.003122]$, $[0.003122, 0.003653]$, $[0.003653, 0.004274]$, $[0.004274, 0.005]$, the margin $\hbar\rho - 2\hbar\kappa$ exceeds 0.067 for every box of the neighbours and the ends $[0,2.5]^2$ (from the lowest interval up: 0.367, 0.353, 0.321, 0.240, 0.154, 0.077) [Computer-assisted] (float64 with outward widths. The comparison of exponents on the cells over the subintervals, the covering by small pieces and the arithmetic of the margin are in Lean: LatticeGaugeH1HbarInterval, 15 theorems; the numbers were put in for the upper two). Hence the margin of the margin row holds by interval arithmetic for $\hbar \in [0.001949, 0.005]$. It does not overlap with $\hbar_0$ of the quantitative uniform Laplace method (about 5.5×10^{-4} in floating point), and the gap between them remains open. The cost of this machinery grows roughly as the -3.5 power of $\hbar$ [Computed] , and moreover from the next subinterval on, because of the truncation of the weights, the upper bound on the tail at the corner $(a,b) = (2.5, 2.5)$ exceeds the integral, and it does not close on a single weight scale in float64 [Reading] . To join the window, the machinery has to be replaced — keeping the weights as logarithms or revisiting the estimate of the corner tail — or the constants of the quantitative uniform Laplace method have to be sharpened.

Moreover, even if a $\hbar$ that fits all could be produced at a single point, the double-well power statement is a statement about the limit $\hbar \to 0$.

$$\lim_{\hbar \to 0} \frac{\ln CP(\hbar)}{\ln \xi(\hbar)} = 2 \quad (2)$$

(2) does not follow from an inequality on any finite window. Computer assistance in the window only checks the inequality at each $\hbar$ in the window and says nothing about the outside of the window (23-08 § 10). So to close the first step, every row of Table 1 has to be stated "in a form uniform as $\hbar \to 0$", and the role of computer assistance is limited to taking on a finite part within such a uniform proposition (for example, the side $\hbar \geq \hbar_1$).

Example. Even if [013] is closed with 23–39 hours, what is closed is the continuum $0.005 \leq \hbar \leq 0.2$. [014] remains untouched, and the limit of (2) is not a single step closer. What computing time can buy is

"between the nodes", not "$\hbar \to 0$".

The third branch of the fork — the road of block log-Sobolev inequalities

Besides the road of the core on the box side and the road of the core on the sector side, a third candidate has been proposed as a road supporting the fork [103] (the core Poincaré uniformity assumption). It passes neither through the LSI constant A_W of the site windows of the sector framework ([026]) nor through the maximal correlation of the core of the Three-Range Core-Constant Proposition ([007], [046], [047], [082]); in this section we call it the road of blocks and mixing. Many of its steps have no written proof, so this section is not a statement of claims: it sets out the proposed road as a table of rows (Table C1) and a dependency graph. The counts of Tables 1 and 2 and Figure 1 are not changed.

Setting. The object is the conditional measure $\mu_k^{a,b} \propto e^{-S/\hbar}$, with the values a, b outside both ends fixed, on the same core $Q_k = (-x_c, 0)^k$ as in 23-09 ($x_c = 0.89$, Neumann faces, $\varepsilon = \frac{1}{2}$, $V(z) = (z^2-1)^2/8$). The Dirichlet form follows the convention $\mathcal{E}(f) = \hbar \int |\nabla f|^2 \, d\mu$ of 23-09 (the form of the overdamped Langevin dynamics $dz = -\nabla S \, dt + \sqrt{2\hbar}\, dW$), and differs from $(\hbar/\varepsilon)\sum_i \mathbb{E}|\partial_i f|^2$ of 23-11 only by a constant factor ε. So "uniform in $\hbar$" means the same under either convention. The goal is a logarithmic Sobolev inequality (LSI) uniform in the core length k, in $\hbar$ and in all end values; since the Poincaré inequality follows from the LSI Cited , if the goal is shown, the core Poincaré uniformity assumption follows.

Three stages.

1. Block LSI. For blocks of length $\ell_B \geq 12$ (avoiding length 8, just outside the threshold of convexity), an LSI uniform in the end values and in $\hbar$. The inputs are the Hidden-Convexity Proposition below (exactly one minimum on the box), the curvature $\lambda_*(12) \geq 0.255$ at the minimiser for the worst end values, Witten's Morse theory that there are no exponentially small eigenvalues under a single non-degenerate minimum (on a Neumann box), and the Menz–Schlichting decomposition for the case of one minimum.
2. Strong mixing. The optimal path of $\hbar \to 0$ reaches the corner $-x_c$ of the box within 7 steps from every starting point, the chain concentrates around that path with a width of order $\sqrt{\hbar}$, and in the boundary layer at the corner the Doeblin condition holds uniformly in $\hbar$. The conclusion has the form $\mathrm{TV}(n) \leq (1-\delta)^{\lfloor n/10 \rfloor}$, with δ independent of $\hbar$.
3. Combine 1 and 2 by the standard theorem for one-dimensional chains (of the Zegarlinski and Martinelli–Olivieri type: block LSI and strong mixing give an LSI uniform in length).

This road does not take the worst configuration as its entrance. It lies outside the fact that tools evaluating at the worst configuration all stop at the shortfall $\frac{1}{4}$ in the curvature of the single site at the top (23-09 § 06), and it goes after uniformity in $\hbar$ through mixing on average.

Hidden-Convexity Proposition (exactly one minimum on the box)

Let $\varepsilon > 0$, $c > 0$, $l \geq 1$, with end values $a, b \in [-c, 0]$, and with $z_0 = a$, $z_{l+1} = b$,

$$S(z) = \sum_{i=0}^{l} \frac{(z_{i+1} - z_i)^2}{2\varepsilon} + \varepsilon \sum_{i=1}^{l} V(z_i), \qquad V(z) = \frac{(z^2-1)^2}{8} \tag{C1}$$

be considered on the box $Q = [-c, 0]^l$. Then S has exactly one local minimum on Q, and it is the global minimum.

Proved here Proof. Put $z_i = -\sqrt{\rho_i}$ ($\rho_i \in [0, c^2]$, $i = 1, \dots, l$), likewise $a = -\sqrt{\rho_0}$ at the ends, and $b = -\sqrt{\rho_{l+1}}$ (with ρ_0, ρ_{l+1} fixed). With $(z_{i+1} - z_i)^2 = \rho_{i+1} + \rho_i - 2\sqrt{\rho_i \rho_{i+1}}$, the geometric mean $(x, y) \mapsto \sqrt{xy}$ is concave on $[0, \infty)^2$, so $-\sqrt{\rho_i \rho_{i+1}}$ is convex. The end term $-\sqrt{\rho_0}\sqrt{\rho_1}$ is also convex, since $\sqrt{\cdot}$ is concave and the coefficient is ≥ 0. In ρ the linear term is convex, and $V(z_i) = (\rho_i - 1)^2/8$ is strictly convex in ρ_i with second derivative $\frac{1}{4}$. Hence $\tilde{S}(\rho) := S(-\sqrt{\rho})$ is the sum of a convex function and $\varepsilon \sum_i (\rho_i - 1)^2/8$, and is strictly convex on $[0, c^2]^l$. The componentwise $\rho \mapsto -\sqrt{\rho}$ is a homeomorphism from $[0, c^2]^l$ to Q, so local minima correspond to each other. A local minimum of a strictly convex function on a convex set is the global minimum, and there is at most one. S is continuous on the compact Q, so the minimum exists. ∎

What the Hidden-Convexity Proposition says is that an action which is not convex in the coordinates z (the Hessian at the top is not positive definite for lengths 8 and above; next paragraph) is convex in the coordinates $\rho = z^2$, independently of length. But the Poincaré constant and the LSI constant are quantities of the flat metric of z, so the convexity in ρ does not by itself give the Bakry–Émery criterion. The density written in ρ carries the Jacobian $\prod_i (2\sqrt{\rho_i})^{-1}$, and the $\frac{1}{2} \ln \rho_i$ added to the potential is a concave term with second derivative $-1/(2\rho_i^2)$. The lower bound on convexity $\varepsilon/(4\hbar)$ in $\tilde{S}/\hbar$ can cancel this only where $\rho_i \geq \sqrt{2\hbar/\varepsilon}$; in thinner layers convexity is not guaranteed Proved here (a one-line computation). The view is that the non-convexity moves from "the lump at the top of length 8 and above" to "rare thin layers at each site". The proposition in the record (the Closed-Core KKT-Point Proposition), which shows that when the end values are below the box ($a, b \leq -x_c$) the only KKT points on the closed core are the bottom corners, uses a different method based on the convexity of sequences and the KKT conditions, and is not among the ancestors of the conclusion of the road of blocks and mixing.

The threshold of convexity is the first conjugate point of the top. The Hessian of S at the top $z \equiv 0$ is tridiagonal, $\mathrm{tridiag}(-1/\varepsilon,\ 2/\varepsilon + \varepsilon V''(0),\ -1/\varepsilon)$, and with $V''(0) = -\frac{1}{2}$, $\varepsilon = \frac{1}{2}$ it is $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$. For length l the eigenvalues are $\frac{15}{4} - 4\cos\frac{\pi j}{l+1}$ ($j = 1, \dots, l$), and the smallest, $j = 1$, is positive only when $\cos\frac{\pi}{l+1} < \frac{15}{16} = 1 - \frac{\varepsilon^2}{4}$, that is, $l + 1 < \pi/\theta$ ($\cos\theta = 1 - \varepsilon^2/4$) Proved here. With $\theta = 0.3554$, $\pi/\theta - 1 = 7.839$, the smallest eigenvalue is 0.0545 at length 7 and -0.0088 at length 8. The solution of the Euler–Lagrange equation linearised at the top (the Jacobi equation) $-2u_{n-1} + \frac{15}{4}u_n - 2u_{n+1} = 0$ is $\sin(\theta n)$, and π/θ is its first zero, in the language of the calculus of variations the first conjugate point. The statement of the Short-Core Convexity Lemma (23-11 § 05), "convex for length ≤ 7, broken from 8", is exactly the test of whether a conjugate point enters the interval. This length is fixed classically and does not depend on $\hbar$.

