Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 26 The crossover band

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

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2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band two roads, the ladder of scales, the breakdown of the band · Kiichi, Shiori and Rin

Entrance — confluence

> This chapter is a guide. § 03 and § 05 are a map of open questions.

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — That the road entering from the direction of time (Parts 23 and 24) and the road entering from the direction of space (Part 25) both arrive at a statement of one type: "it holds at every scale with the same constant". In this part we call the interval of scales that this statement has to cross **the crossover band** (**the band** for short). In the figure of the ladder of scales (Figure 2) we see how far the weak-coupling tools and the strong-coupling tools each reach, and where the gap is. We put the breakdown of the band (the goal, the reduction, the negative lemmas, the electric flux, the place of the non-abelian, β-monotonicity, the renormalisation comparison, the chessboard) into one table, and give a map of the thirteen chapters of Part 26.

Prerequisite chapters — 23-15 § 07, 23-15 § 08 (the one line on uniformity), 24-07 § 01 (the dimensionless coupling), 4-07 (dimensional transmutation), 25-00 (the entrance to the road from the direction of space), 16-09 (the seven stages a proof must contain).

Said first — This page is a map and sets out no new theorem. Figure 2 is a figure of physical estimates, and each row carries a label. "The two roads meet" is a statement at **the level of the goal**. It is not a statement that the **walls at which the roads now stop** are the same (§ 05). No chapter of this part shows the mass gap, or confinement in the continuum limit.

The order of this chapter

1. One type that the two roads reach — "the same constant at every scale"
2. A note on words — the "band" of this part and the "window" of Parts 23 and 24 are different things
3. The ladder of scales — which tool works at which scale (Figure 2)
4. The breakdown of the band — eight components and their labels (Table 1)
5. Confluence at the level of the goal, difference at the level of the wall (Table 2)
6. The map of Part 26 — thirteen chapters, and the one line of 24-07 (Table 3)
7. The table of the five grades and the dependency graph

01

One type that the two roads reach — "the same constant at every scale"

From Part 23 onwards, this book has approached the problem of the mass gap from two directions.

The first is the road from the direction of time (Parts 23 and 24). Around **the diffusive-relaxation-time assumption** — that the time over which fluctuations are smoothed out grows as the square of the correlation length (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption; 23-01 § 05) — we compared the relaxation time on the lattice with the correlation length. The statement aimed at on the four-dimensional lattice is the physical-unit target level of Table 5 in 23-15 § 07: the mass in physical units of the lattice, $m_{\mathrm{lat}}(\beta)/a(\beta)$, tends to a positive finite value. To get there by combining the one-sided bridge (23-01 § 06) with the upper-bound assumption, the constant K of the upper-bound assumption must be uniform as $\beta \to \infty$.

The second is the road from the direction of space (Part 25). We wrap two directions of space on a small torus T^2_L with the 't Hooft twist $n_{12} = 1$, and look at confinement on the remaining plane $\mathbb{R}^2$. If T^2_L is small enough, the twist removes the zero modes, the theory stays weakly coupled up to the scale L, and the vortices can be treated as fractional instantons (25-00, 25-02). The goal of this road is that the string tension $\sigma_{\mathrm{eff}}(L)$ of the two-dimensional theory obtained by shrinking the transverse directions to T^2_L is **positive uniformly in** L, that is,

$$\inf_{L \geq L_0} \sigma_{\mathrm{eff}}(L) \;>\; 0 \qquad \text{(physical units, uniform in the lattice spacing } a\text{)} \tag{1}$$

(the uniform slab tension; the definition is in 26-02).

The two goals have the same form. Neither is "it holds at some scale"; both are statements that "**it holds at every scale, with the same constant**". They ask that the constant does not break down at any intermediate scale, from the lattice spacing a across the physical length $1/\Lambda$ up to large scales. The one line of 23-15 § 08 — what is missing is not the number of steps but uniformity, holding at every scale with the same constant — was a statement of this type written from the side of the first road. We also saw at the end of 23-15 § 07 that it appears in changed form: uniform as $\hbar \to 0$ at the first step (0+1 dimensions), uniform as $\beta \to \infty$ on the four-dimensional lattice.

