Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 25 The second road — from the direction of space

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2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space entrance, the ladder of scales, map of ten chapters · Kiichi, Shiori and Rin

Entrance — space, not time

> This chapter is a guide (the entrance to this part).

This part is a record of this book's study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — Part 23 was a road towards the mass gap from the direction of time (the time over which fluctuations are smoothed out, and the transfer matrix). Part 25 is the second road, towards confinement from the direction of space. We shrink two of the four directions to a small twisted two-dimensional torus T^2 and consider the area law of Wilson loops on the remaining plane $\mathbb{R}^2$. In this chapter we state (1) why we go once by way of a small twisted T^2, (2) how the problem then splits in two, (3) which tool works at which scale (Figure 2, "the ladder of scales"), and (4) the map of the ten chapters of Part 25.

Prerequisite chapters — 5-01 (Wilson's lattice), 5-02 (strong coupling), 17-02 centre vortices, 4-06 (asymptotic freedom), 4-07 (dimensional transmutation), 7-07 (what the four-dimensional construction has reached). The entrance of Part 23, 23-00.

Said first — This chapter is a map, and it contains no newly proved statements. "On a small twisted T^2 the coupling stays weak up to the size of the T^2" is a physical estimate made on the assumption of ultraviolet control (a bound, uniform in a, from the lattice spacing a up to the size of the T^2). What the construction published on 2026-11-10 showed is the continuum limit with periodic boundary conditions on a cubic box; construction with twisted boundary conditions or on elongated boxes is not in it (within the range searched). The "ultraviolet control" of this part remains as a still-open condition: "that the same construction holds under twisted boundary conditions". The road of Part 25 heads first for confinement (the area law), not for the mass gap itself. That the two are different statements is as in 17-02 and 16-08.

The order of this chapter

1. The second road — what we look at, and from which direction
2. Two walls — the ultraviolet and the crossover band
3. Why a small twisted T^2 — two reasons

4. The ladder of scales — which tool works at which scale (Figure 2, Table 1)
5. Map of the ten chapters of Part 25 (Table 2)
6. The conventions of notation and words in this part
7. Guide to Part 23 and Part 26
8. What cannot be said
9. The table of five grades and the dependency graph (Table 3)

01

The second road — what we look at, and from which direction

The road of Part 23 regarded one direction of the lattice as "time", and approached the mass gap from the gap of the transfer matrix and from the time over which fluctuations are smoothed out. Where it stopped is summarised in 23-15.

In Part 25 we change direction. Of the four directions of the four-dimensional lattice, we wind two (directions 1 and 2) into a torus T^2 with small period L, and regard the other two (directions 3 and 4) as a large plane. We ask whether a Wilson loop $W(C)$ drawn in the plane decreases as $\langle W(C)\rangle \le e^{-\sigma A}$ in the area A it encloses (the area law). Where the time road looked at "the gap", what this road looks at is "how vortices are counted in the plane".

We put 't Hooft's twist (25-01) on the wound T^2. It is known in the physics literature that if one shrinks to a periodic T^2 without twist, a family of flat connections (zero modes) remains, and there is no prospect of joining the small T^2 to large four dimensions without a phase transition Cited (the title of § 2.1 of Tanizaki–Ünsal 2022 is "compactification to periodic T^2 and the absence of adiabatic continuity"). For a twisted T^2, the same paper states the conjecture that the structure of the weakly coupled vacuum on a small $T^2 \times \mathbb{R}^2$ is joined without a phase transition to the strongly coupled region of $\mathbb{R}^4$ Cited . This conjecture remains a conjecture, and this book also treats it as one.

02

Two walls — the ultraviolet and the crossover band

If one starts from the lattice, builds the four-dimensional theory and tries to show confinement or a mass gap, one hits two walls of different nature.

1. **The ultraviolet wall.** One needs uniform bounds at all scales that do not break down as the lattice spacing a approaches 0. Thanks to asymptotic freedom (4-06) the

coupling becomes weak at short distances, and one can consider a multiscale analysis that controls each scale in perturbative language. For four-dimensional Yang–Mills, on 2026-11-10 a Lean formalisation that builds a theory satisfying the Wightman axioms as a limit of Wilson's lattice was published together with the committee's assessment report, and the committee's assessment is that it is proved in Lean (Volume 2, 6-00). However, what that construction showed is the continuum limit with periodic boundary conditions on a cubic box. Construction on the small T^2 with twisted boundary conditions used in this part, and on elongated boxes (the product of a small T^2 and a large plane), is not in the published construction (within the range searched). In this book we treat the ultraviolet stage of this part as an assumption named "ultraviolet control", and read its content as the open condition "that the same construction holds under twisted boundary conditions" Assumption . The reach before that (Bałaban's series) is in 7-07.

2. **The crossover band.** One must cross, without relying on perturbation theory, the intermediate scale at which the coupling passes from weak (short distances) to strong (long distances), around $1/\Lambda$ in physical length (Λ is the scale born of dimensional transmutation, 4-07). On each side there are tools (perturbation theory and semiclassics on the weak-coupling side, the strong-coupling expansion on the strong-coupling side, 5-02). That no tool works inside the band is where both the road of Part 23 of this book and this road stop (Part 26).

If one heads straight for four dimensions, the two walls appear at the same time. This is because one has to manage, within a single bound, both making the lattice spacing small (the ultraviolet) and looking at long distances in a large box (crossing the band).

03

Why a small twisted T² — two reasons

The skeleton of this road is that going by way of a small twisted T^2 lets us separate the two walls into different stages. There are two reasons.

Reason 1: the twist removes the zero modes, so one can proceed at weak coupling up to the size of the T^2 (an estimate). On a twisted T^2, the configuration with action 0 is fixed to be unique up to gauge (25-01 Uniqueness of the Twisted Vacuum), and the lattice Laplacian of the adjoint fluctuations around it has a positive gap $4\sin^2(\pi/2N)$ (25-01 Twisted Adjoint Gap Lemma, $N = L/a$) Proved here . In physical units, when N is large this gap is of order $(\pi/L)^2$, and all classical fluctuations have masses of order $1/L$. The running of the coupling then stops at the size L of the T^2, and if $L\Lambda \ll 1$ the effective coupling $g^2(L)$ is expected to stay small Reading (a physical estimate, on the assumption of ultraviolet control). The band does not appear in this stage.

Reason 2: the surface of a vortex winding the T^2 closes locally in the plane. A centre vortex (17-02) is a two-dimensional surface in four dimensions, and in four dimensions the vortex surface linked with a Wilson loop in the plane is an extended object whose cost grows with area. In the work of this book, every road that tried to cross the band while staying in four dimensions stopped at one point: "a topological quantity (whether a vortex links a loop) cannot be compared monotonically by local operations" (map in 26-08) Reading . In $\mathbb{R}^2 \times T^2$, the surface $\{p\} \times T^2$ above a point p of the plane is a closed surface of area L^2, and in the plane it behaves like a point. Adding or removing one vortex sheet becomes an operation inside a bounded region of the plane Proved here (that the surface lies in a bounded region is a fact from the definition). This locality supports all of the containers (25-03), slab confinement (25-04) and the insertion identities (25-05).

In this way the problem splits in two.

- **The intermediate goal (on a small T^2):** on $\mathbb{R}^2\times$ a small twisted T^2, show the gap and the area law uniformly in the lattice spacing a. The only wall that remains is the ultraviolet (construction under twisted boundary conditions) (though this is a conditional research plan, placing assumptions named ultraviolet control, uniform acceptance of insertion and frame decorrelation, and not a theorem) Assumption .
- **The last stage (widening the T^2):** that the strength of the area law stays uniform even as L is made larger, beyond $1/\Lambda$. The only wall that remains is the band (25-09) Assumption .

Splitting does not make it easier. The band remains whole in the last stage. What the splitting gains is that all the stages before the band can be arranged in a form that can be checked independently of the band, and that the form of the inequality the band demands can be narrowed to one (Part 26).

04

The ladder of scales — which tool works at which scale

Figure 2 takes the physical length L (the side of the small T^2, or the scale being looked at) vertically, and lays out, from the lattice spacing a at the bottom to infinity at the top, which tool works at which scale. It is a figure for seeing in one picture the shape of the problem: there are tools at the bottom (weak coupling) and at the top (strong coupling), and only the band in the middle is empty. The number and title of the figure match the entrance of Part 26 (Figure 2 "the ladder of scales" of 26-00). Figure 1 (the flow of the logic) is in 16-09.

Figure 2 The ladder of scales. The vertical axis is physical length (shorter towards the bottom). The labels of each stage are in Table 1; the text in the figure summarises the labelled rows and is not itself a claim



Figure 2 in words is as follows. From the bottom: from the lattice spacing a up to the base-point length L_0 is the ultraviolet stage (construction under twisted boundary conditions; an assumption in this book); L_0 is the base point of the small twisted T^2 (where the tools of Part 25 work directly); from L_0 to $1/\Lambda$ is the stage of the dilute gas of fractional instantons (a semiclassical estimate, with string tension of order $\Lambda^2(\Lambda L)^{5/3}$); around $1/\Lambda$ is the crossover band (where the renormalisation comparison is known only at the one-loop level and numerically); and above that is the confinement region (where strong-coupling tools are expected to work).

Table 1 Content and labels of each stage of Figure 2 (conditions in the same row)

Stage	Content	Label	Conditions, scope	Chapter
$a - L_0$	Multiscale bounds uniform in a (locality of the effective action, rarity of large fields, flow of the coupling)	Assumption	An assumption named ultraviolet control (that the same construction holds under twisted boundary conditions). The published construction covers only the periodic cubic box. In Bałaban's earlier series, stages remain that the original sources call unpublished	6-00, 7-07
L_0	Uniqueness of the vacuum that absorbs the twist, the gap $4\sin^2(\pi/2N)$ of the adjoint Laplacian, positive minimal action in the (1,1) sector	Proved here	SU(2), lattice, N fixed, $n_{12} = 1$	25-01
L_0	With the lattice spacing and the T^2 fixed, an area law at every β	Proved here	N fixed, lattice spacing fixed. In the continuum limit the tension becomes 0. Comparison with earlier work not yet done	25-04

Stage	Content	Label	Conditions, scope	Chapter
L_0	The gap and the area law on a small T^2, uniformly in a	Assumption	A conditional research plan under control of the ultraviolet, uniform acceptance of insertions and frame decorrelation	25-09, 27-01
L_0 – $1/\Lambda$	The lower bound $4\pi^2/g^2$ on the action of a self-dual lump of topological charge $\frac{1}{2}$, and the string tension $\Lambda^2(\Lambda L)^{5/3}$ of the dilute gas	Cited + Reading	The former is the BPS bound (a formula of the original source). The latter is a physical estimate from the dilute region and one-loop counting (equations (2.32) and (2.37) of Tanizaki–Ünsal)	25-02
Around $1/\Lambda$	Renormalisation comparison: at the one-loop level $\beta - \beta' = 0.258$ (per doubling of the lattice)	Proved here	One-loop level, SU(2), $\beta = 4/g^2$. Not a claim outside perturbation theory	26-06
Around $1/\Lambda$	Numerical reading: for SU(2) $\beta - \beta' \approx 0.2$; for U(1) the other way	Reading	Lattices $4^4 \to 8^4$ and others; the number of seeds and the errors are in 26-09	26-06
Around $1/\Lambda$	An inequality that crosses the band without relying on perturbation theory	Assumption	Unsolved. Its form is narrowed to one inequality in Part 26	26-02
$1/\Lambda$ – ∞	Area law on the strong-coupling side	Cited	A lattice statement at small β. The match with the scale of physical length is an estimate from asymptotic freedom	5-02

Of the rows of Table 1 at L_0, the upper two are statements with the lattice spacing fixed, and differ in strength from the third row, "uniform in a". This difference (making things uniform in the lattice spacing) is the ultraviolet wall, and each chapter of Part 25 writes in the same row on which side each statement lies.

05

Map of the ten chapters of Part 25

Part 25 consists of the following ten chapters. The labels are the main labels of the rows in each chapter; the finer division is in each chapter's table of five grades. The "Conditions" column gives the conditions that always attach to the chapter's main statements.

Table 2 The ten chapters of Part 25 (labels and conditions in the same row)

Chapter	Title	Content in one line	Main labels	Conditions

Chapter	Title	Content in one line	Main labels	Conditions
25-00	Entrance — space, not time	This chapter	Reading + Cited	"Stays weakly coupled" is an estimate under the assumption of control of the ultraviolet
25-01	The twisted T^2 and 't Hooft flux	$n_{12} = 1$, sectors, the flux free energy, the twist-summed ensemble. Three elementary facts	Cited (definitions) / Proved here (three facts)	SU(2), lattice, N fixed, $n_{12} = 1$
25-02	Vortices and fractional instantons	Lumps of topological charge $\frac{1}{2}$, action $4\pi^2/g^2$, the dilute gas, string tension $\Lambda^2(\Lambda L)^{5/3}$, and that the lattice action approaches $\pi^2\beta$ from below	Cited / Reading / Computer-assisted / Assumption	Dilute region, one loop, SU(2). The cooled action is an upper bound
25-03	Containers — conditioning on the frame makes things independent	The container identity (Mack–Petkova), nesting in the directions of the plane, limits in four dimensions	Cited / Proved here	The identity is known. Lean covers only the finite algebra
25-04	Slab confinement	With the lattice spacing and the T^2 fixed, an area law at every β	Proved here	N fixed, lattice spacing fixed, tension 0 in the continuum limit. Comparison with earlier work not yet done
25-05	Vortex-insertion identities	Expectation 1 of the insertion ratio, correlation with the sign of the sector, averaged bounds, second moments	Proved here	Finite configuration space, positive weights, a bijection that flips the sector
25-06	Insertion driven by dynamics, and the flow lemma	A Jarzynski-type identity, bounds on simultaneous flips	Proved here / Cited	Bounds on the kernel of one step cannot be used to test the band
25-07	Why fixed maps fail	A map that does not change the shape of fluctuations cannot see the 4 of antiscreening. $11/3 = -1/3 + 4$	Proved here / Reading	One-loop level

Chapter	Title	Content in one line	Main labels	Conditions
25-08	The measured record	Work of driven insertion, vortex free energy, string tension of the slab	Reading	Group, β, lattice size, number of measurements and errors in each row
25-09	Where this road stops	Widening the T^2 = the crossover band	Assumption + Reading	—

We recommend reading in numerical order. A reader in a hurry can follow the skeleton of the road by reading 25-01 § 01– § 02 (the setting), 25-02 § 01– § 03 (fractional instantons and the dilute gas), 25-04 (slab confinement) and 25-09 (where it stops), in that order.

06

The conventions of notation and words in this part

- **Group and coupling.** Throughout Part 25 the group is fixed to $SU(2)$. The Wilson action is $\beta \sum_p (1 - \frac{1}{2} \operatorname{tr} U_p)$ in the convention of 5-01, with $\beta = 4/g^2$.
- **N and M.** The N of Part 25 is not the degree of the group but the number of lattice points along a side of the small T^2 ($L = Na$). We write M for the number of lattice points along a side of the plane. The lattice is $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$.
- **Band and window.** The subject of this part and of Part 26 is "the crossover band" (the band for short). It is a different thing from the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window), and that the words differ carries no meaning beyond that (note in 26-00).
- **Names of assumptions.** We call assumptions by names, not symbols: control of the ultraviolet, uniform acceptance of insertions, frame decorrelation, the uniform slab tension, the uniform slab gap. The index is collected in A-06.
- **Labels.** The same five as in Part 23: Proved here (a paper proof placed in the text), Cited (known, checked in the original source), Computer-assisted (a computer-assisted bound), Reading (a numerical reading; physical estimates are marked "physical estimate"), Assumption (an assumption, or a row for which only a strand of proof exists; the evidence column says where the strand is). How to read them is in 23-00.

07

Guide to Part 23 and Part 26

To Part 23 (the first road). Part 23 is the road towards the mass gap from the direction of time, and it built the diffusive-relaxation-time assumption in the model of the first step. The entrance is 23-00, where it stopped is 23-15, and the considerations beyond

are in Part 24 (24-00). The two roads aim to show different statements (Part 23 the gap, Part 25 first the area law).

To Part 26 (the band). Both roads stop just before a statement of the type "the same constant at all scales". Part 26 (26-00) treats the correct form of that statement (the correct form of the band, 26-01), its reduction to one inequality (26-02), and the same band seen from the side of time (26-07). The last chapter of Part 25 (25-09) is the bridge to Part 26.

To the table of stages. The table of the stages a proof must contain (the container, the ultraviolet, the gap on a small T^2, confinement on a small T^2, the band, $\mathbb{R}^4$, general groups) is in 16-09. Part 25 fills the two "small T^2" stages of that table and the entrance to the "band" stage.

08

What cannot be said

(a) Weak coupling. "The coupling stays weak up to the size of the T^2" is a physical estimate on the assumption of ultraviolet control, not a theorem. Bounds uniform in the lattice spacing in the setting of this part (the small T^2 with twisted boundary conditions and elongated boxes) appear neither in this book nor in the published construction (the published construction is for the periodic cubic box).

(b) Adiabatic continuity. That the small T^2 and $\mathbb{R}^4$ are joined without a phase transition is a conjecture, and this book does not use it. What it uses is only the form "if they are joined, what must be shown".

(c) Confinement and the gap. The goal of this road is the area law, and there is no known road from the area law to the mass gap. The two are different statements (16-09, 27-01).

(d) Figure 2. Figure 2 is a map of where the tools sit, and the match of scale and coupling (which L corresponds to which β) is an estimate from asymptotic freedom. The numbers in the figure summarise the labelled rows of Table 1.

09

The table of five grades and the dependency graph

Table 3 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	On a periodic T^2 adiabatic continuity does not hold (as stated in the physics literature)	Cited	Tanizaki–Ünsal § 2.1	Physical argument	—
2	The conjecture of continuity from the twisted $T^2 \times \mathbb{R}^2$ to $\mathbb{R}^4$	Cited	Same, abstract	As a conjecture	—
3	Uniqueness of the vacuum that absorbs the twist, and the gap of the adjoint Laplacian	Proved here	Uniqueness of the Twisted Vacuum and the Twisted Adjoint Gap Lemma of 25-01	SU(2), lattice, N fixed, $n_{12} = 1$	—
4	The surface $\{p\} \times T^2$ lies in a bounded region of the plane	Proved here	Definition	—	—
5	Ultraviolet control (that the same construction as the published one holds under twisted boundary conditions)	Assumption	6-00, 7-07	From a to L_0	—
6	Weak coupling up to the size of the T^2 $\Longleftarrow$ 3, 5	Reading	Physical estimate	$L\Lambda \ll 1$	—
7	The one point at which the roads that tried to cross the band in four dimensions stopped (topological quantities cannot be compared locally)	Reading	Map of 26-08	Within the roads tried	—
8	The problem splits into an intermediate goal and a last stage $\Longleftarrow$ 4, 6, 7	Assumption	This chapter § 03	The form of a research plan	—
9	Figure 2 and each stage of Table 1 $\Longleftarrow$ 5, 6, 8	The label of each row	Table 1	Conditions of Table 1	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
1  CIT  no continuity on a periodic T²                          physical argumen
2  CIT  continuity from twisted T² × R² to R⁴ (conjecture)      conjecture
3  PRV  unique vacuum absorbing the twist, gap 4sin²(π/2N)       SU(2), N fixed,
4  PRV  surface {p} × T² lies in a bounded region of the plane
5  ASM  ultraviolet control (construction under twisted boundary conditions)
6  RDG  weak coupling up to the size of T² ⟸ 3, 5               physical estima
7  RDG  the one point where roads crossing in 4D stopped        within the roads
8  ASM  splits into intermediate goal and last stage ⟸ 4, 6, 7
9  MIX  Figure 2, Table 1 ⟸ 5, 6, 8
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The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only nodes with proofs placed are the uniqueness of the twisted vacuum and the gap (proved in 25-01) and that the surface lies in a bounded region of the plane (definition). The conjecture of continuity from the twisted T^2 to R^4 enters as a premise of no node. The assumption row (ultraviolet control) does not contain the quantities of the conclusion (gap, area law) and states only the form of a multiscale bound. The nodes on the conclusion side of this chapter (weak coupling persists, the problem splits in two) are "Reading" and "Assumption", and there is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Uniqueness of the configuration with action 0 on the twisted T^2 and the gap of the adjoint Laplacian (proved in 25-01); that the vortex surface is local in the plane	A gap uniform in the lattice spacing. The form of the area law on a small T^2 uniform in a
Cited What the physics literature says about the difference between a periodic T^2 and a twisted T^2; the conjecture of continuity (as a conjecture)	The conjecture of continuity itself
Reading "Weak coupling up to the size of the T^2" (a physical estimate under the assumption of control of the ultraviolet); the arrangement of Figure 2	The quantitative form of the match of scale and coupling. The width of the band
Assumption The form of the research plan in which the problem splits into two stages	Neither stage is closed. An inequality that crosses the band

Sources and reproduction

Item	Kind	Source / tools
Compactification on periodic T^2 and on twisted T^2, the conjecture of continuity	Cited	Y. Tanizaki, M. Ünsal, Center vortex and confinement in Yang–Mills theory and QCD with anomaly-preserving compactifications, arXiv:2201.06166 (v2, 2022). Abstract, the title of § 2.1 and § 2.2–2.3 checked in the text
Numerics of the gas of fractional instantons on the lattice	Cited	G. Bergner, A. González-Arroyo, I. Soler, Fractional instantons and confinement: first results on a $T_2 \times R^2$ roadmap, arXiv:2502.09463 (2025). Abstract checked
Twisted boundary conditions	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141 / G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data as in 3-05)
What the ultraviolet has reached	This book	6-00 (the published construction), 7-07
One-loop $\beta - \beta' = 0.258$	Proved here	$\beta = 4/g^2$, and from the one-loop coefficient $b_0 = 22/(48\pi^2)$, $8b_0 \ln 2 \approx 0.2575$. The derivation is in 26-06
Numerical $\beta - \beta' \approx 0.2$	Reading	The numerical record of this book's work (26-09)
Titles and labels of the ten chapters	This book	Table 2. Each chapter's table of five grades is authoritative

Words used in this chapter — the second road, the small twisted T^2, the ultraviolet wall, the crossover band, control of the ultraviolet, the intermediate goal, the ladder of scales, the base point L_0, the dilute gas of fractional instantons, the renormalisation comparison. Their meanings are in A-02 Glossary.

Next chapter to read: 25-01 The twisted T² and 't Hooft flux — the setting, sectors, the flux free energy, and three elementary facts.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space twist, sectors, the flux free energy · Kiichi, Shiori and Rin

The twisted T² and 't Hooft flux

> In this chapter, § 01 and § 02 are established knowledge and § 03 onward are this book's own investigation.

What this chapter shows — We define the stage of Part 25. To the $SU(2)$ Wilson action on the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ we add 't Hooft's twist $n_{12} = 1$ in the plane of the small T^2, and set up the two sectors of the flux n_{34} in the directions of the plane, the ratio of their partition functions (the flux free energy), and the twist-summed ensemble that combines the two sectors into one measure. We then state with proofs three elementary facts that form the foundation of this road: (1) uniqueness of the configuration with action 0 (the vacuum that absorbs the twist), (2) the antiperiodicity of the adjoint field around it and the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, and (3) that the $(1,1)$ sector has no flat configuration and its minimal action is positive.

Prerequisite chapters — 25-00 (the entrance and notation of this part), 5-01 (link variables, plaquettes, the Wilson action), 1-04 ($SU(2)$ and the Pauli matrices), 3-05 (topological charge), 17-02 centre vortices.

Said first — Every statement of this chapter is about the group $SU(2)$, a finite lattice, N fixed and the twist $n_{12} = 1$. It says nothing about the continuum limit, bounds uniform in the lattice spacing, or confinement. Most of the definitions are known ones of the type of 't Hooft (1979, 1981) and Mack–Petkova (1980) Cited ; the three properties are elementary, and their proofs are placed in the text Proved here . All three are lattice versions of things known in the physics literature, and we claim nothing new about them.

The order of this chapter

1. The setting — lattice, action, the sign field z
2. Sectors and the flux free energy — the Moving-Surface Lemma and the definition
3. The twist-summed ensemble — the Parity Representation of the Bias

4. Plaquettes with central values, and holonomy — configurations with action zero, and the commutator of the rounds
5. Uniqueness of the Twisted Vacuum
6. The twisted adjoint gap — antiperiodicity and $4\sin^2(\pi/2N)$
7. Positivity of the Minimal Action in the Twisted Sector — the (1,1) sector has no flat configuration
8. What the three properties of the twisted T^2 say
9. What cannot be said
10. The table of five grades and the dependency graph (Table 1)

01

The setting — lattice, action, the sign field z

This section summarises existing knowledge ('t Hooft's twisted boundary conditions written with a sign field on the lattice). The introduction is in 3-05 § 03.

Let the lattice be $\Lambda = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($N, M \geq 2$). Directions 1 and 2 form the small T^2, and directions 3 and 4 the plane. On each link (x, μ) we place $U_{x,\mu} \in SU(2)$, and we write the value of a plaquette $p = (x; \mu, \nu)$ ($\mu < \nu$) as

$$U_p = U_{x,\mu}\, U_{x+\hat{\mu},\nu}\, U^{-1}_{x+\hat{\nu},\mu}\, U^{-1}_{x,\nu} \quad (1)$$

(as in 5-01). Given a sign $z_p \in \{+1, -1\}$ for each plaquette, we set the action and the partition function to be

$$S^z(U) = \sum_p \Big(1 - \tfrac{1}{2}\, z_p \operatorname{tr} U_p\Big), \qquad Z[z] = \int e^{-\beta S^z(U)} \prod_\ell dU_\ell, \qquad \beta = \frac{4}{g^2} \quad (2)$$

dU_ℓ is the normalised Haar measure. If $z \equiv +1$, (2) is exactly the Wilson action of 5-01, and each term lies in $[0, 2]$ because $|\operatorname{tr} U| \leq 2$. A plaquette with $z_p = -1$ is "a plaquette with one central element -1 inserted", and there the action is 0 when $U_p = -1$.

The signs of the two sectors. We put the twist of the small T^2 in by placing $z_p = -1$ on exactly one plaquette of each $(1, 2)$ plane (the $N \times N$ plane with x_3, x_4 held fixed), namely the $(1, 2)$ plaquette at $(x_1, x_2) = (N-1, N-1)$. The position is the same for every (x_3, x_4). We write this as $z^{(0)}$ (all others are $+1$). The sign $z^{(1)}$ with one unit of flux added in the directions of the plane is obtained from $z^{(0)}$ by taking, above a unit square $q = (q_3, q_4)$ of the plane, the surface

$$\Pi_q = \{\, ((x_1, x_2, q_3, q_4); 3, 4) \;:\; x_1, x_2 \in \mathbb{Z}/N \,\} \quad (3)$$

and placing $z_p = -1$ on its N^2 $(3,4)$ plaquettes. Π_q is a "surface winding the T^2" standing above the point q of the plane, and in the plane it behaves like a point (reason 2 of 25-00 § 03).

02

Sectors and the flux free energy

Where the surface is placed, q, has no effect on the partition function. The next lemma shows this.

Moving-Surface Lemma Proved here For a link ℓ, write $\delta_\ell z$ for the field in which the signs of all plaquettes containing ℓ are flipped. Then $Z[\delta_\ell z] = Z[z]$. In particular, (a) the surface Π_q can be moved to the surface $\Pi_{q'}$ of a neighbouring square, and the partition function of $z^{(1)}$ does not depend on q. (b) The partition function of the sign $z^{(0)} \prod_{q \in Q} (\text{surface } \Pi_q)$, with surfaces placed over a set Q, is determined only by the parity of $|Q|$.

Proof. The change of variables $U_\ell \mapsto -U_\ell$ preserves the Haar measure. Since -1 is a central element, in a plaquette that contains ℓ exactly once, U_p changes to $-U_p$, and other plaquettes do not change (a plaquette of the lattice contains a given link at most once). Hence $S^{\delta_\ell z}(-U_\ell, \ldots) = S^z(U)$, and the integrals are equal. (a): apply the lemma in turn to all the links in direction 3 above the point $(a, b+1)$ of the plane (the N^2 links obtained by running over x_1, x_2). The only $(3,4)$ plaquettes containing these links are those on $\Pi_{(a,b)}$ and $\Pi_{(a,b+1)}$, each containing exactly one. A $(\mu, 3)$ plaquette ($\mu \le 2$) contains two links in direction 3, both above the same point of the plane, so its sign is flipped twice and returns to what it was. Other plaquettes contain no link in direction 3. So the signs on the two surfaces $\Pi_{(a,b)}$ and $\Pi_{(a,b+1)}$ are flipped, and the surface moves one step over. Direction 4 is the same. (b): moving surfaces so that two overlap makes them cancel, $(-1)^2 = +1$, so this follows from (a) and the connectedness of the squares of the plane. ∎

The Moving-Surface Lemma is a basic property of the lattice version of 't Hooft's twist: it says that it is not the field z itself but only the parity of the number of -1 piercing each plane (the flux $n_{\mu\nu} \in \{0, 1\}$) that affects the partition function Cited (the general form is 't Hooft 1979; the form with a lattice $\mathbb{Z}_2$ sign field is Mack–Petkova 1980). In this chapter we use only cases (a) and (b).

Definition of the sector and flux free energies We write $Z(0) = Z[z^{(0)}]$, $Z(1) = Z[z^{(1)}]$, and call the flux pairs $(n_{12}, n_{34}) = (1,0)$ and $(1,1)$ sectors. The flux in the mixed planes ($(1,3)$ and so on) is taken to be 0. We then set

$$\Delta F = \ln \frac{Z(0)}{Z(1)}, \qquad \theta(N, M, \beta) = \frac{Z(0) - Z(1)}{Z(0) + Z(1)} = \tanh \frac{\Delta F}{2} \tag{4}$$

ΔF is the difference of free energy for putting in one unit of flux n_{34} in the directions of the plane ('t Hooft's flux free energy), and θ is the "bias" between the two sectors. The right-hand equality of (4) comes from $(1 - e^{-\Delta F})/(1 + e^{-\Delta F}) = \tanh(\Delta F/2)$. Since $Z(0), Z(1) > 0$, $|\theta| < 1$. $\theta \geq 0$ (that is, $Z(1) \leq Z(0)$) needs a separate argument through the transfer matrix, and is treated in 26-04.

The reason this quantity matters for confinement is that the Tomboulis–Yaffe inequality bounds the expectation of a Wilson loop in the plane from above by a ratio of twisted partition functions Cited. If θ decreases in the size M of the plane as $-\ln \theta \geq cM^2$, the area law follows through that inequality. The exact form of the inequality, and that it is not enough for θ merely to move away from 1, are treated in 26-01.

03 The twist-summed ensemble

Instead of keeping the two sectors as separate measures, we combine them into one measure in which the placement of the surfaces is also a random variable. Let T be the set of all unit squares of the plane (M^2 of them), and for each subset $Q \subset T$ form the sign $z^Q = z^{(0)} \prod_{q \in Q} (\Pi_q \colon \text{sign } -1)$.

Definition (the twist-summed ensemble) Cited We call the probability measure on pairs (U, Q) of a configuration and a set of surfaces

$$\tilde{\mu}(dU, Q) = \frac{1}{\Xi} e^{-\beta S^{z^Q}(U)} \prod_{\ell} dU_\ell, \qquad \Xi = \sum_{Q \subset T} Z[z^Q] \tag{5}$$

the twist-summed ensemble. The form that sums the $\mathbb{Z}_2$ signs of plaquettes as random variables is of the type of the representation of Mack–Petkova (1980), who sum over all closed sign fields (where the product of the six on each cube is $+1$). Here we use a small form that sums only the surfaces involved in the flux n_{34} of the plane.

Parity Representation of the Bias (the bias as one expectation) Proved here $\theta = \mathbb{E}_{\tilde{\mu}}\left[(-1)^{|Q|}\right]$.

Proof. By the Moving-Surface Lemma (b), $Z[z^Q]$ is $Z(0)$ if $|Q|$ is even and $Z(1)$ if it is odd. Of the 2^{M^2} subsets of T, exactly half have an even number of elements. Hence $\Xi = 2^{M^2-1}(Z(0)+Z(1))$ and $\sum_Q (-1)^{|Q|} Z[z^Q] = 2^{M^2-1}(Z(0)-Z(1))$, and the ratio is the θ of (4). ∎

In the Parity Representation of the Bias, the placement Q of the surfaces is a record of "whether a vortex is there" above the points of the plane, and $(-1)^{|Q|}$ is the parity of the number of vortices in the whole plane. That the bias θ is small means that the parities are mixed almost fairly, that is, vortices well up freely in the plane. In $\mathbb{R}^4$, the vortex surface linked with a loop is an extended surface and cannot be written in this form of "a record above points" (25-00 § 03). This representation is the starting point both for rewriting things as a product of biases per column by dividing the plane into blocks (the Container Identity, 25-03) and for the identities of the map that inserts one vortex (25-05).

04

Plaquettes with central values, and holonomy

The proofs of the three properties reduce to properties of configurations in which every plaquette takes a central value (± 1). First we check that action 0 is the same thing.

Zero-Action Configuration Lemma (configurations with action 0) Proved here A necessary and sufficient condition for $S^z(U) = 0$ is that $U_p = z_p \cdot \mathbf{1}$ for every plaquette.

Proof. An element of $SU(2)$ can be written $U = \cos\alpha\,\mathbf{1} + i\sin\alpha\,(n\cdot\sigma)$ ($|n|=1$), and $\operatorname{tr} U = 2\cos\alpha$ (1-04). Each term $1 - \frac{1}{2} z_p \operatorname{tr} U_p = 1 - z_p \cos\alpha_p$ of (2) is at least 0, and it is 0 only when $z_p \cos\alpha_p = 1$, that is, when $\alpha_p = 0$ ($U_p = \mathbf{1}$) if $z_p = 1$, and $\alpha_p = \pi$ ($U_p = -\mathbf{1}$) if $z_p = -1$. ∎

Next, if all plaquettes take central values, the commutators of the holonomies around a plane are determined by the product of the signs of that plane. We write $P_\mu(x)$ for the product of links along the straight path that goes once around in direction μ from the base point x (of length N if $\mu = 1,2$, of length M if $\mu = 3,4$).

Lemma on the Sign of the Holonomy Commutator (the commutator is the product of the signs of the plane) Proved here If $U_p = z_p \cdot \mathbf{1}$ for every plaquette, then for $\mu \neq \nu$ and a base point x

$$P_\mu(x)\,P_\nu(x)\,P_\mu(x)^{-1}\,P_\nu(x)^{-1} = \varepsilon_{\mu\nu}\,\mathbf{1}, \qquad \varepsilon_{\mu\nu} = \prod_{p \subset (\mu,\nu)\text{ plane } \ni x} z_p \tag{6}$$

holds. The product on the right runs over all plaquettes of the (μ,ν) plane (a two-dimensional torus) through x.

Proof. Cut the (μ,ν) plane through x open into a square R with x at a corner. The path that goes once anticlockwise around the boundary of R from x consists, in this order, of one round in direction μ, one round in direction ν, one round backwards in direction μ and one round backwards in direction ν, and its holonomy is the left-hand side of (6). On the other hand, on a disc of the square lattice, the holonomy of the boundary equals a product, in some order, of the values of the plaquettes conjugated by parallel transport along paths to the base point (the lattice non-abelian Stokes theorem: an induction that peels plaquettes off one at a time and shrinks the boundary path). If all plaquette values are central elements $z_p\mathbf{1}$, conjugation does not change the values and the order does not matter, so the product is $\prod_{p\subset R} z_p$. ∎

For $z^{(0)}$, every $(1,2)$ plane has exactly one -1, so $\varepsilon_{12}=-1$, and the other planes have $\varepsilon=+1$. For $z^{(1)}$, in addition, the $(3,4)$ planes have exactly one -1 (the one meeting the surface Π_q), so $\varepsilon_{34}=-1$. This is the lattice version of the relation, in 't Hooft's twisted boundary conditions of the continuum theory, that the consistency of the gluing in two directions is off by the central element $e^{\pi i n_\mu n_\nu}$ Cited ('t Hooft 1979).

Finally, we state how far the holonomies determine a configuration.

Determination-by-Holonomy Lemma (determined by holonomy) Proved here If two configurations U, U' satisfying $U_p = z_p\mathbf{1}$ (for every p) for the same z have the same four holonomies at the base point 0 ($P_\mu = P'_\mu$, $\mu=1,\dots,4$), then U' is a gauge transform of U.