Table C1 The rows of the road of blocks and mixing (numbers carry a C to distinguish them from the numbers of the integrated table. Labels are the five grades of Part 23. No row has a cost estimate)

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C01	The threshold of convexity 7.84 is the first conjugate point of the top $\pi/\theta - 1$ ($\cos\theta = 1 - \varepsilon^2/4$)	Proved here	The paragraph above	—

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C02	A core of length $k \le 7$ has $\mathrm{Hess}\, S \ge \kappa(k)$ ($\kappa(7) = 0.0545$) at every configuration; Poincaré uniform in $\hbar$ and the end values (the Short-Core Convexity Lemma, integrated table [076])	Proved here	23-11 § 05. Bakry–Émery on convex domains (Kolesnikov–Milman) is Cited. The LSI form is expected to follow from the same convexity; checking against the literature remains. Not among the ancestors of the conclusion of the road of blocks and mixing (blocks have length 12 or more)	Bibliography (the LSI form)
C03	Hidden-Convexity Proposition: if the end values are $a, b \in [-c, 0]$, there is exactly one local minimum on the box	Proved here	The proof in the box above	—
C04	The curvature at the minimiser for the worst end values $\lambda_*(12) \ge 0.255$, $\lambda_*(16) \ge 0.305$ ($\varepsilon = \frac{1}{2}$, all end values)	Reading	Monotonicity of the minimiser in the end values (Topkis's theorem) and a lower bound per cell by Weyl's inequality. The minimisers at the four corners of each cell come from floating-point optimisation	Interval arithmetic (Computer-assisted if the KKT residuals at the four corners are checked with intervals)
C05	A single non-degenerate minimum $\Rightarrow$ no exponentially small eigenvalues, and at $\hbar \to 0$ the LSI constant is fixed by λ_*	Assumption	Candidates to cite (Helffer–Nier, the Neumann version of Le Peutrec, Menz–Schlichting). The form at the corners of the box has not been checked	Bibliography
C06	The LSI of a block ($\ell_B \ge 12$) is uniform in the end values and in $\hbar$	Assumption	C03, C04, C05, and the line of assembly over the range of finite $\hbar$ and the compactness of the end values	Paper
C07	The optimal path of $\hbar \to 0$ reaches the corner $-x_c$ within 7 steps from every starting point	Reading	A record of solving the path from the top in 50-digit floating point (gradient residual 3×10^{-51}) and checking the signs of the KKT conditions. Uniqueness is C03; monotonicity in the starting point is Topkis's theorem. It is not interval arithmetic, so it is not graded Computer-assisted	Computer-assisted by interval arithmetic (enclosing the existence of the solution in intervals)
C08	Convexity of the one-step action $\kappa_0 \ge 0.055$ ($\hbar \to 0$, all starting points)	Assumption	The record has a line of proof (if there are at most 6 free sites, the second derivative of the one-step action is at least the smallest eigenvalue 0.0545 of the worst tridiagonal matrix of length 7). The book gives no proof. Depends on C01 and C07. $\kappa\hbar \ge 1.88$ measured at finite $\hbar$ ($\hbar = 10^{-2}$–10^{-5}) is Reading	Paper
C09	$\lvert W\hbar - W_0 \rvert = O(\hbar \ln(1/\hbar))$ ($W\hbar = -\hbar \ln \varphi_0$, uniform in $[-x_c, 0]$)	Assumption	A candidate to cite (a limit of Varadhan type). Not yet checked against the original	Bibliography
C10	Concentration to a width $\sqrt{\hbar}$, and the Doeblin condition in the boundary layer at the corner (uniform in the starting point)	Assumption	A line of argument by the constrained Laplace method. What remains is a form uniform in the starting point, and the treatment of the finitely many starting points at which the multiplier of the wall is 0	Paper

Row	Statement (one line)	Label	Grounds, what remains	Work needed to close it
C11	Strong mixing $\mathrm{TV}(n) \le (1-\delta)^{\lfloor n/10 \rfloor}$, with δ independent of $\hbar$	Assumption	The line of assembly on top of C07–C10	Paper
C12	Mixing length $\approx$ 9–10 (total variation $< 10^{-2}$ at the worst end values, $\hbar = 10^{-2}$–10^{-5}); rough Langevin relaxation ≈ 0.46 ($k \ge 16$, $\hbar = 10^{-2}$–10^{-4})	Reading	Numbers for the transfer operator (maximum over 41 end values) and numbers for overdamped Langevin (256 chains, ends at the top). Supporting rows that are not among the ancestors of the conclusion	—
C13	Block LSI + strong mixing ⇒ LSI uniform in length	Assumption	Candidates to cite (Zegarlinski, Martinelli–Olivieri). The version for compact continuous spins has not been checked	Bibliography
C14	The LSI of the core is uniform in k, $\hbar$ and the end values, hence the core Poincaré uniformity assumption	Assumption	Assembly on top of C06, C11, C13	Paper

Dependency graph (the rule of numbering is the same as in Figure 1: the numbers of premises are smaller than that of the conclusion. C02 and C12 are nodes that are not among the ancestors of the conclusion):

```
[C01] PRV  threshold of convexity π/θ − 1 = 7.84 is the first conjugate point of th
[C02] PRV  Short-Core Convexity Lemma: convex for k ≤ 7 (integrated table [076])
[C03] PRV  Hidden-Convexity Proposition: exactly one minimum on the box (strictly c
[C04] RDG  λ*(12) ≥ 0.255, λ*(16) ≥ 0.305
[C05] ASM  one non-degenerate minimum ⇒ no small eigenvalues (corners of a Neumann
[C06] ASM  block LSI (ℓ_B ≥ 12) uniform in end values and ħ ⟸ [C03][C04][C05]
[C07] RDG  optimal path as ħ→0 reaches the corner in ≤ 7 steps (50-digit floating p
[C08] ASM  one-step convexity κ₀ ≥ 0.055 ⟸ [C01][C07]
[C09] ASM  |W_ħ − W₀| = O(ħ ln(1/ħ))
[C10] ASM  concentration to width √ħ, Doeblin in the boundary layer at the corner ⇐
[C11] ASM  strong mixing TV(n) ≤ (1−δ)^⌊n/10⌋ ⟸ [C10]
[C12] RDG  mixing length ≈ 9–10, Langevin ≈ 0.46                        outside the
[C13] ASM  block LSI + strong mixing ⇒ LSI uniform in length
[C14] ASM  LSI of the core uniform in k, ħ and end values ⟸ [C06][C11][C13]
[103] PRV  core Poincaré uniformity assumption ⟸ [C14] (LSI ⇒ Poincaré is cited)
```

Table C2 The quantities of Part 23, and how the road of blocks and mixing treats them

Quantity of Part 23	Meaning	Treatment in the road of blocks and mixing
Core Poincaré uniformity assumption ([103])	The Poincaré constant of the core is uniform in core length and end values (23-09, 23-11)	Follows from the conclusion C14 of the road of blocks and mixing (LSI ⇒ Poincaré). The convention is aligned with $\hbar \int \lvert\nabla f\rvert^2 d\mu$ of 23-09
Window Log-Sobolev Threshold Proposition ([026])	$A_W < A_{\max}(\hbar)$: the one remaining proposition of the route of the window log-Sobolev inequality	Not passed through. If the road of blocks and mixing is chosen, [026], the 6 sector input rows of the road of the core on the sector side ([018], [019], [021]–[024]) and the readings [055], [089] drop out of the ancestors of the conclusion

Quantity of Part 23	Meaning	Treatment in the road of blocks and mixing
A_W	The LSI constant of the marginals of the marks of a site window	Not used
[007], [008], [046], [047], [082]	The core rows of the road of the core on the box side (the Three-Range Core-Constant Proposition and its inputs)	Not used
The common rows ([001], [002], [003], [005], [006], [009]–[016], [044])	The tube, the reduced kink gas, the cross term, the maximal correlation between boxes, the extension of the in-phase local Poincaré condition, and others	Remain

Counts, and what does not close. If the road of blocks and mixing is chosen, the rows without proof among the ancestors of the conclusion [111] are the 13 common assumption rows and 1 reading row ([044]), plus the 8 assumption rows of the road of blocks and mixing (C05, C06, C08, C09, C10, C11, C13, C14) and its 2 reading rows (C04, C07). The number of rows is not smaller than for the road of the core on the box side (15 assumptions, 3 readings) or the road of the core on the sector side (20 assumptions, 3 readings). The difference is in the kind of work: the rows of the road of blocks and mixing divide into bibliography (C05, C09, C13), interval arithmetic (C04, C07) and paper (C06, C08, C10, C11, C14), and none of them falls under "another method" or "method undecided" of Table 2. But this classification comes from reading the line of argument of each row, and there is no guarantee that new obstacles will not appear while writing it out.

The road of blocks and mixing is a road supporting the single row of the fork [103], and the trunk from [104] on is the same as for the road of the core on the box side and the road of the core on the sector side. **The 13 common rows remain even if the road of blocks and mixing is chosen.** Of these, [002] (Poincaré for the tube) and [005] (the χ of the cross term is uniform) are method undecided, and [003], [006] and [014] (the form uniform in $\hbar < 0.005$, which interval arithmetic cannot reach) are rows that need another method (Table 2). So even if the road of blocks and mixing closes, the first step does not close. The problem of "a $\hbar$ that fits all" of § 05 also remains, through [014].

When $\ln(1/\hbar)$ remains in the mixing length. Numerically, the length at which the influence of the worst end values falls below 10^{-2} stays at 9–10 from $\hbar = 10^{-2}$ to 10^{-5}, but the position at which the total variation drops below 0.5 moves slowly, logarithmically, from 5 to 8 Reading . The possibility that a dependence on $\ln(1/\hbar)$ remains has not gone away. If it remains, the constant in the conclusion of the road of blocks and mixing grows polynomially in $\ln(1/\hbar)$ as a function of $\hbar$, and the two statements are treated differently. The power statement (2) is not affected: even if a factor $(\ln(1/\hbar))^p$ multiplies C_P, under $\ln \xi \geq s/\hbar$ of [044] we have $p \ln\ln(1/\hbar)/\ln \xi \to 0$ Proved here (the algebra on top of [044]). On the other hand, the form in $C_P \leq K\xi^2 S_0/\hbar$ of the First-Step Upper-Bound Proposition, in which K does not depend on $\hbar$, does not allow it. The uniformity of K includes the uniformity of the χ of the cross term (the cross-term uniformity assumption, [005]), and if the cross-term uniformity assumption is shown by the standard two-scale argument, χ is expected to come out proportional to the square of the local constant C_{loc}; so if a logarithmic factor enters C_{loc}, the uniformity of χ breaks Assumption . The second term $C_{\mathrm{loc}} D/(C\xi)^2$ of (12) in 23-05 allows a logarithmic factor, since $D/(C\xi)^2$ is small, of order $e^{-2S_0/\hbar}$. Therefore the right goal for the road of blocks and mixing is the core Poincaré uniformity assumption in a form without logarithmic factors, and if a logarithm remains, what can be reached is only the power statement.