In four dimensions, between the weak-coupling scales and the strong-coupling scales, there is a scale $L \sim 1/\Lambda$ at which the size of the coupling is of order 1. Λ is a scale born quantum mechanically through dimensional transmutation (4-07), and measured in lattice units it diverges as $a \to 0$. Neither of the two goal statements holds unless it passes through this scale with a uniform constant. In this part we call the interval of scales around $L \sim 1/\Lambda$ **the crossover band**, and we call the statement the two roads reach "a statement that crosses the band".

This confluence also agrees with an arrangement made outside our study. In the division of the continuum tasks (the task of a uniform gap in 16-07), the uniform gap $\inf_L \Delta_L > 0$ is set as one task, and the two roads attack it in two languages. In the table of the seven stages a proof must contain in 16-09, the band corresponds to the "band stage".

02

A note on words — the "band" of this part and the "window" of Parts 23 and 24 are different things

Parts 23 and 24 used the word "window" in at least three senses. **The "band" of this part is different from all of them.** If the words were the same, the two roads would read as if they stopped at the same place, so we separate them here.

Table 0 The "window" and the "band" compared, and notes on other words and symbols

Word	Meaning	Chapters	Relation to the band
The window of sites (the window of A_W, the window mark σ_W, the marginal log-Sobolev input for the window)	An interval of finitely many sites on the path in 0+1 dimensions	23-09	Of finite length in lattice units. Not the band
The window in $\hbar$	The range of $\hbar$ that can be checked with computer assistance	23-07, 23-08, 23-15	An interval of a parameter. Not an interval of scales
The time window (the window factor φ, the In-Window Supremum Proposition)	Test functions restricted to an interval of time	24-04, 24-05	The name of a tool. Not the band
The crossover band (the band)	The interval around the scale $L \sim 1/\Lambda$ at which weak coupling gives way to strong coupling	Part 26	—
The "band" of Parts 23 and 24	The "band statement" of 23-01 § 05 and 23-15 (5) (the statement that a quantity lies in a band of width a constant multiple of the square of a length), ℓ_{band} of 23-09 (a length concerning the summit band), and the "two-dimensional strip" of 24-02 § 03, § 05 and 24-09 (a narrow strip-shaped lattice). The "twisted strip" of the two-dimensional model in 25-07 § 06 also means a strip-shaped lattice	23-01, 23-09, 23-15, 24-02, 24-09, 25-07	A word with another meaning. Not the band of this part (an interval of scales)
Three meanings of θ	The threshold θ of 23-09 (θ_B, θ_σ and others, values that divide configurations into the valley side and the summit side), the sector bias $\theta(R; T_L^2)$ of this part ((4) of 25-01), and the vacuum angle θ of Tanizaki–Ünsal (set to $\theta = 0$ in 25-02)	23-09, 25-01, 25-02	Only the sector bias is a quantity of this part. The other two are different quantities

Word	Meaning	Chapters	Relation to the band
L and L_0	In 26-11, L is the blocking factor and L_0 is the side of the box (as in 7-05)	7-05, 26-11	Different from the L of Parts 25 and 26 (the size of T^2)

There are other clashes of symbols, so in this part we write by the following conventions.

- We call by name the property that the vortex free energy increases monotonically in β — "**β-monotonicity**" — and the inequality comparing the vortex free energy of the doubled lattice with that of the coarse lattice — "**the renormalisation comparison**".
- We call the assumptions only by name: **the base point at small T^2**, **frame decorrelation**, **the lateral-widening inequality** (strong form, weak form), **the uniform slab tension**, **the uniform slab gap**.
- We write the chessboard density as κ_{cb} (to distinguish it from κ of Part 23).

03

The ladder of scales — which tool works at which scale

Figure 2 sets out, in the language of the road from the direction of space, the scale L (the size of the small torus T^2_L, or the size of the box being compared) vertically. The bottom is short distance (the lattice spacing a), the top is long distance. Each row carries a label at its right end. Figure 1 (the flow of the logic) is in 16-09. The number and title of Figure 2 are the same as those of Figure 2 in 25-00.