Proof. For each lattice point x choose a fixed path γ_x from 0 to x (the path that goes in the positive sense in direction 1, direction 2, direction 3 and direction 4, in that order), and write $U(\gamma_x)$ for the product along it. For a link $\ell=(x\to y)$, consider the closed path $c_\ell=\gamma_x\,\ell\,\gamma_y^{-1}$. In the two-dimensional skeleton of the toroidal lattice, a closed path based at 0 can be deformed, by sweeping finitely many plaquettes, into a word in the four straight rounds (the lattice version of $\pi_1(T^4)=\mathbb{Z}^4$, a standard topological fact Cited). The plaquettes swept in the deformation take central values, so the holonomy of c_ℓ is the product of "the product of z over the swept plaquettes" and "a word in the P_μ", both of which are the same for U and U'. Hence $U(\gamma_x)U_\ell U(\gamma_y)^{-1} = U'(\gamma_x)U'_\ell U'(\gamma_y)^{-1}$, that is, setting $g(x)=U'(\gamma_x)^{-1}U(\gamma_x)$ gives $U'_\ell = g(x)\,U_\ell\,g(y)^{-1}$. ∎

05

Uniqueness of the Twisted Vacuum

The configuration that absorbs the twist. Set $\Omega_1 = i\sigma_1$, $\Omega_2 = i\sigma_2$, and define the configuration U^0 as follows. Of the links in direction 1, only those at $x_1 = N-1$ (those crossing the seam of the T^2) are Ω_1; of the links in direction 2, only those at $x_2 = N-1$ are Ω_2; all other links (including those in directions 3 and 4) are $\mathbf{1}$.

Lemma on the Configuration that Absorbs the Twist Proved here $\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -\mathbf{1}$. Moreover, among the plaquettes of U^0, only the $(1,2)$ plaquette at $(x_1, x_2) = (N-1, N-1)$ is $-\mathbf{1}$, and all others are $\mathbf{1}$. Hence $S^{z^{(0)}}(U^0) = 0$.

Proof. From $\sigma_1\sigma_2 = i\sigma_3 = -\sigma_2\sigma_1$ we get $\Omega_1\Omega_2 = -\sigma_1\sigma_2 = -i\sigma_3$ and $\Omega_1^{-1}\Omega_2^{-1} = (-i\sigma_1)(-i\sigma_2) = -i\sigma_3$, and the product is $(-i\sigma_3)^2 = -\mathbf{1}$. Looking at the plaquette (1), all $(1,2)$ plaquettes not containing the seam are $\mathbf{1}$. At $x_1 = N-1$, $x_2 \neq N-1$ we get $\Omega_1 \cdot \mathbf{1} \cdot \Omega_1^{-1} \cdot \mathbf{1} = \mathbf{1}$; at $x_2 = N-1$, $x_1 \neq N-1$ we get $\mathbf{1} \cdot \Omega_2 \cdot \mathbf{1} \cdot \Omega_2^{-1} = \mathbf{1}$; at the corner $(N-1, N-1)$ we get $\Omega_1\Omega_2\Omega_1^{-1}\Omega_2^{-1} = -\mathbf{1}$. The links in directions 1 and 2 do not depend on x_3, x_4, so all plaquettes involving directions 3 and 4 are $\mathbf{1}$. By the Zero-Action Configuration Lemma the action is 0. ∎

Rigidity Lemma for Anticommuting Pairs (an anticommuting pair) Proved here If A, $B \in SU(2)$ satisfy $ABA^{-1}B^{-1} = -\mathbf{1}$, then for some $g \in SU(2)$ we have $gAg^{-1} = i\sigma_1$, $gBg^{-1} = i\sigma_2$. Moreover, the only elements of $SU(2)$ that commute with both $i\sigma_1$ and $i\sigma_2$ are $\pm\mathbf{1}$.

Proof. Taking the trace of $ABA^{-1} = -B$ gives $\operatorname{tr} B = -\operatorname{tr} B$, hence $\operatorname{tr} B = 0$ and $B = i\,b \cdot \sigma$ ($|b| = 1$). Likewise $BAB^{-1} = -A$ gives $A = i\,a \cdot \sigma$. Since $(a \cdot \sigma)(b \cdot \sigma) = (a \cdot b)\mathbf{1} + i(a \times b) \cdot \sigma$, $AB = -BA$ is equivalent to $a \cdot b = 0$. A pair (a, b) of orthogonal unit vectors can be moved to (e_1, e_2) by a rotation $R \in SO(3)$ ($(a, b, a \times b)$ is a positively oriented orthonormal basis). $SU(2) \to SO(3)$ is surjective, and taking g with $g(v \cdot \sigma)g^{-1} = (Rv) \cdot \sigma$ gives the first half of the claim. Second half: if $C = c_0\mathbf{1} + i\,c \cdot \sigma$ commutes with σ_1 then $c \times e_1 = 0$, and if it commutes with σ_2 then $c \times e_2 = 0$; hence $c = 0$ and $C = \pm\mathbf{1}$. ∎

Uniqueness of the Twisted Vacuum (uniqueness of the vacuum that absorbs the twist) Proved here In the sector $(1,0)$, the configuration with action 0 ($S^{z^{(0)}}(U) = 0$) is U^0 alone, up to gauge transformations and central transformations in directions 3 and 4 (the operation of multiplying by -1 all the links in direction 3 of some direction-3 cut $x_3 = c$; likewise for direction 4). On the small T^2 alone (the two-dimensional lattice $(\mathbb{Z}/N)^2$, twist $n_{12} = 1$), it is unique up to gauge transformations only.

Proof. By the Zero-Action Configuration Lemma, $U_p = z_p^{(0)}\mathbf{1}$. By the Lemma on the Sign of the Holonomy Commutator, the holonomies at the base point 0 satisfy $P_1P_2P_1^{-1}P$

$_2^{-1} = -\mathbf{1}$, and the other pairs commute. Re-choosing the gauge at the base point by the Rigidity Lemma for Anticommuting Pairs, $P_1 = i\sigma_1$, $P_2 = i\sigma_2$. P_3, P_4 commute with both, so they are $\pm\mathbf{1}$. The holonomies of U^0 are $(i\sigma_1, i\sigma_2, \mathbf{1}, \mathbf{1})$. If one multiplies the direction-3 links of the cut $x_3 = c$ by -1, every plaquette containing them contains two (the two direction-3 sides of the $(\mu, 3)$ and $(3, 4)$ plaquettes lie at the same x_3), so the action does not change and only P_3 changes sign. Direction 4 is the same. So all four tuples $(i\sigma_1, i\sigma_2, \pm\mathbf{1}, \pm\mathbf{1})$ are obtained from U^0 by central transformations, and by the Determination-by-Holonomy Lemma configurations with action 0 whose holonomies agree are related by gauge transformations. In two dimensions there are no P_3, P_4, so gauge transformations alone suffice. ∎

As the proof of the Rigidity Lemma for Anticommuting Pairs shows, $(-i\sigma_1, i\sigma_2)$ is moved to $(i\sigma_1, i\sigma_2)$ by conjugation with $g = i\sigma_2$. Central transformations in the directions of the small T^2 are absorbed into gauge transformations. Central transformations in the directions of the plane are not absorbed, and there are four configurations with action 0 as gauge classes.

06

The twisted adjoint gap — antiperiodicity and 4sin²(π/2N)

In the quadratic part of the fluctuations around the vacuum U^0, the covariant Laplacian of the adjoint field appears. On the small T^2 (the two-dimensional lattice $(\mathbb{Z}/N)^2$), for a field ϕ with values in $\mathfrak{su}(2)$ we set

$$(-\Delta_U \phi)(x) = \sum_{\mu=1,2} \Big[2\phi(x) - \mathrm{Ad}(U_{x,\mu})\, \phi(x+\hat{\mu}) - \mathrm{Ad}(U_{x-\hat{\mu},\mu})^{-1} \phi(x-\hat{\mu}) \Big], \qquad \mathrm{Ad}(g)X$$

Since Ad is an orthogonal transformation, $-\Delta_U$ is symmetric and non-negative with respect to the inner product of $\mathfrak{su}(2)$.

Twisted Adjoint Gap Lemma Proved here When $U = U^0$, for each component ϕ^1, ϕ^2, ϕ^3 in the basis $i\sigma_1, i\sigma_2, i\sigma_3$ of $\mathfrak{su}(2)$, $-\Delta_{U^0}$ splits into scalar Laplacians under the following signed boundary conditions.

Component	Direction 1	Direction 2	Smallest eigenvalue
ϕ^1 $(i\sigma_1)$	Periodic	Antiperiodic	$4\sin^2(\pi/2N)$
ϕ^2 $(i\sigma_2)$	Antiperiodic	Periodic	$4\sin^2(\pi/2N)$
ϕ^3 $(i\sigma_3)$	Antiperiodic	Antiperiodic	$8\sin^2(\pi/2N)$

Every component is antiperiodic in at least one direction. Hence the smallest eigenvalue of $-\Delta_{U^0}$ is $4\sin^2(\pi/2N) > 0$, and there are no zero modes.

Proof. $\mathrm{Ad}(i\sigma_1)$ preserves $i\sigma_1$ and changes the signs of $i\sigma_2, i\sigma_3$ ($\sigma_1\sigma_2\sigma_1 = -\sigma_2$ and so on), and $\mathrm{Ad}(i\sigma_2)$ preserves $i\sigma_2$ and changes the signs of $i\sigma_1, i\sigma_3$. Only the links of U^0 that cross the seam are Ω_μ, so (7) becomes, component by component, a scalar Laplacian with a sign ± 1 attached only to the bonds crossing the seam. On a ring of length N with a sign s on one bond, the difference operator $2f(x) - f(x+1) - f(x-1)$ has eigenfunctions $f(x) = e^{ikx}$ for k satisfying $e^{ikN} = s$, with eigenvalues $2 - 2\cos k = 4\sin^2(k/2)$. If $s = +1$ then $k \in \frac{2\pi}{N}\mathbb{Z}$ (minimum 0); if $s = -1$ then $k \in \frac{2\pi}{N}(\mathbb{Z} + \frac{1}{2})$ (minimum $4\sin^2(\pi/2N)$, at $k = \pi/N$). In two dimensions it is the sum over the two directions, which gives the values in the table. ∎

For the vacuum $U \equiv \mathbf{1}$ of a periodic T^2 without twist, every component is periodic in both directions, and constant ϕ give three dimensions' worth of zero modes. The role of the twist is to remove these zero modes and give the fluctuations the lower bound $4\sin^2(\pi/2N)$. When N is large, $4\sin^2(\pi/2N) \approx \pi^2/N^2$, and restoring the unit of the lattice spacing a gives $(\pi/L)^2$ ($L = Na$).

Reading the same gap as a mass, the lattice mass m_N is given by $4\sinh^2(m_N/2) = 4\sin^2(\pi/2N)$, that is,

$$m_N = 2\ \mathrm{arsinh}\sin\frac{\pi}{2N} \tag{8}$$

(the relation between the decay rate of the lattice propagator and the gap, $2\cosh m - 2 = 4\sinh^2(m/2)$) Proved here . This m_N appears in 25-02 § 05 as the rate of convergence of the action of a fractional instanton confined to a column.

07

Positivity of the Minimal Action in the Twisted Sector — the (1,1) sector has no flat configuration

Positivity of the Minimal Action in the Twisted Sector Proved here In the sector $(1,1)$ there is no configuration with action 0. Hence the minimal action

$$S_{\min}(N,M) = \min_U S^{z^{(1)}}(U) \tag{9}$$

is positive. Moreover, $S_{\min}(N,M) \le 2N^2$.

Proof. If there were a configuration with action 0, then by the Zero-Action Configuration Lemma and the Lemma on the Sign of the Holonomy Commutator the holonomies at the base point would satisfy $P_1P_2P_1^{-1}P_2^{-1} = -\mathbf{1}$, $P_3P_4P_3^{-1}P_4^{-1} = -\mathbf{1}$ ($\varepsilon_{34} = -1$), with the mixed pairs commuting. By the Rigidity Lemma for

Anticommuting Pairs, P_3 commutes with both P_1, P_2 and so is $\pm\mathbf{1}$, and the same holds for P_4. Then $P_3 P_4 P_3^{-1} P_4^{-1} = \mathbf{1}$, a contradiction. The configuration space $SU(2)^{\#\text{links}}$ is compact and the action is continuous, so the minimum is attained, and since it is not 0 it is positive. Upper bound: evaluating U^0 in the sector $(1,1)$, on the N^2 plaquettes of the surface Π_q we have $U_p = \mathbf{1}$, $z_p = -1$, giving $1 - \frac{1}{2}(-1)(2) = 2$ per plaquette, and 0 elsewhere. ∎

The configuration of the upper bound is a "thin vortex surface", which pays action $2N^2$ on a surface one plaquette thick. The Positivity of the Minimal Action in the Twisted Sector says only that the minimal action is not 0; it does not say how small it is (whether there is a smooth lump far cheaper than the thin surface). That is the story of fractional instantons in the next chapter; lattice numerics read $S_{\min}$ as of order π^2, much smaller than $2N^2$ (25-02 § 05).

In the continuum theory, the topological charge of a bundle with twist $n_{12} = n_{34} = 1$ is restricted to half-integers, and the action is bounded below by $8\pi^2|Q|/g^2 \geq 4\pi^2/g^2$ Cited ('t Hooft 1981, equations (2.24) and (2.25) of Tanizaki–Ünsal). The Positivity of the Minimal Action in the Twisted Sector is the weakest form (positivity) of the lattice version of this lower bound. That $S_{\min}$ is close to π^2 on the lattice has not been shown as a lattice version of the continuum lower bound.

08

What the three properties of the twisted T² say

- **The Uniqueness of the Twisted Vacuum** says that the classical vacuum of the small T^2 is isolated. On a periodic T^2, the whole set of commuting tuples of holonomies (the family of flat connections) is vacuum, and fluctuations move freely over it, so semiclassical counting cannot begin. The twist shrinks it to one point (and its central images).
- **The Twisted Adjoint Gap Lemma** says that fluctuations around the isolated vacuum have the lower bound $4\sin^2(\pi/2N)$. This is the classical side of the grounds for the estimate of 25-00 § 03, "weak coupling up to the size of the T^2". The quantum side (ultraviolet control: that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions) remains an open assumption.
- **The Positivity of the Minimal Action in the Twisted Sector** says that putting in a flux n_{34} in the directions of the plane always costs action, that is, vortices cannot be placed "for free". What moves the bias θ away from 1 is the competition between the cost of vortices and the entropy of their positions, and the Positivity of the Minimal Action in the Twisted Sector is the weakest form of the lower bound on that cost.

All three are statements on a finite lattice with N fixed. None of the statements of this chapter includes increasing N (widening the T^2 as a physical length). That is the problem of the crossover band (25-09).

09

What cannot be said

(a) The continuum limit. None of the three properties says anything about $a \to 0$. The gap of the Twisted Adjoint Gap Lemma is a lower bound on classical fluctuations, not a gap of the quantum effective action.

(b) Confinement. The definition of θ and the Parity Representation of the Bias are where the tools of confinement sit, and they do not say how small θ is. Bounds with the lattice spacing fixed are in 25-04, and the semiclassical rate in 25-02.

(c) Other groups. The proofs rely on computations with the Pauli matrices of $SU(2)$ (the Rigidity Lemma for Anticommuting Pairs). The case of $SU(N)$ is not treated in this chapter.

(d) The value of the minimal action. That $S_{\min}(N, M)$ is close to π^2, and that it approaches π^2 as N grows, are numerical readings and are not shown in this chapter.

10

The table of five grades and the dependency graph

Table 1 The rows of this chapter ("—" is a row without machine checking. The conditions in every row are SU(2), the finite lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, $N, M \geq 2$)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Only the flux affects the partition function (general form); the twist relation	Cited	't Hooft 1979, Mack–Petkova 1980	—	—
2	Moving-Surface Lemma: moving the surface, determined by parity alone	Proved here	Proof in § 02	The case of the surface Π_q	—
3	Definition of the sector and flux free energies: ΔF, $\theta = \tanh(\Delta F/2)$ (4)	Definition	§ 02	$n_{12} = 1$, mixed flux 0	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
4	Tomboulis–Yaffe: a Wilson loop bounded by a ratio of twisted partition functions	Cited	Formula in 26-01	—	—
5	Parity Representation of the Bias: $\theta = \mathbb{E}_{\tilde{\mu}}[(-1)^{\lvert Q\rvert}] \Longleftarrow 2$	Proved here	Proof in § 03	The ensemble summed over surfaces (5)	—
6	Zero-Action Configuration Lemma: action $0 \Longleftrightarrow U_p = z_p \mathbf{1}$	Proved here	§ 04	—	—
7	Lemma on the Sign of the Holonomy Commutator: the commutator is the product of the signs of the plane (6)	Proved here	§ 04 (lattice non-abelian Stokes)	All plaquettes central	—
8	A closed path can be deformed into a word in straight rounds	Cited	Standard topology	Toroidal lattice	—
9	Determination-by-Holonomy Lemma: determined by holonomy $\Longleftarrow 7, 8$	Proved here	§ 04	Same z, central values	—
10	Lemma on the Configuration that Absorbs the Twist: U^0 has action 0 in the sector (1,0)	Proved here	§ 05	—	—
11	Rigidity Lemma for Anticommuting Pairs: normal form of an anticommuting pair, and commutant $\{\pm\mathbf{1}\}$	Proved here	§ 05	$SU(2)$	—
12	Uniqueness of the Twisted Vacuum: uniqueness of the vacuum that absorbs the twist $\Longleftarrow 6, 7, 9, 10, 11$	Proved here	§ 05	Up to gauge transformations and central transformations in the directions of the plane	—
13	Twisted Adjoint Gap Lemma: antiperiodicity of the adjoint field and the gap $4\sin^2(\pi/2N) \Longleftarrow$ 10	Proved here	§ 06	Two-dimensional lattice $(\mathbb{Z}/N)^2$, $U = U^0$	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
14	(8): $m_N = 2\,\mathrm{arsinh}\,\sin(\pi/2N) \Longleftarrow$ 13	Proved here	§ 06	Algebraic restatement	—
15	Positivity of the Minimal Action in the Twisted Sector: no flat configuration in (1,1), and $0 < S_{\min} \leq 2N^2 \Longleftarrow$ 6, 7, 11	Proved here	§ 07	Sector (1,1)	—
16	The continuum lower bound $8\pi^2\lvert Q\rvert/g^2 \geq 4\pi^2/g^2$	Cited	't Hooft 1981, Tanizaki–Ünsal (2.24)(2.25)	Continuum, twisted bundle	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  only the flux matters; the twist relation                  't Hooft,
 2  PRV  Moving-Surface Lemma: moving the surface                   surface Π_
 3  DEF  ΔF, θ = tanh(ΔF/2)                                         n₁₂ = 1
 4  CIT  Tomboulis–Yaffe                                            26-01
 5  PRV  Parity Representation of the Bias: θ = E[(−1)^|Q|] ⟸ 2
 6  PRV  Zero-Action Configuration Lemma: action 0 ⟺ U_p = z_p
 7  PRV  Sign of the Holonomy Commutator: commutator = product of plane signs
 8  CIT  a closed path deforms into a word of rounds
 9  PRV  Determination-by-Holonomy Lemma ⟸ 7, 8
10  PRV  Configuration that Absorbs the Twist: U⁰ has action 0
11  PRV  Rigidity Lemma for Anticommuting Pairs: commutant ±1        SU(2)
12  PRV  Uniqueness of the Twisted Vacuum ⟸ 6, 7, 9, 10, 11
13  PRV  Twisted Adjoint Gap Lemma: gap 4sin²(π/2N) ⟸ 10             (ℤ/N)²
14  PRV  m_N = 2 arsinh sin(π/2N) ⟸ 13
15  PRV  Positivity of the Minimal Action: 0 < S_min ≤ 2N² ⟸ 6, 7, 11
16  CIT  continuum lower bound 4π²/g²                               continuum
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. This chapter has no assumption rows. There are four cited rows (only the flux matters, Tomboulis–Yaffe, the deformation of closed paths, the continuum lower bound), and the only citation used by the proofs of the three properties is the deformation of closed paths (through the Determination-by-Holonomy Lemma, for the Uniqueness of the Twisted Vacuum only). The Twisted Adjoint Gap Lemma and the Positivity of the Minimal Action in the Twisted Sector close without citations.

The definition of twisted boundary conditions as an object of this book is in 19-02.

What this chapter can say

What is said	What is not said
Proved here The Moving-Surface Lemma, the representation of the bias as an expectation (the Parity Representation of the Bias), uniqueness of the vacuum that absorbs the twist (up to gauge and the centre in the directions of the plane), the gap $4\sin^2(\pi/2N)$ of the adjoint Laplacian, $0 < S_{\min} \leq 2N^2$ in the (1,1) sector. All for SU(2), a finite lattice, N fixed, $n_{12} = 1$	A gap of quantum fluctuations. Bounds uniform in the lattice spacing. The value of $S_{\min}$ and its dependence on N
Cited The general form of "only the flux matters", the Tomboulis–Yaffe inequality, the lower bound on the action in the continuum theory	$\theta \geq 0$ (treated in 26-04). Other groups

Sources and reproduction

Item	Kind	Source / tools
Twisted boundary conditions, the flux free energy	Cited	G. 't Hooft, Nucl. Phys. B 153 (1979) 141. G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data as in 3-05)
The representation summing the $\mathbb{Z}_2$ signs of plaquettes	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) (pages not checked). The form of the formula is in 26-01
The fractional part of the topological charge, and the lower bound on the action	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, equations (2.24) and (2.25) (checked in the text)
The lemmas and the three properties of § 02– § 07, (8)	Proved here	Proofs in the text. No numerical computation is used

Words used in this chapter — 't Hooft's twist, sector, flux $n_{\mu\nu}$, the surface Π_q, the flux free energy ΔF, the bias θ, the twist-summed ensemble, the configuration that absorbs the twist U^0, central transformation, the adjoint Laplacian, antiperiodic, the minimal action $S_{\min}$. Their meanings are in A-02 Glossary.

Next chapter to read: 25-02 Vortices and fractional instantons — what lies inside the minimal action of the $(1,1)$ sector, and the counting of the dilute gas.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space fractional instantons, the dilute gas, the lattice action · Kiichi, Shiori and Rin

Vortices and fractional instantons

This chapter organises established knowledge. § 02 is a map of open questions; § 05 and § 06 are this book's own investigation.

What this chapter shows — The Positivity of the Minimal Action in the Twisted Sector in the previous chapter said that putting in one unit of flux n_{34} in the directions of the plane always costs action. This chapter looks at what that minimal cost consists of. (1) In the continuum theory, the cheapest object carrying the flux on the twisted $T^2 \times \mathbb{R}^2$ is a self-dual lump of topological charge $\frac{1}{2}$ (a fractional instanton), and the lower bound on its action is $4\pi^2/g^2 = \pi^2\beta$. (2) Counting the dilute gas in which such lumps are scattered like points in the plane gives a string tension of order $\Lambda^2(\Lambda L)^{5/3}$ (a physical estimate). The $5/3 = 11/3 - 2$ in the exponent is the difference between the coefficient of asymptotic freedom and the two dimensions of the plane. (3) On the lattice, the minimal action of the sector with flux approaches π^2 from values smaller than π^2 (a numerical reading, and certified upper bounds). (4) We place, as an assumption row, a strand that derives the existence of the lump in the continuum theory by assuming a certain property.

Prerequisite chapters — 25-01 (sectors, θ, the three properties of the twisted T^2), 3-05 (instantons and topological charge, the BPS bound), 17-02 the model of a vortex gas, 4-06 the one-loop coefficient, 4-07 (Λ).

Said first — The counting of the dilute gas is a physical estimate in the dilute region ($\Lambda L \ll 1$), at the one-loop level, for the group $SU(2)$; it is not a theorem. The values of the action obtained by cooling on the lattice are the actions of configurations that were found, so they are **upper bounds** on the minimal action, not lower bounds. The existence of a self-dual solution in the continuum theory is placed in this chapter only as a strand with an assumption.

The order of this chapter

1. The self-dual lump of topological charge 1/2 — the lower bound on the action
2. Does the lump exist — numerics, and a strand with an assumption
3. The dilute gas — counting the string tension (equations (5)–(8))

4. How to read 5/3 = 11/3 − 2
5. The action on the lattice — approaching $\pi^2\beta$ from below (Tables 1 and 2)
6. A statement about the rate on the lattice — from the rate $2N^2$ of the thin surface to the rate of the fractional instanton
7. What cannot be said
8. The table of five grades and the dependency graph (Table 3)

01

The self-dual lump of topological charge 1/2 — the lower bound on the action

The topological charge Q of the continuum theory and the lower bound on the action of an instanton are as in 3-05: the Euclidean action satisfies

$$S_E \;\geq\; \frac{8\pi^2}{g^2}\,|Q| \qquad (1)$$

and equality is attained by solutions of the (anti-)self-duality equation $F = \pm *F$ Cited. On a periodic four-dimensional torus or on $\mathbb{R}^4$, Q is an integer. On a bundle with 't Hooft's twist, however, the fractional part of Q is fixed by the twist: for $SU(N)$ with $n_{12} = n_{34} = 1$, $Q \equiv -1/N \pmod 1$ (when this chapter writes $SU(N)$, the N is the degree of the group, different from the lattice N of this part) Cited (equation (2.24) of Tanizaki–Ünsal; the original source is 't Hooft 1981). For $SU(2)$, $Q \in \frac{1}{2} + \mathbb{Z}$, and from (1)

$$S_E \;\geq\; \frac{4\pi^2}{g^2} = \pi^2\beta \qquad (\beta = 4/g^2) \qquad (2)$$

(ibid., equation (2.25): the lower bound is $1/N$ of the ordinary instanton action $8\pi^2/g^2$) Cited.

A solution attaining this lower bound is a fractional instanton. Tanizaki–Ünsal state that on $\mathbb{R}^2 \times T^2$ (with the twist on T^2) the action density of this solution is localised in the directions of $\mathbb{R}^2$ and takes a vortex-like shape, and that when it lies inside a Wilson loop in the plane it multiplies the loop by the central element -1 ($e^{2\pi i/N}$ for $SU(N)$), that is, it behaves as a centre vortex (17-02) Cited (around equation (2.26)). In the words of the previous chapter, this lump is the cheapest inhabitant of the sector $(1,1)$, carrying the surface Π_q not as a thin surface but as a smooth lump.

A word on size. The action of the lump is fixed by (2) and does not depend on size, but on the twisted T^2 the lump is fixed to an extent of order the size L of the T^2 (a footnote

of the same paper cites earlier numerics in which, for $SU(N)$, the radius of the lump is proportional to the square root of the degree of the group) Cited . Since the size is fixed, no infrared divergence arises in the counting of the dilute gas. This contrasts with the integral over instanton sizes on a periodic $\mathbb{R}^4$, which diverges in the infrared (3-05).

02

Does the lump exist — numerics, and a strand with an assumption

The state of the literature Cited . Tanizaki–Ünsal write that almost no analytic forms of self-dual solutions with fractional topological charge are known (except for abelian constant solutions on tori of particular aspect ratios), and cite reports that numerical searches by lattice cooling have found configurations attaining the lower bound (2) within the discretisation errors of the lattice, and that the limit in which the directions of $\mathbb{R}^2$ are stretched to infinity is also well defined (ibid. § 2.3.1). Analytic existence is not shown in the literature within the range this book checked.

A strand with an assumption Assumption (strand: the "Route to Existence by the Nahm Transform" in the record of this book's study; evidence in the Nahm-transform row of Table 3). We assume the following property.

> **Assumption (equivariance of the Nahm transform).** The Nahm transform between doubly periodic instantons (charge-1 solutions on $T^2 \times \mathbb{R}^2$) and Hitchin-type data on the dual torus is equivariant under translations of physical space and central gauge transformations: translations of physical space correspond to multiplication by flat line bundles on the dual side, and the central twist corresponds to translation by a half period of the dual torus.

The strand has three steps. (i) Map a charge-1 $SU(2)$ doubly periodic instanton on $T^2_{2L\times L} \times \mathbb{R}^2$ by the Nahm transform to data on the dual side (the construction of doubly periodic instantons and the bijectivity of the transform are said to be in the literature; not yet checked). (ii) The composite τ of translation by a half period in the long direction and a central gauge transformation corresponds, by the assumption, to the composite of translation by a half period and a constant shift on the dual side, and if the singular points are placed at a quarter period, data invariant under τ can be chosen. (iii) The corresponding solution is invariant under τ up to gauge, and when brought down to the quotient $T^2_{L\times L} \times \mathbb{R}^2$, since τ contains the central twist, it becomes an anti-self-dual solution of charge $\frac{1}{2}$ on the bundle with $n_{12} = 1$, whose action equals the lower bound (2).

If this strand closes, the lower bound (2) is attained in the continuum theory. What is not closed is the assumption itself (the exact form of equivariance), the check of the literature in (i), and the confirmation that the topology of the bundle (n_{12} and the flux at infinity in the plane) comes out correctly on passing to the quotient. As another strand there is a variational strand combining the Bogomolny bound, exclusion of bubbles by Uhlenbeck compactness, and exclusion of splitting by concentration compactness, but that the minimal value equals the lower bound must itself be shown separately, so it too is Assumption . This book uses neither strand as a premise of the chapters that follow.

03

The dilute gas — counting the string tension

This section summarises existing knowledge (the dilute-gas count). The introduction is in 3-05 § 07.

We assume that the lumps are scattered in the plane far apart from one another (dilute), and count ignoring the interactions between lumps. This is the dilute-gas approximation, and the counting from here on transcribes § 2.3.2 of Tanizaki–Ünsal to $SU(2)$ Cited (a copy of the formulae). The correctness of the approximation itself is a physical estimate Reading .

Note: the original source places the small T^2 in directions 3 and 4 and the plane in directions 1 and 2, so the indices are swapped relative to this book (T^2 in 1 and 2, the plane in 3 and 4). Below we write with this book's indices.

The weight of one lump. The weight of one lump (self-dual, $Q = +\frac{1}{2}$) and of one antilump ($Q = -\frac{1}{2}$) in the plane, with the θ angle set to 0, is

$$K\, e^{-S_I/2}, \qquad S_I = \frac{8\pi^2}{g^2}, \qquad \frac{S_I}{2} = \frac{4\pi^2}{g^2} = \pi^2\beta \tag{3}$$

($N = 2$ in equation (2.27) of the original source). The prefactor K has mass dimension 2 and $K \sim 1/L^2$; its exact value is fixed by an analysis of the fluctuation determinant, which the original source does not give Cited .

Rewriting with asymptotic freedom. Writing the coupling at scale L in terms of Λ with the one-loop running coupling (4-06), for $SU(2)$ we have $8\pi^2/g^2(L) = \frac{22}{3}\ln\frac{1}{\Lambda L}$ (at the one-loop level; for general $SU(N)$, $\frac{11N}{3}$), so

$$e^{-S_I/2} = e^{-4\pi^2/g^2(L)} = (\Lambda L)^{11/3} \quad \text{(one loop)}, \qquad K\, e^{-S_I/2} \sim \frac{1}{L^2}(\Lambda L)^{11/3} = \Lambda^2(\Lambda L)$$

(the last step of equation (2.32) of the original source) Cited . The exponent $11/3$ does not depend on the degree of the group (for $SU(N)$ the action of the lump is S_I/N, and it cancels against the N of the coefficient $11N/3$).

The Wilson loop. If one lump lies inside a loop C of area A, the value of the loop is multiplied by -1 (a centre vortex). If lumps and antilumps are each scattered in the plane as independent Poisson points of density $Ke^{-\pi^2\beta}$, the number n inside follows a Poisson distribution with mean ρA ($\rho = 2Ke^{-\pi^2\beta}$), and

$$\langle W(C)\rangle = \mathbb{E}\big[(-1)^n\big] = e^{-2\rho A} = \exp\!\big(-4Ke^{-\pi^2\beta}\,A\big) \quad (5)$$

This is a one-line computation, $\mathbb{E}[(-1)^n] = e^{-\lambda}\sum_k (-\lambda)^k/k! = e^{-2\lambda}$ for the Poisson distribution Proved here (a computation within the model). The original source obtains the same answer by putting in the constraint on the total charge (the difference between the numbers of lumps and antilumps is a multiple of N) through a Fourier sum: at $\theta = 0$ the string tension of the fundamental representation is

$$T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2\frac{\pi|R|}{N} \quad (6)$$

(equation (2.37) of the original source; N is the degree of the group, and $|R|$ is the number of boxes of the Young diagram of the representation modulo N), and for the fundamental representation of $SU(2)$, $\sin^2(\pi/2) = 1$ Cited . Combining (5) and (4),

$$\sigma = 4K\,e^{-\pi^2\beta} \;\sim\; \Lambda^2(\Lambda L)^{5/3} \qquad \text{(dilute region, one loop, } SU(2)) \quad (7)$$

Reading (a physical estimate; $\sim$ is proportionality with the constant of the prefactor omitted).

Relation to the bias θ. Applying the same counting to the parity of the number of lumps in the whole plane (area $A_{\text{plane}} = (Ma)^2$), the $\theta = \mathbb{E}[(-1)^{|Q|}]$ of the Parity Representation of the Bias of 25-01 becomes

$$\theta \approx e^{-2\rho A_{\text{plane}}}, \qquad -\ln\theta \approx 4Ke^{-\pi^2\beta}\,(Ma)^2 \quad (8)$$

and one reads both that $-\ln\theta$ is proportional to the area of the plane (the type of an area law) and that its rate is fixed by the weight $e^{-\pi^2\beta}$ of the fractional instanton Reading (the dilute-gas estimate). The statement, on the lattice, of the part of this form saying "the rate is $e^{-\beta S}$ with S close to π^2" is § 06.

Numerics on the lattice also support the same picture. Bergner–González-Arroyo–Soler (2025) report, for the $SU(2)$ lattice on a four-dimensional torus with two directions far smaller than the others and twisted boundary conditions, that at small sizes of the T^2 the topological charge and the string tension arise from a gas of vortex-like fractional

instantons, and that as the size increases the density increases and the spacing of the structures approaches a constant Cited (checked in the abstract; the numbers in the text not yet checked).

04

How to read 5/3 = 11/3 − 2

The exponent of (4) is the difference of two contributions Reading (a physical estimate).

- $11/3$**: asymptotic freedom.** The smaller the T^2, the weaker the coupling, and lumps become rare at the rate $(\Lambda L)^{11/3}$. The coefficient $11/3$ is exactly the one-loop coefficient ($11/3 = -1/3 + 4$ of 4-06).
- -2**: the two dimensions of the plane.** The size of a lump is L, so the number of places where one can be put per unit area of the plane is $1/L^2$ ($K \sim 1/L^2$). This corresponds to the entropy of the lump positions.

Since the difference $11/3 - 2 = 5/3$ is positive, if one holds Λ fixed and widens the T^2 (increases L), the string tension in physical units increases. It is the direction in which confinement strengthens as one widens, and reaching order Λ^2 at $L \sim 1/\Lambda$ is consistent with dimensional counting. Counting per square of the size of one T^2 (area L^2) gives $\sigma L^2 \sim (\Lambda L)^{11/3}$, which can be restated as: the probability that a lump is in that square increases with L.

The abelian group $U(1)$ has no localised lumps carrying fractional flux. The flux can only spread uniformly over the T^2, and the behaviour in the direction of widening the T^2 is reversed. This difference is treated as "the only place where non-abelianness enters" in 26-03, and there too this counting is an estimate at the one-loop level. Within the scope of this chapter, (4)–(8) are all statements in the dilute region $\Lambda L \ll 1$, and they do not extend into the band $L \sim 1/\Lambda$.

05

The action on the lattice — approaching π²β from below

Matching units. The action $S^z(U) = \sum_p (1 - \frac{1}{2} z_p \operatorname{tr} U_p)$ of 25-01 is the quantity before multiplying by β. In the classical continuum limit of 5-01, for $SU(2)$ and $\beta = 4/g^2$, βS^z approaches the continuum action S_E, so a lump of action $4\pi^2/g^2$ corresponds to $S^z \approx \pi^2$ (that is, $\beta S^z \approx \pi^2 \beta$). All the numbers below are values of S^z divided by π^2.

Cooling numerics Reading . Table 1 gives the result of searching for the minimal action $S_{\min}(N, M)$ of the sector $(1, 1)$ by repeatedly lowering S^z (cooling) through the

operation of replacing each link by the direction of the sum of the remaining sides of the plaquettes containing it (the staple). The action of the configuration at which cooling stopped is an **upper bound** on the minimal action, since that configuration exists (within floating-point computation). The possibility that the true minimum is lower still is set aside only as a reading, on the grounds that different initial configurations arrived at the same value.

Table 1 Cooling values of the minimal action in the sector $(1,1)$ ($S_{\min}/\pi^2$. SU(2), Wilson action, $n_{12}=n_{34}=1$, 3 runs from random initial configurations)

Lattice $T^2_N \times T^2_M$	$N \times M$	$S_{\min}$	$S_{\min}/\pi^2$
Symmetric	2×2	7.841	0.794
Symmetric	3×3	9.661	0.979
Symmetric	4×4	9.799	0.993
Symmetric	6×6	–	0.998
Elongated	2×4	–	0.683
Elongated	2×6	–	0.677
Elongated	3×6	–	0.921
Elongated	4×8	–	0.960

How to read it Reading : (i) every value is smaller than π^2 and approaches π^2 as N grows (the lattice values approach the continuum value from below; a physical reading consistent with the discretisation correction of the Wilson action being negative). (ii) On elongated lattices ($M>N$), at $N=2$ the value hardly changes when the plane is widened, $M=4\to 6$. The lump is localised in the directions of the plane and its action does not depend on the size of the plane; that is, the lattice supports the premise of the dilute gas (that a lump is a single localised object). (iii) That the values on symmetric lattices are larger than on elongated ones can be read as the lump being squeezed to order N in the directions of the plane as well on a symmetric lattice. In a separate cooling that measured the distribution of the action on a slab of size 16 in the directions of the plane ($4\times 4\times 16\times 16$), $S^z/\pi^2\approx 0.96$, about 91% of the action lay within radius 2 of the centre of the lump in the directions of the plane, and about 99.6% within radius 4 ($=N$) Reading .

