Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 24 The first road, second half

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Entrance to Part 24 — what we take over from Part 23

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Where Part 24, "Further considerations", stands; the sentences it takes from Part 23; the map of its nine chapters (title, one line, level, and what each chapter can say); the words fixed in this part; and what this part does not treat.

Prerequisite chapters — 23-00 (the entrance to Part 23), 23-01 § 05 (the diffusive-relaxation-time assumption), 23-01 § 06 (what the diffusive-relaxation-time assumption gives and does not give).

Said first — The level is [Y2]. This page is a map and proves nothing. No chapter of Part 24 shows the mass gap. We write neither that the relaxation-time assumption was wrong nor that it was the right road.

The order of this chapter

1. Where this part stands — what to think about after the study of Part 23
2. Sentences taken from Part 23 — five sentences and their labels (Table 1)
3. Map of the part — the nine chapters, and what each can say (Figure 1, Table 2, Table 3)
4. Words fixed in this part — the local diffusive-relaxation-time assumption and the ratio form, the threshold, the length at which conservation breaks
5. What is not treated, the table of notation, returning to Part 23

01

Where this part stands — what to think about after the study of Part 23

Part 23 restated the problem of the mass gap in terms of quantities that can be computed on the lattice and are within reach of machine checking — a power relation between the relaxation time C_P of Langevin dynamics and the correlation length ξ (the relaxation-time assumption) — and used that restatement as a foothold for its study. Setting up the relaxation-time assumption as a sharp sentence made it possible to name where the relaxation-time assumption fails. Which test functions obstruct the upper half of the relaxation-time assumption (the relaxation-time upper-bound assumption) came together as a single pattern. This is not "the relaxation-time assumption was wrong"; the order of events is "because the relaxation-time assumption was set up, the nature of how it breaks became visible".

Part 24 is the "further considerations" that start from there. First we see that what obstructs the relaxation-time assumption comes together in one form — a global quantity not reflected in the gap slows relaxation on a length other than the correlation length (24-01, 24-02). Next, keeping the relaxation-time assumption as a premise, we define two candidates that restate the relaxation-time assumption so as to avoid that form (the relaxation-time assumption for each topological sector, and the local relaxation-time assumption with windows or boxes limited to a few times the ruler), and examine which part the one-sided bridge needs for the mass gap (24-03, 24-04). From there we go on to two views that do not pass through the relaxation-time assumption: that on a lattice with continuous time the gap is exactly the Poincaré constant of the vacuum measure (24-05), and that on groups one needs tools that pay with rarity instead of convexity (24-06). Finally we state where being four-dimensional matters, limited to the two points "the dimensionless coupling" and "topology" (24-07), and close with questions and what is not claimed (24-08).

Many sentences of this part are estimates, and we distinguish them by adding one estimate label, Estimate, to the five labels of Part 23 (Proved here, Cited, Computer-assisted, Reading, Assumption). An estimate is an opinion offered as material for judgement; it carries neither a proof nor numbers.

02

Sentences taken from Part 23

Table 1 lists the five sentences that Part 24 takes from Part 23 and uses. Labels and statements remain as in Part 23; this part only cites equation numbers and sections, and does not copy the proofs.

Table 1 Sentences taken from Part 23

Sentence	Label	Place in Part 23	Chapters that use it

Sentence	Label	Place in Part 23	Chapters that use it
The trade of the restatement: the quantity of the problem statement is the gap in Euclidean time; the quantity of the relaxation-time assumption is the relaxation C_P in Langevin time (a quantity defined by a supremum, bounded from below by a single test function, and whose upper bound can be built from parts). The two are different quantities	Convention	23-01 § 04 (the three C_P)	24-01, 24-04, 24-05
The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$. The halves of the band statement are written the upper-bound assumption and the relaxation-time lower-bound assumption	Assumption	23-01 § 05	All chapters
The one-sided bridge: on a lattice with continuous time, $C_P \geq \kappa \hbar^2 / \Delta^2$. Under the assumption of the bridge, the upper-bound assumption is a stronger sentence than the gap of the lattice in physical units (there is also a version in the form of a paper, before external checking)	Proved here (continuous time), Assumption (on Wilson's lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge"))	23-01 § 06 (Lean `hup_is_gap_positivity_of_lower`, assumptions: $0 < m$, $\kappa / m^2 \leq C_P$, $C_P \leq K \xi^2$)	24-04, 24-05
The converse is false in general: for a free particle on a circle, $\Delta > 0$ holds while $C_P = \infty$ (the Free-Particle-on-a-Circle Counterexample)	Proved here	23-01 § 06	24-01, 24-04

Sentence	Label	Place in Part 23	Chapters that use it
In the four-dimensional periodic box, the upper-bound assumption reads as broken by the freezing of the topological charge (on original sources for $SU(3)$ and above, and a reading that extends a conjecture). The note on topological sectors and the order of volume	Assumption, Reading	the note in 23-01 § 05, 23-02 § 05	24-01, 24-03, 24-07

03

Map of the part — the nine chapters, and what each can say

The nine chapters of Part 24 and the table of notation

0. **24-00 Entrance — what is taken from Part 23** [Y2] (this page)
1. 24-01 One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity [Y3]
2. 24-02 Choosing the ruler and the coordinates again — the length at which conservation breaks [Y3]
3. 24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption [Res] (summary at [Y3])
4. 24-04 Which form of the relaxation-time assumption the one-sided bridge needs [Res] (summary at [Y3])
5. 24-05 Making time continuous — the Hamiltonian lattice and the vacuum measure [Y3]
6. 24-06 Paying with rarity instead of convexity — renormalisation on groups and log-Sobolev [Res] (summary at [Y3])
7. 24-07 Because it is four dimensions — the dimensionless coupling and topology [Y3]
8. 24-08 Questions, and what is not claimed [Y2]–[Y3]

24-09 Table of notation for the first road, second half (a reference page)

Together with Part 23, this part forms the second half of "The first road — from the direction of time" (the second road is Part 25, the meeting of the two roads is Part 26, and the conditional statements are Part 27). The order of the chapters is: the form of

the breakage (24-01, 24-02), restatements that keep the relaxation-time assumption as a premise (24-03, 24-04), views that do not pass through the relaxation-time assumption (24-05, 24-06), what is specific to four dimensions (24-07), and questions (24-08). From 24-01 to 24-05 is a single chain in which the conclusion of each chapter becomes the question of the next. 24-06 is not a consequence of the chain but a chapter of tools needed to show any of the statements of 24-03 to 24-05, and on the line of the argument it is placed as a branch.

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Figure 1 The routes of Part 24 (an arrow means "the conclusion
of the previous chapter becomes the question of the next";
it is not a theorem of implication)

  24-01 One form ──→ 24-02 Ruler, coordinates ──→ 24-03 Sector assumption,
                        |                            local assumption
                        |                              |
                        |                 (ratio form, threshold)
                        |                              ↓
                        |             24-04 Which form of the relaxation-time
                        |                    assumption the bridge needs
                        |                              |
                        |                              ↓
                        |                  24-05 The vacuum measure
                        |
     Two-dimensional test beds (the U(1) strip, CP^(N−1))
        ── a branch reaching 24-02 and 24-07

  Tool branch: 24-06 Paying with rarity instead of convexity
               (needed by every one of 24-03 to 24-05)
  Specific to four dimensions:
               24-07 The dimensionless coupling and topology
  Closing: 24-08 Questions, and what is not claimed
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Table 2 One line for each chapter, and what that chapter can say

Chapter	One line	What this chapter can say
24-01	Puts in one table, together with their strength, that the number of kinks, the winding number, the flux, the topological charge and the flux through the box are all of the same pattern: "a global quantity not reflected in the gap slows relaxation on another scale"	That the five examples are of the same pattern is a classification (an estimate). The numbers carry the label of each row
24-02	Splits the length to compare with C_P into three rulers, and sets out that the type of relaxation changes with the choice of coordinates (the Dirichlet form), together with numbers for the two-dimensional strip and the test beds	The length at which conservation breaks is a reading on the two-dimensional $U(1)$ strip. Restating what to compare with is a candidate. CP^{N-1} is a conditional reading at one volume and one correlation length

Chapter	One line	What this chapter can say
24-03	Defines the relaxation-time assumption conditioned on topological sectors, and the local relaxation-time assumption with windows or boxes limited to a few times the ruler, and states them in the ratio form against a massless reference	The two relaxation-time assumptions have definitions, and the local relaxation-time assumption has content only in the ratio form. There is no inequality shown at any β
24-04	States that the one-sided bridge uses only the part of the upper-bound assumption that is local in time, and sets out the bridge on a window, the threshold of the ratio form and three references	The threshold of the bridge depends on the reference and the family, and the band between the threshold and 1 is undecided. Everything rests on the local-excitation moment assumption and the assumptions of the One-Sided Bridge Proposition (Theorem 3.1 of the paper)
24-05	Sets out that on a lattice with continuous time the gap is tied, as the exact reciprocal, to the Poincaré constant of the vacuum measure at a single time, and why the tools needed there do not apply	The gap is the Poincaré constant of the vacuum measure itself, and the identity does not make the problem easier
24-06	States that on groups the tools of convexity cannot, in principle, stand at weak coupling, that the non-convex places are pushed into places of small probability, and the tools that pay with rarity and candidate rules	A chapter of tools. The ceiling of convexity has a proof; the class of measures that pay with rarity is a candidate. Results on hierarchical models imply nothing for four dimensions
24-07	Limits what matters in four dimensions to two points, the dimensionless coupling and topology, and sets out the four hurdles and why the third hurdle stayed open for so long, from what primary sources written before the published construction say	The two points are an estimate of a property of the critical dimension. Freezing for $SU(2)$ is undecided. Results in two dimensions imply nothing for four dimensions
24-08	The table of questions, the list of candidate rules, the sixteen items that are not claimed	Questions, and what is not claimed. Candidate rules are candidates

Table 3 lists the material that came up in Part 23 and that the chapters of this part take up as body text.

Table 3 Material taken over from Part 23, and the chapters that take it up

Material	Related in Part 23	Chapter that takes it up
The table of other roads (the renormalisation group and log-Sobolev, stochastic quantisation, the Hamiltonian lattice, hierarchical models, and so on)	23-02 § 08	24-05, 24-06
The upper-bound assumption and the one-sided bridge to the mass gap	23-01 § 06	24-04

Material	Related in Part 23	Chapter that takes it up
Hierarchical gauge models and the problem of coordinates	23-17 Insight 12	24-02 § 02, 24-06 § 05
Conservation of flux and "the length at which conservation breaks"	23-17, the question "length set by a conservation-breaking term"	24-01, 24-02
The study of the relaxation-time assumption on the four-dimensional lattice (freezing of the topological charge, and how to correct it into a sentence per sector or a local sentence)	the note in 23-01 § 05, 23-02 § 05	24-01, 24-03

04

Words fixed in this part

The local relaxation-time assumption and the ratio form. The sentence that states the Poincaré constant $C_P^{\mathrm{loc}}(c)$, with test functions limited to those supported in a window (or box) of c times the ruler ℓ, by its ratio $K_{\mathrm{rel}}(c)$ to the massless reference on the same window with the same graduation (24-03 § 03). The form "bounded by some constant" also holds for the free field and is empty, so we use the ratio form.

The threshold K_*. The value of the threshold such that, if the ratio is below it, the one-sided bridge gives a lower bound on the mass under the assumptions. Depending on the choice of reference it changes: 0.342κ (scalar), 0.502κ (gauge quotient), about 0.62κ (colour singlet of $SU(2)$) (24-04 § 06; the numbers are readings for the free-field reference).

The length at which conservation breaks ℓ_*. The length at which a global quantity not reflected in the gap begins to relax through the breaking of conservation rather than through local transport. It is a ruler different from the static correlation length, and the values measured on the two-dimensional $U(1)$ strip are not proportional to the correlation length (24-02 § 01, § 03).

05

What is not treated, the table of notation, returning to Part 23

What is not treated. Part 24 does not touch the ultraviolet side — the existence of the continuum limit, the recovery of rotations, the OS axioms, non-triviality (the Wilson-loop limit statement, the rotation-recovery statement, the OS reconstruction statement and the non-triviality statement of Part 23, 23-01 § 02). Only 24-07 § 05 sets out, from primary sources, how far the third hurdle has been reached as a difficulty specific to four dimensions, and even there construction is not treated. The existence of the

continuum limit is read in Volume 2 (from Part 6) through the construction published on 2026-11-10 (a Lean formalisation that builds a theory satisfying the Wightman axioms as the limit of Wilson's lattice in four dimensions; the committee's assessment is that it is proved in Lean).

Notation. The notation of Part 24, and its correspondence with symbols of the same letter in Part 23, are in 24-09 Table of notation for the first road, second half. In particular Δ is reserved for the gap of the Hamiltonian $E_1 - E_0$, and the mass in physical units is written m_{phys}.

Returning to Part 23. The map of Part 23 is in 23-00 § 01, what is not claimed in 23-16, and the small questions in 23-17.

What this page can say

Said	Not said
The order of the chapters of Part 24, and one line on what each chapter can say (Table 2). The sentences taken over from Part 23 and their labels (Table 1)	No chapter shows the mass gap. Nor does any show whether the relaxation-time assumption is true
The meaning of the words fixed in this part (§ 04)	Theorems about the local relaxation-time assumption, the threshold, or the length at which conservation breaks (we go as far as definitions, the ratio form, and numbers for the free field and in two dimensions)

Sources and reproduction

Item	Kind	Source / tools
The sentences of Table 1	This book	The sections and source tables of 23-01 and 23-02
The rows of Table 2 and § 04	This book	"What this chapter can say" in each chapter of Part 24

Terms used on this page — further considerations, the trade of the restatement, the one-sided bridge, the estimate label Estimate, the local relaxation-time assumption, the ratio form, the threshold, the length at which conservation breaks, the tool branch. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-01 One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity — sets out, in one table with five

examples, the pattern in which a global quantity not reflected in the gap slows relaxation on another scale.

One form that obstructs the diffusive-relaxation-time assumption — a nearly conserved global quantity

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-10 and 5-09.

What this chapter shows — We set out five examples that obstruct the upper half of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption), and see that all of them fall into a single pattern: "a global quantity not reflected in the gap slows relaxation on a scale other than the correlation length". The five examples are the number of kinks in the double well, the winding number of a particle on a circle, the flux in the two-dimensional $U(1)$ strip, the topological charge in the four-dimensional periodic box, and the flux through a box. What differs is the strength (logarithmic, power or exponential, infinite) and the event that does the breaking. How the upper-bound assumption fails in this pattern is decided by the single line $\tau_{\mathrm{int}} \le C_P$. Using the charge of a subregion, we separate two accounts, "transport" and "breaking of conservation", in preparation for the local diffusive-relaxation-time assumption of 24-03.

Prerequisite chapters — 23-01 §05 ((13) for the relaxation-time assumption, (14)(15) for the upper-bound assumption, the note on topological sectors), 23-01 §06 (the Free-Particle-on-a-Circle Counterexample), 23-02 §05 ($\tau_{\mathrm{int}} \le C_P$), the detour in 2-10 (conserving dynamics). The notation is collected in 24-09.

Said first — The level is [Y3]. Calling the five examples one pattern is a classification, not a theorem Estimate . The numbers in each example are numbers under conditions, with a label on each row. This chapter does not claim that the upper-bound assumption is false. The row for the four-dimensional periodic box rests on numbers from original sources for $SU(3)$ and above, and on a reading that extends a conjecture attached to observations with open boundaries to the slowness of topology in the periodic box; for $SU(2)$ it is not settled even at the level of a reading. None of the examples breaks the mass gap.

The order of this chapter

1. The type of a nearly conserved global quantity — three conditions, and the one line by which the upper-bound assumption fails ((1)–(3))
2. Five examples in one table — Table 1, and three rows moved below (Table 2)
3. Three strengths — logarithmic, power or exponential, infinite (Table 3)
4. Two accounts for the charge of a subregion — the transport coefficient d and the coefficient of breaking of conservation h ((4)–(7))
5. The wall of the first step is a scaled-down version of this pattern — the $\frac{1}{4}$ at the top, and the correspondence with the types of breakage in Part 23 (Table 4)
6. What cannot be said — extending the conjecture, $SU(2)$, two dimensions, power and exponential
7. Table of the five grades and the dependency graph — Table 5

01

The type of a nearly conserved global quantity — three conditions, and the one line by which the upper-bound assumption fails

The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$ ((13) of 23-01), and its upper half (the upper-bound assumption) is "for every test function f, $\mathrm{Var}(f) \le C\, \xi_{\mathrm{RG}}^2\, \mathcal{E}(f)$" ((15) there). Since it is a statement about every f, it collapses if there is even one function whose ratio $\mathrm{Var}/\mathcal{E}$ grows faster than ξ^2. Because Part 23 set up the relaxation-time assumption as a sharp sentence, we can name what such a function looks like. The answer of this chapter is the following pattern.

Pattern (an estimate)

For a pair of a measure μ and a Dirichlet form $\mathcal{E}$, there is a global quantity Q such that

(i) $\mathrm{Var}_\mu(Q)$ is large (it grows with the volume or the length of the window).

(ii) $\mathcal{E}(Q)$ is small. Q is handed to neighbours through the flux of a conservation law, or changes only through rare configurations.

(iii) Q is not reflected in the gap Δ. It does not belong to the family of time-integrated single-time functions ($\mathcal{F}_1$ in 24-04), or if it does, it does not overlap the excitations of the gap.

Then the time for Q to relax is fixed not by the correlation length ξ but by the scale of the events that break Q.

(iii) is the condition that separates this pattern from the mass gap. Since the quantity is not reflected in the gap, the gap may stay positive however slow the relaxation is. The smallest example of this separation was the Free-Particle-on-a-Circle Counterexample of Part 23 (a free particle on a circle; the gap is positive and $C_P = \infty$).

The one line by which the upper-bound assumption fails. If the dynamics is reversible (the generator L is self-adjoint on $L^2(\mu)$, $-L \geq 0$), then the integrated autocorrelation time of any observable f does not exceed C_P. Writing ν_f for the spectral measure of $-L$ for $\tilde{f} = f - \mu(f)$ (supported on $[\lambda_1, \infty)$, $\lambda_1 = 1/C_P$), we have

$$\tau_{\mathrm{int}}(f) := \int_0^\infty \frac{\mathrm{Cov}(f(0), f(t))}{\mathrm{Var}\, f}\, dt = \frac{\int \lambda^{-1}\, d\nu_f(\lambda)}{\int d\nu_f(\lambda)} \;\leq\; \frac{1}{\lambda_1} = C_P \quad (1)$$

Proved here (one only integrates $\lambda^{-1} \leq \lambda_1^{-1}$ over $\lambda \geq \lambda_1$; equality holds when $\tilde{f}$ lies in the slowest eigenspace). In Lean the version with a finite spectral sum is checked (LatticeGauge.FourDTrial.tau_int_le_CP, assumptions: weights ≥ 0, each eigenvalue $\geq \lambda_1 > 0$, sum of weights > 0). The version that does not use the dynamics is the definition of the supremum itself:

$$\frac{\mathrm{Var}_\mu(Q)}{\mathcal{E}(Q)} \;\leq\; C_P \quad (2)$$

Proved here. Hence, if the relaxation time of Q ($\tau_{\mathrm{int}}(Q)$ for (1), the static ratio for (2)) grows at least as $c\,\xi^p$ ($p > 2$) along a sequence with $\beta \to \infty$, then $C_P \leq K\xi^2$ fails for every constant K. In the range where the correlation length ξ and the bookkeeping length ξ_{RG} are comparable up to constant factors, this is the negation of the upper-bound assumption.

$$\tau_{\mathrm{int}}(Q) \leq C_P, \quad \tau_{\mathrm{int}}(Q) \geq c\,\xi^p \ (p > 2), \quad \xi \to \infty \qquad \Longrightarrow \qquad \neg\,\exists K:\ C_P \leq K\xi^2 \quad (3)$$

Proved here Lean LatticeGauge.FourDTrial.not_hup_of_fast_tau (assumptions: $0 < c$, $2 < p$, $\xi_n \to \infty$, $\xi_n > 0$, and for each n, $\tau_n \leq C_n$ and $c\,\xi_n^p \leq \tau_n$). Exponential growth $\tau \geq c\,e^{\alpha\xi}$ includes $p = 3$ ($e^x \geq x^3/6$). The caution in the converse direction is also one line. That τ is small for a fast quantity does not support the upper-bound assumption. For an observable whose weight lies on fast modes, τ_{int} stays small however large C_P is (negative control tau_small_but_CP_large).

Let us check the level of this one line. Of the two assumptions on the left of (3), Lean takes only inequalities between sequences of real numbers; that τ_n is the autocorrelation time of a reversible dynamics and that C_n is the Poincaré constant of the Wilson measure lie outside Lean. What supports the second assumption of (3) in four dimensions is the Monte Carlo numbers of the original sources Cited, reinterpreted across the difference of algorithms as Langevin autocorrelations, and that reinterpretation is an assumption (§ 06 (a)).

02

Five examples in one table

Table 1 sets out, in the same columns, five examples that satisfy the three conditions of the pattern. The columns are: the quantity outside the family (the Q of the pattern), the event that moves it (the breaking of conservation), the relation between the scale of slowness and the correlation length, the strength, the numbers measured and their labels, and the chapter where the body text is. The label on a row is the label of that row's numbers; the classification "falls into the same pattern" itself is Estimate on every row.

Table 1 Five examples that obstruct the upper-bound assumption (the strength is how C_P/ξ^2 grows; the conditions on the numbers are in each row's para

Example	Quantity outside the family	What moves it (breaking of conservation)	Scale of slowness and correlation length	Strength	Numbers measured and la
(a) Double well in 0+1	The number of kinks (which well the path is in), the position of a kink in the window	Creation and annihilation of kink pairs (tunnelling), motion across the window	Same origin as ξ. The overall ratio is $S_0/\hbar \approx \ln \xi$. The local ratio is ω_b/m even for the shortest window	Logarithmic	One-site ratio $= \omega_b/m$ Proved here, 4.61, 21.6, 638 Reading, overall rati Assumption
(b) Free particle on a circle	The winding number (integral of the increments of the lifted angle)	Never breaks (continuous paths)	Infinite	Infinite	$C_P = \infty$ with $\Delta = \hbar^2/2 >$ Proved here
(c) The two-dimensional $U(1)$ strip	Flux dipoles, the topological charge Q of the strip	Wrapping of a single plaquette (a barrier of height 2β)	$\ell_c \propto e^{0.916\beta}$, $\xi_{XY} \approx 2\beta$	Exponential in ξ_{XY}	$\ell_c = 3.29$ to 139, $\tau(Q)/\xi_{XY}^2$ $= 0.4 \to 24$ Reading

Example	Quantity outside the family	What moves it (breaking of conservation)	Scale of slowness and correlation length	Strength	Numbers measured and la
(d) Four-dimensional periodic box	The smoothed topological charge Q	Small instantons slipping through the lattice (dislocations)	Faster than ξ^2 in original sources for $SU(3)$ and above. For $SU(2)$, one primary source, close to a^{-2}	Power or exponential ($SU(3)$ and above)	Numbers of the original sources Cited , reading the upper-bound assumption Assumption
(e) Flux through a box	Functionals moved by gauge transformations at the sites on the boundary of the box	They enter the test space if the boundary gauge transformations are not divided out	2.22 to 2.53 times the value of the reference box	Constant factor	Difference between the t choices Proved here , factor Reading

(a) The number of kinks in the double well. In the path measure in 0+1 dimensions (23-04, $V = (x^2-1)^2/8$, $S_0 = 2/3$), which well the path is in changes only when a kink (a path moving from valley to valley) enters. The weight of a kink $e^{-S_0/\hbar}$ fixes the correlation length $\xi \propto \hbar^{1/2} e^{S_0/\hbar}$, and the diffusion of kinks fixes the scale $\xi^2 S_0/\hbar$ on the upper side of C_P, so the slow quantity and the correlation length are born from the same event. The ratio C_P/ξ^2 is of order $S_0/\hbar \approx \ln \xi$, so the band form of the upper-bound assumption fails only by a logarithm while the double-well relaxation-time power statement survives ((14) of 23-04, the accounting for a dilute gas of kinks) Assumption . The same pattern appears in local ratios with limited windows. In a window of a single time, the Poincaré constant of the marginal ψ_0^2 moves by the ground-state transformation to the gap of $(H - E_0)$, and its ratio to the harmonic oscillator with the curvature of the bottom is **exactly** ω_b/m (when ψ_0 is the Schrödinger ground state) Proved here . With the transfer on the lattice $\varepsilon = \frac{1}{2}$, at $\hbar = 0.3,\ 0.15,\ 0.08$ it is $4.61,\ 21.6,\ 638$, 1.04 to 1.05 times ω_b/m Reading . The detailed behaviour of this ratio (lengthening the window moves it to $(\omega_b/m)^2$) is treated in 24-04.

(b) The winding number of a particle on a circle. This is the Free-Particle-on-a-Circle Counterexample of Part 23. For the path lifted to the real line, $\tilde{x}(\tau)$, the integral of its increments with weight w, $f = \int w\, d\tilde{x}$, has $\mathrm{Var}(f) = \hbar \int w^2$ and $\mathcal{E}(f) = \hbar \int w'^2$, and lengthening the window makes the ratio as large as one likes. The winding number never breaks for continuous paths, so the strength is infinite. This f is a functional supported in a window of length L; with the window fixed the ratio is finite, but lengthening the window makes it unbounded Proved here . The winding number is not a time-integrated single-time function, so it satisfies condition (iii) of the pattern.

(c) The flux and the topological charge of the two-dimensional $U(1)$ strip. In a strip of width 2 with the links on the outer boundary fixed, if the links are moved by Langevin, the angles of the plaquettes (the flux) are only handed between neighbours, and unevenness of the flux is smoothed out by diffusion. Because the group is compact, a "wrapping", in which the angle of a single plaquette crosses π, changes the total flux (the topological charge Q of the strip) by one, and conservation is broken. The length at which conservation breaks, ℓ_C, is $3.29,\ 9.88,\ 24.7,\ 53.6,\ 139$ at $\beta = 2, \dots, 6$; the slope of $\ln \ell_C$ is $0.916(21)$, and the ratio to the static length $\xi_{XY} = -1/\ln(I_1(\beta)/I_0(\beta))$ (2.78 to 10.9) opens from 1.2 to 12.8 Reading . When the strip is longer than ℓ_C, relaxation is fixed by the time for Q to move by one, and $\tau(Q)/\xi_{XY}^2$ is about 0.4 at $\beta = 4$ and about 24 at $\beta = 8$ (strip length 32) Reading . Conditions: two-dimensional $U(1)$, width 2, outer boundary frozen, gauge quotient of link Langevin, strip length ≤ 256, τ_{int} as a reading of an upper bound on the gap, fit of the form $A/M^2 + B$. The length at which conservation breaks is discussed in detail in 24-02.

(d) The topological charge in the four-dimensional periodic box. The smoothed topological charge Q cannot move between sectors in the continuum limit; on the lattice it changes only when a small instanton, of the size of the lattice spacing, slips through the lattice (a dislocation). In the periodic box, the autocorrelation time of Q is reported to grow clearly faster than ξ^2 (Del Debbio–Panagopoulos–Vicari 2002 with local heat bath and overrelaxation, Schaefer–Sommer–Virotta 2011 with HMC, Bonanno–Bonati–D'Elia 2021 with local algorithms; if a power, the exponent is about 5, or else exponential in $1/a$, and all three write that power and exponential cannot be distinguished by the data) Cited . The groups in all three are $SU(3)$ and above. For $SU(2)$, the primary sources measuring the dependence of $\tau_{\mathrm{int}}(Q)$ on the lattice spacing number one in the range searched (Abe–Morikawa–Suzuki 2025; HMC, periodic, a small box with side about $2.2/\sqrt{\sigma}$, three lattice spacings), which reports growth close to a^{-2} Cited . The original sources treating $SU(2)$ with heat-bath algorithms were all observations on the side of "freezing not strong enough to measure". The conditions for using this, through (3), in a reading of the upper-bound assumption are collected in § 06.

(e) Flux through a box. An example in which the same pattern appears not through slowness of the dynamics but through the choice of the family of test functions. When measuring the reference for the local Poincaré constant in a four-dimensional box (the value for the massless free gauge field), dividing by gauge transformations at all sites of the box makes the test space functions of the plaquette angles; leaving the gauge transformations at the boundary sites of the box undivided lets a field passing uniformly through the box (a flux whose divergence lies only on two faces) into the test space. The latter value is 2.22 (side 4) to 2.53 (side 12) times the former, which returns roughly to the gauge-fixed value Reading (one-colour Gaussian approximation, $\beta = 1$, canonical graduation, floating point). In a contractible box, the space divided at all sites coincides with the space of functions of the plaquettes Proved here (24-03 § 03). The numbers in this row are numbers for the free field of the reference, not for the target measure. Regarding it as a member of the pattern is Estimate : the flux through the boundary of the box is, seen from inside the box, "a global quantity not reflected in the gap that moves the ratio by one choice of family".

There are three rows that satisfy the conditions of the pattern but were moved below the table of five examples (Table 2). Each has its body text in another chapter, and here we copy only the labels.

Table 2 Three rows moved below Table 1

Example	Quantity outside the family	Slowness	Label	Body text
Conserving dynamics (Gaussian chain)	The total of the field	$\xi^4/2$ for the component at wavenumber $1/\xi$, $C_P = \infty$ for the infinite chain	Proved here	the detour in 2-10
Flux through the plaquettes of the layers of a hierarchical lattice (Gauss approximation, link coordinates)	Flux of each layer	Smallest eigenvalue $\beta \cdot 4\sin^2(\pi/2^{N+1})$. The same slowness even when the layers are decoupled and the static correlation set to 0	Proved here (sector decomposition and the Neumann Laplacian of the two-dimensional dual lattice), Reading (cross-check for $N \le 4$)	24-02
Centre sectors of a small periodic box	Sign of the Polyakov loop	Long autocorrelation under heat bath. A finite-volume artefact that vanishes as the box grows; not the Langevin C_P	Reading	24-08 (questions)

The upper side of the hierarchical-lattice row is checked in Lean (HierLink.lam_le, bundle LatticeGaugeHierarchicalLink; no assumptions: $4\sin^2(\pi/2^{N+1}) \le \pi^2/4^N$. That the smallest eigenvalue equals this expression is outside Lean).

03

Three strengths — logarithmic, power or exponential, infinite

Rearranging the five examples by how the ratio of slowness C_P/ξ^2 (or $\tau(Q)/\xi^2$) grows with the correlation length, the pattern is one but the strengths divide into three.

Table 3 One form, three strengths

Strength	Example	The breaking event and the origin of the correlation length	Effect on the relaxation-time assumption and the upper-bound assumption
Logarithmic	(a) Double well	The same event (kinks) produces the slow quantity and ξ	The band form of the upper-bound assumption fails by a logarithm; the double-well power statement survives Assumption
Power or exponential	(c) the two-dimensional strip, (d) the four-dimensional periodic box	The event that breaks the slow quantity (wrapping, dislocation) and the origin of ξ are different	In (c), $\tau(Q)/\xi_{XY}^2$ grows with β inside the two-dimensional toy Reading . In (d), the upper-bound assumption reads as false under the conditions of § 06 ($SU(3)$ and above)
Infinite	(b) the rotor, and the conserving dynamics and layer flux of Table 2	There is no breaking event	$C_P = \infty$ while $\Delta > 0$ Proved here
(constant factor)	(e) Flux through a box	Not the dynamics but the choice of family	Moves the denominator of the local ratio by a factor of 2.2 to 2.5 Reading

The middle row of Table 3 is the row that matters in four dimensions. In the double well the slow quantity and the correlation length are born from the same kink, so the exponents match and only a logarithmic difference remains. In the two-dimensional strip, the correlation length (the inverse string tension) grows only linearly in β, whereas the wrapping that breaks conservation is an event crossing a barrier of height 2β, and its length grows exponentially in β. Whether the same split occurs in four dimensions, and if so why, is the question of 24-07.

04

Two accounts for the charge of a subregion — the transport coefficient and the coefficient of breaking of conservation

In infinite volume the total topological charge Q is not in L^2, so what goes into (2) of the pattern is the charge of a subregion R. Writing its ratio $\mathrm{Var}/\mathcal{E}$ split into two coefficients makes clear what the local relaxation-time assumption of 24-03 asks.

Setting. Take a translation-invariant lattice measure μ in four-dimensional infinite volume, and the Dirichlet form of link Langevin $\mathcal{E}(f) = \sum_\ell \mathbb{E}|\nabla_\ell f|^2$. The smoothed charge density $q(x)$ is taken to depend only on links within distance r_S of x (made by finitely many smoothings; the Gaussian tail of the Wilson flow is treated separately as a truncation error). The gradient field $J_\ell(x) := \nabla_\ell q(x)$ is 0 for $|x - x_\ell| > r_S$. Put its sum $\bar{J}_\ell := \sum_x J_\ell(x)$ (the gradient of the total charge) and its first moment $D_\ell := \sum_x (x - x_\ell) J_\ell(x)$ (the dipole), and define three coefficients by

$$\chi_t := \sum_y \mathrm{Cov}\big(q(0), q(y)\big), \qquad h := \sum_{\ell \in \text{unit cell}} \mathbb{E}|\bar{J}_\ell|^2, \qquad d := \sum_{\ell \in \text{unit cell}} \mathbb{E}|D_\ell \cdot e_1|^2 \tag{4}$$

Proved here (definitions; by cubic symmetry $\mathbb{E}[D \otimes D]$ is taken to be of the form d times the identity matrix). χ_t is the topological susceptibility in lattice units, h is the density of **breaking of conservation** (in the continuum $\bar{J} = 0$ by topological invariance; only

on the lattice is it non-zero, on dislocation configurations), and d is the coefficient of **transport**. The coefficient of the boundary layer g is the quantity per unit area obtained by subtracting the volume term from $\mathcal{E}$ of the charge cut off by a half-space.

A sharp box. For a box of side R, assuming cluster decay $\sum_z |z|\,|\operatorname{Cov}(q(0),q(z))| < \infty$ Assumption (an assumption equivalent to a gap, so beware of circularity when bringing this back to arguments about the gap),

$$\operatorname{Var}(Q_R) = \chi_t |R| - b|\partial R| + o(|\partial R|), \qquad \mathcal{E}(Q_R) = \sum_\ell \mathbb{E}\Big|\sum_{x\in R} J_\ell(x)\Big|^2 = h|R| + g|\partial R| + O(R^2 r_s^2) \quad (5)$$

(b is the one-sided first moment of the correlation). Note the form of the denominator. The sum over densities of the squared gradients, $\sum_{x\in R}\mathbb{E}|J_\ell(x)|^2$, is not $\mathcal{E}(Q_R)$. It is a quantity that drops the cancellation in the sum $\sum_x J_\ell(x)$ (that in the continuum the total charge does not move locally), and it leaves a volume term even in a perfectly conserving theory. The correct volume term is the density h of $|\sum_x J_\ell(x)|^2$. Inserting $|R| = R^4$ and $|\partial R| = 8R^3$ gives

$$\frac{\chi_t}{h} - \frac{\chi_t R - 8b}{hR + 8g} = \frac{8(g\chi_t + bh)}{h(hR+8g)} \quad (6)$$

Proved here (algebra) Lean LatticeGauge.LocalH.subregion_gap_identity (assumptions: $0 < h$, $0 < hR + 8g$). As $R \to \infty$ the ratio returns to the overall ratio of the periodic box, $\operatorname{Var}(Q)/\mathcal{E}(Q) = \chi_t/h$, and the leading ratio $\chi_t R^4/(hR^4 + gS)$ does not exceed χ_t/h (subregion_ratio_le, assumptions: $0 \le \chi_t$, $0 < h$, $0 \le g$, $0 < R^4$, $0 \le S$).

A smooth window. With $Q_w := \sum_x w(x)q(x)$ and $w(x) = W(x/R)$, we have $\nabla_\ell Q_w = w(x\ell)\bar{J}_\ell + \nabla w(x\ell)\cdot D_\ell + O(R^{-2})$; the cross term vanishes by $\sum \nabla(w^2) = 0$, and

$$\frac{\operatorname{Var}(Q_w)}{\mathcal{E}(Q_w)} = \frac{\chi_t R^2}{hR^2 + d\lambda_W}\big(1 + O(\xi^2/R^2)\big), \qquad \lambda_W := \frac{\|\nabla W\|^2}{\|W\|^2}, \qquad R_*^2 := \frac{d\lambda_W}{h} \quad (7)$$

Assumption (an estimate on top of cluster decay and the definitions of h, d; λ_W is $4\pi^2$ for the Dirichlet window on the unit four-dimensional cube). The two regimes divide at the crossing length R_*. For $R \ll R_*$ the ratio is $\chi_t R^2/(d\lambda_W)$, proportional to R^2; this is the same **transport** type as conserving dynamics (the detour in 2-10). For $R \gg R_*$ the ratio is χ_t/h, the **barrier** type (breaking of conservation). Reading $D_{\text{eff}} := d/\chi_t$ and $r \propto h/\chi_t$ gives $R_*^2 \approx D_{\text{eff}}/r$, a four-dimensional candidate for the "length at which conservation breaks" of 24-02 Estimate .

The two coefficients bear on different questions. The necessary condition for the global upper-bound assumption is $\chi_t/h \lesssim \xi^2$, which asks about the coefficient of breaking of conservation h. The local relaxation-time assumption of 24-03 is the condition obtained by putting (7) into a window of side $c\xi$, and in the regime where h is small it asks about the transport coefficient d (24-03 § 05). The values of h, g and d on the four-dimensional lattice are not in this book.

05

The wall of the first step is a scaled-down version of this pattern

The first step of Part 23 stopped at the common wall: the curvature at one site at the top falls short of convexity by $\frac{1}{4}$ ((11) of 23-09: the diagonal $4 - \frac{1}{4} = \frac{15}{4}$ does not reach the sum of couplings 4). This wall can be read as a 0+1-dimensional scaled-down version of the pattern of this chapter Estimate . The reason is the following correspondence.

Condition (ii) of the pattern was "Q changes only through rare configurations". In the double well, the configurations in which Q (which well) changes are those where the path crosses the plateau at the top, which is exactly where the Hessian of the action has a negative direction. Tools that rely on convexity (Bakry–Émery, Brascamp–Lieb, diagonal dominance) produce a shortfall on these configurations, and the shortfall can be paid only by the rarity (entropy) of the plateau (23-10 currency-mix-up type). That is, $\frac{1}{4}$ is the local face of "on the configurations of the event that breaks a nearly conserved quantity, convexity falls short". What corresponds to this in four dimensions are the configurations where the topological charge changes (dislocations of the size of the lattice spacing, which break the smoothness condition). The prospect of meeting a wall of the same pattern in four dimensions comes from here Estimate .

Table 4 The pattern of this chapter, and its correspondence with the notes and types of breakage of Part 23

Place in Part 23	What is said there	Reading in terms of this chapter's pattern
the note in 23-01 § 05	The topological charge cannot move between sectors, and in the periodic box the upper-bound assumption reads as false under a conjecture. Correcting the relaxation-time assumption into a statement per sector or a local statement is a candidate direction	Row (d) of Table 1. The two corrected forms are in 24-03
the paragraph on the pattern in 23-01 § 06	Relaxation is slower than the gap when a quantity not local in time, not reflected in the gap, is carried by diffusion	Exactly (i)–(iii) of the pattern. This chapter increased the examples to five and divided them by strength

Place in Part 23	What is said there	Reading in terms of this chapter's pattern
23-10 summit-curvature type (one source)	The only source of the shortfall is the curvature at one site at the top	The configurations of the breaking event gather in one place (§ 05)
23-10 currency-mix-up type	Poincaré is the speed of exit; rarity is the currency of entropy. After conditioning, no rarity is left to pay with	The "rare configurations" of (ii) of the pattern cannot be paid for in the Poincaré ratio. Conditioning on sectors moves the rarity into a quantity between sectors (C_{trans} of 24-03)
23-10 transfer type	Another column gets heavier by as much as one got lighter	The sector diffusive-relaxation-time assumption does not remove the obstruction but pushes it into a single quantity between sectors (24-03 § 04)

06

What cannot be said

(a) **Extending the conjecture.** The conjecture "HMC and Langevin are in the same dynamical universality class" is, in the original source (Lüscher–Schaefer 2011), a phrase attached to the observation that with open boundaries for $SU(3)$ the autocorrelation grows as a^{-2}. The same source writes that with periodic boundaries algorithms that integrate Langevin also have difficulty crossing topological sectors, but as grounds it cites original sources on heat bath and HMC, and has no periodic-boundary numbers of its own Cited . Putting row (d) of Table 1 into (3) to say "the upper-bound assumption reads as false in the periodic box" is **a reading that extends the conjecture to the slowness of topology with periodic boundaries** Assumption . Between the Dirichlet form of heat bath and that of Langevin there is only the one-directional one-link comparison $\mathcal{E}_{\text{HB}} \le c_1 \mathcal{E}_L$, and a lower bound on the Langevin C_P does not follow directly from the slowness of heat bath. No primary source stating the relation between HMC and Langevin as a theorem was found in the range searched.

(b) $SU(2)$ **is undecided.** The groups in the original sources are $SU(3)$ and above. There is one primary source for $SU(2)$, and its numbers lie on the side close to a^{-2}. In a rough semiclassical accounting (taking the action of a dislocation as a constant multiple of the one-loop instanton action), with the constant fitted to the numbers for $SU(3)$, $SU(2)$ falls right near the boundary $z = 2$ (24-07). **For $SU(2)$, whether freezing of the topological charge breaks the relaxation-time assumption is not settled.**

(c) **Two dimensions and four.** Row (c) is numbers for a two-dimensional $U(1)$ toy, and two-dimensional $U(1)$ is a trivial theory in the continuum limit. It says nothing about exponents in four dimensions, non-abelian, $\beta \to \infty$.

(d) **Power or exponential.** The three original sources for row (d) themselves write that "power and exponential cannot be distinguished by the data". The "power or exponential" of Table 3 copies that phrasing.

(e) **The pattern is a classification.** That the five examples fall into one pattern is not a theorem. Row (e) is numbers for the free field of the reference, not for the target measure. The coefficients h, g, d of (4)–(7) are only definitions; there are no four-dimensional values.

(f) **The mass gap.** None of the examples breaks the gap. Condition (iii) of the pattern is what guarantees this, and what this chapter says goes only as far as "there is slowness that is not caught in the form of the upper-bound assumption".

07

Table of the five grades and the dependency graph

We put the rows of this chapter into one table in the format of the five grades (23-00 § 01), with the estimate label Estimate added. The symbols of the rows are the same as the nodes of the dependency graph below.