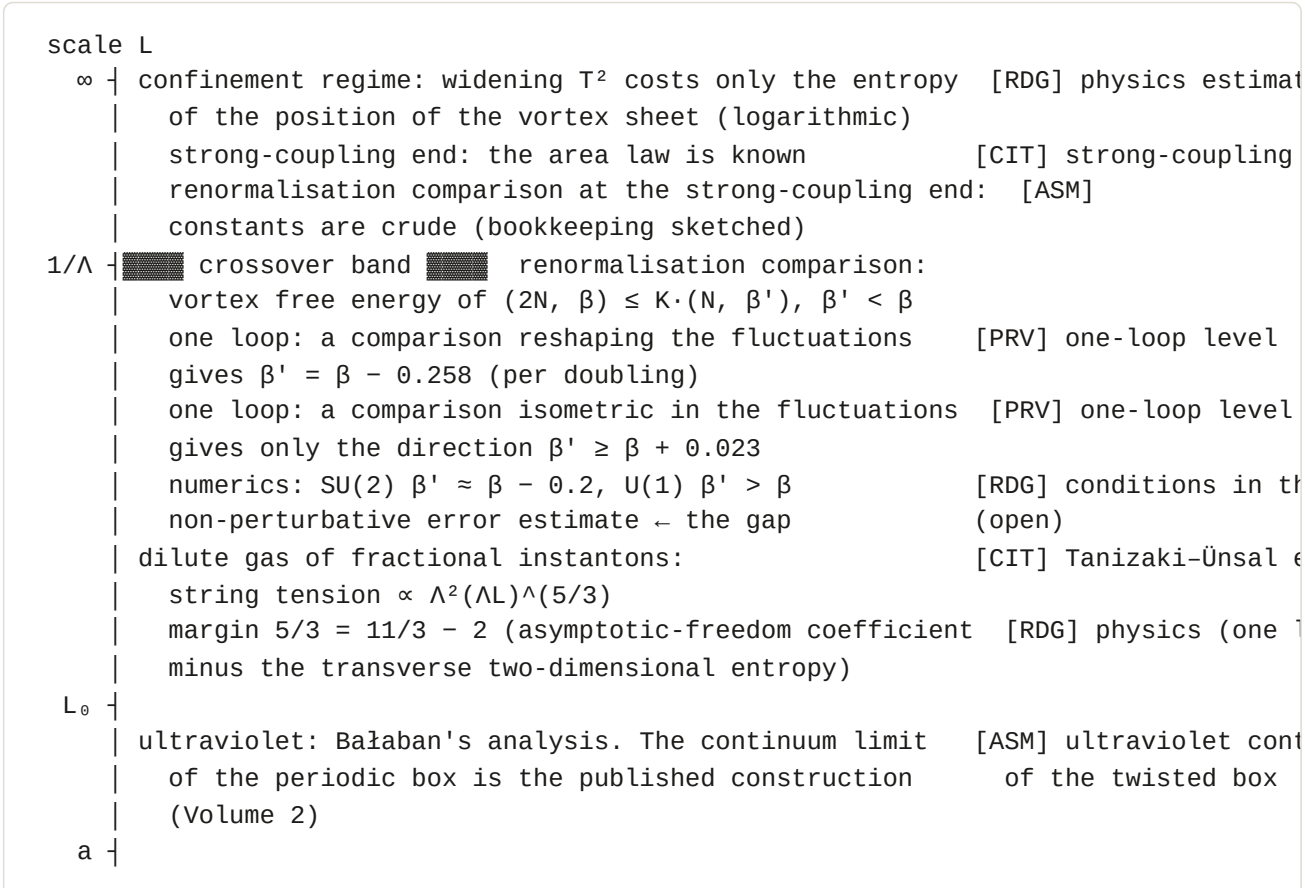

Figure 2 The ladder of scales. The weak-coupling tools (bottom) and the strong-coupling tools (top) each work at their own end; only the non-perturbative estimate in the band (L ~ 1/Λ) is missing. Labels are per row.