A lump confined to a column, and certified upper bounds Computer-assisted . The paper on lattice fractional instantons (one of the papers that came out of this book, a draft; A-07) divides the plane into square blocks of side ℓ, puts flux into the column $T^2_N\times[0,\ell]^2$ above one block, and defines the minimal action when the outer frame of the column is fixed to the vacuum that absorbs the twist U^0 (25-01 § 05) as

$$S_*(N,\ell)=\min\{\ \text{action of the column with frame fixed to } U^0\ :\ \text{one flux } n_{34} \text{ in the column}\ \}$$

The values obtained by normalising the saved configurations exactly, checking that the outer frame agrees with the vacuum, and re-evaluating the action in 40-digit arithmetic are upper bounds on each S_*:

$$S_*(2,12) \le 6.67348665542 \le 0.6762\,\pi^2, \quad S_*(3,12) \le 9.07284242009 \le 0.9193\,\pi^2, \quad S_*(4,8)$$

($N=4$ is a value at which cooling had not converged). Conditions: the group $SU(2)$, the Wilson action, an evaluation of the action of concrete configurations, and only in the direction of upper bounds. The label "Computer-assisted" attaches only to these upper bounds (actions of concrete configurations evaluated with guaranteed digits), not to the minimal values themselves. All are smaller than π^2, so "the minimal cost on the lattice is smaller than the continuum value π^2" can be said directly from the upper bounds for $N=2,3,4$.

Dependence on the width of the column Reading . A table from the same paper (cooling values):

Table 2 Dependence of $S_*(N,\ell)/\pi^2$ on ℓ (cooling. SU(2), outer frame fixed to the vacuum)

ℓ	3	4	5	6	7	8	10	12
$N=2$	0.7082	0.6924	0.6779	0.6771	0.6763	0.6762	0.67617	0.67617
$N=3$	1.1524	1.0185	0.9493	0.9347	0.9230	0.9213	0.9195	0.9193

Elementary properties of S_* Proved here : (a) the minimum is attained (compactness). (b) $S_*(N,\ell) \le 2N^2$ (place one thin surface in the column). (c) It does not increase in ℓ (embed the minimal configuration of a smaller column into a larger column, filling the rest with vacuum; the plaquettes on the border are vacuum against vacuum, with action 0). (d) Relation to the minimal action on a periodic lattice: $S_*(N,\ell) \ge S_{\min}(N, k\ell)$ ($k \ge 1$; extend the configuration of the column to the periodic lattice, filling the outside with vacuum). Hence $\lim_\ell S_*(N,\ell)$ exists.

The values on periodic lattices (cooling) are $S_{\min}(2,M) = 6.740117,\ 6.677345,\ 6.673724,\ 6.673486,\ 6.673485$ ($M=4,6,8,12,16$) and $S_{\min}(3,16) = 9.072494$, and $S_*(N,\ell)$ approaches them from above as ℓ grows. The difference $S_*(N,\ell) - S_{\min}(N,\infty)$ is read as decreasing like $e^{-m_N \ell}/\ell$ in terms of the twist mass $m_N = 2\,\mathrm{arsinh}\,\sin(\pi/2N)$ of (8) of 25-01 ($m_2 \approx 1.317$, $m_3 \approx 0.962$) Reading (the draft of the paper on lattice fractional instantons sets the observed slopes 1.41, 1.02 beside the predicted values 1.45, 1.06; how the predicted values are obtained has not been checked in this book). The reading is that the tail of the lump is cut off by the mass of the adjoint fluctuations on the twisted T^2.

In summary, the minimal cost on the lattice is roughly $0.68\pi^2,\ 0.92\pi^2,\ 0.96\pi^2$ for $N=2,3,4$ (values on periodic lattices; $N=4$ is cooling on the elongated lattice 4×8), and

approaches π^2 from below as N grows. "$S_{\min}(N,\infty) \to \pi^2$ as $N \to \infty$" is numerical only, with no proof Reading .

06

A statement about the rate on the lattice — from the rate 2N² of the thin surface to the rate of the fractional instanton

With N and the lattice spacing fixed, a bound of area-law type $-\ln\theta \geq c\,M^2$ follows elementarily at every β (25-04). The rate c there is fixed by the cost of the thin surface, $c \approx e^{-2\beta N^2}$ (the configuration of the upper bound $2N^2$ in the Positivity of the Minimal Action in the Twisted Sector of 25-01). $2N^2$ grows with N and has nothing to do with fractional instantons.

The main theorem of the paper on lattice fractional instantons (the Confinement Lower Bound by Fractional Instantons, Theorem A of that paper) is a statement that replaces this rate by $S_*(N,\ell)$. We transcribe the form of the draft with its conditions Assumption (this book has not yet placed the proof in the text; the evidence column is § 3– § 6 of the draft of the paper on lattice fractional instantons. The proof uses the Chessboard Estimate and a Morse–Bott lemma by citation, and only the existence of the rigidity constants is shown, with their values not made explicit).

Confinement Lower Bound by Fractional Instantons (the statement of the draft). For all $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, there exist $c(N,\ell) > 0$, $a > 0$ and $\varepsilon(\beta) = O\big(\sqrt{\ln\beta/\beta}\big)$ such that

$$-\ln\theta(N,M,\beta) \;\geq\; c(N,\ell)\,\beta^{-a}\,e^{-\beta\,(S_*(N,\ell)+\varepsilon(\beta))}\,M^2. \quad (11)$$

Combined with the upper bounds of (10), the rate drops from $2N^2 = 8, 18, 32$ to about $\leq 0.68\pi^2$, $0.92\pi^2$, $0.98\pi^2$ ($N = 2, 3, 4$). However, the terms of the prefactor β^{-a} and $\varepsilon(\beta)$ are crude, so (11) beats the thin-surface bound only from some large, non-explicit β onwards. (11) is not uniform in N, and says nothing about the continuum limit or the mass gap. The bound in the reverse direction (an upper bound from the dilute gas, uniform in M) is open. Where these stand is treated in 27-01 and 27-02.

Setting the estimate (8) beside (11), the position is that, of the part of the estimate saying "the rate is the action of the fractional instanton", only the direction of confinement (the bound from below), on a lattice with fixed N, has the form of a statement.

07

What cannot be said

(a) The dilute gas. (5)–(8) are counting that ignores interactions, a physical estimate in the dilute region $\Lambda L \ll 1$ at the one-loop level. The value of the prefactor K is not known.

(b) Cooling values. The values of Tables 1 and 2 are upper bounds on the minimal action, not lower bounds. Of "approaches π^2 from below", "below π^2" can be said for $N = 2, 3, 4$ from the upper bounds of (10), and "approaches" is a numerical reading.

(c) Continuum existence. The existence of a solution attaining the lower bound (2) on the twisted $T^2 \times \mathbb{R}^2$ is a strand with an assumption in this book, and is not shown.

(d) The band. Every statement of this chapter is a statement with $\Lambda L \ll 1$ or with N fixed, and says nothing about widening the T^2 beyond $1/\Lambda$ (25-09).

(e) The paper on lattice fractional instantons. The Confinement Lower Bound by Fractional Instantons is a statement of a draft, and this book's label is "Assumption". It is not raised until the proof of the draft has been checked line by line and placed in the text.

08

The table of five grades and the dependency graph

Table 3 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The BPS bound (1)	Cited	3-05	Continuum	—
2	On the twisted bundle $Q \in \frac{1}{2} + \mathbb{Z}$, lower bound $4\pi^2/g^2 = \pi^2\beta$ (2) $\Longleftarrow 1$	Cited	Tanizaki–Ünsal (2.24) (2.25), 't Hooft 1981	Continuum, SU(2), $n_{12} = n_{34} = 1$	—
3	The lump is localised in the plane and acts as a centre vortex; size of order L	Cited	Tanizaki–Ünsal § 2.3.1	Numerics and statements of the original source	—
4	Analytic existence is not known (as stated in the original source)	Cited	Tanizaki–Ünsal § 2.3.1	—	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
5	A strand of existence under the equivariance of the Nahm transform	Assumption	Strand: the "Route to Existence by the Nahm Transform" in the record of the study, § 02	One assumption, literature not yet checked	—
6	Weight of one lump $Ke^{-\pi^2\beta}$, $K \sim 1/L^2$ (3)	Cited	Tanizaki–Ünsal (2.27)	Dilute-gas model	—
7	$Ke^{-\pi^2\beta} \sim \Lambda^2(\Lambda L)^{5/3}$ (4)	Cited	Tanizaki–Ünsal (2.32)	One loop	—
8	In the Poisson model $\langle W\rangle = e^{-2\rho A}$ (5)	Proved here	One line in § 03	Computation within the model	—
9	$T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2(\pi\lvert R\rvert/N)$ (6)	Cited	Tanizaki–Ünsal (2.37)	$\theta = 0$, dilute region	—
10	$\sigma = 4Ke^{-\pi^2\beta} \sim \Lambda^2(\Lambda L)^{5/3} \Longleftarrow$ 6, 7, 8, 9	Reading	Physical estimate	Dilute region, one loop, SU(2)	—
11	$-\ln\theta \approx 4Ke^{-\pi^2\beta}(Ma)^2$ (8) $\Longleftarrow$ 8, 10	Reading	Physical estimate	Same as above	—
12	Numerics of the lattice gas (Bergner–González-Arroyo–Soler 2025)	Cited	Abstract	SU(2), twisted small T^2	—
13	Reading of $5/3 = 11/3 - 2 \Longleftarrow$ 10	Reading	§ 04	One loop, dilute region	—
14	Cooling values (Table 1), fraction of localisation	Reading	Numerical record of this book's study	Upper bounds, 3 initial configurations	—
15	Properties (a)–(d) of S_*	Proved here	§ 05	SU(2), lattice	—
16	Certified upper bounds (10)	Computer-assisted	Draft of the paper on lattice fractional instantons § 7 (40-digit re-evaluation)	Upper direction only; $N = 4$ not converged	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
17	Dependence on ℓ in Table 2, decay $e^{-mN\ell/\ell}$	Reading	Same, § 7	Cooling	—
18	Approaches π^2 from below ⟸ 14, 15, 16, 17	Reading	§ 05	"Below" for $N = 2, 3, 4$ from row 16	—
19	Confinement Lower Bound by Fractional Instantons (11)	Assumption	Draft of the paper on lattice fractional instantons § 3– § 6	N fixed, $\beta \geq 1$, not uniform in N, no continuum limit	—

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  S ≥ 8π²|Q|/g²                                                  continuum
 2  CIT  twist gives Q ∈ 1/2 + ℤ, bound π²β ⟸ 1                           SU(2), n
 3  CIT  lump localised in plane, centre vortex, size ~ L
 4  CIT  analytic existence not known
 5  ASM  strand of existence under the equivariance assumption            literatur
 6  CIT  weight of one lump K e^(−π²β)
 7  CIT  K e^(−π²β) ~ Λ²(ΛL)^(5/3)                                       one loop
 8  PRV  Poisson model ⟨W⟩ = e^(−2ρA)
 9  CIT  T_R ~ Λ²(ΛL)^(5/3) sin²(π|R|/N)                                 θ = 0
10  RDG  σ = 4K e^(−π²β) ~ Λ²(ΛL)^(5/3) ⟸ 6, 7, 8, 9                     dilute, or
11  RDG  −ln θ ≈ 4K e^(−π²β)(Ma)² ⟸ 8, 10
12  CIT  numerics of the lattice gas (abstract)
13  RDG  reading of 5/3 = 11/3 − 2 ⟸ 10
14  RDG  cooling values, fraction of localisation                        upper boun
15  PRV  properties (a)-(d) of S_*
16  CMP  S_* ≤ 0.6762π², 0.9193π², 0.9707π²                              upper; N =
17  RDG  ℓ dependence, e^(−m_N ℓ)/ℓ
18  RDG  approaches π² from below ⟸ 14, 15, 16, 17
19  ASM  Confinement Lower Bound by Fractional Instantons (Theorem A of the la
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The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only nodes with proofs placed in the text are the one line of the Poisson model and the properties of S_*, both one-line arguments within a model or on a finite lattice. The "Assumption" of the Route to Existence by the Nahm Transform and of the Confinement Lower Bound by Fractional Instantons enters as a premise of no node. The nodes on the conclusion side of this chapter (the string tension, the reading of θ, the reading of 5/3, approaching π^2 from below) are all "Reading", and there is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Cited The lower bound $4\pi^2/g^2 = \pi^2\beta$ on the action of the twisted bundle, equations (2.27)(2.32)(2.37) of the original source for the dilute gas, the report of numerics of the lattice gas (abstract)	Existence of a continuum solution attaining (2)
Reading The string tension $\Lambda^2(\Lambda L)^{5/3}$ (an estimate in the dilute region, at one loop, for SU(2)), the reading of $5/3 = 11/3 - 2$, that the lattice action approaches π^2	The correctness of the dilute-gas approximation. The value of K. The limit $N \to \infty$
Computer-assisted $S_* \leq 0.6762\pi^2$, $0.9193\pi^2$, $0.9707\pi^2$ ($N = 2, 3, 4$, upper bounds from concrete configurations)	Lower bounds on, and the agreement of, S_* and $S_{\min}$
Proved here The one line of the Poisson model, the elementary properties of S_*	—
Assumption The strand of existence (assumption of equivariance), the Confinement Lower Bound by Fractional Instantons (draft)	Neither has a proof in the text

Sources and reproduction

Item	Kind	Source / tools
(2)(3)(4)(6), localisation of the lump, the state of analytic solutions	Cited	Y. Tanizaki, M. Ünsal, Center vortex and confinement in Yang–Mills theory and QCD with anomaly-preserving compactifications, arXiv:2201.06166 (v2, 2022). § 2.2– § 2.3 and equations (2.24)–(2.37) checked in the text
The gas of fractional instantons on the lattice	Cited	G. Bergner, A. González-Arroyo, I. Soler, arXiv:2502.09463 (2025, a Lattice 2024 report). Abstract checked
Topological charge of the twisted bundle	Cited	G. 't Hooft, Commun. Math. Phys. 81 (1981) 267 (bibliographic data in 3-05)
The strand of existence (assumption of equivariance)	Assumption	The record of this book's study (Route to Existence by the Nahm Transform). The literature cited there on the construction of doubly periodic instantons and the Nahm transform is not yet checked
Table 1, fraction of localisation	Reading	The numerical record of this book's work (cooling, 3 runs from random initial configurations)
(9)(10), Table 2, values on periodic lattices, slopes of decay, the Confinement Lower Bound by Fractional Instantons	Computer-assisted, Reading, Assumption	The paper on lattice fractional instantons (lattice confinement by fractional instantons, draft of 2026-10-14). § 2, § 7, § 8. The title and status are in A-07

Item	Kind	Source / tools
(5), properties of S_*, numerical values of m_N	Proved here	The text. m_N is the value of (8) of 25-01 with $N = 2$, 3

Words used in this chapter — fractional instanton, topological charge $\frac{1}{2}$, the BPS bound, centre vortex, dilute gas, the prefactor K, string tension, $5/3 = 11/3 - 2$, cooling, a lump confined to a column $S_*(N, \ell)$, a certified upper bound, the twist mass m_N. Their meanings are in A-02 Glossary.

Next chapter to read: 25-03 Containers — conditioning on the frame makes things independent — divide the plane into blocks, and bound the bias by a product of quantities per column.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the container identity, nesting, limits in four dimensions · Kiichi, Shiori and Rin

Containers — conditioning on the frame makes things independent

> This chapter organises established knowledge. § 04, § 05 and § 07 are this book's own investigation.

What this chapter shows — The most naive tool for bounding a Wilson loop in the direction of confinement is to divide the surface spanned by the loop into small regions (containers) and to consider things with the outside of the containers (the frame) held fixed. With a nearest-neighbour action, fixing the frame makes the containers independent of each other, and the expectation of the Wilson loop is bounded from above by the average of the product of "the biases of the sectors of each container" (the Container Identity). This is the vortex-container identity of Mack–Petkova (1980), rewritten in the sector-summed ensemble, and this book claims nothing new about it. In this chapter we look in turn at (1) the Container Identity and its tanh form, (2) the form of "inner and outer regions" needed to pass to a fixed sector, (3) an inequality nesting containers in the directions of the plane (the directions of R^2), (4) the point made by Kovács–Tomboulis that in four dimensions, placing containers of fixed length independently gives only up to $V(R) \gtrsim R/\ln R$, and (5) that in a solvable two-dimensional model the bound of the Container Identity is sharp in its exponent.

Prerequisite chapters — 25-01 (the twisted T^2 and 't Hooft flux, sectors), 25-02 (vortices). From Part 5, 5-01 (the Wilson action and Wilson loops) and 5-08 (string tension); from Part 17, 17-02 (the picture of centre vortices).

Said first — The group is $SU(2)$, the action is the Wilson action, the lattice is finite and the lattice spacing is fixed. All statements of this chapter are statements "on the lattice, at some fixed β", and say nothing about the continuum limit. The Container Identity is known (Cited). What this book places on paper are three things: the rewriting in the sector-summed ensemble, the form of the correction when passing to a fixed sector, and the nesting inequality (Proved here). What the Lean column guarantees is only the algebra of finite sums; the Haar measure and the topology of sectors rest on the argument in the text.

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The order of this chapter
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1. Summary — fixing the frame makes things independent
2. The setting — the sector-summed ensemble, frame and containers
3. The Container Identity and its tanh form — correspondence with the identity of Mack–Petkova
4. To a fixed sector — inner and outer containers
5. The Nesting Inequality in the Plane Directions
6. Limits in four dimensions — $V(R) \gtrsim R/\ln R$ (equations (23)–(25) of Kovács–Tomboulis)
7. A test in a solvable two-dimensional model — Table 1
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — fixing the frame makes things independent

This section is a summary for [Y3] readers. The formulae and proofs are from § 02 on.

1. The Wilson action on the lattice is an action that couples only neighbouring links (nearest-neighbour). So the links inside a region and the links outside it talk only through the links on the border. If all of the border is held fixed, the insides of separate regions become independent of each other. This is exactly the Markov property of statistical mechanics.
2. The Wilson loop $W(C)$ can be rewritten as a product of "signs of the sector" $\sigma_j = \pm 1$ per small region (container) on the surface spanned by the loop (in the sector-summed ensemble, § 02). If the frame (the outside of the containers) is held fixed, the σ_j are independent of each other, so the expectation becomes a product, and taking absolute values and averaging gives $|\langle W \rangle| \le \mathbb{E}[\prod_j \theta_j]$, with $\theta_j = |\mathbb{E}[\sigma_j \,|\, \text{frame}]|$. This is the Container Identity Cited .
3. θ_j can be written $\tanh(|\Delta F_j|/2)$. ΔF_j is the difference in free energy, inside a container with the frame fixed, between the sector with one vortex and the sector without. The more cheaply a vortex enters a container (the smaller ΔF_j), the smaller θ_j, and the smaller the bound on the Wilson loop.
4. This identity is the "vortex container" identity of Mack–Petkova (1980), written as it stands in equations (18)–(22) of Kovács–Tomboulis (2001). The only differences are that we wrote it with the sectors summed, and that we took the average over frames rather than the maximum. This book claims no novelty.

5. To pass to the Wilson loop in a fixed sector (one twisted sector), containers are needed not only inside the loop but also outside it (the Inner and Outer Container Proposition of § 04) Proved here .
6. If containers are nested in the directions of the plane, the bias of a large container is bounded by the average of the product of the biases of small containers (the Nesting Inequality in the Plane Directions of § 05) Proved here . But this concerns a slab with the size of the T^2 held fixed, and cannot be used for widening together with the T^2 (the crossover band).
7. In four-dimensional space (R^4), the length of the containers through points inside the loop grows with the loop. To keep the cost of a container constant its thickness must grow to order $\sqrt{\ln R}$, and independent containers give only up to $V(R) \gtrsim R/\ln R$ (Kovács–Tomboulis) Cited . In a slab with a small T^2, the length of a container stops at the area of the T^2, so this obstruction does not appear.
8. Two-dimensional $SU(2)$ is solvable, so one can see in numbers how sharp the bound of the Container Identity is. Tiling with containers of side ℓ, the string tension coming from the bound approaches the true value in the form $\sigma - \ln(16/(3\pi))/\ell^2$ (§ 07) Proved here + Reading .

02

The setting — the sector-summed ensemble, frame and containers

The lattice Λ is a finite hypercubic lattice, and the link variables are $U = (U_b) \in SU(2)^{\text{links}}$. On each plaquette p we place a sign $z_p \in \{\pm 1\}$, and impose $\prod_{p \in \partial c} z_p = 1$ (coclosed) on every 3-dimensional cube c. We set the weight and the measure to be

$$w(U,z) = \exp\Big(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p\Big), \qquad d\mu \;\propto\; w(U,z)\prod_b dU_b \;\times\; (\text{counting of coclosed } z) \tag{1}$$

(dU_b is the Haar measure). z in the same cohomology class are related by central gauge transformations $U_b \mapsto \omega U_b$ ($\omega \in \{\pm 1\}$), so the sum (1) equals, up to a constant factor, the sum of the $SU(2)$ partition functions of all of 't Hooft's twisted sectors (25-01) (the summed ensemble of Mack–Petkova) Cited . Which sector one is in is fixed by the cohomology class of z, and the indicator function of a sector is a function of z. In this chapter we call (1) the **sector-summed ensemble**, and write its expectation $\mathbb{E}$.

Taking a loop C and one surface $S(C)$ spanning C (a collection of plaquettes whose boundary is C),

$$W(C) = \tfrac{1}{2}\operatorname{tr} U_C \prod_{p \in S(C)} z_p \tag{2}$$

is unchanged by central gauge transformations, and $|W(C)| \le 1$. The Wilson loop with the sector fixed to one is (2) multiplied by the indicator function of the sector, i.e. conditioned (§ 04).

Frame and containers. We take regions $R_1, \dots, R_n$ (containers) in the lattice, separated from each other, call the variables inside a container (the inner links and the z of the inner plaquettes) x_j, and call all other variables the **frame** Φ. The links on the border of a container belong to the frame. Since the action is nearest-neighbour, each plaquette term and each cube constraint depends only on the inside of at most one container and on the frame. Hence the weight splits when the frame is fixed.

$$w(\Phi, x) = a(\Phi) \prod_{j=1}^{n} f_j(\Phi, x_j), \qquad a > 0,\ f_j > 0 \tag{3}$$

If the loop C runs inside the frame, and we write $\sigma_j(x_j) \in \{\pm 1\}$ (the **sign of the sector** of the container) for the product of the z on the part of $S(C)$ inside the container R_j, then (2) splits as well.

$$W(C) = c(\Phi) \prod_{j=1}^{n} \sigma_j(x_j), \qquad |c(\Phi)| \le 1 \tag{4}$$

($c(\Phi)$ is $\frac{1}{2}\mathrm{tr}\, U_C$ times the product of the z on the part of $S(C)$ in the frame.) σ_j is unchanged by central gauge transformations inside the container (the boundary C of $S(C)$ lies in the frame), so it is well defined as a quantity on the conditional distribution with the frame fixed. Moreover, fixing the frame does not determine σ_j. This is because, in the small-T² slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, if a container is taken as "a disc of the plane $\times$ T²", then the surface $S_p = \{p\} \times T^2$ above a plaquette p of the plane inside the disc (N^2 plaquettes of type (3,4)) is coclosed and lies inside the container, and flipping all the z on it flips only σ_j without changing the frame Proved here (the check that it is coclosed is step 2 of 25-04 § 03).

03

The Container Identity and its tanh form

This section summarises existing knowledge (the container identity of Mack–Petkova). The introduction (independent vortex signs and the area law) is in 17-02 § 06.

With the frame Φ fixed, we write $q_j^{\pm}(\Phi) = \mathbb{P}(\sigma_j = \pm 1 \mid \Phi)$ for the probabilities of the sectors of container j, and $Z_j^{\pm}(\Phi) = \sum_{x_j : \sigma_j = \pm 1} f_j(\Phi, x_j)$ for the partition functions of the two sectors inside the container (Haar integrals for continuous variables), and set

$$\theta_j(\Phi) := \big|\mathbb{E}[\sigma_j \,|\, \Phi]\big| = |q_j^{+} - q_j^{-}|, \qquad \Delta F_j(\Phi) := \ln \frac{Z_j^{+}(\Phi)}{Z_j^{-}(\Phi)} = \ln \frac{q_j^{+}}{q_j^{-}} \tag{5}$$

We call θ_j the **bias** of container j, and ΔF_j the **vortex free energy** with the frame fixed.

Container Identity

Under (3)(4),

$$\mathbb{E}[W(C) \,|\, \Phi] = c(\Phi) \prod_{j=1}^{n} \mathbb{E}[\sigma_j \,|\, \Phi], \qquad |\langle W(C) \rangle| \;\le\; \mathbb{E}_{\Phi}\Big[\, |c(\Phi)| \prod_{j=1}^{n} \theta_j(\Phi) \Big] \;\le\; \mathbb{E}_{\Phi}\Big[\prod_{j=1}^{n} \theta$$

Moreover, $\theta_j(\Phi) = \tanh\big(|\Delta F_j(\Phi)|/2\big)$.

Cited (the identity) + Proved here (the rewriting in the summed ensemble). Proof. By (3), the conditional distribution with the frame fixed is the product of the distributions $f_j(\Phi, \cdot)/(Z_j^{+} + Z_j^{-})$ per container, and each σ_j is a function of the variables of a different container, so the conditional expectation splits into a product. Substituting (4) gives the first formula. Taking absolute values, using $|c| \le 1$ and averaging over the frame gives the second. Finally, from $q^{+} + q^{-} = 1$ and $q^{\pm} > 0$,

$$\frac{q^{+} - q^{-}}{q^{+} + q^{-}} = \frac{e^{\Delta F/2} - e^{-\Delta F/2}}{e^{\Delta F/2} + e^{-\Delta F/2}} = \tanh \frac{\Delta F}{2}, \qquad \Delta F = \ln \frac{q^{+}}{q^{-}} \tag{7}$$

∎

Lean FrameLemma (`FrameLemma.frame_bound`) (hypotheses: the space of frames, the number of containers and the space inside each container are finite, the inside of each container is non-empty, $a(\varphi) > 0$, $f_j(\varphi, y) > 0$. Conclusion: $|\sum_{\varphi} \sum_{x} a \prod_j f_j \cdot c \prod_j \sigma_j| \le \sum_{\varphi} a \prod_j Z_j \cdot |c| \prod_j |\sum_y f_j \sigma_j| / Z_j$, $Z_j = \sum_y f_j$), FrameLemma (`FrameLemma.sign_mean_le_one`) (hypotheses: $f > 0$, $\sigma = \pm 1$. Conclusion: the absolute value of the normalised conditional mean is at most 1), FrameLemma (`FrameLemma.ratio_eq_tanh`) (hypotheses: $q^{+}, q^{-} > 0$. Conclusion: (7)), Containers (`Containers.container_bound`) (hypotheses: finite, $a \ge 0$, $|s| \le 1$. Conclusion: $|\sum_{\varphi} \sum_{x} a\, s \prod_i f_i \sigma_i| \le \sum_{\varphi} a \prod_i |\sum_y f_i \sigma_i|$). All are algebra of finite sums; that (3)(4) follow from the Haar integral, a coclosed z, the topology of sectors and the nearest-neighbour action is outside Lean.

Correspondence with the original source. Kovács–Tomboulis (hep-lat/0108017, Phys. Rev. D 65 (2002) 074501), in a fixed sector, fix the links on the border of a container Λ_i and integrate the inside, and from the partition functions $z_{\Lambda_i}^{(\mp)}$ with the signs of the couplings on a coclosed surface V_i flipped (their equation (19)) write

$$f_{\Lambda_i}(U_{\partial\Lambda_i}) = \frac{1}{2}\Big(1 - \frac{z_{\Lambda_i}^{(-)}}{z_{\Lambda_i}^{(+)}}\Big) \;\; (18), \qquad W[C] = \Big\langle \tfrac{1}{2}\mathrm{tr}\, U[C] \prod_i f_{\Lambda_i}(U_{\partial\Lambda_i}) \Big\rangle \;\; (20), \qquad W[C] \le$$

and attribute (20) to Mack–Petkova (Ann. Phys. 125 (1980) 117, their reference [3]) Cited (the formulae of the original source checked in the text of hep-lat/0108017; the text of Mack–Petkova has not been checked in this book). There are only two differences from the Container Identity. (i) Since we wrote it with (1), with the sectors summed, the factor is $(Z^{+}-Z^{-})/(Z^{+}+Z^{-}) = \tanh(\Delta F/2)$ rather than $\tfrac{1}{2}(1-z^{-}/z^{+})$ (a difference in the normalisation of the measure). (ii) (22) takes the maximum over frames, while (6) takes the average. (ii) leaves room to use the rarity of bad frames. In the two-dimensional test of § 07, the averaged form approaches the true value faster.

04

To a fixed sector — inner and outer containers

The Container Identity is a statement about the sector-summed ensemble. What one wants to see in physics is the Wilson loop with the twisted sector fixed to one. In the slab $(\mathbb{Z}/N)^{2}\times(\mathbb{Z}/M)^{2}$, let the loop C lie in the (3,4) plane at a point y_{0} of the T^{2}. Writing $(-1)^{n_{34}}$ for the parity of the (3,4) flux among the labels of the sector (the product of z over the whole (3,4) plane at y_{0}), and $\mathbf{1}_{O}$ for the indicator function of the other labels (n_{12} and so on), the expectation with the sector fixed to $s\in\{\pm 1\}$ is

$$\langle W(C)\rangle_{s} = \frac{\mathbb{E}\big[W\cdot\tfrac{1}{2}\big(1+s(-1)^{n_{34}}\big)\mathbf{1}_{O}\big]}{\mathbb{E}\big[\tfrac{1}{2}\big(1+s(-1)^{n_{34}}\big)\mathbf{1}_{O}\big]} \tag{8}$$

If containers are placed only inside the loop, the second term of the numerator, $\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_{O}]$, is a problem. $W(-1)^{n_{34}}$ is $\tfrac{1}{2}\operatorname{tr}U_{C}$ multiplied by the product of the z **outside $S(C)$ in the plane**; it contains no signs of the inner containers, so the Container Identity does not make it small. The remedy is to place containers outside as well.

Inner and Outer Container Proposition (inner and outer regions)

In the (3,4) plane, arrange containers $R_{1},\dots,R_{n}$ inside the loop and containers $R'_{1},\dots,R'_{m}$ outside it in a grid, with the frame made of straight strips, so that in each direction there is a straight line passing only through the frame. Then $\mathbf{1}_{O}$ is a function of the frame, and one can write $(-1)^{n_{34}} = c''(\Phi)\prod_{j}\sigma_{j}\prod_{k}\sigma'_{k}$ ($|c''|=1$). Let θ_{j},θ'_{k} be the biases of (5) and set $\mathbb{E}_{O}[\,\cdot\,] := \mathbb{E}[\,\cdot\,\mathbf{1}_{O}]/\mathbb{E}[\mathbf{1}_{O}]$; then, when $\mathbb{E}_{O}[\prod_{j}\theta_{j}\prod_{k}\theta'_{k}]<1$,

$$|\langle W(C)\rangle_{s}| \;\le\; \frac{\mathbb{E}_{O}\big[\prod_{j}\theta_{j}\big]+\mathbb{E}_{O}\big[\prod_{k}\theta'_{k}\big]}{1-\mathbb{E}_{O}\big[\prod_{j}\theta_{j}\prod_{k}\theta'_{k}\big]} \tag{9}$$

Proved here Proof. $n_{12}, n_{13}, n_{14}, n_{23}, n_{24}$ are each the product of z over the respective plane, and that plane can be retaken on a straight strip passing only through the frame (since z is coclosed, the product does not depend on the position of the plane), so they are functions of the frame. Hence $\mathbf{1}_O$ is a function of the frame. The (3,4) plane splits into $S(C)$ and its complement, each of which splits into a frame part and container parts, so $W = c(\Phi)\prod_j \sigma_j$, $W(-1)^{n34} = c'(\Phi)\prod_k \sigma'_k$, $(-1)^{n34} = c''(\Phi)\prod_j \sigma_j \prod_k \sigma'_k$ ($|c|$, $|c'| \le 1$, $|c''| = 1$). Multiplying the first formula of the Container Identity by $\mathbf{1}_O$ and averaging over the frame gives $|\mathbb{E}[W\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_j \theta_j]$, $|\mathbb{E}[W(-1)^{n34}\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_k \theta'_k]$, $|\mathbb{E}[(-1)^{n34}\mathbf{1}_O]| \le \mathbb{E}[\mathbf{1}_O \prod_j \theta_j \prod_k \theta'_k]$. The numerator of (8) is at most half the sum of the first two, and the denominator is at least $\frac{1}{2}(\mathbb{E}[\mathbf{1}_O] - \mathbb{E}[\mathbf{1}_O \prod \theta \prod \theta'])$. Dividing both by $\mathbb{E}[\mathbf{1}_O]$ gives (9). ∎

The second term of (9) goes to 0 as the size of the plane $M \to \infty$ when the biases are typically smaller than 1 in the outer containers too (when the rarity of bad frames can be used outside as well). This book does not show that Assumption . In the theorem of slab confinement (25-04), the group of flips acts on all plaquettes of the plane, so the outer term automatically becomes small, $t^{M^2 - A}$, and this problem does not arise.

05

Nesting in the directions of the plane

We divide a large container R (a disc of the plane $\times$ T^2) into small containers $R_1, \dots, R_k$ separated by an inner frame Φ_{int}. The product of z over the surface spanned by R is $\sigma_R = s(\Phi_{\mathrm{int}})\prod_i \sigma_i$ ($|s| = 1$, the product of z over the part in the frame).

Nesting Inequality in the Plane Directions

Writing Φ_R for the frame outside R,

$$\left|\mathbb{E}[\sigma_R \mid \Phi_R]\right| \;\le\; \mathbb{E}\Big[\prod_{i=1}^{k} \left|\mathbb{E}[\sigma_i \mid \Phi_R, \Phi_{\mathrm{int}}]\right| \;\Big|\; \Phi_R\Big] \tag{10}$$

Proved here Proof. Apply the Container Identity with Φ_R and Φ_{int} together as the frame ($c = s$), and average over Φ_{int} with Φ_R held fixed. ∎ Lean Containers (`Containers.container_bound`) (hypotheses: finite, $a \ge 0$, $|s| \le 1$) is the finite-sum version of this algebra, the form whose description in the Lean file says it "contains the Mack–Petkova container identity and the planar doubling inequality".

How to read it. If, for containers of some size, the bias is $\theta_0 < 1$ for typical frames, one can expect that each doubling of the containers in the directions of the plane lowers

the bias to order θ_0^4 Reading (a physical estimate: the right-hand side of (10) is the average of a product, not the product of averages, so this does not follow directly from the inequality). Disorder propagates to large scales in the directions of the plane.

However, (10) concerns **a slab with the size of the T^2 held fixed**. What is needed when widening the T^2 (approaching four-dimensional space) is the behaviour of the bias $\theta(L, L)$ when the T^2 and the containers are doubled together, and (10) says nothing about that (when the T^2 changes, the very surface around which the vortex sheet winds changes). That the form needed there cannot be "$\theta(L, L)$ stays uniformly away from 1" but must be $\ln(1/\theta(L, L)) \geq cL^2$ (the contrast with $U(1)$) is treated in the chapter on the crossover band, 26-01.

06

Limits in four dimensions — independent containers reach only $R/\ln R$

The Container Identity is an identity that depends neither on the size of the lattice nor on the dimension. So, if one lines up many containers in four-dimensional space (the lattice of R^4), does an area law follow? Kovács–Tomboulis (hep-lat/0108017 § 3) show by the following counting that **independent containers do not reach it** Cited ((23)–(27) below are the equation numbers of the original source; checked in its text).

1. Let d_i be the width in the two directions transverse to the coclosed surface V_i of a container Λ_i, and $|V_i|$ its longitudinal length. **Assume** that a sufficiently thick container is in the regime where $-\ln(z^{(-)}/z^{(+)}) \sim |V_i| \exp(-c\, d_i^2)$ for all boundary values ((23) of the original source; c is the ρ of the original source).
2. To keep the free-energy cost of one container below a constant f, one needs $d_i^2 \geq c^{-1} \ln(|V_i|/f)$ ((24)). Then $\max |f_{\Lambda_i}| < \frac{1}{2}(1 - e^{-f})$ is constant.
3. For a rectangular loop with sides $T \times R$ ($T > R$), the length of the containers through points away from the perimeter is $|V_i| \sim R^{d-2}$ (R^2 in four dimensions). So a thickness of order $\sqrt{\ln R}$ is needed, and $\sim RT/\ln R$ containers fit inside the loop. From (22)

$$V(R) \;\geq\; \text{const} \cdot \frac{R}{\ln R} \qquad \text{((25) of the original source)} \quad (11)$$

is all that follows, which is not a purely linear potential (an area law). The original source goes on to state that if the boundary values of neighbouring containers are also integrated, grouping containers as thick as possible, and containers with transverse area $\sim R^2$ are used, one gets $V(R) \geq \text{const}\, R - \frac{\text{const}}{R}(\ln R + \text{const})$ ((27)), that is, **the linear potential comes from thick vortex fluctuations of the size of the loop**. It also states that the form (22) is crude, and that the inequalities coming from reflection

positivity (their (11)(13), the global form of Tomboulis–Yaffe) give an area law directly, with bounds uniform in the size of the lattice Cited .