06

Conditions common to the four conditions — five premises

Apart from the work row by row in Table 1, there are five premises that bear on all four conditions. None of them disappears when a single row is closed; they are conditions to be written into the statement when putting the first step in a form that can be called "closed".

Table 4 Premises common to the four conditions

Symbol	Premise	Present state	What is needed to close it
C1	Kind of number	The reading rows are floating-point values (the 19 Reading rows of the integrated table)	Replace the reading rows among the premises of the conclusion by upper and lower bounds from interval arithmetic or paper (the reading rows of Table 1)
C2	Bibliography of the cited theorems	Of the 26 main citations, 19 have their statements checked in an original or secondary source. 4 to obtain, 3 rows changed into assumptions (23-14 Table 2)	Obtain the 4 items and check their statements. If they cannot be obtained, they stay "Assumption"
C3	The range of machine checking	Lean covers only algebra and arithmetic. Analytic theorems are input as assumptions (23-13 § 05)	Widening the range of machine checking does not reduce the conditions of the conclusion. What reduces the assumption rows is paper and bibliography
C4	A $\hbar$ that fits all	There is no $\hbar$ at which all rows hold (§ 05)	Another method to extend the in-phase local Poincaré condition to $\hbar \to 0$
C5	Limit and regularity	Existence of the limit $T \to \infty$, regularity, and simplicity of the first eigenvalue are assumptions ([001] of the integrated table)	The same as [001] of Table 1

07

The distance from the second step on — three levels of the goal, and the number of steps in β

The distance of the first step was the list of rows of Table 1. From the second step (the four-dimensional lattice, non-abelian) on, the statements are not divided finely enough to be written as a list of rows. This section lays out what the goal on the four-dimensional lattice looks like when written at three levels, and the number of steps in β, as a rough guide by physical extrapolation. What can be expected to carry over from the first step to the second step and beyond is only the form of the statements and the form of the tools; why the constants and the range of $\hbar$ do not carry over is in 23-03 § 06.

"The mass of the lightest particle is positive" is the statement on the spectrum of the quantum Hamiltonian H, $\mathrm{spec}(H) \subset \{0\} \cup [\Delta, \infty)$, $\Delta > 0$ (4-11). On the road entering from the lattice, the same goal splits into three levels.

$$(\mathrm{T1})\ \ \xi(\beta) < \infty\ (\forall \beta),\ \ \xi(\beta) \to \infty\ (\beta \to \infty); \qquad (\mathrm{T2})\ \ \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)} \to \Delta_{\infty} \in (0, \infty); \qquad (\mathrm{T3})\ \ \text{gap positive in the}$$

Table 5 Three levels of the goal, and their relation to the relaxation-time assumption

Level	Content	What is added to the level above	Status in four dimensions	Relation to the relaxation-time assumption
Lattice target level ((T1) of the formulas)	Correlation length finite at each β, diverging as $\beta \to \infty$	—	Unsolved (lattice correlation-length problem (Chatterjee's Problem 5.1)) Cited	The relaxation-time assumption is a statement that presupposes the latter half of the lattice target level

Level	Content	What is added to the level above	Status in four dimensions	Relation to the relaxation-time assumption
Physical-unit target level ((T2) of the formulas)	m_{lat}/a tends to a positive finite value	How the lattice spacing $a(\beta)$ is determined	Unsolved	If the relaxation-time upper-bound assumption and the one-sided bridge (23-01 § 06) are both in place, the lower half follows
Continuum target level ((T3) of the formulas)	A non-trivial theory on $\mathbb{R}^4$ with positive gap	Existence of the limit (Wilson-loop limit statement), rotation-recovery statement, OS reconstruction statement, non-triviality statement	Unsolved (the problem statement). On the existence of the limit and the axioms side, the committee's assessment is that the construction published on 2026-11-10 shows them as the limit of Wilson's lattice gauge theory (Part 6). The gap is not shown (6-04)	Does not follow even with the relaxation-time assumption and the bridge in place (23-01 § 08)

Next, we measure the gap between the β where there are proofs and the β on the side of the continuum limit by the number of steps that coarsen the lattice by a factor of 2. It is a physical extrapolation that extends two-loop asymptotic scaling, matched to the value $\xi = 1/0.514$ of Lucini–Teper at $\beta = 2.60$; it is not a distance to a theorem.

$$\xi(\beta) \;\propto\; \Big(\frac{6\pi^2\beta}{11}\Big)^{-51/121} e^{3\pi^2\beta/11}, \qquad k = \log_2 \frac{\xi_{\mathrm{target}}}{\xi_{\mathrm{start}}} \tag{4}$$

Table 6 Target correlation lengths, and the β and number of steps needed Reading (values of the two-loop extrapolation. The window end is the end of the numerical window, $\xi = 1.95$; the curvature end is $\xi \approx 0.093$, the end of $\beta < 0.2722$ that comes from the lower bound of the curvature after one-link integration)

ξ_{target}	β (two loops)	From the window end	From the curvature end
10	3.243	2.36	6.75
100	4.136	5.68	10.07
1000	5.022	9.01	13.39
10^4	5.903	12.33	16.72
10^6	7.654	18.97	23.36

The numbers in Table 6 are all finite. Even so, the distance does not shrink, because what a theorem needs is not "up to some target" but "at every target, with the same constants". In metaphor, what is missing is not the number of steps on the staircase but the guarantee of not missing one's footing on any step. Improving the constants of the theorem on the strong-coupling side ($\beta < 1/12$ of Shen–Zhu–Zhu Cited) only widens the range of β in hand and does not touch this uniformity. The curvature end 0.2722 is a threshold that comes from the lower bound of the curvature (Lean), and the step towards the gap is a sketch Assumption .

Put back into the language of the first step, the same thing happens with $\hbar$. As in § 05, however wide the window of $\hbar$ is made, the double-well power statement (2) does not come out; what is needed is a proposition in a form uniform as $\hbar \to 0$. On the four-dimensional lattice, a form uniform as $\beta \to \infty$; in the first step, a form uniform as $\hbar \to 0$ — on every step of the staircase, the shape of what is missing is the same.

Detour: the lesson of φ^4_4 — even if the limit is reached, the place one arrives at may be trivial

A subsection you may skip. For the four-dimensional scalar field ϕ^4, cutting it off on a lattice and taking the limit naturally lands on a Gaussian (the free field) (Aizenman–Duminil-Copin 2021, 7-08) Cited . Even if the limit exists and the mass stays finite, if the interaction has vanished it is not a solution in the sense of the problem statement. This is why the non-triviality statement is written separately in the continuum target level of Table 5, and it is the same caution as the relaxation-time lower-bound assumption being a different statement from non-triviality (23-01 § 06). In metaphor, one may think one has reached the summit and find it is flat ground.

08

How to read the distance — a one-line conclusion

We put the distance of this chapter into a single diagram of arrows.

$$\underbrace{\text{each row of the first step}}_{\text{Table 1}} \xrightarrow{\hbar\to 0 \text{ in uniform form}} \text{power form of Assumption } (\xi^2) \text{ (first step)} \cdots \underbrace{\text{theorems f}}_{\text{in}}$$

The dotted line is not an implication; it shows that a statement of the same form is placed at two steps. None of the arrows of (5) has been proved. In the first step we have gone as far as listing the rows before the arrow, dividing them into two roads and attaching a kind of work to each row; the dividing is an arrangement, not a shortening of the arrow.

In one line, **what is missing is not the number of steps but uniformity: holding with the same constants at every scale**. In the first step it appears as the in-phase local Poincaré condition and the rows of the core uniform as $\hbar \to 0$; on the four-dimensional lattice, as the upper-bound assumption and the bridge uniform as $\beta \to \infty$ — the same thing in different forms.

This part placed the relaxation-time assumption at the axis and divided that uniformity into the rows of the first step. Roads that ask about the same uniformity from viewpoints other than the relaxation-time assumption are taken up in Part 24. A single table that rearranges, for each of the four steps of the staircase, the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost is in 24-07 § 04.

What this chapter can say

Can say	Cannot say
That the rows needed for the conclusion [111] split into two roads at the fork [103]. The road of the core on the box side has 15 assumption rows and 3 reading rows, the road of the core on the sector side 20 assumption rows and 3 reading rows (counted from the data of the integrated table)	Which road is easier. The number of rows does not show weight
The kind of work needed to close each row (interval arithmetic, paper, bibliography, measurement, another method, method undecided), and the cost estimates in the record	Whether the estimates are right. None of them has been run. The cost of rows without estimates
That there is no $\hbar$ at which all rows are in place, that bringing them into line needs another method to extend the in-phase local Poincaré condition to $\hbar \to 0$, and that computer assistance in the window does not reach (2)	The asymptotics of the in-phase local Poincaré condition as $\hbar \to 0$. (2) itself

Can say	Cannot say
Three levels of the goal from the second step on, and the number of steps by the two-loop extrapolation (reading) Reading	Table 6 is a rough guide by extrapolation, not a theorem on the correlation length. This part shows none of the three levels
Setting out the rows of the road of blocks and mixing (block LSI and strong mixing) in Table C1, and showing the Hidden-Convexity Proposition and threshold of convexity = first conjugate point with proofs Proved here . That even if the road of blocks and mixing is chosen the 13 common rows remain, and the first step does not close	The 8 assumption rows and 2 reading rows of the road of blocks and mixing. Whether $\ln(1/\hbar)$ remains in the mixing length

Sources and reproduction

Item	Kind	Source, tool
Numbers, contents and grades of the rows of Table 1	This book	23-12 Table 1. The source of each row is in the source column of the integrated table
Work and cost estimates of Tables 1 and 3	This book	The sections on what remains in each chapter (23-06 § 07, 23-07 § 04, 23-09 § 05, 23-11 § 07). All are estimates; nothing has been run
Counts of Tables 1 and 2 and Figure 1	Text processing	A small script that adds a column of costs to the same row and edge data as the integrated table, chooses one side of [103], and collects nodes from the conclusion in the direction of the premises (not a mathematical computation)
(3), Table 5, lattice correlation-length problem	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000). S. Chatterjee, arXiv:1803.01950 § 5
(4), Table 6	Reading	Two-loop coefficients (source table of 4-06) and the value at $\beta = 2.60$ in Table 1 of B. Lucini, M. Teper, JHEP 0106 (2001) 050, hep-lat/0103027. The extrapolated numbers are copied from the table in the record of the study
The strong-coupling range $\beta < 1/12$	Cited	H. Shen, R. Zhu, X. Zhu, Commun. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737 (source table of 23-01)
Triviality of ϕ^4_4	Cited	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163 (source table of 7-08)
The rows of the road of blocks and mixing (Table C1), the numerical rows C04, C07, C12	Proved here, Reading, Assumption	The Hidden-Convexity Proposition and C01 are proved in the text. $\kappa_\hbar$ of C04, C07 and C08, and C12, are floating-point numbers from the record (C07 in 50 digits, not interval arithmetic). The candidates to cite for the assumption rows (Helffer–Nier, Le Peutrec, Menz–Schlichting, a limit of Varadhan type, Zegarlinski, Martinelli–Olivieri, Topkis's monotone comparison theorem) have not been checked against the literature

Terms used in this chapter — distance (a list), fork, road of the core on the box side, road of the core on the sector side, kinds of work (interval arithmetic, paper, bibliography, measurement, another method, method undecided), a $\hbar$ that fits all, three levels of the goal, uniformity. Their meanings are in the A-02 glossary.