Table 5 Rows on nearly conserved quantities ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Cluster decay $\sum_z \lvert z\rvert \lvert \mathrm{Cov}(q(0), q(z))\rvert < \infty$	Assumption	Premise of (5)	Infinite volume	—
2	A reading that extends the conjecture to the slowness of topology in the periodic box	Assumption	§ 06 (a)	Periodic box	—
3	$\tau(Q)$ faster than ξ^2 in the periodic box for $SU(3)$ and above	Cited	§ 02 (d), three original sources	$SU(3), SU(4), SU(6)$, periodic	—
4	One primary source for $SU(2)$, close to a^{-2}	Cited	§ 02 (d)	HMC, small box, three points	—
5	Overall ratio of the double well $C_P/\xi^2 \approx S_0/\hbar$	Assumption	(14) of 23-04	$\hbar \to 0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
6	ℓ_c and $\tau(Q)/\xi_{XY}^2$ of the strip	Reading	§ 02 (c)	Two-dimensional $U(1)$, $\beta = 2..6$ (also 8 for $\tau(Q)$)	—
7	Reference with the boundary gauge frozen is 2.22 to 2.53 times	Reading	§ 02 (e)	One-colour Gaussian, $\beta = 1$, side ≤ 12	—
8	$\tau_{\mathrm{int}} \leq C_P$ for reversible dynamics	Proved here	(1)	Reversible semigroup	tau_int_le_CP (assumptions: weights ≥ 0, eigenvalues $\geq \lambda_1 > 0$)
9	Free-Particle-on-a-Circle Counterexample: $C_P = \infty$ with $\Delta > 0$	Proved here	(23) of 23-01	Circle, all $\hbar$	rotor_ratio_unbounded (assumption: $0 < r$; only the arithmetic of the skeleton)
10	One-site ratio $= \omega_b/m$	Proved here	§ 02 (a) (ground-state transformation)	Schrödinger ψ_0	one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$; only the algebra of the ratio)
11	The subregion ratio formula (6), and ratio $\leq \chi_t/h$	Proved here	(5)(6) (on top of row 1)	Infinite volume	subregion_gap_identity, subregion_ratio_le (assumptions: positivity)
12	(3): if τ grows as ξ^p ($p > 2$), the upper-bound assumption fails	Proved here	(3)	Sequences with $\xi \to \infty$	not_hup_of_fast_tau (assumptions: $\tau_n \leq C_n$, $c\xi_n^p \leq \tau_n$, $p > 2$)
13	The upper-bound assumption reads as false in the periodic box ($SU(3)$ and above)	Assumption	Reading obtained by putting rows 3 and 2 into row 12	Periodic box	—
14	The five examples are one pattern	Estimate	§ 01– § 03	—	—

```
 1  ASM  cluster decay                                  infinite volume
 2  ASM  conjecture extended to slow topology           periodic box
         in the periodic box
 3  CIT  τ(Q) faster than ξ² for SU(3) and above        periodic box
 4  CIT  one primary source for SU(2) (near a⁻²)        HMC, small box
 5  ASM  overall ratio of the double well ≈ S₀/ħ        ħ → 0
 6  RDG  ℓ_c and τ(Q)/ξ_XY² of the strip                two-dim. U(1)
 7  RDG  factor, reference with frozen boundary gauge   one-colour Gaussian
 8  PRV  τ_int ≤ C_P                                    reversible semigroup
         Lean tau_int_le_CP
 9  PRV  Free-Particle-on-a-Circle Counterexample (23-01)  circle
10  PRV  one-site ratio = ω_b/m                         0+1
11  PRV  subregion ratio (6) ⟸ 1                         infinite volume
         Lean subregion_gap_identity, subregion_ratio_le
12  PRV  if τ ≥ cξ^p (p>2), the upper-bound assumption fails ⟸ 8
         Lean not_hup_of_fast_tau
13  ASM  upper-bound assumption reads false
         in the periodic box ⟸ 2, 3, 12                  SU(3) and up
14  EST  one form ⟸ 5, 6, 7, 9, 10, 13
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How to check: on each "⟸" row, see by eye that the numbers on the right are smaller than the one on the left. There are no circles. The row of the primary source for $SU(2)$ is a premise of no row. That is how the unsettled reading for $SU(2)$ appears on the graph. Among the leaves of the row "the upper-bound assumption reads as false in the periodic box" is an assumption row (the reading that extends the conjecture to the slowness of topology in the periodic box), and this is why the label of that row is "Assumption".

Strictly, the statement "the upper-bound assumption reads as false in the periodic box" contains the quantity of the conclusion C_P, but among its premises, the reading that extends the conjecture and the numbers of the original sources for SU(3) and above do not contain C_P. The numbers of the original sources are numbers for Monte Carlo autocorrelations, and the reading that extends the conjecture is a reading about the universality class of dynamics; the only passage that reaches C_P is the single one through the row τ_int ≤ C_P.

What this chapter can say

Said	Not said
Proved here $\tau_{\mathrm{int}} \leq C_P$ (reversible dynamics), the static lower bound (2), the algebra of (3), the Free-Particle-on-a-Circle Counterexample, the one-site ratio ω_b/m, the subregion ratio formula (6)	A lower bound on $\tau(Q)$ satisfying the second assumption of (3) in four dimensions. Four-dimensional values of h, g, d

Said	Not said
Reading ℓ_c and $\tau(Q)$ for the two-dimensional $U(1)$ strip, the factor for the boundary gauge, the lattice ratios for the double well	All are floating-point and Monte Carlo numbers within conditions, not propositions of upper or lower bounds. The two-dimensional numbers imply nothing for four dimensions
Cited $\tau(Q)$ faster than ξ^2 in the periodic box for $SU(3)$ and above (three original sources), one primary source for $SU(2)$	Power or exponential. Whether $z > 2$ for $SU(2)$
Assumption The upper-bound assumption reads as false in the periodic box ($SU(3)$ and above, on top of a reading that extends a conjecture)	That the upper-bound assumption is false. The same reading for $SU(2)$
Estimate That the five examples are one pattern, and that the wall of the first step is a scaled-down version of this pattern	Turning the classification of the pattern into a theorem

Sources and reproduction

Item	Kind	Source / tools
Autocorrelation of the topological charge in the periodic box ($SU(3)$ and above)	Cited	L. Del Debbio, H. Panagopoulos, E. Vicari, JHEP 08 (2002) 044 § 2; S. Schaefer, R. Sommer, F. Virotta, Nucl. Phys. B 845 (2011) 93 § 4– § 5; C. Bonanno, C. Bonati, M. D'Elia, arXiv:2012.14000 § 1 (text checked)
Primary source for $SU(2)$	Cited	M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 063B03 (checked in the text of § 3.2 and the wording of Figure 6 of arXiv:2501.00286; the numbers in the figure were not read)
Observations for $SU(2)$ with heat-bath algorithms, semiclassical exponents	Cited	A. Athenodorou, M. Teper, JHEP 12 (2021) 082 § 6.2; B. Lucini, M. Teper, JHEP 06 (2001) 050 § 6; B. A. Berg, D. A. Clarke, Phys. Rev. D 97 (2018) 054506 § II (text checked)
The conjecture on the universality class of HMC and Langevin, the sentence on periodic boundaries	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 (abstract, § 3.2– § 3.3 and § 5.2 checked in the text)
The algebra of (1), (3), (6)	Lean	LatticeGauge.FourDTrial (tau_int_le_CP, not_hup_of_fast_tau, tau_small_but_CP_large), LatticeGauge.LocalH (subregion_gap_identity, subregion_ratio_le). The bundles guarantee only the algebra
Numbers for the two-dimensional $U(1)$ strip	Reading	Simulation of the gauge quotient of link Langevin (width 2, outer boundary frozen, strip length ≤ 256) and fits of $A/M^2 + B$. Errors by Madras–Sokal
One-site ratio for the double well, factor for the boundary gauge	Reading	Lattice eigenvalue problem for the transfer kernel ($\varepsilon = \frac{1}{2}$), largest eigenvalue of the compression of the lattice Green function to the box (one-colour Gaussian)

Terms used in this chapter — nearly conserved global quantity, breaking of conservation, the transport coefficient d and the coefficient of breaking of conservation h, the crossing length R_*, wrapping, the topological charge of the strip, flux through a box. The notation is in 24-09.

Next chapter to read: 24-02 Choosing the ruler and the coordinates again — is the length to compare with the static correlation length, or the length at which conservation breaks? The two forms that restate the relaxation-time assumption to avoid the pattern are in 24-03.

Choosing the ruler and the coordinates again — the length at which conservation breaks

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — The diffusive-relaxation-time assumption was a sentence comparing the relaxation time C_P with the square of one length ξ. We split the rulers that could play the role of that ξ into three: the static correlation length, the length at which conservation breaks ℓ_*, and the time for the global quantity to move by one. On a hierarchical lattice we see that, even for the same equilibrium measure, whether the diffusive-relaxation-time assumption is true changes with the coordinates used to move it (the choice of Dirichlet form) ($C_P = 1/\beta$ in tree coordinates, the square of the side in link coordinates). We state that the length at which conservation breaks, measured at five points on the two-dimensional $U(1)$ strip, is not proportional to the static length; that it comes from the same wrapping as one step of the topological charge; and that splitting the relaxation rate into a "raw rate" and a "static fluctuation" reveals the factor the paper prediction had dropped. Finally we set out a candidate correction that compares with $\max(\xi^2,\ \text{the time for the global quantity to move by one})$, and that on the two-dimensional CP^{N-1} test bed the freezing of the topological charge splits with N while the ratios of local quantities did not (conditionally).

Prerequisite chapters — 24-01 (the pattern, the two accounts of (4)–(7), row (c) of Table 1), 23-01 § 05 ((13) for the relaxation-time assumption and the note on topological sectors), 23-01 § 07 (the bookkeeping length ξ_{RG} and the logarithmic factor), the detour in 2-10 (conserving dynamics), "Differences other than constants" in 2-10 (the choice of variables). The notation is collected in 24-09.

Said first — The level is [Y3]. The numbers in this chapter are numbers for the Gauss approximation on a hierarchical lattice and for two-dimensional models (the $U(1)$ strip, CP^{N-1}). Two-dimensional numbers imply nothing about four-dimensional Yang–Mills. "Correcting what to compare with" is a candidate restatement, shown in no model Estimate . This chapter shows neither the mass gap nor whether the relaxation-time assumption is true.

The order of this chapter

1. Three rulers — the static correlation length, the length at which conservation breaks, the time for the global quantity to move by one ((1)(2))
2. Choosing the coordinates again — tree coordinates and link coordinates, conservation of flux (Table 1, (3)–(5))
3. The two-dimensional $U(1)$ strip — the length at which conservation breaks at five points (Table 2, (6))
4. Two faces of the same wrapping — the flux dipole and the topological charge ((7))
5. Raw rate and static fluctuation — the factor the paper prediction had dropped ((8)(9))
6. Correcting what to compare with — $\max(\xi^2, \tau_{\mathrm{glob}})$ ((10))
7. The two-dimensional CP^{N-1} test bed — freezing splits with N, local ratios do not (Table 3, Table 4)
8. What cannot be said
9. Table of the five grades and the dependency graph — Table 5

01

Three rulers — what length is the ξ of the diffusive-relaxation-time assumption?

The relaxation-time assumption is the power statement $\lim \ln C_P / \ln \xi = 2$ ((13) of 23-01), comparing the relaxation time C_P with the square of one length ξ. Part 23 took this ξ to be the correlation length $1/m_{\mathrm{lat}}$ and used the bookkeeping length ξ_{RG} in its place (23-01 § 07). But in the pattern of 24-01, what fixes the slowness is not the correlation length but "the scale of the events that break the global quantity Q". So we split the candidates for what to compare with into three.

(a) The static correlation length ξ. The length over which two-point correlations decay, fixed by the equilibrium measure alone. In four dimensions $\xi = 1/m_{\mathrm{lat}}$. In the two-dimensional $U(1)$ strip, one row of the strip becomes a one-dimensional chain of rotors in axial gauge, and the length over which Wilson loops decay with area (the inverse string tension),

$$\xi_{XY} = -\frac{1}{\ln\bigl(I_1(\beta)/I_0(\beta)\bigr)} \ \approx\ 2\beta \qquad (\beta \text{ large}) \ (1)$$

plays this role (I_ν is a modified Bessel function. The formula is a standard computation, and this book has not checked the original source Assumption).

(b) The length at which conservation breaks ℓ_*. The global quantity (charge) Q_R in a subregion R changes by two routes: the route of transport, flowing out through the boundary (taking time of order R^2), and the route of conservation breaking inside the region (dislocations, wrapping). The length at which the two balance is ℓ_*. The crossing length R_* of (7) in 24-01 was its static version. A candidate definition written with quantities of the measure alone is

$$S := \sup_R \frac{\mathrm{Var}(Q_R)}{\mathcal{E}(Q_R)}, \qquad \ell_* := \inf\Big\{ R : \ \frac{\mathrm{Var}(Q_R)}{\mathcal{E}(Q_R)} \ \geq \ \frac{S}{2} \Big\} \quad (S = \infty \text{ implies } \ell_* = \infty) \ (2)$$

Estimate . Putting in the estimate $\mathrm{Var}/\mathcal{E} \approx \chi_t R^2/(hR^2 + d\lambda_W)$ of (7) in 24-01 gives $S = \chi_t/h$, and the ratio reaches half of it when $hR^2 = d\lambda_W$, that is, $\ell_* = R_*$ Assumption (on top of the estimate (7)). If conservation never breaks ($h = 0$), then $S = \infty$ and $\ell_* = \infty$, which agrees with the flux of the layers in the Gauss approximation of § 02 below (where conservation holds exactly). Strictly, a definition of the form "the supremum of R with $\mathrm{Var}/\mathcal{E} \leq KR^{2}$", comparing the ratio with R^2, does not fix a length: under the estimate (7) the ratio divided by R^2 decreases in R, so the set of R satisfying the condition is a half-line open upwards or empty, and the supremum is infinite or undefined. (2) avoids this by cutting at half the saturation value.

(c) The time for the global quantity to move by one τ_{glob}. A time, not a length. Its representative is the integrated autocorrelation time $\tau_{\mathrm{int}}(Q)$ of the topological charge Q in the periodic box, and for reversible dynamics $\tau_{\mathrm{int}}(Q) \leq C_P$ ((1) of 24-01).

The three can take different values within the same model. In the 0+1-dimensional double well, the slow quantity (the number of kinks) and the correlation length are born from the same kinks, and the three differ only by logarithms (24-01 Table 3). In the two-dimensional strip of § 03, (a) grows linearly in β while (b)(c) grow exponentially, and the three move apart. Fixing the model does not automatically fix which ruler the relaxation-time assumption is a sentence about.

What this section can say: (2) is a candidate definition, and the four-dimensional value of ℓ_* is not in this book. The ξ_{XY} of (1) is used only as the static length of the strip; it does not stand in for the four-dimensional correlation length.

02

Choosing the coordinates again — tree coordinates and link coordinates

The denominator of the Poincaré constant $C_P = \sup \mathrm{Var}(f)/\mathcal{E}(f)$ is a Dirichlet form, and it is not fixed until one decides the variables whose gradients it measures. We saw in Part 2 that changing variables by a constant factor changes C_P by a constant factor ("Differences other than constants" in 2-10: a factor β depending on whether one measures by link angles or by the field). In this section we give an example where the choice of coordinates changes the **type** of growth of C_P.

Stage. A hierarchical lattice replaces the lattice by nested boxes so that one step of renormalisation can be written down. For gauge fields, Ito (1984) built a three-dimensional lattice on which the recursion of the Migdal–Kadanoff approximation holds exactly, and wrote that the extension to higher dimensions is left to the reader Cited . This book defines for itself a four-dimensional version that coarsens by factors of two (each layer a two-dimensional periodic lattice of side $L = 2^N$, with the bonds of level r identified across 4^r layers), and treats only the Gauss approximation on it ($U(1)$, or linearisation to one component of the Lie algebra of $SU(2)$) Assumption (the four-dimensional construction is this book's extension).

Tree coordinates. Taking the values of the plaquettes (the holonomies at the nodes) as variables, in the Gauss approximation the configuration space is a linear subspace S, the equilibrium measure is the density $e^{-\beta|x|^2/2}$ on S, and the Hessian of the Langevin potential is βI on S.

$$\text{tree coordinates (Gauss approximation)}: \qquad \lambda_1 = \beta, \qquad C_P = \frac{1}{\beta} \qquad (\text{every number of levels } N) \ (3)$$

Proved here Proof. It is an Ornstein–Uhlenbeck process with unit metric and drift matrix βI, so the gap of the generator is β. ∎ Lean `HierLink.tree_no_decay` (bundle LatticeGaugeHierarchicalLink; assumption: $0 < \beta$; only the arithmetic that the ratio is 1). Increasing the number of levels, that is, making the lattice however large, does not slow relaxation in these coordinates. Even if the correlation length grows under the non-abelian flow, the coarse levels are expected to stay close to Haar with $C_P = O(1)$ Estimate , and **in tree coordinates the relaxation-time assumption is false or empty**.

Link coordinates. What the relaxation-time assumption of Part 23 names is Langevin in the link variables (23-01 § 04). The link generator in the Gauss approximation is $\beta\, D\Pi D^*$ (D is the exterior derivative from links to plaquettes, Π the projection of the identification), and its smallest positive eigenvalue is

$$\lambda_{\min}(N) = 4\sin^2\frac{\pi}{2^{N+1}}, \qquad 4^N \lambda_{\min}(N) \ \longrightarrow \ \pi^2 \qquad (N \to \infty) \ (4)$$

Proved here (paper: splitting functions of the layers into sectors by the hierarchical projection, the block operator for sector k becomes the Neumann Laplacian of the two-dimensional dual lattice of side 2^{k+1}, and the minimum is the value for sector $k = N -$

1 with side 2^N) Reading (agreeing with the direct full spectrum for $N \le 2$, and with a separate implementation by sectors for $N = 3$, 4). The gap of the generator is $\beta\lambda_{\min}$, which shrinks inversely with the square of the side. Lean `HierLink.lam_le` (no assumptions: $4\sin^2(\pi/2^{N+1}) \le \pi^2/4^N$; an upper bound in the direction of slowness; that $\lambda_{\min}$ is *equal* to this expression is outside Lean), `HierLink.measured_scaled` (with the interval of the measured $\lambda_{\min}(4)$ as assumption, $4^4\lambda_{\min} \in (9.83, 9.84)$).

Table 1 Coordinates and gap (Gauss approximation, one component of $U(1)$, this book's four-dimensional extension; the gap

N	Side $L = 2^N$	Tree coordinates	Link coordinates (metric of the original lattice)	$4^N\lambda_{\min}$	Link coordinates (metric per class)	Gauge quotient on the four-dimensio
1	2	1	2.000000	8.0000	2.000000	2.0000
2	4	1	0.5857864	9.3726	0.8768944	0.5857864
3	8	1	0.1522409	9.7434	0.2575235	—
4	16	1	0.0384294	9.8379	0.0667663	—

Labels of Table 1: the tree-coordinate column is Proved here by (3). Link coordinates (metric of the original lattice) are Proved here Reading , the closed formula (4) agreeing with computation. The metric per class (moving identified bonds as one variable) is computation only Reading , 1.0–1.74 times the metric of the original lattice. The gauge quotient on the four-dimensional original lattice agrees with the metric of the original lattice to 10^{-9} for $N \le 2$ Reading . The lowest eigenvector lies, for all N, in the same sector (the contrast between the four top-level groups of layers) Reading .

The slowness comes from conservation of flux. When links move, the values of the plaquettes (the flux) are only handed between neighbours.

$$\frac{d}{dt}F = d\Big(\frac{d}{dt}A\Big), \qquad \sum_{p\in\text{block}} Fp = 0 \quad \text{(when the boundary bonds are fixed; Stokes)} \tag{5}$$

Unevenness of the flux is smoothed out only by diffusion, which takes time of the square of the side (the conserving dynamics of Part 2, the model B type, the detour in 2-10). The decisive point is the last column of Table 1. If the layers are decoupled and every bond made an independent variable per layer, then statically the plaquettes are almost independent (only the constraint on the total) and the correlation length is 0, yet the smallest eigenvalue is $4\sin^2(\pi/L)$ and the same L^{-2} type remains Proved here . Hence **this slowness does not come from the correlation length**. The static correlation of plaquettes in the Gauss approximation decays masslessly, of order $-\beta^{-1}R^{-4}$; β does not flow (the Gauss-approximation face of the coupling being dimensionless in four dimensions, 24-07), and the correlation length is only cut off by the volume Proved here .

Consequence. The relaxation-time assumption is not a sentence about the measure alone but **a sentence about the pair (measure, Dirichlet form)** Estimate . Tree coordinates and link coordinates have the same equilibrium measure, yet in the former C_P does not depend on the number of levels, and in the latter it grows as the square of the side. In Part 2, "C_P changes by a constant factor with the coordinates" did not change the power of the growth, but here the power itself (N^0 versus 4^N) changes. The relaxation-time assumption of Part 23 names Langevin in the link variables, so it already meets this point. Reinterpreting heat-bath or overrelaxation numbers as C_P of link Langevin is an assumption (2-10 § "Do not identify heat bath with Langevin"). And even in link coordinates, conservation holds exactly in the Gauss approximation, so ℓ_* of (2) is infinite, and comparing "$C_P \propto L^2$" with the correlation length has no meaning. What should be compared is a quantity outside the Gauss approximation: at what length the compactness or non-abelianness of the group breaks the conservation of flux. The next section is the first measurement of it.

What this section can say: hierarchical models lack translation and rotation invariance, and imply nothing about four-dimensional Yang–Mills. The four-dimensional construction is this book's extension of the three-dimensional construction of the original source. The numbers are numbers for the Gauss approximation, with $N \le 4$.

03

The two-dimensional $U(1)$ strip — the length at which conservation breaks at five points

Model. We cut one layer of the previous section into a narrow strip. Plaquettes (x, y), $x = 0, \dots, M-1$, $y = 0, 1$ (width $W = 2$); all links on the outer boundary are fixed at angle 0, and the inner links are moved by the Langevin dynamics $d\theta_\ell = -\partial_\ell S\, dt + \sqrt{2}\, dB_\ell$ for the action $S = -\beta\sum_p \cos\varphi_p$ (unit metric). Gauge transformations at inner vertices do not change the plaquette angles φ_p, so on gauge-invariant functions the link generator reduces to a diffusion on φ, whose metric is the Neumann Laplacian of the dual graph of the plaquettes Proved here . The equilibrium measure is the product of independent $e^{\beta\cos\varphi}$ per plaquette conditioned on $\sum_p \varphi_p \equiv 0 \pmod{2\pi}$, and there is almost no static correlation between plaquettes.

In the Gauss approximation conservation holds exactly. With $\sin\varphi \to \varphi$, the smallest gap is $\beta \cdot 4\sin^2(\pi/2M) \approx \beta\pi^2/M^2$, inversely proportional to the square of the strip length Proved here (Lean `ConsLen.neumann_gap_lower` , `ConsLen.neumann_gap_upper` , bundle LatticeGaugeConservationLength; assumption: $1 \le M$; $4/M^2 \le 4\sin^2(\pi/2M) \le \pi^2/M^2$. That the gap equals this expression is outside Lean).

If the group is compact, conservation breaks. An event in which the angle of a single plaquette crosses $\pm\pi$ (a wrapping) changes by one the sum of the angles reduced to $(-\pi, \pi]$,

$$Q := \frac{1}{2\pi}\sum_p [\varphi_p] \;\in\; \mathbb{Z} \quad (6)$$

(the topological charge of the strip, an integer because the outer boundary is fixed). A wrapping is a rare event crossing a barrier of height 2β. We measure two quantities: $\lambda_X := 1/\tau_{\mathrm{int}}(X)$ for the flux dipole $X = \sum_p w_p[\varphi_p]$ ($w_p = \cos(\pi(x+\frac{1}{2})/M)$, the slowest mode of the Gauss approximation), and $\lambda_Q := 1/\tau_{\mathrm{int}}(Q)$ for the topological charge. For reversible dynamics $\tau_{\mathrm{int}} \le C_P$ ((1) of 24-01), so both are read as **upper bounds on the gap**. The length at which conservation breaks is defined by fitting $\lambda_X = A/M^2 + B$ and setting $\ell_c := \sqrt{A/B}$ (the length at which the diffusion rate A/M^2 equals the saturation value B).

Table 2 Five points for the two-dimensional $U(1)$ strip (width 2, outer boundary frozen, gauge quotient of link Langevin, strip length $M \le 256$; λ_Q is a weighted mean over M with $\mathrm{Var}\, Q \ge 0.5$; $E := \mathbb{E}[\varphi_p]^2$; errors by Madras–Sokal and the covariance of the fit)

β	ξ_{XY}	ℓ_c	Paper estimate	λ_Q	$\tau(Q)$	$\pi/\sqrt{E\lambda_Q}$	$\rho = \ell_c/\frac{\pi}{\sqrt{E\lambda_Q}}$	ℓ_c/ξ_{XY}
2	2.78	3.29(20)	3.53	0.769(21)	1.30	4.10	0.80(5)	1.2
3	4.75	9.88(39)	8.28	0.224(13)	4.46	10.0	0.985(50)	2.1
4	6.82	24.7(9)	19.6	0.0579(29)	17.3	23.9	1.03(5)	3.6
5	8.87	53.6(3.1)	47.7	0.0127(10)	79	58.5	0.92(6)	6.0
6	10.90	139(14)	118	0.00276(32)	362	139	1.00(11)	12.8

Every label in Table 2 is Reading . "Paper estimate" is the value, by numerical integration, of a capacity estimate with a test function (the length at which the two terms of an upper bound become equal, using a test function whose return from wrapping is optimised by capacity; for large β, $e^{\beta}\sqrt{\pi/(6\beta)}$) Assumption (an estimate on top of the independence approximation), and is of the same order as the measured ℓ_c.

Slopes. With a weighted straight line over $\beta = 2, \dots, 6$:

- The slope of $\ln \ell_c$ is $a = 0.916(21)$. Split into intervals: $0.988(34)$ for $\beta = 2$ to 4, $0.834(46)$ for 4 to 6.
- The slope of $-\ln \lambda_Q$ is $b = 1.355(18)$. Split into intervals: $1.286(28)$ and $1.520(57)$ (difference 3.7σ).
- The slope of ln of the combination $\pi/\sqrt{E\lambda_Q}$ is $0.884(9)$, and 0.879 to 0.884 on every interval.

Reading Lean `ConsExp.lc_slope_2_6` (bundle LatticeGaugeConservationExponent; assumptions: $\ell_c(2) \in [3.09, 3.49]$, $\ell_c(6) \in [125, 153]$, the measured 1σ intervals; conclusion: $a \in [3/4, 1]$), `ConsExp.lamQ_slope_2_6` (with the 1σ intervals of $\lambda_Q(2)$ and $\lambda_Q(6)$ as assumptions, $b \in [5/4, 3/2]$). The bundles guarantee only interval arithmetic.

Not proportional to the static length. The static length $\xi_{XY} \approx 2\beta$ is linear in β and grows only about fourfold over the five points. The length at which conservation breaks, $\ell_c \propto e^{0.92\beta}$, grows 42-fold, and the ratio ℓ_c/ξ_{XY} opens from 1.2 to 12.8 Reading . Of the three rulers of § 01, (a) and (b) move apart exponentially within the same model. When the strip is longer than ℓ_c, relaxation stops depending on the strip length and is fixed by the time for Q to move by one.

Conditions: two-dimensional $U(1)$, width 2, outer boundary frozen, gauge quotient of link Langevin with unit metric, $\beta = 2, \dots, 6$, strip length $M \le 256$, τ_{int} as a reading of an upper bound on the gap, fit of the form $A/M^2 + B$ (for $\beta = 2$, $\chi^2/\mathrm{dof} = 3.3$, and the form does not fit). Nothing is said about the exponent as $\beta \to \infty$.

04

Two faces of the same wrapping — the flux dipole and the topological charge

The ρ column of Table 2 shows that the two quantities are one process. The relaxation of the flux dipole X falls as M^{-2} at the rate $\mu_1(M)/E$ ($\mu_1 \approx \pi^2/M^2$), with the Gauss stiffness replaced by $1/E$, while the relaxation rate λ_Q of the topological charge Q does not depend on the strip length. The length at which the time for diffusion to cross a length ℓ equals the time for Q to move by one is

$$\frac{\pi^2}{E\,\ell^2} = \lambda_Q \quad\Longleftrightarrow\quad \ell = \frac{\pi}{\sqrt{E\,\lambda_Q}} \quad (7)$$

Proved here (algebra) Lean `ConsExp.lc_eq_iff` (assumptions: $0 < E$, $0 < \lambda$, $0 < \ell$), and its ratio ρ to the measured ℓ_c is 0.80 to 1.03 over the five points Reading . **The length at which conservation breaks reads as the distance the flux crosses by diffusion while the topological charge moves by one.** The flux dipole departing from M^{-2} and the topological charge relaxing are two faces of the same wrapping.

A single point at $\beta = 8$ supplements this reading. Up to strip length 64, X keeps M^{-2} (ratio to the Gauss stiffness 0.86 to 1.03), but at strip length 32, $\tau_{\mathrm{int}}(Q) = 5.3(1.7) \times 10^3$ (run length 10^6, $\mathrm{Var}\, Q = 0.145$), and a slow mode at about $1/380$ of the value for X appears Reading . Because $\mathrm{Var}\, Q$ is small (a sector where $Q \neq 0$ is rare), this point is not included in the slopes.

What this section can say: ℓ_c is the length of the bend in the relaxation of X, not the length of a bend in the gap itself (including Q, the upper bound on the gap at $\beta = 4$ already saturates at $M \approx 8$ to 16). We do not claim that the topological charge of the strip is the same thing as the four-dimensional topological charge.

05

Raw rate and static fluctuation — the factor the paper prediction had dropped

The paper prediction before measuring was "$\lambda_Q \propto \beta e^{-2\beta}$" (the Kramers rate for the barrier 2β), so the slope b should have been near 2. The measured $b = 1.355(18)$ is gentler, and moreover split before and after β (bottom of Table 2). To find where the split comes from, we divide the relaxation rate into two factors. Q can be regarded as a chain of ± 1 jumps with a restoring force, so from the total jump rate $WM\nu$ (ν is the rate of wrapping per plaquette) and the variance $\mathrm{Var}\, Q \approx WM\, E/(4\pi^2)$ comes

$$\lambda_Q = \kappa \cdot \frac{2\pi^2\, \nu}{E}, \qquad E = \mathbb{E}\,[\varphi_p]^2 \quad (8)$$

this form Assumption (the chain picture; κ is a correction for the raw count. It is a different quantity from the comparison constant κ of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (24-09)). The ν in the numerator is the **raw rate** (a quantity of the dynamics), and the E in the denominator is the **static fluctuation** (a quantity of the measure). Taking logarithms gives

$$\Delta \ln \lambda_Q = \Delta \ln \nu - \Delta \ln E \quad (9)$$

Proved here (algebra) Lean `ConsExp.lamQ_log_split` (assumptions: $0 < \kappa$, each quantity positive).

Counting the raw number of wrappings ν_{raw} (the number of times $\pm\pi$ is crossed), we found the following Reading .

- $\kappa = \lambda_Q E/(2\pi^2 \nu_{\mathrm{raw}})$ is constant at $0.167,\ 0.165,\ 0.158,\ 0.167$ for $\beta = 3, 4, 5, 6$ (the raw count also counts back-and-forth trips near π, so the effective crossings are about $1/6$ of it).
- The slope of the raw rate $-d \ln \nu_{\mathrm{raw}}/d\beta$ is $1.73,\ 1.75,\ 1.79$ on the intervals $3 \to 4,\ 4 \to 5,\ 5 \to 6$, agreeing within 1–3% with the slope $2 - 1/\beta$ of the Kramers rate $\beta e^{-2\beta}$ ($1.71,\ 1.78,\ 1.82$ at the midpoints of the intervals).
- The slope of the static factor $\ln(1/E)$ is $0.38,\ 0.27,\ 0.21$, decreasing with β (approaching the Gauss value $1/E \approx \beta$).

From (9), $b = (\text{slope of the raw rate}) - (\ln(1/E)\ \text{slope})$, and since the latter decreases with β, b increases with β. This is what the split really is. **The paper prediction had dropped the static factor $1/E$.** The corrected form is $\lambda_Q \propto \beta e^{-2\beta}/E$ (for large β, $\propto \beta^2 e^{-2\beta}$, $b \to 2 - 2/\beta$), in the same direction as the measured b Estimate . In the combination $\pi/\sqrt{E\lambda_Q}$, E cancels and only $\nu^{-1/2}$ remains, so the slope is half the slope of the raw rate (about 0.88) and constant over all intervals. That is the third slope at the bottom of Table 2.

Strictly, this split is a check added after seeing the split in the data; it was not among the pre-registered predictions. Also, nothing can be said from these five points about the exponent as $\beta \to \infty$ (whether $b \to 2$).

What to keep as a view Estimate : the relaxation rate of a global quantity splits into the product of "the rate of events" (dynamics, barrier) and "the static fluctuation that measures how much one event moves the global quantity" (measure). The one-loop accounting that the original sources write for the four-dimensional topological charge (24-07 § 02) also counts only the exponent of the density of events. Whether the static factor moves the reading of the exponent has not been checked in four dimensions.

06

Correcting what to compare with — $\max(\xi^2, \tau_{\mathrm{glob}})$

In the strip, measuring $\tau(Q)$ with the static length ξ_{XY} as the ruler, $\tau(Q)/\xi_{XY}^2$ is about 0.4 at $\beta = 4$ and about 24 at $\beta = 8$ (strip length 32) Reading . Since $C_P \geq \tau_{\mathrm{int}}(Q)$, C_P/ξ_{XY}^2 also grows with β. "$C_P \lesssim \xi^2$", with only the static correlation length as the ruler, tends to fail with β inside this toy (row (c) of the pattern in 24-01). So we put forward, as a candidate, a form that adds the time of the global quantity to what is compared with.

$$C_P \asymp \max\left(\xi^2,\ \tau_{\mathrm{glob}}\right), \qquad \tau_{\mathrm{glob}} := \left(\text{the time for the global quantity to move by one}\right) \quad (10)$$

Estimate . This is one direction for correcting the relaxation-time assumption. The other direction is to restrict the side of test functions to families local in time and space (24-03 § 02), where, as long as the box is smaller than ℓ_*, the time of the global quantity is expected not to enter (§ 07).

(10) has three weaknesses. (i) If τ_{glob} is defined by $\tau_{\mathrm{int}}(Q)$ of the same dynamics, (10) is close to a restatement of $\tau_{\mathrm{int}}(Q) \le C_P$ and has no content. To give it content, τ_{glob} must be fixed independently of the dynamics. Candidates are the split of § 05 (barrier rate and static fluctuation, $\tau_{\mathrm{glob}} \approx E/(2\pi^2\kappa\nu)$), or making the diffusion time ℓ_*^2/D from the static ℓ_* of (2) Estimate . (ii) The strip numbers see only the lower side of (10), $C_P \ge \tau_{\mathrm{glob}}$, and not a single number bounds C_P from above. At $\beta = 4$, $\tau(Q) \approx 17$ against $\xi_{XY}^2 \approx 46$, and whether C_P is on the side of ξ_{XY}^2 has not been measured. (iii) In four dimensions the candidate for τ_{glob} is the topological charge in the periodic box, whose measurements are the heat-bath and HMC numbers of the original sources, and reinterpreting them for Langevin is an assumption (24-01 § 06).

What this section can say: (10) is a candidate restatement, shown in no model. Nor do we propose replacing the relaxation-time assumption of Part 23 with (10). The order of the account is: because the relaxation-time assumption was set up sharply, we could name the places where a single thing to compare with is not enough.

07

The two-dimensional CP^{N-1} test bed — freezing splits with N, local ratios do not

Why this model. The two-dimensional CP^{N-1} model shares two points with four-dimensional Yang–Mills — a dimensionless coupling, and instantons with no scale (24-07) — and lets one vary N, which plays the role of the size of the group. There are primary sources reporting that in the periodic box the autocorrelation of the topological mode grows like $\exp(1.72\,\xi)$ for $\xi = 2$ to 6 ($N = 21$, Hasenbusch 2017), and worsens with N (Del Debbio–Manca–Vicari 2004, Berni–Bonanno–D'Elia 2019) Cited . It is also reported that with open boundaries the growth returns to $z \approx 1$ Cited .

Setting. The action is Hasenbusch's form $S = -\beta N \sum_{x,\mu}(\bar{z}_{x+\mu} z_x \lambda_{x,\mu} + \mathrm{c.c.} - 2)$ ($z_x \in \mathbb{C}^N$, $|z_x| = 1$, $\lambda \in U(1)$), and the topological charge is the sum of plaquette angles $Q = \frac{1}{2\pi}\sum \theta_{\mathrm{plaq}}$. The dynamics is heat bath (sites and links) mixed with n_{ov} overrelaxation steps; it is **not Langevin**. The autocorrelations measured here are in units of heat-bath sweeps, and reinterpreting them as the Langevin C_P is an assumption (2-10). The local quantities are three for a box of side R: the energy E_B, the topological charge of the box q_B, and the "magnetisation" of the box m_B.

The two-dimensional reference differs in character from four dimensions. The reference of the ratio form K_{rel} of 24-03 (the box value for the massless free field) diverges in two dimensions not only with the box size R but logarithmically in the lattice side L (the constant part of the continuum kernel $-\frac{1}{2\pi}\ln|z|$ sits on the box average) Proved here . Removing the box average makes it finite, about 0.0548 by extrapolation in $1/R^2$ Reading . Moreover, even for mean-zero quantities the free-field Langevin autocorrelation grows with $\ln L$ Reading (in a box with $R = 16$, from 36 to 119 as the lattice side goes from 64 to 1024). So the ratios of this test bed are **ratios of autocorrelations with the lattice side fixed at $L = 64$**, $K_{\mathrm{rel}}(c; \mathrm{qty}) := \tau(\mathrm{qty}_B(c\xi))/\tau^{\mathrm{free}}(D_B(c\xi))$ (D_B is the first Neumann mode of the free-field box, with the same algorithm and the same L) and $K_{\mathrm{glob}} := \tau(Q^2)/\tau^{\mathrm{free}}(D_B(4\xi))$, quantities different from the static ratios of 24-03.

Positive control. With two runs (run length 70000 sweeps) at $N = 21$, $\beta = 0.70$, $L = 44$, $n_{\mathrm{ov}} = 6$, it passed with $\tau_{\mathrm{int}}(Q^2) = 160(26)$ (0.80 times 199(18) of Hasenbusch's Table II, inside the band [139, 259] fixed in advance), an energy difference of -9×10^{-6}, and a ξ difference of -0.1% Reading .

Table 3 Core pairs and pure heat-bath pairs ($L = 64$, periodic, $\xi \approx 4$; K is three values at $c = 1, 2, 4$; one run each, some runs shortened from the registered length)

N	β	n_{ov}	ξ_{2nd}	$\tau(Q^2)$	Changes of Q / measurements	K_{glob}	$K_{\mathrm{rel}}(E_B)$	$K_{\mathrm{rel}}(m_B)$	$K_{\mathrm{rel}}(q_B)$
21	0.70	6	3.738(14)	154(46)	527 / 30600	Undefined (the reference τ^{free} is 0.33–0.54)			
2	1.140	6	4.25(8)	0.50	93123 / 99000	Same as above			
21	0.70	0	3.76(6)	3874(2810)	120 / 41400	55	0.32, 0.11, 0.038	0.68, 0.59, 0.51	1.71, 1.74, 2.27
10	0.80	0	4.62(8)	70.9(10.3)	24347 / 81000	1.01	0.35, 0.13, 0.049	0.83, 0.78, 0.57	1.51, 1.11, 0.83
2	1.140	0	4.89(24)	0.73	133493 / 144000	0.010	0.26, 0.084, 0.030	0.66, 0.41, 0.25	0.21, 0.062, 0.022

Every label in Table 3 is Reading . $\tau(Q^2)$ for $N = 21$, $n_{\mathrm{ov}} = 0$ has a run length of only about 11τ and is treated as a lower bound.

What can be read.

1. **The freezing of the topological charge splits with N.** At $\xi \approx 4$, $\tau(Q^2)$ for $N = 21$ against $N = 2$ is 154 against 0.50 with overrelaxation (about 300 times), and 3874 against 0.73 with pure heat bath (about 5000 times; the former treated as a lower bound) Reading .
2. **The ratios of local quantities are small regardless of N.** For the pure heat-bath pairs, K_{rel} for the energy and the magnetisation is almost the same at $N = 2, 10, 21$ (0.03–0.35 for E_B, 0.25–0.83 for m_B), and the ratios of one-step Rayleigh quotients (the ratio of $1/(1-\rho(1))$ to its free-field value) are all 1.1 or below Reading . The verdict at the freezing point was on the side of "freezing of topology does not break the local ratios up to box 4ξ". Conditions: $L = 64$, a single point $\xi \approx 4$, one run each, a post-hoc cell (below). There is only one freezing point, ($N = 21$, $\xi \approx 4$), and $K_{\mathrm{glob}} = 55$ has a 1σ width of 15–95 from the error of one run, so whether it exceeds the decision threshold 30 is within that error. **It is a weak reading at one point.**

3. **ℓ_* grows with N.** Table 4 lists the length at which the autocorrelation of the topological charge of the box catches up with the global value.

Table 4 Autocorrelation of the topological charge of the box $\tau(q_B(R))$ and the global $\tau(Q)$ (pure heat bath, $L = 64$)

N	$R = 2$	4	8	16	32	$\tau(Q)$	$\tau(q_B(32))/\tau(Q)$	Exponent for $R \to 2R$
21	3.08	9.82	38.9	160	738	4071	0.18	1.67, 1.98, 2.04, 2.21
10	3.01	8.72	24.7	58.4	113	222	0.51	1.54, 1.50, 1.24, 0.95
2	0.98	1.21	1.39	1.51	1.53	1.55	0.99	0.30, 0.20, 0.12, 0.02

Reading For $N = 2$ the autocorrelation of the box charge saturates at the global value at $R \approx 8$ to 16; for $N = 10$ it begins to bend at $R = 32$; for $N = 21$ it still grows as R^2 at $R = 32$. The box charge takes time R^2 through transport across the boundary (the route (b) of § 01), and the length at which it catches up with the global value is the reading of ℓ_*. **In this model and this range, freezing does not break the ratios for boxes smaller than ℓ_*, and rides only on quantities on scales beyond ℓ_*** Estimate . That in Table 3 only $K_{\mathrm{rel}}(q_B)$ of the box charge splits with N ($0.022 \to 0.83 \to 2.27$ at $c = 4$) we read as a reflection of this difference in catching-up length. In the language of 24-03, the local diffusive-relaxation-time assumption asks about the transport coefficient d and the global relaxation-time assumption about the coefficient of breaking of conservation h (24-03 § 05), so this is one two-dimensional face of the prospect that local ratios are unaffected by freezing. It also agrees in direction with the conclusion of 24-04 that what the one-sided bridge needs is the local side (24-04), but the ratios here are ratios of autocorrelations, not the ratio of the bridge's threshold (a static ratio).

What should be disclosed.

- **The registered verdict could not be defined.** The verdict registered in advance was the ratio for dynamics mixed with overrelaxation ($n_{\mathrm{ov}} > 0$), but overrelaxation makes the autocorrelation of the free-field reference oscillate, and $\tau^{\mathrm{free}}(D_B)$ came out at 0.33 to 0.54, below 1. By the rule that a ratio with a denominator below 2 is not defined (added before the main runs), all the registered ratios were "undefined". Readings 2 and 3 above are numbers for pure heat-bath pairs added as a separate cell after the registration was fixed.
- **The dynamical exponent $z_Q(N)$ cannot be read.** For each N there is only one point in ξ. The decision script printed a negative exponent for $N = 21$, but the second point was an artefact in which the script returned a small autocorrelation for a series where the topological charge never changed (only rounding error), and it was discarded.
- **The topological charge for $N = 2$ carries an ultraviolet label.** For $N = 2$ the two definitions of the topological susceptibility (the sum of plaquette angles and the triangle definition of Berg–Lüscher) differed by 13–14%, and for $N = 10, 21$ by 0.1–3% Reading . We understand that the lattice topological susceptibility for $N = 2$ (the same model as $O(3)$) is known to have an ultraviolet problem Assumption (from memory; the original source was not obtained). $N = 2$ may fail to freeze because it is on the side where the lattice scale (the ultraviolet) fixes Q Estimate . If so, the speed for $N = 2$ is not "weak freezing" but a shadow of "Q not being a topological charge in the continuum sense", and the analogy with $SU(2)$ (24-07) carries this reservation.
- Some runs were shortened from the registered length (because of an upper limit on the time of one run), and the statistics are rough. The negative control with open boundaries was not run.

What this section can say: numbers for the two-dimensional CP^{N-1}, this action and these algorithms (heat bath and overrelaxation), $L = 64$, a single point $\xi \approx 4$; nothing is said about four dimensions, $SU(N)$, the relaxation-time assumption or the mass gap.

08

What cannot be said

(a) Two dimensions and four. The numbers of § 03– § 07 are numbers for the two-dimensional $U(1)$ strip and the two-dimensional CP^{N-1}. Two-dimensional $U(1)$ is a trivial theory in the continuum limit, and neither set of numbers implies anything about exponents in four dimensions, non-abelian, $\beta \to \infty$.

(b) Readings of upper bounds. λ_X and λ_Q are readings as upper bounds on the gap (lower bounds on C_P), not the gap itself. ℓ_c is the length of the bend in the relaxation of the flux dipole and depends on the form of the fit $A/M^2 + B$.

(c) Candidates. The ℓ_* of (2) and the thing to compare with of (10) are candidates, shown in no model. The strip numbers see only the lower side of (10).

(d) The hierarchical lattice. The results are numbers for the Gauss approximation, and the four-dimensional construction is this book's extension of the three-dimensional construction of the original source. Hierarchical models lack translation and rotation invariance and imply nothing for four-dimensional Yang–Mills.

(e) Conditions of the test bed. The CP^{N-1} ratios are ratios of autocorrelations of heat-bath dynamics, conditional quantities with $L = 64$ fixed, and different from the static ratios of 24-03. For each N there is one point in ξ, one freezing point, and the verdict on local

ratios is a weak reading at one point on a cell added after registration; the registered verdict could not be defined. $z_Q(N)$ has not been read.

(f) Afterthought. The split into raw rate and static fluctuation in § 05 is a check added after seeing the split.

(g) The analogy with $SU(2)$. Using the split of freezing with N as a two-dimensional analogy of the borderline reading for $SU(2)$ is an estimate, and $N = 2$ carries an ultraviolet label (24-07).

09

Table of the five grades and the dependency graph

Table 5 Rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	The Gauss approximation (abelian linearisation), and the four-dimensional extension of the hierarchical lattice	Assumption	§ 02	$N \le 4$	—
2	$\lambda_1 = \beta$ in tree coordinates ((3))	Proved here	§ 02	Gauss approximation, all N	`HierLink.tree_no_decay` (assumption: $0 < \beta$; arithmetic of the ratio)
3	$\lambda_{\min}(N) = 4\sin^2(\pi/2^{N+1})$ in link coordinates ((4))	Proved here	Sector decomposition of § 02	Gauss approximation	`HierLink.lam_le` (no assumptions; only the direction of the upper bound)
4	Cross-check for $N \le 4$, metric per class, gauge quotient on the original lattice	Reading	Table 1	$N \le 4$ (gauge quotient $N \le 2$)	`HierLink.measured_scaled` (measured interval as assumption)
5	L^{-2} remains even with the layers decoupled (conservation of flux)	Proved here	(5), Table 1	Gauss approximation	—
6	The relaxation-time assumption is a sentence about the pair (measure, Dirichlet form)	Estimate	§ 02	—	—
7	Gauss gap of the strip $\beta \cdot 4\sin^2(\pi/2M)$	Proved here	§ 03	Width 2, outer boundary frozen	`ConsLen.neumann_gap_lower/upper` (assumption: $1 \le M$)
8	$\tau_{\mathrm{int}} \le CP$	Proved here	(1) of 24-01	Reversible dynamics	`LatticeGauge.FourDTrial.tau_int_le_CP` (assumptions: weights ≥ 0, eigenvalues $\ge \lambda_1 > 0$)
9	Five points of the strip: ℓ_c, λ_Q, slopes a, b	Reading	Table 2	Conditions of § 03	`ConsExp.lc_slope_2_6`, `lamQ_slope_2_6` (1σ intervals as assumptions)
10	Algebra of (7)	Proved here	§ 04	—	`ConsExp.lc_eq_iff` (assumptions: $0 < E, \lambda, \ell$)
11	$\rho = 0.80$ to 1.03 (two faces)	Reading	Table 2	Five points	—
12	Algebra of the split (9)	Proved here	§ 05	—	`ConsExp.lamQ_log_split` (assumptions: $0 < \kappa$, positivity)
13	The raw rate within 1–3% of the Kramers slope, κ constant (added afterwards)	Reading	§ 05	$\beta = 3..6$	—
14	The paper prediction had dropped $1/E$	Estimate	§ 05	Strip	—
15	$\tau(Q)/\xi_{XY}^2$ goes $0.4 \to 24$	Reading	§ 06	$\beta = 4, 8$	—
16	Compare with $\max(\xi^2, \tau_{\mathrm{glob}})$ ((10))	Estimate	§ 06	—	—
17	Passing the positive control for CP^{N-1}	Reading	§ 07	$N = 21$, $L = 44$	—
18	$\tau(Q^2)$ splits with N	Reading	Table 3	$L = 64$, $\xi \approx 4$	—
19	Local ratios are small regardless of N	Reading	Table 3	Post-hoc cell, one point, weak	—
20	The length at which the box charge catches up with the global one grows with N	Reading	Table 4	$L = 64$	—
21	$N = 2$ may fail to freeze because it is on the side where the ultraviolet fixes Q	Estimate	§ 07	—	—
22	Freezing rides only on quantities on scales beyond ℓ_*	Estimate	§ 07	This model, this range	—

What the machine-checking column guarantees is only arithmetic, algebra and comparisons of intervals. That eigenvalues equal closed formulas, the spectral representation of $\tau_{\mathrm{int}} \le CP$, the fits and the Monte Carlo lie outside Lean (23-13).

Dependency graph (premise numbers are smaller than conclusion numbers):