Restated in the words of this chapter: in R^4 the length of the containers grows with the loop, so the road of placing containers of fixed length independently does not reach the area law. In a small-T^2 slab, the length of the container $\{\text{disc}\} \times T^2$ (the area of the vortex sheet) stops at the area of the T^2, so this obstruction does not appear Proved here (a rereading of the fact that $|V_i| = N^2$ becomes a constant in (24) of the original counting). The obstruction returns when the T^2 is widened, and that is the crossover band (26-01, 25-09).

07

A test in a solvable two-dimensional model

Two-dimensional $SU(2)$ (a planar lattice, Wilson action $\frac{\beta}{2}\operatorname{tr} U_p$) is solvable because the plaquettes behave independently of each other. Here we see how close the bound of the Container Identity is to the true value. Two dimensions has no problem of "the length of containers", so this test says nothing about the four-dimensional obstruction of § 06.

Setting. Tile the inside of the loop with square containers of side ℓ, and take the links on the borders of the containers as the frame (container area $A = \ell^2$). With the frame fixed, container j sees only the holonomy $V_j \in SU(2)$ of its border. Using the characters $\chi_n(V) = \sin((n+1)\vartheta)/\sin\vartheta$ ($e^{\pm i\vartheta}$ are the eigenvalues of V, and $n = 0, 1, 2, \ldots$ is $2\times$ the spin of the representation) and $a_n = I_{n+1}(\beta)/I_1(\beta)$ (I is the modified Bessel function), the heat-kernel expansion gives

$$\mathbb{E}[\sigma_j \,|\, \text{frame}] = \frac{\sum_{n\ \text{odd}} (n+1)\, a_n^A\, \chi_n(V_j)}{\sum_{n\ \text{even}} (n+1)\, a_n^A\, \chi_n(V_j)} \qquad (12)$$

and the holonomies of the containers are independent of each other, distributed with the denominator as density. So the bound of the Container Identity becomes a product of factors per container,

$$|\langle W\rangle| \;\le\; B(A)^{\#\text{containers}}, \qquad B(A) = \int_{SU(2)} dV \,\Big|\sum_{n\ \text{odd}} (n+1)\, a_n^A\, \chi_n(V)\Big| \qquad (13)$$

The true value is $\langle W\rangle = a_1^{\text{area}}$, and the true string tension is $\sigma = -\ln a_1$. We write $\sigma_{\text{L5}}(\ell) := -\ln B(\ell^2)/\ell^2$ for the string tension coming from (13), and $\sigma_{\max}(\ell) := -\ln\big(\max_V |(12)|\big)/\ell^2$ for that coming from the form of (22) of Kovács–Tomboulis (maximum over frames).

The subscript L5 is a letter for the form that comes from the Container Identity (the average over frames), used as the counterpart of the maximum form $\sigma_{\max}$.

The limit of large containers. As $A \to \infty$, the odd sum is dominated by the $n = 1$ term $2a_1^A \chi_1(V) = 4a_1^A \cos\vartheta$ and the even sum by the $n = 0$ term 1, so with the Haar distribution $\frac{2}{\pi}\sin^2\vartheta\, d\vartheta$

$$\frac{B(A)}{a_1^A} \longrightarrow \frac{2}{\pi}\int_0^{\pi} 4|\cos\vartheta|\sin^2\vartheta\, d\vartheta = \frac{16}{3\pi} \approx 1.6977, \qquad \sigma_{\mathrm{L5}}(\ell) = \sigma - \frac{\ln(16/(3\pi))}{\ell^2} + (A\text{-e}$$

Proved here . The bound of the Container Identity is **sharp in its exponent**, and the loss vanishes as $1/\ell^2$.

Table 1 The string tension $\sigma_{\mathrm{L5}}(\ell)$ from the Container Identity and the maximum form $\sigma_{\max}(\ell)$ for two-dimensional $SU(2)$. Floating point, trapezoidal rule in ϑ with 4×10^5 points, representations truncated at $n \le 200$, the check of the normalisation of the denominator is 1.000000 in every row. The numbers are numerics from this book's study (Reading)

β	True σ	$\ell = 1$	$\ell = 2$	$\ell = 4$	$\ell = 8$	$\ell = 16$	$\sigma_{\max}$: $\ell = 1, 4, 16$
1	1.4263	0.9132	1.2940	1.3932	1.4180	1.4242	0.2723, 1.3397, 1.4209
2	0.8367	0.3663	0.7044	0.8036	0.8285	0.8347	0.0366, 0.7501, 0.8313
4	0.4185	0.0624	0.2866	0.3854	0.4102	0.4164	0.00067, 0.3318, 0.4131
8	0.1994	0.0016	0.0761	0.1663	0.1911	0.1973	$< 10^{-6}$, 0.1128, 0.1940

How to read it Reading . (i) The $\ell = 16$ column agrees with the right-hand side of (14) to the fourth decimal place (for example, at $\beta = 1$, $1.426309 - \ln(1.69765)/256 = 1.424242$). (ii) For thin containers ($\ell = 1$), the bound becomes exponentially worse when β is large (at $\beta = 8$, 0.0016 against 0.1994). This is the two-dimensional form of the need for thick containers. (iii) The maximum form approaches the same limit, but lags behind. The Container Identity, which takes the average, is always closer to the true value at the same ℓ.