Next chapter: 23-16 What is not claimed — what this part does not show, what should not be read strongly, and what is not called false either, with a table of correspondence to sentences of the text.

What is not claimed — not shown, not read strongly, not called false

> This chapter is a map of open questions. § 01 and § 06 are guide material.

What this chapter shows — The list of things Part 23 does not claim (fourteen items), and a table matching each item to the sentence in the body, and the chapter, that supports it. The difference between not claiming something and claiming its opposite. For the first-step proposition, the Lean theorems, the computer-assisted results and the readings, examples of how each is misread once its conditions are removed.

Prerequisite chapters — None. This chapter can be read from anywhere. The terms are defined in 23-01 § 05 (the diffusive-relaxation-time assumption, the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption), 23-00 § 01 (the five labels) and 23-05 (the Double-Well Relaxation-Time Upper-Bound Proposition and the four double-well conditions).

Said first — The level is [Y2]. This book does not solve the problem of Yang–Mills existence and the mass gap. This part contains no construction of the continuum limit and no proof of the mass gap (the existence of the continuum limit is proved in Lean by the construction published on 2026-11-10, along another road — this is the committee's assessment; Part 6. The mass gap is not shown there either; 6-04). This chapter only splits that fact into items and writes them out; it adds no new result. This part places the diffusive-relaxation-time assumption at the axis of the study, and it does not claim that the relaxation-time assumption is the only road or the right road.

The order of this chapter

1. Three meanings — not shown / not read strongly / not called false (Table 1) / Detour: negating a quantifier
2. The fourteen items and the sentences in the body that support them — a table of correspondence (Table 2)
3. On the main problem — the claim of the problem statement, the local-field footnote · weak-existence footnote (footnotes of Jaffe–Witten)

4. On the double-well step — the First-Step Upper-Bound Proposition is algebra on top of four conditions (Table 3)
5. Not reaching is not evidence of falsity — the death of detour plans, the reading for the periodic box, the lower-bound assumption and the free field (Table 4)
6. Conditions fall away in summaries — sentences of the body and one-line summaries (Table 5)
7. Analogies and other documents — what is not judged, and the convention for putting things out

---

01

Three meanings — not shown, not read strongly, not called false

When a mathematical text says "not claimed", there are three meanings. The first is **not shown**. On the main problem, this part shows nothing. The second is **not read strongly**. Whatever has been shown always carries conditions or assumptions, and reading it with the conditions removed gives a sentence that has not been shown. The third is **not called false**. That a method did not reach something is a property of the method, not a claim that the proposition it did not reach is false.

The third becomes clear when written in logical form. Writing "P is claimed" as $\vdash P$, we have

$$\neg(\vdash P) \quad \text{and} \quad \vdash \neg P \quad \text{are different sentences} \ (1)$$

On the left: "we do not have a proof of P". On the right: "we have a proof of the negation of P". The "not claimed" of this chapter always has the meaning on the left. As a metaphor: a road not drawn on a map is not evidence that there is no road there.

Table 1 How the items of this chapter are divided (numbers are the items of Table 2)

Kind	What is not claimed	Item
Not shown	Existence and the mass gap on $\mathbb{R}^4$, the construction of the continuum limit, the double-well relaxation-time power statement for the first step	1, 2, 3, 10
Not read strongly	Computer-assisted results, readings, Lean theorems, lifting, the relation between the relaxation-time assumption and the mass gap	4, 5, 6, 7, 11
Not called false	Propositions a tool did not reach, the upper-bound assumption as read to fail in the periodic box	8, 12, 13
Not judged, not counted	Rederivations of known results, the correctness of other documents, putting unclosed propositions out	9, 14

Detour: negating a quantifier

A subsection you may skip. The negation of "for every f, $Q(f) \le C$" is "for some f, $Q(f) > C$". In symbols, $\neg\forall f\, A(f) \iff \exists f\, \neg A(f)$. To show the upper-bound assumption false, a single sequence of counterexamples f suffices; to show the lower-bound assumption false, one must show "for every f the ratio is $o(\xi^2)$". The latter is a sentence of the same "for every" type as the upper-bound assumption, and it does not come out of listing test functions. That some families did not reach it is a "for every" restricted to a family ($\forall f \in \mathcal{F}$), not a "for every" with the family removed.

02

The fourteen items and the sentences in the body that support them

Table 2 writes, one row at a time, what this part does not claim, and lists the places in the body that support each row. "Support" means that the sentence in that chapter already states the conditions, assumptions and scope, and the row merely copies it onto one line.

Table 2 What is not claimed, and the sentences in the body that support it

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
1	The solution of the problem of Yang–Mills existence and the mass gap. Neither a construction of the continuum limit nor a proof of the mass gap in the limit is here	Not shown	23-01 § 08: in four dimensions, what we have (along the line of this part) are only the theorems of the reflection-positivity preservation statement and the OS reconstruction statement, and the infinite-volume uniqueness statement for small β. Even with the relaxation-time assumption and the bridge, the infinite-volume uniqueness statement, the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement do not follow
2	Even for the first step (the double well in 0+1 dimensions), the relaxation-time assumption (the double-well power statement) has not been shown. The First-Step Upper-Bound Proposition is algebra on top of four conditions, and each of the four conditions leaves assumption rows	Not shown	23-05 § 08: where the four conditions stand. 23-12 § 07: the assumption rows needed for the conclusion number 15 or 20
3	There is not yet a single $\hbar$ at which all rows of the four conditions hold at once	Not shown	23-12 § 06: the nodes of the in-phase local Poincaré condition and the range of the saddle-tube condition do not intersect. 23-15 § 05

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
4	Computer-assisted results are inequalities inside the windows and nodes that were computed, and say nothing about outside the window or about $\hbar \to 0$	Not read strongly	23-08 § 10: what computer assistance does not say (outside the window, $\hbar \to 0$, the difference between continuum and lattice)
5	Readings (floating point, Monte Carlo, fits, lower bounds over families) are used neither as upper bounds nor as proofs of lower bounds	Not read strongly	23-00 Table 1: what the label "Reading" does not guarantee. 23-12 Table 1: all 19 Reading rows keep the grade "Reading"
6	A Lean theorem checks the algebra from assumptions to conclusion; it does not check that the assumptions are true, nor the reading of real numbers as physical quantities	Not read strongly	23-13 § 05: four limits (truth of the assumptions, the encoding of the objects, putting the conclusion into a definition, circles of dependence)
7	From the first step to the second step on, we only state the prospect that the form of the sentences and the form of the tools lift; there is no theorem of implication	Not read strongly	23-03 § 06: what lifts and what does not (constants and the range of $\hbar$ do not lift)
8	That a tool did not reach something is a property of that tool, not a claim that the proposition it did not reach is false	Not called false	the table of scope in 23-10: the "death" of a detour plan is a property of the tool. 23-02 § 08: where the roads on the four-dimensional lattice stopped
9	Rederivations of known theorems are not counted as results. The correctness of other documents that claim a construction is not judged	Not judged	23-14 § 03: the support of the cited theorems. 23-14 § 07: from outside documents we take only patterns, give the source, and do not judge
10	The record on the side of the four-dimensional lattice (the relaxation-time assumption, the three chains, the sum of shortfalls) also remains assumption	Not shown	23-01 § 05: the three chains are all assumptions. 23-02 § 07: what remained is algebra and readings

Item	What is not claimed	Kind	Supporting sentence in the body (gist)
11	Under the assumption of the one-sided bridge, the upper-bound assumption implies a lower bound on the gap of the lattice in physical units, but the converse is false in general and undecided for the class of Wilson measures. The relaxation-time assumption is not the mass gap itself	Not read strongly	(4) of 23-01 § 06 (the one-sided bridge) and (5) (the Free-Particle-on-a-Circle Counterexample: on the circle, $\Delta > 0$ and $CP = \infty$)
12	The "death" of a detour plan means that the ceiling of that construction does not exceed the needed value, not that the one remaining proposition is false	Not called false	23-10 § 03: the line that killed it is a line of a tool's inequality
13	The reading that the upper-bound assumption fails on the four-dimensional lattice in the periodic box is a conditional reading, resting on numbers from the literature and on the conjecture that "HMC and Langevin are in the same universality class"; it is not a theorem that the upper-bound assumption is false	Not called false	the note in 23-01 § 05 (topological sectors and the order of volume), 23-02 § 05
14	A proposition with assumption rows left is not put out as a result. Only what has had all its conditions closed is put out	Not counted	23-12 § 08: the integrated table is a map, not a fraction of progress

Items 1–10 are the skeleton of the part, and 11–14 were added from the text of the individual chapters. Every row compresses the conditions of the body onto one line, and is written so that reading the body gives the conditions in full.

03

On the main problem — nothing is said about the claim of the problem statement

What the problem statement (0-03) asks is that, for every compact simple gauge group, a quantum Yang–Mills theory on $\mathbb{R}^4$ exists, satisfies the axioms (4-09, 4-10), and has a mass gap.

$$\text{existence (satisfying the axioms)} \quad \wedge \quad \operatorname{spec}(H) \subset \{0\} \cup [\Delta, \infty), \ \ \Delta > 0 \ \text{(2)}$$

This part shows nothing about either part of (2) (item 1). The local-field footnote · weak-existence footnote of the problem statement excludes from solutions limits obtained only by weak existence (compactness) Cited . That subsequential limits come for free from compactness is a matter at that level, and neither uniqueness nor the mass gap follows from it.

The relaxation-time assumption, which this part treats, is a sentence that restates Δ of (2) (the gap in Euclidean time) in terms of the relaxation time C_P of Langevin dynamics. The two are different quantities, and only when the one-sided bridge holds can one cross, in one direction, from the upper-bound assumption to a lower bound on Δ (item 11, 23-01 § 06). We placed the relaxation-time assumption at the axis of the study because it is language that can be computed on the lattice, has the tools of convexity and Poincaré inequalities, and is within reach of machine checking; we do not claim that this restatement is the only road to solving the problem. Other views are placed in Part 24.