```
 1  ASM  Gauss approx., 4D extension of hier. lattice      N ≤ 4
 2  PRV  λ₁ = β in tree coordinates ⟸ 1                     Lean tree_no_decay
 3  PRV  λ_min(N) in link coordinates ⟸ 1                   Lean lam_le (upper bound)
 4  RDG  cross-check for N ≤ 4 ⟸ 3                          Lean measured_scaled
 5  PRV  L⁻² even with layers decoupled ⟸ 1
 6  EST  the relaxation-time assumption is a sentence
         about a pair (measure, form) ⟸ 2, 3, 5
 7  PRV  Gauss gap of the strip                             Lean neumann_gap_lower/upper
 8  PRV  τ_int ≤ C_P ((1) of 24-01)                         Lean tau_int_le_CP
 9  RDG  five points of the strip ℓ_c, λ_Q, a, b ⟸ 7, 8     Lean lc_slope_2_6, lamQ_slope_2_6
10  PRV  algebra of (7)                                     Lean lc_eq_iff
11  RDG  ρ = 0.80–1.03 ⟸ 9, 10
12  PRV  the split (9)                                      Lean lamQ_log_split
13  RDG  raw rate and κ (added afterwards) ⟸ 9
14  EST  the paper prediction dropped 1/E ⟸ 12, 13
15  RDG  τ(Q)/ξ_XY² goes 0.4 → 24 ⟸ 8, 9
16  EST  compare with max(ξ², τ_glob) ⟸ 6, 11, 15
17  RDG  positive control for CP^{N−1}
18  RDG  τ(Q²) splits with N ⟸ 17
19  RDG  local ratios small regardless of N ⟸ 17          post-hoc cell, one point
20  RDG  catching-up length grows with N ⟸ 17
21  EST  is N = 2 on the side where UV fixes Q? ⟸ 18
22  EST  freezing rides only on quantities beyond ℓ_* ⟸ 19, 20
```

How to check: see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The conclusion nodes of this chapter (the relaxation-time assumption is a sentence about a pair of measure and form; the correction of what to compare with; freezing rides only on quantities beyond the length at which conservation breaks) are all "Estimate", and there is neither a four-dimensional node nor a node for the mass gap. The two-dimensional readings (the nodes from the five points of the strip onwards) are premises of none of the four-dimensional rows.

What this chapter can say

Said	Not said
Proved here On the hierarchical lattice in the Gauss approximation, the gap in tree coordinates is β and the gap in link coordinates is $\beta \cdot 4\sin^2(\pi/2^{N+1})$, and the slowness comes from conservation of flux	The gap of non-linear, non-abelian hierarchical lattices. Implications for four-dimensional Yang–Mills
Reading The length at which conservation breaks, ℓ_c, at five points of the two-dimensional $U(1)$ strip with slope $0.916(21)$; that it is not proportional to the static length; $\ell_c \approx \pi/\sqrt{E\lambda_Q}$	The exponent as $\beta \to \infty$. The bend of the gap itself. The four-dimensional ℓ_*
Reading The raw rate follows the Kramers slope, and the split comes from the static factor $1/E$ (added afterwards)	The static factor in the rate of the four-dimensional topological charge
Estimate The three rulers, the candidate definition (2) of ℓ_*, comparing with $\max(\xi^2, \tau_{\mathrm{glob}})$	A sentence shown in any model. The right way to correct the relaxation-time assumption
Reading In two-dimensional CP^{N-1} ($L = 64$, $\xi \approx 4$), freezing splits with N, local ratios are small regardless of N (post-hoc cell, one point, weak), and the catching-up length grows with N	$z_Q(N)$. The registered verdict (could not be defined). Implications for four dimensions and $SU(N)$

Sources and reproduction

Item	Kind	Source / tools
Hierarchical gauge lattices	Cited	T. Ito, Commun. Math. Phys. 95 (1984) 247 (§ 1 checked in the text; writes that the extension to higher dimensions is left to the reader). The four-dimensional version is this book's extension
Comparison values for the hierarchical Gaussian free field	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474 § 4.1.2 (text checked)
Algebra of (3)(4), (5), (7)(9)	Proved here, Lean	Proofs in the text. Bundles LatticeGaugeHierarchicalLink (`HierLink.*`), LatticeGaugeConservationLength (`ConsLen.*`), LatticeGaugeConservationExponent (`ConsExp.*`). The bundles guarantee only arithmetic, algebra and comparisons of intervals
Numbers of Table 1	Reading	All eigenvalues of the dense matrix of the class model ($N \le 2$), a separate implementation per sector ($N \le 4$), the gauge quotient on the constraint surface of the four-dimensional original lattice ($N \le 2$)
Numbers of Table 2	Reading	Simulation of link Langevin on the strip (step $\beta\,dt \approx 0.04$, strip length ≤ 256). Positive control: τ of the Gauss dynamics agrees with the closed formula within 1σ. Negative control: moving the outer links too gives $O(\beta)$ independent of M. Errors by Madras–Sokal, with block averages as a separate route
The formula for ξ_{XY}, the Kramers rate	Assumption	Used as standard computations (original sources not checked)

Item	Kind	Source / tools
The action, topological charge and positive-control values for CP^{N-1}	Cited	M. Hasenbusch, Phys. Rev. D 96 (2017) 054504, arXiv:1706.04443 (equations (1), (10)–(12) and Table II checked in the text). B. Berg, M. Lüscher, Nucl. Phys. B 190 (1981) 412 (the original source for the triangle definition, bibliographic data only)
Dependence on N of the freezing of the topological mode	Cited	L. Del Debbio, G. M. Manca, E. Vicari, Phys. Lett. B 594 (2004) 315 (text), M. Berni, C. Bonanno, M. D'Elia, Phys. Rev. D 100 (2019) 114509 (text), M. Campostrini, P. Rossi, E. Vicari, Phys. Rev. D 46 (1992) 2647 (abstract)
Numbers of Tables 3 and 4	Reading	Simulation of heat bath ($O(3)$ heat bath on sites, von Mises heat bath on links) and overrelaxation, autocorrelation with Wolff's window. The free-field reference uses the same algorithm and the same L
The ultraviolet problem of the lattice topological susceptibility for $O(3)$	Assumption	From memory (original source not obtained)

Terms used in this chapter — the three rulers, the length at which conservation breaks ℓ_* (ℓ_c), the Dirichlet form (coordinates), tree coordinates and link coordinates, conservation of flux, wrapping, the topological charge of the strip, the flux dipole, raw rate and static fluctuation, the time for the global quantity to move by one, two-dimensional test beds. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-03 Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption — two forms that restate the relaxation-time assumption to avoid the pattern. What is specific to four dimensions (why the two lengths split) is treated in 24-07.

2026-10-09 · chapter Yang–Mills and the mass gap Part 24 The first road, second half local Poincaré constant, ratio form, sector relaxation-time assumption · Kiichi, Shiori and Rin

Two ways to repair the relaxation-time assumption — the sector assumption and the local assumption

> This chapter is this book's own investigation. The established knowledge it builds on is in 2-10 and 5-09.

What this chapter shows — We define two forms that restate the diffusive-relaxation-time assumption so as to avoid the pattern of 24-01 (a nearly conserved global quantity). The local relaxation-time assumption is a sentence about the Poincaré constant $C_P^{\mathrm{loc}}(c)$ over functionals supported in a window of length $c\ell$; it is fixed by the marginal law on the window alone, and returns unconditionally to the global C_P as $c \to \infty$. But the form "$C_P^{\mathrm{loc}} \le K(c)\ell^2$ for some $K(c)$" holds even for the massless free field and is empty, so it must be stated by the ratio K_{rel} to the massless value on the same box. The sector relaxation-time assumption is a sentence about the Poincaré constant in each sector of the smoothed topological charge; it pushes the obstruction to the relaxation-time assumption into a single quantity between sectors, C_{trans}, but it does not become a bridge to the mass gap. Seen through the charge of a subregion, the local relaxation-time assumption asks about the transport coefficient d, and the global relaxation-time assumption about the coefficient of breaking of conservation h.

Prerequisite chapters — 24-01 (the pattern, and the two accounts of (4)–(7)), 23-01 § 05 ((13)–(15) for the relaxation-time assumption and the relaxation-time upper-bound assumption, the note on topological sectors), 23-01 § 06 (the setting of the One-Sided Bridge Proposition (Theorem 3.1 of the paper)), 23-11 § 02 (the Box-Reduction Proposition and the closed box C_{box}). Notation in 24-09.

Said first — The level is [Res]; only § 01 is written to be readable at [Y3]. The core of this chapter is definitions, and **neither of the two relaxation-time assumptions has been shown at any β**. What has been shown are properties of the definitions (being a quantity of the marginal on the window, not being dense, monotonicity, the limit), an inequality for the decomposition of the variance by sectors, and values for the free-field reference. The definition on the four-dimensional Wilson measure is a copy of the definition on the path measure with continuous time, and the existence of the infinite-volume measure is a premise row.

The order of this chapter

1. Summary — a section readable at [Y3]
2. Definition of the local relaxation-time assumption — the algebra of time-averaged functionals $\mathcal{D}$, the window families $\mathcal{D}_c$, the Local Poincaré Constant Properties Proposition ((1)–(4))
3. The ratio form and the gauge-quotient reference — the emptiness of the absolute form, K_{rel}, gauge-invariant references, the note on 0+1 ((5)–(7), Table 1)
4. The sector relaxation-time assumption — cuts at regular values, the reflecting form, the Sector Poincaré Constant Proposition and the sandwich, three weaknesses ((8)–(11))
5. Open boxes and closed boxes, two coefficients — local asks d, global asks h, Table 2 (the order of volume and $a \to 0$)
6. Table of the five grades — Table 3 / dependency graph
7. What cannot be said

01

Summary

A section readable at [Y3]. The relaxation-time assumption is a sentence about "every test function", so if the relaxation of a quantity of the pattern of 24-01 (a winding number, the topological charge) grows faster than ξ^2, that alone breaks the upper half (the upper-bound assumption) ((3) of 24-01). Such a quantity is global, local neither in time nor in space. So we restrict the test functions in two ways and restate the relaxation-time assumption.

The local relaxation-time assumption. Restrict the test functions to those depending only on configurations inside a window of length $c\ell$ (in four dimensions, a spacetime box of side $c\ell$). ℓ is a ruler given from outside, independently of the gap; in four dimensions it is the bookkeeping length ξ_{RG}. The supremum $C_P^{\mathrm{loc}}(c)$ is fixed by the marginal distribution on the window alone, increases monotonically as the window widens, and returns exactly to the global C_P as $c \to \infty$. But in a four-dimensional box, C_P^{loc} is finite, of order the square of the side of the box, even for the massless free field. Merely satisfying "$C_P^{\mathrm{loc}} \le K\ell^2$" says nothing, so we ask that the ratio K_{rel} to the massless value on the same box be sufficiently smaller than 1.

The sector relaxation-time assumption. In the periodic box, divide configurations into sectors by the value of the smoothed topological charge, and measure the Poincaré constant in each sector. Splitting the variance into within sectors and between sectors, the global C_P is sandwiched: above by the sum of "the worst constant within a sector" and "the quantity between sectors C_{trans}", and below by C_{trans}. Assuming the sector relaxation-time assumption, the upper-bound assumption becomes the same as $C_{\mathrm{trans}} \lesssim \xi^2$. The obstruction does not vanish; it only moves into one quantity.

What is asked. Writing the ratio for the charge of a subregion with two coefficients ((7) of 24-01), the local relaxation-time assumption needs the **transport** coefficient of the charge d not to be too small, and the global upper-bound assumption needs the coefficient of **breaking of conservation** of the charge h not to be too small. In four dimensions, the freezing of the topological charge acts on the h side, so the local relaxation-time assumption is expected not to be affected by freezing Estimate . But the local relaxation-time assumption does not become a part of the chain that builds the upper-bound assumption; it is a separate sentence (the dependency graph of § 06). That what the one-sided bridge needs is the side local in time (under the assumptions of the one-sided bridge) is treated in the chapter on the bridge, 24-04.

02

Definition of the local diffusive-relaxation-time assumption

Setting. We use the same setting as the one-sided bridge of Part 23 ((17) of 23-01): a finite spatial lattice, the configuration space of one time $\mathcal{Q}$ (a finite-dimensional connected Riemannian manifold with metric M), $H = -\frac{\hbar^2}{2}\Delta_M + V$ with the ground-state assumption ((A1) in the paper; discrete spectrum, simple ground state $\psi_0 > 0$, $\Delta = E_1 - E_0 > 0$), μ the stationary path measure with $T = \infty$ (one-time distribution ψ_0^2, transition $e^{-\tau A}$, $A = (H - E_0)/\hbar$), and $m = \Delta/\hbar$. We take the collection of test functions to be the following **algebra of time-averaged functionals**.

$$\mathcal{D} := \Big\{ f = \Phi\Big(\int h_1(\tau) F_1(q(\tau))\, d\tau, \dots, \int h_k(\tau) F_k(q(\tau))\, d\tau \Big) \Big\}, \qquad \mathcal{E}_G(f) = \hbar \int \mathbb{E}\big\langle d_{q(\tau)} f,$$

Here $\Phi \in C_b^\infty(\mathbb{R}^k)$, $F_i \in C^\infty(\mathcal{Q})$ (up to polynomial growth for $\mathcal{Q} = \mathbb{R}^n$), and the weights $h_i \in L^\infty(\mathbb{R})$ have **bounded support**. $d_{q(\tau)} f = \sum_i \partial_i \Phi \cdot h_i(\tau)\, dF_i(q(\tau))$, and the metric G satisfies the metric comparison assumption of the one-sided bridge ((A3) in the paper). The global Poincaré constant is defined as $C_P := \sup_{f \in \mathcal{D},\, \mathcal{E}(f) > 0} \mathrm{Var}(f)/\mathcal{E}(f)$ **over** $\mathcal{D}$. The test functions of the one-sided bridge, $fT = \int_0^T F(q)\, d\tau$, belong to this $\mathcal{D}$.

Definition of the local Poincaré constant

For a ruler of length $\ell > 0$ and $c > 0$, put the window $W_c := [0, c\ell]$ and

$$\mathcal{D}_c := \{ f \in \mathcal{D} : \text{ every weight } h_i \text{ has support in } W_c \}, \qquad C_P^{\mathrm{loc}}(c) := \sup_{f \in \mathcal{D}_c,\, \mathcal{E}(f) > 0} \frac{\mathrm{Va}}{\mathcal{E}}$$

ℓ is not the length fixed by the gap, $\xi = \hbar/\Delta$, but a length given from outside (in four dimensions, the bookkeeping length ξ_{RG}).

ℓ is not fixed by the gap because, when a lower bound on the gap is derived from the local relaxation-time assumption in 24-04, the argument would be circular if the gap entered the length of the window. Since the measure is stationary, the position of the window does not affect the value (the interval $[a, a + c\ell]$ gives the same value).

Local Poincaré Constant Properties Proposition

(i) $C_P^{\mathrm{loc}}(c)$ is fixed only by the marginal law of the path on the window, $\mu|_{W_c}$, and the Dirichlet form inside the window.

(ii) $\mathcal{D}_c$ is not dense in $L^2(\mu)$.

(iii) If $c \le c'$, then $C_P^{\mathrm{loc}}(c) \le C_P^{\mathrm{loc}}(c') \le C_P$.

(iv) $\lim_{c \to \infty} C_P^{\mathrm{loc}}(c) = C_P$. This limit needs no assumption.

Proved here Proof. (i) $f \in \mathcal{D}_c$ is fixed only by the values of the path inside the window (the path is continuous, and $\int h F(q)\, d\tau$ is the limit of Riemann sums over points inside the window). For $\tau \notin W_c$, $d_{q(\tau)} f = 0$, so $\mathcal{E}(f) = \hbar \int_{W_c} \mathbb{E}\langle \cdots \rangle\, d\tau$, and both the variance and the Dirichlet form are expectations under the marginal law on the window.

(ii) The closure of $\mathcal{D}_c$ lies in $L^2(\mathcal{F}_{W_c})$, the space of functions measurable with respect to the σ-algebra $\mathcal{F}_{W_c}$ generated by the path inside the window. The value $F(q(\tau_0))$ at a time τ_0 outside the window (F non-constant) is not measurable with respect to $\mathcal{F}_{W_c}$: with $s > 0$ the distance from the end of the window, the conditional expectation is of the form $\psi_0^{-1} e^{-sA}(F\psi_0)$, which under the ground-state assumption does not coincide with F itself.

(iii) If $c \le c'$, then $\mathcal{D}_c \subseteq \mathcal{D}_{c'} \subseteq \mathcal{D}$, and the supremum over a subset is smaller.

(iv) The weights in $\mathcal{D}$ have bounded support, so taking the window as $[-c\ell/2, c\ell/2]$ by stationarity, $\mathcal{D} = \bigcup_c \mathcal{D}_c$. If $b < C_P$, there is $f \in \mathcal{D}$ whose ratio exceeds b, and that f lies

in $\mathcal{D}_c$ for some c, so $C_P^{\mathrm{loc}}(c) > b$. Together with the monotonicity of (iii), the limit is C_P . ∎

Lean LatticeGauge.LocalH.local_le_global (assumptions: bounded above, non-empty, subset) and local_approaches_global (assumptions: $S = \bigcup_c S_c$, bounded above, non-empty; conclusion: every b smaller than $\sup S$ is exceeded by $\sup S_c$ for some c) check only the part of (iii)(iv) about suprema of sets of real numbers. The path measure, $\mathcal{D}$ and $\mathcal{E}$ are outside Lean.

(iv) holds without assumptions because C_P was defined over $\mathcal{D}$. The assumption "local functions form a core of the form" is not needed for this limit. The question of the core remains only when one identifies $C_P(\mathcal{D})$ with the gap of the generator of the equation of stochastic quantisation (the SPDE of Langevin dynamics) (the gap of the generator of the closure of the form on $\mathcal{D}$ is $1/C_P(\mathcal{D})$, and whether it coincides with the generator of the SPDE is the question of the core).

The local relaxation-time assumption. On the definition of the local Poincaré constant, we write the local relaxation-time assumption at two levels.

$$(\xi^2)_c^{\mathrm{loc}}: \ \lim \frac{\ln C_P^{\mathrm{loc}}(c)}{\ln \ell} = 2, \qquad (\xi^2)\!\uparrow_c^{\mathrm{loc}}(K): \ C_P^{\mathrm{loc}}(c) \le K\,\ell^2 \quad (3)$$

(left: the power statement for families with $\ell \to \infty$; right: the absolute form of its upper half). Two things follow in one line from the Local Poincaré Constant Properties Proposition Proved here . First, the global upper-bound assumption $C_P \le K\ell^2$ implies $(\xi^2)\!\uparrow_c^{\mathrm{loc}}(K)$ with the same constant for every c ((iii)). Second, if $(\xi^2)\!\uparrow_c^{\mathrm{loc}}(K)$ holds for every c with **a K uniform in c**, it returns by (iv) to the global upper-bound assumption. So the local relaxation-time assumption has content different from the global upper-bound assumption only when stated with **a single c** fixed.

A version local in space too. On an infinite spatial lattice (or a spatial lattice containing the box), take the product $B \times W_c$ of a spatial box B and a window, and define in the same form the family $\mathcal{D}_{B,c}$, with F_i restricted to single-time functions depending only on variables in B, and $C_P^{\mathrm{loc}}(B,c)$. (i)–(iv) of the Local Poincaré Constant Properties Proposition hold with the same proof, and the marginal in (i) is the law of the field on $B \times W_c$. The local relaxation-time assumption on the four-dimensional Wilson measure copies this definition onto a spacetime box B_c of side $c\ell$ ($\ell = \xi_{\mathrm{RG}}$), treating time and space alike, and the σ-algebra generated by the links in the box; the Dirichlet form is that of link Langevin, $\mathcal{E}(f) = \mathbb{E}\sum_\ell |\nabla_\ell f|^2$. What is presupposed at the copying stage is the existence and uniqueness of the infinite-volume Wilson measure (the infinite-volume uniqueness statement of Part 23) Assumption .

03

The ratio form and the gauge-quotient reference

The absolute form is empty. The right-hand form of (3), $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$, holds on a four-dimensional spacetime box, local in space too, even for the massless free field. Writing $C_P^{\mathrm{loc},0}(R)$ for the local Poincaré constant of the infinite-volume massless free scalar field (one component, canonical graduation) on a spacetime box B_R of side R,

$$\frac{R^2}{4\pi^2} \;\le\; C_P^{\mathrm{loc},0}(R) \;\le\; R^2 \qquad (4)$$

Proved here . The gradient Poincaré constant of a Gaussian measure equals the operator norm of the covariance compressed to the subspace spanned by the family (a transcription to subspaces of the Covariance-Compression Lemma (Lemma 2.2 of Hairer–Stuart–Voss–Wiberg) of Hairer–Stuart–Voss–Wiberg 2005) Cited , so $C_P^{\mathrm{loc},0}(R) = \|P_B(-\Delta)^{-1}P_B\|$. In four dimensions $(-\Delta)^{-1}$ is finite (transient) and its kernel is continuous, of the type $1/(4\pi^2|z|^2)$, so the Schur estimate $\sup_x \sum_{y\in B} G(x-y) \le \int_{|z|<2R} d^4z/(4\pi^2|z|^2) = R^2$ gives the upper side, and the value with the Dirichlet ground state of the box as test function gives the lower side. Hence, reading $\ell = R/c$, $(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ holds with $K = c^2$, and **the form "$C_P^{\mathrm{loc}}(c) \le K(c)\ell^2$ for some $K(c)$" says nothing about the gap**. This is the negative control of this definition (the massless free field). In numbers, $\Gamma(R) := C_P^{\mathrm{loc},0}(R)/R^2$ is $0.0802,\ 0.0768,\ 0.0756,\ 0.0750,\ 0.0747,\ 0.0744$ at $R = 4, 6, 8, 10, 12, 16$; the correction is $1/R^2$ with no logarithmic correction, and $\Gamma_\infty = 0.0740$ Reading (floating point, numerical integration of the heat kernel on the infinite lattice, extrapolation from $R \le 16$; within relative $2\cdot 10^{-3}$ of an independent route via periodic lattices). The massless field itself satisfies $K(c) = 0.0740\,c^2$.

One line of contrast with lower dimensions. For massless fields in two spacetime dimensions the infrared diverges and $C_P^{\mathrm{loc}} = \infty$; localising only to a time window on a finite spatial lattice, the constant mode gives $C_P^{\mathrm{loc}} \approx T/(2m)$ $(mT \ll 1)$, which diverges as $m \to 0$. A massless field has a finite local constant in spacetime dimension three or more when localised in space too, and the four-dimensional setting of Part 24 is exactly that case Proved here .

Definition of the ratio form

Put the ratio of the local Poincaré constant of the target measure μ (the Wilson measure, or the path measure in continuous time) on the box $B_{c\ell}$ to the value of the massless reference with a Dirichlet form on the same lattice, the same box and the same graduation as

$$K_{\mathrm{rel}}(c) := \frac{C_P^{\mathrm{loc}}(B_{c\ell};\mu)}{C_P^{\mathrm{loc},0}(c\ell)}, \qquad (\xi^2)\uparrow_c^{\mathrm{rel}}(K) : \ K_{\mathrm{rel}}(c) \le K \quad (\beta \ge \beta_0 \text{ uniformly}, \ \ell = \xi_{\mathrm{RG}}$$

The local relaxation-time assumption is this sentence at **a single** c; it is not required uniformly in c.

It corresponds to the constant of the absolute form by $K(c) = K_{\mathrm{rel}}\,\Gamma\,c^2$ Proved here . In the ratio form the constant K becomes an absolute number independent of the box size, and "sufficiently smaller than 1" has meaning.

Why it is not required uniformly in c. For the free field of mass m the ratio is a function of mR alone, $r(mR)$, and is $0.932,\ 0.825,\ 0.623,\ 0.359,\ 0.0772$ at $mR = 0.5,\ 1,\ 2,\ 4,\ 12$ Reading (agreeing within 0.3% for boxes of side 4 to 12). When the box is smaller than the correlation length ($mR \to 0$) the ratio tends to 1. An asymptotically free theory is close to a free field at short distance, so a sentence requiring $K < 1$ uniformly down to $c \to 0$ is expected to be false Estimate . For a field whose action is a convex perturbation $\frac{1}{2}\phi(-\Delta)\phi + \sum U(\phi_x)$ (U convex), the Brascamp–Lieb inequality gives $K_{\mathrm{rel}} \le 1$ Assumption (the statement of the inequality in the original source has not been checked in this part). What the bridge needs is not that the ratio is below 1 but that it is below the threshold K_*, and the threshold moves with the choice of reference: 0.342κ (scalar), 0.502κ (gauge quotient, one-colour linearisation), $\approx 0.62\kappa$ (colour singlet of $SU(2)$, leading term). The threshold, and that the band $[K_*, 1)$ is undecided, are treated in 24-04.

The reference for gauge fields is taken on the gauge quotient. For four-dimensional gauge fields, the value of the reference changes with the test space in which it is measured. The target Wilson measure is gauge invariant, and the Poincaré constant of a non-gauge-invariant function is infinite along gauge orbits, so the numerator of the ratio is finite only on gauge-invariant functions. So the reference in the denominator is also measured on gauge-invariant functions of the same box.

$$V_F := \{\text{gradients of functions of the plaquette angles of the box}\} = d^*\big(\ell^2(P_R)\big), \qquad V_Q :=$$

If the box is contractible, $V_F = V_Q$ Proved here (the cubical complex is contractible with zero first homology, so cycles in the graph of the box are boundaries of 2-chains of plaquettes). In non-contractible regions (with holes), $V_F \subsetneq V_Q$. Gauge-fixed references (Feynman gauge and so on) depend on the fixing parameter α, with $\Gamma_Q \le \Gamma_{\text{Landau}} \le \Gamma_{\alpha=1}$, and moreover $\Gamma_\alpha \ge \alpha/(\lambda_D^{\mathrm{int}} R^2)$, so they become infinite as $\alpha \to \infty$ (removing the fixing) Proved here . The numbers are in Table 1.

Table 1 References for the four-dimensional box (one-colour Gaussian approximation, $\beta = 1$, canonical graduation, floating point, extrapolation from $R \le 16$; λ_W is the Dirichlet value of the box within the test

space times R^2)

Reference	Test space	Γ_∞	$\lambda_{W,\infty}$	As a reference
Scalar (one component)	All functions on the box	0.0740 Reading	$4\pi^2 = 39.5$	Not used for gauge fields
Feynman gauge ($\alpha = 1$)	All functions of the links of the box	0.0740 Reading	39.5	Depends on α; not intrinsic
Gauge quotient	$V_Q = V_F$ (functions of plaquette angles)	0.0277 Reading	72.0 Reading	Independent of α. This one is used
Variant with the boundary gauge frozen	Divided only by gauge transformations at inner sites	2.22 to 2.53 times the gauge quotient (side 4 to 12) Reading	—	Not gauge invariant (flux through the box enters)

The last row of Table 1 is (e) of Table 1 in 24-01. Gauge transformations at the boundary sites of the box also act on links inside the box, so gauge-invariant observables of the box (Wilson loops and so on) lie in the space divided at all sites. **"Dividing by gauge transformations at all sites of the box" is part of the definition**, and leaving it out moves the numbers by a factor of 2.2 to 2.5. Every reference is a number for a free field; this book has not measured a single ratio for the target Wilson measure.

Note on 0+1. In 0+1 dimensions there is no finite massless reference (for a free particle $C_P^{\mathrm{loc}} = \infty$). The reference always has a scale, and for the double well we take the harmonic oscillator with the curvature of the bottom (angular frequency ω_b). Its ratio for the shortest window (a single time) is **exactly** ω_b/m (when ψ_0 is the Schrödinger ground state) Proved here :

$$K_{\mathrm{rel}}(\text{single time}) = \frac{\varepsilon/(2m)}{\varepsilon/(2\omega_b)} = \frac{\omega_b}{m} \quad (7)$$

(ε is the time step. The numerator is the Poincaré constant of the marginal ψ_0^2, which moves to the gap of $(H - E_0)$ by the ground-state transformation. This identity is the same as the formula for the vacuum measure in 24-05.) Lean LatticeGauge.LocalHDoubleWell.one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$; only the algebra of the ratio). In the double well, tunnelling gives $m \ll \omega_b$, so the ratio does not become 1 even for short windows; on the lattice $\varepsilon = \frac{1}{2}$ it is 4.61 ($\hbar = 0.3$) to 638 ($\hbar = 0.08$) Reading . Hence "the ratio tends to 1 at short distance" is not a fact that follows from the ratio form but **an assumption depending on a property of the theory, asymptotic freedom (being close to a free field at short distance)** Assumption . The three regimes of the ratio as the window is lengthened are treated in 24-04.

04

The sector diffusive-relaxation-time assumption

Setting. Take the periodic box $\Lambda_L = (\mathbb{Z}/L)^4$, the configuration space $\mathcal{M} = G^{\text{links}}$ (a compact connected manifold with a bi-invariant metric), the Wilson measure μ (smooth positive density), and $\mathcal{E}(f) = \int |\nabla f|^2 d\mu$. For the topological charge we use the lattice charge Q_s smoothed by the Wilson flow (flow time s) or by finitely many smoothings. The smoothing map is smooth and the charge density is a polynomial in matrix elements, so $Q_s \in C^\infty(\mathcal{M})$.

Definition of sectors, and of the form reflecting at the walls

Take the cuts $\theta_n \in (n-1, n)$ at **regular values** of Q_s (by Sard's theorem almost all values are regular, so they can be taken arbitrarily close to $n - \frac{1}{2}$).

$$I_n := \{\theta_n \le Q_s < \theta_{n+1}\}, \quad \mu_n := \mu(\,\cdot\,|I_n), \quad \mathcal{E}_n(f) := \int_{I_n} |\nabla f|^2 d\mu_n \ \ (f \in H^1(I_n)),$$

(only for n with $\mu(I_n) > 0$). No boundary condition is imposed on $\mathcal{E}_n$. Its generator is Langevin reflecting at the walls (Neumann).

Because the cut is at regular values, the closure of a sector is a compact manifold with smooth boundary, and the walls are smooth hypersurfaces of codimension 1. Cutting by a geometrically defined integer-valued charge (piecewise constant and discontinuous), the form can be defined as H^1 on open sets, but the regularity of the walls is unknown and (i) below cannot be said. This is why this chapter uses a smoothed charge.

Sector Poincaré Constant Proposition (sector Poincaré constants and the decomposition of the variance)

(i) $C_P^{\text{sec}}(n) < \infty \iff I_n$ is connected.

(ii) With N the sector index, for smooth f,

$$\mathcal{E}(f) = \sum_n \mu(I_n)\, \mathcal{E}_n(f), \qquad \operatorname{Var} f = \sum_n \mu(I_n) \operatorname{Var}_{\mu_n} f + \operatorname{Var}\big(\mathbb{E}[f \mid N]\big) \quad (9)$$

(iii) For the quantity between sectors $C_{\text{trans}} := \sup_f \operatorname{Var}(\mathbb{E}[f|N])/\mathcal{E}(f)$,

$$C_{\text{trans}} \ \le \ C_P \ \le \ \sup_n C_P^{\text{sec}}(n) + C_{\text{trans}} \quad (10)$$

(iv) $C_P^{\text{sec}}(n) \le C_P$ does not hold in general.

Proved here Proof. (i) The closure of a sector is compact with smooth boundary and the density is positive and smooth, so the resolvent of the Neumann operator is compact (Rellich's theorem Assumption : the statement in the original source has not been checked in this part). The multiplicity of the zero eigenvalue equals the number of connected components, so a positive gap is the same as connectedness. (ii) The walls have measure 0, so the Dirichlet form splits into a sum over sectors, and the variance is the sum of the expectation of the conditional variance and the variance of the conditional expectation. (iii) The left is $\mathbf{Var}(\mathbb{E}[f|N]) \le \mathbf{Var}\, f$; the right uses the Poincaré inequality in each sector on the first term of (9) and closes with $\sum_n \mu(I_n)\mathcal{E}_n = \mathcal{E}$. (iv) Functions on sectors may jump at the walls (they are not in $H^1(\mathcal{M})$), so if conditioning creates a new bottleneck, $C_P^{\mathrm{sec}}(n) > C_P$ is possible. ∎ Lean LatticeGauge.LocalH.sector_split (assumptions: weights ≥ 0, the inequality in each sector, the two formulas of (9), $\mathbf{Var}(\mathbb{E}[f|N]) \le C_{\mathrm{trans}}\mathcal{E}$; conclusion: the right of (10)), trans_le_total (assumptions: weights and sector variances ≥ 0, (9); conclusion: the inequality for the numerator on the left of (10)). The identity (9) itself, Rellich, and the measure are outside Lean.

The sector relaxation-time assumption and its corollary. The sector relaxation-time assumption is the sentence that, in a box with fixed physical side, the Poincaré constant within a sector is bounded by the square of the correlation length.

$$(\xi^2)\uparrow^{\mathrm{sec}}: \quad L = \lambda\,\xi_{\mathrm{RG}} \text{ gives } \sup_n C_P^{\mathrm{sec}}(n) \le K(\lambda)\,\xi_{\mathrm{RG}}^2 \qquad \Longrightarrow \qquad \big(\,(\xi^2)\uparrow \iff C_{\mathrm{trans}} \lesssim$$

Proved here (from (10); $\lesssim$ is up to replacing constants). The sector relaxation-time assumption pushes the obstruction to the relaxation-time assumption **exactly** into the quantity between sectors C_{trans}. It does not remove the obstruction but only moves it into one quantity, the same form as the transfer type in 23-10. The physical reading of C_{trans} is the rate of dislocations, and statically $C_{\mathrm{trans}} \ge \mathbf{Var}(\mathbb{E}[Q_s|N])/\mathcal{E}(Q_s)$.

It does not connect to the bridge. What follows from (10) and the one-sided bridge ((17) of 23-01, with corrections for finite T in periodic time) goes only as far as $\kappa/m^2 \lesssim \sup_n C_P^{\mathrm{sec}}(n) + C_{\mathrm{trans}}$, and no gap comes out unless C_{trans} is bounded Proved here . The Langevin gap within a sector, $1/C_P^{\mathrm{sec}}(n)$, is a fifth quantity of time, and the conditioned measure μ_n has no transfer operator (the constraint on the total charge breaks the Markov property in time). A "physical gap within a sector" of the type of the one-sided bridge does not stand by definition; to set it up one would need a treatment of superpositions of θ vacua, which has not been formulated Estimate .

Three weaknesses. (a) *The walls of conditioning can create artificial bistability.* In the core on the box side of Part 23, two local minima appeared on the walls of conditioning the interval to "exactly this interval", giving rise to a bistability that was

not physical metastability (23-11 § 06). The walls of sectors lie on dislocation configurations of small measure, so we expect the harm to be small, but this has not been checked Estimate . That it is unknown whether sectors are connected in (i) of the Sector Poincaré Constant Proposition is the extreme form of this weakness. (b) *When the physical volume is made large, the obstruction remains even with the sector fixed.* With β fixed and $L \to \infty$, the charge of a half box can move even within a sector I_n. With the coefficients of (5) in 24-01, the conditional variance is of order $\chi_t V/4$ and the Dirichlet form of order $hV/2 + g\,|\mathrm{face}|$, and the ratio tends to $\chi_t/(2h)$ as $V \to \infty$, the same order as the overall ratio of the periodic box χ_t/h Assumption (on the definitions of h, g and the accounting for the conditional variance of a normal distribution with the sum of the two halves fixed). Fixing the sector stops only the one mode of the total charge; the modes of the charges of subregions remain. Hence the sector relaxation-time assumption is expected to have content different from the upper-bound assumption only when the box is smaller than the crossing length (subregions within a sector are in the transport regime), that is, **to have meaning only in the order in which the physical volume is fixed** Estimate . (c) *The path measure with $T = \infty$ has no sectors.* The four-dimensional charge is a quantity integrated over time, and for $T = \infty$ it is not in L^2.

Correspondence with the splitting of the first step. (9) is the same splitting as the splitting of the variance in the first step (local Poincaré within a phase, and transport between phases, 23-05 § 01), with phase ↔ sector and kink ↔ dislocation Estimate . It is one concrete destination of "the two-scale splitting is expected to lift" in 23-03 § 06. But in the first step, the time for transport between phases (diffusion of kinks) is read to be of order $\xi^2 S_0/\hbar$ ((14) of 23-04 Assumption), and fits into the form of the double-well relaxation-time power statement. The four-dimensional quantity between sectors C_{trans} is exactly a quantity of the pattern of 24-01, and there is no prospect yet that it fits into the same form.

05

Open boxes and closed boxes, two coefficients

Local asks about the transport coefficient; global asks about the coefficient of breaking of conservation. The charge Q_w in a smooth window of (7) in 24-01 belongs to the local family of a box of side $R = c\xi$. Putting (7) into the local relaxation-time assumption (absolute form), the necessary condition is $\chi_t c^2\xi^2/(hc^2\xi^2 + d\lambda_W) \le K\xi^2$, and in the regime where h is small (the crossing length R_* much longer than $c\xi$),

$$\frac{\chi_t (c\xi)^2}{d\,\lambda_W} \le K\xi^2 \iff \chi_t c^2 \le K\,d\,\lambda_W, \qquad \text{that is} \qquad D_{\mathrm{eff}} := \frac{d}{\chi_t} \ge \frac{c^2}{K\lambda_W} \quad (12)$$

Proved here (algebra) Lean LatticeGauge.LocalH.transport_requirement (assumptions: $0 < d$, $0 < \lambda_W$, $0 < \xi$). In the ratio form, K is read as $K_{\mathrm{rel}}\Gamma c^2$. On the other hand, the necessary condition for the global upper-bound assumption is (2)(6) of 24-01 with $R \to \infty$,

$$\frac{\chi t}{h} \lesssim \xi^2 \quad (13)$$

Assumption (on top of (5) of 24-01). (12) and (13) bear on different coefficients. **The local relaxation-time assumption asks about the transport coefficient of the charge d, and the global relaxation-time assumption about the coefficient of breaking of conservation of the charge h.** The freezing of the four-dimensional topological charge can be read as the appearance of h vanishing quickly in the continuum limit (dislocation configurations becoming rare), so what it affects is the side of (13). The prospect that the local relaxation-time assumption is unaffected by freezing comes from here Estimate . There are no four-dimensional lattice values for d and h.

Closed boxes and open boxes. The Box-Reduction Proposition of Part 23 (23-11 § 02) is a reduction bounding the global C_P by a box constant C_{box}, and C_{box} was the constant of a conditioned measure **with the boundary values fixed** (with sup over the boundary values). We call this a closed box. The box of the local relaxation-time assumption is a marginal box without fixing the boundary, and we call it an open box. In four dimensions, if the boundary links are fixed, the integer part of the topological charge inside the box is expected to change only through dislocations inside the box (in the continuum, the charge of the box is the sum of the Chern–Simons value on the boundary and a winding number) Estimate . Hence for closed boxes of size $\chi t (c\xi)^4 \gtrsim 1$, C_{box} is expected to inherit the time of the dislocation barrier Estimate . In open boxes the charge flows in and out through the boundary, and relaxation is of the transport type of (12) Assumption .

The difference between closed and open boxes has the same form as the difference between periodic and open boundaries. Open boundaries remove the topological obstruction only inside a finite box with fixed physical side (the autocorrelation time of Q^2 with open boundaries grows roughly in proportion to the length in the time direction, § 5.3 of Lüscher–Schaefer 2011 Reading). This is the transport type of (12) restated in the language of boxes. A reading of the consequence Estimate : if, under the reading that extends the conjecture (§ 06 (a) of 24-01), the upper-bound assumption reads as false in four dimensions, then even when the local relaxation-time assumption holds, the four-dimensional version of the Box-Reduction Proposition fails at the stage of closed boxes. The place where it breaks is not "relaxation inside the box" but "the passage between sectors inside a closed box".

Table 2 The order of volume and $a \to 0$, and the corrections expected to work

Order	Taken first	Corrections expected to work	Corrections expected not to work	Label
$a \to 0$ with the physical side fixed	Continuum limit of a finite box	Open boundaries, the sector relaxation-time assumption	—	Cited (original sources on open boundaries), Estimate
Volume to infinity first (the order of this book)	Infinite-volume lattice measure	The local relaxation-time assumption (open boxes)	Open boundaries, the sector relaxation-time assumption	Assumption (accounting for the charge of a subregion), Estimate

The original source writes that the scaling of the autocorrelation of algorithms implementing Langevin ($1/a^2$ in lattice units) "is expected to hold on infinite lattices and on finite lattices with open boundaries", and excludes transitions changing topology with periodic boundaries from the analysis as lattice artefacts (the footnote of Lüscher–Schaefer, Non-renormalizability of the HMC algorithm, 2011) Cited . The second row of Table 2 points the same way as this wording, but the object of the original source is the autocorrelation of local quantities, not the local relaxation-time assumption itself.

06

Table of the five grades

We put the rows of this chapter into one table in the format of the five grades, with Estimate added. The symbols of the rows are the same as the nodes of the dependency graph, and one row of 24-01 (the row of cluster decay) is carried over.

Table 3 Rows on the sector relaxation-time assumption and the local relaxation-time assumption ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Existence and uniqueness of the infinite-volume Wilson measure	Assumption	The infinite-volume uniqueness statement of Part 23	$\beta \geq \beta_0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	Rellich's theorem (compactness of the Neumann resolvent)	Assumption	Sector Poincaré Constant Proposition (i) (the statement in the original source has not been checked)	Smooth boundary	—
3	Poincaré constant of a Gaussian measure = norm of the compressed covariance	Cited	(4), Hairer–Stuart–Voss–Wiberg 2005	Gaussian	—
4	Local Poincaré Constant Properties Proposition (i)–(iv)	Proved here	Proof in § 02	Continuous time, finite spatial lattice, $T = \infty$	local_le_global, local_approaches_global (assumptions: bounded above, non-empty, $S = \bigcup S_c$; only the supremum part)
5	That the proof of the Local Poincaré Constant Properties Proposition goes through in the same way for boxes of the four-dimensional Wilson measure	Assumption	End of § 02 (the treatment of domains is not written out)	Four dimensions, infinite volume	—
6	Properties of the four-dimensional local Poincaré constant	Proved here	On rows 1, 4, 5	Four dimensions	—
7	The absolute form holds with $K = c^2$ for the massless free field	Proved here	(4)	Four-dimensional spacetime, local in space too	—
8	$\Gamma_\infty = 0.0740$, $\Gamma_{Q,\infty} = 0.0277$, $\lambda_W^Q = 72.0$, 2.22 to 2.53 times with the boundary frozen	Reading	§ 03, Table 1	Free field, one-colour Gaussian, $\beta = 1$, $R \le 16$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
9	$V_F = V_Q$ in contractible boxes; gauge-fixed references depend on α and diverge as $\alpha \to \infty$	Proved here	(6)	Contractible box	—
10	Ratio for the shortest window $= \omega_b/m$	Proved here	(7)	0+1, Schrödinger ψ_0	one_slice_ratio (assumptions: $0 < \varepsilon, m, \omega$)
11	The ratio tends to 1 at short distance	Assumption	The note on 0+1 in § 03 (depends on asymptotic freedom)	Four dimensions	—
12	Local relaxation-time assumption (ratio form): $K_{\mathrm{rel}}(c) \le K$ at a single c	Assumption	(5)	$\beta \ge \beta_0$	—
13	Sector Poincaré Constant Proposition (i): sector constant finite $\iff$ sector connected	Proved here	§ 04 (on top of row 2)	Periodic box, cuts at regular values	—
14	The sandwich (10)	Proved here	(9)(10)	Periodic box	sector_split, trans_le_total (assumptions: weights ≥ 0, the inequality in each sector, the two formulas of (9))
15	Corollary (11): under the sector relaxation-time assumption, the upper-bound assumption $\iff C_{\mathrm{trans}} \lesssim \xi^2$	Proved here	(11)	Box with fixed physical side	—
16	What comes out of sectors goes only as far as $\kappa/m^2 \lesssim \sup_n C_P^{\mathrm{sec}} + C_{\mathrm{trans}}$	Proved here	§ 04 (on top of the one-sided bridge)	Finite box	—
17	Cluster decay (row 1 of Table 5 in 24-01)	Assumption	(5) of 24-01	Infinite volume	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
18	(12): local asks d; (13): global asks h	Proved here	(12) (algebra), (13) on top of row 17	Four dimensions	transport_requirement (assumptions: $0 < d, \lambda_W, \xi$)
19	Ratio for half boxes within a sector $\to \chi t/(2h)$	Assumption	Weakness (b) of § 04	$L \to \infty$ with β fixed	—
20	Alternating maximal-correlation assumption for four-dimensional boxes: maximal correlation < 1 between families of closed boxes, uniform in β	Assumption	Premise of the four-dimensional version of the Box-Reduction Proposition	Four dimensions, families of 2^4 colours	—
21	Gap assumption for closed four-dimensional boxes: $C_{\mathrm{box}}(c\xi_{\mathrm{RG}}) \le K'\xi_{\mathrm{RG}}^2$ for closed boxes	Assumption	§ 05 (the row where the topological barrier is expected to enter)	Four dimensions	—
22	Four-dimensional multi-family box reduction $C_P \le C(c)\, C_{\mathrm{box}}/(1 - c_{\mathrm{alt}})$	Assumption	Four-dimensional version of the one-dimensional Box-Reduction Proposition (not written out)	Four dimensions	—
23	Relaxation-time upper-bound assumption	Proved here	Algebra on top of rows 20, 21, 22	Four dimensions	—

Dependency graph

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 1  ASM  infinite-volume Wilson measure exists, unique        β ≥ β₀
 2  ASM  Rellich's theorem                                     smooth
 3  CIT  Gaussian C_P = norm of compressed covariance          Gaussia
 4  PRV  Local Poincaré Constant Properties Proposition
         (continuous time)                                    T = ∞
         Lean local_le_global, local_approaches_global (supremum part
 5  ASM  its proof goes through in 4D boxes                    four di
 6  PRV  properties of the 4D local C_P ⟸ 1, 4, 5
 7  PRV  vacuity of the absolute form (4) ⟸ 3                   4D spa
 8  RDG  reference numbers (Γ_∞, Γ_Q, λ_W^Q,
         factor with frozen boundary)                         free fi
 9  PRV  V_F = V_Q in contractible boxes; fixing depends on α  contra
10  PRV  ratio for the shortest window = ω_b/m                 0+1
         Lean one_slice_ratio
11  ASM  ratio → 1 at short distance (asymptotic freedom)      four di
12  ASM  local relaxation-time assumption (ratio form, one c) β ≥ β₀
13  PRV  sector constant finite ⟺ connected ⟸ 2                regu
14  PRV  sandwich C_trans ≤ C_P ≤ sup C^sec + C_trans          periodi
         Lean sector_split, trans_le_total
15  PRV  under the sector assumption, the upper-bound
         assumption ⟺ C_trans ≲ ξ² ⟸ 14
16  PRV  bridge inequality from sectors ⟸ 14 (+ one-sided bridge)
17  ASM  cluster decay (24-01)                                 infinit
18  PRV  local asks d, global asks h ⟸ 17
         Lean transport_requirement
19  ASM  ratio of half boxes in a sector → χ_t/(2h)
20  ASM  alternating max. correlation of 4D boxes < 1        four dim
21  ASM  gap of closed 4D boxes C_box ≤ K'ξ_RG²
         ← the row the topological barrier is expected to enter
22  ASM  4D multi-family box reduction (not written out)
23  PRV  upper-bound assumption ⟸ 20, 21, 22
```

How to check: on each "⟸" row, see by eye that the numbers on the right are smaller than the one on the left. There are no circles. The leaves marked "Assumption" number ten (the existence of the infinite-volume measure, Rellich's theorem, the transcription to four dimensions of the proof of the properties of the local Poincaré constant, the reading of the ratio at short distance, the local relaxation-time assumption, cluster decay, the ratio of half boxes, the alternating maximal correlation of four-dimensional boxes, the gap of closed four-dimensional boxes, the four-dimensional multi-family

box reduction); the "Cited" leaf is the row of the Gaussian Poincaré constant, and the "Reading" leaf is the row of the reference numbers.

Two things are read from this graph. First, **the local relaxation-time assumption enters nowhere in the chain that builds the upper-bound assumption.** A reduction of the type of the Box-Reduction Proposition uses closed boxes (the gap assumption for closed four-dimensional boxes), whereas the boxes of the local relaxation-time assumption are open boxes. The local relaxation-time assumption is not a part of the upper-bound assumption but a separate sentence. Where the local relaxation-time assumption is used is in the graph of the bridge in 24-04. Second, the place where the upper-bound assumption is expected to break in four dimensions is concentrated in the single row of the gap assumption for closed four-dimensional boxes. The statement of this row contains only the constant of closed boxes C_{box} and not the global C_P, so it is not of the form that puts the conclusion among the premises.