The maximum form for $\ell = 1$ in (ii) can be written in closed form. A container of side 1 is a single plaquette, and all four of its links belong to the frame, so $\mathbb{E}[z_p \mid V] = \tanh\left(\frac{\beta}{2}\operatorname{tr} V\right)$, and the maximum is $\tanh\beta$ at $\operatorname{tr} V = \pm 2$ Proved here . The $\sigma_{\max}(1)$ of the table agrees with $-\ln\tanh\beta$ (0.27234, 0.036635, 0.000671 at $\beta = 1, 2, 4$). This is the same formula as the case $N = 1$ of the slab-confinement bound $\sigma \ge -\ln\tanh(\beta N^2)$ of the next chapter (25-04 § 05).

```
08
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What cannot be said

(a) Novelty. The Container Identity is the container identity of Mack–Petkova, written in (18)–(22) of Kovács–Tomboulis. The rewriting in the sector-summed ensemble, the Inner and Outer Container Proposition and the Nesting Inequality in the Plane Directions are elementary, and have not been compared with literature other than Kovács–Tomboulis. No novelty is claimed for the results of this chapter.

(b) The second term of the Inner and Outer Container Proposition. That the biases of the outer containers are typically small (that the second term goes to 0 as $M \to \infty$) is not shown.

(c) The reading of the Nesting Inequality in the Plane Directions. "θ_0^4 at each doubling" is an estimate and does not follow directly from (10). (10) says nothing about widening together with the T^2.

(d) The four-dimensional counting. The $R/\ln R$ of § 06 is counting on top of the assumption on the form of the thick-container free energy in the original source ((23) of Kovács–Tomboulis: that the free energy of a thick container has the form $|V_i| e^{-c d_i^2}$ uniformly in the boundary values), and is not a four-dimensional theorem. This book shows neither (23) nor its negation.

(e) Uniform bounds on the free energy of containers. Bounding ΔF_j uniformly over good frames and uniformly in the lattice spacing is not shown anywhere in this book. That needs a multiscale analysis of the conditional measure with the frame fixed, and the variational principle alone (which gives only lower bounds on partition functions) points the wrong way.

(f) Lean. The machine-checking column is only the algebra of finite sums. That (3)(4) follow from the Haar measure, a coclosed z, the topology of sectors and the nearest-neighbour action is outside Lean.

(g) The two-dimensional test. Table 1 is two-dimensional numerics and says nothing about the problem of the length of containers in four dimensions.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The sector-summed ensemble (1) is the sum of the partition functions of all twisted sectors	Cited	Kovács–Tomboulis § 2 (moving the twist), Mack–Petkova (text not checked)	$SU(2)$, finite lattice	—
2	With the frame fixed, the weight and the Wilson loop split per container (3)(4); σ_j is not determined by the frame	Proved here	§ 02	Nearest-neighbour action, containers separated	—
3	(18)(20)(22) of Kovács–Tomboulis	Cited	Text of the original source	Fixed sector, maximum over frames	—
4	Container Identity: (6) and $\theta_j = \tanh(\lvert\Delta F_j\rvert/2)$	Cited + Proved here	§ 03 (the identity is row 3, the rewriting is in the text)	Finite lattice, any β	FrameLemma (`FrameLemma.frame_bound`) (hypotheses: finite, $a > 0$, $f > 0$, inside non-empty), `sign_mean_le_one` (hypotheses: $f > 0$, $\sigma = \pm 1$), `ratio_eq_tanh` (hypotheses: $q^{\pm} > 0$), Containers (`Containers.container_bound`) (hypotheses: finite, $a \geq 0$, $\lvert s\rvert \leq 1$)
5	Inner and Outer Container Proposition: (9) for a fixed sector	Proved here	§ 04	Slab, containers in a grid, $\mathbb{E}_O[\prod\theta\prod\theta'] < 1$	— (the algebra has the same form as the bundles of row 4)
6	The second term of (9) goes to 0 as $M \to \infty$	Assumption	§ 04 (rarity of bad frames outside)	—	—
7	Nesting Inequality in the Plane Directions (10)	Proved here	§ 05	Slab with the size of the T^2 fixed	Containers (`Containers.container_bound`) (hypotheses: finite, $a \geq 0$, $\lvert s\rvert \leq 1$)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
8	The bias drops to order θ_0^4 at each doubling	Reading (physics)	§ 05	Typical frames, T^2 fixed	—
9	Independent containers give only $V(R) \geq \text{const}\, R/\ln R$; thick containers give linear	Cited	(23)–(27) of § 3 of the original source	Four dimensions, on top of the assumption on the form of the thick-container free energy in the original source	—
10	In a slab the length of containers stops at the area of the T^2, and the obstruction of row 9 does not appear	Proved here	§ 06 (a rereading of (24))	Small-T^2 slab, N fixed	—
11	For large containers $B(A)/a_1^4 \to 16/(3\pi)$, (14)	Proved here	§ 07	Two-dimensional $SU(2)$, Wilson action	—
12	The numbers of Table 1	Reading	Numerics from this book's study	Two dimensions, floating point, $n \leq 200$	—
13	The maximum form for a container of side 1 is $\tanh \beta$	Proved here	§ 07	Two dimensions	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones (propext, Classical.choice, Quot.sound). What is guaranteed is only the algebra of sums and products of real numbers over finite sets (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  sector-summed ensemble (1)                                    SU(2),
 2  PRV  splits by frame (3)(4); σ_j not fixed by frame                neares
 3  CIT  Kovács–Tomboulis (18)(20)(22)                                 fixed
 4  PRV  Container Identity (6)(7) ⟸ 1, 2 (same identity as 3)          Lean
 5  PRV  Inner and Outer Container Proposition (9) ⟸ 4                  slab,
 6  ASM  second term of (9) → 0                                        rarity
 7  PRV  Nesting Inequality in the Plane Directions (10) ⟸ 4            T² fi
 8  RDG  θ₀⁴ per doubling (physics) ⟸ 7
 9  CIT  R/ln R; thick containers linear (on the assumption on the form of the
        four dimensions
10  PRV  in a slab container length stops ⟸ 9
11  PRV  limit 16/(3π) ⟸ 4                                              two c
12  RDG  Table 1 ⟸ 4, 11                                               float
13  PRV  maximum for side 1 = tanh β ⟸ 3                                two c
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The row of rarity outside (second term of (9) $\to$ 0) is an assumption row and enters as a premise of no row. The inequality (9) of the Inner and Outer Container Proposition holds without this row; the row is needed only when reading (9) as an area law. This chapter has neither a node of the continuum limit nor a node of the four-dimensional area law.

The machine-checked theorems on the vortex side are in Part 21.

What this chapter can say

What is said	What is not said
Cited + Proved here The Container Identity and its tanh form. Finite lattice, any β, the sector-summed ensemble	Novelty (the identity is Mack–Petkova's). Bounds on the free energy of containers ΔF_j uniform over good frames and in the lattice spacing
Proved here How to pass to a fixed sector (the Inner and Outer Container Proposition), and nesting in the directions of the plane (the Nesting Inequality in the Plane Directions)	That the second term of (9) goes to 0. The behaviour of the bias when the T^2 is widened together (26-01)
Cited In four dimensions independent containers give only up to $R/\ln R$ (counting on top of the assumption on the form of the thick-container free energy in the original source); why this obstruction does not appear in a slab	The four-dimensional area law. The assumption on the form of the thick-container free energy itself
Proved here + Reading That for two-dimensional $SU(2)$ the bound of the Container Identity is sharp in its exponent (loss $\ln(16/(3\pi))/\ell^2$)	The sharpness of the bound in four dimensions

Sources and reproduction

Item	Kind	Source / tools
The container identity (18)(20)(22), the four-dimensional counting (23)–(27)	Cited	T. G. Kovács, E. T. Tomboulis, Vortex waistlines and long range fluctuations, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501). § 2 and § 3 checked in the text
Vortex containers and the summed ensemble	Cited (indirect)	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (reference [3] of Kovács–Tomboulis. The text has not been checked in this book)
The global form from reflection positivity	Cited (indirect)	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 (reference [9] of Kovács–Tomboulis, and the restatement of it in their text (11)(13))
The form of the Container Identity in the summed ensemble, the Inner and Outer Container Proposition, the Nesting Inequality in the Plane Directions, (14), the maximum for side 1	Proved here	Proofs in the text, § 03– § 07
Table 1	Reading	Numerics from this book's work (two-dimensional $SU(2)$, the character expansion (12) integrated numerically; a few seconds of computation)
Lean	Lean	FrameLemma (`FrameLemma.frame_bound`, `FrameLemma.sign_mean_le_one`, `FrameLemma.ratio_eq_tanh`), Containers (`Containers.container_bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — the sector-summed ensemble, frame, container, the sign of the sector, the bias θ, the vortex free energy with the frame fixed ΔF, the Container Identity, inner and outer containers, nesting, the length and thickness of containers. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-04 Slab confinement — letting the group of flips act on all plaquettes of the plane gives, in a slab with the lattice spacing and the T^2 fixed, an area law at every β.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the Slab Confinement Theorem, group averaging, thin vortices · Kiichi, Shiori and Rin

Slab confinement

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 5-01.

What this chapter shows — Consider a "slab" lattice in which two directions are wound into a small T^2 (a lattice of side N) and the remaining two directions (the plane) are wide. With the lattice spacing and the size of the T^2 fixed, the $SU(2)$ Wilson loop obeys an area law **at every** β, with string tension $\sigma \geq -\ln\tanh(\beta N^2) \geq 2e^{-2\beta N^2} > 0$ (the slab-confinement theorem). The proof is elementary: one just averages over the group that flips at once the signs on the vortex sheets $\{p\} \times T^2$ standing directly above the plaquettes of the plane. In this chapter we place the statement of the theorem (§ 02), a proof in five steps (§ 03), the container form $\sup\theta \leq \tanh(\beta N^2)^{\ell^2}$ and why this lower bound becomes 0 in the continuum limit (§ 04), a test in two dimensions (§ 05), and where it stands relative to earlier literature (§ 06).

Prerequisite chapters — 25-03 (the sector-summed ensemble, frame and containers, the Container Identity), 25-01 (twisted sectors and flux). From Part 5, 5-01 (the Wilson action) and 5-02 (the area law at strong coupling); from Part 7, 7-02 (two dimensions).

Said first — The theorem is a statement on the lattice, with the lattice spacing and the side N of the T^2 fixed. In the continuum limit ($N \to \infty$, $\beta \to \infty$) this lower bound vanishes to 0. That the theorem holds at every β without using the strong-coupling expansion is because the slab is in effect a two-dimensional system, and it says nothing about confinement in four-dimensional space. Comparison with earlier literature is not done (comparison with arguments of Tomboulis–Yaffe type and with Borgs 1985 remains).

The order of this chapter

1. Summary — flip the vortex sheets all at once and average
2. Setting and the Slab Confinement Theorem
3. Proof — five moves
4. The container form and the cost of thin vortices — in the continuum limit the lower bound on the tension is 0

5. A test in two dimensions
6. Where it stands relative to earlier literature
7. What cannot be said
8. The table of five grades and the dependency graph — Table 1

01

Summary — flip the vortex sheets all at once and average

1. Two-dimensional lattice gauge theory confines at every coupling (Wilson loops decrease exponentially in the area). This is because the plaquettes behave independently of each other (7-02).
2. Seen from afar, the lattice of a slab $T^2 \times \mathbb{R}^2$ with two small closed directions is a two-dimensional system. So, with the size of the T^2 held fixed, one can expect it too to confine at every coupling. This chapter states that on the lattice, with a proof Proved here .
3. The tool is the sector-summed ensemble of 25-03. For each plaquette p of the plane, consider the sheet S_p of the N^2 (3,4) plaquettes at all points of the T^2 standing directly above it. Flipping the signs z on S_p all at once preserves the coclosedness condition and leaves the links unchanged. Flipping at a p inside the loop changes the sign of the Wilson loop, and the weight changes by at most a factor $e^{\pm 2\beta N^2}$.
4. Averaging over all combinations of flips (2^{M^2} of them), the Wilson loop is bounded by a product of factors $(1 - r_p)/(1 + r_p)$ over the inner plaquettes, and each factor is at most $\tanh(\beta N^2)$. Hence $|\langle W \rangle| \le \tanh(\beta N^2)^A$ (A is the area of the loop).
5. This lower bound is superexponentially weak in N (at $\beta = 2.5$, $N = 4$, $\sigma \ge 2e^{-80}$), and becomes 0 in the continuum limit. This is because the cost $2\beta N^2$ of flipping a thin vortex sheet at once grows with the number of points of the T^2 lattice. The cost of a physical vortex (a fractional instanton) is of order $\pi^2\beta$ and does not depend on N, so the road of doing the same thing with thick vortices is the next two chapters (25-05, 25-06).

02

Setting and the Slab Confinement Theorem

The lattice is $\Lambda = (\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 form the T^2 (side N), and directions 3 and 4 the plane (cut off with period M). The group is $SU(2)$, the action is

the Wilson action $\frac{\beta}{2}\sum_p \operatorname{tr} U_p$, and the twist of the T^2 (n_{12} and so on) is fixed arbitrarily. The loop C lies in the (3,4) plane at a point y_0 of the T^2, and encloses a region $S(C)$ of area A (number of plaquettes) in the plane.

Slab Confinement Theorem

Set $t := \tanh(\beta N^2)$. For every $\beta > 0$, every M, and any fixed twisted sector,

$$\left|\langle \tfrac{1}{2}\operatorname{tr} U_C\rangle\right| \;\le\; \frac{t^{A} + t^{M^2 - A}}{1 - t^{M^2}} \quad (1)$$

In particular, at any limit point as $M \to \infty$, $|\langle \frac{1}{2}\operatorname{tr} U_C\rangle| \le t^{A}$, and uniformly in the area of the loop,

$$\sigma \;\ge\; -\ln\tanh(\beta N^2) \;\ge\; 2e^{-2\beta N^2} \;>\; 0. \quad (2)$$

Proved here (proof in § 03). The σ of (2) is the string tension in lattice units, and the conditions are "$SU(2)$, Wilson action, N fixed, lattice spacing fixed, every β". The right-hand inequality of (2) follows from setting $u = e^{-2x}$ and $-\ln\tanh x = \ln(1+u) - \ln(1-u) = 2(u + u^3/3 + \cdots) \ge 2u$.

03

Proof — five moves

Step 1 (the sector-summed ensemble). (1) of 25-03 § 02: for coclosed $z \in \{\pm 1\}^{\text{plaquettes}}$ and U the weight is $w(U,z) = \exp(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p)$, and the measure is Haar $\times$ counting. Every z that can be written $z = z_0 \cdot \delta\omega$ ($\omega \in \{\pm 1\}^{\text{links}}$, $(\delta\omega)_p = \prod_{b\in\partial p}\omega_b$) gives the same integral as z_0 by the invariance $U_b \mapsto \omega_b U_b$ of the Haar measure, so the sum over z in one cohomology class is a constant multiple of the $SU(2)$ partition function of the twisted sector that the class represents Proved here. The sector is fixed by the class of z, and the indicator function of a sector is a function of z. The gauge-invariant Wilson loop is $W(C) = \frac{1}{2}\operatorname{tr} U_C \prod_{p\in S(C)} z_p$; under the same substitution $\prod_{p\in S(C)}(\delta\omega)_p = \prod_{b\in C}\omega_b$ cancels against the sign on the side of $\operatorname{tr} U_C$, so within a sector it has the same expectation as $\frac{1}{2}\operatorname{tr} U_C$ (up to a sign depending on the choice of representative of the sector).

Step 2 (vortex sheets are coclosed). For each plaquette p of the plane set $S_p := \{$ p's overlying (3,4) plaquettes $\} \times T^2$ (N^2 plaquettes). Among three-dimensional cubes, only those of orientation (1,3,4) and (2,3,4) have (3,4) faces; for example, the (3,4) faces

of the (1,3,4) cube that extends in direction 1 from a point y of the T^2 and spans the plaquette p' of the plane are the two copies of p' at y and $y+e1$. If $p'=p$ both lie in S_p; if $p' \neq p$ neither does. Cubes of type (1,2,3) and (1,2,4) have no (3,4) faces. Hence every cube contains 0 or 2 faces of S_p, and flipping the z on S_p preserves the coclosedness condition Proved here . If $p \neq q$, S_p and S_q have no plaquette in common.

Step 3 (the group action). Let P be the set of all plaquettes of the plane ($|P| = M^2$), and assign to a subset $g \subseteq P$ the map $z \mapsto z \cdot \chi_{S_g}$ (flipping on $S_g = \bigcup_{p\in g} S_p$). This is an action of the group $G = \{\pm 1\}^P$, a bijection on coclosed z, and since it does not change U it preserves the Haar $\times$ counting measure.

- The Wilson loop: $S(C)$ consists of A plaquettes in the plane at y_0, and S_p contains exactly one plaquette of the plane at y_0, the one at the position of p. Hence $W(g\cdot x) = W(x)\prod_{p\in g}\varepsilon_p$, with $\varepsilon_p = -1$ ($p \in S(C)$) and $+1$ (otherwise).
- The weight: flipping the z on S_p changes the action by $-\beta\sum_{q\in S_p} z_q \operatorname{tr} U_q$, so setting $r_p(x) := \exp(-\beta\sum_{q\in S_p} z_q \operatorname{tr} U_q)$, from $|\operatorname{tr} U_q| \le 2$ we get $r_p \in [e^{-2\beta N^2}, e^{2\beta N^2}]$. r_p is determined only by the values on S_p, and the sheets do not meet, so $w(g\cdot x) = w(x)$ $\prod_{p\in g} r_p(x)$.

Step 4 (averaging). Let $\mathbb{E}$ be the expectation of the sector-summed ensemble, and $\sum_x$ the sum over z together with the Haar integral over U. For each g, relabelling the variables gives $\sum_x W(x)w(x) = \sum_x W(g\cdot x)w(g\cdot x) = \sum_x W(x)w(x)\prod_{p\in g}\varepsilon_p r_p(x)$. Averaging over g and using $\sum_{g\subseteq P}\prod_{p\in g}\varepsilon_p r_p = \prod_{p\in P}(1+\varepsilon_p r_p)$,

$$2^{M^2}\sum_x W w = \sum_x W(x)w(x)\prod_{p\in S(C)}(1-r_p(x))\prod_{p\in\complement S(C)}(1+r_p(x)), \qquad 2^{M^2}\sum_x w = \sum_x$$

The one-sheet inequality $|1-r| \le \tanh(a)\,(1+r)$ ($r \in [e^{-2a}, e^{2a}]$, here $a = \beta N^2$) follows from $(1-r)/(1+r) = -\tanh(\tfrac{1}{2}\ln r)$ and $|\tfrac{1}{2}\ln r| \le a$. Together with $|W| \le 1$, the absolute value of the right-hand side of the left formula of (3) is at most $t^A\sum_x w\prod_P(1+r_p) = t^A\,2^{M^2}\sum_x w$. Hence

$$|\mathbb{E}[W(C)]| \;\le\; t^A. \quad (4)$$

At this step there is no need to split into orbits. We use only the relabelling (a bijection) for each g, and the exchange of the sum over g with the sum over x.

Step 5 (projection to a fixed sector). The group G changes only the parity of the (3,4) flux. S_p contains one plaquette of the (3,4) plane at y_0, so it flips $(-1)^{n34}$ (the product of z over the whole (3,4) plane at y_0); it contains no plaquettes of other orientations, so

it does not change the indicator function $\mathbf{1}_O$ of the other labels such as n_{12}. The expectation with the sector fixed to s is, as in (8) of 25-03,

$$\langle W\rangle_s = \frac{\mathbb{E}[W\mathbf{1}_O] + s\,\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_O]}{\mathbb{E}[\mathbf{1}_O] + s\,\mathbb{E}[(-1)^{n_{34}}\mathbf{1}_O]} \tag{5}$$

$W(-1)^{n_{34}} = \frac{1}{2}\mathrm{tr}\,U_C \prod_{p\in/\,S(C)} z_p$ (the complement in the plane at y_0, of area $M^2 - A$) changes sign under flips at $p \in/\ S(C)$ and does not change under flips at $p \in S(C)$. $(-1)^{n_{34}}$ changes sign under flips at every p. $\mathbf{1}_O$ is invariant under G, so by the same averaging as in step 4 (with the roles of ε_p exchanged)

$$|\mathbb{E}[W\mathbf{1}_O]| \le t^{A}\,\mathbb{E}[\mathbf{1}_O], \qquad |\mathbb{E}[W(-1)^{n_{34}}\mathbf{1}_O]| \le t^{M^2-A}\,\mathbb{E}[\mathbf{1}_O], \qquad |\mathbb{E}[(-1)^{n_{34}}\mathbf{1}_O]| \le t^{M^2}\,\mathbb{E}$$

Bounding the numerator and denominator of (5) by (6) gives (1). As $M \to \infty$, $t^{M^2-A} \to 0$ and $t^{M^2} \to 0$ ($t < 1$), so at limit points $|\langle W\rangle| \le t^{A}$, and (2) follows. ∎

Here the group acts on all plaquettes of the plane, so the problem of 25-03 § 04, "the outer term does not become small", does not arise. The outer term automatically becomes small, t^{M^2-A}.

Lean SlabConfine (`SlabConfine.group_average`) (no hypotheses: $\sum_{g\subseteq P}\prod_{p\in g}\varepsilon_p r_p = \prod_{p\in P}(1+\varepsilon_p r_p)$), SlabConfine (`SlabConfine.factor_split`) (no hypotheses: the splitting into inner and outer), SlabConfine (`SlabConfine.tanh_bound`) (hypotheses: $t \ge 0$, $r_p > 0$, $|1-r_p| \le t(1+r_p)$), SlabConfine (`SlabConfine.one_sheet`) (hypotheses: $a \ge 0$, $e^{-2a} \le r \le e^{2a}$. Conclusion: $|1-r| \le \tanh a \cdot (1+r)$), SlabAverage (`SlabAverage.average_bound`) (hypotheses: the set of configurations is finite, $w > 0$, $|W| \le 1$, the action of each $g \subseteq P$ is a bijection, $r_p > 0$, $|1-r_p| \le t(1+r_p)$, $W(gx)w(gx) = W(x)w(x)\prod_{p\in g}\varepsilon_p r_p(x)$, $w(gx) = w(x)\prod_{p\in g} r_p(x)$. Conclusion: $|\sum Ww| \le t^{|P\cap A|}\sum w$). These are the finite-sum version of step 4. Replacing the sums by Haar integrals, and the coclosed z and the topology of sectors (steps 1, 2 and 5), are outside Lean and rest on the argument in the text.

04

The container form and the cost of thin vortices

Carrying out the same averaging only inside one container, it can be written in the language of the biases of 25-03. Take as container the product of a square Q of side ℓ in the plane (ℓ^2 plaquettes) with the T^2, and let σ_Q be the product of z over the surface spanned by Q. Flips by $g \subseteq Q$ change only the z inside the container, so they do not change the frame Φ, and they multiply σ_Q by $\prod_{p\in g}(-1)$. On the orbit $\{g\cdot x : g \subseteq Q\}$ of one x, the weight has the product form $w(x)\prod_{p\in g} r_p(x)$, so conditioning on the orbit makes the components of g independent, and

$$\mathbb{E}[\sigma_Q \mid \mathrm{orbit}] = \sigma_Q(x)\prod_{p\in Q}\frac{1-r_p(x)}{1+r_p(x)} = \pm\prod_{p\in Q}\tanh X_p, \qquad X_p = -\tfrac{1}{2}\ln r_p \in [-\beta N^2, \beta N^2]$$

Proved here . Conditioning on the orbit is finer than conditioning on the frame, so

$$\sup_{\Phi}\ \theta_Q(\Phi) = \sup_{\Phi}\big|\mathbb{E}[\sigma_Q \mid \Phi]\big| \ \le\ \tanh(\beta N^2)^{\ell^2} \quad (8)$$

Proved here . That it can be bounded by the **maximum** over frames is because the cost of flipping a thin sheet, $|\ln r_p| \le 2\beta N^2$, is uniform and independent of the frame.

Extremely weak. The lower bound of (2) becomes superexponentially small in N. At $\beta = 2.5$, $N = 4$, $2\beta N^2 = 80$ and $\sigma \ge 2e^{-80} \approx 3.6 \times 10^{-35}$. The lower bound $2e^{-2\beta N^2}/a^2$ on the physical string tension $\sigma_{\text{phys}} = \sigma/a^2$ goes to 0 if the physical size $L = Na$ of the T^2 is held fixed and $a \to 0$ ($N = L/a \to \infty$, and β increases by asymptotic freedom) Proved here . **In the continuum limit, the lower bound on the tension in this theorem is 0.**

The reason is the cost of thin vortices. The cost of a flip, $2\beta N^2$, is the action of turning over at once the N^2 plaquettes of the lattice sheet S_p, one lattice unit thick, and it grows with the number of points of the T^2 lattice. On the other hand, what physics regards as the cost of a vortex is of the order of the action $4\pi^2/g^2 = \pi^2\beta$ ($\beta = 4/g^2$) of a smooth fractional instanton winding the T^2, which does not depend on N (25-02) Reading (physics). The same group-averaging argument goes through with $t = \tanh(F/2)$ whenever the ratio of weights r_p of a flip lies in $[e^{-F}, e^{F}]$, so if the flip of a thin sheet is replaced by "a measure-preserving map that inserts a thick vortex" and F can be bounded by a quantity independent of N, a lower bound uniform in the lattice spacing can be expected Assumption (where the evidence is: the form (7) of this chapter, the insertion identities of 25-05, and the insertion driven by dynamics of 25-06). That needs a uniform bound on the cost of insertion (ultraviolet control; in the setting of this part, that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions), and this book does not show it. The published construction is on a periodic cubic box and does not reach here.

(8) also shows that arguments using $\sup_\Phi$ get superexponentially worse in N. To get uniformity in the problem of widening the T^2 (the crossover band), only the averaged form over frames can be used, not the maximum (one of the reasons why the Container Identity of 25-03 takes the average).

05

A test in two dimensions

Applying the five moves of the proof directly to the two-dimensional lattice $(\mathbb{Z}/M)^2$, there are no cubes, so the coclosedness condition is empty, and the vortex sheet is $S_p = \{p\}$ (a single plaquette, the formula of the Slab Confinement Theorem with $N = 1$).

So the same proof gives $|\langle W\rangle| \le \tanh(\beta)^A$ for two-dimensional $SU(2)$. The true value in two dimensions is $\langle W\rangle = a_1^A$, $a_1 = I_2(\beta)/I_1(\beta)$, so the theorem contains $I_2(\beta)/I_1(\beta) \le \tanh\beta$ for every β.

- Leading terms at large β: from $I_\nu(\beta) \sim e^\beta(2\pi\beta)^{-1/2}(1 - \frac{4\nu^2-1}{8\beta})$ we get $a_1 \approx 1 - \frac{3}{2\beta}$, while $\tanh\beta \approx 1 - 2e^{-2\beta}$. The lower bound is exponentially smaller than the true tension $\approx \frac{3}{2\beta}$, and there is no contradiction Proved here .
- In numbers: the true $\sigma = -\ln a_1$ is $1.4263,\ 0.8367,\ 0.4185,\ 0.1994$ at $\beta = 1, 2, 4, 8$, and the lower bound of the theorem $-\ln\tanh\beta$ is $0.2723,\ 0.0366,\ 0.00067,\ 2.25\times10^{-7}$ Reading (floating point).
- This lower bound agrees exactly with "the maximum form for containers of side 1", $\sigma_{\max}(1)$, in Table 1 of 25-03. This is because in a container of side 1 only the z of one plaquette is inside, $\mathbb{E}[z_p \mid V] = \tanh(\frac{\beta}{2}\operatorname{tr} V)$, and its maximum is $\tanh\beta$ Proved here (25-03 § 07). The slab theorem has the same form as the bound "containers one unit thick, bounded by the maximum over frames".

Measurements of Wilson loops on four-dimensional slabs (Creutz ratios, comparison with and without twist) are placed in the chapter of the measured record, 25-08.

06

Where it stands relative to earlier literature

The method of the Slab Confinement Theorem (averaging, as a group, the signs on central sheets winding the compact directions) is of the same lineage as the vortex free-energy arguments of Mack–Petkova and Tomboulis–Yaffe, and amounts to bounding directly by the action the free energy of a container $\{p\}\times T^2$ one unit thick Cited (the correspondence of lineage is by comparison with the framework of Kovács–Tomboulis hep-lat/0108017 § 2– § 3). Since the idea is classical, the same bound may already exist in the literature, or as folklore. Two comparisons remain.

- Arguments of Tomboulis–Yaffe type (Commun. Math. Phys. 100 (1985) 313): inequalities that bound a Wilson loop by a ratio of twisted partition functions (as restated in Kanazawa arXiv:0808.3442). We have not checked whether a bound of the same form on a slab is written out explicitly.
- Borgs (Nucl. Phys. B 261 (1985), the area law of spatial Wilson loops in lattice gauge theory at high temperature): recorded as work that showed, by a cluster expansion, an area law for spatial Wilson loops at sufficiently high temperature. This book has not checked the text of the original source (bibliographic data from secondary records). The expected differences from the Slab Confinement Theorem are three —

non-abelian $SU(2)$, every β with N fixed, and an explicit and elementary bound — but the comparison is not done.

Within this book, the Slab Confinement Theorem reappears in the next chapter as a special case (thin insertion) of the insertion identities (25-05 § 06).

07

What cannot be said

(a) The continuum limit. The lower bound of (2) becomes 0 as $a \to 0$. The Slab Confinement Theorem says nothing about a tension uniform in the lattice spacing.

(b) Four-dimensional space. The Slab Confinement Theorem is a statement with the size N of the T^2 fixed, and the lower bound vanishes as $N \to \infty$ (approaching four-dimensional space). It says nothing about confinement in four dimensions. The stage of widening the T^2 is the crossover band (25-09, 26-01).

(c) Improvement to thick vortices. "Bounding F independently of N with a map of thick vortices" in § 04 is a provisional strand; there is no proof, including of the uniform bound on the cost of insertion.

(d) Earlier literature. The comparison with the two items of § 06 is not done. No novelty is claimed for the results of this chapter.

(e) Lean. The machine-checking column is only the algebra of finite sums (the group-average identity, the tanh bound, the one-sheet inequality). The Haar measure of $SU(2)$, the coclosed z, and the topology of sectors and the projection (steps 1, 2 and 5) are outside Lean.

(f) Existence of the limit. The existence of the limiting measure as $M \to \infty$ is not treated in this chapter; the statement is an upper bound at all limit points.

08

The table of five grades and the dependency graph

Table 1 The rows of this chapter ("—" is a row without machine checking)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The sector-summed ensemble is the sum of the sector partition functions; the correspondence of Wilson loops (move 1)	Proved here	§ 03	$SU(2)$, finite lattice	—
2	The vortex sheets S_p are coclosed and do not meet each other (move 2)	Proved here	§ 03	Slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$	—
3	Under a flip W is multiplied by ε_p and the weight by $r_p \in [e^{-2\beta N^2}, e^{2\beta N^2}]$ (move 3)	Proved here	§ 03	Wilson action	—
4	Group average (3) and $\lvert\mathbb{E}[W]\rvert \le t^A$ (4)	Proved here	§ 03 move 4	Sector-summed ensemble	SlabAverage (`SlabAverage.average_bound`), SlabConfine (`SlabConfine.group_average`, `SlabConfine.tanh_bound`, `SlabConfine.one_sheet`) (hypotheses as at the end of § 03)
5	Projection to a fixed sector (5)(6)	Proved here	§ 03 move 5	The group acts on all plaquettes of the plane	—
6	Slab Confinement Theorem: (1) and $\sigma \ge -\ln\tanh(\beta N^2) \ge 2e^{-2\beta N^2}$	Proved here	§ 02, § 03	$SU(2)$, Wilson action, N fixed, lattice spacing fixed, every β	— (only the finite-sum core, row 4)
7	Container form (7), $\sup_\Phi \theta_Q \le \tanh(\beta N^2)^{\ell^2}$ (8)	Proved here	§ 04	Container of side $\ell \times \mathrm{T}^2$	—
8	In the continuum limit ($L = Na$ fixed, $a \to 0$) the lower bound is 0	Proved here	§ 04	That β does not decrease as $a \to 0$	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
9	The cost of a physical vortex is of order $\pi^2\beta$, independent of N	Reading (physics)	25-02	Dilute region, semiclassical	—
10	A map inserting thick vortices gives a lower bound uniform in the lattice spacing	Assumption	Strand: the form (7) of § 04, and 25-05, 25-06	Under ultraviolet control	—
11	Applied to two dimensions, $I_2(\beta)/I_1(\beta) \le \tanh\beta$; agrees with the maximum form for containers of side 1	Proved here	§ 05	Two-dimensional $SU(2)$	—
12	The numbers of § 05	Reading	§ 05	Floating point	—
13	The method is of the lineage of the vortex free energy of Mack–Petkova and Tomboulis–Yaffe	Cited	Comparison with Kovács–Tomboulis § 2– § 3	—	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is the algebra of sums over a finite set of configurations; moves 1, 2 and 5 and the Haar integral are outside Lean (23-13).

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  summed = sum of sector partition fns; W matches   SU(2), finite latti
 2  PRV  vortex sheets S_p coclosed, disjoint              slab
 3  PRV  W times ε_p, weight times r_p, |ln r_p| ≤ 2βN²
 4  PRV  group average (3)(4) ⟸ 2, 3                       Lean average_bound
 5  PRV  to a fixed sector (5)(6) ⟸ 1, 4
 6  PRV  Slab Confinement Theorem (1)(2) ⟸ 5                N fixed, a fixed,
 7  PRV  container form (7)(8) ⟸ 3
 8  PRV  lower bound 0 in continuum limit ⟸ 6              L = Na fixed
 9  RDG  cost of physical vortex π²β (physics)             dilute, semiclassic
10  ASM  uniform bound with thick vortices ⟸ 7, 9           under ultraviolet
11  PRV  to two dimensions; matches maximum form ⟸ 6
12  RDG  numbers of §05 ⟸ 11                                floating point
13  CIT  lineage of vortex free energy
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The premises of the Slab Confinement Theorem are only the rows of the five moves of § 03, and the assumption row (uniform bound with thick vortices) enters as a premise of no row. Its conclusion is "a lower bound uniform in the lattice spacing", and does not contain the quantity of the Slab Confinement Theorem (a lower bound with N fixed).

The machine-checked theorems on the vortex side are in Part 21.

What this chapter can say

What is said	What is not said
Proved here For the $SU(2)$ Wilson action on the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, with N and the lattice spacing fixed, $\sigma \geq -\ln \tanh(\beta N^2) \geq 2e^{-2\beta N^2}$ at every β and in every twisted sector (the Slab Confinement Theorem)	A tension uniform in the lattice spacing. The tension when the T^2 is widened (the crossover band). Novelty (comparison not done)
Proved here The container form $\sup_\Phi \theta Q \leq \tanh(\beta N^2)^{\ell^2}$; that the lower bound becomes 0 in the continuum limit	A uniform bound on the cost of inserting a thick vortex (the thick-vortex row is an assumption)
Proved here + Reading Consistency in two dimensions ($I_2/I_1 \leq \tanh\beta$, agreement with the maximum form for containers of side 1)	Measurements on four-dimensional slabs (25-08)

Sources and reproduction

Item	Kind	Source / tools
The Slab Confinement Theorem and the five moves of the proof, the container form (7)(8), application to two dimensions	Proved here	Proofs in the text, § 03– § 05
The lineage of the arguments on vortex containers and free energy	Cited	T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501), § 2– § 3 checked in the text. G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 and E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313 through that paper's citations (texts not checked)
Restatement of the Tomboulis–Yaffe inequality	Cited	T. Kanazawa, arXiv:0808.3442
The area law of spatial Wilson loops at high temperature	Comparison not done	C. Borgs, Nucl. Phys. B 261 (1985) (bibliographic data from secondary records; text not checked)
Numbers in two dimensions	Reading	Floating-point values of I_2/I_1 and $\tanh\beta$ (the same computation as 25-03 Table 1)
Lean	Lean	SlabConfine (`SlabConfine.group_average`, `SlabConfine.factor_split`, `SlabConfine.tanh_bound`, `SlabConfine.one_sheet`), SlabAverage (`SlabAverage.average_bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — slab, the vortex sheet S_p, the group of flips, group averaging, the cost of thin vortices, the bias of a container θ_Q, the Slab Confinement Theorem. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-05 Vortex-insertion identities — if the flip of a thin sheet is replaced by a general "bijection that flips the sector", what can be said unconditionally.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the insertion ratio, the tanh form, second moments · Kiichi, Shiori and Rin

Vortex-insertion identities

This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 1-08.

What this chapter shows — In 25-04 we obtained slab confinement by averaging over maps that flip thin vortex sheets. In this chapter we list what can be said **unconditionally** when that map is replaced by a general "bijection that flips the sector" T (a map that inserts one vortex). We use only three things: a finite configuration space, positive weights, and a bijection that flips the sign of the sector. From these follow (1) the mean of the insertion ratio ρ is 1, and $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$; (2) the bias of the sector is bounded by the fluctuation of the insertion ratio: $4\mathbb{E}[\sigma]^2 \le \mathrm{Var}(\bar{\rho})$; (3) passing to an involution, a bounded form $\mathbb{E}[H] = \mathbb{E}[H\tanh(\Delta S/2)]$ and a form averaged over positions; (4) the second-moment form and the Paley–Zygmund form that follow from $\mathbb{E}[H\sqrt{\bar{\rho}}] = 0$. All are elementary identities and Cauchy–Schwarz, and their finite-sum versions are checked in Lean.

Prerequisite chapters — 25-03 (the sector-summed ensemble, the sign of the sector, the bias), 25-04 (slab confinement). From Part 1, 1-08 (probability and measure, Cauchy–Schwarz).

Said first — Every theorem of this chapter is a statement under three conditions — a finite configuration space, positive weights, and a bijection that flips the sector (a finite version of a Haar-preserving map) — and the row of each theorem states those conditions. To pass to the continuous configuration space of $SU(2)$ one must replace the sums by Haar integrals and have the map preserve the measure (or put the Jacobian into the weight), and that part is outside Lean. The results are elementary, and comparison with earlier literature is not done. Nothing about confinement follows from these identities alone. What is missing is written in § 07 and § 08.

The order of this chapter

1. Summary — the balance of a map that inserts one vortex
2. The setting — three conditions
3. The Mean Insertion-Ratio Identity, and the bias estimate by fluctuation

4. Bounded forms — the Bounded Insertion Identity, the fluctuation relation, the average over positions
5. Second-moment forms — the square-root identity, the Paley–Zygmund form, many cells
6. Rederiving slab confinement
7. What it means, and what is missing — Table 1
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — the balance of a map that inserts one vortex

1. In the sector-summed ensemble (25-03), whether a vortex passes through a region is expressed by a sign $\sigma = \pm 1$. A map T that inserts (or removes) one vortex flips σ.
2. We call the ratio of weights when a vortex is inserted in a configuration x the insertion ratio $\rho(x)$. The smaller ρ is than 1, the more expensive it is to insert a vortex.
3. The mean of the insertion ratio is always exactly 1 (the Mean Insertion-Ratio Identity). This is nothing but a balance equation: "relabelling configurations by a map does not change the total sum of weights". From the same balance also follows $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$ Proved here .
4. Combining the two, the bias of the sector $|\mathbb{E}[\sigma]|$ is at most half the square root of the fluctuation (variance) of the insertion ratio (the Bias Estimate by Insertion-Ratio Fluctuation). If one inserts at many positions and averages the insertion ratio, the fluctuation can become small. Averaging over positions plays the role of entropy Proved here .
5. The insertion ratio can take large values (heavy tails), so we also prepare a bounded form that is easy to handle. $\tanh(\Delta S/2)$ always lies in $[-1, 1]$, and $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ holds (the Bounded Insertion Identity). $1 - \mathbb{E}[\tanh]$ is twice the mean acceptance rate of a Monte Carlo move that inserts a vortex (the Insertion Fluctuation Relation Lemma) Proved here .
6. The form using $\sqrt{\rho}$ has a second moment that is always at most 1, so there is no worry about tails (the Square-Root Insertion Identity, the Paley–Zygmund-Type Bias Estimate) Proved here .
7. Taking the flip of a thin vortex sheet (25-04) as the insertion, the slab-confinement theorem comes out as a special case of this chapter (§ 06).

8. These are not a proof of confinement. An area law needs the bias to be small in many regions at once (large deviations) and a bound on the cost of insertion uniform in the lattice spacing (the ultraviolet), and both are outside this chapter.

02

The setting — three conditions

Throughout this chapter we place the following three.

- **Finite configuration-space condition** X (a finite set).
- **Positive-weight condition** $w : X \to (0, \infty)$. The expectation is $\mathbb{E}[f] = \sum_x w(x) f(x) / \sum_x w(x)$.
- **Sector-flipping bijection condition**: a sign of the sector $\sigma : X \to \{\pm 1\}$ and a bijection $T : X \to X$ with $\sigma(Tx) = -\sigma(x)$.

The sector-flipping bijection condition is a finite version of "a map that preserves the Haar measure and flips the sign of the sector of some region" in the sector-summed ensemble (for example, the composite of flipping the z of a vortex sheet with a smooth left multiplication supported inside the region). Being an involution, commuting with one another, and worst-case bounds are not used in § 03. To pass from the finite configuration-space condition to the continuous configuration space $(SU(2)^{\text{links}}$ and coclosed z), one must replace the sums by Haar and counting integrals, and T must preserve the measure (if not, the Jacobian must be put into the weight). The proofs go through with the same relabelling of variables, but that version is outside Lean.

We set the insertion ratio to be

$$\rho(x) = \begin{cases} w(Tx)/w(x) & (\sigma(x) = +1) \\ w(T^{-1}x)/w(x) & (\sigma(x) = -1) \end{cases} \quad (1)$$

The orientation is chosen so that a vortex is inserted by T into configurations with $\sigma = +1$, and configurations with $\sigma = -1$ are taken back by T^{-1}. In terms of the action $S = -\ln w$, $\rho = e^{-\Delta S}$.

03

The Mean Insertion-Ratio Identity, and the bias estimate by fluctuation

Mean Insertion-Ratio Identity

Under the finite configuration-space condition, the positive-weight condition and the sector-flipping bijection condition,

$$\mathbb{E}[\rho] = 1, \qquad \mathbb{E}[\rho\,\sigma] = -\mathbb{E}[\sigma]. \tag{2}$$

Proved here Proof. Set $X_\pm = \{\sigma = \pm 1\}$. T maps X_+ to X_- and X_- to X_+, and since it is a bijection, $T(X_+) = X_-$ and $T^{-1}(X_-) = X_+$. Hence

$$\sum_{x \in X_+} w(x)\rho(x) = \sum_{x \in X_+} w(Tx) = \sum_{y \in X_-} w(y), \qquad \sum_{x \in X_-} w(x)\rho(x) = \sum_{x \in X_-} w(T^{-1}x) =$$

Adding gives $\sum w\rho = \sum w$, and multiplying by σ and adding gives $\sum w\rho\sigma = \sum_{X_-} w - \sum_{X_+} w = -\sum w\sigma$. ∎

Lean InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`) (hypotheses: X is finite, $w > 0$, T is a bijection of X, $\sigma(Tx) = -\sigma(x)$, $\sigma = \pm 1$. Conclusion: $\sum w\rho = \sum w$, $\sum w\rho\sigma = -\sum w\sigma$). No involution, commutativity or measure structure is used.

Bias Estimate by Insertion-Ratio Fluctuation (the bias of the sector is bounded by the fluctuation of the insertion ratio)

Under the finite configuration-space condition and the positive-weight condition, take a sign of the sector σ and a non-empty finite family $T_1, \dots, T_n$ of bijections that flip it (each T_k satisfying the sector-flipping bijection condition), and let $\bar\rho = \frac{1}{n}\sum_k \rho_k$ be the mean insertion ratio. Then

$$4\,\mathbb{E}[\sigma]^2 \;\le\; \mathbb{E}\big[(\bar\rho - 1)^2\big] = \mathrm{Var}(\bar\rho), \qquad \text{that is} \quad |\mathbb{E}[\sigma]| \le \tfrac{1}{2}\sqrt{\mathrm{Var}(\bar\rho)}. \tag{3}$$

Proved here Proof. Applying the Mean Insertion-Ratio Identity to each T_k and averaging gives $\mathbb{E}[\bar\rho] = 1$, $\mathbb{E}[\bar\rho\sigma] = -\mathbb{E}[\sigma]$. Hence $\mathbb{E}[(\bar\rho - 1)\sigma] = -2\mathbb{E}[\sigma]$. From Cauchy–Schwarz and $\sigma^2 = 1$, $4\mathbb{E}[\sigma]^2 \le \mathbb{E}[(\bar\rho - 1)^2]\mathbb{E}[\sigma^2] = \mathrm{Var}(\bar\rho)$. ∎

Lean InsertionIdentity (`Insertion.sector_bias_le`) (hypotheses: X is finite, the index set of the family is finite and non-empty, $w > 0$, each T_k is a bijection with $\sigma(T_k x) = -\sigma(x)$, $\sigma = \pm 1$. Conclusion: $4(\sum w\sigma)^2 \le (\sum w)\sum w(\bar\rho - 1)^2$).

To apply it to containers (25-03), use the Bias Estimate by Insertion-Ratio Fluctuation on the conditional measure with the frame fixed. A bijection that moves only the inside of a container does not change the frame, so $\theta_j(\Phi) = |\mathbb{E}[\sigma_j \mid \Phi]| \le \frac{1}{2}\sqrt{\mathrm{Var}(\bar\rho \mid \Phi)}$ Proved here . The reading is that the factor $\tanh(|\Delta F_j|/2)$ of the Container Identity is bounded from above by the fluctuation of the mean insertion ratio of insertions at many positions inside the container.

04

Bounded forms — tanh and the average over positions

The insertion ratio $\rho = e^{-\Delta S}$ can take large values, so $\mathrm{Var}(\bar{\rho})$ can be large. We therefore switch to a bounded factor. From the T of the sector-flipping bijection condition we build the map oriented by the sector

$$\tau(x) = \begin{cases} Tx & (\sigma(x) = +1) \\ T^{-1}x & (\sigma(x) = -1) \end{cases} \quad (4)$$

and τ is an **involution** ($\tau \circ \tau = \mathrm{id}$) (if $\sigma(x) = +1$, then $\tau x = Tx \in X_-$, so $\tau(\tau x) = T^{-1}Tx = x$; the converse is the same) Proved here . We set $\Delta S = S \circ \tau - S$, $\rho = w \circ \tau / w = e^{-\Delta S}$, and

$$t(x) := \frac{w(x) - w(\tau x)}{w(x) + w(\tau x)} = \tanh \frac{\Delta S(x)}{2} \in (-1, 1) \quad (5)$$

with these definitions.

Bounded Insertion Identity

Under the finite configuration-space condition and the positive-weight condition, if τ is an involution of X and H is a function odd under τ ($H \circ \tau = -H$), then $\mathbb{E}[H] = \mathbb{E}[H\,t]$.

Proved here Proof. Since $1 - t = 2w \circ \tau / (w + w \circ \tau)$, $\mathbb{E}[H] - \mathbb{E}[Ht]$ is proportional to $2 \sum_x g(x) / \sum w$ with $g(x) = w(x)H(x)w(\tau x)/(w(x) + w(\tau x))$. $g(\tau x) = -g(x)$ (H is odd and the rest is symmetric under $x \leftrightarrow \tau x$), so relabelling the sum by τ gives $\sum g = -\sum g$, hence 0. ∎

Lean BoundedInsertion (`Bounded.single`) (hypotheses: X is finite, τ is a bijection with $\tau \circ \tau = \mathrm{id}$, $w > 0$, $H \circ \tau = -H$. Conclusion: $\sum wHt = \sum wH$).

Insertion Fluctuation Relation Lemma (the fluctuation relation and the acceptance rate)

In the setting of the Bounded Insertion Identity, for every real s, $\mathbb{P}(\Delta S = s) = e^{s}\,\mathbb{P}(\Delta S = -s)$. In particular $\mathbb{E}[t] \geq 0$, and

$$1 - \mathbb{E}[t] = 2\,\mathbb{E}\left[\frac{1}{1 + e^{\Delta S}}\right] = 2 \times (\text{mean Barker acceptance rate of an insertion move}). \quad (6$$

Proved here Proof. Relabelling gives $\mathbb{E}[g] = \mathbb{E}[(g \circ \tau)\,\rho]$ for any g. Since $\Delta S \circ \tau = -\Delta S$, taking $g = \mathbf{1}\{\Delta S = s\}$ gives $g \circ \tau = \mathbf{1}\{\Delta S = -s\}$, and on the set $\{\Delta S = -s\}$ we have $\rho =$

$e^{-\Delta S} = e^{s}$, which gives the first formula. Pairing s with $-s$ gives $\mathbb{P}(s)\tanh\frac{s}{2} + \mathbb{P}(-s)\tanh\frac{-s}{2} = \mathbb{P}(-s)(e^{s}-1)\tanh\frac{s}{2} \geq 0$, so $\mathbb{E}[t] \geq 0$. Finally $1 - \tanh(\Delta S/2) = 2/(1+e^{\Delta S})$. ∎ This lemma has no Lean check.

How to read (6). A Barker-type move is a Monte Carlo move that "accepts the configuration with a vortex inserted with probability $w(\tau x)/(w(x)+w(\tau x))$". If insertions are almost always refused (the cost is high), $\mathbb{E}[t] \approx 1$; if they pass often, $\mathbb{E}[t]$ moves away from 1. The first formula has the same form as what non-equilibrium statistical mechanics calls Crooks' fluctuation relation (comparison with the original source, and the version driven by dynamics, are in 25-06).

Many cells and the average over positions. For each of mutually separated cells c, take insertion involutions $\tau_{c,p}$ at positions $p \in P$ in the cell (P a non-empty finite set), build $t_{c,p}$ by (5), and set $\bar{t}_c = \frac{1}{|P|}\sum_p t_{c,p}$.

Position-Averaged Insertion Estimate (a bounded estimate averaged over positions)

Under the finite configuration-space condition and the positive-weight condition, assume that each $\tau_{c,p}$ is an involution, H is odd under every $\tau_{c,p}$, and the factors of other cells do not move ($t_{d,q}\circ\tau_{c,p} = t_{d,q}$ if $c \neq d$). Then

$$\mathbb{E}[H] = \mathbb{E}\Big[H\prod_c \bar{t}_c\Big], \qquad |H| \leq 1 \text{ implies } |\mathbb{E}[H]| \leq \mathbb{E}\Big[\prod_c |\bar{t}_c|\Big]. \tag{7}$$

Proved here Proof. First, for $\prod_c t_{c,p_c}$ with one position chosen from each cell, an induction adding cells one at a time gives $\mathbb{E}[H\prod_c t_{c,p_c}] = \mathbb{E}[H]$: at the step adding cell d, the product so far $G = \prod_{c\neq d} t_{c,p_c}$ does not move under τ_{d,p_d}, so HG is odd and the Bounded Insertion Identity applies. Expanding $\prod_c \bar{t}_c$ gives the average over the $|P|^{\#\text{cells}}$ choices, hence the first formula. Taking absolute values with $|H| \leq 1$ gives the second. ∎ Each factor lies in $[-1,1]$, and no moment assumptions are needed.

Lean BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (hypotheses: X, the set of cells and the set of positions P are finite, P is non-empty, every $\tau_{c,p}$ is an involution, $w > 0$, H is odd under every $\tau_{c,p}$, $t_{d,q}\circ\tau_{c,p} = t_{d,q}$ if $c \neq d$, and for the latter also $|H| \leq 1$. Conclusion: (7)). One step of the induction is BoundedInsertion (`Bounded.multi`).

05

Second-moment forms

The tanh form is bounded, but it does not capture the exponential average (the free energy) as it is. Another bounded form is $\sqrt{\rho}$.

Square-Root Insertion Identity (the Bhattacharyya form)

Under the finite configuration-space condition and the positive-weight condition, if τ is an involution, $H \circ \tau = -H$ and $\rho = w \circ \tau / w$, then $\mathbb{E}[H\sqrt{\rho}] = 0$.

Proved here Proof. $w\sqrt{\rho} = \sqrt{w \cdot w \circ \tau}$ is symmetric under $x \leftrightarrow \tau x$ and H is antisymmetric, so $\sum_x w(x)H(x)\sqrt{\rho(x)}$ changes sign when relabelled by τ, and is 0. ∎

Lean SecondMoment (`SecondMoment.sqrt_identity`) (hypotheses: X is finite, τ is a bijection with $\tau \circ \tau = \mathrm{id}$, $w > 0$, $H \circ \tau = -H$. Conclusion: $\sum wH\sqrt{w \circ \tau / w} = 0$).

Paley–Zygmund-Type Bias Estimate

Under the finite configuration-space condition and the positive-weight condition, take an involution τ_p for each p of a non-empty finite set P, and let H be odd under every τ_p with $|H| \le 1$. Setting $Y := \frac{1}{|P|}\sum_p \sqrt{\rho_p}$ (a weighted count of insertions that flip the sector),

$$\mathbb{E}[H]^2\,\mathbb{E}[Y^2] \;\le\; \mathbb{E}[Y^2] - \mathbb{E}[Y]^2, \qquad \text{that is} \quad \mathbb{E}[H]^2 \le 1 - \frac{\mathbb{E}[Y]^2}{\mathbb{E}[Y^2]}, \qquad \text{and} \;\; \mathbb{E}[Y^2] \le 1.$$

Proved here Proof. Averaging the Square-Root Insertion Identity gives $\mathbb{E}[HY] = 0$. Setting $C = \mathbb{E}[Y^2]$, $B = \mathbb{E}[Y]$ gives $\mathbb{E}[H(C - BY)] = C\,\mathbb{E}[H]$. From Cauchy–Schwarz and $H^2 \le 1$, $C^2\mathbb{E}[H]^2 \le \mathbb{E}[(C - BY)^2] = C^2 - 2CB^2 + B^2C = C(C - B^2)$. Dividing by $C > 0$ ($\rho_p > 0$) gives the first formula. The last inequality follows from $Y^2 \le \frac{1}{|P|}\sum_p \rho_p$ (Cauchy–Schwarz) and, for involutions, $\mathbb{E}[\rho_p] = \sum_x w(\tau_p x)/\sum w = 1$. ∎

Lean SecondMoment (`SecondMoment.pz_bound`) (hypotheses: X and P are finite, P is non-empty, every τ_p is an involution, $w > 0$, H is odd under every τ_p, $|H| \le 1$. Conclusion: $(\sum wH)^2(\sum wY^2) \le (\sum w)\big((\sum wY^2)(\sum w) - (\sum wY)^2\big)$, which normalised is the first formula of (8)). $\mathbb{E}[Y^2] \le 1$ has no Lean check.

(8) says that the more concentrated Y is in the sense of second moments (the closer $\mathbb{E}[Y^2]$ is to $\mathbb{E}[Y]^2$), the smaller the bias. The second moment is always at most 1, so there is no worry about heavy tails as in the variance form of ρ.

Many cells. We also place a form that handles products of insertion ratios over cells.

Many-Cell Second-Moment Estimate

Under the finite configuration-space condition and the positive-weight condition, with $|H| \le 1$, take a function $r_c : X \to \mathbb{R}$ (an averaged insertion ratio) for each c of

a finite set of cells C, and assume **oddness under insertion** — for every $S \subseteq C$, $\mathbb{E}[H \prod_{c \in S} r_c] = (-1)^{|S|}\mathbb{E}[H]$. Then

$$\mathbb{E}\Big[H \prod_{c \in C}(1 - r_c)\Big] = 2^{|C|}\,\mathbb{E}[H], \qquad 4^{|C|}\,\mathbb{E}[H]^2 \;\le\; \mathbb{E}\Big[\prod_{c \in C}(1 - r_c)^2\Big]. \tag{9}$$

Proved here Proof. Expand $\prod_c(1 - r_c) = \sum_{S \subseteq C}(-1)^{|S|}\prod_{c \in S} r_c$; using the assumption each term becomes $\mathbb{E}[H]$, and there are $2^{|C|}$ terms. Cauchy–Schwarz and $H^2 \le 1$ give the second formula. ∎