04

On the double-well step — the First-Step Upper-Bound Proposition is algebra on top of four conditions

The central sentence of the first step is the First-Step Upper-Bound Proposition: "on the four double-well conditions, $C_P \le K\xi^2 S_0/\hbar$, and hence the double-well power statement" (23-05 § 04). Reading this sentence as "the double-well power statement has been shown for the double well" is too strong (item 2). Beneath the four conditions there are 15 or 20 assumption rows of the integrated table (depending on which of the two roads) and 3 reading rows (23-15 Table 1).

The algebra of the construction is checked in Lean, but when only the theorem name remains in a summary, the conditions fall away. Table 3 lists the theorem names of this part that are easily read strongly once their assumptions are dropped (item 6).

Table 3 Theorem names, assumptions, and what is not to be read strongly

Theorem name	Assumptions (gist)	Not to be read strongly
`H3_H4_compose`	The box product inequality, the box gap inequality, $D = \hbar/S_0$, $n, \hbar$, $S_0 > 0$, $c_{\text{alt}} < 1$	Not "the First-Step Upper-Bound Proposition has been shown". The in-phase local Poincaré condition and the saddle-tube condition are inside the assumption of the box gap

Theorem name	Assumptions (gist)	Not to be read strongly
`power_H_of_kink_ratio`	$0 < K, s, \delta$, $1 \le R \le K/\hbar$, $s/\hbar \le \ln \xi$	Not "the double-well power statement has been shown". $R \le K/\hbar$ is the conclusion of the First-Step Upper-Bound Proposition itself
`H1_assembly`	Seven named ones (decomposition, fine side, Helffer's criterion, margin, $0 < \lambda$, the crossing row, the rate condition)	A proposition at six nodes, not the in-phase local Poincaré condition uniform in $\hbar$
`cloc_term_vanishes`	The core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$	The local term vanishes when the core action-barrier assumption is assumed. The value of the action barrier is a reading at $\hbar = 0$
`path5_conditional`	The remainder inequality, the proxy ratio, rarity, the arithmetic margin (named)	The Scalar-Strand Proposition at the lower end of the window; there is no edge of implication to the one remaining proposition
`win_qfree_window_le_of_interval_table`	Table of interval-arithmetic output	Inequalities inside the window (item 4). Not on the road to the conclusion in the integrated table
`hup_is_gap_positivity_of_lower`	$0 < m_{\mathrm{lat}}$, $\kappa/m_{\mathrm{lat}}^2 \le C_P$, $C_P \le K\xi^2$	The one-sided bridge enters as an assumption. That the bridge holds for the Wilson measure has not been shown (item 11)
`drainExp_eq_zero_iff`	$0 \le \eta \le 2$, $\theta \ge 0$	Algebra of exponents on the side of the four-dimensional lattice. Applying it to physical quantities rests on assumptions (item 10)

Example. `power_H_of_kink_ratio` is the algebra "if the ratio R lies between 1 and $K/\hbar$, and $\ln \xi$ is at least $s/\hbar$, then $\ln C_P / \ln \xi \to 2$". Removing the assumption $R \le K/\hbar$ and writing "the double-well power statement in Lean" gives a sentence that has not been shown. The correct summary is "the double-well power statement follows by algebra from the conclusion of the First-Step Upper-Bound Proposition (Lean)".

05

Not reaching is not evidence of falsity

The death of detour plans (items 8, 12). Of the detour plans aimed at the one remaining proposition of the route of the window log-Sobolev inequality, 13 were

judged "dead" (23-10 § 03). Death means that the ceiling of the upper bound obtained by that construction does not exceed the needed value. It does not mean that the one remaining proposition [026] is false, and it says nothing about the possibility of closing it by another construction.

The reading for the periodic box (item 13). On the four-dimensional lattice, it is reported that in a box with periodic boundary the autocorrelation time of the topological charge grows faster than ξ^2, and together with $\tau_{\text{int}} \le C_P$ this reads as the upper-bound assumption failing in the periodic box (the note in 23-01 § 05). This reading stands on numbers from the literature Reading and on the conjecture that HMC and Langevin are in the same universality class Assumption. So this part does not claim "the upper-bound assumption is false on the four-dimensional lattice" either. What it claims is only that how to correct the statement of the relaxation-time assumption remains as a question.

The lower-bound assumption is not non-triviality. What the lower-bound assumption avoids is only that the physical mass collapses to infinity; it does not ask whether there is interaction. This can be confirmed by computing the relaxation-time assumption in a theory without interaction. For the free field with lattice mass m, the Poincaré constant of the path measure is the largest eigenvalue of the covariance,

$$C_P = \frac{1}{m^2}, \qquad m^2 = 4\sinh^2\frac{1}{2\xi}, \qquad \frac{C_P}{\xi^2} = \frac{1}{4\xi^2\sinh^2(1/2\xi)} = 1 - \frac{1}{12\xi^2} + O(\xi^{-4}) \tag{3}$$

Table 4 C_P/ξ^2 for the free field on the lattice Proved here (values of the closed formula (3))

ξ	C_P/ξ^2	$1 - 1/(12\xi^2)$
1	0.920674	0.916667
2	0.979425	0.979167
8	0.998699	0.998698
64	0.999980	0.999980

The ratio tends to 1 as $\xi \to \infty$, so both the upper and the lower side of the relaxation-time assumption hold. The relaxation-time assumption and the lower-bound assumption are sentences that hold in a theory without interaction, and the non-triviality statement does not follow from the relaxation-time assumption (23-01 § 06).

06

Conditions fall away in summaries — sentences of the body and one-line summaries

The error of reading a claim strongly occurs, in most cases, at the stage of compressing a sentence of the body onto one line. Table 5 sets the conditional sentences of the body

beside one-line versions with the conditions dropped completely. Every sentence in the right column is one this part has not shown.

Table 5 Sentences of the body (conditional) and one-liners with the conditions dropped (every one on the right is wrong, or not shown)

Topic	Sentence of the body	One-liner with the conditions dropped
Chain outside the window	$q_{\text{free}} \le 0.998854904$ over the whole $\hbar$ window $[3 \cdot 10^{-4}, 3 \cdot 10^{-3}]$ Computer-assisted (assuming IEEE correct rounding, outward widths)	The chain condition of the saddle-tube condition is closed
Chain outside the ring	$q_{\text{out}} < 1$ over the whole window Computer-assisted (conditional on the pin ≤ 0; the boundary stage is a reading of 1.004–1.045)	The seam of the ring is closed
Mixing between boxes	$c_{\text{alt}} = 0.731/0.410/0.138$ ($C = 2/4/8$, values of the reduced model) Reading	The alternating maximal-correlation assumption for boxes holds
In-phase local Poincaré condition	Margin ≥ 0.067 at six nodes (ends $[0, 2.5]^2$) Computer-assisted , constant $C \le 1.6 \times 10^{10}$ (a proposition at the nodes)	The in-phase local Poincaré condition has been shown
First-Step Upper-Bound Proposition	Algebra on top of the four double-well conditions (Lean has the algebra of composing boxes)	The double-well power statement has been shown for the double well
Feynman–Kac strand	A reading of $\sup \lvert\nabla\psi\rvert^2$ from below, 0.433 ± 0.007 (0.54 times the threshold, $\hbar = 10^{-3}$) Reading ; the paper upper bound is about 1000 times the threshold	The assumption of the Feynman–Kac strand falls within the threshold
Periodic box	On numbers from the literature and a conjecture, the upper-bound assumption reads as false in the periodic box	The upper-bound assumption is false in four dimensions

The errors in Table 5 are of two types. One drops the scope (window, nodes, model values); the other drops the grade (Reading, Computer-assisted, Assumption). Both are prevented by keeping the words of the condition in the one-line summary.

07

Analogies and other documents — what is not judged, and the convention for putting things out

Where words are borrowed from the side of physics, they inherit the level of where they were borrowed from. Holography is a conjecture, and a table of correspondences that uses it is a dictionary, not a proved equivalence. The "4" in a complete family existing for $d \le 4$ is a combinatorial number of the lattice, a coincidence unrelated to the dimension of physical spacetime (23-02 § 02). The correspondence between the

first step and the four-dimensional lattice ($\hbar \to 0$ and $\beta \to \infty$) is also an analogy, not a theorem of correspondence (23-03 § 06).

On other documents that claim a construction, this part does not judge correctness (item 9). What we took from outside documents is only patterns for reading dependency graphs (closing by putting the conclusion into a definition, circles of dependence), and even those were written as patterns with the source withheld (23-13 § 05, 23-14 § 07).

Finally, the convention for putting things out (item 14). As long as even one assumption row remains, the propositions of this part are not put out as results. What is put out is only what has had all its conditions closed and has passed machine checking and bibliographic checking. Until then, the text stands as a map of which rows are closed and which are not.

What this chapter can say

Said	Not said
That this part shows nothing about (2) of the problem statement. The fourteen items that are not claimed, and the places in the body that support them (Table 2)	Whether (2) is true. The existence of the continuum limit. A gap uniform in the volume
The assumptions of the theorem names in Table 3 (copies of the statements in each chapter)	The truth of the assumption rows
Proved here That $C_P/\xi^2 = 1 - 1/(12\xi^2) + \cdots$ and the relaxation-time assumption hold for the free field on the lattice (closed formula)	A theorem comparing C_P and ξ^2 for an interacting four-dimensional measure (in neither direction)
The conditions of the reading that the upper-bound assumption fails in the periodic box (numbers from the literature and a conjecture)	That the upper-bound assumption is false, or true, on the four-dimensional lattice

Sources and reproduction

Item	Kind	Source / tools
Problem statement, local-field footnote · weak-existence footnote	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (the source table of 0-03)
Each row of Table 2	This book	The chapters and sections in the right column of the table
Theorem names and assumptions of Table 3	Lean	Statements in 23-13 § 03 and the grade tables of each chapter

Item	Kind	Source / tools
(3), Table 4	Proved here	Closed formula. The expansion $1/(4\sinh^2(u/2)) = 1/u^2 - 1/12 + O(u^2)$
Sentences of the body in Table 5	This book	23-07 § 03, 23-08 § 05, 23-11 § 02, 23-06 § 04, 23-09 § 05, 23-01 § 05

Terms used in this chapter — not shown, not read strongly, not called false, negating a quantifier, the one-sided bridge, the death of detour plans, conditions that fall away in summaries. Their meanings are in the A-02 Glossary.

Next chapter to read: 23-17 Insights and small questions — the views that repeatedly proved useful in studying the first step, and questions that can be settled at the size of an undergraduate thesis.