07

What cannot be said

(a) The two relaxation-time assumptions are **candidate definitions; at no β is there an inequality shown for the target gauge theory**. What has been shown are properties of the definitions (the Local Poincaré Constant Properties Proposition and the Sector Poincaré Constant Proposition) and values of the free-field reference. Nor do we claim "the right way to correct" the relaxation-time assumption.

(b) The local relaxation-time assumption on the four-dimensional Wilson measure is a copy of the definition on the path measure in continuous time, and the existence of the infinite-volume measure and the treatment of domains (that the proof of the Local Poincaré Constant Properties Proposition goes through in the same way for four-dimensional boxes) are premise rows.

(c) The accounting for the charge of a subregion (transport in open boxes, barrier in closed boxes) is an estimate on top of cluster decay and the formal coefficients h, g, d, with no four-dimensional lattice values.

(d) The four-dimensional multi-family box reduction has not been written. The harm of sector walls, and how the sector relaxation-time assumption breaks when the physical volume is made large, are estimates.

(e) There is no evidence that the local relaxation-time assumption holds in four dimensions. There is only the accounting "freezing acts on the h side, and the local relaxation-time assumption asks about d".

(f) Every number in Table 1 is a number for the free field of the reference (one-colour Gaussian, $\beta = 1$, floating point, extrapolation); not a single ratio K_{rel} for the target

measure has been measured. The threshold depends on the reference and on the family of test observables, and the bridge says nothing about fields in the band $[K_*, 1)$ (24-04).

(g) "The ratio tends to 1 at short distance" is an assumption depending on asymptotic freedom; in the 0+1 double well this reading does not hold ((7)).

What this chapter can say

Said	Not said
Proved here The definition of the local Poincaré constant and the Local Poincaré Constant Properties Proposition (a quantity of the marginal on the window, not dense, monotone, returns unconditionally to C_P as $c \to \infty$; in continuous time, on a finite spatial lattice, $T = \infty$, under the assumptions of the one-sided bridge)	The treatment of domains on the four-dimensional Wilson measure (the copy). The question of the core (identification with the generator of the SPDE)
Proved here The local relaxation-time assumption in absolute form holds even for the massless free field on a box in four-dimensional spacetime, local in space too ($K = c^2$)	That the local relaxation-time assumption in ratio form holds for the target theory
Proved here That the family of plaquette functions and the gauge quotient coincide for contractible boxes; that gauge-fixed references depend on the parameter; the ratio for the shortest window ω_b/m	References for $SU(2)$ at finite β. Values for families of large four-dimensional loops
Reading Reference numbers $\Gamma_\infty = 0.0740$, $\Gamma_{Q,\infty} = 0.0277$, $\lambda_W^Q = 72.0$, 2.22 to 2.53 times with the boundary frozen	Not propositions of upper or lower bounds. Numbers for free fields, not for the target measure
Proved here The definition of sectors (cuts at regular values, the reflecting form), finite $\Longleftrightarrow$ connected (on top of Rellich), the sandwich (10), corollary (11), the bridge inequality from sectors	Whether sectors are connected. Estimates of C_{trans}. The treatment of θ vacua
Estimate That the local relaxation-time assumption is expected not to be affected by freezing; that closed boxes are expected to inherit the barrier; that the sector relaxation-time assumption is expected to have meaning only in the order with the physical volume fixed	None of them is a theorem

Sources and reproduction

Item	Kind	Source / tools
Poincaré constant of a Gaussian measure = norm of the covariance	Cited	The Covariance-Compression Lemma of M. Hairer, A. M. Stuart, J. Voss, P. Wiberg (2005) (equality for linear functionals)

Item	Kind	Source / tools
Autocorrelation of Q^2 with open boundaries and the length in the time direction; the expected scaling of Langevin algorithms	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 § 5.3; M. Lüscher, S. Schaefer, JHEP 04 (2011) 104 § 1 and footnote (text checked)
Local Poincaré Constant Properties Proposition (iii) (iv), (10) of the Sector Poincaré Constant Proposition, the algebra of (12), the ratio (7)	Lean	LatticeGauge.LocalH (local_le_global, local_approaches_global, sector_split, trans_le_total, transport_requirement), LatticeGauge.LocalHDoubleWell (one_slice_ratio). The bundles guarantee only suprema of sets of real numbers and algebra
Reference numbers of Table 1, $\Gamma(R)$, $r(mR)$	Reading	Largest eigenvalue (Lanczos) of the compression to the box of the lattice Green function (numerical integration of the heat kernel), and an independent route via periodic lattices (removing the zero mode, extrapolation in $1/L^2$). The gauge quotient by two routes: projection via the Neumann Laplacian of the graph of the box, and projection via a dense singular value decomposition
Lattice ratios for the double well	Reading	Lattice eigenvalue problem for the transfer kernel (symmetric product with time step $\varepsilon = \frac{1}{2}$)

Terms used in this chapter — the algebra of time-averaged functionals $\mathcal{D}$, window, the local Poincaré constant $C_P^{\mathrm{loc}}(c)$, the ratio form K_{rel}, the massless reference and Γ, the gauge-quotient reference, sectors and the reflecting form, the quantity between sectors C_{trans}, open boxes and closed boxes, the transport coefficient d and the coefficient of breaking of conservation h. The notation is in 24-09.

Next chapter to read: 24-04 Which form of the relaxation-time assumption the one-sided bridge needs — the threshold K_* at which a lower bound on the gap comes out of the local relaxation-time assumption in ratio form, and the size of box at which the bridge sees the mass.

Which form of the relaxation-time assumption the one-sided bridge needs

This chapter is this book's own investigation. The established knowledge it builds on is in 5-09 and 2-10.

What this chapter shows — Which part of the diffusive-relaxation-time assumption the one-sided bridge of Part 23 uses, and nothing more. On the family of time-integrated single-time functions, the supremum of the Poincaré constant is fixed exactly by the gap, so what the bridge needs is only the part of the relaxation-time upper-bound assumption that is local in time. That a lower bound on the gap in finite spatial volume follows from the local relaxation-time assumption with a limited window; that in infinite volume the local relaxation-time assumption must be stated as a ratio to a massless reference; that its threshold changes with the choice of reference (scalar, gauge quotient, colour singlet of $SU(2)$); and that the bridge sees the mass only when the box is at least 4–5 times the correlation length. In the 0+1-dimensional double well, the ratio does not become 1 even for the shortest window.

Prerequisite chapters — 23-01 § 05 ((13) for the diffusive-relaxation-time assumption, (14)(15) for the upper-bound assumption), 23-01 § 06 ((17) of the one-sided bridge, the identity (18), the bridge (19), (20) of the time-step comparison assumption ((D) in the paper "one-sided bridge")), 23-01 § 06 (5) (the Free-Particle-on-a-Circle Counterexample), 24-03 (definitions of the local relaxation-time assumption and the ratio form). The notation is collected in 24-09.

Said first — The level is [Res]; only the summary in § 01 is [Y3]. This chapter does not show the mass gap. Neither the relaxation-time assumption, nor the upper-bound assumption, nor its local version has been shown at any β. What we show is the logical form of which part of which assumption the bridge uses, and values for free-field references. The window argument consists of sentences about the stationary path measure with $T = \infty$ on a lattice with continuous time and a finite spatial lattice; the infinite-volume version further rests on the local-excitation moment assumption and on the infinite-volume ground-state transformation identity. The values of the references are readings from floating-point computation and extrapolation.

The order of this chapter

1. Summary [Y3] — the bridge uses only the part of the upper-bound assumption local in time
2. The one-sided bridge and the family local in time — the supremum is exactly ξ^2; correspondence with the paper ((1)–(3))
3. Limiting the window — monotonicity of the window factor, the In-Window Supremum Proposition, the lower bound in finite volume ((4)–(10), Table 1)
4. "Some $K(c)$" is empty — the massless free field as a negative control, and the ratio form ((11)(12))
5. The bridge in infinite volume and the threshold — the local-excitation moment assumption, transfer to the ratio ((13)–(16))
6. Three references — scalar, gauge quotient, colour singlet (Table 2); the box at which the bridge sees the mass
7. Note on 0+1 dimensions — even for the shortest window the ratio is ω_b/m ((17), Table 3)
8. Which part of which assumption each conclusion needs — Table 4
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 5

01

Summary [Y3] — the bridge uses only the part of the upper-bound assumption local in time

In 23-01 § 06 of Part 23, we showed the one-sided bridge $C_P \geq \kappa\hbar^2/\Delta^2$ for the path measure on a lattice with continuous time. The test function is the single one $f_T = \int_0^T F(q(\tau))\,d\tau$, the time integral of a function of the configuration at one time, $F = \psi_1/\psi_0$. Setting this one function beside $C_P \leq C\xi_{\mathrm{RG}}^2$ of the upper-bound assumption gives a lower bound on the gap of the lattice in physical units, $m_{\mathrm{phys}}^2 \geq \kappa/C$ ((19) of 23-01). The question of this chapter is which part of the upper-bound assumption is used there.

The answer comes in three stages.

First stage. On the family $\mathcal{F}_1$ of "functions of one time averaged with time weights", the supremum of $\mathrm{Var}\,/\mathcal{E}$ is exactly $\xi^2 = \hbar^2/\Delta^2$ in the canonical graduation Proved here . Hence the upper-bound assumption restricted to this family is the same as "the gap of the lattice in physical units is at least $C^{-1/2}$". The content of the upper-bound assumption beyond the gap lies only outside this family — in functions not local in time, such as winding numbers and the topological charge (the Free-Particle-on-a-Circle Counterexample of 23-01, the pattern of 24-01). The bridge uses only the part of the upper-bound assumption local in time.

Second stage. From the local relaxation-time assumption with the time window limited to length $c\ell$ (ℓ is a ruler given from outside, independently of the gap; in four dimensions it is the bookkeeping length ξ_{RG}), too, a lower bound on the gap in finite spatial volume follows, $m_{\mathrm{phys}} \geq \sqrt{1/c^2 + \kappa/K} - 1/c$ Proved here . As $c \to \infty$ it returns to the $\sqrt{\kappa/K}$ of Part 23.

Third stage. Going local in space too and on to infinite volume, a sentence of the form "the local Poincaré constant in the box is at most $K(c)\ell^2$" becomes empty, because the four-dimensional massless free field also satisfies it for some $K(c)$ Proved here . So we state it by the ratio K_{rel} to the value of the massless free field (the reference) on the same box with the same graduation. Under the local-excitation moment assumption, if the ratio is below the threshold $K_* = \kappa/(\lambda_W \Gamma)$, a lower bound on the gap follows Proved here (algebra on top of the assumption). The threshold changes with the choice of reference: 0.342κ for the scalar, 0.502κ for the gauge quotient (one-colour linearisation), and about 0.62κ for the colour singlet of $SU(2)$ (leading term, second chaos; a test value, so a lower value of the usable threshold) Reading . Examining free fields with mass, the bridge sees the mass only when the side of the box is at least about 4.2 times (scalar) to 5.2 times (gauge quotient) the correlation length Reading . For fields whose ratio lies in the band $[K_*, 1)$ between the threshold and 1, this bridge says nothing.

Finally we place a note on 0+1 dimensions. In the double well, even for the shortest window (a single time), the ratio is exactly ω_b/m and does not become 1 Proved here . "At short distance the ratio is close to 1" is not a fact that follows from the ratio form; in four dimensions it is an assumption depending on asymptotic freedom Assumption .

As a metaphor, the bridge looks only at "the time for the hand of the clock to go round once", not at the slowness of "the gear that counts how many times the hand has gone round". The upper-bound assumption is a sentence about the whole, gear included, and what the bridge needs is only the part about the hand.

02

The multidimensional one-sided bridge and the family local in time — the supremum is exactly ξ^2

The setting is the same as the one-sided bridge in 23-01 § 06. Space is a finite lattice, the configuration space at one time is Q, the Hamiltonian is $H = -\frac{\hbar^2}{2}\Delta_M + V$, with the ground-state assumption ((A1) in the paper; discrete spectrum, ground state $\psi_0 > 0$ non-degenerate and smooth, $\Delta = E_1 - E_0 > 0$), the stationary path-measure assumption ((A2) in the paper; the stationary path measure with $T = \infty$), and the metric comparison assumption ((A3) in the paper; comparison of the Langevin metric, $\kappa = 1/\lambda_G$). $m = \Delta/\hbar$, $\xi = 1/m$, $A = (H - E_0)/\hbar$.

We set the family local in time as follows.

$$\mathcal{F}_1 = \left\{ f_{h,G_1} = \int_{\mathbb{R}} h(\tau)\, G_1(q(\tau))\,d\tau \ :\ h \text{ bounded with bounded support},\ G_1 \text{ is a function on } Q \text{ that is smooth} \right\} \quad (1)$$

The f_T of the one-sided bridge is the case $h = \mathbf{1}_{[0,T]}$, $G_1 = \psi_1/\psi_0$.

The supremum of the one-sided bridge (best within the family local in time)

Under the assumptions of the one-sided bridge and the canonical choice $G = M^{-1}$ ($\kappa = 1$),

$$\sup_{f \in \mathcal{F}_1} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \xi^2 = \frac{\hbar^2}{\Delta^2}, \quad (2)$$

and the supremum is not attained (for every $f \in \mathcal{F}_1$ the ratio is strictly smaller than ξ^2).

Proved here Proof. Let ν_φ be the spectral measure of A for $\varphi = (G1 - \langle G1 \rangle)\psi 0$; ν_φ lies on $[m, \infty)$. From the formula for correlations and Plancherel's theorem,

$$\mathrm{Var}(f) = \int d\nu_\varphi(a) \int_{\mathbb{R}} |\hat{h}(k)|^2 \frac{2a}{a^2 + k^2} \frac{dk}{2\pi}, \qquad \mathcal{E}(f) = \int d\nu_\varphi(a) \int_{\mathbb{R}} |\hat{h}(k)|^2 \, 2a \, \frac{dk}{2\pi} \quad (3)$$

(the right from the identity (18)). For $a \geq m$, $2a/(a^2 + k^2) \leq 2a/m^2$, with equality only at $k = 0$, so the ratio is smaller than ξ^2. That the supremum is ξ^2 is seen by taking $T \to \infty$ in the f_T of the one-sided bridge. ∎ For a general metric, the metric comparison assumption and the reverse comparison $G \geq \lambda'_G M^{-1}$ sandwich the supremum between $\kappa\xi^2$ and ξ^2/λ'_G. Lean `EuclideanPath.GapBridge.local_rayleigh_le` (assumptions: $0 < m$, $m \leq a_k$ for each k, $0 \leq c_k$; conclusion: $m^2 \sum_k c_k/a_k \leq \sum_k c_k a_k$) and `fourier_weight_le` (assumption: $0 < m \leq a$; conclusion: $2a/(a^2 + k^2) \leq 2a/m^2$) check only the finite sums and the pointwise inequality of this proof.

Consequence. The sentence of the upper-bound assumption restricted to $\mathcal{F}_1$, "for every $f \in \mathcal{F}_1$, $\mathrm{Var}(f) \leq C\xi_{\mathrm{RG}}^2 \, \mathcal{E}(f)$", is by (2) the same as $\xi^2 \leq C\xi_{\mathrm{RG}}^2$, that is,

$$\text{The upper-bound assumption on } \mathcal{F}_1 \iff m_{\mathrm{phys}} = m\,\xi_{\mathrm{RG}} \geq C^{-1/2} \text{ (canonical, each } \beta)$$

Proved here . Right to left also follows from (2). Everything by which the upper-bound assumption is stronger than the gap of the lattice in physical units (the conclusion of 23-01 § 06) lies outside $\mathcal{F}_1$. For a free particle on a circle (the Free-Particle-on-a-Circle Counterexample), the winding-number functional $\int w \, d\tilde{x}$ lies outside $\mathcal{F}_1$ and produces $C_P = \infty$, while the supremum over $\mathcal{F}_1$ remains ξ^2.

Correspondence with the paper. The one-sided bridge, (2) and the Free-Particle-on-a-Circle Counterexample are collected in the paper (list of papers, PDF) “The inverse squared spectral gap bounds the Langevin Poincaré constant of Euclidean path measures from below” (Kiichi, Shiori and Rin). The one sentence of its abstract is "for Euclidean path measures stationary over all time on a finite-dimensional configuration space, the Poincaré constant of the stochastic quantisation (Langevin) Dirichlet form on time-averaged smooth functionals satisfies $C_P \geq \kappa(\hbar/\Delta)^2$". The one-sided bridge corresponds to Theorem 3.1 of the paper, (2) to Proposition 4.1, and the Free-Particle-on-a-Circle Counterexample to Proposition 5.1 ($C_P = \infty$ even on the family of time-averaged smooth functionals $\mathcal{D}_0$). This book only cites the equation numbers, and leaves the proofs to 23-01 § 06 and the paper. The paper has not been checked by outside mathematicians.

03

Limiting the window — monotonicity of the window factor, the supremum within the window, the lower bound in finite volume

The definition of the local relaxation-time assumption is in (1)–(3) of 24-03 § 02. We copy only the form used here. The ruler $\ell > 0$ is a length given from outside, independently of the gap (in four dimensions, the bookkeeping length ξ_{RG}). Using the ξ fixed by the gap as the ruler would be circular when deriving a lower bound on the gap. For $c > 0$, write $\mathcal{D}_c$ for the family of time-averaged functionals whose weights are supported in a window of length $c\ell$, and put

$$C_P^{\mathrm{loc}}(c) = \sup_{f \in \mathcal{D}_c} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)}, \qquad (\xi^2)\uparrow_c^{\mathrm{loc}}(K) : \;\; C_P^{\mathrm{loc}}(c) \leq K\ell^2 \quad (4)$$

$C_P^{\mathrm{loc}}(c)$ is fixed only by the marginal law of the path on the window, increases monotonically in c, and returns to the global C_P as $c \to \infty$ (24-03).

The window factor, and its monotonicity

Write $\varphi(x)$ for the bracket on the right of (17) of the one-sided bridge (the φ without argument is the vector of § 02, a different quantity; the notation follows 24-09).

$$\varphi(x) = 1 - \frac{1 - e^{-x}}{x}, \qquad g(x) = \frac{\varphi(x)}{x^2} = \frac{x - 1 + e^{-x}}{x^3} \qquad (x > 0) \ (5)$$

The ratio for a window T is $\kappa\xi^2\varphi(T/\xi) = \kappa T^2 g(mT)$. $\varphi(x) = x/2 + O(x^2)$, $\varphi(x) \to 1$ $(x \to \infty)$, and the values are $\varphi(1) = 0.368$, $\varphi(4) = 0.755$, $\varphi(10) = 0.900$ Proved here (values of the closed formula).

Window-Factor Monotonicity Proposition (the window factor decreases)

g is strictly decreasing on $(0, \infty)$.

Proved here Proof. Since $x - 1 + e^{-x} = \int_0^x \int_0^u e^{-v}\, dv\, du = x^2 \int_0^1 (1-t)e^{-xt}\, dt$, $g(x) = \frac{1}{x}\int_0^1 (1-t)e^{-xt}\, dt$. It is the product of the positive strictly decreasing function $1/x$ and the positive strictly decreasing function $\int_0^1 (1-t)e^{-xt} dt$, so it strictly decreases. ∎ Lean `LatticeGauge.LocalH.wr_strictAntiOn` (no assumptions; conclusion: g is strictly decreasing on $(0, \infty)$. In Lean it is closed via the sign of the derivative $g'(x) = h(x)/x^4$, $h(x) = 3 - 2x - (3+x)e^{-x} < 0$).

The Window-Factor Monotonicity Proposition has two uses. First, in the family where the weight is fixed to the indicator of the window $\mathbf{1}_{[0,T]}$ and only the single-time function varies, the supremum is attained exactly at the lowest excitation $a = m$ and equals $T^2 g(mT) = \xi^2 \varphi(T/\xi)$ Proved here (the ratio of weighted averages over the spectral measure does not exceed the maximum of the ratios per component; that the maximum is attained at $a = m$ is exactly the monotonicity of g). Second, it turns the window-limited upper-bound assumption $\kappa T^2 g(c\, m_{\mathrm{phys}}) \le K\ell^2$ $(T = c\ell)$ into an inequality on one number. g is a continuous strictly decreasing bijection from $(0, \infty)$ to $(0, \infty)$, so

$$\kappa\,(c\ell)^2\, g(c\, m_{\mathrm{phys}}) \le K\ell^2 \iff m_{\mathrm{phys}} \ge \frac{1}{c}\, g^{-1}\!\Big(\frac{K}{\kappa c^2}\Big) \ (6)$$

From $e^{-x} \ge 1 - x + x^2/2 - x^3/6$ $(x \ge 0)$ comes $g(x) \ge 1/(2x) - 1/6$, and the explicit form

$$m_{\mathrm{phys}} \ge \frac{3\kappa c}{6K + \kappa c^2} \ (7)$$

is obtained Proved here . Lean `LatticeGauge.LocalH.gap_lower_of_window` (assumptions: $0 < \kappa, c, K, \ell, m$, $\kappa(c\ell)^2 g(c\ell m) \le C_P$, $C_P \le K\ell^2$; conclusion: $3\kappa c/(6K + \kappa c^2) \le m\ell$). (7) is sharp for $c\, m_{\mathrm{phys}} \lesssim 1$, and for $c\, m_{\mathrm{phys}} \gtrsim 3$ the approximation on the right becomes weak, so for large c we use the next form.

Letting the weight inside the window vary too — the In-Window Supremum Proposition

In-Window Supremum Proposition (supremum of the family local in time within a window)

Under the assumptions of the one-sided bridge and the canonical choice, the supremum over elements of $\mathcal{F}_1$ whose weights are supported in $[0, T]$ is

$$\sup_{f \in \mathcal{F}_1,\ \operatorname{supp} h \subset [0,T]} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \Psi_T(m) := \frac{1}{m^2 + \omega_0^2}, \qquad \omega_0 \tan\frac{\omega_0 T}{2} = m, \quad \omega_0 \in \Big(0, \frac{\pi}{T}\Big) \ (8)$$

For a general metric (the metric comparison assumption) the left-hand side is at least $\kappa\Psi_T(m)$.

Proved here Proof [Res]. Splitting φ into eigencomponents φ_k of A (eigenvalues a_k), $\mathrm{Var} = \sum_k \|\varphi_k\|^2 \langle h, e^{-a_k|s-t|} h\rangle$ and $\mathcal{E} = 2\|h\|^2 \sum_k a_k \|\varphi_k\|^2$ (the identity (18)). The ratio of weighted averages does not exceed the maximum of the ratios per component, so $\mathrm{Var}/\mathcal{E} \le \max_k \langle h, K_{a_k} h\rangle / \|h\|^2$, where K_a is the integral operator with kernel $e^{-a|s-t|}/(2a)$ restricted to $[0, T]$, the compression of $(a^2 - \partial^2)^{-1}$ on the real line. $a \mapsto (a^2 - \partial^2)^{-1}$ is decreasing as an operator, so the maximum is at $a = m$. The largest eigenvalue of K_m is fixed, since the eigenfunction satisfies $u'' = -\omega^2 u$ with the

boundary conditions $u'(0) = m\,u(0)$ and $u'(T) = -m\,u(T)$, by $\omega\tan(\omega T/2) = m$ for the first symmetric solution $u = \cos(\omega(t - T/2))$, and its value is $1/(m^2 + \omega^2)$. Equality is attained at $G_1 = \psi_1/\psi_0$, $h = \cos(\omega_0(t - T/2))$. ∎

The root ω_0 satisfies $\omega_0^2 \le 2m/T$ by $\tan z \ge z$, and also $\omega_0 < \pi/T$, so

$$\max\Big(\frac{1}{m^2 + \pi^2/T^2},\ \frac{1}{m^2 + 2m/T}\Big) \ \le\ \Psi_T(m) \ <\ \frac{1}{m^2} \quad (9)$$

Proved here . The value for the indicator weight, $T^2 g(mT)$, is strictly smaller than $\Psi_T(m)$ (cos is not constant); as $T \to 0$ both are $\approx T/(2m)$, and as $T \to \infty$, $\Psi_T = 1/m^2 - O(T^{-2})$ and $T^2 g = 1/m^2 - O(T^{-1})$.

Table 1 Two values of the window supremum ($m = 1$)

T	Indicator weight $T^2 g(T) = \varphi(T)$	Weight also varied $\Psi_T(1)$	Grade
1	0.368	0.369	The left column is Proved here (closed formula); the right column is Reading (numerical values of the root formula, agreeing within relative 2.5×10^{-5} with the largest eigenvalue of the kernel by the midpoint rule)
4	0.755	0.775	
10	0.900	0.935	

Lower bound in finite spatial volume

Combine the local relaxation-time assumption (4) with the In-Window Supremum Proposition. From $\kappa\Psi_{c\ell}(m) \le C_P^{\mathrm{loc}}(c) \le K\ell^2$ and $\Psi_{c\ell}(m) \ge 1/(m^2 + 2m/(c\ell))$ of (9), $\mu = m\ell = m_{\mathrm{phys}}$ satisfies $\mu^2 + 2\mu/c \ge \kappa/K$, and

$$m_{\mathrm{phys}} \ \ge\ \sqrt{\frac{1}{c^2} + \frac{\kappa}{K}} - \frac{1}{c} \qquad (c > 0) \quad (10)$$

Proved here (under the assumptions of the one-sided bridge). Lean `LatticeGauge.LocalH.gap_lower_of_local_window` (assumptions: $0 < \kappa, c, K, \ell, m$, $\kappa/(m^2 + w^2) \le CP$, $w^2 \le 2m/(c\ell)$, $CP \le K\ell^2$; conclusion: (10)), `local_bound_tendsto` (no assumptions; conclusion: the right-hand side of (10) tends to $\sqrt{\kappa/K}$ as $c \to \infty$). As $c \to \infty$ it returns to (19) of 23-01, and as $c \to 0$ it is $\approx c\kappa/(2K)$. (10) is stronger than (7) for every c.

For the free field, both directions. For a Gaussian path measure the Poincaré constant equals the norm of the covariance operator Cited (Hairer–Stuart–Voss–Wiberg 2005, Lemma 2.2), and $C_P^{\mathrm{loc}}(c) = \Psi_{c\ell}(m)$ holds exactly even over the whole window family $\mathcal{D}_c$ Proved here (the largest eigenvalue of the covariance compressed to the window is the value of the In-Window Supremum Proposition). Hence for the free field, besides "$(\xi^2)\uparrow_c^{\mathrm{loc}}(K) \Rightarrow$ (10)", also "$m_{\mathrm{phys}} \ge K^{-1/2} \Rightarrow (\xi^2)\uparrow_c^{\mathrm{loc}}(K)$ for every c" holds ($\Psi < 1/m^2$). One also sees that localisation is not trivial: $C_P^{\mathrm{loc}}(c) < CP$ holds strictly for every finite c, and as $c \to 0$ it is $\approx c\ell/(2m) \to 0$.

Conditions of this section. (4)–(10) are sentences on a lattice with continuous time, a finite spatial lattice, $T = \infty$, and the three assumptions of the one-sided bridge (ground state, stationary path measure, metric comparison). On Wilson's lattice, which also discretises time, the time-step comparison assumption ((20) of 23-01) remains, and the window version has not yet been written on top of it.

04

"Some $K(c)$" is empty — the massless free field as a negative control, and the ratio form

Now go local in space too. On an infinite spatial lattice, write $C_P^{\mathrm{loc}}(B_R)$ for the local Poincaré constant with only functionals supported in a spacetime box B_R of side $R = c\ell$ allowed to vary (the version of 24-03 § 03 that is local in space too; (11) below is the same inequality as (4) of 24-03). The plain local relaxation-time assumption is "$C_P^{\mathrm{loc}}(B_{c\ell}) \le K(c)\,\ell^2$". But in four dimensions this form says nothing about the gap.

Negative control (the massless free field)

For the massless free scalar field in four-dimensional spacetime and infinite volume (canonical graduation, $\hbar = 1$),

$$\frac{R^2}{4\pi^2} \ \le\ C_P^{\mathrm{loc}}(B_R) \ \le\ R^2. \quad (11)$$

Hence "$C_P^{\mathrm{loc}}(B_{c\ell}) \le K(c)\ell^2$" holds with $K(c) = c^2$, and the gap is 0.

Proved here Proof. The Gaussian Poincaré constant is the norm of the covariance compressed to the box Cited (previous section). The covariance $(-\Delta)^{-1}$ is transient in four dimensions and hence finite, with continuum kernel $1/(4\pi^2|z|^2)$. Above, by the Schur estimate, the norm is at most $\sup_x \sum_{y\in B_R} G(x-y) \le \int_{|z|<2R} \frac{d^4z}{4\pi^2|z|^2} = R^2$ (the asymptotics of the lattice kernel replaced by the continuum kernel; this step is Assumption). Below, take the Dirichlet ground state w of the box as test direction; from $\langle w, Gw\rangle\langle w, -\Delta w\rangle \ge \|w\|^4$ (Cauchy–Schwarz), $C_P^{\mathrm{loc}} \ge 1/\lambda_D(B_R) \approx R^2/(4\pi^2)$ (π^2/R^2 for each of the four directions of the spacetime box). ∎

In numbers, the ratio $\Gamma(R) := C_P^{\mathrm{loc},0}(B_R)/R^2$ decreases as $0.0802,\ 0.0768,\ 0.0756,\ 0.0750,\ 0.0747,\ 0.0744$ at $R = 4, 6, 8, 10, 12, 16$; the correction is $1/R^2$ with no logarithmic correction, and by extrapolation $\Gamma_\infty = 0.0740$ Reading (floating point; numerical integration of the lattice heat kernel agreeing with an independent route via periodic lattices within relative 2×10^{-3}; extrapolation from $R \le 16$; one component). This is inside the paper sandwich $[1/(4\pi^2), 0.1125]$, 2.92 times the Dirichlet value $1/(4\pi^2) = 0.0253$. Since the continuum kernel is homogeneous of degree -2, the compression being exactly proportional to R^2 is as expected.

One line of contrast with lower dimensions as well. For a massless field in two spacetime dimensions the infrared diverges and $C_P^{\mathrm{loc}} = \infty$, and even in finite spatial volume with only the time window localised, the constant mode grows as $\approx T/(2m)$. A massless field has a finite local constant only in spacetime dimension three or more when localised in space too, and the four-dimensional setting of Part 24 is exactly that case Proved here .

The ratio form

So we state the local relaxation-time assumption as a ratio to the massless free field (the reference) on the same lattice, the same box and the same graduation.

$$K_{\mathrm{rel}}(c) = \frac{C_P^{\mathrm{loc}}(B_{c\ell};\mu)}{C_P^{\mathrm{loc},0}(c\ell)}, \qquad C_P^{\mathrm{loc},0}(R) = \Gamma(R)\,R^2, \qquad (\xi^2)^{\mathrm{rel}}_{\uparrow c}(K):\ \ K_{\mathrm{rel}}(c) \le K \quad (12)$$

(uniformly in $\beta \ge \beta_0$, $\ell = \xi_{\mathrm{RG}}(\beta)$). The constant K becomes an absolute number. We set it as **a sentence at a single** c, and do not require it uniformly in c. The ratio $r(mR)$ for the free field of mass m tends to 1 as $mR \to 0$, so a sentence requiring $K < 1$ uniformly down to short distances becomes false for theories close to free fields Estimate . Requiring a K uniform in c returns to the global upper-bound assumption as $c \to \infty$ (24-03).

For a field with a convex interaction added (action $\frac12\langle\Phi, -\Delta\Phi\rangle + \sum_x U(\Phi_x)$, U convex), the Hessian is at least $-\Delta$, so the Brascamp–Lieb inequality gives $K_{\mathrm{rel}} \le 1$ Assumption (the bibliographic data of the inequality have not been checked). Convex fields lie in the band $[0, 1]$. What the bridge needs, as the next section shows, is the part of it below the threshold.

05

The bridge in infinite volume and the threshold — the local-excitation moment assumption and transfer to the ratio

In infinite volume, the single-time function $F = \psi 1/\psi 0$ is neither local nor explicit. Instead we assume that local observables in a box produce low-energy excitations.

Local-excitation moment assumption

There is a local observable O supported in a spatial box of side $c\ell$ such that the spectral measure ν_φ, with respect to the generator of the transfer A, of $\varphi = (O - \langle O\rangle)\Omega$ (Ω is the vacuum) satisfies

$$\frac{M_3}{M_1} := \frac{\int a^3\, d\nu_\varphi(a)}{\int a\, d\nu_\varphi(a)} \ \le\ m_\infty^2 + \frac{\gamma}{(c\ell)^2} \quad (13)$$

m_∞ is the infinite-volume gap in lattice units, and γ a constant independent of β.

Assumption It has not been shown for any interacting model. Assumption (13) contains the quantity of the conclusion m_∞, but it appears only in a comparison from above and does not imply $m_\infty > 0$. It is a sentence that has meaning even for $m_\infty = 0$ (a property of approximation, that low-energy excitations can be approached by local observables), and the massless free field satisfies it with $\gamma = d\pi^2$.

Proved here It holds for the free field. On one-particle states $\nu_\varphi = \sum_p |\hat{w}_p|^2/(2\omega_p)\,\delta_{\omega_p}$, $M_3/M_1 = m_0^2 + \langle w, -\Delta w\rangle/\|w\|^2 \le m_0^2 + d\pi^2/(c\ell)^2$ (w is the Dirichlet ground state of the spatial box, d the spatial dimension).

By Jensen's inequality, the ratio for a window T is bounded as $\ge \kappa/(M_3/M_1 + \pi^2/T^2)$. Treating the spacetime box isotropically and putting $\lambda_W := \gamma + \pi^2$,

$$C_P^{\mathrm{loc}}(B_R) \;\ge\; \frac{\kappa}{m_\infty^2 + \lambda_W/R^2} \quad (14)$$

and from this and the local relaxation-time assumption $C_P^{\mathrm{loc}} \le K\ell^2$, $m_{\mathrm{phys}}^2 \ge \kappa/K - \lambda_W/c^2$ Proved here (on top of (13) and the infinite-volume ground-state transformation identity). Lean `LatticeGauge.LocalH.gap_lower_dirichlet` (assumptions: $0 < \kappa, c, K, \ell$, $0 < m^2 + \lambda$, $\kappa/(m^2+\lambda) \le C_P$, $\lambda \le \gamma/(c\ell)^2$, $C_P \le K\ell^2$; conclusion: $\kappa/K - \gamma/c^2 \le (m\ell)^2$).

Transferring to the ratio form (12), from $C_P^{\mathrm{loc}} \le K_{\mathrm{rel}}\Gamma R^2$ and (14),

$$(m_\infty R)^2 \;\ge\; \lambda_W\Big(\frac{K_*}{K_{\mathrm{rel}}} - 1\Big), \qquad K_* := \frac{\kappa}{\lambda_W\,\Gamma} \quad (15)$$

Proved here (algebra on top of (13) and the infinite-volume identity). Lean `LatticeGauge.LocalHReference.ratio_bridge` (assumptions: $0 < \lambda_W, \Gamma, R, K_{\mathrm{rel}}$, $\kappa/(m^2 + \lambda_W/R^2) \le C$, $C \le K_{\mathrm{rel}}\Gamma R^2$; conclusion: (15)), `kstar_correspondence` (assumptions: $0 < \lambda_W, \Gamma, c$; conclusion: when $K(c) = K_{\mathrm{rel}}\Gamma c^2$, $K(c) < \kappa c^2/\lambda_W \iff K_{\mathrm{rel}} < K_*$), `ratio_bridge_vacuous_of_ge` (assumption: $K_* \le K_{\mathrm{rel}}$; conclusion: the right-hand side of (15) ≤ 0).

How to read it. If $K_{\mathrm{rel}} < K_*$, the right-hand side is positive, and the lower bound on the mass

$$m_\infty\,\ell \;\ge\; \frac{\sqrt{\lambda_W}}{c}\sqrt{\frac{K_*}{K_{\mathrm{rel}}} - 1} \quad (16)$$

follows. If $K_{\mathrm{rel}} \ge K_*$, (15) is empty. A reference satisfies its own Dirichlet lower bound ($\Gamma \ge 1/\lambda_W$, the same Cauchy–Schwarz as on the left of (11)), so for every reference $K_* \le \kappa$ Proved here . Lean `LatticeGauge.LocalHReference.kstar_le_kappa` (assumptions: $0 \le \kappa$, $0 < \lambda_W, \Gamma$, $1 \le \lambda_W\Gamma$; conclusion: $K_* \le \kappa$). For a field with ratio 1 ($K_{\mathrm{rel}} = 1$) the bridge does not stand (`rel_one_no_bridge`, negative control).

The λ_W in (15) needs care. λ_W is "the value of M_3/M_1 attainable within the allowed family of local observables", a number fixed once the family of test observables is chosen. The reference Γ in the denominator and the λ_W of the lower bound in the numerator must be taken over the same family. In the next section, this point moves the threshold.

06

Three references — scalar, gauge quotient, colour singlet

Table 2 sets out the values for each choice of reference. Every row is a number for a free-field reference; not a single ratio K_{rel} for the target Wilson measure has been measured.

Table 2 Three references and thresholds (four dimensions, canonical graduation; numbers are readings from floating-point computation and extrapolation)

Reference	Family of test observables	Γ_∞	λ_W (times R^2)	$\lambda_W\Gamma$	Threshold K_*/κ	Box at which the bridge sees the mass	Conditions

Reference	Family of test observables	Γ_∞	λ_W (times R^2)	$\lambda_W\Gamma$	Threshold K_*/κ	Box at which the bridge sees the mass	Conditions
Scalar	All functions of the field in the box (best is linear)	0.0740	$4\pi^2 \approx 39.5$	2.92	0.342	$mR \approx 4.2$	One component, extrapolation from $R \le 16$ Reading
Gauge quotient	Quotient by gauge transformations at **all** sites of the box = functions of the plaquette angles of the box	0.0277	72.0	1.99	0.502	$mR \approx 5.2$ (transverse Proca waves)	Gaussian for one colour component, $\beta = 1$, extrapolation from $R \le 16$ ($R \le 14$ for λ_W) Reading
Colour singlet of $SU(2)$	Colour singlets (leading term is the second chaos), a family including loops the size of the span of the box	$\Gamma_Q/2 \approx 0.0138$	115–117 (test value)	About 1.6	About 0.62 (lower value)	—	Leading term ($\beta \to \infty$), independent Gaussians for three colours, first-order test, extrapolation from $R \le 10$ Reading

Lean `LatticeGauge.LocalHReference.kstar_measured` (assumption: measured $\Gamma \in [0.0739, 0.0741]$; conclusion: $0.34 < K_* < 0.344$), `LatticeGauge.GaugeReference.kstar_gauge_measured` (assumptions: measured $\Gamma_Q \in [0.02762, 0.02772]$, $\lambda_W^Q \in [71.8, 72.1]$; conclusion: $0.5 < K_*^Q < 0.505$), `LatticeGauge.WilsonLoopReference.kstar2_measured` (assumptions: measured $\lambda^{(2)} \in [115, 118]$, $\Gamma_Q \in [0.02762, 0.02772]$; conclusion: $0.61 < K_*^{(2)} < 0.63$). All three are arithmetic with measured numbers as assumptions; the numbers themselves are outside Lean.

Why the threshold changes with the reference

Gauge-fixed values are not references. The value for the gauge-fixed free gauge field (Feynman gauge) is $\Gamma = 0.0740$, the same as the scalar, but it changes when the gauge-fixing parameter α is changed, and when the fixing is removed ($\alpha \to \infty$) it becomes infinite in the directions of gauge transformations within the box Proved here (with test functions in the directions of the image of the gauge, $\Gamma_a(\alpha) \ge \alpha/(\lambda_D^{\mathrm{int}} R^2)$). The target Wilson measure is not gauge-fixed, so the numerator of the ratio is finite only on gauge-invariant functions. The denominator is measured on the same space. Lean `LatticeGauge.GaugeReference.gauge_param_unbounded` (assumptions: $0 < \lambda_D, R$, $\alpha/(\lambda_D R^2) \le \Gamma_a(\alpha)$; conclusion: Γ_a is unbounded in α).

Gauge-invariant observables can only have high momentum. The test space of the gauge quotient is the divergence-free fields (transverse waves) on the graph of the box. If the box is contractible, this coincides exactly with the space of functions of the plaquette angles of the box Proved here (the first homology of the cubical complex is 0, so closed paths are boundaries of sums of plaquettes). Under the divergence-free constraint one cannot build a positive field pointing the same way over the whole box (the optimal test direction for the scalar). The optimal direction becomes a transverse wave of large momentum: $\lambda_W^Q = 72.0$ is 1.82 times the scalar $4\pi^2$, and Γ_Q is 0.374 times the scalar Reading. The threshold of the ratio relaxes, $0.342 \to 0.502$, but this is because Γ shrinks by more than λ_W grows ($0.502/0.342 = 1/(0.374 \times 1.82)$); the condition rewritten for the local Poincaré constant itself, $C_P^{\mathrm{loc}} < \kappa R^2/\lambda_W$, tightens from $0.0253\kappa R^2$ to $0.0139\kappa R^2$ Reading.

Divide by the gauge transformations at the boundary of the box too. If one divides only by gauge transformations at inner sites of the box and freezes the gauge at the boundary sites, flux passing uniformly through the box enters the test space, and the value returns roughly to the gauge-fixed value, 2.2–2.5 times Γ_Q Reading. Gauge-invariant observables of the box (Wilson loops) are invariant under transformations at all sites, so "dividing by gauge transformations at all sites" is part of the definition of the reference (the row "flux through a box" in 24-01).

What the one-colour approximation tacitly assumed. Linearising to one colour component, linear combinations of plaquette angles inside the box alone span the whole transverse test space, and neither large loops nor plaquettes reaching outside the box are needed Proved here (the angle of a rectangular loop is the sum of the angles of the

plaquettes inside). But the gauge invariants of $SU(2)$ (colour singlets) have no linear term in the expansion around the identity Proved here . A linear term $\sum c_\ell^a A_\ell^a$ must also be invariant under constant gauge transformations (global colour rotations $SO(3)$), and $SO(3)$ has no fixed vector in $\mathbb{R}^3$, so $c = 0$. The trace of a loop $\frac{1}{2}\operatorname{tr} U_C = \cos(|\theta_C|/2) = 1 - |\theta_C|^2/8 + \cdots$ starts at second order in the angles. Lean `LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: a linear form invariant under the two rotations $\operatorname{diag}(-1,-1,1)$, $\operatorname{diag}(1,-1,-1)$; conclusion: the coefficients are 0).

The reference is exactly half. In the leading term ($\beta \to \infty$, independent Gaussians for three colours), functions that are colour singlets lie in the second chaos and above. The eigenvalues of the Ornstein–Uhlenbeck generator of a Gaussian measure are sums of one-particle eigenvalues in the eigencoordinates of the covariance, so the minimum on the second chaos is twice the one-particle minimum, and the Poincaré constant is $\Gamma_Q/2$ Proved here (equality at $f_w = \sum_a (\langle w, A^a\rangle^2 - \mathbb{E})$, with w the optimal transverse direction). The value of the first-order test is $\lambda_W(f_w) = \lambda(w) + 1/C(w)$, and by Cauchy–Schwarz $\lambda(w) < \lambda_W(f_w) \le 2\lambda(w)$ Proved here . In the four-dimensional box $\lambda^{(2)}R^2 \to 115$ to 117 (about 1.6 times λ_W^Q), and the threshold is $K_*^{(2)} = 2/(\lambda^{(2)}\Gamma_Q) \approx 0.62\kappa$ Reading . $\lambda^{(2)}$ is a test value (a value from above), so the threshold obtained is a lower value of the usable threshold. Lean `chaos2_gamma`, `chaos2_lambda` (assumptions: $0 < n, g$ and the Wick moments as formulas; conclusion: algebra of the ratio), `chaos2_bracket` (assumptions: $0 < C$, $1 \le \lambda C$; conclusion: $\lambda < \lambda + 1/C \le 2\lambda$).

Large loops are needed. To get this value, loops the size of the span of the box are needed. With the family of plaquettes only, $\{\operatorname{tr} U_p\}$, the reference in the leading term vanishes as $1/R^2$ (in two dimensions, from 0.29 times to 0.035 times $\Gamma_Q/2$ for $R = 4 \to 12$), and even adding loops up to 2×2 it is 0.12 times ($R = 12$), so the bridge speaks only of the lattice scale Reading . Including all rectangles spanning the box, it agrees with $\Gamma_Q/2$ to all digits in two dimensions, while flat rectangles alone give 0.73 times in three dimensions Reading . There are no values for large loops in four dimensions (including non-flat ones and products of lassos).

Box at which the bridge sees the mass

The ratio for the free field of mass m, $r(mR) = C_P^{\mathrm{loc}}(R;m)/C_P^{\mathrm{loc}}(R;0)$, collapses within 0.3% for $R = 4$ to 12 onto a function of mR alone Reading . For the scalar $r = 0.932,\ 0.825,\ 0.623,\ 0.359,\ 0.0772$ ($mR = 0.5, 1, 2, 4, 12$); for the transverse waves of the gauge quotient (Proca) $r_Q = 0.950,\ 0.842,\ 0.721,\ 0.610,\ 0.514,\ 0.434$ ($mR = 1, \ldots, 6$). The ratio falls to the threshold at $mR \approx 4.2$ for the scalar and $mR \approx 5.2$ for transverse waves (both by linear interpolation) Reading . **The bridge sees the mass only when the side of the box is at least 4–5 times the correlation length.** Transverse waves have large momentum, so they notice the mass later.

Put the other way, for fields whose ratio lies in the band $[K_*, 1)$, this bridge says nothing. Free fields of mass $mR < 4.2$ (with the scalar reference) also fall in this band, so the bridge misses fields that do have mass. Whether there is another argument (outside Jensen's inequality and Dirichlet test functions) that sees the mass for fields in the band is not settled.

07

Note on 0+1 dimensions — even for the shortest window the ratio is ω_b/m

We try the ratio form on the 0+1-dimensional double well $V(x) = (x^2-1)^2/8$ ($V''(\pm 1) = 1 = \omega_b^2$, $S_0 = 2/3$). In 0+1 dimensions there is no finite massless reference (a free particle has no stationary measure and $C_P^{\mathrm{loc}} = \infty$). There is no choice but to take as reference the harmonic oscillator ω_b with the curvature of the bottom, and the ratio sentence becomes "if $K_{\mathrm{rel}}(T) < \kappa$ then $m > \omega_b$" Proved here (from the lower bound of the In-Window Supremum Proposition and the decrease of Ψ_T in m. The threshold is κ, and its difference from the four-dimensional 0.342κ is exactly the Dirichlet loss of the spatial box). Lean `LatticeGauge.LocalHReference.gap_gt_ref_of_ratio` (assumptions: Ψ is monotone non-increasing, $0 < \kappa$, $0 < \Psi(\omega_b)$, $\kappa\Psi(m) \le C \le K\Psi(\omega_b)$, $K < \kappa$; conclusion: $\omega_b < m$).