Lean MultiCell (`MultiCell.bound`) (hypotheses: X and C are finite, $w > 0$, $|H| \le 1$, the equalities of "oddness under insertion" above for every $S \subseteq C$. Conclusion: the second formula of (9)) and MultiCell (`MultiCell.expand_identity`) (the first formula). Within Lean, "oddness under insertion" is **an equality placed as a hypothesis**. For a single bijection it follows from the pushforward equality $\sum w\,(w \circ T/w)\,F = \sum w\,F \circ T^{-1}$ (MultiCell (`MultiCell.pushforward`), hypotheses: finite, $w > 0$), but that the equality holds when insertions in separate cells are composed depends on the commutativity of the maps and the invariance of the ratios, and is outside Lean.

06

Rederiving slab confinement

Take the thin insertion of 25-04. Let the cells be the plaquettes $p \in S(C)$ of the plane inside the loop, with one position ($|P| = 1$), and let τ_p be the flip of z on the vortex sheet S_p. τ_p is itself an involution, and since it does not change the links it preserves the Haar $\times$ counting measure. $H = W(C)$ is odd under each τ_p, and $t_p = (1 - r_p)/(1 + r_p)$ is determined only by the values on S_p, so it does not move under τ_q for $q \neq p$. From the Position-Averaged Insertion Estimate,

$$|\mathbb{E}[W(C)]| \;\le\; \mathbb{E}\Big[\prod_{p \in S(C)} |t_p|\Big] \;\le\; \tanh(\beta N^2)^A \tag{10}$$

($|t_p| \le \tanh(\beta N^2)$ is the one-sheet inequality of 25-04) Proved here . This is exactly (4) of move 4 of 25-04, and the projection to a fixed sector is the same as move 5 of 25-04. The Slab Confinement Theorem is the special case "thin insertion, one position" of the Position-Averaged Insertion Estimate. (10) on the continuous configuration space is the same argument with sums replaced by integrals, and is outside Lean.

07

What it means, and what is missing

The average over positions is entropy. The $\mathrm{Var}(\bar{\rho})$ of the Bias Estimate by Insertion-Ratio Fluctuation is expected to be of order $\approx VK/n$, where K is the number of

positions, among the n, within reach of each other's correlations and V is the variance of one insertion ratio Reading (physics). Increasing the number of positions n plays the role of entropy, and the decay of correlations between positions (the mass gap) plays the role of keeping K small. For insertions of the smooth fractional-instanton type, $V \sim e^{S_{\text{inst}}}$ is expected, the area of the region needed is proportional to V, and the string tension is of order $\sigma \sim e^{-S_{\text{inst}}}/L^2$, the same form as the formula for a dilute vortex gas Reading (physics, a semiclassical estimate; not a proof).

Table 1 Comparison of the three forms (factor per cell)

Form	Factor	Strengths	Weaknesses
Variance of the insertion ratio (Bias Estimate by Insertion-Ratio Fluctuation)	$1-\bar{\rho}$	Captures the free energy (the exponential average) as it is	Heavy tails. The second moment can be large
tanh (Position-Averaged Insertion Estimate)	$\bar{t}_C$	Bounded. Meaning as an acceptance rate (Insertion Fluctuation Relation Lemma)	Not an exponential average, so it captures the entropy of vortex shapes poorly
$\sqrt{\rho}$ (Paley–Zygmund-Type Bias Estimate)	Y	Second moment always at most 1, and it can capture the entropy of shapes too	Needs bounds on the overlaps between positions and shapes

Collapsing onto a chessboard. If there is reflection positivity in the directions of the plane, chessboard estimates collapse things into a single translation-invariant quantity, $\mathbb{E}[\prod_{c\in\text{inside}} |\bar{t}_c|] \le \Lambda^n$ with $\Lambda := (\mathbb{E}[\prod_{\text{all cells}} |\bar{t}_c|])^{1/\#\text{cells}}$, and if $\Lambda < 1$ an area law follows; this is a strand Assumption (where the evidence is: the treatment of reflection positivity is in 26-04 and 26-05. The same structure is in (13)(21) of Kovács–Tomboulis Cited).

What is missing. (i) The large-deviation stage: $\Lambda < 1$ needs the event that insertions are often refused in all cells at once to be exponentially rare in the number of cells. (ii) The ultraviolet stage: the lower bound on the acceptance rate must be uniform in the lattice spacing, and with thin flips (25-04) it gets worse with N. With smooth insertions, it is known that fixed maps pay an extra cost in the ultraviolet (25-07). (iii) An upper bound on second moments: an upper bound on the V of the Bias Estimate by Insertion-Ratio Fluctuation. This book shows none of the three.

A negative control. Four-dimensional $U(1)$ does not confine at weak coupling, so if the bounds of this chapter are applied there, they must stop working somewhere (for example, V becoming large). This check has not yet been done in this book.

> 08

What cannot be said

(a) Confinement. None of the identities and inequalities of this chapter implies confinement. (i)–(iii) of § 07 remain.

(b) The continuous configuration space. All the theorems are finite versions under the finite configuration-space condition, the positive-weight condition and the sector-flipping bijection condition; the version on the Haar measure of $SU(2)$ is the same argument with the measure preservation of the maps (or the Jacobian) added, but it is outside Lean.

(c) The hypothesis of the Many-Cell Second-Moment Estimate. The equalities of "oddness under insertion" are not shown in this chapter for maps composed of insertions in separate cells (in Lean too they are hypotheses).

(d) Novelty. All are elementary combinations of relabelling of variables and Cauchy–Schwarz, and comparison with earlier literature is not done. The summed ensemble and the container identity lie within the framework of Mack–Petkova, the form bounding Wilson loops by twisted partition functions within Tomboulis–Yaffe, and products on a chessboard within Kovács–Tomboulis. The combination of insertion averaged over positions with Cauchy–Schwarz is recorded as a standard technique in proofs of Mermin–Wagner type, and is said to appear in gauge theory in Chatterjee (arXiv:2602.00436) as well (not yet checked). No novelty is claimed for the results of this chapter.

(e) The estimates of § 07. $\mathrm{Var}(\bar{\rho}) \approx VK/n$, $V \sim e^{S_{\mathrm{inst}}}$ and $\sigma \sim e^{-S_{\mathrm{inst}}}/L^2$ are physical estimates, and none is a proof.

(f) The negative control. That the bounds stop working for four-dimensional $U(1)$ has not yet been checked.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter (conditions: the finite configuration-space condition, the positive-weight condition, the sector-flipping bijection condition)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Mean Insertion-Ratio Identity: $\mathbb{E}[\rho] = 1$, $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$	Proved here	§ 03	The finite configuration-space, positive-weight and sector-flipping bijection conditions. Uses neither involution nor commutativity	InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`) (hypotheses: the finite configuration-space, positive-weight and sector-flipping bijection conditions, $\sigma = \pm 1$)
2	Bias Estimate by Insertion-Ratio Fluctuation: $4\mathbb{E}[\sigma]^2 \leq \mathrm{Var}(\bar{\rho})$	Proved here	§ 03	The finite configuration-space and positive-weight conditions, each T_k satisfies the sector-flipping bijection condition, the family is finite and non-empty	InsertionIdentity (`Insertion.sector_bias_le`) (hypotheses: as left)
3	The τ of (4) is an involution	Proved here	§ 04	Sector-flipping bijection condition	—
4	Bounded Insertion Identity: $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$	Proved here	§ 04	The finite configuration-space and positive-weight conditions, τ an involution, H odd	BoundedInsertion (`Bounded.single`) (hypotheses: as left)
5	Insertion Fluctuation Relation Lemma: $\mathbb{E}[t] \geq 0$, the acceptance rate (6)	Proved here	§ 04	The finite configuration-space and positive-weight conditions, τ an involution	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
6	Position-Averaged Insertion Estimate: (7)	Proved here	§ 04	The finite configuration-space and positive-weight conditions, involutions, H odd, factors of other cells do not move, $\lvert H\rvert \le 1$	BoundedInsertion (`Bounded.averaged`, `Bounded.averaged_bound`) (hypotheses: as left)
7	Square-Root Insertion Identity: $\mathbb{E}[H\sqrt{\bar{\rho}}] = 0$	Proved here	§ 05	The finite configuration-space and positive-weight conditions, involution, H odd	SecondMoment (`SecondMoment.sqrt_identity`) (hypotheses: as left)
8	Paley–Zygmund-Type Bias Estimate: the first formula of (8)	Proved here	§ 05	The finite configuration-space and positive-weight conditions, a family of involutions, H odd, $\lvert H\rvert \le 1$	SecondMoment (`SecondMoment.pz_bound`) (hypotheses: as left)
9	$\mathbb{E}[Y^2] \le 1$	Proved here	§ 05	Involutions	—
10	Many-Cell Second-Moment Estimate: (9)	Proved here	§ 05	The finite configuration-space and positive-weight conditions, $\lvert H\rvert \le 1$, "oddness under insertion" assumed	MultiCell (`MultiCell.bound`, `MultiCell.expand_identity`) (hypotheses: as left. "Oddness under insertion" is a hypothesis in Lean too)
11	(10) of slab confinement via thin insertion	Proved here	§ 06	Slab, N fixed, the sector-summed ensemble	— (the core is the bundle of row 6)
12	$\mathrm{Var}(\bar{\rho}) \approx VK/n$, $\sigma \sim e^{-S_{\mathrm{inst}}/L^2}$	Reading (physics)	§ 07	Dilute region, semiclassical	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
13	The product structure of the chessboard	Cited	(13)(21) of Kovács–Tomboulis	Under reflection positivity	—
14	Collapsing onto Λ by the chessboard; an area law if $\Lambda < 1$	Assumption	Strand: § 07, 26-04, 26-05	Under reflection positivity in the directions of the plane	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is only the algebra of sums over finite sets (23-13). In every chapter Lean names are written in the form "file (namespace.theorem)". The theorems of the namespace Insertion are in the file InsertionIdentity, and those of Bounded in the file BoundedInsertion.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  E[ρ] = 1, E[ρσ] = −E[σ]                        the three conditions
 2  PRV  4E[σ]² ≤ Var(ρ̄) ⟸ 1                            Lean sector_bias_le
 3  PRV  the oriented τ is an involution                bijection condition
 4  PRV  E[H] = E[H t] ⟸ 3                               Lean Bounded.single
 5  PRV  fluctuation relation, acceptance rate ⟸ 3
 6  PRV  average over positions (7) ⟸ 4                  Lean averaged_bound
 7  PRV  E[H√ρ] = 0 ⟸ 3                                  Lean sqrt_identity
 8  PRV  Paley–Zygmund (8) ⟸ 7                           Lean pz_bound
 9  PRV  E[Y²] ≤ 1 ⟸ 1, 3
10  PRV  many cells (9) (oddness under insertion assumed)   Lean MultiCell.bou
11  PRV  slab confinement via thin insertion ⟸ 6     slab, N fixed
12  RDG  VK/n, e^(−S_inst)/L² (physics) ⟸ 2
13  CIT  product structure of the chessboard
14  ASM  collapse to Λ; area law if Λ < 1 ⟸ 6, 13   under reflection positivi
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. The only assumption row is the chessboard row (collapse to Λ; area law if $\Lambda < 1$), and it enters as a premise of no row. The "oddness under insertion" of the many-cell estimate is a hypothesis within the statement of the theorem, and does not come from another row. This chapter has neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Under the finite configuration-space, positive-weight and sector-flipping bijection conditions, $\mathbb{E}[\rho] = 1$, $\mathbb{E}[\rho\sigma] = -\mathbb{E}[\sigma]$, $4\mathbb{E}[\sigma]^2 \leq \mathrm{Var}(\bar{\rho})$	An upper bound on $\mathrm{Var}(\bar{\rho})$ (a bound on the second moment V)
Proved here For involutions and odd H: the tanh form, the fluctuation relation, the average over positions, the $\sqrt{\rho}$ form, the Paley–Zygmund form	The large-deviation stage (rarity of the event of simultaneous refusal in all cells). The ultraviolet stage (a lower bound on the acceptance rate uniform in the lattice spacing)
Proved here The slab-confinement theorem is a special case of thin insertion	Confinement when the T^2 is widened. The negative control of four-dimensional $U(1)$
Assumption The strand of collapsing onto a single quantity Λ by the chessboard	A proof of $\Lambda < 1$

Sources and reproduction

Item	Kind	Source / tools
The Mean Insertion-Ratio Identity, the Bias Estimate by Insertion-Ratio Fluctuation, the Bounded Insertion Identity, the Insertion Fluctuation Relation Lemma, the Position-Averaged Insertion Estimate, the Square-Root Insertion Identity, the Paley–Zygmund-Type Bias Estimate, the Many-Cell Second-Moment Estimate, (10)	Proved here	Proofs in the text, § 03– § 06. The same content as the theorems of the draft paper "Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds" (the paper on vortex-insertion identities in A-07)
The summed ensemble, the container identity	Cited (indirect)	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117 (as in 25-03; text not checked)
The product structure of the chessboard (13)(21)	Cited	T. G. Kovács, E. T. Tomboulis, arXiv:hep-lat/0108017 (Phys. Rev. D 65 (2002) 074501)
The combination of the average over positions with Cauchy–Schwarz (an example in gauge theory)	Comparison not done	S. Chatterjee, arXiv:2602.00436 (bibliographic data from a checked record; the correspondence of theorems not compared)

Item	Kind	Source / tools
Lean	Lean	InsertionIdentity (`Insertion.mean_one`, `Insertion.mean_rho_sigma`, `Insertion.sector_bias_le`), BoundedInsertion (`Bounded.single`, `Bounded.multi`, `Bounded.averaged`, `Bounded.averaged_bound`), SecondMoment (`SecondMoment.sqrt_identity`, `SecondMoment.pz_bound`), MultiCell (`MultiCell.pushforward`, `MultiCell.expand_identity`, `MultiCell.bound`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — insertion map, the insertion ratio ρ, the involution oriented by the sector τ, the tanh factor t, acceptance rate, the average over positions, the $\sqrt{\rho}$ form, the Paley–Zygmund form, oddness under insertion. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-06 Driven insertion and the flow lemma — if, instead of applying the map at once, the vortex is inserted slowly by heat-bath dynamics, the Bounded Insertion Identity takes the form of the work W.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space driven insertion, Crooks and Jarzynski, the flow lemma · Kiichi, Shiori and Rin

Driven insertion and the flow lemma

This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 2-08.

What this chapter shows — In 25-05 we obtained $\mathbb{E}[H] = \mathbb{E}[H \tanh(\Delta S/2)]$ from a map τ that inserts a vortex at once. For thin vortices the cost of insertion ΔS is large, proportional to the area, and the bound is almost useless. In this chapter, instead of inserting at once, we consider "driven insertion", which updates only the inside of the container by heat bath while flipping the couplings of the vortex sheet little by little, and show (1) that on the space of paths the combination of time reversal and sector flip is an involution, with weight ratio e^{-W} (W is the work along the path) (the Crooks relation); (2) that from this follow $\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$ (the driven identity) and the Jarzynski equality; (3) for the probability a that one step of a stationary kernel flips the sector, $\theta \le 1 - a$ (the flow lemma), and flip probability $\le 1 - \theta^2$; and (4) that moving containers simultaneously gives $\mathbb{E}\prod \theta \le \mathbb{P}(\text{none flips})$. Finally we record that for a one-step kernel this guarantee reaches only $\theta_0 \sim \frac{1}{2}$, and cannot be used to test the crossover band (26-01).

Prerequisite chapters — 25-03 (containers, frame, the bias of the sector θ, the Container Identity), 25-05 (the Bounded Insertion Identity, the Insertion Fluctuation Relation Lemma, the Position-Averaged Insertion Estimate), 26-01 (the correct form of the band $\ln(1/\theta(L,L)) \ge cL^2$). From Part 2, 2-08 (Gibbs measures, free energy), and from Part 5, 5-07 (heat bath, detailed balance).

Said first — All theorems are statements under "a finite configuration space, positive weights, and kernels that move only the inside of the container", and each row states those conditions. Passing to the continuous configuration space of $SU(2)$ is the same as in 25-05 § 02 (replace the sums by Haar integrals) and is outside Lean. The Crooks relation and the Jarzynski equality are known results of non-equilibrium statistical mechanics; here we write out their proofs in order to apply them to the sector-summed ensemble of the lattice. Confinement follows from none of the inequalities of this chapter.

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The order of this chapter
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1. Summary — inserting a vortex slowly
2. The setting — the driving procedure and the space of paths
3. The Crooks involution and the driven identity — the Jarzynski Equality, the acceptance rate, many cells
4. Two limits — inserting at once, inserting quasi-statically
5. The Cut-Measure Formula
6. The Flip-Probability Lemma and the Flow Lemma
7. The Simultaneous-Flip Estimate, and the limits of this guarantee (Table 1)
8. What cannot be said
9. The table of five grades and the dependency graph — Table 2

01

Summary — inserting a vortex slowly

1. The insertion of 25-05 was an operation that flips at once the signs of the couplings on a vortex sheet. Its cost ΔS is, for a thin vortex sheet, proportional to the number of plaquettes on the sheet (the area N_T^2 of the T^2), and the factor of the bound $\tanh(\Delta S/2)$ is almost 1.
2. So we split the coupling into m steps and flip it little by little, updating only the inside of the container by heat bath at each step. The surrounding fluctuations relax to fit the new coupling, so the work paid W goes down.
3. On the space of paths, the map "follow time backwards and flip the sector" is an involution, and the weight ratio is e^{-W} (the Crooks Relation) Cited . Applying the Bounded Insertion Identity of 25-05 to this involution gives $\mathbb{E}[H] = \mathbb{E}[H\,\tanh(W/2)]$ Proved here .
4. From the same relation, independently of the number of steps m of the procedure, $\mathbb{E}[e^{-W}\,|\,\text{frame}] = Z_-/Z_+$ (the Jarzynski equality) Cited . With 0 steps, $W = \Delta S$ and we are back to 25-05.
5. For any stationary kernel that flips the sector, the probability a that the sector flips in one step is at most $1-\theta$. In other words $\theta \le 1-a$ (the flow lemma) Proved here . There is also a weaker form, $a \le 1-\theta^2$ Proved here .
6. With the frame fixed the containers are independent, so running kernels in all containers simultaneously gives $\mathbb{E}\prod_t \theta_t \le \mathbb{P}(\text{no container flips})$ Proved here . The right-hand side is a quantity measurable by Monte Carlo, and it estimates an upper bound on the Wilson loop in the correct direction.

7. However, the form of the band needs $a \geq 1 - e^{-cL^{\prime}}$, that is, a kernel that brings the overlap of the forward and reverse paths to 1 exponentially in the area; the a of the one-step kernels measured is about 0.6–0.73 (acceptance $\min(1, e^{-W})$), and what this guarantee can say goes only up to $\theta \leq \theta_0$ (a constant less than 1) Reading . It cannot be used to test the form of the band, $\theta \leq e^{-cL^2}$. Its proper use is a lower bound on the slab tension with the T^2 fixed.

02

The setting — the driving procedure and the space of paths

We use the sector-summed ensemble of 25-03 and 25-04. The weight on the $SU(2)$ link variables U and the ± 1 plaquette variables z (coclosed) is $w(U, z) = \exp\left(\frac{\beta}{2} \sum_p z_p \operatorname{tr} U_p\right)$. For a plaquette p of the $(3,4)$ plane of the slab $T^2_{N_T} \times (\mathbb{Z}/M)^2$, flipping all the z of the vortex sheet $S_p = \{p\} \times T^2$ flips the sign of the sector seen by a Wilson loop enclosing p (25-04). We take a container $D_p \times T^2$ around p (D_p is a disc in the $(3,4)$ plane), and call the variables outside the container the **frame** Φ.

Throughout this chapter we place the following three.

- **Finite configuration-space condition** X. The continuous case is as noted in 25-05 § 02.
- **Positive-weight condition** $w = e^{-S} > 0$.
- **Kernel moving only inside the container**: the kernels K below update only the variables inside the container and do not change the frame.

Interpolating the coupling. For $\lambda \in [0,1]$, let S_λ be the action in which only the couplings of the plaquettes on S_p are multiplied by $1 - 2\lambda$. $S_0 = S$, and $S_1(x) = S(Rx)$ (R is the flip of z on S_p). S_λ is gauge invariant.

Kernels. Let K_λ be a heat-bath transition kernel that moves only the links inside the container and is reversible (detailed balance) with respect to e^{-S_λ}:

$$e^{-S_\lambda(x)} K_\lambda(x, y) = e^{-S_\lambda(y)} K_\lambda(y, x). \quad (1)$$

The forward procedure. $\lambda_k = k/m$ ($k = 0, \dots, m$). Starting from a configuration x_0 in sector $+$, at each step $k = 1, \dots, m$: "change the coupling $\lambda_{k-1} \to \lambda_k$, add the resulting change of action to the work, then update $x_{k-1} \to x_k$ by K_{λ_k}". The work is

$$W = \sum_{k=1}^{m} \left[S_{\lambda_k}(x_{k-1}) - S_{\lambda_{k-1}}(x_{k-1}) \right]. \quad (2)$$

Rx_m, obtained by finally applying R, is a configuration in sector $-$. From configurations in sector $-$, the same is done by the reverse procedure mapped by R (λ going $1 \to 0$). If $m = 0$ there is no update, and $W = S_1(x_0) - S_0(x_0) = \Delta S$, the single insertion of 25-05.

The space of paths. Let $\Omega = \{(s, x_0, \dots, x_m)\}$ ($s = \pm$ is the sector of the starting point), and set the weight of a path starting in sector $+$ to be

$$\hat{W}(+, x_0, \dots, x_m) = e^{-S_0(x_0)} \prod_{k=1}^{m} K_{\lambda_k}(x_{k-1}, x_k), \tag{3}$$

The weight of a path starting in sector $-$ is defined by the product of the same form for the reverse procedure. The rows of the kernels sum to 1, so projecting the measure on Ω to the starting point gives back the original measure. Hence the expectation of a quantity determined by the starting point alone does not change when computed on Ω Proved here .

The involution. $\iota(+, x_0, \dots, x_m) = (-, Rx_m, \dots, Rx_0)$: follow the path backwards and flip the sector (and conversely for paths starting at $-$). $\iota \circ \iota = \mathrm{id}$. By (1), paths of positive weight are mapped to paths of positive weight.

03

The Crooks involution and the driven identity

Crooks Relation

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), for every path ω of positive weight,

$$\frac{\hat{W}(\iota\omega)}{\hat{W}(\omega)} = e^{-W(\omega)}, \qquad W \circ \iota = -W. \tag{4}$$

Cited A relation of this form is due to Crooks (source table); here we write out the standard computation adapted to the lattice procedure. Proof. Let $\omega = (+, x_0, \dots, x_m)$. The weight of the reversed path, brought back to the coordinates of $+$ using that $S\lambda$ maps to $S_{1-\lambda}$ under R, is $e^{-S_1(x_m)} \prod_{k=1}^{m} K_{\lambda_k}(x_k, x_{k-1})$. Replacing each factor by $K_{\lambda_k}(x_{k-1}, x_k)\, e^{S_{\lambda_k}(x_k) - S_{\lambda_k}(x_{k-1})}$ using (1), the exponent of the ratio is

$$S_0(x_0) - S_1(x_m) + \sum_{k=1}^{m} \left[S_{\lambda_k}(x_k) - S_{\lambda_k}(x_{k-1}) \right] = \sum_{k=1}^{m} S_{\lambda_{k-1}}(x_{k-1}) - \sum_{k=1}^{m} S_{\lambda_k}(x_{k-1}) =$$

(shift the summation index by one, and let $S_0(x_0)$ cancel against the $k = 1$ term and $-S_1(x_m)$ against the $k = m$ term). Computing (2) along the reversed path, the same terms appear with opposite signs, so $W \circ \iota = -W$. ∎

The Crooks Relation says that "the involution with weight ratio $e^{-\Delta S}$" of (5) of 25-05 has been replaced, on the space of paths, by "an involution with weight ratio e^{-W}". So the Bounded Insertion Identity of 25-05 can be used as it is.

Driven Insertion Identity (the driven identity)

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), if $H = h(\Phi)\,\sigma$ (h is a function of the frame alone, σ is the sign of the sector of the container), then

$$\mathbb{E}[H] = \mathbb{E}_\Omega\big[H \tanh(W/2)\big]. \quad (5)$$

Proved here Proof. Regard H as a function on Ω evaluated at the starting point. The kernels and R do not change the frame, so h is invariant under ι and σ flips, giving $H \circ \iota = -H$. For the weight $\hat{W}$ and the involution ι on Ω, the factor of (5) of 25-05 is $(\hat{W} - \hat{W} \circ \iota)/(\hat{W} + \hat{W} \circ \iota) = (1 - e^{-W})/(1 + e^{-W}) = \tanh(W/2)$ by (4). From the Bounded Insertion Identity of 25-05, $\mathbb{E}_\Omega[H] = \mathbb{E}_\Omega[H \tanh(W/2)]$. The left-hand side is determined by the starting point alone, so it equals $\mathbb{E}[H]$ (§ 02). ∎

Lean: the core is BoundedInsertion (`Bounded.single`) (hypotheses: a finite set, an involution, $w > 0$, H odd. Conclusion: $\sum wHt = \sum wH$) applied to the finite space of paths Ω (restricted to paths of positive weight). The construction of the space of paths and the Crooks Relation are outside Lean.

Jarzynski Equality

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), independently of the number of steps m,

$$\mathbb{E}\big[e^{-W} \,\big|\, \Phi,\ \sigma = +\big] = \frac{Z_-(\Phi)}{Z_+(\Phi)}, \quad (6)$$

Here $Z_\pm(\Phi)$ is the partition function of sector $\pm$ with the frame Φ fixed.

Cited Due to Jarzynski (source table). Proof. From (4), $\sum_\omega \hat{W}(\omega) e^{-W(\omega)} = \sum_\omega \hat{W}(\iota\omega)$. The right-hand side is the total weight of the paths starting in sector $-$, which is $Z_-(\Phi)$ since the rows of the kernels sum to 1. Dividing the left-hand side by the total weight $Z_+(\Phi)$ of paths in sector $+$ gives (6). ∎

(6) is a quantity that does not depend on the speed of the procedure (the number of steps m), and can be used as a numerical check. On the other hand, the average of e^{-W} is dominated by rare small W, so with few samples the estimate fluctuates strongly (that is why some rows of the table in 25-08 have negative $-\ln\langle e^{-W}\rangle$).

Driven Insertion Acceptance-Rate Corollary (the acceptance rate)

In the setting of the Driven Insertion Identity, $1-\mathbb{E}_\Omega[\tanh(W/2)] = 2\,\mathbb{E}_\Omega\big[1/(1+e^{W})\big]$. The right-hand side is twice the mean acceptance rate of a move that accepts the endpoint of a driven path by the Barker-type rule.

Proved here One just takes the expectation of $1-\tanh(W/2) = 2/(1+e^{W})$. It is the Insertion Fluctuation Relation Lemma of 25-05 with ΔS replaced by W.

Many cells. For each of mutually separated cells c (containers $D_c \times T^2$), choose a position P_c uniformly and independently from a set of positions P (a non-empty finite set), and perform driven insertion at the chosen position. The path of each cell reads only its own container and its surroundings in the starting configuration.

Many-Cell Driven Estimate

Under the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container and (1), if the containers are separated from each other (no plaquette straddles the insides of two containers), $H = h(\Phi)\prod_c \sigma_c$ and $|H|\le 1$, then

$$|\mathbb{E}[H]| \;\le\; \mathbb{E}\Big[\prod_c \frac{1}{|P|}\sum_{p\in P}\mathbb{E}\big[\,|\tanh(W_{c,p}/2)|\;\big|\;x_0\big]\Big]. \tag{7}$$

Proved here Proof. Build the enlarged space consisting of the starting point x_0, the choice of positions (P_c) and the path of each cell. The involution ι_d of cell d follows the path of cell d backwards and replaces the inside of container d of the starting point by the flip of the endpoint of the path. Since the containers are separated, the weight with the frame fixed splits into a product over containers, and ι_d does not change the path weights or works of other cells. So the weight ratio of ι_d is e^{-W_d}, and the factors $\tanh(W_c/2)$ of the other cells ($c \neq d$) do not move under ι_d. The proof of the Position-Averaged Insertion Estimate of 25-05 (the induction adding cells one at a time) goes through as it is, giving $\mathbb{E}[H] = \mathbb{E}[H\prod_c \tanh(W_{c,P_c}/2)]$. Taking absolute values, and using that with the starting point fixed the paths of the cells and the choices of positions are independent, gives (7). ∎

Unlike the Position-Averaged Insertion Estimate of 25-05, (7) has the absolute value inside the average over positions. It is weaker by the amount that it does not use cancellation of signs between positions.

04

Two limits — inserting at once, inserting quasi-statically

Zero steps. At $m = 0$, $W = \Delta S$, and the Driven Insertion Identity is the Bounded Insertion Identity of 25-05 Proved here . For thin vortices ΔS is proportional to the area of the T^2, and $\tanh(W/2)$ is almost 1.

The quasi-static limit. As $m \to \infty$ (driving slowly), W is expected to approach the conditional free energy of the container with the frame fixed, $\Delta F_D(\Phi) = \ln\big(Z_+(\Phi)/Z_-(\Phi)\big)$ Reading (physics; the standard estimate of thermodynamic integration, which requires the kernel to mix well enough at each λ. Not proved in this book). The factor of the Driven Insertion Identity then becomes $\tanh(\Delta F_D/2) = \theta_D(\Phi)$, and we return to the Container Identity of 25-03 (the Mack–Petkova identity). The reading is that the length m of the procedure connects the crude bound of inserting at once with the exact conditional free energy.

The numerical picture. For $SU(2)$, $\beta = 2.3$, the slab $2 \times 2 \times 8 \times 8$, twist $n_{12} = 1$, disc radius 2, 6 configurations $\times$ 4 positions, the mean work at $m = 0, 4, 16$ steps was 10.7, 6.5, 2.9, and the mean of $\tanh(W/2)$ was 0.990, 0.968, 0.688 Reading . Increasing the number of steps lowers the work and moves the factor away from 1. The records for other β, N_T and container radii are collected in 25-08. If the container is smaller than the vortex, the work does not come down fully even as the number of steps increases (same table).

05

The formula of the cut measure

Inside the container $D \times T^2$, let μ_{cut} be the measure (with the frame Φ fixed) in which only the plaquette terms on the flip sheet $S = \{p_0\} \times T^2$ are removed from the action, and set $X := \sum_{p \in S} z_p \operatorname{tr} U_p$ (z is the representative of sector $+$). The weight on the sheet is $e^{(\beta/2) z_p \operatorname{tr} U_p}$, and in sector $-$ the z on S are flipped, so

Cut-Measure Formula

$$\frac{Z_{\pm}(\Phi)}{Z_{\mathrm{cut}}(\Phi)} = \mathbb{E}_{\mathrm{cut}}\left[e^{\pm\beta X/2}\right], \qquad \theta(\Phi) = \frac{\left|\mathbb{E}_{\mathrm{cut}}[\sinh(\beta X/2)]\right|}{\mathbb{E}_{\mathrm{cut}}[\cosh(\beta X/2)]}. \tag{8}$$

Proved here The first formula is from the definition; the second just puts the first into $\theta = |Z_+ - Z_-|/(Z_+ + Z_-)$ and uses $e^a \mp e^{-a} = 2\sinh a,\ 2\cosh a$.

Lean FlipLemma (`FlipLemma.cut_formula`) (hypotheses: real-valued weights v on a finite set (non-negativity not used either), real-valued X, a real b. Conclusion: $\sum ve^{bX/2} - \sum ve^{-bX/2} = 2\sum v\sinh(bX/2)$, and the equality in sum form for cosh). What is guaranteed is only the algebra of sums.

μ_{cut} has no sectors. $\theta(\Phi)$ can be read as "the strength with which the information of the frame reaches the sheet, crossing the cut measure over the size of the container". Driven insertion can also be set up in the symmetric form $+ \to \text{cut measure} \leftarrow -$ (compatible with the involution of the Crooks Relation).

06

The flip-probability lemma and the flow lemma

Let μ_Φ be the conditional measure with the frame Φ fixed, $\sigma \in \{\pm 1\}$ the sign of the sector of the container, $m_\Phi = \mathbb{E}[\sigma \mid \Phi]$, $\theta(\Phi) = |m_\Phi|$. Let K be a Markov kernel that moves only the inside of the container, and $(X, X') \sim \mu_\Phi \otimes K$ (X drawn from μ_Φ and advanced one step by K gives X').

Flip-Probability Lemma

If K is **stationary** with respect to μ_Φ ($\sum_x \mu_\Phi(x)K(x,y) = \mu_\Phi(y)$), then

$$\mathbb{P}\big(\sigma(X') \mathrel{/=} \sigma(X)\big) \ \le\ 1 - m_\Phi^2 = 1 - \theta(\Phi)^2. \tag{9}$$

Proved here Proof. Since $\sigma = \pm 1$, $\mathbb{P}(\sigma(X') \mathrel{/=} \sigma(X)) = \sum_{x,y} \mu_\Phi(x)K(x,y)(\sigma(x) - \sigma(y))^2 /4$. Bound it above by $(a-b)^2 \le 2(a-m)^2 + 2(b-m)^2$, and use that the rows sum to 1 and stationarity to turn both sums into $\sum_x \mu_\Phi(x)(\sigma(x) - m)^2 = 1 - m^2$, giving $\frac{1}{2}(1-m^2) + \frac{1}{2}(1-m^2)$. ∎ Reversibility is not used.

Lean FlipLemma (`FlipLemma.flip_le`) (hypotheses: a finite set, $w \ge 0$, $\sum w = 1$, $K \ge 0$, rows sum to 1, stationarity $\sum_x w_x K_{xy} = w_y$, $\sigma^2 = 1$. Conclusion: $\sum_{x,y} w_x K_{xy}(\sigma_x - \sigma_y)^2/4 \le 1 - (\sum w\sigma)^2$).

Flip-Probability Corollary for Positive Kernels

If K is reversible with respect to μ_Φ and positive as an operator on $L^2(\mu_\Phi)$ ($\langle f, Kf\rangle \geq 0$), the right-hand side of (9) drops to $(1-\theta^2)/2$.

Proved here Proof. $\mathbb{P}(\text{flip}) = (1 - \langle\sigma, K\sigma\rangle)/2$ (inner product in μ_Φ). Setting $\tilde\sigma = \sigma - m$, since the rows of K sum to 1 and it is reversible, $\langle\sigma, K\sigma\rangle = \langle\tilde\sigma, K\tilde\sigma\rangle + m^2 \geq m^2$. ∎ This corollary has no Lean check.

The next form is sharper, and is the one mainly used in this chapter.

Flow Lemma

In the setting of the Flip-Probability Lemma (stationary K), letting $a(\Phi)$ be the probability that the sector flips in one step,

$$a(\Phi) \;\leq\; 1 - \theta(\Phi), \qquad \text{that is} \qquad \theta(\Phi) \;\leq\; 1 - a(\Phi). \tag{10}$$

Proved here Proof. Take the flow $A = \mathbb{P}(\sigma(X) = +, \sigma(X') = -)$ from $+ \to -$ and the flow B from $- \to +$. By stationarity the distribution of the sector of X' is the same as that of X, so $A = B$. Also $A \leq \mathbb{P}(\sigma(X) = +) = (1+m)/2$, $B \leq (1-m)/2$. Hence $a = A + B = 2A \leq 2\min\big((1+m)/2, (1-m)/2\big) = 1 - |m|$. ∎ Since $\theta \leq 1$, $1 - \theta \leq 1 - \theta^2$, and (10) is a sharper form than (9).

Lean FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (hypotheses: a finite set, $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationarity, the values of σ are 1 or -1. Conclusion: $\sum_{x,y} w_x K_{xy} (\sigma_x - \sigma_y)^2/4 \leq 1 - |\sum w\sigma|$).

How to use it. If one can build a stationary kernel that flips the sector with flip probability at least c, then $\theta \leq 1 - c$. A kernel that proposes a driven insertion and accepts the endpoint with probability $\min(1, e^{-W})$ is reversible (hence stationary) with respect to μ_Φ by the Crooks Relation, and its flip probability is the acceptance rate. The acceptance rate is at least $\mathbb{P}(W \leq w_0)\, e^{-w_0}$, so there is no need to control the average of e^{-W}; it is enough that "cheap work occurs with positive probability" Proved here .

07

Simultaneous flips, and the limits of this guarantee

Take a finite set S of mutually separated containers, and let $\theta_t(\Phi)$ be the bias of the sector of each container $t \in S$ (the frame Φ is the variables outside all containers).

Simultaneous-Flip Estimate

Under the finite configuration-space condition and the positive-weight condition, if the containers are separated from each other, and for each container t one takes a stationary kernel K_t that moves only the inside of that container and moves all containers one step simultaneously, then

$$\mathbb{E}\Big[\prod_{t\in S}\theta_t(\Phi)\Big] \;\le\; \mathbb{P}\big(S:\text{no container flips}\big). \tag{11}$$

Proved here Proof. With the frame fixed, the insides of the containers are conditionally independent (the conditional independence of 25-03), and the kernels also move independently per container, so $\mathbb{P}(\text{none flips}\,|\,\Phi)=\prod_t(1-a_t(\Phi))$. By the Flow Lemma each factor is at least $\theta_t(\Phi)$. Take expectations. ∎

The container identity of 25-03 had the form $|\langle W(C)\rangle|\le\mathbb{E}\prod_t\theta_t$ (for the containers inside C). Together with (11), an upper bound on the Wilson loop is bounded **in the correct direction** by a quantity measurable by Monte Carlo, $\mathbb{P}(\text{none flips})$ Proved here .

Limits (an important caution). What (10) gives is $\theta\le 1-a$. The flip probability a of a one-step kernel (a kernel that proposes one driven insertion and decides whether to accept it) is about 0.6–0.73 in the range measured with kernels of acceptance $\min(1,e^{-W})$ Computed (values recomputed from the raw W of Tables 1 and 2 of 25-08. The acceptance-rate columns of the tables are means of the Barker-type $1/(1+e^{W})$, and their ceiling $1/(1+e^{\Delta F})\le\frac12$ belongs to the acceptance rule, not to the kernel Paper Lean). So what can be said for containers of size L is $\theta\le\theta_0$ (with θ_0 a constant less than 1) at best. Even if this held uniformly in L, all that follows is

$$\sigma \;\ge\; \frac{\ln(1/\theta_0)}{L^2} \;\longrightarrow\; 0 \qquad (L\to\infty) \tag{12}$$

and this is exactly the "weak form" of 26-01 (a form that $U(1)$ also satisfies) Proved here . To show the correct form of the band $\theta(L,L)\le e^{-cL^2}$ by (10), one needs a kernel with $a\ge 1-e^{-cL^2}$, that is, a kernel that flips the sector almost surely Proved here (a rewriting of (10)). The acceptance rate is sandwiched by the overlap O of the measures of the forward and reverse paths ($e^{-\Delta F}O\le a_F\le\min(e^{-\Delta F},O)$, and for the mixed kernel $(1-\theta)O\le a\le 1-\theta$) Paper Lean (LatticeGaugeAcceptanceSandwich (overlap_sandwich, mixture_sandwich); the assumption is the Crooks-type relation $P_F e^{-W}=e^{-\Delta F}P_R$), and $a\ge 1-e^{-cL^2}$ demands $O\ge 1-e^{-cL^2}$, in terms of dissipation of the order $D\lesssim e^{-2cL^2}$ Reading (Pinsker's inequality read in reverse). There is no prospect of building that with a one-step kernel.

Therefore, **the guarantee from simultaneous flips cannot be used as a numerical test of the crossover band.** The record measured with this tool (25-08 § 06) cannot be read as a numerical picture of the band either. Its proper uses are a lower bound on the slab tension with the size L of the T^2 fixed, and an estimate from above of the chessboard density κ_{cb} (26-05).

Table 1 What the bounds of this chapter can and cannot say (conditions: the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container)

Bound	Form	What can be said	What cannot be said
Driven identity (the Driven Insertion Identity, the Many-Cell Driven Estimate)	$\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$	Increasing the number of steps can move the factor away from 1 (numerics)	That the upper bound on the factor is uniform in L
Jarzynski Equality	$\mathbb{E}[e^{-W}] = Z_-/Z_+$	A numerical check	Precision with few samples (dominated by rare events)
Flip probability (the Flip-Probability Lemma)	$a \le 1-\theta^2$	$\theta \le \sqrt{1-a}$	— (weaker than the Flow Lemma)
Flow Lemma	$\theta \le 1-a$	A lower bound on the slab tension with the T^2 fixed	For a one-step kernel only up to $\theta_0 \sim \frac{1}{2}$. The form of the band e^{-cL^2} does not follow
Simultaneous-Flip Estimate	$\mathbb{E}\prod\theta \le \mathbb{P}(\text{no flip})$	An estimator in the correct direction for an upper bound on the Wilson loop	A numerical test of the band

08 What cannot be said

(a) Confinement. None of the identities and inequalities of this chapter implies confinement. One would need the factors of the Driven Insertion Identity and the Many-Cell Driven Estimate to stay away from 1 uniformly in L and the lattice spacing, or a kernel with a close to 1, and this book shows neither.

(b) The quasi-static limit. That $W \to \Delta F_D$ as $m \to \infty$ is the estimate of § 04; this book has no proof of it, including bounds on how the kernels mix.

(c) The continuous configuration space. All theorems are finite versions under the finite configuration-space condition. The version on the Haar measure of $SU(2)$ needs the reversibility of the heat-bath kernels and the measure on the space of paths to be rewritten for the continuous case, and is outside Lean.

(d) The limit of one-step kernels. The a of the kernels measured (0.6–0.73 with acceptance $\min(1, e^{-W})$) is a reading of measurements, not a theorem. The conclusion that the form of the band is not reached rests not on this reading but on the rewriting

of (10) that $a \geq 1 - e^{-cL^{\angle}}$ is needed (the row "only the weak form follows" of Table 2). That the acceptance rates of the tables of 25-08 stay below $\frac{1}{2}$ is the ceiling of the Barker-type rule. Conversely, if there were a kernel with a close to 1, (10) could be used as it is, but no construction of one is known.

(e) Novelty. The Crooks Relation and the Jarzynski Equality are known, and the application to the sector-summed ensemble, the Flip-Probability Lemma, the Flow Lemma and the Simultaneous-Flip Estimate are elementary. Techniques that raise the acceptance rate with non-equilibrium proposals that flip a sector are known in the Monte Carlo literature as tempered transitions and non-equilibrium candidate methods (not yet checked). No novelty is claimed for the results of this chapter.

09

The table of five grades and the dependency graph

Table 2 The rows of this chapter (conditions: the finite configuration-space condition, the positive-weight condition, the kernel moving only inside the container)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The marginal of the starting point on the space of paths is the original measure	Proved here	§ 02	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, rows of the kernels sum to 1	—
2	Crooks Relation: $\hat{W} \circ \iota / \hat{W} = e^{-W}$, $W \circ \iota = -W$	Cited	§ 03, Crooks	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, detailed balance (1)	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
3	Driven Insertion Identity: $\mathbb{E}[H] = \mathbb{E}_\Omega[H \tanh(W/2)]$	Proved here	§ 03	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, (1), $H = h(\Phi)\sigma$	BoundedInsertion (`Bounded.single`) applied to Ω (hypotheses: finite, involution, $w > 0$, H odd. The construction of Ω is outside Lean)
4	Jarzynski Equality (independent of the number of steps)	Cited	§ 03, Jarzynski	The finite configuration-space and positive-weight conditions, the kernel moving only inside the container, (1), frame fixed	—
5	Driven Insertion Acceptance-Rate Corollary: $1 - \mathbb{E}[\tanh(W/2)] = 2\times$ acceptance rate	Proved here	§ 03	The setting of row 3	—
6	Many-Cell Driven Estimate: (7)	Proved here	§ 03	Containers separated, $\lvert H\rvert \le 1$, positions chosen independently and uniformly	— (the type of induction is BoundedInsertion (`Bounded.averaged`) of 25-05)
7	$W \to \Delta F_D(\Phi)$ as $m \to \infty$	Reading (physics)	§ 04	The kernels mix well enough at each λ	—
8	The work goes down with the number of steps ($\beta = 2.3$, $2 \times 2 \times 8 \times 8$)	Reading	§ 04, 25-08	$SU(2)$, 6 configurations $\times$ 4 positions	—
9	Cut-Measure Formula (8)	Proved here	§ 05	Frame fixed, terms of the flip sheet removed	FlipLemma (`FlipLemma.cut_formula`) (hypotheses: a finite set, real-valued v, X, b)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
10	Flip-Probability Lemma: flip probability $\leq \frac{1-\theta}{2}$	Proved here	§ 06	Stationary kernels (reversibility not needed)	FlipLemma (`FlipLemma.flip_le`) (hypotheses: $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationary, $\sigma^2 = 1$)
11	Flip-Probability Corollary for Positive Kernels: $(1 - \theta^2)/2$ for positive kernels	Proved here	§ 06	Reversible and a positive operator	—
12	Flow Lemma: $\theta \leq 1 - a$	Proved here	§ 06	Stationary kernels (reversibility not needed)	FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (hypotheses: $w \geq 0$, $\sum w = 1$, $K \geq 0$, rows sum to 1, stationary, $\sigma = \pm 1$)
13	The kernel with acceptance $\min(1, e^{-W})$ is stationary, acceptance rate $\geq \mathbb{P}(W \leq w_0) e^{-w_0}$	Proved here	§ 06	The setting of row 2	—
14	Simultaneous-Flip Estimate: $\mathbb{E}\prod \theta_t \leq \mathbb{P}(\text{no flip})$	Proved here	§ 07	Containers separated, a stationary kernel per container	— (the core is row 12)
15	The a of a one-step kernel is 0.6–0.73 in the range measured (acceptance $\min(1, e^{-W})$; the Barker-type columns of the tables are below the ceiling $\frac{1}{2}$ of the rule)	Reading	§ 07, 25-08	The range of β, N_T and containers measured	—
16	What follows from (10) is $\sigma \geq \ln(1/\theta_0)/L^2$ (the weak form). The form of the band needs $a \geq 1 - e^{-cL^2}$	Proved here	§ 07	When only $\theta \leq 1 - a$ is used	—

The bundles in the machine-checking column are those confirmed, by re-checking on this book's side, to contain no `sorry` and to use axioms forming a subset of the three standard ones. What is guaranteed is only the algebra of sums over finite sets (23-13).

`flip_le` and `cut_formula` are in the file FlipLemma, and `flip_le_one_sub_abs` is in the file FluxLemma.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  marginal of start on paths = original measure   rows sum to 1
 2  CIT  weight ratio e^(−W), W∘ι = −W                    detailed balance (1)
 3  PRV  E[H] = E[H tanh(W/2)] ⟸ 1, 2                      Lean Bounded.single
 4  CIT  E[e^(−W) | frame] = Z₋/Z₊ ⟸ 2
 5  PRV  1 − E[tanh(W/2)] = 2 × acceptance ⟸ 3
 6  PRV  many cells (7) ⟸ 2                                containers separated
 7  RDG  m → ∞ gives W → ΔF_D (physics) ⟸ 4
 8  RDG  work goes down with the number of steps          β = 2.3, 2×2×8×8
 9  PRV  Cut-Measure Formula                              Lean cut_formula
10  PRV  flip probability ≤ 1 − θ²                        Lean flip_le
11  PRV  (1 − θ²)/2 for positive kernels ⟸ 10
12  PRV  θ ≤ 1 − a                                        Lean flip_le_one_sub_
13  PRV  min(1, e^(−W)) kernel stationary, lower bound on acceptance ⟸ 2
14  PRV  E∏θ ≤ P(no flip) ⟸ 12                             containers separated
15  RDG  a of one-step kernels is 0.6–0.73 (Metropolis type)   range measured
16  PRV  only the weak form; band needs a ≥ 1 − e^(−cL²) ⟸ 12
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. This chapter has no assumption (Assumption) rows. The reading of the acceptance of one-step kernels is not made a premise of the row "only the weak form follows" (that row is a rewriting that follows from $\theta \le 1 - a$ alone, and the reading of the acceptance is a judgement of how far it can be used). There is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Proved here Under the finite configuration-space condition, the positive-weight condition and the kernel moving only inside the container, the driven identity $\mathbb{E}[H] = \mathbb{E}[H \tanh(W/2)]$ and its form for many cells	An upper bound on the factor $\tanh(W/2)$ uniform in L and the lattice spacing
Cited The Crooks relation and the Jarzynski equality (written out for the lattice procedure)	A proof of the quasi-static limit $W \to \Delta F_D$
Proved here For stationary kernels, $a \le 1 - \theta^2$, $\theta \le 1 - a$, and simultaneous flips $\mathbb{E}\prod\theta \le \mathbb{P}(\text{no flip})$	A construction of a kernel with a close to 1. The form of the band $\theta \le e^{-cL^2}$

What is said	What is not said
Reading The one-step kernels measured are far from $a \geq 1 - e^{-cL^2}$ (cannot be used to test the band)	The same theorems on the continuous configuration space (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The Crooks relation (path weight ratio e^{-W})	Cited	G. E. Crooks, “The entropy production fluctuation theorem and the nonequilibrium work relation for free energy differences”, Phys. Rev. E 60 (1999) 2721, arXiv:cond-mat/9901352 (bibliographic data checked on the arXiv page. The original source of the Markov-chain form not checked)
The Jarzynski equality	Cited	C. Jarzynski, “A nonequilibrium equality for free energy differences”, Phys. Rev. Lett. 78 (1997) 2690, arXiv:cond-mat/9610209 (bibliographic data checked on the arXiv page)
The Driven Insertion Identity, the Driven Insertion Acceptance-Rate Corollary, the Many-Cell Driven Estimate, the Cut-Measure Formula, the Flip-Probability Lemma, the Flip-Probability Corollary for Positive Kernels, the Flow Lemma, the Simultaneous-Flip Estimate	Proved here	Proofs in the text, § 03– § 07. The Driven Insertion Identity has the same content as the driven identity of the draft paper "Exact centre-vortex insertion identities in the twist-summed ensemble, and bounded, averaged and driven confinement bounds" (the paper on vortex-insertion identities in A-07). The Flow Lemma has the same content as the flux bound of the paper on the reduction of the band in A-07
The container identity	Cited (indirect)	G. Mack, V. B. Petkova (as in 25-03)
Tempered transitions, non-equilibrium candidate methods	Comparison not done	The Monte Carlo literature (bibliographic data not checked)
Numerics (§ 04)	Reading	The same measurement as the first row of Table 1 of 25-08
Lean	Lean	FlipLemma (`FlipLemma.flip_le` , `FlipLemma.cut_formula`), FluxLemma (`FluxLemma.flip_le_one_sub_abs`), BoundedInsertion (`Bounded.single`). What the bundles guarantee is only the algebra of finite sums

Words used in this chapter — driven insertion, the number of steps m, the work W, the space of paths Ω, the Crooks involution ι, quasi-static, the cut measure, the flip probability, the flow lemma, simultaneous flips. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-07 Why fixed maps fail — why driven insertion is cheap and a fixed map applied at once is expensive, explained at the one-loop level as the difference of "whether the shape of the fluctuations is changed".

Why fixed maps fail

> This chapter is this book's own investigation. The established knowledge it builds on is in 4-06 and 4-04.

What this chapter shows — As seen in 25-05 and 25-06, a fixed map that inserts a vortex at once is expensive, and driven insertion that inserts it slowly by heat bath is cheap. In this chapter we explain the difference in the form of a principle. (1) Bounds from below on the sector ratio Z_-/Z_+ (from above on the sector free energy ΔF) are obtained by the Gibbs variational principle with a map, and the gap is exactly a relative entropy (the principle of direction). (2) In a Gaussian model, the loss of a map that does not change the shape of the fluctuations (an isometric map) does not decrease as the number of degrees of freedom grows, while reshaping the fluctuations with a map of determinant 1 brings the loss down to $\Delta F^2/n$. (3) In the one-loop count $\frac{11}{3} = -\frac{1}{3} + 4$ for gluons in a background magnetic field, the $+4$ of antiscreening comes from a spin term that is traceless in the polarisations, so **at the one-loop level** a comparison by isometric maps can give only the direction opposite to asymptotic freedom, $\beta' \geq \beta + 0.023$, while a map that reshapes the fluctuations reproduces $\beta - \beta' = 0.258$. (4) We set side by side that fixed pointwise maps pay an extra cost in the ultraviolet (numerics in a two-dimensional model), and that if the residual relative entropy grows with the area, no area law follows.

Prerequisite chapters — 25-05 (insertion identities), 25-06 (driven insertion), 26-01 (the correct form of the band). From Part 4, 4-06 (asymptotic freedom, the one-loop coefficient) and 4-04 (renormalisation); from Part 2, 2-08 (free energy, relative entropy).

Said first — All claims of § 03– § 05 are statements **at the one-loop (Gaussian) level**. This chapter makes no non-perturbative claims. Bounding the error of maps that reshape fluctuations at intermediate coupling and large fields is an open problem, and that is where the core of the crossover band (Part 26) lies. The extra ultraviolet cost of § 06 is a numerical reading in a two-dimensional model.

The order of this chapter

1. Summary — whether the shape of the fluctuations is changed

2. The principle of direction — upper bounds come from variation (the Transport Identity)
3. The Gaussian model — the loss of isometric maps, and shaping the fluctuations (Table 1)
4. The one-loop table 11/3 = −1/3 + 4 — the spin term does not appear in first-order traces (Table 2)
5. Consequences for the renormalisation comparison — isometric versus shaping comparison (Table 3)
6. The extra ultraviolet cost of fixed maps — Table 4
7. Deterministic squeezing and the area law — the relative-entropy obstruction of the residual, and additivity
8. What cannot be said
9. The table of five grades and the dependency graph — Table 5

01

Summary — whether the shape of the fluctuations is changed

1. Confinement needs the sector with a vortex not to be too expensive compared with the sector without, that is, a bound **from above** on the sector free energy $\Delta F = \ln(Z_+/Z_-)$ (to make $\theta = \tanh(\Delta F/2)$ small).
2. What gives a bound in that direction is the Gibbs variational principle with a map T that carries the measure of sector $+$ to sector $-$: $\Delta F = \mathbb{E}_+[S \circ T - S - \ln J_T] - \mathrm{KL}$. Discarding the KL (relative entropy) gives an upper bound Proved here .
3. The discarded KL is the loss of the bound. Computing in a Gaussian model, the loss of maps that do not change the shape of the fluctuations (the identity map, group translations, gauge transformations, decimation) does not decrease as degrees of freedom are added. A map that stretches and shrinks the fluctuations while keeping volume brings the loss down to $\Delta F^2/n$ (n is the number of degrees of freedom) Proved here .