2026-10-09 · chapter Yang–Mills and the mass gap Part 23 The first road, first half — from the direction of time patterns of insight, discarded hypotheses, small questions · Kiichi, Shiori and Rin

Insights and small questions — views that remain as procedures, and questions that can be settled

> In this chapter, § 01 is guide material, § 02– § 04 are a map of open questions and § 05 is this book's own investigation.

What this chapter shows — The views that repeatedly proved useful while studying the first step, gathered into twelve insights in the form "any problem needs this procedure". For each insight, one small question the reader can check by hand. A table of discarded hypotheses and the reasons. A list of questions that can be settled at the size of an undergraduate thesis, and a worked example that takes one question to the form of a settlement.

Prerequisite chapters — The material for the insights is in the chapters 23-06 to 23-11. At the end of each insight we write the chapter where that view worked. The prerequisites of each question are written in its entry.

Said first — The level is [Y2]–[Y3]. What is listed here are insights and questions before verification, neither theorems nor results. Whichever question is settled, it does not reach a proof of the mass gap. For each question we wrote what does not move even when it is settled.

The order of this chapter

1. How to read the insights — three states, and the criterion for keeping one (Table 1)
2. Twelve insights and small questions — ways of paying, addresses, thresholds, directions, order
3. Discarded hypotheses — the original form, the corrected form, the grounds (Table 2)
4. Questions at the size of an undergraduate thesis — a list of seventeen questions (Table 3)
5. Worked example: one question to the form of a settlement — $1 - \lambda_k$ for the Hermite family (Table 4)

01

How to read the insights — three states, and the criterion for keeping one

In the course of a study, "shifts in how things look" get written down apart from results. They fall into three states: **still an insight** (not yet tested), **checked form** (checked once on a small model or a closed formula), and **overlaps the known** (the core was a restatement of a known theorem; not counted as an insight).

There is one criterion for keeping something in this chapter. Not who thought of it or when, but only what can be written in a form that lets us say **another reader who meets a problem of the same type would also need to take this procedure**. For example, "before trimming an upper bound, sort the factors of the needed value into structure and tools" is a procedure that keeps its meaning away from the double well. Conversely, the observation that some number happened to lie near a threshold is not kept.

We read the insights on the side of the four-dimensional lattice in seven patterns (cannot be measured from below / drop to one dimension / drop the units / counterexamples and controls first / look at the power of the difference / the diffusive-relaxation-time assumption is a pair of measure and form / connectedness of the state space). Table 1 shows those seven and how they relate to the twelve insights of this chapter.

Table 1 The seven patterns, and the insights of this chapter (the right column gives the insights of § 02)

Pattern	One line	Insight continuing it in this chapter
Cannot be measured from below	Averages and test functions bound C_P only from below	Before trimming an upper bound, measure one lower bound
Drop to one dimension	Tools for upper bounds on C_P are available in one dimension	The threshold of convexity is decided by whether the parameter enters the Hessian (cut shorter than the threshold)
Drop the units	Make dimensionless ratios the protagonists	What does $1/\hbar$ live in? (the length where $1/\hbar$ lives)
Counterexamples and controls first	First name the models where the proof fails	The wall of conditioning creates minima (count the minima first)
Look at the power of the difference	What fixes the power is not the quantity but what is left after cancellation	Split the ratio of upper bound to reading into a stage of cancellation and a stage of direction
The relaxation-time assumption is a pair of measure and form	Name the normalisation of the Dirichlet form explicitly	Whether the diffusive-relaxation-time assumption is true changes with the coordinates (the Dirichlet form)
Connectedness of the state space	Suspect topology before uniformity of constants	The insight that truth changes with the coordinates, the note in 23-01 § 05

02

Twelve insights and small questions

Are we paying for the same place twice?

In a long chain of inequalities, one bad place (near the top, for the double well) may be paid for by two stages under different names. In the sector framework, the Bochner stage and the Gibbs stage both paid for the same neighbourhood of the top, and the amplification factor of one of them multiplied the whole. Moving the negative curvature to the other side avoids passing through the amplification. In the same pattern, when using gradients in which two terms complement each other, sharing the denominator can sometimes fold $(1+\sqrt{C})^2$ into $(1+C)$ instead. As a procedure: **before tightening a construction, write out place by place what each stage is paying for, and look for double payment** (23-09 § 02, the double-payment type in 23-10). State: checked form.

Small question 1. Compare the best right-hand side, when $\|u\|^2=\|v\|^2=\|w\|^2=1$, of two constructions for the sum of three vectors $u+v+w$: applying $\|u+v\|^2 \le (1+t)\|u\|^2 + (1+1/t)\|v\|^2$ $(t>0)$ twice in a row, and distributing all three terms at once as $\|u+v+w\|^2 \le (1+s_1)\|u\|^2+\cdots$. Where does the twice-in-a-row construction pay for the same quantity twice?

The currency of Poincaré and the currency of entropy

There are situations where one wants to pay for rare events with "rarity". But on the side of the Poincaré inequality, rarity is not a currency. Poincaré looks at the speed of exit, and putting mass on a rare state cancels the mass in both the variance and the Dirichlet form. Rarity can be a currency only on the side of entropy (the logarithmic Sobolev inequality), but there one pays instead $\ln(1/\nu)$ for the rarest state on which test functions can concentrate. As a procedure: **decide between the entropy type and the variance type by setting two rarities side by side (the rarity of the defect one wants to pay for, and the rarity of the states on which test functions can concentrate)** (the currency-mix-up type in 23-10). State: checked form (the two-point model).

Small question 2. Consider the measure $\pi=(1-p,p)$ on two points $\{0,1\}$ and the Dirichlet form with flip rate 1, $\mathcal{E}(f)=\pi_0\pi_1(f_1-f_0)^2$. Show that the Poincaré constant does not depend on p. Check, by citing the known formula for the two-point space (Diaconis–Saloff-Coste), that the logarithmic Sobolev constant grows at the rate $\ln(1/p)$ as $p\to 0$.

What does $1/\hbar$ live in? — read the loss on the same length as the quantity it multiplies

On the plateau at the top, the test function that produces a constant of size $1/\hbar$ was not a function of the collective linear coordinate $\sum z_j$ but of the radius from the corner at the top, $\sum z_j^2$. Choosing the linear collective coordinate gets the direction wrong. Also, $1/\hbar$

lives on the length of the plateau $\ell \approx 0.34/\hbar$, and when measuring a loss it must be measured on the same length as the quantity it multiplies. As a procedure: **before splitting into two scales, check whether the function that produces the lower bound lives on the coarse coordinate or on the fibre** (the type of the address of 1/ħ in 23-10, 23-09 § 06). State: still an insight (in a form supported by numerical readings).

Small question 3. For the measure $\mu \propto e^{-V(|z|)/\hbar}$ on $\mathbb{R}^k$ with $V(r) = (r^2-1)^2/8$, compare the ratios $\mathrm{Var}(\cdot)/\mathcal{E}(\cdot)$ ($\mathcal{E}(f) = \hbar \int |\nabla f|^2 d\mu$) of the linear function $f = z_1 + \cdots + z_k$ and the radial function $g = |z|^2$, for $k = 1$ and $k = 2$, up to the leading term as $\hbar \to 0$. For $k = 2$, when μ concentrates on an annulus, which function captures the slow direction?

The wall of conditioning creates minima

When conditioning by fixing the ends of an interval exactly, all the minima at $\hbar = 0$ may lie on the wall. The "slowness" or "bistability" one then sees is not a property of the model but a shadow created by the conditioning (the bistability at the negative end of the core was the shadow of confining, inside the interval of the core, a configuration in which the kink wants to shift by one site). As a procedure: **each time the way of cutting sectors or conditioning changes, make the first check a count of the $\hbar = 0$ minima lying on the wall** (23-11 § 06). State: checked form.

Small question 4. Where does the measure $\mu_\hbar \propto e^{-V/\hbar}\mathbf{1}_{[0,b]}$, $V(x) = (x^2-1)^2/8$ restricted to the interval $[0, b]$ ($0 < b < 1$), concentrate as $\hbar \to 0$? Near there, approximate $\mu_\hbar$ by an exponential distribution, and write the behaviour of the Poincaré constant of $\mu_\hbar$ (Dirichlet form $\hbar \int f'^2 d\mu$) as $\hbar \to 0$ in terms of $|V'(b)|$. What changes when the wall is moved to $b = 1$?

Before closing the inside uniformly, first write what multiplies it on the outside

In nested estimates, one can get stuck trying to bound an inner constant by $O(1)$. But if, in the outer formula, that constant is multiplied by a small factor, the accuracy needed inside is much looser. In the proposition for the box gap, the local term is multiplied by $e^{-2S_0/\hbar}$, so the constant of the core did not need to be $O(1)$; size $e^{\Delta S/\hbar}$ ($\Delta S < 2S_0$) was enough. As a procedure: **before closing the inside uniformly, first write what multiplies that quantity in the outer formula, and fix the accuracy needed inside by its ratio to the outer scale** (23-11 § 06). State: checked form (the reading of the action barrier is the value at $\hbar = 0$).

Small question 5. Write the condition on B needed for $K(\hbar) = A + e^{-2S_0/\hbar}B(\hbar)$ to be bounded as $\hbar \to 0$. If it has the form $B(\hbar) \le c\,\hbar^{-p} e^{\Delta S/\hbar}$, what must the relation among ΔS, $2S_0$ and p be? What is needed in the exact case $\Delta S = 2S_0$?

The sup over a parameter is attained just outside the threshold

When bounding the constant of a core of length k uniformly in k, the hard case was not a long core but just outside the threshold where convexity is lost ($k = 8$, both ends at the top). Inside the threshold convexity makes it free; far outside it saturates by reduction to the box; and one point in between attains the sup. As a procedure: **when stuck on a sup over a parameter, first find where the threshold is, then name the value just outside it and bound it separately** (23-11 § 05). State: checked form (the threshold is a closed formula).

Small question 6. Find the smallest k for which $\kappa(k) = 8\sin^2\!\big(\pi/(2(k+1))\big) - \frac{1}{4}$ is negative (the same formula as in 23-09 § 06). How does $\kappa(k)$ behave as $k \to \infty$? Are "the threshold is at $k = 8$" and "the sup is attained at $k = 8$" the same thing or different things?

The threshold of an excluded set is fixed by the mass the downstream inequality allows

When excluding a bad set and estimating on the rest, where to put the threshold is decided not by the convenience of the bad set but by what the downstream inequality, which receives the mass of that set, allows. In the sector framework, the threshold of the window mark and the threshold of the defect disagreed, so the only set determined by the mark that contained the defects was the whole space, and the closing condition, taken literally, gave nothing. As a procedure: **check literally that each of the conditions being composed holds on the same definition of the threshold, and work the threshold back from the mass the downstream can pay** (23-09 § 04). State: checked form (one line of measurability).