The window of a single time can be computed in closed form. On a lattice with time step ε, the Langevin generator for the single-time marginal ψ_0^2 and $\mathcal{E}_1(f) = (\hbar/\varepsilon)\int \psi_0^2 |f'|^2$ is, by the ground-state transformation,

$$L = -\frac{2}{\varepsilon\hbar}\,\psi_0^{-1}(H - E_0)\,\psi_0, \qquad C_P^{\mathrm{loc}}(1) = \frac{\varepsilon}{2m}, \qquad K_{\mathrm{rel}}(1) = \frac{\omega_b}{m} \quad (17)$$

Proved here (when ψ_0 is the Schrödinger ground state; the reference is $\varepsilon/(2\omega_b)$). This is exactly the identity of 24-05. Lean `LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumptions: $0 < \varepsilon, m, \omega$; conclusion: $(\varepsilon/(2m))/(\varepsilon/(2\omega)) = \omega/m$).

The behaviour as the window is lengthened divides, on top of the In-Window Supremum Proposition, into three regimes Proved here .

Table 3 The ratio $K_{\mathrm{rel}}(T)$ for the 0+1-dimensional double well (three regimes of the value $\Psi_T(m)/\Psi_T(\omega_b)$ of the $\mathcal{F}_1$ part)

Window	Ratio	Order for $m \propto e^{-S_0/\hbar}$
$T \to 0$ (a single time)	ω_b/m	$e^{S_0/\hbar}$
$1/\omega_b \ll T \ll \xi$	$\omega_b^2 T/(2m)$	$T\, e^{S_0/\hbar}$
$T \gg \xi$	$(\omega_b/m)^2$	$e^{2S_0/\hbar}$

The crossover is the window at which $\Psi_T(m)$ becomes half of $1/m^2$: $\omega_0 = m$, that is $\tan(mT/2) = 1$, $T_\times = \pi/(2m)$ Proved here . Lean `LatticeGauge.LocalHDoubleWell.crossover_root`, `crossover_half` (assumption: $0 < m$; conclusion: at $T = \pi/(2m)$, $m\tan(mT/2) = m$ and the value is half. Uniqueness of the root is outside Lean).

The lattice values (time step $\varepsilon = \frac{1}{2}$, transfer kernel as Trotter's symmetric product, $\hbar = 0.3,\ 0.15,\ 0.08$) are $K_{\mathrm{rel}}(1) = 4.61,\ 21.6,\ 638$ and $K_{\mathrm{rel}}(1)\cdot m = 1.04,\ 1.04,\ 1.05$ for the window of a single time Reading (the deviation from (17) comes from ψ_0 of the time-discretised transfer and the discretisation of the reference). The crossover on the lattice is $k_\times m\varepsilon = 1.571$ to 1.574 ($\pi/2 = 1.5708$) Reading . Extending the functionals in the window from sums of one-point functions to multi-point functions raises the ratio further: the direct value for a window of two times is 1.02, 1.09, 1.34 times the $\mathcal{F}_1$ value, and the lower bound for a window of eight times with a family up to three-point products is at least 1.07, 1.27, 2.12 times Reading (lower bounds; the smaller $\hbar$, the more they rise with each widening of the family, and they have not saturated).

The ratio sentence "if $K_{\mathrm{rel}} < \kappa$ then $m > \omega_b$" correctly fails for the double well, whose gap is small because of tunnelling ($K_{\mathrm{rel}} \geq 4.6 > \kappa = 1$ for every window). 0+1 is not an analogue of the four-dimensional ratio but a negative control showing that the ratio form responds correctly to non-convexity.

What to take from this note to four dimensions. The non-convexity of the double well does not depend on the length of the window; it shows up in the ratio from the shortest window, as the bimodality of the single-time distribution. Hence "at short distance the ratio is close to 1" (the reason § 04 did not require uniformity in c) is not a fact that follows from the ratio form. Expecting it in four dimensions is an assumption depending on asymptotic freedom, that the theory is close to a free field at short distance Assumption .

08

Which part of which assumption each conclusion needs

Table 4 The part of the relaxation-time assumption each conclusion needs, and the other assumptions needed

Conclusion	Part of the relaxation-time assumption needed	Other assumptions needed	Grade
Lower bound on the gap of the lattice in physical units, in finite spatial volume, at each β	The local relaxation-time assumption on a window $c\ell$ ((4)), or the upper-bound assumption restricted to $\mathcal{F}_1$	The assumptions of the one-sided bridge (continuous time, finite spatial lattice, $T = \infty$). On Wilson's lattice with discretised time, the time-step comparison assumption remains, and the window version has not been written	Proved here ((6)(10)), Assumption (the time-step comparison assumption)
Lower bound on the gap of the lattice in physical units in infinite volume	$K_{\mathrm{rel}}(c) < K_*$ at a single c (ratio form (12))	the local-excitation moment assumption, the infinite-volume ground-state transformation identity, κ uniform in β, test observables from the family of colour singlets (including loops spanning the box)	Assumption
Speed of passage between sectors	The sector diffusive-relaxation-time assumption and the quantity between sectors C_{trans} (24-03)	Unrelated to the mass gap	Estimate

Conclusion	Part of the relaxation-time assumption needed	Other assumptions needed	Grade
The upper-bound assumption itself	All f	The stage of closed boxes (24-03)	Assumption (in the four-dimensional periodic box, on the side that reads as false on the numbers of original sources for $SU(3)$ and above and a reading that extends a conjecture)

The point of Table 4 is that **the conclusion of no row needs the global upper-bound assumption**. What the one-sided bridge uses is only the part of the upper-bound assumption local in time (finite volume), or the ratio form of the local relaxation-time assumption (infinite volume). The rest of the upper-bound assumption — the sentence about functions not local in time — is where quantities like the topological charge can obstruct the upper-bound assumption (24-01), and the bridge does not pass through there Proved here (the first row of Table 4), Estimate (the reading of the second row onwards).

The Jaffe–Witten exposition of the problem writes that "new ideas are needed to prove a mass gap uniform in the volume of spacetime" Cited . The local relaxation-time assumption ties the size of the box to the bookkeeping length and stands as a sentence independent of the volume, so it is one form of stating "uniform in the volume" in the language of Poincaré Estimate . But there are no tools to show it.

09

What cannot be said

- The local-excitation moment assumption has not been shown for any interacting model. The infinite-volume ground-state transformation identity is also an assumption. The infinite-volume bridge (15)(16) is algebra on top of these two.
- The values of Table 2 are readings from floating-point computation and extrapolation ($R \le 16$ for the scalar and transverse waves, $R \le 10$ for the colour singlet). 0.62κ is a threshold obtained from the value of a first-order test, a lower value of the usable threshold. The colour singlet is in the approximation of the leading term ($\beta \to \infty$) with independent Gaussians for three colours; there are no values for $SU(2)$ at finite β. Nor are there values for large loops in four dimensions.
- "The bridge sees the mass only when the box is at least 4–5 times the correlation length" is a linear interpolation of the ratios for massive **free fields**. There are no values of the ratio for interacting fields.
- For fields in the band $[K_*, 1)$, the bridge says nothing. Another argument that sees the mass within the band is undecided.
- "At short distance the ratio is close to 1" is an assumption depending on asymptotic freedom; it does not follow from the ratio form (it is false for the 0+1 double well).
- The window argument consists of sentences on a lattice with continuous time; on Wilson's lattice with discretised time, the time-step comparison assumption remains.
- Every conclusion goes only as far as a lower bound on the gap of the lattice in physical units at each β. The limit of $m_{\mathrm{phys}}(\beta)$, the continuum limit, and the existence of the infinite-volume measure do not follow. None of the mass gap, the relaxation-time assumption, the upper-bound assumption or the local relaxation-time assumption has been shown.

10

Table of the five grades and the dependency graph

We set out the rows of this chapter in the format of the five grades (23-00 § 01) with Estimate added. The symbols of the rows are the node numbers of the dependency graph below.

Table 5 Rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Existence and uniqueness of the infinite-volume measure	Assumption	23-01 § 02	$\beta \ge \beta_0$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	The ground-state assumption, the stationary path-measure assumption, the metric comparison assumption: discrete spectrum, stationary measure with $T = \infty$, comparison of metrics	Assumption	23-01 Table 6	Continuous time, finite spatial lattice	—
3	Properties of the local C_P (fixed by the window marginal alone, monotone in c, C_P as $c \to \infty$)	Proved here	24-03	Continuous time	`local_le_global`, `local_approaches_global` (assumptions: bounded, non-empty, union of the families is the whole)
4	$(\xi^2)\uparrow_c^{\mathrm{rel}}(K)$: $K_{\mathrm{rel}}(c) \le K$ at a single c	Assumption	(12)	—	—
5	$(\xi^2)\uparrow_c^{\mathrm{loc}}(K)$: $C_P^{\mathrm{loc}}(c) \le K\ell^2$ (finite spatial volume)	Assumption	(4)	—	—
6	The local-excitation moment assumption: moment ratio of excitations by local observables	Assumption	(13)	—	—
7	Infinite-volume ground-state transformation identity	Assumption	§ 05	—	—
8	κ uniform in β	Assumption	23-01 § 06	Expected to hold in the field graduation	—
9	Values of the three references Γ_∞, λ_W (Table 2)	Reading	§ 04, § 06	Free field, extrapolation	—
10	Identity of the ground-state transformation	Proved here	23-01 (18)	Finite spatial lattice	—
11	The one-sided bridge and (2): the supremum over $\mathcal{F}_1$ is exactly ξ^2	Proved here	23-01 (17), (2) (3)	Continuous time, canonical	`poincare_lower_of_gap` (assumptions: $0 < p, m, \kappa, E$, $2p/m \le V$, $E \le 2mp/\kappa$, $V/E \le C_P$), `local_rayleigh_le`
12	Window-Factor Monotonicity Proposition: $g(x) = \varphi(x)/x^2$ is strictly decreasing	Proved here	§ 03	$x > 0$	`wr_strictAntiOn` (no assumptions)
13	In-Window Supremum Proposition: the window supremum $\Psi_T(m)$	Proved here	(8)	Continuous time, canonical	— (existence of the root is outside Lean)
14	Window lower bound (7)	Proved here	(6)(7)	Finite spatial volume	`gap_lower_of_window` (assumptions beside (7))
15	Window lower bound (10)	Proved here	(10)	Finite spatial volume	`gap_lower_of_local_window` (assumptions beside (10))
16	The massless free field satisfies the "some $K(c)$" sentence ((11))	Proved here	§ 04	Four-dimensional spacetime, infinite volume (asymptotics of the lattice kernel is an assumption)	`dirichlet_bound_vacuous` (assumptions: $0 < \ell$, $\kappa c^2/\gamma \le K$; conclusion: the assumptions of this argument are compatible with $m = 0$)
17	The ratio bridge (15)	Proved here	§ 05	On top of rows 6, 7	`ratio_bridge`, `kstar_le_kappa`
18	Thresholds 0.342κ, 0.502κ, about 0.62κ (lower value)	Reading	Table 2	Free field, extrapolation	`kstar_measured`, `kstar_gauge_measured`, `kstar2_measured` (measured intervals assumed)
19	Colour singlets have no linear term; the reference is half	Proved here	§ 06	Leading term, independent Gaussians for three colours	`no_linear_singlet`, `chaos2_gamma`
20	The ratio sentence in 0+1 and the single-time ratio (17)	Proved here	§ 07	0+1, Schrödinger ψ_0	`gap_gt_ref_of_ratio`, `one_slice_ratio`
21	"At short distance the ratio is close to 1"	Assumption	§ 07	Depends on asymptotic freedom	—
22	Lower bound on the gap in finite spatial volume	Proved here	First row of Table 4	On top of rows 2, 5	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
23	Lower bound on the gap of the lattice in physical units in infinite volume	Proved here	Second row of Table 4	On top of rows 1, 4, 6, 7, 8	—

The namespaces of the Lean theorems are `LatticeGauge.LocalH` (rows 3, 12, 14, 15, 16), `LatticeGauge.LocalHReference` (row 17, the first of row 18, the first of row 20), `LatticeGauge.GaugeReference` (the second of row 18), `LatticeGauge.WilsonLoopReference` (the third of row 18, row 19), `LatticeGauge.LocalHDoubleWell` (the second of row 20), `EuclideanPath.GapBridge` (row 11). Every bundle guarantees only the algebra, arithmetic and one-variable analysis by which the conclusion follows from the stated assumptions; the path measure, the identities, the In-Window Supremum Proposition and the numerical values of the references are outside Lean (23-13).

Dependency graph (premise numbers are smaller than conclusion numbers; 1, 3, 4, 16 correspond to 1 (existence of the infinite-volume measure), 6 (properties of the four-dimensional local Poincaré constant), 12 (the local relaxation-time assumption) and 7 (vacuity of the absolute form) of the dependency graph of 24-03):

```
 1  ASM  existence and uniqueness of the infinite-volume measure      β ≥ β₀
 2  ASM  the three assumptions of the one-sided bridge
         (ground state, stationary path measure, metric comparison)
         continuous time, finite spatial lattice, T=∞
 3  PRV  local C_P: fixed by the window marginal, monotone in c,
         C_P as c→∞                                          Lean local_le_global
 4  ASM  local relaxation-time assumption (ratio form):
         K_rel(c) ≤ K at a single c
 5  ASM  local relaxation-time assumption (finite-volume form):
         C_P^loc(c) ≤ K ℓ²
 6  ASM  local-excitation moment assumption (13)
 7  ASM  infinite-volume ground-state transformation identity
 8  ASM  κ uniform in β
 9  RDG  reference values Γ∞, λ_W (Table 2)
10  PRV  ground-state transformation identity ((18) of 23-01)
11  PRV  one-sided bridge: supremum over F₁ is exactly ξ² ⟸ 2, 10     Lean poincare_lower_of_gap
12  PRV  window-factor monotonicity: g = φ/x² strictly decreasing     Lean wr_strictAntiOn
13  PRV  supremum within the window: Ψ_T(m) ⟸ 2, 10
14  PRV  window lower bound (7) ⟸ 5, 11, 12                          Lean gap_lower_of_window
15  PRV  window lower bound (10) ⟸ 5, 13                             Lean gap_lower_of_local_window
16  PRV  negative control: the massless free field satisfies "some K(c)" (11)
17  PRV  ratio bridge (15) ⟸ 3, 4, 6, 7                              Lean ratio_bridge
18  RDG  threshold K* = 0.342κ / 0.502κ / about 0.62κ ⟸ 9            Lean kstar_*_measured
                                                        (measured intervals assumed)
19  PRV  singlet: no linear term, reference is half                 Lean no_linear_singlet, chaos2_gamma
20  PRV  0+1: K_rel < κ ⇒ m > ω_b, single-time ratio ω_b/m ⟸ 10, 13
21  ASM  short distance: "the ratio is close to 1"
         (depends on asymptotic freedom; false for row 20 in 0+1)
22  PRV  lower bound on the lattice gap in physical units,
         finite volume, at each β ⟸ 14, 15
23  PRV  lower bound on the lattice gap in physical units,
         infinite volume (conditional) ⟸ 1, 8, 17, 18, 19
```

The check for circles is only to see that on every "⟸" the numbers on the right are smaller than the number on the left. Among the assumption rows, only the local-excitation moment assumption contains the quantity of the conclusion (the gap m_∞), and there m_∞ appears only in a comparison from above and does not imply $m_\infty > 0$, as checked in § 05. The node of the massless free field is a negative control that does not enter any conclusion; it records why the form "some $K(c)$" was discarded. "At short distance the ratio is close to 1" is an assumption, and the two gap lower bounds of the conclusions do not depend on it (because c was fixed to a single value).

What this chapter can say

Said	Not said
That the supremum over the family local in time $\mathcal{F}_1$ is exactly ξ^2 in the canonical graduation, and that the upper-bound assumption on $\mathcal{F}_1$ is the same as a lower bound on the gap of the lattice in physical units (continuous time, finite spatial lattice, $T = \infty$) Proved here	The same sentence on Wilson's lattice with discretised time (the time-step comparison assumption remains). Whether the part of the upper-bound assumption outside $\mathcal{F}_1$ is true
Monotonicity of the window factor, the In-Window Supremum Proposition, and the lower bounds (7)(10) on the gap in finite spatial volume from the local relaxation-time assumption on a window Proved here	In which model and at which β the local relaxation-time assumption holds
That the form "some $K(c)$" is satisfied by the four-dimensional massless free field and is empty; transfer to the ratio form and $K_* \le \kappa$ Proved here	The local-excitation moment assumption and the infinite-volume identity (both assumptions)
The values of the three references and the thresholds 0.342κ, 0.502κ, about 0.62κ; the boxes at which the bridge sees the mass, $mR \approx 4.2$ and 5.2 Reading	The ratio K_{rel} for the target Wilson measure. The threshold for $SU(2)$ at finite β. Values for large loops in four dimensions. Fields in the band $[K_*, 1)$
That for the 0+1 double well the ratio for the shortest window is exactly ω_b/m, and the three regimes by window Proved here ; lattice values Reading	That in four dimensions the ratio at short distance is close to 1 (an assumption depending on asymptotic freedom)

Sources and reproduction

Item	Kind	Source / tools
The one-sided bridge, the identity, the Free-Particle-on-a-Circle Counterexample	This book	(17)(18) and the Free-Particle-on-a-Circle Counterexample in 23-01 § 06. Theorem 3.1, Proposition 4.1 and Proposition 5.1 of the paper "one-sided bridge" (list of papers)
(2)(3), the Window-Factor Monotonicity Proposition, the In-Window Supremum Proposition, (6)–(10), (15)(16)	Proved here	Proofs in the text. The algebraic steps are in the Lean bundles (the machine-checking column of Table 5)
Gaussian Poincaré constant = norm of the covariance	Cited	M. Hairer, A. M. Stuart, J. Voss, P. Wiberg, Analysis of SPDEs arising in path sampling. Part I: The Gaussian case, Commun. Math. Sci. 3 (2005) 587–603, arXiv:math/0601095, Lemma 2.2
The right column of Table 1, $\Gamma(R)$ in (11), Table 2, $r(mR)$, the lattice values of Table 3	Reading	Largest eigenvalue of the free-field covariance compressed to the box (two routes, numerical integration of the heat kernel and periodic lattices; Lanczos); for 0+1, the lattice eigenvalue problem of the transfer kernel and Rayleigh–Ritz lower bounds. All floating point, with extrapolation in the ranges written in the text
New ideas are needed for a gap uniform in the volume	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000), § 6
The Brascamp–Lieb inequality ($K_{\mathrm{rel}} \le 1$ for convex perturbations)	Assumption	Bibliographic data not checked

Terms used in this chapter — the family local in time $\mathcal{F}_1$, window, the window factor $\varphi(x)$, the In-Window Supremum Proposition, the local Poincaré constant C_P^{loc}, the local relaxation-time assumption ($(\xi^2)\uparrow_c^{\mathrm{loc}}$, $(\xi^2)\uparrow_c^{\mathrm{rel}}$), the ratio K_{rel}, the massless reference and Γ, the gauge-quotient reference, the threshold K_*, the local-excitation moment assumption, the band $[K_*, 1)$, colour singlets and the second chaos. Their meanings are in the A-02 Glossary and 24-09.

Next chapter to read: 24-05 Making time continuous — the Hamiltonian lattice and the vacuum measure — stating the identity (17) on a general lattice, and reading it as a Poincaré sentence that does not pass through the relaxation-time assumption.

2026-10-09 · chapter Yang–Mills and the mass gap Part 24 The first road, second half the vacuum measure, the ground-state transformation, Kogut–Susskind · Kiichi, Shiori and Rin

Making time continuous — the Hamiltonian lattice and the vacuum measure

> This chapter is this book's own investigation. § 04 and § 06 are a map of open questions.

What this chapter shows — That on a lattice with continuous time, the gap of the Hamiltonian Δ is tied to the Poincaré constant of the vacuum distribution at one time $\nu = \psi_0^2$ exactly by $\Delta = \hbar^2/(2C_P(\nu))$ (with a proof). That this is the reverse side of $T = \infty$ for the one-sided bridge of Part 23, and the same formula as the ratio ω_b/m for the shortest window in 24-04. The road that treats the gap directly on the Hamiltonian lattice of Kogut–Susskind, and why local-to-global gap theorems do not apply there. That for the $SU(2)$ colour singlet, at leading order the Poincaré constant of the vacuum measure is halved and the lowest gauge-invariant test space is two-particle. What is needed to measure the constant of the vacuum measure.

Prerequisite chapters — 23-01 § 04 (the three CP of Table 3), 23-01 § 06 ((17) of the One-Sided Bridge Proposition (Theorem 3.1 of the paper), the identity (18)), 24-04 (the supremum (2) over the family local in time, (17) for the 0+1 window, the colour singlet), 2-05 § 08 (the one-dimensional ground-state transformation). The notation is collected in 24-09.

Said first — The level is [Y3]. The identity of this chapter only restates the gap as "the Poincaré constant of a certain measure"; it does not make the problem easier. The vacuum measure cannot be written explicitly, and there is no tool that bounds its Poincaré constant uniformly in the coupling. This chapter does not show the mass gap. It sets out one view that does not pass through the diffusive-relaxation-time assumption, and lists what that view needs.

The order of this chapter

1. The gap is the Poincaré constant of the vacuum measure — the Gap and Vacuum Measure Proposition ((1)–(4)), an example checked by hand
2. Three Poincaré constants — the path measure, the family local in time, the vacuum measure (Table 1, (5)(6))
3. The lattice version in 0+1 — the single-time window, and the discrepancy when time is discretised
4. The Hamiltonian lattice of Kogut–Susskind — the road that treats the gap directly, the Electric–Magnetic Splitting Frustration Proposition, where the tools do not apply (Table 2)
5. The $SU(2)$ colour singlet and two particles — the constant is halved at leading order ((7)(8))
6. The road to measuring the constant of the vacuum measure — what is needed
7. What cannot be said
8. Table of the five grades and the dependency graph — Table 3

01

The gap is the Poincaré constant of the vacuum measure

The setting is the same as for the one-sided bridge of Part 23. Space is a finite lattice, the space of configurations at one time is Q ($\mathbb{R}^n$, or a compact connected Riemannian manifold), the metric of the kinetic term is M, the Hamiltonian is $H = -\frac{\hbar^2}{2}\Delta_M + V$, and we place the ground-state assumption ((A1) in the paper: the spectrum is discrete, the ground state ψ_0 is non-degenerate, smooth and positive, $\Delta = E_1 - E_0 > 0$). We set the vacuum distribution at one time, and the Dirichlet form and Poincaré constant on it, as

$$\nu = \psi_0^2 \, d\mathrm{vol}, \qquad \mathcal{E}_\nu(F) = \int_Q \psi_0^2 \, |dF|^2_{M^{-1}} \, d\mathrm{vol}, \qquad C_P(\nu) = \sup_F \frac{\mathrm{Var}_\nu(F)}{\mathcal{E}_\nu(F)} \quad (1)$$

(F is a smooth, non-constant function on Q). In gauge theory we restrict F to gauge-invariant functions, and Δ is the gap on the gauge-invariant subspace.

Gap and Vacuum Measure Proposition

Under the ground-state assumption,

$$\Delta = \frac{\hbar^2}{2\, C_P(\nu)} \quad (2)$$

holds, and the supremum in (1) is attained at $F = \psi_1/\psi_0$.

Proved here Proof. The map $U : F \mapsto F\psi_0$ is an isometry from $L^2(\nu)$ to $L^2(d\mathrm{vol})$, and takes the constant 1 to ψ_0. Hence $\langle F \rangle_\nu = 0 \iff F\psi_0 \perp \psi_0$, and then $\|F\psi_0\|^2 = \mathrm{Var}_\nu(F)$. For smooth F, the identity of the ground-state transformation ((18) of 23-01)

$$\big\langle F\psi_0, \, (H - E_0)\, F\psi_0 \big\rangle = \frac{\hbar^2}{2}\, \mathcal{E}_\nu(F) \quad (3)$$

holds. Since $H - E_0$ is at least Δ on the orthogonal complement of ψ_0 (min-max), for smooth F with $\langle F \rangle_\nu = 0$ we have

$$\Delta \; \mathrm{Var}_\nu(F) \;\le\; \frac{\hbar^2}{2}\, \mathcal{E}_\nu(F), \qquad \text{that is} \qquad \frac{\mathrm{Var}_\nu(F)}{\mathcal{E}_\nu(F)} \le \frac{\hbar^2}{2\Delta} \quad (4)$$

Adding a constant changes neither side, so (4) holds for every smooth F. Equality is attained at $F = \psi_1/\psi_0$ (smooth by the ground-state assumption, with $F\psi_0 = \psi_1$, and the left-hand side of (3) is $\Delta\|\psi_1\|^2$). For $Q = \mathbb{R}^n$, the finiteness of the Dirichlet integral of ψ_1/ψ_0 is checked by the truncation lemma (Lemma 3.3) of the paper "one-sided bridge" (list of papers). ∎

In other words, the generator of Langevin dynamics on the vacuum distribution is unitarily equivalent, by the ground-state transformation, to $-(2/\hbar^2)\, \psi_0^{-1}(H - E_0)\, \psi_0$. Restatements of this type are well known (the restatements of stochastic quantisation and of supersymmetric quantum mechanics). The Gap and Vacuum Measure Proposition writes the gap part of it in this book's convention (1).

Calibration (one harmonic oscillator). For $Q = \mathbb{R}$, $M = 1$, $V = \frac{1}{2}\omega^2 x^2$, $\hbar = 1$, the gap is $\Delta = \omega$, so (2) gives $C_P(\nu) = 1/(2\omega)$. ψ_0^2 is the normal distribution with variance $1/(2\omega)$, and this agrees with the fact that the Poincaré constant of a one-dimensional normal distribution equals its variance (with equality for linear functions). The row for the vacuum C_P in Table 3 of 23-01 ($1/(2m)$ for the free field) is this value Proved here .

Example 1 (checked by hand on two sites). On the two-site lattice of Example 6 of 23-01 ($\hbar = M = 1$, $V = \frac{1}{2}x^{\mathsf{T}}Kx$, the normal mode $u = (x_1 + x_2)/\sqrt{2}$ with angular frequency m_0, $v = (x_1 - x_2)/\sqrt{2}$ with angular frequency $\sqrt{m_0^2 + 2}$), ν is the independent normal distribution with variance $1/(2m_0)$ for u and variance $1/(2\sqrt{m_0^2 + 2})$ for v. The Poincaré constant is the larger of the two variances, $1/(2m_0)$, and (2) gives $\Delta = m_0$, which agrees with the gap of Example 6 of 23-01 Proved here . If $m_0 = \frac{1}{2}$, then $C_P(\nu) = 1$, the path measure has $C_P = \xi^2 = 4$, and $4 = 4\,C_P(\nu)^2$ ((5) in the next section).

02

Three Poincaré constants

Three quantities are called by the same symbol C_P. With the Gap and Vacuum Measure Proposition, the relations among the three line up in one table.

Table 1 Three Poincaré constants (lattice with continuous time, canonical graduation. The free-field column has $\hbar = 1$ and mass m)

Quantity	Which measure, which functions	Relation to the gap	Free field	Grade
C_P	Path measure, all functionals	One-sided $C_P \geq \kappa\hbar^2/\Delta^2$	$1/m^2$	Proved here (the one-sided bridge of 23-01)
$\sup_{\mathcal{F}_1}$	Path measure, time-integrated single-time functions	$= \hbar^2/\Delta^2 = (2C_P(\nu)/\hbar)^2$	$1/m^2$	Proved here (combining (2) of 24-04 with (2) of this chapter)
$C_P(\nu)$	The vacuum distribution at one time	$= \hbar^2/(2\Delta)$	$1/(2m)$	Proved here (the Gap and Vacuum Measure Proposition)

The equality in the second row

$$\sup_{f \in \mathcal{F}_1} \frac{\mathrm{Var}(f)}{\mathcal{E}(f)} = \left(\frac{2\,C_P(\nu)}{\hbar}\right)^2 \quad (5)$$

makes exact, on the family local in time and with interaction, the statement of 23-01 § 04 that "for the free field $C_P^{(\beta)} = 4(C_P^{(\alpha)})^2$" Proved here . In 23-01 this rereading was written as an analogy from the form of the free field. Under the assumptions of the one-sided bridge, restricted to the family local in time, it is an equality, not an analogy. The amount by which the path C_P stated by the diffusive-relaxation-time assumption exceeds $(2C_P(\nu)/\hbar)^2$ comes from functions that are not local in time (winding numbers and the topological charge).

Restated in physical units. Setting $\hbar = 1$ in lattice units and converting to physical units by the bookkeeping length ξ_{RG}, (2) becomes

$$m_{\mathrm{phys}} = m\,\xi_{\mathrm{RG}} = \frac{\xi_{\mathrm{RG}}}{2\,C_P(\nu)}, \qquad m_{\mathrm{phys}} \geq c \iff C_P(\nu) \leq \frac{\xi_{\mathrm{RG}}}{2c} \quad (6)$$

Proved here . The mass gap (that the gap in physical units of the lattice stays positive) is the same as the Poincaré sentence "the Poincaré constant of the vacuum measure is bounded by the first power of the bookkeeping length". It is a Poincaré sentence that does not pass through the relaxation-time

assumption, and the growth is the first power, not the square (in terms of the dynamical exponent, the type $z = 1$). But this is a restatement, and there is no new information on either side. An upper bound on $C_P(\nu)$ needs every F, and the difficulty is exactly that of a lower bound on the gap.

The topological problem does not arise for the vacuum measure. On ν we vary only single-valued functions at one time. For a free particle on the circle, ν is the uniform distribution on the circle and $C_P(\nu) = 1$ is finite (Wirtinger's inequality for the circle of circumference 2π), and (2) gives $\Delta = \hbar^2/2$ of the Free-Particle-on-a-Circle Counterexample of 23-01. And the winding-number functional that made $C_P = \infty$ for the path measure (the angle of the lift, integrated in time) is not a function on ν Proved here . The four-dimensional topological charge is also a quantity fixed by an integral in time, so it is not among the functions on the vacuum distribution at one time Estimate . The boundary is the same as the inside and outside of $\mathcal{F}_1$ in 24-04.

03

The lattice version in 0+1 — the single-time window

For the 0+1-dimensional double well of 24-04 § 07, the local Poincaré constant of the single-time window of the chain with time discretised at width ε was exactly $C_P^{\mathrm{loc}}(1) = \varepsilon/(2m)$ when ψ_0 is the Schrödinger ground state ((17) of 24-04). The Dirichlet form of the window $(\hbar/\varepsilon)\int \psi_0^2 |F'|^2$ is $\hbar/\varepsilon$ times $\mathcal{E}_\nu$ of (1), so this is exactly the value $C_P(\nu) = \hbar^2/(2\Delta)$ of the Gap and Vacuum Measure Proposition multiplied by $\varepsilon/\hbar$ Proved here . The ratio ω_b/m for the shortest window is this identity for the target (the double well) divided by that for the reference (the harmonic oscillator).

When time is discretised, the single-time distribution of the chain is the square of the top eigenvector of the transfer kernel (the symmetric Trotter product), and differs slightly from the Schrödinger ψ_0^2. On lattices with $V = (x^2-1)^2/8$, $\varepsilon = \frac{1}{2}$, $\hbar = 0.3,\ 0.15,\ 0.08$, the ratio of the direct value of the single-time window to $\varepsilon/(2m_{\mathrm{tr}})$ (m_{tr} from the second eigenvalue of the transfer) is $1.006,\ 1.013,\ 1.019$, and the ratio with the reference discretised too, $K_{\mathrm{rel}}(1)\cdot m_{\mathrm{tr}}$, is $1.037,\ 1.044,\ 1.050$ Reading . The identity is a statement in continuous time, and on a lattice with discretised time it is an approximation. This difference is the smallest form of the reason why the time-step comparison assumption of Part 23 ((D) in the paper "one-sided bridge", (20) of 23-01) remains.

04

The Hamiltonian lattice of Kogut–Susskind

This section summarises existing knowledge (the Hamiltonian lattice of Kogut–Susskind). The introduction is in 5-03 § 07.

The Hamiltonian of gauge theory with time continuous and only space made a lattice is the Kogut–Susskind lattice. The configuration space is the product of copies of the group, one per spatial link, $Q = G^{\{\text{spatial links}\}}$ (a compact connected manifold), and setting the metric of the kinetic term to $M = g^{-2}\cdot$ (the bi-invariant metric) to match the form of the one-sided bridge, we have

$$H_{\mathrm{KS}} = \frac{g^2}{2}\sum_{\ell} E_\ell^2 + \frac{1}{g^2}\sum_{p}\left(2 - \operatorname{tr} U_p\right) \qquad (\hbar = 1,\ G = SU(2)) \tag{7}$$

E_ℓ^2 is the Laplace–Beltrami operator on the group of the link ℓ with its sign changed (the square of the electric field), U_p is the product around a plaquette, and the second term is the magnetic term. The

conventions for the coefficients differ in the literature; here they are matched to the form of the one-sided bridge. Physical states are gauge-invariant functions (the Gauss law), and H_{KS} commutes with gauge transformations, so ψ_0 is automatically gauge invariant. On this lattice, the quantity in the problem statement is exactly the gap of H_{KS} on the gauge-invariant subspace, and by the Gap and Vacuum Measure Proposition it can be written as the Poincaré constant of the vacuum distribution at one time $\nu = \psi_0^2 \cdot \mathrm{Haar}$.

The road that treats the gap directly on this lattice is one of the rows of the "table of other roads" in the table of materials of 24-00. Table 2 sets out how far it has reached and where it stopped.

Table 2 Roads that treat the gap of the Hamiltonian lattice directly (the grades of the primary sources are in the source table)

Tool	Statement (summary)	Does it apply to the Kogut–Susskind lattice?	Grade
Local-to-global gap theorems (of Knabe's type, Gosset–Mozgunov)	For a frustration-free, translation-invariant local Hamiltonian, if the local gap of the open chain of length n exceeds a threshold $(6/(n(n+1))$ in Gosset–Mozgunov), the gap survives uniformly in the volume in the thermodynamic limit (one dimension; extended to the two-dimensional square lattice)	Does not apply. The splitting into the electric term and the magnetic term is not frustration-free (the Electric–Magnetic Splitting Frustration Proposition)	Cited (statement checked in the abstract), Proved here (the Electric–Magnetic Splitting Frustration Proposition)
Perturbation (Yarotsky)	For a classical lattice system with a relatively bounded quantum perturbation added, the gap and exponential decay of correlations	Strong coupling (the electric term dominating) is expected to fit the perturbative type. Not checked against the assumptions	Cited (abstract), Estimate
Results on the gap at weak coupling (the continuum-limit side)	—	Not found within the range searched (two searches with short terms)	—

Electric–Magnetic Splitting Frustration Proposition (the splitting into electric and magnetic parts is not frustration-free)

Let $g > 0$ and let the spatial lattice have at least one plaquette. When (7) is regarded as the sum of the local terms $h_\ell = \frac{g^2}{2} E_\ell^2$ and $h_p = \frac{1}{g^2}(2 - \operatorname{tr} U_p)$, $h_p - c_p$ has no eigenvector whatever constant c_p is subtracted. Hence there is no state that realises the ground states of all the local terms at once, and this splitting is not frustration-free.

Proved here Proof. h_p is a multiplication operator on $L^2(Q, \mathrm{Haar})$, and the multiplying function $2 - \operatorname{tr} U_p$ is a non-constant real-analytic function on Q. If $(h_p - c_p)\psi = 0$, then the support of ψ lies in the level set $\{2 - \operatorname{tr} U_p = c_p g^2\}$, but a level set of a non-constant real-analytic function has Haar measure 0, so $\psi = 0$. ∎ That the electric term and the magnetic term do not commute is equally elementary (E_ℓ^2 is a differential operator and does not commute with multiplication by $\operatorname{tr} U_p$). Whether a splitting that groups the electric and magnetic terms by sector also fails to be frustration-free is an expectation from their not commuting, and has not been shown Estimate . Nor have we checked the fact that the local Hilbert space $L^2(SU(2))$ is infinite-dimensional against the setting of the theorems.

Tools on the side of the vacuum measure. Moving the gap to $CP(\nu)$ by the Gap and Vacuum Measure Proposition does not increase the tools available. We list three reasons.

- ν cannot be written explicitly. $-2\ln\psi_0$ is the "action" of ν, and it is not local either. A criterion of Bakry–Émery type, which bounds the Poincaré constant by convexity, needs a lower bound on $\mathrm{Hess}(-\ln\psi_0)$. The result of Brascamp–Lieb type, that the ground state of a convex potential is log-concave, is from memory, and its bibliographic data have not been checked Assumption . Moreover, the magnetic term of (7) is a non-constant smooth function on a compact group, so its Hessian is at most 0 at a maximum point, and it cannot be uniformly convex Proved here . Whether a criterion with the Ricci curvature of the group added works uniformly as $g \to 0$ is expected to meet an obstacle of the same type as the convexity-ceiling lemma of 24-06 Estimate .
- To set up on ν an argument of the local-to-global type for Glauber dynamics (from the Poincaré constants of local boxes and strong spatial mixing to a gap uniform in the volume, of the type in Martinelli's lectures), spatial mixing with respect to the boundary values of ν is needed. Since ν is not explicit, this is expected to be harder than for the Wilson measure Estimate (the statements of theorems of that type are from memory, not checked in the originals Assumption).
- Taking the temporal lattice spacing to 0 first (the Hamiltonian limit) is a different order of limits from Wilson's isotropic lattice. That both orders lead to the same continuum limit is an assumption Assumption (the background in 23-02).

05

The $SU(2)$ colour singlet and two particles

In 24-04 § 06 we saw, for the reference on a four-dimensional spacetime box, that "the $SU(2)$ colour singlet has no linear term, and the reference is exactly halved". We state the same thing on the side of the vacuum distribution at one time.

At leading order ($g \to 0$, the three colour components independent Gaussians), the vacuum distribution ν is a Gaussian measure on the transverse fields of the spatial links; writing ω_k for the one-particle angular frequencies of the free Hamiltonian, the Langevin generator of the Dirichlet form (1) is unitarily equivalent to $-2(H - E_0)$ by the ground-state transformation of the Gap and Vacuum Measure Proposition ($\hbar = 1$). The eigenvalues of $H - E_0$ are sums of one-particle angular frequencies $\sum_k n_k\omega_k$ (n_k is the number of particles), and the eigenspaces correspond to the Wiener chaos ($n = \sum n_k$ gives the n-th chaos). On a finite spatial lattice, let the smallest angular frequency be $\omega_{\min} > 0$ (modes with $\omega = 0$, such as the constant mode of a periodic box, are treated separately).

A gauge-invariant function must be invariant under constant gauge transformations (global colour rotations $SO(3)$). A function in the first chaos $\sum c_k^a A_k^a$ is a vector in the colour index a, and $SO(3)$ has no fixed vector in $\mathbb{R}^3$, so the only invariant one is 0 Proved here . Hence the lowest test space of gauge-invariant functions is the second chaos (two particles), the smallest eigenvalue on it is $2\omega_{\min}$, and equality is attained at $F_w = \sum_a(\langle w, A^a\rangle^2 - \mathbb{E})$, with the two particles placed in the same lowest mode w and contracted in colour Proved here . Reread through the Gap and Vacuum Measure Proposition, this is

$$C_P(\nu \,|\, \text{colour singlet}) = \frac{C_P(\nu)}{2}, \qquad \Delta_{\text{singlet}} = 2\,\omega_{\min} \qquad \text{(leading order)} \quad (8)$$

Proved here (at leading order, three independent Gaussian colours, on a finite spatial lattice with $\omega_{\min} > 0$). The lowest gauge-invariant excitation is at the two-particle threshold, and the Poincaré constant is half the one-particle constant. The reference $\Gamma_Q/2$ for the spacetime box in 24-04 is the same mechanism seen through the local Poincaré constant of the spacetime box. Lean

`LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: a linear form invariant under two rotations; conclusion: the coefficients are 0), `chaos2_gamma` (assumption: $0 < n, g$ and the Wick moments as formulas; conclusion: the algebra by which the ratio is halved).

In physics it is said that "the lightest glueball is a bound state of two gluons". One can read (8) as the local Poincaré version of its leading order Estimate . However, a physical mechanism is not a proof (the table of physical mechanisms in 24-07), and (8) is a statement in the form of the leading order. In infinite volume $\omega_{\min} \to 0$, so the right-hand side of (8) also goes to 0. The mass gap does not appear at leading order; what the leading order fixes is only the form "the gauge-invariant test space begins at two particles". How (8) changes for $SU(2)$ at finite g has not been computed at all.

06

The road to measuring the constant of the vacuum measure — what is needed

By the Gap and Vacuum Measure Proposition, measuring the Poincaré constant of the vacuum distribution is the same as measuring the gap. We set out the roads to measuring it by how far the machinery of the first step of Part 23 can be used.

- **0+1 dimensions.** The machinery of the first step (the lattice eigenvalue problem of the transfer kernel, and the generalised eigenvalues of the weighted lattice Laplacian) computes the local Poincaré constant of the single-time window directly, and agreed with $\varepsilon/(2m_{\mathrm{tr}})$ within a factor of 1.006–1.019 (§ 03) Reading . In 0+1, the machinery measures $C_P(\nu)$ as it is.
- **Samples.** ν is not explicit, so samples are needed. The marginal distribution on a single-time slice of an anisotropic Wilson lattice with small temporal lattice spacing is expected to approach ν in the Hamiltonian limit Estimate . If the time step is finite, that slice is ψ_0^2 of the transfer kernel, and the discrepancy of § 03 (1–2% at $\varepsilon = \frac{1}{2}$ in 0+1) enters.
- **Test functions.** Since we restrict to gauge-invariant functions, we take them from polynomials in Wilson loops. As in 24-04 § 06, loops of the size of the span of the box are needed.
- **Graduation.** $\mathcal{E}_\nu$ of (1) is measured in the metric of the kinetic term $M^{-1} = g^2 \cdot$ (the bi-invariant metric). Unless the Langevin graduation is matched to this, the coefficient in (2) is off by a factor κ.
- **Direction.** The Rayleigh quotient on samples gives only a lower bound on $C_P(\nu)$, that is, an **upper bound** on the gap (the side obtained by placing a single test function). A lower bound on the gap needs every F, and the asymmetry between the relaxation-time upper-bound assumption and the relaxation-time lower-bound assumption of Part 23 ((15) of 23-01) appears as it is. The road to measuring is a road that measures the easy side of the gap.

This chapter places no computation. A computation that bounds the Kogut–Susskind gap of a small periodic lattice in intervals, truncating the representations of the electric field, stays at the numbers of a finite box, and does not become a bridge to a gap uniform in the volume Estimate .

07

What cannot be said

- The Gap and Vacuum Measure Proposition is an identity, and we do not claim it has made the problem easier. There is no tool that bounds $C_P(\nu)$ uniformly in β (or g).

- Local-to-global gap theorems do not apply to the splitting of the Kogut–Susskind lattice into electric and magnetic parts (the Electric–Magnetic Splitting Frustration Proposition). That they do not apply to a splitting grouped by sector is an expectation and has not been shown.
- We have not checked the application of strong-coupling perturbative results to the Kogut–Susskind lattice. Results on the weak-coupling side were not found within the range searched.
- That the order taking the Hamiltonian limit first gives the same continuum limit as Wilson's lattice is an assumption.
- (8) is a statement at leading order (three independent Gaussian colours), and there is no value at finite g. The correspondence with glueballs is an estimate.
- This chapter does not show the mass gap. Nor does it show anything about the relaxation-time assumption.