4. The coefficient of asymptotic freedom $\frac{11}{3}$ splits at the one-loop level into an orbital part $-\frac{1}{3}$ and a spin term $+4$. The spin term is traceless in the polarisations, so it does not appear in bounds from maps that do not change the shape of the fluctuations (first-order traces) Proved here (at the one-loop level).
5. As a result, at the one-loop level, a renormalisation comparison built with isometric maps can give only $\beta' \geq \beta + 0.023$ per doubling (the same direction as $U(1)$), while a

map that reshapes the fluctuations reproduces the $\beta - \beta' = 0.258$ of asymptotic freedom Proved here (at the one-loop level).

6. In a two-dimensional model, fixed pointwise maps paid an extra cost growing faster than linearly in the lattice size N_T in the twisted direction. In trials that redraw the fluctuations around the new background, this growth disappears Reading .
7. If the residual KL of a deterministic map grows with the area of the sheet, no area law follows from the variational bound Proved here (under that assumption). The role of the transport road lies in a qualitative target weaker than the form of the band itself.

02

The principle of direction — upper bounds come from variation

Let X_+, X_- be the sets of configurations of sectors $+$ and $-$, $\mu_\pm = e^{-S}/Z_\pm$ the Gibbs measures on each, and set the sector free energy to be $\Delta F = \ln(Z_+/Z_-)$ (the same orientation as ΔF_D of 25-06). The bias of the sector is $\theta = \tanh(|\Delta F|/2)$, so what is needed in the direction of confinement is an **upper bound** on ΔF, in other words a **lower bound** on Z_-/Z_+.

Transport Identity (the transport identity, the Gibbs variational principle)

If $T : X_+ \to X_-$ is a measurable bijection with Jacobian J_T relative to the reference measure (Haar $\times$ counting; Lebesgue in the Gaussian model), and $T_\# \mu_+$ is the pushforward of μ_+ by T, then

$$\Delta F = \mathbb{E}_+\big[S \circ T - S - \ln J_T\big] - \mathrm{KL}\big(T_\# \mu_+ \,\big\|\, \mu_-\big). \quad (1)$$

In particular $\Delta F \le \mathbb{E}_+[S \circ T - S - \ln J_T]$, with equality only when $T_\# \mu_+ = \mu_-$ (T carries μ_+ exactly to μ_-).

Proved here A standard computation. Proof. The density of $\nu = T_\# \mu_+$ is $\nu(y) = \mu_+(x)/J_T(x)$ at $y = Tx$. Hence

$$\mathrm{KL}(\nu \| \mu_-) = \mathbb{E}_+\big[\ln \mu_+(x) - \ln J_T(x) - \ln \mu_-(Tx)\big] = \mathbb{E}_+\big[S(Tx) - S(x) - \ln J_T(x)\big] - \ln Z$$

Rearranging gives (1). The inequality follows from $\mathrm{KL} \ge 0$. ∎ On a finite configuration space (the finite configuration-space condition of 25-05), if T is a bijection then $J_T = 1$, and the first term of the right-hand side of (1) is the $\mathbb{E}_+[\Delta S]$ of 25-05.

The other direction. The structural tools used in this book do not bound ΔF from above. Reflection positivity gives $Z_- \le Z_+$ ($\theta \ge 0$, $\Delta F \ge 0$), but that is a **lower bound** on ΔF, the direction opposite to confinement (26-04) Proved here . Chessboard estimates replace $\mathbb{E}\prod\theta$ by a single density κ_{cb}, but saying $\kappa_{cb} < 1$ still needs an upper bound on ΔF (26-05). The Tomboulis–Yaffe inequality carries a bound on θ to the Wilson loop, and does not bound θ itself (26-01). Within the range of this book's tools, upper bounds on ΔF have come only from variation of the type (1) (trial maps, trial measures) and from explicit constructions Reading (a judgement arranging the list of tools, not a theorem). We call this the **principle of direction**. A candidate tool for crossing the band must first of all give a bound in this direction.

03

The Gaussian model — isometric maps and reshaping maps

We look at how large the KL of (1) is in a Gaussian model where everything can be computed. Take two Gaussian measures on $\mathbb{R}^n$, $\mu_0 = N(0, \Delta_0^{-1})$ and $\mu_B = N(0, \Delta_B^{-1})$ (actions $S_0 = \frac{1}{2}x^{\mathsf{T}}\Delta_0 x$, $S_B = \frac{1}{2}x^{\mathsf{T}}\Delta_B x$; Δ_0, Δ_B positive definite). μ_0 plays the role of "the fluctuations of sector $+$ (no background)", and μ_B that of "the fluctuations of sector $-$ (around the vortex background B)". The exact value is

$$\Delta F = -\ln\frac{Z_B}{Z_0} = \tfrac{1}{2}\ln\det(\Delta_B\Delta_0^{-1}) = \tfrac{1}{2}\sum_k \ln\lambda_k, \quad (2)$$

λ_k are the eigenvalues of $\Delta_0^{-1/2}\Delta_B\Delta_0^{-1/2}$. Putting a linear map T into (1), the bound, using $M := \Delta_0^{-1/2}T^{\mathsf{T}}\Delta_B T\Delta_0^{-1/2}$, is

$$\mathbb{E}_0\big[S_B\circ T - S_0\big] - \ln|\det T| = \tfrac{1}{2}(\operatorname{tr} M - n) - \ln|\det T| \quad (3)$$

(from $\mathbb{E}_0[x^{\mathsf{T}}Ax] = \operatorname{tr}(A\Delta_0^{-1})$).

A **map isometric for the fluctuations** is a linear map with $T^{\mathsf{T}}\Delta_0 T = \Delta_0$ (it does not change the shape of the fluctuations of μ_0, i.e. it preserves μ_0). The identity map, linearisations of group translations and gauge transformations, and the linearisation of decimation (the operation of discarding some of the fine degrees of freedom and taking the rest as they are as degrees of freedom of the coarse lattice) belong to this class (for the treatment of decimation, see § 08 (b)).

Isometric-Map Loss Proposition (the loss of isometric maps is not diluted by degrees of freedom)

If T is isometric for the fluctuations, the bound (3) is $\frac{1}{2}\sum_k(\lambda_k - 1)$ independently of T, and its difference from the exact value (2) is

$$\tfrac{1}{2}\sum_k \big(\lambda_k - 1 - \ln\lambda_k\big) \;\geq\; 0. \tag{4}$$

Proved here Proof. $O := \Delta_0^{1/2} T \Delta_0^{-1/2}$ is an orthogonal matrix satisfying $O^{\mathsf{T}} O = 1$, and $M = O^{\mathsf{T}} (\Delta_0^{-1/2} \Delta_B \Delta_0^{-1/2}) O$. Hence $\operatorname{tr} M = \sum_k \lambda_k$ and $|\det T| = 1$. The difference is the sum of $t - 1 - \ln t \geq 0$. ∎ (4) is exactly the relative entropy $\mathbf{KL}(\mu_0 \| \mu_B)$ of the two Gaussian measures. If the background changes the fluctuations in only a few directions (finitely many λ_k different from 1), (4) does not decrease as n is made large.

Fluctuation-Shaping Proposition (shaping with a map of determinant 1 gives loss $\Delta F^2/n$)

Taking the infimum of the bound (3) over all linear maps with $\det T = 1$, with $r := \det(\Delta_B \Delta_0^{-1}) = e^{2\Delta F}$,

$$\inf_{\det T = 1} \Big(\tfrac{1}{2}(\operatorname{tr} M - n) \Big) = \tfrac{n}{2}\big(r^{1/n} - 1\big), \qquad 0 \;\leq\; \tfrac{n}{2}\big(e^{2\Delta F/n} - 1\big) - \Delta F \;\leq\; \frac{\Delta F^2}{n}\, e^{2|\Delta}$$

Proved here Proof. If $\det T = 1$ then $\det M = r$, and conversely any positive definite M with $\det M = r$ is realised by $T = \Delta_B^{-1/2} M^{1/2} \Delta_0^{1/2}$ (times an orthogonal matrix; $\det T = r^{-1/2} r^{1/2} = 1$). By the arithmetic–geometric mean inequality $\operatorname{tr} M \geq n (\det M)^{1/n} = n r^{1/n}$, with equality at $M = r^{1/n} \cdot 1$. The bound on the difference follows, with $u = 2\Delta F/n$, from $e^u - 1 - u \leq \frac{u^2}{2} e^{|u|}$. ∎

The map of the Fluctuation-Shaping Proposition preserves volume ($\ln J_T = 0$), stretches and shrinks the fluctuations in the few directions the background changes, and spreads that change of volume thinly over the many remaining directions, $\Delta F/n$ each. **What is needed is not having a Jacobian (entropy), but changing the shape of the fluctuations.**

Table 1 An example in the Gaussian model: eigenvalues 0.2, 0.3, 3, 4 (all the rest 1)

Bound	$n = 10$	$n = 100$	$n = 10^4$
Exact value (2)	-0.16425	-0.16425	-0.16425
Isometric map (Isometric-Map Loss Proposition)	$+1.75$	$+1.75$	$+1.75$
Optimal map of determinant 1 (Fluctuation-Shaping Proposition)	-0.1616	-0.1640	-0.16425

Proved here (values of the closed formulae (2), (4), (5)). When small and large eigenvalues are mixed, the first-order trace $\frac{1}{2}\sum(\lambda_k - 1)$ and $\frac{1}{2}\sum \ln\lambda_k$ can have opposite signs. In this example the change of sector is exactly **cheap** ($\Delta F < 0$), yet the bound of the isometric map says it is expensive ($+1.75$). The spin term of the next section has exactly this form.

04

The one-loop table 11/3 = −1/3 + 4

Place a constant abelian magnetic field B in the (1,2) plane as background, and consider the fluctuations of the $SU(2)$ gluons around it in Feynman gauge. The operator of the quadratic action of the fluctuations is

$$M_B = -D^2\,\delta_{\mu\nu} - 2iF_{\mu\nu} \quad (6)$$

where the first term is the orbital part (the square of the covariant derivative) and the second term is the spin term. The ghost operator is $-D^2$. The heat-kernel traces (per volume, omitting the common factor $1/(16\pi^2 s^2)$ and the group factor $C_A = 2$ = the number of charged components), as functions of the proper time s, are

$$\text{vector (4 polarisations)}: \ \frac{Bs}{\sinh Bs}\big(2 + 2\cosh 2Bs\big), \qquad \text{ghosts}: \ -2\,\frac{Bs}{\sinh Bs} \quad (7)$$

With $x = Bs$, using $x/\sinh x = 1 - x^2/6 + O(x^4)$ and $\cosh 2x = 1 + 2x^2 + O(x^4)$ and picking out the coefficient of x^2 gives the following table.

Table 2 Breakdown of the coefficient of $B^2 s^2$ (at the one-loop level)

Part	Form of the heat-kernel trace	Coefficient of $B^2 s^2$	Meaning
Orbital (4 orbital degrees of freedom of the vector − 2 of the ghosts)	$\frac{Bs}{\sinh Bs}\,(4-2)$	$-\frac{1}{3}$	Diamagnetic (screening). The same type as a charged particle of $U(1)$
Spin term	$\frac{Bs}{\sinh Bs}\,2(\cosh 2Bs - 1)$	$+4$	Paramagnetic (antiscreening). Peculiar to the non-abelian case
Total	Sum of (7)	$+\frac{11}{3}$	The one-loop coefficient of asymptotic freedom (the $\frac{11}{3}$ of 4-06)

Proved here (at the one-loop level. The expansion is a two-line computation: orbital $2(1 - x^2/6) = 2 - x^2/3$, spin $(1 - x^2/6)\cdot 4x^2 = 4x^2 + O(x^4)$.) In the conventions of this count, the coefficient $+\frac{11}{3}$ is the direction of asymptotic freedom and $-\frac{1}{3}$ the opposite direction.

First-Order Vanishing of the Spin Term (the spin term does not appear in first-order traces — at the one-loop level)

At the Gaussian level, when comparing the operator without background $M_0 = -\partial^2\delta_{\mu\nu}$ with the M_B of (6), the bound (1) from a map isometric for the fluctuations is the first-order trace $\frac{1}{2}\mathrm{tr}\big((M_B - M_0)M_0^{-1}\big)$, and the contribution of the spin term is

exactly 0. This bound is at least the exact value for the orbital part alone, $\frac{1}{2}\ln \det(M_{\mathrm{orb}} M_0^{-1})$.

Proved here (at the one-loop level) Proof. M_0^{-1} is proportional to $\delta_{\mu\nu}$ in the polarisation indices, and $F_{\mu\nu}$ is antisymmetric, so $\mathrm{tr}_{\mu\nu}(F_{\mu\nu}) = 0$. Hence the first-order trace of the spin term is 0, the bound equals $\frac{1}{2}\mathrm{tr}((M_{\mathrm{orb}} - M_0)M_0^{-1})$, and by the inequality $t - 1 \geq \ln t$ of the Isometric-Map Loss Proposition it is at least $\frac{1}{2}\ln \det(M_{\mathrm{orb}} M_0^{-1})$. ∎ The spin term takes effect at second order ($\cosh 2Bs - 1 = O(B^2)$ in Table 2) and gives the $+4$ of antiscreening. A bound that looks only at first-order traces loses it completely. (Traces and determinants of infinite-dimensional operators are statements under the regularisation usually done in one-loop computations.)

05

Consequences for the renormalisation comparison

The renormalisation comparison of Part 26 (26-06) aims to bound from above the vortex free energy of $(2N, \beta)$, with the whole lattice doubled, by that of the original size (N, β'), and to say $\beta' < \beta$ (the direction in which the coupling becomes stronger, asymptotic freedom). At the one-loop level, the coefficient c around the vortex background fixes the shift of the coupling per doubling, and with $\beta = 4/g^2$ and $b_0 = c_A/(16\pi^2)$, $\beta - \beta' = 8b_0 \ln 2$.

Isometric versus Shaping Comparison Proposition (comparison of the two maps — at the one-loop level)

(a) If the bound (1) is built with maps isometric for the fluctuations, the coefficient that can be captured at the one-loop level is at most $c = -\frac{1}{3}$, and only $\beta' \geq \beta + 0.023$ per doubling can be given (the same as $U(1)$, the direction opposite to asymptotic freedom).
(b) A map that shapes the fluctuations while preserving volume (the Fluctuation-Shaping Proposition) reproduces the exact $\frac{1}{2}\ln\det$ with error $O(\Delta F^2/n)$, and gives $c = \frac{11}{3}$, $b_0 = 22/(48\pi^2)$, and $\beta - \beta' = 0.258$ per doubling.

Proved here (at the one-loop level) Proof. (a) follows from the First-Order Vanishing of the Spin Term and the fact that the bound is an upper bound (the Transport Identity). The number is $8 \cdot \frac{-1/3 \cdot 2}{16\pi^2} \ln 2 = -\frac{0.2575}{11} \approx -0.0234$. (b) applies the Fluctuation-Shaping Proposition to the fluctuations shell by shell, and the number is $8 \cdot \frac{22}{48\pi^2} \ln 2 \approx 0.2575$.

∎

Table 3 Shift of the coupling per doubling $\beta - \beta'$ ($SU(2)$, $\beta = 4/g^2$)

From what	$\beta - \beta'$	Grade	Conditions written in the same row
The exact one-loop coefficient $\frac{11}{3}$	0.258	Proved here	One-loop level
Bound from a map shaping the fluctuations (Isometric versus Shaping Comparison Proposition (b))	0.258 (error $O(\Delta F^2/n)$)	Proved here	One-loop level, Gaussian bound
Bound from an isometric map (Isometric versus Shaping Comparison Proposition (a))	≤ -0.023	Proved here	One-loop level. The same direction as $U(1)$
Test doubling the whole lattice ($4^4 \to 8^4$, 1–3 seeds per β)	$+0.16$ to $+0.25$	Reading	$\beta = 2.5$ to 3.0. The range at each β is as in 26-06 Table 4 (for example $+0.13$ to $+0.19$ at $\beta = 2.7$, $+0.19$ to $+0.5$ at $\beta = 2.5$). At $\beta = 2.3$, $\gtrsim 0.1$. Small lattices
The same test for $U(1)$ ($(N, M) = (4, 8) \to (8, 16)$, $\beta = 1.5$)	< -0.2	Reading	Opposite direction ($\beta' > 1.7$, off the curve). 48 replicas, minimum exchange acceptance 0.37. One seed (26-06 Table 2)

The attempt to build a renormalisation comparison with Migdal–Kadanoff-type decimation (Tomboulis 2007) was criticised on the point that the same argument also flows to strong coupling for $U(1)$ Cited (Ito–Seiler, Kanazawa; source table).

Decimation belongs on the side that does not change the shape of the fluctuations, so the Isometric versus Shaping Comparison Proposition (a) can be read as giving a one-loop-level explanation of this criticism Reading (physics; an interpretation in which a one-loop computation pins down the gap in a non-perturbative argument, not a proof of the criticism itself).

A screening criterion. A candidate method for crossing the band must have **a linearisation of the trial map that is non-isometric for the fluctuations** — this is a necessary condition at the one-loop level Proved here (a restatement of the Isometric versus Shaping Comparison Proposition (a); at the one-loop level). The view that the same condition is needed at the non-perturbative level is Reading (physics). The driven insertion of 25-06 is cheap because the heat bath relaxes the shape of the fluctuations to fit the new coupling, which is consistent with this criterion Reading (physics).

> 06

The extra ultraviolet cost of fixed maps

The Isometric-Map Loss Proposition said that "the loss of isometric maps is not diluted by degrees of freedom". Numerics show that with fixed pointwise maps on the lattice, the loss moreover **grows** as the lattice spacing is made finer.

A two-dimensional model. Place the two-dimensional $O(3)$ model (spins $s_x \in S^2$, action $-\beta \sum s_x \cdot s_y$) on the strip $\mathbb{Z} \times (\mathbb{Z}/N_T)$, with a twist of a π rotation about the z axis in the periodic direction. The classical vacua are the north and south poles, and the action of the kink joining them (the two-dimensional version of the fractional instanton) approaches $2\pi\beta$ from below. It is a light test bed for the small twisted T^2 in four dimensions.

Table 4 The cost of kink insertion on the twisted two-dimensional strip (action in units with β removed; the extra cost is $\beta \times (\mathbb{E}[\Delta S] - \text{classical value})$)

Trial	β	$N_T = 2$	$N_T = 4$	$N_T = 8$	Conditions
Extra cost of the fixed pointwise rotation (the minimal rotation from vacuum to kink at each point)	2	-2.1	$+4.1$	$+14.7$	Standard deviations at $N_T = 8$ are 4.6 ($\beta = 2$) and 2.6 ($\beta = 4$). The number of configurations is not in the record
Same as above	4	-2.0	$+4.6$	$+16.2$	

Reading The extra cost hardly depends on β and grows faster than linearly in N_T. This is the opposite of the expected "decrease by $\ln N_T$ due to the running coupling".

On the other hand, in trials that do not carry the fluctuations over on the old background but **redraw them from the Gaussian measure around the new background (the kink)**, the difference of the bound becomes the ratio of one-loop determinants $\frac{1}{2} \ln[\det' H_{\text{kink}} / \det H_{\text{vac}}]$. Its value (with the two collective coordinates removed) changes only logarithmically, $-3.68, -2.94, -2.43, -2.11, -1.88$ at $N_T = 3, 4, 6, 8, 10$, and the quantity including the measure of the rotational and translational collective coordinates was almost constant, -4.91 to -4.85 for $N_T = 4$ to 12 Reading (numerical computation of eigenvalues; no error estimate). The faster-than-linear growth has disappeared.

A four-dimensional slab. For $SU(2)$, $\beta = 2.3$, the slab $2 \times 2 \times 8 \times 8$, an insertion that spreads the flux smoothly in a fixed colour direction (σ_3) raised the cost instead of lowering it: the mean of ΔS was 21.9 at spreading radius 1 and 36.0 at radius 2 Reading . This can be read as being because, in configurations without gauge fixing, the field in a fixed direction does not line up with the colour direction at each point Reading (physics). On the same slab, driven insertion (25-06) brought the mean work down to 2.9 with 16 steps.

How to read it. Fixed pointwise maps move the fluctuations "on a fixed background", so they fail to capture the difference of entropy of the fluctuations on the new background. The picture is that the loss of the Isometric-Map Loss Proposition piles up for each fine degree of freedom of the lattice Reading (physics; whether it is proportional to N_T is a reading from three numerical points, and there is no general bound). The average of insertions (25-05), driven insertion (25-06) and redrawing the fluctuations are all tools for avoiding this cost.

07

Deterministic squeezing and the area law

The correct form of the band was $\theta(L,L) \le e^{-cL^2}$ (26-01). Since $\theta = \tanh(\Delta F/2) \le \Delta F/2$, showing this variationally requires the upper bound on $\Delta F(L,L)$ to be as small as order $2e^{-cL^2}$.

Relative-Entropy Obstruction to the Area Law (if the residual KL grows with the area, no area law follows)

If, for a map T, the residual $\mathrm{KL}(T_{\#}\mu_+ \| \mu_-)$ of (1) is $\ge c'A$ ($c' > 0$) in the area A of the vortex sheet, then the bound $\mathbb{E}_+[S \circ T - S - \ln J_T]$ from (1) is at least $c'A$, and the upper bound on θ derived from it is at least $\tanh(c'A/2)$, approaching 1 as $A \to \infty$. In particular no bound of the form $\theta \le e^{-\sigma A}$ follows.

Proved here Proof. From (1), $\mathbb{E}_+[\cdots] = \Delta F + \mathrm{KL} \ge \mathrm{KL} \ge c'A$ (taking the orientation of the sectors so that $\Delta F \ge 0$). $\tanh$ is monotone. ∎ This is a statement under an assumption, and whether the assumption holds must be checked map by map.

Additivity of Relative Entropy over Containers (for independent containers the residual KL adds up)

If, with the frame fixed, the insides of containers $D_1, \dots, D_k$ are conditionally independent (25-03) and the map is a product over containers $T = T_1 \times \cdots \times T_k$, then $\mathrm{KL}(T_{\#}\mu_+ \| \mu_- \mid \Phi) = \sum_j \mathrm{KL}(T_{j\#}\mu_{+,j} \| \mu_{-,j} \mid \Phi)$. If each term is at least $c_0 > 0$, the residual is proportional to the number of containers, that is, to the area.

Proved here The standard fact that the relative entropy of a product measure is the sum of the relative entropies of the factors. ∎

That for fixed maps (not adapted to the fluctuations) the residual of each container is bounded below by $c_0 > 0$, so that the assumption of the Relative-Entropy Obstruction to the Area Law holds, is the shared judgement of the study from two viewpoints,

constructive field theory and probability Reading (physics, an estimate; the same goes for the judgement that the ceiling of triangular maps is of perimeter type). Under this judgement the transport road is not a tool for showing the form of the band itself, but a tool for a weaker qualitative target — $\Delta F(L, L) \to 0$ (at any speed). $\Delta F(L, L) \to 0$ is enough to rule out the Coulomb phase of $U(1)$ (where $\Delta F(L, L)$ approaches a positive constant) and the deconfined phase of $\mathbb{Z}_2$, but it yields neither confinement nor a mass gap Reading (physics; numerics in 26-06).

08

What cannot be said

(a) The non-perturbative level. The First-Order Vanishing of the Spin Term, the Isometric versus Shaping Comparison Proposition and the top three rows of Table 3 are all statements at the one-loop (Gaussian) level. Bounding the error of maps that shape the fluctuations at intermediate coupling and large fields (local relative entropy near the vortex sheet, and higher loops) is an open problem, and this book does not show it.

(b) The classification of isometric maps. The identity map and the linearisations of group translations and gauge transformations are isometric by definition. Putting decimation on the isometric side is a classification in the sense that it carries the fluctuations of the kept degrees of freedom over as they are; decimation is not a bijection, so this is not a literal application of the Isometric-Map Loss Proposition.

(c) The principle of direction. "Upper bounds on ΔF come only from variation and explicit constructions" is a judgement about the list of this book's tools, not a theorem.

(d) The extra ultraviolet cost. Table 4 is three numerical points in a two-dimensional model, and does not show that four-dimensional gauge theory grows in the same way. In four dimensions cancellations through Ward identities are said to be possible, and whether they occur depends on gauge fixing and the covariance of the map.

(e) The assumption of the Relative-Entropy Obstruction to the Area Law. That the residual KL of fixed maps is proportional to the area can be said only in the product case of the Additivity of Relative Entropy over Containers, assuming that the residual of each container is bounded below; for general maps it is an estimate.

(f) Novelty. The Gibbs variational principle, the relative entropy of Gaussian measures and the one-loop computation of the background field are all standard. No novelty is claimed for the view that lines them up as a screening criterion of "whether the shape of the fluctuations is changed". For scalar fields, Polchinski-type transport maps are known (Bauerschmidt–Bodineau–Dagallier); no example for gauge fields was found in the range searched (not yet checked).

09

The table of five grades and the dependency graph

Table 5 The rows of this chapter (§ 03– § 05 are all at the one-loop (Gaussian) level)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	Transport Identity: $\Delta F = \mathbb{E}_+[S \circ T - S - \ln J_T] - \mathrm{KL}$	Proved here	§ 02 (standard)	Measurable bijection, Jacobian defined	—
2	Reflection positivity gives only $\Delta F \geq 0$ (the direction opposite to confinement)	Proved here	§ 02, 26-04	Reflection directions with the twist fixed	—
3	The principle of direction: upper bounds on ΔF only from variation and explicit constructions	Reading	§ 02	A judgement about the list of this book's tools	—
4	Isometric-Map Loss Proposition: $\frac{1}{2}\sum(\lambda_k - 1 - \ln \lambda_k)$ (not diluted in n)	Proved here	§ 03	Gaussian model, $T^{\mathsf{T}}\Delta_0 T = \Delta_0$	—
5	Fluctuation-Shaping Proposition: the optimum over maps of determinant 1, and loss $\leq (\Delta F^2/n)e^{2\lvert\Delta F\rvert/n}$	Proved here	§ 03	Gaussian model, linear maps	—
6	Values of the example of Table 1	Proved here	§ 03	Eigenvalues 0.2, 0.3, 3, 4	—
7	Table 2: $\frac{11}{3} = -\frac{1}{3} + 4$ ($+4$ is the spin term)	Proved here	§ 04	One-loop level, constant abelian background, Feynman gauge	—
8	First-Order Vanishing of the Spin Term: the spin term does not appear in first-order traces	Proved here	§ 04	One-loop level	—
9	Isometric versus Shaping Comparison Proposition (a): isometric comparison gives $\beta' \geq \beta + 0.023$	Proved here	§ 05	One-loop level, per doubling	—
10	Isometric versus Shaping Comparison Proposition (b): shaping comparison gives $\beta - \beta' = 0.258$	Proved here	§ 05	One-loop level, per doubling	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
11	$\beta - \beta'$ of the doubling test (positive for $SU(2)$, negative for $U(1)$)	Reading	§ 05, 26-06	Small lattices, a few values of β, 1–3 seeds	—
12	Migdal–Kadanoff interpolation does not distinguish $U(1)$ (criticism)	Cited	Ito–Seiler, Kanazawa	On the argument of Tomboulis 2007	—
13	A one-loop-level explanation of row 12	Reading (physics)	§ 05	One-loop level	—
14	Screening criterion: the linearisation must be non-isometric for the fluctuations	Proved here	§ 05	A necessary condition at the one-loop level (Reading non-perturbatively)	—
15	The extra cost of fixed pointwise maps is at least linear in N_T	Reading	§ 06	Two-dimensional $O(3)$, twisted strip, $N_T = 2, 4, 8$	—
16	Redrawing the fluctuations makes the growth disappear	Reading	§ 06	Same, $N_T = 3$ to 12, no error estimate	—
17	In four dimensions, insertion spread in a fixed direction raises the cost	Reading	§ 06	$SU(2)$, $\beta = 2.3$, $2 \times 2 \times 8 \times 8$	—
18	Relative-Entropy Obstruction to the Area Law: if the residual KL is $\geq c' A$, no area law follows	Proved here	§ 07	Assuming a lower bound on the residual KL	—
19	Additivity of Relative Entropy over Containers: for independent containers the KL adds up	Proved here	§ 07	With the frame fixed the containers are independent, and the map is a product	—
20	For fixed maps the residual of each container is bounded below	Reading (physics)	§ 07	Estimate	—
21	$\Delta F(L, L) \to 0$ rules out the Coulomb phase and the deconfined $\mathbb{Z}_2$ phase	Reading (physics)	§ 07, 26-06	Yields neither confinement nor a gap	—

This chapter has no machine-checked bundles.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  ΔF = E₊[S∘T − S − ln J] − KL                          bijection, Jacobia
 2  PRV  reflection positivity gives only ΔF ≥ 0                26-04
 3  RDG  principle of direction (judgement on the tools) ⟸ 1, 2
 4  PRV  loss of isometric maps not diluted in n ⟸ 1             Gaussian
 5  PRV  shaping with det 1, loss ~ ΔF²/n ⟸ 1                    Gaussian
 6  PRV  example of Table 1 ⟸ 4, 5
 7  PRV  11/3 = −1/3 + 4                                       one-loop level
 8  PRV  spin term absent from first-order trace ⟸ 4, 7          one-loop level
 9  PRV  isometric comparison β' ≥ β + 0.023 ⟸ 1, 8              one-loop level
10  PRV  shaping comparison β − β' = 0.258 ⟸ 5, 7                one-loop level
11  RDG  sign of doubling test (SU(2) +, U(1) −)               small lattices
12  CIT  criticism of Migdal–Kadanoff interpolation
13  RDG  one-loop explanation of row 12 (physics) ⟸ 9, 12
14  PRV  screening criterion: non-isometric ⟸ 9                  one-loop level
15  RDG  extra cost of fixed maps (two dimensions)
16  RDG  redrawing removes the growth (two dimensions)
17  RDG  fixed-direction insertion is expensive (four-dimensional slab)
18  PRV  residual KL ≥ c'A gives no area law ⟸ 1                  with assumption
19  PRV  KL adds up over independent containers
20  RDG  residual of fixed maps bounded below per container (estimate) ⟸ 19
21  RDG  role of ΔF(L,L) → 0 (physics) ⟸ 18, 20
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. There are no assumption (Assumption) rows. The conclusion of the Relative-Entropy Obstruction to the Area Law applies to fixed maps only when the estimate "for fixed maps the residual of each container is bounded below" is correct, and that distinction is expressed by the grade (Reading) of the last row (the role of $\Delta F(L,L) \to 0$). This chapter has no node of confinement, no node of the mass gap, and no non-perturbative node.

What this chapter can say

What is said	What is not said
Proved here The transport identity (the Gibbs variational principle), and that the loss of its bound is a relative entropy	Bounds on the error of maps reshaping the fluctuations at the non-perturbative level
Proved here In the Gaussian model, the loss of isometric maps is not diluted, and the loss of reshaping maps is $\Delta F^2/n$	Literal extension to operations that are not bijections, such as decimation

What is said	What is not said
Proved here At the one-loop level $\frac{11}{3} = -\frac{1}{3} + 4$, the isometric comparison gives $\beta' \geq \beta + 0.023$, the reshaping comparison $\beta - \beta' = 0.258$	The renormalisation comparison at intermediate coupling (the core of the band in Part 26)
Reading Fixed pointwise maps pay an extra cost in the ultraviolet (three points in a two-dimensional model)	The same growth in four-dimensional gauge theory
Proved here If the residual KL is proportional to the area, no area law follows (with an assumption)	That the assumption holds for fixed maps in general

Sources and reproduction

Item	Kind	Source / tools
The seven propositions of § 02– § 07, Table 1, Table 2	Proved here	Proofs and computations in the text, § 02– § 07. The First-Order Vanishing of the Spin Term, the Isometric versus Shaping Comparison Proposition and Table 2 have the same content as the section on the renormalisation comparison in the draft paper (the paper on the reduction of the band in A-07)
The one-loop coefficient of asymptotic freedom	Cited	As in 4-06
The argument by Migdal–Kadanoff interpolation	Cited	E. T. Tomboulis, arXiv:0707.2179
Its criticism	Cited	K. R. Ito, E. Seiler, arXiv:0711.4930. T. Kanazawa, arXiv:0805.2742 (checking as in 26-06)
Polchinski-type transport maps (scalar fields)	Comparison not done	Bauerschmidt–Bodineau–Dagallier (bibliographic data not checked)
The numerical rows of Table 3	Reading	26-06 (the test doubling the whole lattice)
Table 4 and the two-dimensional numerics of § 06	Reading	Computations on the twisted strip of the two-dimensional $O(3)$ model done in this book's work (cooled kinks, the pointwise rotation map, eigenvalues of the Hessian). The error estimates are rough
The four-dimensional slab numerics of § 06	Reading	The same measurement as the numerical section of the paper on vortex-insertion identities in A-07

Words used in this chapter — the transport identity, the Gibbs variational principle, the principle of direction, maps isometric for the fluctuations, reshaping the fluctuations, the spin term, antiscreening, the residual relative entropy, the screening criterion. Their meanings are in A-02 Glossary (the words of Part 25 are to be added).

Next chapter to read: 25-08 The measured record — the numerics of Part 25 (the work of driven insertion, the vortex free energy when the T^2 is widened, the string tension of slabs) gathered in one place, with conditions.

The measured record

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-03 and 5-07.

What this chapter shows — We collect the numerical record of Part 25 in one place. There are three series: (1) the work and acceptance rate of driven insertion (25-06), (2) the vortex free energy of the global flux when the twisted T^2 is widened (with a $U(1)$ control), and (3) the string tension of slabs (Creutz ratios). Finally we annotate the measurement of simultaneous flips, and why it cannot be used as a test of the band.

Prerequisite chapters — 25-06 (driven insertion, the flow lemma), 25-03 (containers, the bias of the sector θ), 26-02 (the lateral-widening inequality, monotonicity of the string tension). From Part 5, 5-07 (Monte Carlo) and 5-08 (string tension, Creutz ratios).

Said first — The numbers of this chapter are **all numerical readings** (label "Reading"), and none is a proof. Each row states the group, β, the size of the T^2 N_T, the size of the plane M, the number of measurements and the error. Where the error is not in the record, we write "not in the record". The lattices are small, and β takes a few values. The rows with $N_T = 2$ are a region where the T^2 is only 2 in lattice units and lattice artefacts are large.

The order of this chapter

1. Summary — three series
2. Common setting and how to read it
3. The work of driven insertion — Tables 1 and 2
4. The vortex free energy when the T^2 is widened — Tables 3, 4 and 5
5. The string tension of slabs — Tables 6 and 7
6. The measurement of simultaneous flips, and why it cannot be used as a test of the band — Tables 8 and 9
7. What cannot be said
8. The table of five grades and the dependency graph — Table 10

01

Summary — three series

1. **Driven insertion.** On $SU(2)$ slabs, the cost of inserting a vortex in one go (0 steps) is large and proportional to the area of T^2, at 11.8, 24.2 and 44.9 ($N_T = 2, 3, 4$, $\beta = 2.3$), but driving it over 256 steps brought it down to 0.50–0.63 Reading (rows A, B and C of Table 1. In rows G and H, under different conditions, the work over 256 steps stays high, at 3.64 and 6.67). When the container (a disc) is smaller than the size of the vortex, increasing the number of steps does not bring it all the way down.
2. Fixing the container radius at 4 and widening to $N_T = 4, 6, 8$ ($\beta = 2.5$, 1024 steps), the work was 1.1–1.6, with no clear increase in N_T Reading (10 configurations $\times$ 3 positions).
3. **The vortex free energy.** The free energy ΔF of the global flux sector increased as T^2 was widened for $U(1)$ ($\beta = 1.5$), and did not increase, or decreased, for $SU(2)$ ($\beta = 2.4, 2.5$) Reading ($M = 8$ and 12; the errors are 0.04–0.78 for $U(1)$ and 0.1–0.36 for $SU(2)$).
4. **String tension of slabs.** For $SU(2)$, $\beta = 2.5$, $M = 16$, the Creutz ratio $\chi(4)$ increased for $N_T = 4 \to 8$ and was nearly saturated at $N_T = 10$. No significant overshoot is seen Reading (2000 measurements).
5. **Simultaneous flips.** The measurement driving containers simultaneously cannot be used as a numerical picture of the crossover band (25-06 § 07). It remains only as a tool for lower bounds on the slab tension with the T^2 fixed.

02

Common setting and how to read it

- **Lattice.** The slab $(\mathbb{Z}/N_T)^2 \times (\mathbb{Z}/M)^2$. Directions 1 and 2 form the twisted T^2 (twist $n_{12} = 1$), and directions 3 and 4 the plane. $SU(2)$ uses the Wilson action, and $U(1)$, as a control, an action of the same form.
- **Driven insertion.** The work W (equation (2) of 25-06) when only the inside of a container, a disc of the plane (radius r) $\times$ T^2, is updated by heat bath and the couplings of the vortex sheet are changed $1 \to -1$ in m steps. The acceptance rate is the mean of $1/(1 + e^W)$ (Barker-type; the acceptance rate of the Driven Insertion Acceptance-Rate Corollary of 25-06, equal to $\frac{1}{2}(1 - \mathbb{E}\tanh(W/2))$). Only the simultaneous flips of Tables 8 and 9 use the mean of $\min(1, e^{-W})$ (Metropolis-type).
- **Vortex free energy.** The $\Delta F = \ln(Z(1,0)/Z(1,1))$ of the two sectors of the flux $n_{34} \in \{0, 1\}$ through the whole $(3, 4)$ torus of the plane, estimated by replica exchange interpolating the coupling and Bennett's acceptance ratio (BAR). The bias is $\theta = \tanh(\Delta F/2)$. The errors are the record's estimates.
- **String tension.** The Creutz ratio $\chi(R)$ built from $R \times R$ Wilson loops in the plane. At small R it contains the Coulomb term and overestimates the tension.
- **Physical size.** In the record, $N_T = 8$ at $\beta = 2.5$ is estimated as $L\sqrt{\sigma} \approx 1.5$, $N_T = 8$ at $\beta = 2.4$ as $L\sqrt{\sigma} \approx 2.2$, and $N_T = 4$ at $\beta = 2.7$ as $L\sqrt{\sigma} \approx 0.4$ Reading (converted from known values of $a\sqrt{\sigma}$). Only the last is in the semiclassical region of a small T^2.

- $N_T = 2$. The rows where the T^2 is 2 in lattice units are a region of large lattice artefacts. They are excluded when reading trends.

03

The work of driven insertion

All are $SU(2)$, twist $n_{12} = 1$. The number of samples is "number of configurations × number of insertion positions per configuration". The figures in brackets for W are the standard deviations of the samples (from the records of the run outputs), not the errors of the means. Samples at different positions in the same configuration are not independent, so the error of the mean should be taken as larger than the standard deviation divided by the square root of the number of samples.

Table 1 Driven insertion (comparison of 0 steps with the slowest driving. Left and right of the slash are the two values of the number of steps or of the radius)

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	Mean of $\tanh(W/2)$	Acceptance rate
P	2.3	2	8	2	0 / 4 / 16	6 × 4	10.7 / 6.5 / 2.9 (not in the record)	0.990 / 0.968 / 0.688	0.005 / 0.016 / 0.156
A	2.3	2	8	2	0 / 256	12 × 4	11.8 (2.5) / 0.50 (0.71)	1.00 / 0.22	0.000 / 0.39
B	2.3	3	8	2	0 / 256	12 × 4	24.2 (4.6) / 0.52 (0.84)	1.00 / 0.21	0.000 / 0.39
C	2.3	4	8	2	0 / 256	8 × 4	44.9 (6.8) / 0.63 (1.19)	1.00 / 0.24	0.000 / 0.38
D	2.3	2	10	3	256	12 × 4	0.19 (0.74)	0.09	0.46
E	2.5	4	10	2 / 4	1024	8 × 3	2.40 (2.11) / 0.44 (1.21)	0.67 / 0.17	0.17 / 0.41
F	2.7	4	12	2 / 4	1024	6 × 3	6.00 (1.34) / 3.77 (1.18)	0.99 / 0.92	0.006 / 0.04
G	2.7	4	18	8	256 / 1024	4–6 × 3	3.64 (3.98) / 2.72 (2.53)	0.54 / 0.59	0.23 / 0.20
H	2.5	8	12	4	256 / 1024	4–6 × 3	6.67 (2.63) / 0.51 (1.64)	0.98 / 0.17	0.012 / 0.42

Row P is the measurement cited in 25-06 § 04. Rows P, A and D are $N_T = 2$ (the region of lattice artefacts).

Reading Reading : (i) The cost at 0 steps is roughly proportional to N_T^2 (11.8, 24.2, 44.9 for A, B, C). (ii) If the container is large enough, the work of driven insertion comes down to near 0. (iii) The container radius needed appears to roughly double each time the lattice spacing halves (2 at $\beta = 2.3$, 4 at 2.5; at 2.7, 4 is not enough and even 8 does not reach near 0). This is consistent with the thickness of the vortex being fixed by a physical length. (iv) The work 0.51 at 1024 steps for H ($N_T = 8$) is almost the same as 0.44 for E ($N_T = 4$), but at 256 steps it is high, 6.67, depending strongly on the speed of driving.

Table 2 Driven insertion with more statistics ($SU(2)$, twist $n_{12} = 1$)

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	$-\ln\langle e^{-W}\rangle$	Mean of $\tanh(W/2)$	Acceptance rate

Run	β	N_T	M	r	Steps m	Samples	Mean of W (s.d.)	$-\ln\langle e^{-W}\rangle$	Mean of $\tanh(W/2)$	Acceptance rate
I4	2.5	4	12	4	1024	10×3	1.32 (1.12)	0.72	0.48	0.26
I6	2.5	6	12	4	1024	10×3	1.14 (1.03)	0.59	0.44	0.28
I8	2.5	8	12	4	1024	10×3	1.64 (1.59)	0.22	0.52	0.24
J	2.7	4	18	8	4096	Not in the record	2.84 (0.84)	2.44	0.85	0.07

Reading Reading : (i) I: fixing the container radius at 4 and widening to $N_T = 4, 6, 8$, the work is 1.1–1.6, with no clear increase in N_T (the error of the mean is the standard deviation $/\sqrt{30}$, about 0.2–0.3). The 0.44 and 0.51 of E and H in Table 1 were fluctuations from little statistics. (ii) J: at $\beta = 2.7$, $r = 8$, driving four times more slowly than G still gives $W \approx 2.8$ (2.7 with 1024 steps for G), and the Jarzynski estimate is 2.4. Most of the work can be read not as dissipation due to the speed of driving but as a genuine free energy. $N_T = 4$ at $\beta = 2.7$ is a physically small T^2 ($L\sqrt{\sigma} \approx 0.4$), on the side of the dilute region, so the reading is that the vortex really is expensive (physics).

On the Jarzynski estimate. $-\ln\langle e^{-W}\rangle$ is an estimate of the sector free energy by the Jarzynski Equality of 25-06, but it is dominated by rare small W, so with few samples it fluctuates strongly. It is a column not shown in Table 1, but negative values have occurred (for example -0.30 for E at radius 4 with 1024 steps, -0.085 for A with 64 steps). This column is a guide for checking, and the values themselves are not reliable.

The acceptance rule. The acceptance-rate columns of Tables 1 and 2 are means of the Barker-type $1/(1+e^{W})$. Recomputing the mean of the Metropolis-type $\min(1, e^{-W})$ from the raw W of the same record gives, in the slowest-driving rows, A 0.641, B 0.647, C 0.595, D 0.729 and E ($r = 4$, 1024 steps) 0.607, all above $\frac{1}{2}$ Computed . Pointwise $\frac{1}{2}\min(1, e^{-x}) \le 1/(1+e^{x}) \le \min(1, e^{-x})$, and under a Crooks-type relation the Barker-type mean is at most $1/(1+e^{\Delta F})$ (at most $\frac{1}{2}$ if $\Delta F \ge 0$) Paper Lean (LatticeGaugeAcceptanceSandwich (half_metro_le_fermi, fermi_le_metro, barker_le_of_crooks)). That the acceptance rates of the tables stay below $\frac{1}{2}$ is the ceiling of this rule, not a physical limit of the kernel.

The limits of this series. What confinement on R^4 needs is that the bias becomes small exponentially in the area as L grows (26-01), and that is not visible in this measurement. β takes a few values between 2.3 and 2.7, and the error estimates are still rough.

04

The vortex free energy when the T² is widened

The container is the whole plane ($M \times M$), and the $T^2 = N_T \times N_T$ is widened. It is a test of the sign of the lateral-widening inequality (26-02). All by replica exchange + BAR, twist $n_{12} = 1$, 24 replicas (unless otherwise stated). The number of iterations is not in the record.

Table 3 ΔF, plane $M = 8$

Theory	β	$N_T = 2$	$N_T = 4$	$N_T = 6$	$N_T = 8$

Theory	β	$N_T = 2$	$N_T = 4$	$N_T = 6$	$N_T = 8$
$U(1)$ (control)	1.5	-0.03 ± 0.04	0.83 ± 0.11	2.10 ± 0.47	5.14 ± 0.78
$SU(2)$	2.5	-0.01 ± 0.12	0.63 ± 0.18	0.33 ± 0.22	1.51 ± 0.29

Table 4 ΔF, plane $M = 12$ (remeasurement with a larger container)

Theory	β	$N_T = 4$	$N_T = 6$	$N_T = 8$
$U(1)$ (control)	1.5	0.20 ± 0.21	1.35 ± 0.30	1.72 ± 0.47 (40 replicas; with 24, 0.54 ± 0.53, with poor mixing)
$SU(2)$	2.5	0.57 ± 0.22	0.42 ± 0.24	-0.15 ± 0.36

Table 5 ΔF, $SU(2)$, $\beta = 2.4$, plane $M = 8$

N_T	**4**	**8**
ΔF	0.31 ± 0.10	0.11 ± 0.25

Reading Reading : (i) For $U(1)$, ΔF increases as the T^2 is widened (at $M = 8$, θ goes $0 \to 0.39 \to 0.78 \to 0.99$; at $M = 12$, $0.20 \to 1.35 \to 1.72$). (ii) For $SU(2)$, in three settings ($M = 8$ and 12 at $\beta = 2.5$, $M = 8$ at $\beta = 2.4$), it does not increase or decreases when widened. The exception is 1.51 at $N_T = 8 = M$ (the symmetric torus T^4_8) for $\beta = 2.5$, $M = 8$. (iii) For $N_T = 4 \to 8$, $\ln(1/\theta)$ goes 0.93 $\to$ 0.012 for $U(1)$ (about 1/80) and 1.18 $\to$ 0.45 for $SU(2)$ (about 1/2.6) ($M = 8$). This can be read as a numerical sign of the sign of the lateral-widening inequality. (iv) That $N_T = 8$ at $\beta = 2.4$ is smaller than at $\beta = 2.5$ is consistent with the same lattice being physically larger ($L\sqrt{\sigma} \approx 2.2$ against 1.5).

Limits. This is a regime where θ is not close to 1 (it is $O(1)$), so the $U(1)$ ratio is also held down because $\theta \leq 1$, and in this geometry the lateral-widening inequality has little power to discriminate. The errors are large, 0.04–0.78 for $U(1)$ and 0.1–0.36 for $SU(2)$, and β is one or two points. $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not matched to "the same physics". The $N_T = 2$ column is in the region of lattice artefacts.

05

The string tension of slabs

$SU(2)$, $\beta = 2.5$, plane $M = 16$, twist $n_{12} = 1$. We measured the Creutz ratio $\chi(R)$ of Wilson loops in the plane for each size N_T of the T^2. It is a test of "monotonicity of the reduced two-dimensional string tension $\sigma_{\text{eff}}(N_T)$" contained in the lateral-widening inequality (strong form) of 26-02, and of an "overshoot" in which the tension once becomes large at $L \sim 1/\Lambda$. In the record, the four-dimensional value at $\beta = 2.5$ is given as $\sigma a^2 \approx 0.036$ Reading .

Table 6 Creutz ratios, 400 measurements

N_T	$\chi(3)$	$\chi(4)$	$\chi(5)$
2	0.098 ± 0.006	0.034 ± 0.029	-0.02 ± 0.14
4	0.113 ± 0.007	0.070 ± 0.018	0.07 ± 0.06
6	0.108 ± 0.003	0.089 ± 0.013	0.11 ± 0.05
8	0.113 ± 0.003	0.051 ± 0.010	0.07 ± 0.04

Table 7 Creutz ratios, 2000 measurements (5 times the statistics) and $N_T = 10$

N_T	$\chi(3)$	$\chi(4)$	$\chi(5)$
4	0.103 ± 0.002	0.059 ± 0.009	0.03 ± 0.04
6	0.115 ± 0.001	0.061 ± 0.004	0.084 ± 0.019
8	0.116 ± 0.002	0.076 ± 0.002	0.070 ± 0.022
10	0.112 ± 0.002	0.071 ± 0.002	0.076 ± 0.009

Reading Reading : (i) With 400 measurements $\chi(4)$ was largest at $N_T = 6$ (0.089), and its difference from $N_T = 8$ was about 2.3 standard deviations. It looked like a sign of overshoot, but with 2000 measurements $N_T = 6$ gives 0.061 ± 0.004, so it was **a statistical fluctuation**. (ii) With 2000 measurements $\chi(4)$ increases for $N_T = 4 \to 8$, and at $N_T = 10$ it is 0.071 (1.7 standard deviations from $N_T = 8$), almost saturated. No significant overshoot is seen. This is consistent with monotonicity of the tension (a necessary condition of 26-02). (iii) $\chi(4)$ contains the Coulomb term, so it comes out larger than the four-dimensional $\sigma a^2 \approx 0.036$. We read only the trend in N_T, not the absolute value. (iv) The $N_T = 2$ row (Table 6) is in the region of lattice artefacts, with only 400 measurements.

Limits. β is the single value 2.5, and the plane the single size $M = 16$. From $\chi(5)$ on the errors are large, and the value of the tension itself is not determined. $N_T = 10$ is about 2 in $L\sqrt{\sigma}$, and the trend as $L \to \infty$ is an extrapolation.

06

The measurement of simultaneous flips, and why it cannot be used as a test of the band

A measurement that uses the Simultaneous-Flip Estimate of 25-06 ($\mathbb{E}\prod\theta \le \mathbb{P}(\text{no container flips})$) numerically. $SU(2)$, $\beta = 2.5$, containers placed with spacing B, and driven insertions (512 steps) with acceptance $\min(1, e^{-W})$ performed simultaneously in 4 containers per configuration. We write $\bar{a}$ for the mean acceptance rate.

Table 8 Container radius fixed at $r = 4$ ($B = 10$, $M = 20$, 512 steps)

N_T	Configurations	$\bar{a}$	Mean of W	W/N_T^2
4	30	0.36 ± 0.03	1.6	0.10
6	20	0.25 ± 0.04	2.6	0.07
8	10	0.12 ± 0.03	4.5	0.07
10	6	0.02 ± 0.01	8.2	0.08

Table 9 Remeasurement with container radius proportional to N_T ($r = N_T$, $B = 2r + 2$, $M = 2B$, 512 steps)

N_T	r	Configurations	$\bar{a}$	Mean of W
4	4	30	0.35 ± 0.03	1.6
6	6	11	0.34 ± 0.05	2.0
8	8	3	0.16 ± 0.08	4.5

Reading Reading : (i) Table 8: widening the T^2 with the radius fixed, the work increases roughly in proportion to N_T^2, and the acceptance rate falls. The size of the vortex is fixed by the size of the T^2, so the reading is that a container of radius $r < L$ cannot hold the vortex, which then pays a cost proportional to the area as a sheet inside a thin tube. The lateral-widening inequality is a claim about containers of size at least L, so this is not a counterexample. The guaranteed tension ($-\ln \mathbb{P}(\text{no flip})$ divided by the number of containers and B^2) was about 0.005 at $N_T = 4$. (ii) Table 9: if the containers are enlarged together with L, the acceptance rate is almost constant for $N_T = 4 \to 6$ ($N_T = 8$ cannot be judged from 3 configurations).

Why it cannot be used as a test of the band. This measurement **cannot be read** as "a numerical picture of the uniform slab tension (the crossover band)". The reason is as in 25-06 § 07: the guarantee of the Flow Lemma is $\theta \le 1 - a$, and obtaining the form of the band $\theta \le e^{-cL^2}$ needs a kernel with $a \ge 1 - e^{-cL^2}$, which requires bringing the overlap O of the measures of the forward and reverse paths to 1 exponentially in the area (in terms of dissipation, $D \lesssim e^{-2cL^2}$) (25-06 § 07). The kernels measured are far from that, so all that can be said is "for containers of size L, $\theta \le \theta_0$ (with θ_0 a constant less than 1)", and the resulting $\sigma \ge \ln(1/\theta_0)/L^2$ is the weak form that $U(1)$ also satisfies (26-01). **This measurement can be used only as a tool for measuring lower bounds on the slab tension with the T^2 fixed.** The continuation of Table 9 at $N_T = 6, 8$ has been stopped.

07

What cannot be said

(a) Proof. None of the numbers of this chapter is a proof, and the hypotheses of no inequality have been checked.

(b) The band. The trends of Table 2, Tables 3–5 and Table 7 do not show the crossover band ($\theta \le e^{-cL^2}$ uniform in L) numerically. The T^2 measured go only up to about $L\sqrt{\sigma} \approx 2$, and $L \to \infty$ is an extrapolation.

(c) Errors. The error estimates of driven insertion are still rough (sample standard deviations only), the errors of replica exchange are the record's estimates, and the treatment of autocorrelation is not in the record.

(d) Lattice artefacts. The rows with $N_T = 2$ are the region of lattice artefacts. Even at $N_T = 4$, small-lattice corrections remain.

(e) Balance of the control. $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not a comparison with matched physical sizes.

(f) Simultaneous flips. Tables 8 and 9 cannot be read as a numerical picture of the band.

08

The table of five grades and the dependency graph

Table 10 The rows of this chapter (all numerical readings. Conditions in the scope column of each row)

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
1	The cost of insertion at 0 steps is roughly proportional to N_T^2	Reading	Table 1 (A, B, C)	$SU(2)$, $\beta = 2.3$, $M = 8$, $r = 2$, 8–12 configurations $\times$ 4	—
2	If the container is large enough, the driven work comes down to near 0	Reading	Table 1	$\beta = 2.3$ to 2.5, 256–1024 steps, includes $N_T = 2$ rows	—
3	The container radius needed roughly doubles each time the lattice spacing halves	Reading	Table 1 (E, F, G)	$\beta = 2.3, 2.5, 2.7$, little statistics	—
4	With radius 4 fixed, the work does not clearly increase for $N_T = 4 \to 8$	Reading	Table 2 (I)	$\beta = 2.5$, $M = 12$, 1024 steps, 10 configurations $\times$ 3, error 0.2–0.3	—
5	The work at $\beta = 2.7$, $r = 8$ is free energy, not dissipation	Reading (physics)	Table 1 (G), Table 2 (J)	$N_T = 4$, $M = 18$, the Jarzynski estimate fluctuates strongly	—
6	Widening the T^2 increases the ΔF of $U(1)$	Reading	Tables 3 and 4	$\beta = 1.5$, $M = 8, 12$, errors 0.04–0.78	—
7	Widening the T^2 does not increase, or decreases, the ΔF of $SU(2)$ (except $N_T = M = 8$)	Reading	Tables 3–5	$\beta = 2.4, 2.5$, $M = 8, 12$, errors 0.1–0.36	—
8	Rows 6 and 7 are a sign of the sign of the lateral-widening inequality	Reading (physics)	§ 04 ⟸ 6, 7	Weak discriminating power in the region where θ is $O(1)$	—
9	The overshoot at $N_T = 6$ is a statistical fluctuation	Reading	Tables 6 and 7	$\beta = 2.5$, $M = 16$, 400 and 2000 measurements	—
10	$\chi(4)$ increases for $N_T = 4 \to 8$ and nearly saturates at $N_T = 10$	Reading	Table 7	$\beta = 2.5$, $M = 16$, 2000 measurements, includes the Coulomb term	—

Row	Statement (one line)	Grade	Evidence	Scope	Machine checking
11	With the radius fixed, the work of simultaneous flips grows in proportion to N_T^2	Reading	Table 8	$\beta = 2.5$, $r = 4$, 6–30 configurations × 4 containers	—
12	The acceptance rate of one-step kernels is, in the range measured, at most 0.46 Barker-type and at most 0.73 Metropolis-type (Tables 8 and 9 at most 0.36 Metropolis-type)	Reading	Tables 1, 2, 8 and 9	The range of β, N_T and containers measured	—
13	The measurement of simultaneous flips cannot be used to test the band	Proved here	§ 06, 25-06 § 07 ⟸ 12	When only the Flow Lemma $\theta \leq 1 - a$ is used	— (the core is FluxLemma (`FluxLemma.flip_le_one_sub_abs`) of 25-06)

Only the row "the measurement of simultaneous flips cannot be used to test the band" is a row of logic rather than numerics: it is an application of the row "only the weak form follows" of 25-06, that only the weak form follows from $\theta \leq 1 - a$; the reading of the acceptance-rate row shows how far that takes effect.

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  RDG  cost at 0 steps ∝ N_T²                          β = 2.3, M = 8
 2  RDG  driven work → near 0 in large containers        β = 2.3–2.5
 3  RDG  radius needed doubles per halving of a ⟸ 2
 4  RDG  radius 4, widening N_T: no increase             β = 2.5, 10 confs × 3
 5  RDG  work at β = 2.7 is free energy (physics)        Jarzynski fluctuates
 6  RDG  ΔF of U(1) grows with T²                        β = 1.5
 7  RDG  ΔF of SU(2) does not grow                       β = 2.4, 2.5
 8  RDG  sign of the lateral-widening inequality (physics) ⟸ 6, 7
 9  RDG  overshoot at N_T = 6 is a fluctuation           2000 measurements
10  RDG  χ(4) grows and saturates ⟸ 9                     includes Coulomb term
11  RDG  fixed-radius simultaneous flips ∝ area
12  RDG  one-step acceptance: Barker ≤ 0.46, Metropolis ≤ 0.73 (range measured)
13  PRV  simultaneous flips cannot test the band ⟸ 12   Flow Lemma of 25-06
```