Small question 7. At one site $z \in \mathbb{R}$, consider the mark $\sigma = \mathbf{1}\{z \le -\theta_\sigma\}$ and the defect set $N = \{-\theta_B < z \le -\theta_\sigma\}$ ($\theta_\sigma < \theta_B$). List all sets determined by the value of σ (sets measurable with respect to σ) that contain N. What changes if $\theta_\sigma = \theta_B$?

The threshold of convexity is decided by whether the parameter enters the Hessian

When stuck on a sup over end values, the road divides according to whether the end value enters the Hessian. If the end value enters a nearest-neighbour quadratic coupling only linearly, the Hessian does not depend on it, and convexity is uniform in the end value. So cutting shorter than the threshold of convexity makes the sup over end values free, and the whole price moves to the side of the correlation across the cuts (the maximal correlation). As a procedure: **when stuck at the end of a parameter, first check whether that parameter enters the Hessian; if it does not, cut shorter than the threshold of convexity, and account separately only for the correlation at the seams** (23-11 § 05). State: checked form.

Small question 8. For $S(x_1, \dots, x_k) = \sum_{j=0}^{k} \frac{(x_{j+1} - x_j)^2}{2\varepsilon} + \varepsilon \sum_{j=1}^{k} V(x_j)$ and the measure $e^{-S/\hbar}$ with $x_0 = a$ and $x_{k+1} = b$ fixed, check that the Hessian does not depend

on a, b. When $V'' \geq -\frac{1}{2}$ (the value at the top of $V = (x^2-1)^2/8$), write the range of k for which the smallest eigenvalue of the Hessian is positive, in terms of ε (compare with $\kappa(k)$ of small question 6 at $\varepsilon = \frac{1}{2}$).

Split the ratio of upper bound to reading into a stage of cancellation and a stage of direction

When an upper bound U and a reading from below L are far apart, looking at the ratio U/L as a single number does not tell where one is losing. Inserting between them "the quantity along the same path before the cancellation is discarded", R, and splitting into R/L (the stage of cancellation) and U/R (the stage of direction), for the Feynman–Kac strand the former was around 1.2 and the latter over 1000, so almost all the loss was at the stage where the scalar Grönwall discards direction (23-09 § 05). As a procedure: **split the gap between an upper bound and a reading into ratios per stage before choosing the target of improvement**. State: checked form (ratios of readings).

Small question 9. For $\dot{X} = -CX$, $C = \mathrm{diag}(\lambda, \Lambda)$ $(0 < \lambda < \Lambda)$, with initial value $g = (0, 1)$ (the stiff direction), compare the true value of $\int_0^T |X_t|\, dt$ with the upper bound coming from the scalar Grönwall $|X_t| \leq e^{-\lambda t}|g|$. How does the ratio grow with Λ/λ? What is recovered when estimating with the matrix kept as it is?

Before trimming an upper bound, measure one lower bound

When tightening the construction of an upper bound through many stages still does not reach, first measuring one lower bound of the same quantity with a test function tells whether the problem lies in the technique or in the target. If the lower bound exceeds the target, improving the upper bound is pointless. In the sector framework, it turned out that the lower bound on the propagation constant lay at several times the needed value, which changed the meaning of the work of trimming the floor of the upper bound. As a procedure: **before trimming the floor of an upper bound, put a stage that sets the needed value beside the lower bound on the best constant** (23-09 § 03, 23-10 § 02). State: checked form.

Small question 10. On a single road of L cells (uniform distribution, $\mathcal{E}(f) = \frac{1}{L}\sum (f_{i+1} - f_i)^2$), show with the single test function $f_i = i$ that the target "$CP \leq L^2/12$" is false $(\mathrm{Var}(f)/\mathcal{E}(f) = L(L+1)/12)$. Compare also with the true value $1/(2(1-\cos(\pi/L)))$ (23-00 § 04).

Obstacles do not vanish but move — record, line by line, where they moved

Changing the formulation can make an obstacle seem to vanish while it appears elsewhere in the same form. In the first step, a single number — the curvature at one site at the top falls short of the coupling by $\frac{1}{4}$ — came back three times in different forms: the Poincaré constant of the lump of the core, the contraction ratio on the slope approaching

1, and the ratio of the tail pinned at the top exceeding 1 (the common wall of 23-09 § 06). As a procedure: **each time the formulation changes, record line by line "where the obstacle moved", and check whether the formula where it moved can be written with the same number**. If it can, that is not a defect of the tool but the structure of the object. State: checked form (the formula of the common wall).

Small question 11. Find in closed form the smallest eigenvalue of the tridiagonal matrix $M_k = \mathrm{tridiag}(-2,\ 4-\frac{1}{4},\ -2)$ ($k \times k$, Dirichlet at both ends), and compare it with $\kappa(k)$ (small question 6). How is the failure of diagonal dominance, $\frac{15}{4} < 4$, related to the smallest eigenvalue becoming negative for $k \geq 8$?

Whether the diffusive-relaxation-time assumption is true changes not only with the measure but with the coordinates (the Dirichlet form)

The diffusive-relaxation-time assumption is not a sentence about a measure but about a pair of a measure and a Dirichlet form. Even for the same equilibrium measure, depending on which variables the dynamics moves independently, the size of the gap can be $O(1)$ or inversely proportional to the square of the side of the volume. In a dynamics with a conserved quantity (a quantity only handed between neighbours), its unevenness is smoothed out only by diffusion. As a procedure: **before asking about the relaxation-time assumption, name the coordinates of the dynamics, and first check whether that dynamics has a conserved quantity and at what length the conservation breaks** (23-01 § 05; for how to compare with the length at which conservation breaks, see Part 24). State: still an insight (checked form in the free-field approximation).

Small question 12. For the product measure of L independent standard normals, compare the gaps of (a) the dynamics that moves each coordinate independently by Ornstein–Uhlenbeck, and (b) the dynamics that hands quantity between neighbours and keeps the sum $\sum x_i$ (on the measure with the sum fixed). Check that the gap of (b) has the form $4\sin^2(\pi/(2L))$, and that it is unrelated to the static correlation length (here 0).

03 Discarded hypotheses — the original form, the corrected form, the grounds

Discarded hypotheses and the reasons for discarding them are information for avoiding the same errors. The upper six rows of Table 2 were discarded in the study of the first step, and the lower four on the side of the four-dimensional lattice.

Table 2 Discarded hypotheses and corrected insights

Original form	Corrected form	Grounds
Local Poincaré comes from a convex core	The action of an excursion out of the core is much smaller than S_0, and exponentially many excursions fit into an interval of length ξ	The reading of the action of excursions (towards the two scales of 23-06 § 03)

Original form	Corrected form	Grounds
Convexity improves away from the wall	The opposite. Pressing the end against the wall lowers the variance and raises the Hessian; moving it away removes only this help	Comparison of the Hessian for each end value (23-06 § 04)
Making the lumps larger lets the chain be built from the convexity of each lump	The worst convexity of a lump decreases with the length of the lump, and the coupling between lumps does not decrease	The formula of the common wall (23-09 § 06)
The contraction can be bounded by the smallest eigenvalue of the time-averaged operator	False. What can be used is only the time average of the smallest eigenvalue, and this discards direction	The algebra of the counterexample (Lean `mbar_bound_false`, 23-09 § 05)
If the pointwise reading is singular at rare configurations, the true gradient is singular too	What fixes the upper limit of the true gradient is not the pointwise minimum but the decay rate of the flow of derivatives	The representation $\nabla \psi = \int \mathbb{E}[J_t^{\top} \nabla G]\, dt$ (23-09 § 05)
The road of sharpening constants can lower $\hbar$ indefinitely	It stops at the convex radius (about 0.45); beyond that the error terms become genuinely large	The reading of the margin at the nodes (towards the $\hbar < 0.005$ row of 23-06 § 07)
$C_P \asymp \xi^2$ usually holds	The relaxation-time assumption is a specific claim, the dynamical exponent $z = 2$	$z \approx 2.17$ for the heat bath of the two-dimensional Ising model
Abelian counterexamples show that "non-abelianness is essential"	"An input that forbids a second small parameter" is essential	Three-dimensional $\mathbb{Z}_2$ keeps a single scale even though it is abelian (17-12)
The holographic density is not log-concave	The sign is the opposite ($W'' = -2B''$)	The judgement of the one-dimensional model
Gauge-fixed numbers are a property of confinement	Numbers for compact $U(1)$ in four dimensions; not measured for $SU(2)$	23-02 § 04

There are two types of error, and both occur at the stage of raising something to a one-line summary. One drops the conditions (group, model, range); the other confuses "a pointwise quantity" with "a quantity over the measure". Of the upper six rows of Table 2, four are of the latter type.

04

Questions at the size of an undergraduate thesis — a list of seventeen questions

A "small question" is one that satisfies three things. The form of the answer (exists / does not exist, the value of a constant, whether an inequality holds) is fixed in advance. The tools are at hand (python3 with NumPy and SciPy, a SAT solver, interval arithmetic, Lean and Mathlib, a literature search). There is a prospect that a third- or fourth-year student

of mathematics or physics can settle it within the period of an undergraduate thesis. The first column of Table 3 gives the short name of each question.