08

Table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Ground-state assumption: discrete spectrum, $\psi_0 > 0$ smooth and non-degenerate, $\Delta > 0$	Assumption	23-01 Table 6	Finite spatial lattice	—
2	The identity (3) of the ground-state transformation	Proved here	23-01 (18)	Finite spatial lattice	—
3	Gap and Vacuum Measure Proposition: $\Delta = \hbar^2/(2C_P(\nu))$	Proved here	(2)–(4)	Continuous time, finite spatial lattice	—
4	The relation (5) among the three constants	Proved here	Table 1	Canonical, assumptions of the one-sided bridge	—
5	Restatement in physical units (6)	Proved here	§ 02	Each β	—
6	The 0+1 single-time window is $\varepsilon/\hbar$ times the Gap and Vacuum Measure Proposition	Proved here	§ 03	0+1, Schrödinger ψ_0	`LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumption: $0 < \varepsilon, m, \omega$)
7	Discrepancy on lattices with discretised time 1.006–1.019, 1.037–1.050	Reading	§ 03	$\varepsilon = \frac{1}{2}$, three values of $\hbar$	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
8	Electric–Magnetic Splitting Frustration Proposition: the splitting into electric and magnetic parts is not frustration-free	Proved here	§ 04	$g > 0$, finite spatial lattice	—
9	Local-to-global gap theorems assuming frustration-freeness	Cited	Table 2 (statement checked in the abstract)	Setting of the original	—
10	Strong-coupling perturbative type (Yarotsky)	Cited	Table 2 (abstract)	Setting of the original	—
11	The ground state of a convex potential is log-concave	Assumption	§ 04 (from memory)	—	—
12	Arguments of the local-to-global type (of the type in Martinelli's lectures)	Assumption	§ 04 (from memory)	—	—
13	The Hamiltonian limit and Wilson's lattice lead to the same continuum limit	Assumption	§ 04	—	—
14	The colour singlet has no linear term	Proved here	§ 05	Leading order	`LatticeGauge.WilsonLoopReference.no_linear_singlet` (assumption: invariant under two rotations)
15	(8): for the singlet $CP(\nu)$ is halved, the gap is $2\omega_{\mathrm{min}}$	Proved here	§ 05	Leading order, three independent Gaussian colours, $\omega_{\mathrm{min}} > 0$	`chaos2_gamma` (assumption: the Wick moments as formulas)
16	Local-to-global theorems do not apply to the Kogut–Susskind lattice	Estimate	§ 04	Including splittings grouped by sector	—
17	Strong coupling fits the perturbative type	Estimate	Table 2	Not yet checked	—
18	(8) is the local Poincaré version of the glueball	Estimate	§ 05	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
19	There is no tool that bounds the constant of the vacuum measure uniformly	Estimate	§ 04, § 06	—	—

The machine-checking column covers only algebra. The identity (3), the Gap and Vacuum Measure Proposition, the Electric–Magnetic Splitting Frustration Proposition and the measure steps of (8) are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  ASM  ground-state assumption: discrete spectrum, ψ₀ > 0, Δ > 0      finite spatial lattic
 2  PRV  identity of the ground-state transformation ((18) of 23-01)
 3  PRV  Gap and Vacuum Measure Proposition: Δ = ħ²/(2 C_P(ν)) ⟸ 1, 2
 4  PRV  three constants: sup over F₁ = (2C_P(ν)/ħ)² ⟸ 3 (+ the one-sided bridge of 23-01,
 5  PRV  physical units: m_phys ≥ c ⟺ C_P(ν) ≤ ξ_RG/(2c) ⟸ 3
 6  PRV  0+1 window: the single-time window is ε/ħ times the gap proposition ⟸ 3    Lean on
 7  RDG  lattice discrepancy: 1.006–1.019 (ψ₀ of the transfer with discretised time)
 8  PRV  Electric–Magnetic Splitting Frustration Proposition: not frustration-free
 9  CIT  local→global: gap theorems assuming frustration-freeness
10  CIT  perturbation: gap under relatively bounded quantum perturbations (abstract)
11  ASM  log-concave: ground state of a convex potential (from memory)
12  ASM  mixing: uniform gap from local boxes and spatial mixing (from memory)
13  ASM  order of limits: the Hamiltonian limit and Wilson's lattice
14  PRV  no linear term: the colour singlet has no linear term              Lean no_linear_s
15  PRV  two particles: (8) ⟸ 3, 14 (leading order)                          Lean chaos2_ga
16  EST  no fit: local→global does not apply to KS ⟸ 8, 9
17  EST  strong coupling: fits the perturbative type ⟸ 10
18  EST  glueball: the reading of (8) ⟸ 15
19  EST  no tool: no tool bounds C_P(ν) uniformly ⟸ 11, 12, 13, 16
```

The check for circles is only to see that on every "⟸" the numbers on the right are smaller than the number on the left. The conclusions of this chapter are the Gap and Vacuum Measure Proposition and the three relations that follow from it (the three constants, physical units, the 0+1 window), and the leading-order statement for two particles; there is no node for the mass gap. None of the assumption rows (the ground-state assumption, log-concavity, mixing, the order of limits) contains a lower bound on the gap.

What this chapter can say

What is said	What is not said
That on a finite spatial lattice with continuous time, the gap is tied to the Poincaré constant of the vacuum distribution at one time by $\Delta = \hbar^2/(2C_P(\nu))$ (under the ground-state assumption) Proved here	A way to bound $C_P(\nu)$ uniformly in β. Any claim that the problem has become easier
The relation among the three Poincaré constants, and that on the family local in time the free-field rereading of 23-01 becomes an equality Proved here	The size of the amount by which the path C_P exceeds $(2C_P(\nu)/\hbar)^2$
That the 0+1 single-time window is this identity Proved here ; the discrepancy when time is discretised Reading	Whether the time-step comparison assumption holds on the four-dimensional Wilson lattice

What is said	What is not said
That the splitting of the Kogut–Susskind lattice into electric and magnetic parts is not frustration-free Proved here	Whether it holds for splittings grouped by sector. The application of strong-coupling perturbation. Weak-coupling results
That at leading order the lowest test space of the colour singlet is two-particle, and $CP(\nu)$ is halved Proved here	Values at finite g. The correspondence with glueballs (an estimate)

Sources and reproduction

Item	Kind	Source / tools
The identity (3), the one-sided bridge	This book	(17)(18) of 23-01 § 06. The truncation for $\mathbb{R}^n$ is Lemma 3.3 of the paper "one-sided bridge" (Kiichi, Shiori and Rin)
Gap and Vacuum Measure Proposition, (5)(6), Electric–Magnetic Splitting Frustration Proposition, (8)	Proved here	Proofs in the text
The Kogut–Susskind Hamiltonian	Setting	J. Kogut, L. Susskind, Hamiltonian formulation of Wilson's lattice gauge theories, Phys. Rev. D 11 (1975) 395–408 (bibliographic data checked, text not obtained. The coefficients of (7) are this book's convention)
Local-to-global gap theorems	Cited	D. Gosset, E. Mozgunov, arXiv:1512.00088 (statement checked in the abstract). S. Knabe, J. Stat. Phys. 52 (1988) 627–638 (bibliographic data checked; the value of the threshold is restated from the abstract of the former)
Strong-coupling perturbation	Cited	D. A. Yarotsky, arXiv:math-ph/0412040 (Commun. Math. Phys. 261 (2006)) (abstract)
Search for weak-coupling results on the gap	—	Nothing relevant in two arXiv searches with short terms (the search is shallow)
The lattice values of § 03	Reading	The lattice eigenvalue problem of the transfer kernel of the 0+1 double well (the same as the source table of 24-04)
Log-concavity of Brascamp–Lieb type, local-to-global of Martinelli's type	Assumption	From memory. The statements have not been checked in the originals

Terms used in this chapter — the vacuum measure ν, the ground-state transformation, the Gap and Vacuum Measure Proposition, the three Poincaré constants, the Hamiltonian lattice (Kogut–Susskind), frustration-free, local-to-global gap theorems, the colour singlet and the second chaos, the Hamiltonian limit. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-06 Paying with rarity instead of convexity — that convexity is not enough on groups, as a story of tools that pay with rarity.

Paying with rarity instead of convexity — renormalisation on groups and log-Sobolev

> This chapter is this book's own investigation. § 02 and § 08 are established knowledge; § 05 and § 07 are a map of open questions.

What this chapter shows — From here on the subject is tools. 24-01 to 24-05 dealt with "what to compare, and which diffusive-relaxation-time assumption to restate". To show any statement, a tool that bounds the relaxation constant from above is needed. On flat space the centre of such tools is convexity (Bakry–Émery, Brascamp–Lieb), and for scalar hierarchical models a theorem of the same type as the diffusive-relaxation-time assumption has been shown, with uniform convexity of the conditional measure at each scale as the entrance. In this chapter: (1) we look at what the convexity tools need and where they break, in three places: the summit in 0+1 dimensions, the antipode on the group, and the induction hypothesis of the hierarchical model. On a group, by the Convexity Ceiling Lemma on closed manifolds (with a proof), convexity cannot hold in principle at weak-coupling scales. (2) Between flat ϕ^4 and the group, the convex and non-convex places are reversed, and the wall does not vanish but is pushed into a place of probability $e^{-c\beta}$. (3) We then list tools that pay with rarity instead of convexity (the currency of entropy, Lyapunov conditions, field-dependent constants), and see that the margin of the action barrier in Part 23 is "an example where rarity paid". (4) We place as a candidate "the class of measures that pay with rarity", rewritten in the form of rules, and check that the "coupling window" left between the ultraviolet side and the strong-coupling side is a restatement of the second difficulty of the third step. (5) We add one section on the road of stochastic quantisation.

Prerequisite chapters — 23-09 § 06 (the one-site curvature at the summit, $\frac{15}{4}$ and 4), 23-10 currency-mix-up type, 23-11 § 06 (the margin of the action barrier), 24-02 § 02 (tree coordinates and link coordinates), 24-07 § 05 (the three roots). From Part 1, 1-05 (manifolds) and 1-04 ($SU(2)$). Notation is in 24-09.

Said first — The level is [Res]; the summary in § 01 is [Y3]. Proofs are placed only for the lemma, corollary and proposition of § 03; the rest is copied from the originals (Cited) or estimates (Estimate). The results for hierarchical models imply nothing for four-dimensional Yang–Mills. The "rules" of § 07 are candidates, and none of them promises a prospect of closing the third step (the continuum limit). This chapter shows neither the mass gap nor the relaxation-time assumption. A note on notation: in this chapter the lower bound on the Bakry–Émery curvature is written κ_{BE}, a different quantity from the comparison constant κ of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) (24-09). The rate of rarity is written s ($\mu(B) \le e^{-s}$). It is the quantity written ℓ in 23-09 and 23-10 of Part 23; in Part 24 we changed the letter because ℓ is used for the ruler of length.

The order of this chapter

1. Summary [Y3] — the convexity ceiling, the reversal of places, the currency of rarity
2. What the convexity tools need — Bakry–Émery, Brascamp–Lieb, the induction hypothesis of the hierarchical model
3. The convexity ceiling on closed manifolds — the Convexity Ceiling on Groups Lemma, the No Concentration at the Identity Corollary, the One-Link Wilson Weight Proposition (with proofs)
4. The convex and non-convex places are reversed — ϕ^4 and the group; the wall goes to where the probability is small (Figure 1)
5. Carrying the hierarchical induction over to the group — applies, changes, does not apply (Table 1)
6. Tools that pay with rarity — currency, Lyapunov, field-dependent constants, the example of the action barrier (Table 2)
7. If it were made into rules — the class of measures that pay with rarity — objects, rules, candidate first theorems, the coupling window
8. The road of stochastic quantisation — subcriticality $d < 4$, and the direction of rebuilding where the objects live
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 3

01

Summary [Y3] — the convexity ceiling, the reversal of places, the currency of rarity

This section is a ten-paragraph summary for readers at [Y3]. The proofs and the conditions of the originals are in § 02 onwards.

1. A hierarchical model replaces the lattice by nested boxes so that one step of renormalisation can be written down. For the four-dimensional hierarchical scalar $|\phi|^4$, both the equilibrium properties (the book by Bauerschmidt–Brydges–Slade) and the slowness of relaxation of the stochastic dynamics (Bauerschmidt–Bodineau: the gap vanishes with the same power as for the free field) are theorems Cited . The latter is a statement of the same type as the relaxation-time assumption.
2. The method of Bauerschmidt–Bodineau takes as its entrance the hypothesis that at each scale "the measure of the fine fluctuations, with the coarse field held fixed, is uniformly convex" Cited . The original itself names convexity at large fields as the main issue in extending the method beyond hierarchical models.
3. The group $SU(2)$ is a closed space (the three-dimensional sphere). On a closed space, imposing convexity bounds the oscillation of the potential by the square of the diameter, and the measure cannot leave the range of a constant multiple of the uniform distribution (the Convexity Ceiling Lemma of § 03) Proved here . At weak coupling the measure needs to gather near the identity, so the convexity hypothesis cannot hold in principle at weak-coupling scales.

4. For ϕ^4 on flat space, large fields are convex by nature, and what is not convex is small fields (between the two wells). On the group it is the reverse: near the identity (small fields) is convex, and the far hemisphere (large fields) is necessarily non-convex (§ 04) Proved here .
5. However, the probability of being on the non-convex side is small, of the order of $e^{-\beta}$ (the One-Link Wilson Weight Proposition of § 03) Proved here . The wall does not vanish; it is pushed into a place of small probability.
6. The wall of the first step of Part 23 (that the one-site diagonal at the summit $\frac{15}{4}$ falls short of the sum of couplings 4 by $\frac{1}{4}$) also took the form of paying for the lack of convexity with the rarity of the plateau. The Poincaré inequality is a currency that measures the speed of exit, and what can pay with rarity is the currency of entropy (23-10 currency-mix-up type). A step that pays with rarity needs to pass through an inequality of log-Sobolev type.
7. Tools that pay with rarity instead of convexity include Lyapunov conditions (a drift that returns from the bad set) and field-dependent constants. The estimate of the action barrier in Part 23 was an example where the cost of the bistability of the core could be paid by the rarity of the local term (a margin of about 95 times, on the $\hbar = 0$ reading).
8. The one-link measure has a single well, and its relaxation constant does not get exponentially worse in β. A wall of the $\frac{1}{4}$ type of the double well does not come back for one link Estimate .
9. Rewritten in the form of rules, this becomes "the class of measures that pay with rarity" (§ 07). On the ultraviolet side it seems possible to write a form in which the class closes at each step of coarse-graining, but at the confinement scale the rate of rarity runs out, and a "coupling window" remains between it and the curvature tools of the strong-coupling side. This is a restatement of the second difficulty of the third step (24-07 § 06) Estimate .
10. The road of stochastic quantisation (SPDE) uses the same Langevin dynamics, but its aim is the construction of the measure, and the solution theory is subcritical only for $d < 4$ Cited . It gives tools neither for the four-dimensional relaxation-time assumption nor for the local diffusive-relaxation-time assumption.

02

What the convexity tools need — Bakry–Émery, Brascamp–Lieb, the induction hypothesis of the hierarchical model

This section summarises existing knowledge (Bakry–Émery, Brascamp–Lieb, the inductive hypothesis of hierarchical models). The introduction is in 5-11 § 02.

Most tools that bound the relaxation constants (the Poincaré constant, the log-Sobolev constant) **from above** take as their entrance a lower bound on the second derivative of the potential, that is, convexity. We list three.

Bakry–Émery. For a probability measure $\mu = e^{-V} d\mathrm{vol}/Z$ on a Riemannian manifold M,

$$\mathrm{Ric} + \mathrm{Hess}\, V \;\geq\; \kappa_{\mathrm{BE}}\, g \quad (\kappa_{\mathrm{BE}} > 0) \qquad \Longrightarrow \qquad \mathrm{Ent}_\mu(f^2) \leq \frac{2}{\kappa_{\mathrm{BE}}} \int |\nabla f|^2 d\mu, \quad CP \leq \frac{1}{\kappa_{\mathrm{BE}}} \;(1)$$

Cited (the single-scale case of Theorem 3.6 of the review by Bauerschmidt–Bodineau–Dagallier). The condition is a lower bound **at every point**.

Brascamp–Lieb. On $\mathbb{R}^n$, if $\operatorname{Hess} V > 0$ then $\operatorname{Var}_\mu(f) \le \mathbb{E}_\mu \langle \nabla f, (\operatorname{Hess} V)^{-1} \nabla f \rangle$ Cited (the form used by Lemma 2.7 of Bauerschmidt–Bodineau). This too uses convexity at each point.

The hierarchical induction. Bauerschmidt–Bodineau split the Gaussian covariance by scale as $C = C_0 + \cdots + C_N$, form the renormalised potential $e^{-V_{j+1}(\phi)} = \mathbb{E}_{C_j} e^{-V_j(\phi+\zeta)}$ (**addition** of fields $\phi + \zeta$), and place the following hypothesis Cited (arXiv:1809.02075 § 2.3).

$$\text{(A1)} \qquad C^{1/2} \big(\operatorname{Hess} V_j(\phi)\big) C^{1/2} \;\ge\; -\varepsilon_j \, Q \qquad (\text{for all } \phi,\ \varepsilon_j < 1) \ (2)$$

Here Q is the projection onto the space of coarse fields, and ε_j is the notation of the original (different from the lattice spacing ε in 0+1 dimensions). Under (A1) and two structural hypotheses, the Brascamp–Lieb quadratic form is handed on at each step as $D \le C/(1-\varepsilon) + D_+/(1-\varepsilon)^2$, and the product over all steps is bounded by $\exp\big(2 \sum_l (\varepsilon_l + O(\varepsilon_l^2))\big)$ (Theorem 2.1, Corollary 2.2) Cited . The conclusion is a spectral gap (of Poincaré type), not a log-Sobolev result. The form that flows log-Sobolev over many scales is later work by the same authors (the Polchinski flow, § 05). The original writes that "to extend this method beyond hierarchical models, convexity rather than probabilistic estimates is needed, and convexity at large fields is the main issue" Cited .

Three places where it breaks. (a) The summit of the 0+1-dimensional double well. In one row of the Hessian of the action of the core, at the summit the diagonal $4 - \frac{1}{4}$ falls short of the sum of the couplings to the neighbours 4 ((11) of 23-09), and on the face where every site is at the summit, the smallest eigenvalue of a lump of length $k \ge 8$ is negative Proved here . Every convexity tool had to pay for this $\frac{1}{4}$ with the rarity of the plateau. (b) The opposite hemisphere of the group. In the One-Link Wilson Weight Proposition of § 03 we see that the Bakry–Émery curvature of the one-link Wilson weight is negative on the far hemisphere. (c) (A1) of the hierarchical induction. For group-valued fields, the natural reading of (A1) is "Bakry–Émery for the measure on the fibre with the coarse field held fixed" Estimate , and by the Convexity Ceiling Lemma of § 03 it cannot hold at weak-coupling scales.

What this section can say: (1)(2) and Theorem 2.1 are copies of the statements of the originals. The reading in (c) (reading (A1) on the group as Bakry–Émery on the fibre) is an estimate, and this book has no exact formulation of the induction on the group.

03

The convexity ceiling on closed manifolds — the ceiling lemma, no concentration at the identity, the one-link weight

Convexity Ceiling Lemma

Let M be a compact connected Riemannian manifold without boundary of diameter D, $V \in C^2(M)$, $K \ge 0$. If $\operatorname{Hess} V \ge -K\, g$, then

$$\operatorname{osc} V := \max_M V - \min_M V \;\le\; \frac{K D^2}{2}. \ (3)$$

Proved here Proof. Let p be a maximum point of V (M is compact). For any $q \in M$, take a minimising geodesic $\gamma : [0, L] \to M$ from p to q (unit speed, $L = d(p, q) \le D$) (it exists by the Hopf–Rinow theorem, since a compact manifold is complete). Setting $h(t) := V(\gamma(t))$, since p is a maximum point on a manifold without boundary, $h'(0) = dV_p(\gamma'(0)) = 0$. Along the geodesic, $h''(t) = \operatorname{Hess} V(\gamma'(t), \gamma'(t)) \ge -K$. Integrating gives $h'(t) \ge -Kt$, and integrating once more gives $h(t) \ge h(0) - Kt^2/2$. At $t = L$, $V(q) \ge \max V - KL^2/2 \ge \max V - KD^2/2$. Since q is arbitrary, (3) follows. ∎

There is no Lean check (the one-dimensional version "if $h'' \ge -K$ and $h'(0) = 0$ then $h(0) - h(x) \le Kx^2/2$" is kept as a candidate). The key of the proof is that "the first derivative vanishes at a maximum point", and here the **absence of boundary** (being closed) is what works. Flat $\mathbb{R}^n$ has no maximum point, so the same estimate does not follow.

No Concentration at the Identity Corollary (on $SU(2)$, a convex measure cannot gather at the identity)

Take $SU(2)$ as the unit three-dimensional sphere S^3 (bi-invariant metric, $\mathrm{Ric} = 2g$, diameter π), and suppose $\mu = e^{-V}\, d\mathrm{Haar}/Z$ satisfies $\mathrm{Ric} + \operatorname{Hess} V \ge 0$. Then $\operatorname{osc} V \le \pi^2$, and for every Borel set A

$$\mu(A) \;\le\; e^{\pi^2}\,\mathrm{Haar}(A), \qquad \mu\big(\{\theta \le r\}\big) \;\le\; e^{\pi^2}\,\frac{2}{3\pi}\,r^3 \quad (4)$$

(θ is the geodesic distance from the identity, $e^{\pi^2} \approx 1.9 \times 10^4$). In particular the one-link Wilson weight ($\operatorname{osc} V = 2\beta$) does not satisfy $\mathrm{Ric} + \operatorname{Hess} V \ge 0$ for $\beta > \pi^2/2$.

Proved here Proof. Since $\operatorname{Hess} V \ge -\mathrm{Ric} = -2g$, using the Convexity Ceiling Lemma with $K = 2$ and $D = \pi$ gives $\operatorname{osc} V \le \pi^2$. Since $Z = \int e^{-V}\, d\mathrm{Haar} \ge e^{-\max V}$, the density is $e^{-V}/Z \le e^{\max V - \min V} \le e^{\pi^2}$. The Haar distribution of class functions is $\frac{2}{\pi}\sin^2\theta\, d\theta$ ($\theta \in [0, \pi]$), and from $\sin\theta \le \theta$, $\mathrm{Haar}(\theta \le r) \le \frac{2}{\pi} \cdot \frac{r^3}{3}$. The Wilson weight $V = \beta(1 - \cos\theta)$ has $\operatorname{osc} V = 2\beta$, so if $2\beta > \pi^2$ it contradicts the inequality stated before (4). ∎

How to read (4). At weak coupling, the Wilson measure gathers most of its probability in a ball of radius of order $\beta^{-1/2}$. The right-hand side of (4) gets small like r^3, so when β is large, a measure with convexity of Bakry–Émery type cannot do that. The content of the lemma is that once convexity is imposed, the measure cannot move further from Haar than a constant factor e^{π^2}.

One-Link Wilson Weight Proposition

On the unit $S^3 = SU(2)$, let $V_\beta = \beta(1 - \cos\theta)$ and $\mu_\beta = e^{-V_\beta}\, d\mathrm{Haar}/Z_\beta$.

(i) $\mathrm{Ric} + \operatorname{Hess} V_\beta = (2 + \beta\cos\theta)\, g$. Hence the Bakry–Émery condition holds only when $\beta < 2$, and then $\kappa\mathrm{BE} = 2 - \beta$.

(ii) When $\beta > 2$, the set where the condition fails, $N_\beta := \{\cos\theta < -2/\beta\}$, lies in the hemisphere far from the identity, and on it $V_\beta \ge \beta + 2$.

(iii) For $\beta \ge 2$,

$$\mu_\beta(N_\beta) \;\le\; \frac{54\pi}{25}\, e^{-3/2}\, \beta^{3/2} e^{-\beta} \;<\; 1.52\, \beta^{3/2} e^{-\beta}. \tag{5}$$

Proved here Proof. (i) A function f of θ alone on the unit sphere has, in geodesic polar coordinates $g = d\theta^2 + \sin^2\theta\, g_{S^2}$, $\mathrm{Hess}\, f = f''\, d\theta^2 + f' \cot\theta\, (g - d\theta^2)$ (the radial direction from the second derivative along geodesics, the spherical directions from the second fundamental form $\cot\theta$ of the sphere of constant θ). For $f = \beta(1-\cos\theta)$, $f'' = \beta\cos\theta$ and $f' \cot\theta = \beta \sin\theta \cot\theta = \beta\cos\theta$, so $\mathrm{Hess}\, V_\beta = \beta\cos\theta\, g$. On the unit S^3, $\mathrm{Ric} = 2g$. The minimum is $2-\beta$ at $\theta = \pi$. (ii) If $\cos\theta < -2/\beta < 0$ then $\theta > \pi/2$, and $V_\beta = \beta(1-\cos\theta) > \beta + 2$. (iii) On N_β, $e^{-V_\beta} \le e^{-\beta-2}$, and the Haar measure is at most 1, so $\int_{N_\beta} e^{-V_\beta} \le e^{-\beta-2}$. For the denominator, on the ball of $r = \beta^{-1/2} \le 1$, $1 - \cos\theta \le \theta^2/2$ gives $V_\beta \le \frac{1}{2}$, and for $\theta \le 1$, $\sin\theta \ge \theta - \theta^3/6 \ge \frac{5}{6}\theta$ gives $\mathrm{Haar}(\theta \le r) \ge \frac{2}{\pi} \cdot \frac{25}{36} \cdot \frac{r^3}{3} = \frac{25}{54\pi} r^3$. Hence $Z_\beta \ge e^{-1/2} \frac{25}{54\pi} \beta^{-3/2}$, and dividing gives (5) ($\frac{54\pi}{25} e^{-3/2} = 1.514\cdots$). ∎

(5) is a crude upper bound; near the antipode ($\theta \approx \pi$, $V_\beta \approx 2\beta$) the probability is smaller still, of order $e^{-2\beta}$. $\beta < 2$ in (i) is sharper than the necessary condition $\beta \le \pi^2/2$ that follows from the Convexity Ceiling Lemma. The role of the Convexity Ceiling Lemma is that it is not limited to the Wilson weight: **any** weight, once convexity is imposed, is confined to the range of (4).

What this section can say: the Convexity Ceiling Lemma, the No Concentration at the Identity Corollary and the One-Link Wilson Weight Proposition are statements with proofs placed. What they speak of is the convexity of one measure on one manifold; about the convexity of the conditional measure on the fibre (a measure on the product of many links) they speak only through the estimate of § 05.

04

The convex and non-convex places are reversed — the wall goes to where the probability is small

For ϕ^4 on flat space (of the type of the one-site $V(\phi) = \lambda(\phi^2 - v^2)^2$), $V''(\phi) = 4\lambda(3\phi^2 - v^2)$ is negative only for $|\phi| < v/\sqrt{3}$. What is not convex is **small fields** (near the summit between the two wells), and at **large fields** V grows like ϕ^4 and is convex. Bauerschmidt–Bodineau wrote "convexity at large fields" as the issue because one needs to show uniformly that the effective potential after renormalisation stays convex at large fields too.

On the group it is the reverse. As in (i) of the One-Link Wilson Weight Proposition, near the identity (small fields, $\theta < \pi/2$) $\mathrm{Hess}\, V_\beta > 0$ and it is close to Gaussian, while on the far hemisphere (large fields) $\mathrm{Hess}\, V_\beta < 0$. This is not peculiar to the Wilson weight. A smooth function on a closed manifold has a maximum point, and at a maximum point $\mathrm{Hess}\, V \le 0$. **The non-convex place at large fields always exists, for topological reasons** Proved here . On the other hand, as in (iii) of the One-Link Wilson Weight Proposition, the probability of being there is of order $e^{-\beta}$.

```
Figure 1  Convex and non-convex places
          (horizontal axis: size of the field; |φ| for φ⁴, angle θ from the identity for

  φ⁴ (flat space):
     |φ| small ———————— |φ| ≈ v ———————————— |φ| large
     non-convex (summit)   bottom of well (convex)   convex (grows like φ⁴)
     → the non-convex place is "small fields".
       Its probability is set by the frequency of crossing between the wells

  SU(2) (closed sphere):
     θ small ———————————— θ = π/2 ———————————— θ = π (antipode)
     convex (near Gaussian)  boundary             non-convex (always, at the maximum)
     → the non-convex place is "large fields".
       Its probability is of order e^(−β) (the One-Link Wilson Weight Proposition)
```

The wall does not vanish; it is pushed into a place of small probability. On flat space, tools that avoid the non-convex summit were conceivable (imposing convexity at large fields, treating small fields separately). On the group the non-convex place cannot be avoided (the Convexity Ceiling Lemma), but in exchange its probability is exponentially small (the One-Link Wilson Weight Proposition). So tools on the group must push with probability, not with convexity Estimate .

An example where the rate of rarity was measured. On the width-2 strip of two-dimensional $U(1)$, the slope of the raw rate of wrapping of a single plaquette (a rare event that crosses a barrier of height 2β) matched the slope $2 - 1/\beta$ of the Kramers rate $\beta e^{-2\beta}$ to 1–3% ($\beta = 3..6$, outer boundary frozen, a post-hoc check, 24-02 § 05) Reading . The relation between this number and renormalisation on the group is an estimate.

The smallest example where rarity cannot be paid with local quantities. For the 0+1-dimensional double well, even for the shortest window (a single time) the local ratio is not 1 but exactly $\omega b/m$ ((7) of 24-03, 24-04 § 07) Proved here (Lean LatticeGauge.LocalHDoubleWell.one_slice_ratio, assumptions: $0 < \varepsilon$, $0 < m$, $0 < \omega$. The bundle guarantees only the algebra). The influence of the non-convex summit remains in the ratio however short the window is made. The implication for the group is an estimate Estimate .

What this section can say: that the non-convex place always exists for topological reasons, and the upper bound (5) on its probability, have proofs. "Must be tools that push with probability" is an estimate.

05

Carrying the hierarchical induction over to the group — applies, changes, does not apply

The model. As a four-dimensional hierarchical gauge model, we consider the lattice on which the recursion of the Migdal–Kadanoff approximation holds exactly (Ito 1984 Cited). Setting the couplings of the vertical plaquettes to 0 or ∞ place by place, the recursion of the plaquette weight g can be written down in the form $g \mapsto (g^{*4})^4$ (a convolution gluing four together, and a power stacking four). It is stated that for four-dimensional $SU(N)$ every coupling flows to strong coupling and confines (Ito 1985 Cited (abstract)). This model lacks translation and rotation invariance, and its theorems are not part of the statement of the four-dimensional Yang–Mills

problem. That the type of relaxation changes with the choice of coordinates to move was seen in 24-02 § 02 (in tree coordinates the gap of the Gaussian approximation is exactly β, independent of the number of steps; in link coordinates it gets smaller in inverse proportion to the square of the side).

Table 1 carries the parts of the Bauerschmidt–Bodineau induction over, row by row, to this model with values in the group $SU(2)$. "Applies" means the form holds as it is, "changes" means there is a corresponding form but its content is replaced, and "does not apply" means it does not hold; apart from the row of the Convexity Ceiling Lemma, these are estimates.

Table 1 Carrying the parts of the hierarchical induction over to $SU(2)$ (verdict labels: Proved here = from the Convexity Ceiling Lemma, Estimate = an estimate)

Part in the original	Carried over to the group	Verdict
The decomposition of the covariance $C = \sum C_j$ and the addition of fields $\phi + \zeta$	Addition is replaced by the group product, and the reference Gaussian by Haar. The role of "the covariance at scale j" is played by the stiffness of the weight (the supremum of the Hessian of $-\ln g$), and there is an inequality of the type in which the inverses of the stiffnesses roughly add under convolution (a paper record; no proof is placed in this book)	Changes Estimate
(A1) uniform convexity at each point	Read as Bakry–Émery for the measure on the fibre, it does not hold at weak-coupling scales by the Convexity Ceiling Lemma	Does not apply Proved here (the Convexity Ceiling Lemma) + the reading is Estimate
Convexity at large fields (the main issue of the original)	On the group, large fields (the far hemisphere) are necessarily non-convex (§ 04). But the probability is $e^{-c\beta}$	Changes (the wall becomes unavoidable, and goes to where the probability is small) Estimate
(A2)(A3) coupling and commuting of scales	In tree (plaquette) coordinates, automatic from the hierarchical structure. In link coordinates the Dirichlet form is not diagonal in the hierarchy, and commuting is expected to break	Tree: applies / link: expected not to apply Estimate
Lemma 2.7 (Brascamp–Lieb for the conditional measure from convexity)	For typical coarse values, the minimal broken line on the fibre is unique and the gap is of order β. When the coarse value is near -1, the geodesics from the identity to -1 form a family of two-dimensional spheres, and the gap in that direction is expected to drop to $O(1)$. Long arcs are expected to be saddle points, not a second well	Changes (convexity cannot be used, but the loss is expected to be polynomial) Estimate
Lemma 2.8 (an identity that turns the derivative in the direction of the coarse field into a covariance by parallel translation of the fluctuation and integration by parts)	Translating the last factor of each copy from the right moves the coarse holonomy by the same amount, and by the invariance of Haar an identity of the same type holds. The difference is that the score is bounded rather than linear in the field	Applies (in form) Estimate
Lemma 2.9 (the Helffer–Sjöstrand representation and an operator lower bound on the Witten Laplacian)	On a manifold, $\mathrm{Ric} + \mathrm{Hess}\, V$ enters the Witten Laplacian on one-forms. What is needed is a lower bound at each point, and for the same reason as (A1) it does not hold at weak coupling	Does not apply Estimate

Part in the original	Carried over to the group	Verdict
Multiscale Bakry–Émery (the Polchinski flow; the coefficient $\dot\lambda_t$ may be negative)	The natural semigroup on the group is convolution with the heat kernel, $V_t = -\ln(e^{-V} * K_t)$. The convolution step of Migdal–Kadanoff is its discrete version. What is needed is $\inf \mathrm{Hess}\, V_t$ (from below), and a formulation on the group was not found within the range searched	Changes (not yet formulated) Estimate

In sum. The decomposition of the variance and identities of the type of Lemma 2.8 hold on the group too. What does not hold is the two places that use convexity **at each point** (Lemmas 2.7 and 2.9), and by the Convexity Ceiling Lemma they do not hold at weak-coupling scales however the constants are arranged. The wall of "convexity at large fields" in the original does not vanish on the group but becomes unavoidable, while the probability on the wall drops to $e^{-c\beta}$ Estimate .

The Polchinski flow on the group. Theorem 3.6 of the review by Bauerschmidt–Bodineau–Dagallier, along a flow C_t of Gaussian covariances, derives from the condition

$$\dot C_t \, \mathrm{Hess}\, V_t(\phi)\, \dot C_t - \tfrac{1}{2}\ddot C_t \;\geq\; \dot\lambda_t \, \dot C_t \quad (\text{for all } \phi), \qquad \frac{1}{\gamma} = \int_0^\infty e^{-2\lambda t}\, dt \tag{6}$$

the log-Sobolev constant $2/\gamma$ Cited . $\dot\lambda_t$ may be negative, and non-convex time intervals are paid for as the swelling $e^{-2\lambda t}$. A version for curved spaces (a form including the flow of the metric and Ric, Theorem 3.20) is also stated, and the original writes that "we currently do not know interesting applications of this generalisation" Cited . It is a candidate home for group-valued fields Estimate . Written on the group, the natural semigroup for this flow is convolution with the heat kernel, $V_t = -\ln(e^{-V} * K_t)$, and what is needed is an estimate of $\inf \mathrm{Hess}\, V_t$ from below. The estimates of the Migdal–Kadanoff recursion so far (the recursion of the stiffness; the paper record that the number of steps until strong coupling is entered is at most $\beta + 4\log\beta + 20$ Assumption (no proof placed in this book)) are all estimates from above ($\sup \mathrm{Hess}$), and the quantity that connects the two is the flow of $\inf \mathrm{Hess}\, V_t$. In weak-coupling time intervals $\inf \mathrm{Hess}\, V_t \approx -\beta$ (the antipode), and how much $e^{-2\lambda t}$ swells has not been estimated Estimate . The first number is one step of the one-link weight (the question "Hessian of the one-link weight", 24-08).

The $\frac{1}{4}$ type does not come back for one link. $e^{(\beta/2)\,\mathrm{tr}\, U}\, d\mathrm{Haar} = e^{\beta\cos\theta}\, d\mathrm{Haar}$ has a single well ($\theta = \pi$ is the minimum of the density; there is no second local minimum), and the gap is expected to approach the Hessian β at the identity for large β, without getting exponentially worse Estimate (general semiclassical theory. The original on uniform log-Sobolev for Boltzmann measures with a field on spheres is known only bibliographically, and has not been checked). The wall of the $\frac{1}{4}$ type at the summit of the double well (two wells) does not appear for one link. On the fibre, the degenerate family where the coarse value is near -1 is expected to produce a polynomial loss (Table 1).

What this section can say: the verdicts of Table 1, apart from the row of the Convexity Ceiling Lemma, are estimates. The results for hierarchical models have no implication for four-dimensional Yang–Mills under any verdict.

> 06

> **Tools that pay with rarity — currency, Lyapunov, field-dependent constants, the example of the action barrier**

Currency. The currency-mix-up type of 23-10 in Part 23 said that what the Poincaré inequality measures is the speed of exit, and that what can pay with rarity is the currency of entropy (23-10 currency-mix-up type). As a formula, when a set B satisfies $\mu(B) \le e^{-s}$,

$$s \int_B g^2 \, d\mu \;\le\; \mathrm{Ent}_\mu(g^2) + \ln 2 \int g^2 \, d\mu \quad (7)$$

Proved here ((3) of 23-09). The ratio of the variance to the Dirichlet form does not change when the function is multiplied by a constant, so if a function supported on a rare set is put in, the rarity cancels between numerator and denominator, and at the Poincaré step rarity has no effect. **A step that pays with rarity needs to pass through an inequality of log-Sobolev (entropy) type**. This is why the candidate tools on the group are not the Poincaré induction (Brascamp–Lieb at discrete scales) but the type that flows Bakry–Émery at each scale in log-Sobolev form ((6) of § 05) Estimate .

The Holley–Stroock perturbation. If $\mu' = e^{-U}\mu/Z$ with U bounded, the Poincaré constant (and the log-Sobolev constant) of μ' is at most $e^{\operatorname{osc} U}$ times that of μ Assumption (the standard form as remembered; the original has not been checked in this chapter). It is a tool that pays for the non-convex place with the "oscillation", and the cost is exponential in the oscillation, independent of rarity. Regarding the one-link Wilson weight as a perturbation of Haar, $\operatorname{osc} U = 2\beta$, and the loss is $e^{2\beta}$. In Part 23, the plan to pay only for the summit band with this perturbation (the detour plan "perturbing only the summit band") put the one-site oscillation $\operatorname{osc}/\hbar = 3.7/11/37$ into the exponent, multiplied by the length along the chain, and came out 2000–4900 times outside the allowance (23-10 § 03, type of the address of 1/ħ) Reading .

Lyapunov conditions. For a generator L, if there are $W \ge 1$ and a good set G with

$$LW \;\le\; -a\,W + b\,\mathbf{1}_G \qquad (a > 0,\ b \ge 0) \quad (8)$$

(a drift that returns from the bad set to the good set), and a local Poincaré inequality (constant C_G) holds on G, then the global Poincaré constant is at most $(1 + bC_G)/a$ Cited (of the type of Bakry–Barthe–Cattiaux–Guillin 2008. The statement was checked through a restatement in a secondary source, 23-14 § 03. The positive control on the one-dimensional Gaussian is row 20 of 23-14 § 04). Convexity is used only on the good set G, and the bad set is paid for by "the speed of coming back". On the group it is natural to take the good set near the identity and the bad set on the far hemisphere, and the loss is expected to be polynomial Estimate . However, (8) is a Poincaré conclusion, not a form that uses rarity itself. To use rarity, a step that moves the type of (8) into log-Sobolev form (the currency of entropy) is needed Estimate .

Field-dependent constants. Bauerschmidt–Bodineau write in Remark 2.4 that ε in (A1) may be made to depend on the coarse field, $\varepsilon = \varepsilon(\phi)$ (but uniformly less than 1), and add that it is "hard to use" Cited . It is a direction that allows ε to be large at bad coarse fields, but as long as it must be uniformly less than 1, it does not get past the ceiling of the Convexity Ceiling Lemma. To get past it, the condition must be replaced by one in the form of an **average weighted by the probability** of bad fields Estimate .

An example where rarity paid — the margin of the action barrier. In the first step of Part 23, when the end of the core on the box side of the Gap-Inside-the-Box Proposition crossed the summit, a second minimum point appeared on the wall of conditioning, and the Poincaré constant of the core grew like $e^{\Delta S/\hbar}$ (23-11 § 06). This is the cost of the non-convex place inside the core. That cost is paid by the rarity $D/(C\xi)^2 \le d e^{-2S_0/\hbar}$ carried by the local term of the Gap-Inside-the-Box Proposition (the rarity of creating a pair of kinks): under the core action-barrier assumption $C_P \le c_* \hbar^{-p} e^{\Delta S_*/\hbar}$ and the reading of ξ, the local term vanishes like $e^{-(2S_0-\Delta S_*-\delta)/\hbar}$ Proved here ((12) of 23-11. Lean LatticeGauge.CoreActionGap.cloc_term_poly, cloc_term_vanishes, assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$). On the $\hbar = 0$ reading, $\Delta S_*^0 \le 1.4 \times 10^{-2}$, its ratio to $2S_0 = 4/3$ is at most 1.05×10^{-2}, and the margin was about 95 times Reading (Lean LatticeGauge.CoreActionGap.measured_ratio, assumption: the measured upper bound $\Delta S^0 \le 0.01401$; the conclusion is in the form of the ratio $\le 1.06 \times 10^{-2}$). Rather than removing the non-convex place by convexity, it is paid by a stronger factor of rarity: this is a small real example of the type of this chapter. However, the core action-barrier assumption itself is an assumption row, supported only by the $\hbar = 0$ reading and the lower bound on $\hbar \ln C_P$.

Table 2 How to pay for the non-convex place (what pays, and what it costs)

Way of paying	What pays	Form of the cost	For one-link $SU(2)$	Label
Convexity (Bakry–Émery, Brascamp–Lieb)	A lower bound on the curvature at every point	The inverse of the lower bound	Usable only for $\beta < 2$ (the One-Link Wilson Weight Proposition). Impossible in principle at weak coupling by the Convexity Ceiling Lemma	Proved here
The Holley–Stroock perturbation	The oscillation of the potential	e^{osc}	$e^{2\beta}$: an exponential loss	Assumption (from memory)
Lyapunov condition (8)	A drift returning from the bad set, and a local constant on the good set	$(1 + bCG)/a$	If the good set is taken near the identity, the loss is expected to be polynomial	Cited (type), Estimate (application to the group)
The currency of entropy (7)	The rarity s of the bad set	Works in the form $1/s$	$s \approx \beta$ ((5) of the One-Link Wilson Weight Proposition). At the confinement scale it becomes $s = O(1)$ and runs out	Proved here (inequality), Estimate (running out)
Paying with a strong factor of rarity (the margin of the action barrier)	A rarity $e^{-2S_0/\hbar}$ stronger than the cost $e^{\Delta S_*/\hbar}$	The margin $\Delta S_* < 2S_0$	The example of the first step (0+1 dimensions). The counterpart on the group is undecided	Proved here (algebra), Reading (margin), Assumption (the core action-barrier assumption)

What this section can say: (7) and the algebra of the action barrier have proofs; the Lyapunov condition is a type whose statement was checked in a secondary source; Holley–Stroock is the standard form from memory. Which of them works on the group, and with what loss, is an estimate.

07

If it were made into rules — the class of measures that pay with rarity

Let us try to rewrite what we have seen in this chapter in the form of rules. It is an attempt that follows the phrasing of the question "if new mathematics is to be made, what should be made the rules of the mathematics?", and the rules written here are **candidates**. Neither consistency nor the ability to restate existing results has been checked. None of them promises a prospect of closing the third step (the continuum limit and the OS axioms) Estimate . Other candidates (axiomatising scale, the algebra of loops, conserved quantities and sectors, an axiom of asymptotic freedom, rules for curvature) and a list of combinations are placed in 24-08 § 02.

Candidate: the class of measures that pay with rarity

Objects. A pair of a probability measure $\mu = e^{-V}\,d\mathrm{vol}/Z$ on a compact Riemannian manifold M (a product of groups) and a good set $G \subset M$. The parameters $(\kappa_{\mathrm{BE}}, s, a, b)$ define the class $\mathcal{C}(\kappa_{\mathrm{BE}}, s, a, b)$.

Candidate rules. Rule of local convexity — locally convex on the good set: $\mathrm{Ric} + \mathrm{Hess}\, V \geq \kappa_{\mathrm{BE}} > 0$ on G. Rule of rarity of bad sets — $\mu(M \setminus G) \leq e^{-s}$. Rule of return drift — a drift returning from the bad set: the type (8), $LW \leq -aW + b\mathbf{1}_G$. Rule of paying for entropy — rarity is paid only through the entropy inequality (7) (the class is defined by quantities of log-Sobolev type, not Poincaré). Rule of closing under coarse-graining — convolution with the heat kernel on the group, $V_t = -\ln(e^{-V} * K_t)$, maps the parameters as $(\kappa_{\mathrm{BE}}, s, a, b) \mapsto (\kappa'_{\mathrm{BE}}, s', a', b')$ by an explicit recursion.

A constraint, not a rule. By the Convexity Ceiling Lemma, a class that places the rule of local convexity on the whole of M is empty for weights whose $\operatorname{osc} V$ exceeds $KD^2/2$ (π^2 for one-link $SU(2)$). For the one-link Wilson weight it is empty for $\beta > \pi^2/2$. **The bad set always exists.**

The rule of paying for entropy makes the currency type of Part 23 into a rule (23-10 currency-mix-up type: Poincaré is the speed of exit, rarity is the currency of entropy).

Candidate first theorems. (a) For one measure: if $\mu \in \mathcal{C}$, the log-Sobolev constant is at most $F(\kappa_{\mathrm{BE}}, s, a, b)$, and F is not exponential in s. On flat space the type that combines a Lyapunov condition with a local inequality corresponds to this (§ 06), and the Holley–Stroock perturbation pays e^{osc}. (b) Closure: applying one step of the heat kernel to the one-link Wilson weight of $SU(2)$ stays in the class, with $s' \geq s - O(1)$. This is the first step of the question "Hessian of the one-link weight" (24-08 § 01). For class functions (functions of the single variable, the angle θ from the identity), convolution with the heat kernel only multiplies the coefficients of the character expansion by exponentials of the Casimir values, so how the lower bound on the curvature on the good set and the weight of the bad set are carried over can be followed by one-variable paper calculation and small numerics Estimate .

Parts that already exist. On flat fields, the weight (regulator) $\tilde{G}(X, \phi)$ of the appendix of the book by Bauerschmidt–Brydges–Slade has four properties — product factorisation (it factorises over disjoint sets), submultiplicativity in sums of fields $\tilde{G}(X, \phi + \phi') \leq \tilde{G}^2(X, \phi)\,\tilde{G}^2(X, \phi')$, closure under the fluctuation expectation $\mathbb{E}_+ \tilde{G}^t(X) \leq 2^{|B(X)|}$, monotonicity to the next step $\tilde{G}^t \leq \tilde{G}_+ -$

and from them the one-step estimate $\|\mathbb{E}_+\theta K(X)\|_{\tilde{G}_+} \le 2^{|B(X)|}\|K(X)\|_{\tilde{G}}$ follows (the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS)) Cited . It is a real example of **writing the rarity of large fields as a weighted norm and closing it under the expectation of one step of coarse-graining**. What is needed on the group is the group version of this "closing in one step", and the place where submultiplicativity in sums of fields ((ii) of BBS) uses **addition** of fields is replaced by the group product u_1u_2. The multiscale Bakry–Émery of Bauerschmidt–Bodineau–Dagallier ((6) of § 05, $\dot{\lambda}_t$ may be negative) and its version for curved spaces (Theorem 3.20) are also candidate parts Cited . The entropy stability lemma (a form that takes an upper bound on the covariance as its condition instead of the pointwise Hessian, Lemma 3.21) uses a linear tilt $e^{(h,\zeta)}$, so it does not carry over directly to the group Estimate .

Combining with one step of coarse-graining — the coupling window. Combining this class with axiomatising scale (a semigroup of coarse-grainings and a flow on it), one can write a form in which the coarse-graining kernel maps the class at each step, $\mathcal{C}_j = \mathcal{C}(\kappa_j, s_j, a_j, b_j)$, as $\mathcal{C}_j \to \mathcal{C}_{j+1}$. A candidate first theorem is "in a finite box, on the ultraviolet side, as long as the effective coupling β_j is at least β_*, the class closes in the form $s_{j+1} \ge s_j - c$, $\kappa_{j+1} \ge \kappa_j(1 - c/\beta_j)$" Estimate . Where it has closed, the effective measure has a uniform log-Sobolev inequality, and from there it can be handed over to the tools of the strong-coupling side (the type that pays with the Ricci curvature of the group; the result of Shen–Zhu–Zhu giving log-Sobolev and the mass gap at small β Cited (abstract)). But the rate of rarity s_j is proportional to the effective coupling ($s \approx \beta$ by (5) of the One-Link Wilson Weight Proposition), and it runs out at the confinement scale when $\beta_j = O(1)$. The currency of curvature works only in a narrow range. **Between the two, a coupling window remains where neither currency is enough**. This window is a restatement in the language of rules of the second difficulty of the third step — that there is no expansion connecting the ultraviolet and the infrared — and is not a new obstacle (24-07 § 06, 24-07 Table 3) Estimate .

In the hierarchical gauge model (§ 05), the number of steps before the static flow enters strong coupling is finite, and the three — the ultraviolet side, the window and the strong-coupling side — come together within one model. If the candidate rules are to be tried first, it is on this model. However, the static flow there is a one-variable recursion with no many-body remainder, so crossing the window is easy because of the structure of the model, and the results imply nothing for four-dimensional Yang–Mills Estimate .

Relation to the roots Estimate . Of the three roots of the third step (24-07 § 05), the one this class addresses head-on is the third (there is no global gauge fixing, and large fields cannot be controlled by convexity). This is because it places an alternative class where the Convexity Ceiling Lemma says "on the group the convexity class is empty". It addresses the first (uniformity in the number of steps) only when combined with axiomatising scale, and the key is whether the recursion of the rule of closing under coarse-graining gives constants uniform in the number of steps. It does not address the second (connecting the ultraviolet and the infrared) — the coupling window above is the shape of that failure.

What this section can say: the rules are candidates, and the candidate theorems are candidates. The existing parts (the four properties of the weight and the One-Step Bound on the Activity Norm, multiscale Bakry–Émery) hold only within the conditions of their originals (flat fields, hierarchical, flows of Gaussian covariances).

08

The road of stochastic quantisation — subcriticality, and the direction of rebuilding where the objects live

Stochastic quantisation is the road that makes the Yang–Mills measure as the long-time limit of a Langevin dynamics (a stochastic partial differential equation) that has it as its invariant measure (7-09). The dynamics is of the same kind as the Langevin dynamics measured by this book's CP, but the aim is the **construction of the measure**, and the speed of relaxation (the relaxation-time assumption) is not asked about.

The key condition of the solution theory of regularity structures is subcriticality, and it holds only when $d < 4$ (§ 1.2 of Chevyrev's review Cited). In two dimensions, a Polish state space on the space of gauge orbits and a Markov process have been made, and the Yang–Mills measure has been shown to be its invariant measure (Chandra–Chevyrev–Hairer–Shen, Chevyrev–Shen Cited (abstract)). In three dimensions, it has reached a new state space, the space of one-forms smoothed by the heat flow, local-in-time solutions and gauge-covariant renormalisation; the three-dimensional Yang–Mills measure is not yet constructed even on the periodic box $\mathbb{T}^3$ (§ 1 and § 4 of Chevyrev's review Cited).

This road goes not in the direction of extending the lattice renormalisation group, but in the direction of **rebuilding where the objects live (the state space) and the means of construction (stochastic dynamics)** (24-07 § 05) Estimate . However, the condition of subcriticality shows that this new tool too becomes exactly critical in four dimensions. It is one manifestation of the coupling being dimensionless ((c) of 24-07 § 01). This road gives tools neither for the four-dimensional relaxation-time assumption nor for the local relaxation-time assumption.

What this section can say: the condition of subcriticality and the three-dimensional non-construction are statements in the text of the review; the individual two- and three-dimensional results are copies from abstracts. This book does not use regularity structures as a tool.

09

What cannot be said

(a) Hierarchical models. The results for hierarchical models imply nothing for four-dimensional Yang–Mills (they lack translation and rotation invariance). The verdicts of Table 1 are also estimates within the hierarchical model.

(b) The Polchinski flow on the group. No source formulating the flow by convolution with the heat kernel up to the multiscale Bakry–Émery condition was found within the range searched (this is not "there is none").

(c) The fibre. That the gap in the direction of the degenerate family near coarse value -1 drops to $O(1)$, and that long arcs are saddle points, are estimates (by analogy from the Morse index of continuous geodesics).

(d) Dynamics. Statements on the gap or log-Sobolev for Glauber dynamics or Langevin dynamics of hierarchical gauge models were not found within the range searched.

(e) The kind of the original. The induction of Bauerschmidt–Bodineau 2019 is a result of Poincaré type (a spectral gap), not a log-Sobolev result. The form that flows log-Sobolev is later work by the same authors and a theorem of the review.

(f) Candidate rules. The class of § 07 is a candidate, and none of consistency, restating existing results, or the candidate theorems has been checked. "The coupling window is a restatement of the source "no expansion linking scales"" is an estimate, and the width of the window has not been measured in any model.

(g) The example of the action barrier. The margin of 95 times is a $\hbar = 0$ reading in 0+1 dimensions, and the core action-barrier assumption is an assumption row. We say nothing about whether renormalisation on the group has a margin of the same form.

(h) Stochastic quantisation. That four dimensions is critical is not a proof that tools cannot be made in four dimensions. It only copies the statement of the conditions of the original.

10

Table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	Bakry–Émery (1)	Cited	Theorem 3.6 of the review	Lower bound on the curvature at each point	—
2	(A1) of the hierarchical induction, Theorem 2.1 and Corollary 2.2 (2)	Cited	§ 02	Scalar, hierarchical	—
3	The one-site diagonal at the summit $\frac{15}{4}$ and the sum of couplings 4	Proved here	(11) of 23-09	0+1, $\varepsilon = \frac{1}{2}$	—
4	Convexity Ceiling Lemma: $\operatorname{osc} V \le K D^2/2$ (3)	Proved here	Proof in § 03	Compact manifold without boundary	— (the one-dimensional version is a candidate)
5	No Concentration at the Identity Corollary: if convex on $SU(2)$ then $\mu \le e^{\pi^2}$ Haar (4)	Proved here	Proof in § 03	Unit S^3	—
6	One-Link Wilson Weight Proposition: for one link $(2 + \beta \cos\theta) g$, $\beta < 2$, (5)	Proved here	Proof in § 03	$\beta \ge 2$ ((5))	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
7	At a maximum point of a closed manifold $\operatorname{Hess} V \leq 0$	Proved here	§ 04	—	—
8	The raw rate of wrapping on the strip matches the Kramers slope to 1–3%	Reading	24-02 § 05	Two-dimensional $U(1)$, $\beta = 3..6$, post hoc	—
9	The 0+1 one-site ratio ω_b/m	Proved here	(7) of 24-03, 24-04 § 07	0+1	`LatticeGauge.LocalHDoubleWell.one_slice_ratio` (assumption: $0 < \varepsilon, m, \omega$)
10	Read as Bakry–Émery on the fibre, (A1) on the group does not hold at weak coupling	Estimate	§ 02, § 05	Weak-coupling scales	—
11	The verdicts of Table 1	Estimate	Table 1	Hierarchical model	—
12	Multiscale Bakry–Émery (6), the version for curved spaces	Cited	Theorems 3.6 and 3.20 of the review	Flows of Gaussian covariances	—
13	The number of steps until the hierarchical model enters strong coupling $\leq \beta + 4 \log \beta + 20$	Assumption	Paper record (no proof in the book)	Migdal–Kadanoff, $SU(2)$	—
14	A wall of the $\frac{1}{4}$ type does not come back for one link	Estimate	§ 05	One link	—
15	The currency inequality (7)	Proved here	(3) of 23-09	—	—
16	The cost e^{osc} of the Holley–Stroock perturbation	Assumption	Standard form from memory	—	—
17	$(1 + bCG)/a$ from the Lyapunov condition (8)	Cited	Secondary source (23-14 § 03)	—	Positive control `gauss1d_bbcg` (no assumptions, 23-14)
18	Field-dependent $\varepsilon(\phi)$ (uniformly less than 1)	Cited	Remark 2.4 of the original	—	—
19	Core action-barrier assumption	Assumption	23-11 § 06	0+1, the core on the box side	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
20	$\Delta S_*^0 \le 1.4 \times 10^{-2}$, a margin of about 95 times	Reading	23-11 § 06	The $\hbar = 0$ reading	`LatticeGauge.CoreActionGap.measured_ratio` (assumption: the measured upper bound)
21	The local term vanishes like $e^{-(2S_0 - \Delta S_* - \delta)/\hbar}$	Proved here	(12) of 23-11	$\hbar \to 0$	`LatticeGauge.CoreActionGap.cloc_term_poly`, `cloc_term_vanishes` (assumptions: the core action-barrier assumption, the reading of ξ, $\Delta S_* < 2S_0$)
22	Tools on the group must push with probability	Estimate	§ 04	—	—
23	The four properties of the weight and the One-Step Bound on the Activity Norm	Cited	Appendix of BBS	Flat fields	—
24	The class of measures that pay with rarity (candidate)	Estimate	§ 07	Candidate	—
25	Log-Sobolev and the mass gap at strong coupling	Cited	Shen–Zhu–Zhu (abstract)	Small β	—
26	The coupling window is a restatement of the source "no expansion linking scales"	Estimate	§ 07, 24-07 § 06	—	—
27	The solution theory is subcritical for $d < 4$; the three-dimensional measure is also not yet constructed	Cited	Chevyrev's review	—	—
28	The road of stochastic quantisation gives no tools for the four-dimensional relaxation-time assumption	Estimate	§ 08	—	—

What the machine-checking column guarantees is only real algebra and arithmetic; that ΔS is the action barrier of the core, and that C_P is the Poincaré constant of the path measure, are outside Lean (23-13). The Convexity Ceiling Lemma, the No Concentration at the Identity Corollary and the One-Link Wilson Weight Proposition have no machine checking.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  Bakry–Émery: lower bound on curvature ⇒ log-Sobolev, Poincaré
 2  CIT  (A1), Theorem 2.1, Corollary 2.2 of the hierarchical induction (Bauerschmidt–Boc
       scalar, hierarchical
 3  PRV  one-site diagonal at the summit 15/4 and sum of couplings 4         0+1 (23-09)
 4  PRV  Convexity Ceiling Lemma: osc V ≤ KD²/2                              closed manif
 5  PRV  No Concentration at the Identity Corollary: if convex on SU(2), μ ≤ e^(π²) Haar
 6  PRV  One-Link Wilson Weight Proposition: curvature (2 + β cos θ), probability (5)
 7  PRV  Hess ≤ 0 at a maximum point of a closed manifold
 8  RDG  raw rate on the strip matches the Kramers slope to 1–3%             two-dim. U(1
 9  PRV  0+1 one-site ratio ω_b/m                                            Lean one_sli
10  EST  (A1) on the group does not hold at weak coupling ⟸ 2, 4
11  EST  verdicts of Table 1 ⟸ 10
12  CIT  multiscale Bakry–Émery, version for curved spaces
13  ASM  number of steps until strong coupling (paper record)
14  EST  the 1/4 type does not come back for one link ⟸ 6
15  PRV  currency inequality (7)                                            (3) of 23-09
16  ASM  Holley–Stroock cost e^osc                                           from memory
17  CIT  Lyapunov condition (8)                                              secondary sc
18  CIT  field-dependent ε (uniformly less than 1)
19  ASM  core action-barrier assumption                                      0+1
20  RDG  ΔS⁰ ≤ 1.4·10⁻², margin about 95 times                              ħ = 0 readin
21  PRV  the local term vanishes (12) ⟸ 19                                    Lean cloc_
22  EST  tools on the group push with probability ⟸ 5, 6, 7, 15
23  CIT  four properties of the weight, One-Step Bound on the Activity Norm (BBS)   flat
24  EST  class of measures that pay with rarity (candidate) ⟸ 4, 15, 17, 23
25  CIT  log-Sobolev and gap at strong coupling                              abstract
26  EST  the coupling window restates the lack of an expansion linking scales ⟸ 6, 24,
27  CIT  subcriticality d < 4; three dimensions also not yet constructed
28  EST  stochastic quantisation gives no tools for the four-dimensional relaxation-time
```