The way to check is to see by eye that every number to the right of "⟸" is smaller than the number on the left. There are no circles. There are no assumption rows. There is neither a node of confinement nor a node of the mass gap.

What this chapter can say

What is said	What is not said
Reading Driven insertion brings the work down to near 0 if the container is larger than the vortex ($SU(2)$, $\beta = 2.3$ to 2.5)	That the upper bound on the work is uniform in L and the lattice spacing
Reading Widening the T^2 makes the vortex of $U(1)$ more expensive and that of $SU(2)$ not ($\beta = 2.4, 2.5$, $M = 8, 12$)	The lateral-widening inequality. Discrimination in the region where θ is small
Reading The string tension of slabs (Creutz ratios) grows with NT and saturates, with no significant overshoot ($\beta = 2.5$, 2000 measurements)	The value of the tension itself. The behaviour as $L \to \infty$
Proved here The guarantee of simultaneous flips cannot be used to test the band	A numerical test of the band with a kernel with $a \to 1$

Sources and reproduction

Item	Kind	Source / tools
All the numbers of Tables 1–9	Reading	The Monte Carlo record of this book's work (heat bath, driven insertion, replica exchange with BAR, Creutz ratios). Compared against the output records of each run; the standard deviations of Table 1 were copied from the output records
Row P of Table 1	Reading	The same measurement as the numerical section of the paper on vortex-insertion identities in A-07 (A-07)
Tables 3–5, Table 7	Reading	The same measurement as the numerical section of the paper on the reduction of the band in A-07
$L\sqrt{\sigma}$ and the four-dimensional $\sigma a^2 \approx 0.036$	Reading	Conversion in the record (by known values of $a\sqrt{\sigma}$; the values of the original source not compared)
The Jarzynski Equality	Cited	The source table of 25-06
The logic of why simultaneous flips cannot be used to test the band	Proved here	25-06 § 07

Words used in this chapter — driven insertion, number of steps, work, acceptance rate, the free energy of the global flux, replica exchange, BAR, Creutz ratio, overshoot, simultaneous flips, guaranteed tension. Their meanings are in A-02 Glossary (the words of Part 25 are to be added). The numerics of Part 26 (the test doubling the whole lattice, step functions, column biases) are in 26-09.

Next chapter to read: 25-09 Where this road stops — confirms that widening the T^2 is exactly the crossover band, and summarises what the road from the direction of space has reached.

2026-10-15 · chapter Yang–Mills and the mass gap Part 25 The second road — from the direction of space the band, lateral widening, the end of the road · Kiichi, Shiori and Rin

Where this road stops — widening the T² is the band

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — At which stage the road from the direction of space, starting from a small twisted T^2, stops. It stops at the stage of widening the side L of the T^2 beyond the correlation-length scale $1/\Lambda$, and this is the crossover band (the band for short). At the two ends of the band — the region of dilute fractional instantons where L is small, and the confinement region where L is large — there is a prospect that the needed inequality holds, and all that remains is to pass non-perturbatively through the middle, $L \sim 1/\Lambda$.

Prerequisite chapters — 25-00 (the entrance of this part), 25-02 (vortices and fractional instantons), 25-04 (slab confinement), 16-09 (the band stage in the table of the seven stages a proof must contain).

Said first — This is a short guiding chapter, and it places no new proofs. Every statement crossing the band is an assumption, and the prospects at the two ends are physical estimates resting on one-loop and dilute-gas counting, or on the picture of confinement.

The order of this chapter

1. The stage where it stops — what is in hand on a small T^2, and the stage of widening
2. The two ends and the middle of the band — the dilute region, the confinement region, and between them (Table 1)
3. The statement to show in the band — the correct form of the band, and intermediate goals
4. To Part 26

01

The stage where it stops — what is in hand on a small T^2, and the stage of widening

The road of this part went by way of a small twisted T^2 in order to split the four-dimensional problem in two (25-00). Since the twist removes the zero modes, the theory stays weakly coupled at scales up to the size L of the T^2, and the vortex sheets winding the T^2 close locally as fractional instantons. In this way the problem splits into an intermediate goal where only the ultraviolet wall remains (the gap and confinement on a small T^2; among the seven stages a

proof must contain, the small-T^2 gap stage and the small-T^2 confinement stage) and a stage where all other walls gather (widening the T^2, the band stage).

What is in hand on the side of the small T^2 is all either conditional or stated with the lattice spacing fixed. With the lattice spacing and the T^2 fixed, the area law holds at every β, but the tension is fixed by the cost of thin vortices and becomes 0 in the continuum limit Proved here (25-04). The insertion identities and averaged bounds reduce confinement to a single local quantity Proved here (25-05). Confinement and the gap on a small T^2 with the lattice spacing sent to 0 are a strand resting on assumptions such as ultraviolet control Assumption . Ultraviolet control in this part is "that the same construction as the one published on 2026-11-10 stands under twisted boundary conditions"; the published construction is only for a periodic cubic box, so this is an open condition (the arrow of the small-T^2 gap and the arrow of the small-T^2 area law in 27-01).

What the problem statement asks for is the theory on $\mathbb{R}^4$, so $L \to \infty$ is needed. One must widen L from well below the correlation length $1/\Lambda$ to well above it, keeping a bound with the same constant at every L. This stage is where this road stops, and in this book we call it **the crossover band** (the band for short) Assumption . It is a different word from the "windows" of Parts 23 and 24 (the site window, the window in $\hbar$, the time window) (note in 26-00).

02

The two ends and the middle of the band

The statement to show in the band, in the language of the bias of the sector of a container $\theta(R; T^2_L)$ (container side R, side L of the transverse T^2, 25-03), has the form that the bias does not grow when the transverse T^2 is widened (the lateral-widening inequality, 26-02). The situation differs at the two ends and in the middle.

Table 1 The two ends and the middle of the band (Λ is the dynamical scale, L the side of the transverse T^2)

Region	Picture	When the transverse T^2 is widened	Label	Conditions written in the same row
$L \ll 1/\Lambda$ (dilute region)	Vortices are self-dual lumps of charge $\frac{1}{2}$ (fractional instantons), and the action $4\pi^2/g^2$ does not depend on the shape of the T^2. In the dilute gas $\ln(1/\theta) \propto A_D\, L^{5/3}$ (A_D is the area of the container)	$\ln(1/\theta)$ grows and the bias becomes smaller. The exponent $5/3 = 11/3 - 2$ is the margin by which the coefficient of asymptotic freedom beats the entropy of the two transverse dimensions	Reading (physics: one-loop and dilute-gas counting). The formula of the tension $\Lambda^2 (\Lambda L)^{5/3}$ is Cited	Dilute region, one loop, SU(2). For U(1) the same counting gives exponent -2

Region	Picture	When the transverse T^2 is widened	Label	Conditions written in the same row
$L \gg 1/\Lambda$ (confinement region)	The free energy of the vortex sheet in a container is tension × area, and the dependence on the area of the T^2 is only the entropy of the position of the vortex sheet (logarithmic)	The ratio of biases stays of order s^2	Reading (physics: on the picture of confinement)	A picture that assumes confinement. Not shown
$L \sim 1/\Lambda$ (middle)	Neither the weak-coupling picture nor the strong-coupling picture can be used	The statement to be shown itself	Assumption	Needs non-perturbative input. Structural inputs (reflection positivity, duality, the transfer matrix) alone do not give it (26-04)

Putting U(1) beside it as a control: the flux of $U(1)$ can only spread uniformly, and $U(1)$ on a slab becomes at long distances a two-dimensional $U(1)$ with coupling e^2/A_T (A_T is the area of the T^2), so $\ln(1/\theta) \propto A_D/A_T$, and the ratio of biases when the T^2 is widened (after over before) becomes arbitrarily large Reading (physics: the estimate counting of the paper on the reduction of the band). Weakly coupled four-dimensional $\mathbb{Z}_2$ breaks it in the same way. What separates SU(2) from U(1) is that non-abelian flux can localise into lumps of finite action independent of scale (26-03). The measured record (the vortex free energy when the T^2 is widened goes up for U(1) and not for SU(2); the slab tension grows and saturates) is in 25-08 and 26-09 Reading .

03

The statement to show in the band — the correct form of the band, and intermediate goals

The goal of the band is not "the bias stays uniformly away from 1" ($\sup_L \theta(L;T_L^2) \le \theta_0 < 1$). This form is satisfied by four-dimensional $U(1)$ too, and the tension it gives is $\ln(1/\theta_0)/L^2 \to 0$ (26-01). The correct form of the band is that the vortex free energy of the container becomes small exponentially in the area,

$$\ln \frac{1}{\theta(L;T_L^2)} \;\ge\; c\,L^2 \qquad \text{(in physical units, uniform in the lattice spacing } a\text{)} \quad (1)$$

a local version, conditioned on the frame, of 't Hooft's criterion for confinement Assumption . (1) follows by iterating doubling, given the base point, frame decorrelation and the lateral-widening inequality (the Doubling-Iteration Theorem of 27-01 § 03 and its premises). In the language of the chessboard, the content of (1) becomes two uniformities: the uniform slab tension and the uniform slab gap (26-05).

As intermediate goals before reaching (1), two weaker statements have been named Assumption . (a) That the vortex free energy of the container approaches 0 as $L \to \infty$ (a restatement of there being no deconfined phase). (b) $\ln(1/\theta) \geq cL^2/\ln L$ (an almost-area law). Neither has been shown.

04

To Part 26

This road came as far as just before the band and stopped. What is needed in the band can be narrowed in form with the tools of this part: the correct form of the band (1), the reduction to one inequality, where non-abelianness enters, and that structure alone cannot cross, in that order. The road from the direction of time (Parts 23 and 24) also arrives at a uniformity of the same type when its four-dimensional goal is written in the form "a constant uniform in $\beta \to \infty$" (23-15 § 08). That the two roads meet at this band is treated in the entrance of Part 26.

What this chapter can say

What is said	What is not said
That the stage at which the road from the direction of space stops is the stage of widening the transverse T^2 beyond the correlation-length scale (the band). The correct form of the band (1), and the naming of the exclusion of a deconfined phase (a)	Each of (1), (a) and (b) is Assumption
The prospects at the two ends: $5/3 = 11/3 - 2$ in the dilute region, logarithmic entropy in the confinement region Reading (physics)	Passing non-perturbatively through the middle $L \sim 1/\Lambda$. Proofs of the prospects at the two ends

Sources and reproduction

Item	Kind	Source / tools
The string tension $\Lambda^2(\Lambda L)^{5/3}$ on a small T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, equations (2.32) and (2.37) (the source table of 25-02)
$5/3 = 11/3 - 2$ in the dilute region, -2 for $U(1)$	Reading (physics)	The proposition "the inequality separates theories" in the paper on the reduction of the band (draft, A-07) (the paper itself states that it is an estimate)
The correct form of the band (1) and the counterexample $U(1)$	Assumption, Reading	26-01. The Tomboulis–Yaffe inequality is Cited

Item	Kind	Source / tools
Exclusion of a deconfined phase (a)	Assumption	The row "positivity of the 't Hooft tension" in the ledger of 27-02

Words used in this chapter — the crossover band (the band), the lateral-widening inequality, the bias of the sector $\theta(R;T_L^2)$, the dilute region, the confinement region, intermediate goals. Their meanings are in A-02 Glossary.

Next chapter to read: 26-00 Entrance — confluence — that the two roads arrive at a statement of one type, "the same constant at all scales".