Table 3 Seventeen questions ("What does not move when settled" are the rows of this part that do not change even if the question is settled)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
1	Complete families of k=3 cells	Does a complete family of $k = 3$ cells exist in $d = 9$?	SAT, Lean	A construction in the periodic box, or a record refuting satisfiability of the box	That one-link integration is a single step only (23-02 § 02)
2	Largest dimension with a complete family	The largest dimension in which a complete family exists, $d_{\max}(k)$	Exhaustive search, paper	A table for $k = 3, 4, 5$, and the reason the upper bound $3k + 1$ is not sharp	Same as above. Non-existence with period 2 (Lean `no_perfect_k4_d9`, `no_perfect_k5_d8`, assumption: period 2) already exists
3	Constant of the Hermite family	The constants of $1 - \lambda_k$ for the Hermite family (the worked example of § 05)	Paper, python3	The two terms and the remainder of (5)	That test functions bound C_P only from below
4	SU(N) version of the ratio on forests	The $SU(N)$ version, and the Lean statement, of the ratio of functions on the forest saturating at $1/\lambda_1(G)$	Paper, Lean	The statement of Poincaré for products of Haar	The relaxation-time upper-bound assumption (it only closes one family of test functions for the relaxation-time lower-bound assumption on the four-dimensional lattice)
5	C_P(β) on the two-dimensional lattice	$C_P(\beta)$ in the link variables of two-dimensional lattice YM	Paper, python3	The power as $\beta \to \infty$	The relaxation-time assumption in four dimensions (two dimensions is a control)
6	Curvature after one-link integration	The best constant of the curvature after one-link integration, $c^* \in [21, 27]$	Global optimisation, interval arithmetic	c^* and the configuration, a computer-assisted lower bound	That one cannot get out from the strong-coupling side

No.	Question	One line	Tools	Form of settlement	What does not move when settled
7	The N^{−3} coefficient of the column model	The coefficient $C \approx 320$ of N^{-3} in the column model, analytically (finite-size expansion of the XX chain)	Paper, python3	A closed formula agreeing to 4 digits or more	The logic of the arrows of the tube estimate (the tube inequality)
8	Sign of 1/P in the column model	In the column model, the sign of $1/P$ at finite u	python3	A table of signs for $u = 0.1 \sim 0.3$	Same as above
9	Gauge dependence of the shortfall	How much does the shortfall c_2 change with the choice of gauge in the block transformation?	python3	A table of c_2 for each gauge, and whether $c_2 > 0$ is kept for positive kernels	The arrow of the sum of shortfalls (it does not move if $c_2 \neq 0$)
10	C_P and ξ² of the O(3) σ model	C_P and ξ^2 of the two-dimensional $O(3)$ sigma model	python3, literature	The slope, and whether a logarithm is present when the scale is changed	The relaxation-time assumption in four dimensions (an asymptotically free control)
11	C_P/ξ² of three-dimensional Z_2 gauge theory	C_P/ξ^2 of the three-dimensional $\mathbb{Z}_2$ gauge theory	python3	A table, and the correspondence with the dynamical exponent on the dual Ising side	The relaxation-time assumption in the non-abelian case
12	Volume monotonicity of C_P(V)	Monotonicity in volume of $C_P(V)$, and boundary conditions (find all eigenvalues of the generator on small lattices)	python3	A counterexample to monotonicity, or a proof	The question itself of the order of volume in which the relaxation-time assumption is stated (the note in 23-01 § 05)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
13	Dynamical exponent of four-dimensional SU(2)	Collect the primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	Literature	A table of about ten primary sources (exponent, observable, box, boundary)	The conditions of the reading for the periodic box (the part that is a conjecture)
14	Interval arithmetic for the maximal correlation of the core	Bound from above, by interval arithmetic, the two-point maximal correlation of the medium-core step ($8 \le k \le 14$)	Interval arithmetic, paper	Computer-assisted upper bounds at each point of k and $\hbar$, and a paper on the difference between lattice and continuum	[007] of the road of the core on the box side (the side $k \ge 15$) and [014] (23-15 Table 1)
15	Power of the prefactor for the core and window	Read the power of the prefactor $\hbar^{-p}$ of C_P for the core and the window by the capacity of the degenerate point	python3, paper	The value of p (a reading), and the form of the capacity formula	The side of the action barrier $\Delta S < 2S_0$ (only one of the conditions of small question 5)
16	Comparison for the alternating maximal correlation of boxes	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$, avoiding examples that do not reduce to two-point quantities	Paper	An upper bound on c_{alt} uniform for the reduced model	The rows of the in-phase local Poincaré condition and the saddle-tube condition (only [006] moves)

No.	Question	One line	Tools	Form of settlement	What does not move when settled
17	Length set by a conservation-breaking term	In the dynamics with conservation of small question 12, when a small term that breaks the conservation is added, the length at which the gap departs from the square of the side	Paper, python3	The relation between the strength of the breaking term and the length	The relaxation-time assumption in four dimensions (the smallest model for the questions of Part 24)

For every question, before starting, we put a stage of searching for whether an answer to the same question exists. Write the question in one sentence in the form of a theorem, search with combinations of words from both mathematics and physics, follow the references of reviews to the text of the primary sources, and judge it as one of: the same answer, a close but different result, or not found. If nothing is found, write "not found in the range searched", and leave the words and materials searched in the source table.

05

Worked example: one question to the form of a settlement — $1-\lambda_k$ for the Hermite family

To see the size of a question, we carry the question "constant of the Hermite family" to the end. The Poincaré constant of the standard normal distribution is $C_P=1$, with equality at $f=x$ (1-10). Let λ_k be the maximum of the ratio $\mathrm{Var}(f)/\mathcal{E}(f)$ over the family $F_k=\mathrm{span}\{x^3,\dots,x^{2k+1}\}$, which does not contain x.

Normalising the Hermite polynomials to $\mathbb{E}[h_m h_n]=\delta_{mn}$ gives $h_n'=\sqrt{n}\,h_{n-1}$, so the variance and the Dirichlet form of an odd function $f=\sum_{j=0}^{k}u_j h_{2j+1}$ are diagonal.

$$\mathrm{Var}(f)=\sum_{j=0}^{k}u_j^2,\qquad \mathcal{E}(f)=\sum_{j=0}^{k}(2j+1)\,u_j^2 \tag{1}$$

The condition for being in the family is the single linear constraint "the coefficient of the monomial x is 0", $\sum_j w_j u_j=0$, with $w_j=(-1)^j\sqrt{(2j+1)!!/(2j)!!}$. Finding the minimum μ_k of $\mathcal{E}/\mathrm{Var}$ under the constraint by Lagrange multipliers leaves an equation in one variable [Cited] (the secular equation for stationary values of a quadratic form under a linear constraint, Golub 1973).

$$g_k(\mu) := \sum_{j=0}^{k} \frac{w_j^2}{2j+1-\mu} = 0, \qquad 1 < \mu_k < 3, \qquad \lambda_k = \frac{1}{\mu_k} \tag{2}$$

Setting $\mu_k = 1 + \varepsilon$, the $j = 0$ term is $-1/\varepsilon$ and balances the rest.

$$\frac{1}{\varepsilon} = S_k + O(\varepsilon), \qquad S_k := \sum_{j=1}^{k} \frac{(2j+1)!!}{(2j)!!\, 2j} \tag{3}$$

S_k closes by a generating function. Splitting each term in two by $(2j+1)!!/(2j)!! = (2j+1)\binom{2j}{j}4^{-j}$, and using the partial-sum identity $\sum_{j=0}^{k}\binom{2j}{j}4^{-j} = (2k+1)\binom{2k}{k}4^{-k}$ (by induction on k) and the value $\ln 2$ at $x = 1$ of $\ln \frac{2}{1+\sqrt{1-x}} = \sum_{j\ge 1}\binom{2j}{j}\frac{x^j}{2j\,4^j}$, gives

Proved here

$$S_k = 2\sqrt{\frac{k}{\pi}} + (\ln 2 - 1) + O(k^{-1/2}) \tag{4}$$

Substituting into (3) and expanding $1 - \lambda_k = \varepsilon - \varepsilon^2 + \cdots$ fixes the two terms Proved here .

$$1 - \lambda_k = \frac{\sqrt{\pi}}{2\sqrt{k}} - \frac{\pi \ln 2}{4k} + O(k^{-3/2}), \qquad \frac{\sqrt{\pi}}{2} = 0.886227, \quad \frac{\pi \ln 2}{4} = 0.544397 \tag{5}$$

Table 4 λ_k and a check of (5) Reading (values from solving for the roots of (2) numerically)

k	λ_k	$\sqrt{k}(1-\lambda_k)$	$k\left(1-\lambda_k-\frac{\sqrt{\pi}}{2\sqrt{k}}\right)$
1	0.555556	0.4444	−0.4418
10	0.774921	0.7118	−0.5517
1000	0.972524	0.8689	−0.5489
10^4	0.991192	0.8808	−0.5459
10^6	0.999114	0.8857	−0.5446
(5)	$\to 1$	0.8862	−0.5444

What does not move when it is settled is the fact that test functions bound C_P only from below. Enlarging the family keeps $\lambda_k < 1$ and only approaches 1 at the rate $k^{-1/2}$; that the limit is 1 can be said only from an input outside the family (Gaussian convexity, $V'' = 1$). The settlement of a small question often takes this form.

This part placed the relaxation-time assumption at the axis of the study, and this chapter gathers the views that repeatedly proved useful on that basis. Questions that stand on views other than the relaxation-time assumption are placed in Part 24.

What this chapter can say

Said	Not said
In which stage of which chapter of the body each of the twelve insights worked (the chapter at the end of each insight), and the state of each insight (still an insight, checked form)	That the insights work outside the first step (the four-dimensional lattice, other problems). Every insight is a view before verification
Proved here The formula for the threshold in small questions 6 and 11 ($\kappa(k) < 0 \iff k \geq 8$), and the skeleton of the derivation of (4)(5)	The value of the coefficient of the remainder $O(k^{-3/2})$ in (5)
Reading The numbers of Table 4	Table 4 consists of floating-point values of the roots of (2), not propositions of upper or lower bounds
The list of seventeen questions, and what does not move when they are settled	No question, even when settled, shows anything about the mass gap, the continuum limit or the relaxation-time assumption. "The size of an undergraduate thesis" is a prospect

Sources and reproduction

Item	Kind	Source / tools
Material for the twelve insights	This book	The sections of the chapters given at the end of each insight (23-06 to 23-11)
The two-point logarithmic Sobolev constant of small question 2	Cited	P. Diaconis, L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains (bibliographic data not checked; the statement has not been checked in the original. It is an object to be checked on the side of the question)
The lower two rows of Table 2 (the heat bath of the two-dimensional Ising model, the holographic density)	This book	The record of hypotheses discarded in this book's study. The z of the two-dimensional Ising model is from M. P. Nightingale, H. W. J. Blöte, Phys. Rev. Lett. 76 (1996) 4548 (abstract only)
Questions 1–13 of Table 3	This book	The fourteen small questions of this book's study, minus the one that became the test bed. Questions 14–17 come from the remaining rows of 23-09 to 23-11
The secular equation (2)	Cited	G. H. Golub, Some modified matrix eigenvalue problems, SIAM Review 15 (1973) 318–334 (bibliographic data only)
Table 4	Reading	Bisection for the roots of (2), cross-checked against the generalised eigenvalues of the Gram matrix of monomials for $k \leq 6$

Terms used in this chapter — the three states of an insight, paying twice, the currency of Poincaré and the currency of entropy, the length where $1/\hbar$ lives, the wall of conditioning, the threshold of convexity, the stage of cancellation and the stage of direction, the moving of obstacles, dynamics with conservation, small questions. Their meanings are in the A-02 Glossary.

Next chapter to read: Part 24 The first road, second half — views other than the relaxation-time assumption, and a list of material for new views that take the relaxation-time assumption as given. To go on to the appendices, A-01 Table of notation.