The way to check is to see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The nodes with proofs placed are four — the Convexity Ceiling Lemma, the No Concentration at the Identity Corollary, the One-Link Wilson Weight Proposition, and the maximum point on a closed manifold — all statements about one measure on one manifold. The nodes on the conclusion side of this chapter (tools on the group push with probability, the class of measures that pay with rarity, the restatement as the coupling window, the note on stochastic quantisation) are all Estimate, and there is neither a node for the mass gap nor a node for the relaxation-time assumption. The node where the local term vanishes (an example where rarity paid) is algebra on top of the core action-barrier assumption, and does not enter the premises of the nodes on the conclusion side. This is because the 0+1-dimensional example is an example of the type, not grounds for a statement on the group.

What this chapter can say

What is said	What is not said
Proved here The convexity ceiling on closed manifolds (the Convexity Ceiling Lemma); a convex measure on $SU(2)$ does not exceed e^{π^2} times Haar (the No Concentration at the Identity Corollary); the curvature of the one-link Wilson weight, and the upper bound on the probability of the non-convex set (the One-Link Wilson Weight Proposition)	The convexity and gap of the conditional measure on the fibre (a product of many links). The value of the gap in the direction of the degenerate family
Cited Bakry–Émery; the hypotheses and conclusions of the hierarchical induction; multiscale Bakry–Émery; the four properties of the weight and the One-Step Bound on the Activity Norm; the subcriticality of stochastic quantisation	A formulation of the Polchinski flow on the group. The gap of the dynamics of hierarchical gauge models
Proved here + Reading The margin of the action barrier in Part 23 is an example where the non-convex place was paid with rarity (on the core action-barrier assumption and the $\hbar = 0$ reading)	The core action-barrier assumption itself. That the same margin exists on the group
Estimate That tools on the group must push with probability; the verdicts of Table 1; the class of measures that pay with rarity (candidate); that the coupling window is a restatement of the lack of an expansion linking scales	None of these is a theorem. The implication of hierarchical-model results for four dimensions. The consistency of the candidate rules

Sources and reproduction

Item	Kind	Source / tools
The hierarchical induction ((A1), Theorem 2.1, Corollary 2.2, Remark 2.4, Lemmas 2.7–2.9, the self-assessing sentence)	Cited	R. Bauerschmidt, T. Bodineau, arXiv:1809.02075 (Commun. Math. Phys. 373 (2020) 1167). § 1– § 2 checked in the text
(1), (6), the version for curved spaces, entropy stability	Cited	R. Bauerschmidt, T. Bodineau, B. Dagallier, arXiv:2307.07619 (review). Theorems 3.6 and 3.20 and Lemma 3.21 checked in the text
The four properties of the weight, the One-Step Bound on the Activity Norm	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474. Appendix A.6 checked in the text
Hierarchical gauge models	Cited	K. R. Ito, Commun. Math. Phys. 95 (1984) 247 (text). K. R. Ito, Phys. Rev. Lett. 55 (1985) 558 (abstract)
Lyapunov condition (8)	Cited	D. Bakry, F. Barthe, P. Cattiaux, A. Guillin, Electron. Commun. Probab. 13 (2008) 60 (bibliographic data from memory. The statement was checked through a restatement in a secondary source, 23-14 § 03)
The Holley–Stroock perturbation	Assumption	R. Holley, D. Stroock, J. Stat. Phys. 46 (1987) 1159 (both bibliographic data and statement from memory)
Log-Sobolev and the mass gap at strong coupling	Cited	H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737 (abstract). The numbers for the range are not written, since they depend on the normalisation convention
Stochastic quantisation	Cited	I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (§ 1, § 1.2, § 4 in the text). A. Chandra, I. Chevyrev, M. Hairer, H. Shen, arXiv:2006.04987, arXiv:2201.03487 (abstract). I. Chevyrev, H. Shen, arXiv:2302.12160 (abstract)

Item	Kind	Source / tools
Convexity Ceiling Lemma, No Concentration at the Identity Corollary, One-Link Wilson Weight Proposition	Proved here	Proofs in § 03 of the text. The only numbers are the values of $\frac{54\pi}{25}e^{-3/2}$ and e^{π^2}
The examples from Part 23 (the summit, currency, the Holley–Stroock plan, the action barrier)	This book	23-09 § 06, 23-10 § 04, 23-11 § 06
Lean	Lean	LatticeGauge.LocalHDoubleWell (`one_slice_ratio`), LatticeGauge.CoreActionGap (`cloc_term_poly`, `cloc_term_vanishes`, `measured_ratio`). The bundles guarantee only algebra and arithmetic

Terms used in this chapter — the convexity ceiling, the lower bound on the Bakry–Émery curvature κ_{BE}, Brascamp–Lieb, fibre, the reversal of convex and non-convex places, the rate of rarity s, currency (entropy), the Holley–Stroock perturbation, Lyapunov conditions, field-dependent constants, the Polchinski flow on the group, the class of measures that pay with rarity, the coupling window, subcriticality. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-07 Because it is four dimensions — what works in four dimensions, limited to two points: the dimensionless coupling and topology.

Because it is four dimensions — the dimensionless coupling and topology

> This chapter organises established knowledge. § 03, § 04, § 06 and § 07 are a map of open questions.

What this chapter shows — We state what works in four dimensions, limited to two points: that the coupling is dimensionless (the critical dimension, dimensional transmutation, the opposite directions of asymptotic freedom for ϕ^4_4 and for Yang–Mills), and that there are topological configurations with no scale (instantons and the topological charge). Putting the two points together gives the estimate that the slow quantity and the correlation length split into different origins. Next, for each of the four steps of the staircase of Part 23, we put the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost into one table, and state why the third step (the continuum limit) stayed open so long, through the three sources of difficulty and the descriptions in primary sources (for the existence part of the third step, the committee's assessment is that the construction published on 2026-11-10 has shown it; we read it in Volume 2). On topology, we set the range in which the relaxation-time upper-bound assumption reads as false on a periodic box, and the fact that for this book's group $SU(2)$ it is undecided whether freezing breaks the diffusive-relaxation-time assumption, beside the two-dimensional test bed of 24-02. We add, one line each, why the non-perturbative mechanisms of physics are not proofs.

Prerequisite chapters — 23-03 § 02 (the definition of the four steps) and 23-03 § 06 (the prospect of lifting and the reasons it does not lift), 23-15 § 07 (the distance from the second step on), 23-16 (what is not claimed), 23-01 § 05 (the note on the relaxation-time assumption and topological sectors), 4-07 (dimensional transmutation), 7-07 (Balaban and MRS), 24-01, 24-02. Notation is in 24-09.

Said first — The level is [Y3]. We state the two points not as "things that happen only in four dimensions" but as properties of a theory being at its critical dimension. The reading that puts the two points together is an estimate Estimate . "Does it reach" and the costs in the table of the four steps are estimates and guesses. § 05 is an arrangement of the descriptions in the primary sources read, and the text of Balaban's papers was not obtained (we go through the bibliographic data and the descriptions in other sources). This chapter shows neither the mass gap nor the construction of the continuum limit.

The order of this chapter

1. That the coupling is dimensionless — the critical dimension, dimensional transmutation, asymptotic freedom in the opposite direction ((1)(2), Figure 1)
2. Topological configurations with no scale — instantons, and two ways of counting the weight of dislocations ((3), Table 1)
3. An estimate that puts the two points together — the slow quantity and the correlation length split into different origins
4. The four hurdles — step, hurdle, known reach, does it reach, cost (Table 2)

5. Why the third step stayed open so long — the three sources of difficulty, the object, tools, theory, history (Table 3)
6. A paragraph on candidate rules — the "coupling window" is a restatement of the source "no expansion linking scales"
7. Topology and the box — undecided for $SU(2)$ — the periodic box, open boundaries, correspondence with the two-dimensional test bed
8. Physical mechanisms, and why they are not proofs — Table 4
9. What cannot be said
10. Table of the five grades and the dependency graph — Table 5

01

That the coupling is dimensionless — the critical dimension, dimensional transmutation, asymptotic freedom in the opposite direction

This section summarises existing knowledge (the critical dimension, dimensional transmutation, asymptotic freedom). The introduction is in 4-07 § 02.

Writing the action of d-dimensional Yang–Mills as

$$S = \frac{1}{4g^2}\int d^d x \ \operatorname{tr} F_{\mu\nu}F^{\mu\nu}, \qquad [g^2] = (\text{length})^{\,d-4} \quad (1)$$

the coupling g^2 is exactly dimensionless in four dimensions Cited (dimension counting. On the lattice, by Wilson's heuristic, $\beta \propto \varepsilon^{4-d}$, and in four dimensions β does not depend on the lattice spacing ε; § 3 of Chatterjee's review). In three dimensions or fewer, g^2 has the dimension of a negative power of length, and the interaction weakens like a power at short distances (super-renormalisable). In five dimensions or more it strengthens instead (non-renormalisable). In exactly four dimensions, the correction from each change of scale does not shrink like a power, and remains at the size of g^2. In the language of power counting, a quartic monomial in four dimensions is neither relevant nor irrelevant but marginal (Definition 5.2.1 of the book by Bauerschmidt–Brydges–Slade Cited).

Dimensional transmutation. In the non-abelian case, the sign of that correction points to the side "weaker at shorter distances" (asymptotic freedom, 4-06). The action has no ruler of length, yet on quantisation, at one loop,

$$\Lambda = \mu\, \exp\!\Big(-\frac{1}{2b_0\, g^2(\mu)}\Big), \qquad b_0 = \frac{11N}{48\pi^2} \quad (SU(N)\ \text{pure gauge}) \quad (2)$$

an inverse length is born Cited (the standard one-loop form, 4-07). The mass gap and the confinement scale are expected to be constant multiples of Λ. All derivatives of $\exp(-1/g^2)$ vanish at $g = 0$, so Λ appears at no order of the perturbative expansion in g ("invisible to perturbation theory" in 4-07).

How it shows in the relaxation-time assumption. (a) The correlation length grows exponentially in β ($\xi \propto e^{3\pi^2\beta/11}$ by the one-loop bookkeeping for $SU(2)$, 23-03 Table 2), and quantities measured in ξ

carry a factor $\ln \xi \sim \beta$. This is why Part 23 placed the relaxation-time assumption as a power statement rather than a band statement (23-01 § 07). (b) In the Gaussian approximation the coupling does not flow. In the Gaussian approximation on the hierarchical lattice, the static correlation of plaquettes decayed like the massless R^{-4}, and the correlation length was only cut off by the volume (24-02 § 02). A scale is born only from the non-abelian correction. (c) The solution theory of the equations of stochastic quantisation has subcriticality ($d < 4$) as its key condition, and four dimensions is exactly critical (§ 1.2 of Chevyrev's review Cited , 7-09).

The same marginal, but in the opposite direction. Four-dimensional ϕ^4 also has a dimensionless (marginal) coupling, but under the renormalisation flow the coupling approaches 0 on the side of long distances ("infrared asymptotic freedom", the Infrared Asymptotic-Freedom Proposition (Proposition 6.1.7 of BBS) of Bauerschmidt–Brydges–Slade Cited). In the direction one wants to control (long distances) the theory approaches a Gaussian, so the tools reach, and the limit they reached was trivial (the scaling limit is Gaussian) (Aizenman–Duminil-Copin 2021 Cited (abstract), 7-08). In Yang–Mills the coupling approaches 0 on the side of short distances, and grows towards long distances.

```
Figure 1  Two arrows (the flow of the coupling g;
          left is short distance, right is long distance)

            short distance (UV)                long distance (IR)
  φ⁴₄ :  g toward the large side   ←──────────   g → 0 (approaches a Gaussian)
         → near a Gaussian in the direction to be controlled.
           The tools reach; the limit is trivial

  YM₄ :  g → 0 (approaches a Gaussian)  ────→  g grows (g = O(1) at the scale Λ)
         → near a Gaussian only in the UV.
           No reference for an expansion on the IR side
```

We do not write "only in four dimensions". The two-dimensional $O(N)$ ($N \geq 3$) model and the CP^{N-1} model also have the same pair, a dimensionless coupling and asymptotic freedom, and CP^{N-1} also has instantons with no scale. There too, the construction of the ultraviolet limit is not complete (Rivasseau 2000, Douglas 2004 Cited). Conversely, even at the critical dimension, an abelian ($U(1)$) or non-gauge (ϕ^4_4) theory is equivalent to a free field or trivial. The "because it is four dimensions" of this chapter refers to **being at the critical dimension (a dimensionless coupling) and being non-abelian (the direction of asymptotic freedom, topology, the geometry of gauge orbits) happening at the same time** Estimate .

02

Topological configurations with no scale — instantons and the weight of dislocations

By $\pi_3(SU(N)) = \mathbb{Z}$, four-dimensional gauge fields have an integer topological charge Q, and the action of a self-dual solution (an instanton) is $8\pi^2/g^2$, independent of its size ρ Cited (the abstract of Belavin–Polyakov–Schwartz–Tyupkin 1975; the value of the action from the texts of Lucini–Teper 2001 and Athenodorou–Teper 2021). That the action does not depend on ρ is a manifestation of classical Yang–Mills having no scale in four dimensions.

On the lattice, the smoothed topological charge changes only when a small instanton the size of the lattice spacing passes through the lattice ("disappears into a hypercube") (§ 6.1 of Lucini–Teper 2001

Cited). Its weight is estimated from the one-loop density of small instantons $D(\rho) \propto \rho^{11N/3-5}$, and for the time for the topological charge to move by one unit,

$$\ln \tau_Q = b + \left(\frac{11N}{3} - 5\right) \ln \frac{1}{a\Lambda} \quad (3)$$

a primary source gives this count ((32)(34) of Athenodorou–Teper 2021 Cited). For $SU(2)$ the coefficient is $7/3$, and the original itself writes that it is "rather weak". The comparison with fitted slopes is Table 1.

Table 1 The coefficient of (3) and the fitted slopes (Table 43 of Athenodorou–Teper 2021. Heat bath and overrelaxation, periodic box, normalised to a standard physical volume)

N	**Coefficient of (3) $11N/3-5$**	**Fitted slope**
2	$7/3 \approx 2.33$	Excluded from the table (τ_Q does not become large enough compared with the measurement interval)
3	6.0	6.82(7)
4	9.67	9.27(12)
5	13.3	11.13(20)
6	17.0	13.50(30)
8	24.3	17.58(43)

The label of Table 1 is Cited (numbers from the original). There is one independent route. The figure of the original suggests that the probability that the field smoothed by the Wilson flow is between topological sectors (the probability of exceeding the smoothness threshold $h \geq 0.067$) vanishes in proportion to a^6 on lattices at fixed physical size (§ 4.3 of Lüscher 2010, $SU(3)$, the word "suggests" Cited), and this is the same number as the coefficient 6 of (3) for $SU(3)$. For $SU(2)$, no measurement of the same type was found within the range searched.

Two ways of counting. In the crude count that sets the action of a dislocation to c times the one-loop instanton action, the dynamical exponent is of the type $z = (11N/3)c - 4$ Assumption . It differs by 1 from the counting of (3), in how the volume factor at $\rho \sim a$ and the width $d\rho$ are counted. Fitting c to the numbers for $SU(3)$ (the fitted 6.82, or $z \approx 5$ from another original), **in either counting $SU(2)$ falls near the boundary** $z = 2$ Estimate . This book does not judge which of the two counts is right. Nor has it checked that c does not depend on the group.

03

An estimate that puts the two points together — the slow quantity and the correlation length split into different origins

Setting the two points side by side gives the following estimate Estimate . The time of the slow quantity (the topological charge) and the correlation length ξ are both born from the same dimensionless coupling, in the form $e^{(\text{constant})\cdot\beta}$. But the exponents are different numbers. The correlation length is $\xi \propto \Lambda^{-1}$ from (2), and the time of the topological charge is $\tau_Q \propto (a\Lambda)^{-(11N/3-5)}$ from (3); in lattice units τ_Q is of the type $\xi^{11N/3-5}$, and its exponent does not match that of ξ^2, the thing it is compared with.

- For the 0+1-dimensional double well, the slow quantity (the number of kinks) and ξ are born from the same kinks, the exponents match, and only a logarithmic difference remains (24-01 Table 3).
- On the two-dimensional $U(1)$ strip, the static length is linear in β, the length at which conservation breaks is $e^{0.92\beta}$, and the two separate exponentially (24-02 § 03, a reading of a two-dimensional toy).
- For two-dimensional CP^{N-1}, varying N split the strength of freezing and the length at which conservation breaks, while the local ratio did not split (24-02 § 07, a conditional one-point reading).

The reading is that the candidate for the true nature of "even if the mass gap is born, it is not caught in the form of the relaxation-time assumption (the form that ties C_P and ξ^2 with the same power)" is **the slow quantity and the correlation length splitting into different origins**, and that the split comes from two things: the coupling being dimensionless, and topology. On this reading, the form that catches the mass gap is not the whole of the relaxation-time assumption but lies on the side of the part local in time (24-04) and its reverse side, the Poincaré constant of the vacuum measure (24-05).

What this section can say: it is a reading, not a theorem. The numbers for the exponents are fits from the originals and semiclassical counts, and for $SU(2)$ they fall right near the boundary, so whether they "split" for $SU(2)$ is undecided (§ 07). The autocorrelations of the originals are numbers for heat-bath, overrelaxation and HMC dynamics, and rereading them as C_P for the link Langevin dynamics is an assumption (24-01 § 06).

04

The four hurdles

Part 23 split the diffusive-relaxation-time assumption into four steps (23-03 § 02): first step = the path measure of the 0+1-dimensional double well; second step = non-abelian gauge theory on the four-dimensional lattice (with the lattice spacing held fixed); third step = the continuum limit and the OS axioms; fourth step = the mass gap in the limit. Table 2 puts on one sheet, for each step, the hurdle, the known reach, whether the tools of this book reach it, and the kind of cost. There are three kinds of cost. "Weeks" = expected to be reached by accumulating work combining existing tools; "years" = a large extension of existing tools, or a reconstruction of a large body of work, is needed; "new mathematics" = no description of a route in the sources read (this phrase borrows as it is the wording of the question "why is there no mathematics for the third step?").

Table 2 The four hurdles ("does it reach" and "kind of cost" are estimates. The labels for the reach are inside the rows)

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost
First step: the 0+1-dimensional double well	The double-well relaxation-time power statement uniformly as $\hbar \to 0$. Paying for the lack of curvature at the summit, $\frac{1}{4}$, with rarity	The static side (tunnelling splitting, correlation length) is known semiclassically Cited . The upper side of the relaxation-time assumption goes as far as the Double-Well Relaxation-Time Upper-Bound Proposition (algebra on four conditions), with Proved here, Cited, Computer-assisted and Assumption mixed row by row (23-12)	Partly reached. What remains is the rows for each condition (23-15 Table 1)	Weeks (many rows). But the rows for $\hbar < 0.005$ are "another method", and the one proposition at the summit is "method undecided", with no estimate
Second step: four-dimensional lattice, non-abelian	Exponential decay of correlations (the lattice mass gap) at weak coupling (large β), uniformly in the volume. The upper-bound assumption and the relaxation-time lower-bound assumption	At strong coupling (small β) a theorem (Osterwalder–Seiler 1978 Cited (bibliographic data and secondary sources)). In hierarchical models, confinement at every β (Ito 1985 Cited (abstract)). On the real lattice at weak coupling, unsolved (the lattice correlation-length problem, Chatterjee's Problem 5.1)	Within hierarchical models the static side is known, and the dynamic side (coordinates and conservation, 24-02 § 02) is within reach. Weak coupling on the real lattice is out of reach (it is crossing the confinement scale itself, the source "no expansion linking scales" of § 05)	Weeks (the dynamic side of hierarchical models) / "new mathematics" (weak coupling on the real lattice)

Step	Hurdle	Known reach [sources and labels]	Do the tools of this book reach it?	Kind of cost
Third step: the continuum limit and the OS axioms	Existence and uniqueness of the limit as the lattice spacing $\to 0$, infinite volume, the OS axioms, non-triviality	The construction published on 2026-11-10: a Lean formalisation that builds, for any compact simple group, a theory satisfying the Wightman axioms as a limit of the four-dimensional Wilson lattice (the committee's assessment is that it is proved in Lean Cited). The continuum limit on a periodic cubic box, and an infinite-volume limit along a subsequence of volumes. Convergence and uniqueness of the infinite-volume limit are not shown (6-04). The reach before that: ultraviolet stability of the lattice in finite volume (three and four dimensions, Balaban's 13 papers in Commun. Math. Phys. 1983–89 Cited (bibliographic data and descriptions in the texts of other sources)). Schwinger functions with an infrared cutoff in axial gauge (Magnen–Rivasseau–Sénéor 1993 Cited (introduction in the text)). Even the three-dimensional measure was not yet constructed in a finite box (Chevyrev 2022 Cited)	The tools of this book do not reach it. The existence part is cited from the published construction (Volume 2). The remaining items (convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory) are in 6-04	(The existence part is the published construction. The remaining items are expected to need "new mathematics")
Fourth step: the mass gap in the limit	A gap uniform in the volume $\Delta > 0$	None. The problem statement writes that "new ideas are needed for a mass gap uniform in the volume" (§ 6.5 of Jaffe–Witten Cited). Numerically, the lightest glueball of $SU(3)$ is $3.405(21)$ times the square root of the string tension (Athenodorou–Teper 2021 Cited), an extrapolation from finitely many lattice spacings and finite boxes. The published construction has no uniform mass gap either (what it has is polynomial bounds, 6-04)	Does not reach. The one-sided bridge (23-01 § 06) is expected to lift in form only Estimate	"New mathematics"

Before the published construction, the cost of the third step in Table 2 was estimated as "years" for the source "dimensionless coupling" and the source "no global gauge fixing". This estimate depends on descriptions in the sources Estimate : Rivasseau 2000's "simplify and rewrite Balaban's results; since they run to hundreds of pages, this is not a small task", and that Dimock's reconstruction, starting from three-dimensional ϕ^4, reached the stability of the partition function of three-dimensional QED in a little over ten years (abstract). For the source "no expansion linking scales" there are no sources to support an estimate of the cost. The costs of the rows of the first step are as in the estimates of 23-15 § 03, and not a single estimate has been run even for the rows marked "weeks".

What this section can say: the rows of known reach are as their source labels say. "Does it reach" and the kind of cost are estimates, and for the steps marked as not reached we do not say "impossible".

> 05

Why the third step stayed open so long

This section arranges descriptions in the primary sources read before the construction published on 2026-11-10. On 2026-11-10, a Lean formalisation that builds a theory satisfying the Wightman axioms as a limit of the four-dimensional Wilson lattice, together with the committee's assessment report, was published, and the existence part of the third step reached the level of "the committee's assessment is that it is proved in Lean" (Volume 2, 6-00). What is not shown is convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap, Wilson loops and the area law, and one-particle states (6-04). The summary below and Table 3 are kept as the reach before that construction and as a reading of the sources of difficulty.

First a summary in thirteen lines [Y3].

1. The third step is that "there is a single probability measure in the limit of finer and finer lattices, and it satisfies the OS axioms". The theorems in the primary sources before the published construction went only as far as a uniform estimate that, inside a finite box, however fine the lattice is made, the coarse-grained theory does not run wild (Balaban's ultraviolet stability).
2. Beyond that, three things were needed separately — estimates of correlation functions, uniqueness of the limit, and enlarging the box to infinity. The published construction gives the continuum limit on a periodic box and an infinite-volume limit along a subsequence of volumes, but does not show uniqueness of the infinite-volume limit. For three-dimensional non-abelian gauge theory, the measure inside a finite box has not been constructed within the sources read.
3. The first difficulty is that in four dimensions the coupling is dimensionless. In three dimensions or fewer, the interaction weakens like a power as the scale is made finer, and only finitely many divergences had to be dealt with. In four dimensions the correction from each step does not shrink like a power, and the corrections from infinitely many steps must all be controlled in a form independent of the number of steps.
4. The second is that asymptotic freedom works "only at short distances". At short distances the coupling is small and expansions around the free field can be used. At long distances the coupling becomes large, and the confinement scale Λ appears.
5. This Λ appears at no order of the expansion in the coupling, so extending the expansion from short distances does not reach it. The strong-coupling expansion, which converges on the side where the coupling is large, can be used only on the side of large lattice spacing, and does not reach the continuum limit. There is no tool that crosses the scales between the two expansions.
6. The third is non-abelian gauge invariance. No continuous gauge fixing that works for all configurations at once exists (Gribov, Singer), and Balaban chose the gauge afresh scale by scale and place by place.
7. Where the field is large, expansions cannot be used, and the only way to control it is by its being "unlikely". On a group, large fields are necessarily non-convex, so methods that rely on convexity cannot be used, and methods that pay with rarity are needed (24-06).
8. Magnen–Rivasseau–Sénéor (1993) constructed the Schwinger functions in axial gauge, keeping a cutoff at long distances. They wrote in their introduction that the reasons for not removing the cutoff were that the coupling becomes large and enters the non-perturbative effects of confinement, and that there is no easy solvable model around which to expand.
9. In lattice numerics, the mass of the lightest glueball of $SU(3)$ has been extrapolated to the continuum limit as about 3.41 times the square root of the string tension.

10. But the numbers are extrapolations from finitely many lattice spacings and finite boxes, and do not show that the limit exists, that it is unique, or uniformity in the size of the box.
11. Field theories with non-trivial interaction in four dimensions are thought to be limited to asymptotically free ones. The ϕ^4 theory has been shown to be trivial in four dimensions (its scaling limit is Gaussian).
12. As a result, before the published construction there was not a single "interacting theory already constructed" in four dimensions to compare with, and Yang–Mills was sought as its first example.
13. The walls this book met in the first step (the double well) and the second step (the hierarchical gauge model) — paying for the lack of convexity at the summit with rarity, the convexity hypothesis not holding on the group, global quantities not reflected in the gap — are small versions of the third difficulty of the third step. In the third step, that "rarity" ceases to be rare at long distances Estimate .

Three sources of difficulty

Table 3 The sources of difficulty (the column of facts is as its source labels say; "what stops" is an estimate)

Source	Fact (descriptions in primary sources)	What stops	Correspondence with the walls of this book
Source: dimensionless coupling — the coupling is dimensionless, and every scale works with the same weight	In four dimensions g is dimensionless and the quartic term is marginal (§ 01). The models constructed are only super-renormalisable ones in two and three dimensions and the massive Gross–Neveu model in two dimensions (Rivasseau 2000 Cited). $d = 2, 3$ is considerably easier than $d = 4$ (Chevyrev 2022 Cited)	The corrections of infinitely many steps need to be summed uniformly in the number of steps. Asymptotic freedom gives $g_j^2 \sim 1/(bj)$, and the sum converges only logarithmically	Placing the relaxation-time assumption as a power statement (the factor $\ln \xi \sim \beta$)
Source: no expansion linking scales — there is no expansion connecting the ultraviolet and the infrared	The sentence in the introduction of MRS 1993 (below). "The weak-coupling expansion of the ultraviolet limit is to some extent in contradiction with the low-temperature expansion"; "techniques for gluing together different expansions and different regions of the renormalisation group should be developed" (Rivasseau 2000 Cited). The strong-coupling expansion converges only on the side of large lattice spacing (Chatterjee Cited)	The control of Balaban and MRS works only at steps where g_j is small, and there is no theorem usable at the steps that cross the scale Λ^{-1}	In the first step the summit configurations were rare, and there was a currency called rarity. In the infrared, configurations where the Gaussian approximation does not work become typical, and that currency is absent

Source	Fact (descriptions in primary sources)	What stops	Correspondence with the walls of this book
Source: no global gauge fixing — non-abelian gauge invariance	There is no continuous global gauge fixing (Gribov 1978, Singer 1978 Cited (bibliographic data only; the content from memory)). Balaban fixed the gauge locally and scale by scale on regular configurations, and defined averages of group-valued fields separately (titles of the papers). Convexity at large fields is the main issue in going beyond hierarchical models (Bauerschmidt–Bodineau Cited)	Large fields cannot be controlled by convexity, and can only be paid for with rarity	The convexity ceiling on the group (the lemma of 24-06). The convex and non-convex places are the reverse of flat ϕ^4

The relation among the three sources Estimate : the source "dimensionless coupling" is "how many steps there are" (infinitely many, each with the same weight), the source "no expansion linking scales" is "at which step the expansion breaks off" (the confinement scale), and the source "no global gauge fixing" is "what is lacking within one step" (convexity and a global gauge). Balaban handled the lack of global gauge fixing at the ultraviolet steps in finite volume, and handled the source "dimensionless coupling" in the range where g_j is small. None of the sources read has touched the lack of an expansion linking scales.

Where Balaban and MRS stopped

Balaban. The gauge-theory series is 13 papers in Commun. Math. Phys. (1983–1989, including one on estimates of lattice Green's functions), about 426 pages when the pages are added up Cited (counted from the bibliographic data; the texts were not obtained). Combining the descriptions in the texts of other sources, the reach is as follows. In finite volume on the lattice, the effective action on the unit lattice stays bounded after arbitrarily many repetitions of the block-spin transformation (ultraviolet stability). From this it should follow that limits of gauge-invariant quantities such as smoothed Wilson loops exist along **subsequences** by compactness, but **the limit is not necessarily unique** (the summary in the introduction of MRS 1993 Cited). No source includes estimates of correlation functions, uniqueness of the limit, infinite volume, the OS axioms or the mass gap in Balaban's reach.

MRS 1993. What they showed is the construction of the Schwinger functions of pure $SU(2)$ four-dimensional Yang–Mills with a fixed infrared cutoff and no ultraviolet cutoff, in a regularised axial gauge, in the trivial topological sector, and that the Slavnov identities hold non-perturbatively. The authors themselves write that they do not claim to have given the proof of convergence in every detail, and that it is a detailed sketch Cited (introduction checked in the text; the body of the proof not read). The reason they stopped is the following sentence of the introduction (this book's translation).

> We do not investigate invariance under large gauge transformations, or non-trivial topological effects such as instantons. Nor do we try to remove the infrared cutoff, because that leads to large values of the coupling constant, to the so-called non-perturbative effects, which probably correspond to confinement. These problems are for the moment beyond the reach of our constructive methods, in particular because there is no easy solvable model of these phenomena around which to expand.

The "we" of the quotation refers to the authors of MRS. Since the infrared cutoff is not removed, the introduction goes on, the OS axioms as a whole cannot be examined either.

The object, tools, theory, history

The object is undefined. The problem statement is "prove that a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap"; for the existence part it requires axioms as strong as Wightman's or OS, and in a footnote it says that weak existence by compactness alone is not counted as a solution (Jaffe–Witten Cited). The construction itself is the content of the problem; it is not in the form of asking about properties of an object already constructed. Douglas's report of 2004 also writes that "there is no satisfactory mathematical definition of quantum Yang–Mills theory" Cited . Moreover, the space on which to place the measure is not fixed. The natural space is the quotient of everything by gauge orbits, which for the non-abelian case is a non-linear space, and even the construction of the state space is not trivial. Chevyrev's review writes that already in three dimensions the Gaussian free field is too singular to be restricted to lines, and the space of holonomies that could be used in two dimensions cannot be used Cited . The objects up to the second step (finite-dimensional probability measures on the lattice) are defined, and there is mathematics that asks about their properties. Before the published construction, for the object of the third step, even on which space and with which topology to place it was undecided. One step before "there are no numbers" there was "there is no object" Estimate . The published construction gives the object as a table of correlations (Schwinger functions) and an OS space, and does not give a probability measure for the continuum theory (6-04).

The tools are perturbation around a Gaussian reference. Most of the models constructed are in the weak-coupling region, made with cluster expansions, Mayer expansions and their multiscale versions (Rivasseau 2000 Cited). The analysis of one step of the renormalisation group relies on perturbation in powers around the field being 0 and on Taylor approximation, which is a good approximation only when the field is small (the book by Bauerschmidt–Brydges–Slade Cited). The solution theory of stochastic quantisation uses the fact that, under subcriticality ($d < 4$), the solution is a perturbation of the stochastic heat equation, which has the Gaussian free field as its invariant measure (Chevyrev Cited). All of them take as input "close to a Gaussian measure at each scale". The asymptotic freedom of Yang–Mills gives this input **only at short distances**. At long distances, the grounds for starting from a Gaussian reference run out. For four-dimensional ϕ^4, on the contrary, it approaches a Gaussian at long distances, so the same tools work, and the conclusion was triviality. **In four dimensions, the theories the tools reach become trivial, and the theory that should be non-trivial gives the tools no input** Estimate .

There is no theory that crosses infinitely many scales with neither a small parameter nor convexity. The multiscale theorems within the range read all place a small parameter or convexity at each step: a small coupling (hierarchical and Euclidean ϕ^4_4), uniform convexity of the conditional measure at each step (Bauerschmidt–Bodineau), a convergent perturbative expansion (two-dimensional Gross–Neveu, bibliographic data only), a small g_j and the rarity of large fields (Balaban, MRS) Cited . For group-valued fields, the convexity hypothesis cannot hold in principle at weak-coupling scales (the lemma of 24-06: on a compact manifold of diameter D, if $\mathrm{Hess}\, V \geq -K$ then $\mathrm{osc}\, V \leq KD^2/2$). **Gauge invariance forbids this control** Estimate : a gauge-invariant function is constant along orbits, so it cannot have convexity ($\mathrm{Hess}\, V > 0$) in the direction of the orbits. If one tries to remove the orbit directions by fixing the gauge, there is no continuous global fixing. Balaban's local fixing and the seam between the two gauges of MRS both work only in the region of small fields. The Riemannian geometry of the orbit space has been described (Singer 1981, bibliographic data only),

but a general theory handling measures on it that are neither Gaussian nor perturbations of one is not in the sources read.

History. The introduction of MRS writes that Balaban's work is "very impressive but hard to access; the results are scattered over many publications, and it is very difficult to check the consistency of all the arguments", and Rivasseau 2000 lists rewriting it among the open problems Cited . Dimock's reconstruction writes that "others have been slow to adopt this method", starts from three-dimensional ϕ^4 (text), and has reached an estimate of the partition function of three-dimensional QED uniform in the lattice spacing (abstract) Cited . A reconstruction for the four-dimensional non-abelian case was not in the sources read before the published construction. Of the intermediate goals, the committee's assessment is that construction on a finite box T^4 was given by the published construction as the continuum limit on a periodic box. The three-dimensional measure is open within the sources read. There are also moves to make different mathematics: the formulation in the language of Wilson loops and probability (Chatterjee), and the road that makes a Markov process on gauge orbits by stochastic quantisation and poses the construction of the invariant measure as an open problem (Chandra–Chevyrev–Hairer–Shen, Chevyrev's review) Cited . Both go in the direction not of extending the lattice renormalisation group but of rebuilding where the objects live (the state space) or the means of construction (stochastic dynamics). However, the key condition $d < 4$ of the latter shows that this new tool too becomes exactly critical in four dimensions.

What this section can say: an arrangement of descriptions in primary sources, and the matching to sources of difficulty (an estimate). Balaban's texts were not obtained, so the description of his reach goes through the texts of other sources. The contents of Gribov and Singer are from memory; only the bibliographic data were checked.

06

A paragraph on candidate rules

The list of candidates for what to place as rules if the "new mathematics" in the table of § 04 were to be made is dealt with in 24-08. Here we write only one Estimate . Combining axiomatising scale (a semigroup of coarse-grainings and a flow on it) with the class of measures that pay with rarity (locally convex on the good set, the bad set rare, a drift returning from the bad set, rarity paid only through the entropy inequality), it seems possible on the ultraviolet side to write a form in which the class closes at each step of coarse-graining. On the strong-coupling side, tools of the type that pay with the curvature of the group work in a narrow range. **Between the two, a "coupling window" remains where neither currency is enough.** It is the range where the rate of rarity, proportional to the effective coupling, runs out at the confinement scale, and the currency of curvature is not yet enough. This window is a restatement in the language of rules of the lack of an expansion linking scales (that there is no expansion connecting the ultraviolet and the infrared), and is not a new obstacle. The hierarchical gauge model is a place where the three — the ultraviolet side, the window, and the strong-coupling side — come together within one model, but the results there imply nothing for four-dimensional Yang–Mills.

07

Topology and the box — undecided for this book's group $SU(2)$

The periodic box. For reversible dynamics $\tau_{\mathrm{int}}(Q) \le C_P$ ((1) of 24-01), so if the autocorrelation of the topological charge grows faster than ξ^2, the upper-bound assumption breaks ((3) of 24-01. Lean

`LatticeGauge.FourDTrial.not_hup_of_fast_tau`, assumptions: $\tau_n \le C_n$, $c\,\xi_n^p \le \tau_n$, $p > 2$, $\xi_n \to \infty$. Negative control `z_two_compatible_with_hup`, no assumptions: $\tau \propto \xi^2$ is compatible with the upper-bound assumption). On the periodic box, originals for $SU(3)$ and above report this, so under the reading that extends the conjecture that HMC and Langevin are in the same dynamic universality class (in the original, a phrase attached to observations with open boundaries) to the topological slowness on the periodic box, **the upper-bound assumption reads as false on the periodic box** Assumption (the note in 23-01 § 05, 24-01 § 06).

Open boundaries rescue only finite boxes. With open boundaries, inside a box with fixed physical side, the growth of the autocorrelation goes back to a^{-2}. But if the spatial side of the box is held fixed and only the length in the time direction is extended, the autocorrelation time of Q^2 grows nearly in proportion to the length: $42.1(2.5)$, $113(6)$, $148(10)$ (lattices with time-direction lengths 24, 48, 80) (§ 5.3 of Lüscher–Schaefer 2011 Cited). This is because the charge enters and leaves only by transport through the boundary, and in this book's order, which takes the volume to infinity first, it is expected not to be a rescue Assumption (24-03 § 05 and Table 2).

For $SU(2)$ there is one primary source, on the a^{-2} side. Within the range searched, there is one primary source that measured the dependence on the lattice spacing of the autocorrelation of the topological charge in pure $SU(2)$ gauge theory (Abe–Morikawa–Suzuki 2025: HMC, periodic, a small box of physical side about $2.2/\sqrt{\sigma}$, three lattice spacings, reporting that it fits a^{-2} fairly well Cited). The heat-bath originals are all observations on the side where freezing in $SU(2)$ does not become strong enough to measure, and $SU(2)$ was excluded in Table 1 as well. Both counts of § 02 place $SU(2)$ near the boundary $z = 2$. **Hence, for this book's group $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption.** The primary source leans to the boundary or below it Reading .

Correspondence with the two-dimensional test bed of 24-02 Estimate . For two-dimensional CP^{N-1} at $\xi \approx 4$, freezing split by N (strongly frozen at $N = 21$, not frozen at $N = 2$), while the ratio of local quantities did not split (24-02 § 07, a conditional one-point reading). That the strength of freezing splits with N, which plays the role of the size of the group, can be read as a two-dimensional analogue of $SU(2)$ falling on the boundary in four dimensions while $SU(3)$ and above clearly cross it. The two-dimensional count in the primary source is also of the type "$z \sim N$", and it writes that the same argument applies to four-dimensional $SU(N)$ (Del Debbio–Manca–Vicari 2004 Cited). However, $N = 2$ carries an ultraviolet flag (the two definitions of the topological susceptibility differ by 13–14%), and the reason $N = 2$ does not freeze may be that it is on the side where the lattice scale determines the topological charge. The analogy carries this reservation, and **the two-dimensional results imply nothing for four dimensions**.

What this section can say: "the upper-bound assumption reads as false on the periodic box" is a statement on top of the originals for $SU(3)$ and above and a reading that extends a conjecture; we do not claim that the upper-bound assumption is false. For $SU(2)$ it is undecided.

08

Physical mechanisms, and why they are not proofs

There are many physical mechanisms that explain the mass gap and confinement. This book uses none of them as tools of proof. Table 4 lists, one line each, what each mechanism says and why it is

not a proof. The last row is an estimate that brings the leading-order statement of 24-05 of this book close to the language of physics, and its character differs from the other rows.

Table 4 Physical mechanisms, and why they are not proofs (labels: Cited = statement checked in the text or abstract, Assumption = from memory)

Mechanism	What it says	Why it is not a proof (one line)	Origin and label
The area law at strong coupling	On the strong-coupling lattice, Wilson loops decrease with area, and there are no free quarks	The strong-coupling limit has no rotation (Euclidean) invariance, and does not extend to the side of the continuum limit (weak coupling)	Wilson 1974 Cited (abstract). The strong-coupling theorem (Osterwalder–Seiler 1978) is Assumption (from memory)
Dual superconductivity (abelian projection, condensation of monopoles)	Gauge-fixing the non-abelian part first leaves $N-1$ abelian gauge fields and monopoles, and if the monopoles condense, there is confinement	Condensation has not been shown, and the projection depends on the choice of gauge. The examples where confinement is described by condensation are supersymmetric theories with matter fields plus a perturbation, and do not apply to pure gauge theory	't Hooft 1981, Seiberg–Witten 1994 Cited (abstract), Jaffe–Witten § 6.6 Cited
Centre vortices	Vortices with values in the elements of the centre carry the string tension	Lattice numerics on the maximal centre gauge and projection (the word "suggest"), with neither inequalities nor statements about the limit. The algebra of 't Hooft 1978 classifies phases, but does not decide which phase one is in	Del Debbio–Faber–Greensite–Olejník 1997, 't Hooft 1978 Cited (abstract)
The dilute instanton gas	Approximates the vacuum by a semiclassical sum around the topological solutions of action $8\pi^2/g^2$	Plausible only at small ρ; as ρ grows, (anti-)instantons overlap and the effective action changes. The premise of the approximation breaks down at the scale where the gap is born	BPST 1975, 't Hooft 1976, Callan–Dashen–Gross 1978 Cited (abstract), Lucini–Teper 2001 § 6 Cited (text). The infrared divergence of the integral over sizes is Assumption (from memory)
Large N and duality with strings	Planar diagrams dominate, and $1/N$ plays the role of the string coupling constant	A rearrangement of the perturbation series, or a conjecture, with no way back to finite N. The problem statement too, while saying "for sufficiently large N it may be a starting point", writes that there is no effective road to gauge theories without fermions	't Hooft 1974, Witten 1979 Cited (abstract), Jaffe–Witten § 6.6 Cited
Curvature of the orbit space	From the curvature of the space of gauge orbits and the gauge invariance of the wave functional, excitations have finite energy	Feynman himself writes that it is a qualitative, suggestive argument, and does not discuss 3+1 dimensions. Singer's is a geometric description, with no statement of a lower bound on the Hamiltonian. The language is closest to Bakry–Émery (a gap from a lower bound on curvature) Estimate	Singer 1981, Feynman 1981 Cited (abstract), Jaffe–Witten § 6.6 Cited

Mechanism	What it says	Why it is not a proof (one line)	Origin and label
(This book's estimate) The glueball is the two-particle colour singlet	On the side of the vacuum measure, the lowest gauge-invariant test space for $SU(2)$ is not in one particle but in two particles (the second chaos) (24-05 § 05)	A statement at leading order ($g \to 0$, three independent Gaussian colours), saying nothing about values at finite g. In infinite volume the right-hand side at leading order also goes to 0. The correspondence with glueballs is itself an estimate	(7)(8) of 24-05 Proved here (leading order), the correspondence Estimate

What this section can say: "what it says" in each row is a copy from the abstract or text of the original, and we judge none of the mechanisms right or wrong.

09

What cannot be said

(a) The candidate for the true nature. "The slow quantity and the correlation length split into different origins" of § 03 is an estimate. Neither the value of the ratio c for the action of a dislocation nor its independence of the group has been checked. $SU(2)$ falls near the boundary in either of the two counts.

(b) Mechanisms. This book uses none of the mechanisms of Table 4 as tools of proof.

(c) Two dimensions. We do not say what the results for two-dimensional models (the $U(1)$ strip, CP^{N-1}) imply in four dimensions.

(d) We do not say "only in four dimensions". The two points were stated as properties of being at the critical dimension.

(e) Table 2. "Does it reach" and the kind of cost are estimates and guesses; "new mathematics" refers to the absence of a description of a route in the sources read, and does not mean impossible.

(f) Sources. § 05 is an arrangement within the range of primary sources read, and the texts of Balaban, Gribov, Singer and Osterwalder–Seiler were not obtained. A theory written as "there is none" means "there is none in the sources read", and § 05 is an arrangement of sources from before the publication of 2026-11-10.

(g) $SU(2)$. For this book's group $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption. Nor do we claim that the upper-bound assumption is false.

10

Table of the five grades and the dependency graph

Table 5 The rows of this chapter ("—" marks rows without machine checking)

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
1	In four dimensions g^2 is dimensionless and the quartic term is marginal ((1))	Cited	§ 01	—	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
2	Dimensional transmutation (2) (one loop)	Cited	4-07	One loop	—
3	The scaling limit of ϕ^4_4 is Gaussian	Cited	Aizenman–Duminil-Copin (abstract)	Nearest neighbour, lattice cutoff	—
4	Balaban's ultraviolet stability (finite volume, lattice, subsequences)	Cited	Bibliographic data and texts of other sources	Three and four dimensions, finite volume	—
5	The MRS construction (infrared cutoff, axial gauge, $SU(2)$, trivial sector, sketch)	Cited	Introduction in the text	Fixed infrared cutoff	—
6	The three-dimensional measure is also not yet constructed; $d < 4$ is the key for the solution theory	Cited	Chevyrev's review	—	—
7	$8\pi^2/g^2$, $D(\rho) \propto \rho^{11N/3-5}$, (3)	Cited	§ 02	One loop	—
8	The fits of Table 1 ($6.82(7)$ for $SU(3)$)	Cited	Table 1	Heat bath and overrelaxation, periodic	—
9	One primary source for $SU(2)$ (close to a^{-2})	Cited	§ 07	HMC, small box, three points	—
10	With open boundaries the autocorrelation of Q^2 is proportional to the length in the time direction	Cited	§ 07	$SU(3)$, box with fixed physical side	—
11	If $\tau \geq c\xi^p$ ($p > 2$), the upper-bound assumption breaks	Proved here	(3) of 24-01	Sequences with $\xi \to \infty$	`LatticeGauge.FourDTrial.not_hup_of_fast_tau` (assumptions: $\tau_n \leq C_n$, $c\xi_n^p \leq \tau_n$, $p > 2$), negative control `z_two_compatible_with_hup` (no assumptions)
12	A reading that extends the conjecture to topological slowness on the periodic box	Assumption	24-01 § 06	Periodic box	—

Row	Statement (one line)	Grade	Grounds	Scope	Machine checking
13	The upper-bound assumption reads as false on the periodic box ($SU(3)$ and above)	Assumption	§ 07	Periodic box	—
14	In the order that takes the volume to infinity first, open boundaries are no rescue	Assumption	24-03 § 05	Infinite volume	—
15	In the two counts, $SU(2)$ is near the boundary $z = 2$	Estimate	§ 02	Semiclassical	—
16	Whether freezing breaks the relaxation-time assumption for $SU(2)$ is undecided	Estimate	§ 07	—	—
17	The slow quantity and the correlation length split into different origins (the estimate of the two points)	Estimate	§ 03	—	—
18	The two points are properties of the critical dimension (we do not say only in four dimensions)	Estimate	§ 01	—	—
19	The three sources of difficulty and their relation	Estimate	§ 05, Table 3	Within the sources read	—
20	The "coupling window" is a restatement of the lack of an expansion linking scales	Estimate	§ 06	—	—
21	"Does it reach" and the costs in the table of the four steps	Estimate	Table 2	—	—
22	The split in N in two dimensions is an analogue of the boundary for $SU(2)$	Estimate	§ 07	Two dimensions; an ultraviolet flag on $N = 2$	—

What the machine-checking column guarantees is only the algebra of sequences of reals; that τ_n is the autocorrelation time of a reversible dynamics, and that C_n is the Poincaré constant of the Wilson measure, are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  in four dimensions g² is dimensionless, marginal
 2  CIT  dimensional transmutation Λ = μ e^(−1/(2b₀g²))          one loop
 3  CIT  the limit of φ⁴₄ is Gaussian
 4  CIT  Balaban's UV stability (subsequences, no uniqueness)    finite volume
 5  CIT  MRS 1993 (IR cutoff, axial gauge, SU(2), sketch)
 6  CIT  3D measure also not yet constructed, d < 4 is the key
 7  CIT  8π²/g², D(ρ) ∝ ρ^(11N/3−5), (3)
 8  CIT  SU(3) fit 6.82(7)                                      heat bath, periodic
 9  CIT  one primary source for SU(2) (close to a⁻²)            HMC, small box
10  CIT  open boundaries: τ(Q²) proportional to time length     box with fixed physical sid
11  PRV  τ ≥ cξ^p (p > 2) ⇒ the upper-bound assumption breaks   Lean not_hup_of_fast_tau
12  ASM  reading that extends the conjecture to the periodic box
13  ASM  the upper-bound assumption reads as false on the periodic box ⟸ 8, 11, 12   SU(3)
14  ASM  infinite volume first: open boundaries no rescue ⟸ 10
15  EST  SU(2) is near the boundary z = 2 ⟸ 7, 8
16  EST  for SU(2), whether freezing breaks the relaxation-time assumption is undecided ⟸
17  EST  slow quantity and correlation length split into different origins ⟸ 1, 2, 7
18  EST  properties of the critical dimension ⟸ 1, 3
19  EST  three sources of difficulty ⟸ 1, 2, 4, 5, 6
20  EST  the coupling window restates the lack of an expansion linking scales ⟸ 19
21  EST  "does it reach" and costs in the four-step table ⟸ 4, 5, 6, 19
22  EST  the split in N in 2D is an analogue of the SU(2) boundary ⟸ 15, 16
```

The way to check is to see by eye that on every "⟸" the numbers on the right are smaller than the number on the left. There are no circles. The conclusion nodes of this chapter (from "the upper-bound assumption reads as false on the periodic box" onwards) are Assumption or Estimate, and there is neither a node for the mass gap nor a node for the continuum limit. The node for the primary source for $SU(2)$ enters only the node "whether freezing breaks the relaxation-time assumption is undecided", not the premises of the node for the periodic box. This is how the undecided reading for $SU(2)$ looks on the graph (the same form as the dependency graph of 24-01).

What this chapter can say

What is said	What is not said
Cited That the coupling is dimensionless in four dimensions; dimensional transmutation; the triviality of ϕ^4_4; the instanton action and the one-loop density; the fits of the originals	A mathematical proof of the relation between β and the lattice spacing. That a constant multiple of Λ is the mass gap
Cited The reach of the third step (the continuum limit on a periodic box of the construction published on 2026-11-10, at the level of the committee's assessment; before that, Balaban's ultraviolet stability and the MRS construction with an infrared cutoff), and the sentence in the introduction giving why MRS stopped	A check of Balaban's texts. Convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, a uniform mass gap (absent from the published construction too)
Assumption The upper-bound assumption reads as false on the periodic box ($SU(3)$ and above, a reading that extends a conjecture)	That the upper-bound assumption is false. The same reading for $SU(2)$

What is said	What is not said
Estimate The estimate of the two points, the three sources of difficulty, the coupling window, "does it reach" and the costs in the table of the four steps, the two-dimensional analogue	None of these is a theorem. The implication of two-dimensional results for four dimensions

Sources and reproduction

Item	Kind	Source / tools
The problem statement and the status report	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (§ 6, § 6.5, § 6.6 and footnote 2 checked in the text). M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem (2004) (text)
The current state of constructive field theory	Cited	V. Rivasseau, Constructive field theory and applications: perspectives and open problems, arXiv:math-ph/0006017 (II and VI checked in the text. The journal volume and pages from memory). S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (§ 3, § 5, § 6 in the text). I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (§ 1, § 3, § 4 in the text)
Balaban's series	Cited (bibliographic data)	T. Balaban, Commun. Math. Phys. 89 (1983) 571; 95 (1984) 17; 96 (1984) 223; 98 (1985) 17; 99 (1985) 75; 99 (1985) 389; 102 (1985) 255; 102 (1985) 277; 109 (1987) 249; 116 (1988) 1; 119 (1988) 243; 122 (1989) 175; 122 (1989) 355 (bibliographic data checked, texts not obtained. Pages counted from the bibliographic data)
MRS 1993	Cited	J. Magnen, V. Rivasseau, R. Sénéor, Construction of YM_4 with an infrared cutoff, Commun. Math. Phys. 155 (1993) 325–383 (introduction checked in the text; the quotation is this book's translation)
Reconstruction	Cited	J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335 (introduction in the text). The same author's overview of three-dimensional QED, arXiv:2204.07201 (abstract)
Power counting, infrared asymptotic freedom, large fields	Cited	R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474 (Definition 5.2.1, the Infrared Asymptotic-Freedom Proposition, § 7 in the text). R. Bauerschmidt, T. Bodineau, arXiv:1809.02075 (introduction and § 2 in the text)
The triviality of ϕ^4_4	Cited	M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163–235 (abstract. The source table of 7-08)
The one-loop form of (2)	Cited	The source tables of 4-06 and 4-07
Instantons, (3), Table 1	Cited	A. A. Belavin, A. M. Polyakov, A. S. Schwartz, Yu. S. Tyupkin, Phys. Lett. B 59 (1975) 85 (abstract). A. Athenodorou, M. Teper, JHEP 12 (2021) 082 (§ 6.2, (32)(34), Tables 42 and 43 in the text). B. Lucini, M. Teper, JHEP 06 (2001) 050 (§ 6 in the text). M. Lüscher, JHEP 08 (2010) 071 (§ 4.3 in the text)
Autocorrelation of the topological charge, open boundaries	Cited	M. Lüscher, S. Schaefer, JHEP 07 (2011) 036 (§ 3, § 5.3 in the text). M. Abe, O. Morikawa, H. Suzuki, PTEP 2025 063B03 (the text of § 3.2 and of Figure 6 in the text; the numbers in the figure were not read). L. Del Debbio, G. M. Manca, E. Vicari, Phys. Lett. B 594 (2004) 315 (text)
Global obstruction to gauge fixing	Cited (bibliographic data only)	V. N. Gribov, Nucl. Phys. B 139 (1978) 1. I. M. Singer, Commun. Math. Phys. 60 (1978) 7. Contents from memory

Item	Kind	Source / tools
The mechanisms of Table 4	Cited	K. G. Wilson, Phys. Rev. D 10 (1974) 2445. G. 't Hooft, Nucl. Phys. B 190 (1981) 455. N. Seiberg, E. Witten, Nucl. Phys. B 426 (1994) 19. L. Del Debbio, M. Faber, J. Greensite, S. Olejník, Phys. Rev. D 55 (1997) 2298. G. 't Hooft, Nucl. Phys. B 138 (1978) 1. G. 't Hooft, Phys. Rev. D 14 (1976) 3432. C. G. Callan, R. F. Dashen, D. J. Gross, Phys. Rev. D 17 (1978) 2717. G. 't Hooft, Nucl. Phys. B 72 (1974) 461. E. Witten, Nucl. Phys. B 160 (1979) 57. I. M. Singer, Phys. Scripta 24 (1981) 817. R. P. Feynman, Nucl. Phys. B 188 (1981) 479 (all checked in the abstract). K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440 (bibliographic data only)
Glueball numbers	Cited	A. Athenodorou, M. Teper, JHEP 12 (2021) 082 (Tables 24 and 33)
The one-line algebra by which the upper-bound assumption breaks	Lean	LatticeGauge.FourDTrial (`not_hup_of_fast_tau`, `z_two_compatible_with_hup`). The bundle guarantees only the algebra of sequences of reals

Terms used in this chapter — critical dimension, marginal, dimensional transmutation, infrared asymptotic freedom, instanton, dislocation, the four hurdles, the three sources of difficulty, ultraviolet stability, the coupling window, dual superconductivity, centre vortex. Their meanings are in A-02 Glossary and 24-09.

Next chapter to read: 24-08 Questions, and what is not claimed — questions that can be settled at the size of an undergraduate thesis, the list of candidate rules, and what Part 24 does not claim.

Questions, and what is not claimed

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — We put the questions left by the chapters of Part 24 into one table (done, partly done, not yet, new; with what moves if it is settled and what does not move even if it is settled), list candidate rules in answer to the question "if new mathematics is to be made, what should be placed as rules?", and write what Part 24 does not claim as sixteen rows, matching each to the sentence of which chapter of the text supports it.

Prerequisite chapters — None. This chapter can be read from anywhere. The three meanings (not shown, not read strongly, not called false) are used as in 23-16 § 01. Notation is in 24-09.

Said first — The level is [Y2]–[Y3]. This chapter adds no new results. The table of questions is questions, and contains no claims. The candidate rules are candidates, and none of them promises a prospect of closing the third step (the continuum limit and the OS axioms). The "what is not claimed" of Part 23 (23-16) applies to this part as it is.

The order of this chapter

1. The table of questions — nineteen questions, their status, what does not move even if settled (Table 1)
2. The list of candidate rules — six candidates and three combinations, relation to the sources of difficulty (Tables 2 and 3)
3. What is not claimed — sixteen items and the sentences of the text that support them (Table 4)
4. Relation to Part 23

01

The table of questions — questions that can be settled at the size of an undergraduate thesis

Table 1 lists the questions left by the chapters of Part 24. "Size" is the kind of work needed to settle it (paper, reading, computation, Lean), not an estimate of time. In each row we always wrote "what does not move even if it is settled". Even if a question is settled, the mass gap problem does not move. "Done" in the status means that the text of this part has answered the question (with conditions), and the label of the answer is in the chapter on the right.

Table 1 Questions of Part 24 (status: done, partly done, not yet, new [a question added while writing this part])

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
1	**Comparing local excitations of the free field**: for the free field, the comparison of excitation energies of observables supported in the box (computing the local-excitation moment assumption)	Paper	Done: in the moment form, it holds for the free field with $\gamma = d\pi^2$	The form of the coefficient of the local bridge	The local-excitation moment assumption for interacting models	24-04 § 05
2	**Monotonicity of the window factor**: monotonicity of the window factor $\varphi(x)/x^2$	Elementary, Lean	Done: strictly decreasing for $x > 0$ (Lean `wr_strictAntiOn`, no assumptions)	The explicit form of the window threshold	Nothing (only the form of the constant)	24-04 § 03
3	**Sign of the Polyakov loop**: whether the static ratio $\mathbf{Var}/\mathcal{E}$ catches the slowness, for the sign of the Polyakov loop on a small periodic box (a quantity of a slow sector whose answer is known)	Small computation	Not yet	A positive control for the method of static test functions	The verdict on the topological charge	24-01 § 02

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
4	**U(1) gap on two-dimensional layers**: for compact $U(1)$ on a two-dimensional layer, whether the gap keeps decreasing like M^{-2} or saturates as the strip is lengthened (the non-linear value of the length at which conservation breaks)	Computation	Done: on the width-2 strip, outer boundary frozen, five points $\beta = 2..6$, $\ell_c \propto e^{0.916\beta}$ (a reading)	The non-linear value of the length at which conservation breaks	Four-dimensional ℓ_*	24-02 § 03
5	**Data on topological-charge autocorrelation**: a primary source on the autocorrelation of the topological charge in pure $SU(2)$ gauge theory (if none, a small measurement at fixed physical volume)	Reading, computation	Partly done: one primary source (HMC, small box, close to a^{-2}). The small measurement is not yet done	The label of the $SU(2)$ row (at the boundary, or clearly faster than ξ^2)	The conjecture on the relation between HMC and Langevin	24-07 § 07
6	**Many-family version of the Two-Projection Lemma**: a many-family version of the Two-Projection Lemma (the family of 2^d colours for a d-dimensional box), and its condition on maximal correlation	Paper	Not yet	The node of reduction of the four-dimensional box	An estimate of maximal correlation uniform in β	24-03 § 06
7	**Hessian of the one-link weight**: on $SU(2)$, the flow of $\inf \operatorname{Hess} V_t$ for the one-link Wilson weight convolved with the heat kernel, and of the weight of the bad set (one variable, class functions)	Paper + small computation	Not yet (the first step of the class of measures that pay with rarity)	The first numbers of the Polchinski flow on the group; whether the class closes in one step	Lattices that are not hierarchical; closing uniformly over many steps	24-06 § 07
8	**Time-step comparison for the transfer operator**: for Wilson's transfer operator, show the time-step comparison assumption ((D) in the paper "one-sided bridge": the comparison of increments at neighbouring times with gradients, uniform in β) restricted to functions cut off at low energy	Paper	Not yet	Carrying the finite-volume window bridge to Wilson's lattice with discretised time	Infinite volume	24-04 § 08
9	**Lean for τ_{int} and C_P**: for finite-state reversible Markov chains, close in Lean $\tau_{\text{int}} \le C_P$ and the identity of the ground-state transformation (the algebra of the Gap and Vacuum Measure Proposition), defined from probabilistic objects	Lean	Not yet	An example of formalisation with the objects themselves as types (a check of a different form from 23-13)	Path measures, continuum fields	24-05 § 01
10	**Rereading the modes of CP^{N-1}**: reread the autocorrelations of local modes and topological modes in two-dimensional CP^{N-1} through the eyes of the local diffusive-relaxation-time assumption (index in A-06)	Reading, computation	Partly done: the literature has been read. The test-bed measurement goes only as far as a conditional reading at one volume and one correlation length	Material for the test bed	Four dimensions	24-02 § 07

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
11	**Loop families of two-dimensional SU(2)**: for two-dimensional $SU(2)$ (plaquettes independent), find λ_W and Γ of families of loops at finite β	Computation	New	The β-dependence of the threshold for the colour singlet	Four-dimensional values	24-04 § 06
12	**Values of large four-dimensional loops**: on a four-dimensional box, the values for families of loops the size of the span (including non-planar ones and products of lassos)	Computation	New	The four-dimensional value of the colour-singlet reference	The ratio for the target measure	24-04 § 06
13	**Reduction to kink positions**: for the 0+1 double well, sandwich the local ratio by a reduction to one dimension with the position of the kink as coordinate	Paper + small computation	New	The constant of the motion that crosses the window	The four-dimensional ratio	24-04 § 07
14	**The single point β=7 of the band**: add one point $\beta = 7$ for the strip, and see whether the slope of the raw rate keeps matching $2 - 1/\beta$ and whether the slope of ℓ_c settles	Computation	New	The reliability of the slopes for the two-dimensional strip	Four dimensions	24-02 § 05
15	**Mass in the band between the threshold and 1**: for fields whose ratio lies in the band $[K_*, 1)$ between the threshold and 1, whether there is another argument that sees the mass while staying in the ratio form (outside Jensen and Dirichlet test functions)	Paper	New	Handling of the undecided band	Values of the ratio for interacting fields	24-04 § 06
16	**Window bridge on a time-step lattice**: on Wilson's lattice with discretised time, write the window version of the bridge (the In-Window Supremum Proposition and the lower bound in finite volume) under the time-step comparison assumption	Paper	New	The lattice version of the window bridge	Infinite volume	24-04 § 03
17	**Frustration-free sector splitting**: whether the Kogut–Susskind Hamiltonian becomes frustration-free when it is split with terms grouped by sector	Paper	New	Whether local-to-global gap theorems apply	The gap at weak coupling	24-05 § 04
18	**Extending the CP^{N-1} test bed**: on the CP^{N-1} test bed, add other random seeds and the remaining pairs of N, and read the exponent of the slowness of the topological mode at a second correlation length	Computation	New	The reliability of the reading that freezing splits by N	Four dimensions	24-02 § 07

No.	Question	Size	Status	What moves if settled	What does not move even if settled	From which chapter
19	**Conservation length of subregion charge**: measure the definition (2) of the length at which conservation breaks (the smallest R at which the ratio reaches half its saturation value) with the charge of a four-dimensional subregion	Computation	New	The first value of the four-dimensional length at which conservation breaks	Whether the diffusive-relaxation-time assumption is true	24-02 § 01

If any question is settled, the label of one row of one chapter moves. Whether the mass gap, the diffusive-relaxation-time assumption or the relaxation-time upper-bound assumption is true does not move when any question is settled.

02

The list of candidate rules — "if new mathematics is to be made, what should be placed as rules?"

The considerations of Part 24 also ran into the question "if new mathematics is to be made, what should be made the rules of the mathematics?" (24-07 § 06). Like Boolean algebra, which threw away the meaning of propositions and kept only the rules of equality, how could one assemble a system that places properties of renormalisation and the continuum limit as rules? Table 2 is the list of candidates. **Every row is a candidate**, and neither consistency nor the ability to restate existing results has been checked Estimate . The comparison with Boolean algebra is a metaphor for how to assemble, not a mathematical claim.

Every candidate can be written in the same form: remove one assumption of an existing framework, and place a replacement rule instead. We condensed what is removed and what is placed into one line per row, and added one question and what does not move even if that question is settled.

Table 2 Candidate rules (one line each, one question) Estimate

Rule	Candidate rule (one line)	Question (one)	What does not move even if settled
Rule of axiomatising scale	A semigroup of coarse-grainings, locality, gauge covariance, contracting and marginal directions, weights for large fields	On the hierarchical $SU(2)$ model, which of the five rules apply and which do not	Non-hierarchical lattices, the continuum limit
Rule of the loop algebra	Bounded, the Mandelstam identities, positivity and reflection positivity, the loop equations (throw away the connection, keep the values of loops)	On the two-dimensional $SU(2)$ lattice, whether the state satisfying the four rules is determined uniquely at finite β	Definition of continuum loops in three dimensions and above
Rule of conserved quantities and sectors	Decomposition of the variance, the sandwich $C_{\mathrm{trans}} \le CP \le \sup C_P^{\mathrm{sec}} + C_{\mathrm{trans}}$, local windows, two lengths (24-03 § 04)	Whether C_{trans} in the sandwich can be written as a quantity of the reduced chain between sectors	Construction (the third step)
Rule of the rarity class	Local convexity on the good set, rarity of the bad set, a returning drift, the currency of entropy (24-06 § 07)	How the parameters of the class are carried over when the one-link Wilson weight is convolved once with the heat kernel (the question "Hessian of the one-link weight")	Closing uniformly over many steps
Rule of an asymptotic-freedom axiom	Changes of scale, a monotone coupling, a free field in the ultraviolet, dimensional transmutation (nothing is placed in the infrared)	Whether it follows from the rule of approaching a free field in the ultraviolet that the local ratio approaches 1 at short distances	The infrared, the mass gap
Curvature rule	Bakry–Émery on the space of gauge orbits	On the orbit space of one plaquette, whether the curvature term of the quotient widens the range of strong coupling	Weak coupling
Rule of axiomatising scale + rule of the rarity class	A rarity class closed under one step of coarse-graining	On the hierarchical $SU(2)$ model, how wide the coupling window is that remains between how the class closes on the ultraviolet side and the strong-coupling tools	The four-dimensional window
Rule of conserved quantities and sectors + rule of the rarity class	A rarity class with sectors (of Eyring–Kramers type, sandwiching by the larger of the constant within sectors and the rate of transitions between sectors)	On the two-dimensional $U(1)$ strip, whether the upper bound on the gap can be sandwiched by the larger of "within sectors" and "the rate of transitions"	The four-dimensional topological charge
Rule of the loop algebra + rule of conserved quantities and sectors	Putting sectors into the algebra of loops (reading the decomposition of the set of states into extreme points as sectors)	Whether the non-uniqueness of solutions of the loop rules corresponds to there being several phases (sectors)	Uniqueness at weak coupling

Table 3 gives the relation to the three sources of difficulty (24-07 Table 3: controlling infinitely many steps uniformly in the number of steps [source: dimensionless coupling]; that there is no expansion connecting the ultraviolet and the infrared [source: no expansion linking scales]; that there is no global gauge fixing and large fields cannot be controlled by convexity [source: no global gauge fixing]).

Table 3 Candidate rules and the three sources of difficulty ("works" means the language of the rule can handle that source, not that it gets past the source)

Candidate	Source: dimensionless coupling (uniform in the number of steps)	Source: no expansion linking scales (ultraviolet and infrared)	Source: no global gauge fixi
Rule of axiomatising scale	Works (it is the language itself)	Does not work (leaves the region when the coupling is $O(1)$)	Half (the weights on the gr formulated)
Rule of the loop algebra	Does not work (goes back to the definition of continuum loops)	Works in form (the same rules at every β). Whether the solution is unique is undecided	Works best (no gauge fixin
Rule of conserved quantities and sectors	Does not work	Does not work	Only gives a name
Rule of the rarity class	Only when combined with the rule of axiomatising scale	Does not work (the rate of rarity runs out)	Works head-on
Rule of an asymptotic-freedom axiom	Only copies it out	Does not work (leaves the infrared empty)	Does not work
Curvature rule	Does not work	Does not work (only on the strong-coupling side)	Gives a meaning

Reading the table Estimate . There is not a single "works" in the column of the source "no expansion linking scales". Only the rule of the loop algebra has rules of the same form at every coupling, but whether the solution of the rules is unique has been replaced by the very content of the lack of an expansion linking scales. The coupling window that appears with the rule of axiomatising scale + the rule of the rarity class (24-06 § 07) is this lack of an expansion linking scales restated in the language of rules.

Where the Boolean-algebra type is possible Estimate . Replacing continuous objects by discrete rules is already half there at the level of the **algebra** of observables: the lattice (replacing a continuous field by group elements on edges), the Mandelstam identities and loop equations for loops, the integers of sectors, the one-variable recursion of hierarchical models. But the rules that decide the third step are **inequalities**, not equalities: positivity, rarity, convexity, estimates uniform in the number of steps. The loop equations (equalities) do not determine a unique solution; what narrows the solutions is positivity (infinitely many inequalities). Put as a metaphor, what the third step needs is closer to the type of measure theory (rules of inequality such as σ-additivity, and rules of limits) than to Boolean algebra.

What is not claimed about the candidate rules. (i) None of the candidates promises a prospect of closing the third step. (ii) The comparison with Boolean algebra is a metaphor, and "the rules of the third step are inequalities" is also an estimate. We have not shown that the rules of Yang–Mills cannot be written as equalities. (iii) The correspondences between candidates and existing results (for example, between the form of the stable manifold of the rule of axiomatising scale and ultraviolet stability in finite volume) are all restatements not checked in the texts of the originals. (iv) "The hierarchical model is a place where the three — the ultraviolet side, the window and the strong-coupling side — come together" is a way of putting it within the range searched, with no implication for four dimensions. (v) A candidate written as not reaching is not evidence of being false.

03

What is not claimed — sixteen items and the sentences of the text that support them

Table 4 writes, one row each, what Part 24 does not claim, and lists the places in the text that support each row. "Support" means that the sentence of that chapter already writes the conditions, assumptions and scope, and this row merely copies it into one line. The Kind column is the three meanings of 23-16 (not shown, not read strongly, not called false).

Table 4 What Part 24 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (summary)
1	Solution of the problem of the existence of Yang–Mills and the mass gap. The continuum limit, recovery of rotations, the OS axioms and non-triviality are not touched (for the existence part, the construction published on 2026-11-10 is read in Volume 2)	Not shown	24-00 § 05: what is not dealt with. 24-07 § 04: the third and fourth steps are not reached by the tools of this book
2	That the relaxation-time assumption is false. The statement that the upper-bound assumption does not hold on the four-dimensional periodic box is a statement on top of the numbers of the originals for $SU(3)$ and above, and a reading that extends a conjecture attached to observations with open boundaries (that HMC and Langevin are in the same dynamic universality class) to the topological slowness on the periodic box	Not called false	(a)(b) of 24-01 § 06, 24-07 § 07
3	The sector diffusive-relaxation-time assumption and the local relaxation-time assumption are candidate definitions, not shown at any β. Nor do we claim "the right way to fix" the relaxation-time assumption	Not shown	24-03 § 07
4	The one-sided bridge (including the local version) is an inequality on a lattice with continuous time, and on Wilson's lattice with discretised time the time-step comparison assumption remains. The local version further rests on the local-excitation moment assumption	Not read strongly	24-04 § 08 (Table 4), 24-04 § 09

Item	What is not claimed	Kind	Supporting sentence of the text (summary)
5	"The gap = the Poincaré constant of the vacuum measure" is an identity, and we do not claim it has made the problem easier	Not read strongly	24-05 § 01, 24-05 § 07
6	The definition of the length ℓ_* at which conservation breaks, and the restatement of what to compare with, $CP \asymp \max(\xi^2, \tau_{\mathrm{glob}})$, are candidates, not shown in any model	Not read strongly	24-02 § 06, 24-02 § 08
7	The results for hierarchical models and two-dimensional models imply nothing for four-dimensional Yang–Mills	Not read strongly	(a) of 24-06 § 09, 24-02 § 08
8	The non-perturbative mechanisms of physics (confinement, instantons, centre vortices, duality) are not used as tools of proof in this part	Not shown	24-07 § 08 (Table 4)
9	The two points of "because it is four dimensions" are estimates as properties of the critical dimension, and we do not claim they happen only in four dimensions	Not read strongly	24-07 § 01, (d) of 24-07 § 09
10	The "what is not claimed" of Part 23 applies to this part as it is	—	23-16 § 02
11	The local relaxation-time assumption goes only as far as the definition and the ratio form. At no β has the local relaxation-time assumption been shown for the target gauge theory. What has been shown is properties of the definition and values for the free-field reference	Not shown	24-03 § 03, 24-04 § 04
12	The thresholds of the local bridge (0.342κ, 0.502κ, about 0.62κ) depend on the choice of reference and on the family of test observables. For fields whose ratio lies between the threshold and 1, the bridge says nothing	Not read strongly	24-04 § 06 (Table 2)
13	For the 0+1 double well, the local ratio is not 1 even for the shortest window. Taking "the ratio approaches 1 at short distances" in four dimensions is an assumption depending on asymptotic freedom	Not read strongly	24-04 § 07, (7) of 24-03
14	The length at which conservation breaks was measured only on the width-2 strip of two-dimensional $U(1)$ (outer boundary frozen, $\beta = 2..6$). Nothing is said about four dimensions, the non-abelian case, or the exponent as $\beta \to \infty$	Not read strongly	24-02 § 03 (Table 2)
15	For $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption. There is one primary source, and its numbers are on the side close to a^{-2}	Not called false	24-07 § 07, (b) of 24-01 § 06
16	The results of the two-dimensional CP^{N-1} test bed imply nothing for four-dimensional Yang–Mills (the ratio is a ratio of conditional autocorrelations at one fixed volume, a different quantity from the static ratio)	Not read strongly	24-02 § 07

The word "conjecture" in item 2 is, in the original, attached to observations with open boundaries, so its application to the periodic box is written as "an extended reading". What is not claimed about the candidate rules (§ 02) is placed at the end of § 02.

04

Relation to Part 23

The fourteen items of "what is not claimed" of Part 23 (23-16) apply to this part as they are. We do not list them again here. In particular, two of them — that not reaching is not evidence of being false (items 8 and 12 of 23-16), and that conditions fall away in summaries (23-16 § 06) — apply equally when reading the estimate sentences of this part (label "Estimate").

What this chapter can say

What is said	What is not said
The list of questions of Part 24, and what does not move even if each is settled (Table 1)	The answer to any of the questions. The "size" of a question is the kind of work, not an estimate of time
The list of candidate rules, and the estimate of their relation to the three sources of difficulty (Tables 2 and 3)	The consistency of any candidate, or its prospect for the third step
The sixteen items that Part 24 does not claim, and the places in the text that support them (Table 4)	Whether the proposition of each item is true

Sources and reproduction

Item	Kind	Source / tools
Each row of Table 1	This book	The chapter and section in the right-hand column

Item	Kind	Source / tools
The candidates and sources of difficulty of Tables 2 and 3	Estimate	24-06 § 07, 24-07 § 05, § 06. The sources of the existing results the candidates draw on are in the source table of each chapter (the four properties of the weight from the book by Bauerschmidt–Brydges–Slade, the two-dimensional loop equations from the summary in § 6 of Chatterjee's review, log-Sobolev at strong coupling from the abstract of Shen–Zhu–Zhu)
Each row of Table 4	This book	The chapter and section in the right-hand column
The three meanings	This book	23-16 § 01

Terms used in this chapter — the status of a question (done, partly done, not yet, new), what does not move even if settled, candidate rules, combinations, the three sources of difficulty, the coupling window, not shown / not read strongly / not called false. Their meanings are in A-02 Glossary and 24-09.

Next page to read: 24-09 Table of notation for the first road, second half. To go back to the map of Part 24, 24-00.

Table of notation for the first road, second half

This chapter is a guide to notation and terms.

What this page shows — A list of the symbols used in Part 24. For each symbol, its meaning, where it is defined, its unit, and whether it is the same quantity as the symbol with the same letter in Part 23 or a different one.

Prerequisite chapters — None. The notation of Part 23 is in 23-01 § 01 and A-01 Table of notation.

Said first — Quantities are counted in lattice units (the lattice spacing is 1). Langevin time has the dimension of length squared (diffusion), and the value of C_P measured in physical units is $a^2 C_P$. Numbers are values under the conditions of the chapter of definition, and their labels are in that chapter.

01

List of symbols

Table 1 Symbols of Part 24 ("Same" = the same quantity as the same letter in Part 23; "New" = a symbol added in Part 24)

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ℓ	A ruler of length given from outside (independent of the gap). In four dimensions the bookkeeping length; for the 0+1 double well $1/\omega_b = 1$	24-03 § 02	Length / $a\ell$	New (in four dimensions the same quantity as the bookkeeping length of 23-01 § 07)
ξ	Correlation length $1/m_{\mathrm{lat}}$. In 0+1, $\hbar/\Delta$	(1) and (17) of 23-01	Length / $a\xi$	Same

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ξ_{RG}	The bookkeeping length (the length fixed by counting the steps that coarsen the lattice) $= 1/a$	23-01 § 07	Length / 1	Same
ξ_{XY}	The static length of the two-dimensional $U(1)$ strip, $-1/\ln(I_1(\beta)/I_0(\beta))$ (the inverse of the string tension)	24-01 § 02, 24-02	Length	New
$R = c\ell$	Side of the local window or box	24-03 § 03	Length	New
c	The ratio of the window length to the ruler	(2) of 24-03	Dimensionless	New
$\mathcal{D}$	The algebra of time-averaged functionals (weights with bounded support)	(1) of 24-03	—	New (fixes the collection over which the supremum in (7) of 23-01 is taken)
$\mathcal{D}_c$	The subfamily of $\mathcal{D}$ whose weights have support in the window $[0, c\ell]$	(2) of 24-03	—	New
$\mathcal{F}_1$	The family of time-integrated single-time functions	24-04	—	A name given to the family of test functions of the One-Sided Bridge Proposition (Theorem 3.1 of the paper) of 23-01
C_P	The Poincaré constant $\sup \mathrm{Var}/\mathcal{E}$ (path measure)	(7) of 23-01. In Part 24 the supremum is taken over $\mathcal{D}$	Length squared (Langevin time) / $a^2 C_P$	Same (the path C_P of 23-01 Table 3)

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
$C_P^{\mathrm{loc}}(c)$	The local Poincaré constant (the supremum over $\mathcal{D}_c$)	(2) of 24-03	Length squared	New
$C_P^{\mathrm{loc},0}(R)$	The value of the massless reference in the same box with the same graduation, $= \Gamma(R)R^2$	(4)(5) of 24-03	Length squared	New
$K(c)$	The constant of the local diffusive-relaxation-time assumption in absolute form, $C_P^{\mathrm{loc}}(c) \le K(c)\ell^2$	(3) of 24-03	Dimensionless	The local version of C in (14) of 23-01 (K in Lean)
K_{rel}	The ratio $C_P^{\mathrm{loc}}/C_P^{\mathrm{loc},0}$	(5) of 24-03	Dimensionless	New
Γ	The coefficient of the value for the reference box, $C_P^{\mathrm{loc},0}(R)/R^2$. $\Gamma_\infty = 0.0740$ for the scalar, $\Gamma_Q = 0.0277$ for the gauge quotient	24-03 § 03, Table 1	Dimensionless	New
λ_W	The Dirichlet value of the box within the test space multiplied by R^2 ($4\pi^2$ for the scalar, 72.0 for the gauge quotient). For a window function W, $\|\nabla W\|^2/\|W\|^2$	(7) of 24-01, 24-03 Table 1	Dimensionless	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
κ	The constant comparing the Langevin metric with the metric of the kinetic term (the metric comparison assumption ((A3) in the paper)). A different quantity from the correction κ to the raw rate on the strip in 24-02 § 05 (which is neither κ_BE nor this κ)	The one-sided bridge of 23-01	Dimensionless	Same
κ_{BE}	The lower bound on the Bakry–Émery curvature. A different quantity from the comparison constant κ of the one-sided bridge	24-06 § 01	Inverse length squared	New
s	The rate of rarity (the exponent with which the measure of the bad set B falls, as $\mu(B) \le e^{-s}$). The same quantity as ℓ of Part 23	24-06 § 01 (formulas: (7) of § 06, the rule of rarity of bad sets of § 07)	Dimensionless ($-\ln \mu(B)$; in the 0+1-dimensional example, the order of the action difference divided by $\hbar$; for one link of the group, $s \approx \beta$)	The rarity exponent ℓ of 23-09
K_*	The threshold of the bridge $\kappa/(\lambda_W \Gamma)$	24-04	Dimensionless	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
ℓ_c	The length at which conservation breaks on the two-dimensional $U(1)$ strip (the bend in the relaxation of the flux dipole)	24-01 § 02, 24-02	Length	New
ℓ_*	The length at which conservation breaks (the general name. The candidate definition is (2) of 24-02: the smallest R at which the ratio reaches half its saturation value)	24-02	Length	New
R_*	The crossing length $\sqrt{d\lambda_W/\hbar}$ (the boundary between the regimes of transport and of the barrier)	(7) of 24-01	Length	New
λ_Q	The relaxation rate of the topological charge of the strip, $1/\tau_{\mathrm{int}}(Q)$	24-02	Inverse Langevin time	New
τ_{int}	Integrated autocorrelation time	(1) of 24-01, 23-02 § 05	Langevin time (in the originals, numbers of sweeps or molecular-dynamics time)	Same
m	The gap as an inverse time, $m = \Delta/\hbar$. In four dimensions m_{lat}	(1) and (17) of 23-01	Inverse length	Same (m_{lat})

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
m_{phys}	The mass in physical units $m\ell$ ($m_{\mathrm{lat}}\xi_{\mathrm{RG}}$ in four dimensions)	(12)(16) of 23-01	Dimensionless (physical units)	Same
Δ	The gap of the Hamiltonian $E_1 - E_0$	The one-sided bridge of 23-01	Energy	**Restricted to one meaning** (§ 02 below)
ω_b	The angular frequency at the bottom of the double well $\sqrt{V''(\pm 1)} = 1$	(7) of 24-03, 24-04	Inverse time	New (a quantity of the model of 23-04)
S_0	The kink action $2/3$	23-04 § 03	Action	Same
$\varphi(x)$	The window factor $1 - (1 - e^{-x})/x$	The bracket in (17) of 23-01, 24-04	Dimensionless	Same (different from the plaquette angle φ_p)
φ_p	The plaquette angle of the two-dimensional strip	24-01 § 02, 24-02	Angle	New
Q, Q_s	The topological charge, the smoothed topological charge. On the two-dimensional strip $Q = \sum_p [\varphi_p]/2\pi$	24-01 § 02, 24-03 § 04	Integer (Q_s is real)	The same as Q in the note of 23-01 § 05
χ_t, h, d, g	The topological susceptibility, the density of breaking of conservation, the transport coefficient, the coefficient of the boundary layer	(4)(5) of 24-01	Lattice units (h, d, g depend on the graduation of $\mathcal{E}$)	New

Symbol	Meaning	Where defined	Unit (lattice) / physical reading	Correspondence with Part 23
$C_P^{\mathrm{sec}}(n)$, C^{trans}	The Poincaré constant of a sector, the quantity between sectors	(8)(10) of 24-03	Length squared	New

02

Symbols with the same letter but different meanings

Δ. In 23-01 of Part 23, Δ was used in two meanings: $\Delta(\beta) = m_{\mathrm{lat}}/a$ of (1) (the mass in physical units) and $\Delta = E_1 - E_0$ of the one-sided bridge (the gap of the Hamiltonian). In Part 24, Δ is **restricted to the meaning $E_1 - E_0$ of the one-sided bridge**, and the mass in physical units is written m_{phys}.

K. (14) of 23-01 writes the constant of the relaxation-time upper-bound assumption as C, and the Lean theorems write the same constant as K. $K(c)$ in Part 24 is its local version, K_{rel} is the constant of the ratio form, and K_* is the threshold of the bridge. The three are different quantities.

φ. The window factor $\varphi(x)$ carries an argument, and the plaquette angle of the two-dimensional strip is φ_p with a subscript.

Q. The four-dimensional topological charge and the topological charge of the two-dimensional strip are written with the same letter. Both are "global integers that do not show in the gap", examples of the Q of the type of 24-01. The two-dimensional numbers do not imply anything for four dimensions.

Symbols with λ. λ_1 (the Langevin gap $= 1/C_P$, 23-01), λ_W (the Dirichlet value of the box) and λ_Q (the relaxation rate of the topological charge of the strip) are different quantities.

Next chapter to read: 25-00 Entrance — space, not time.