Just three points on how to read the figure.

(i) The shift of β per doubling. "β′ = β − 0.258" is the value coming from the coefficient $11/3 = -1/3 + 4$ of the one-loop heat kernel of fluctuations in a background magnetic field; in the convention $\beta = 4/g^2$ it is $\beta - \beta' = 8b_0 \ln 2$, with $b_0 = 22/(48\pi^2)$. A comparison that reshapes the fluctuations reproduces this value. In a comparison isometric in the fluctuations (the identity map, translation, gauge transformation, Migdal–Kadanoff-type decimation), the spin term $+4$ does not contribute to the first-order estimate, and only $\beta' \geq \beta + 0.023$ on the $-1/3$ side (the same direction as U(1)) comes out. Both are **statements at the one-loop level**, not non-perturbative comparisons. The breakdown into tables and the computation are in 25-07; the comparison itself is treated in 26-06.

(ii) The conditions of the numerical "β′ ≈ β − 0.2". This is a value read from measurements of the vortex free energy ΔF for $SU(2)$ with twist $n_{12} = 1$, by replica exchange and Bennett's acceptance-ratio method: the whole lattice was doubled ($N_T \times M = 4 \times 8 \to 8 \times 16$ and $4^4 \to 8^4$), and the value on the large lattice was fitted to the β' curve of the small lattice. For $\beta = 2.5$–3.0, with 1–3 seeds, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ (the width of the reading is of order ± 0.05–0.1; judging from the differences between seeds, the errors are rather underestimated). For $U(1)$ at $\beta = 1.5$, $4 \times 8 \to 8 \times 16$, $\beta - \beta' < -0.2$ (the opposite direction). The lattices are small and the statistics few, so this is a reading of the direction, not a determination of the value [Reading]. The table is in 26-09.

(iii) The gap. The strong-coupling end and the one-loop end are each held by a different tool. In the band in the middle, there is no tool that estimates the error of the comparison non-perturbatively (at intermediate coupling, large fields included). The "▒" row of the figure is the subject of this whole part.

04

The breakdown of the band — eight components and their labels

Table 1 lists, in the language of the road from the direction of space, the components involved in the statement that crosses the band. Labels are per row, and conditions are written in the same row. The labels mean what they mean in the five grades of Part 23: [Proved here] a proof set out on paper; [Cited] known, checked in the original source; [Computer-assisted]; [Reading] a numerical reading (physical estimates are noted "physics"); [Assumption]. The Lean column is outside the labels and lists names and assumptions.

Table 1 The breakdown of the band (labels per row; "—" marks a row with no machine checking)

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter
Goal	"The sector bias θ stays away from 1 uniformly in the scale" is not enough. What is needed is $\ln(1/\theta(L,L)) \geq cL^2$ (an area exponent). To the area law by the Tomboulis–Yaffe inequality	Proved here (the logic of insufficiency), Cited (Tomboulis–Yaffe), Reading (the U(1) count, physics)	The U(1) row: Coulomb phase, two-dimensional U(1) at long distance, an estimate	—	26-01
Reduction	Under the three of the base point, frame decorrelation and the lateral-widening inequality, the goal follows from the doubling iteration $\theta(2R) \leq K\theta(R)^4$	Proved here (conditional)	Under the three assumptions. $L \geq L_{\min}(a)$, with no lattice artefacts. On a dyadic sequence of scales	`Doubling.iterate`, `decay` (only the algebra of the iteration)	26-02
Negative lemma	Reflection positivity, the centre and duality identities, and the positivity of the transfer matrix hold in the same way for $\mathbb{Z}_2$ and U(1), so the lateral-widening inequality does not follow from them alone	Proved here	In the form "from this set of inputs alone"	—	26-04
Electric flux	$\theta = Y(1)/Y(0) \geq 0$ (the ratio of the partition functions with electric flux 1 and 0; a lattice version of 't Hooft duality)	Proved here	For $\theta \leq 1$, the positivity of the twisted partition function is used as an assumption	`FluxDuality` (the algebra of the spectral form)	26-04
The place of the non-abelian	In the dilute regime $\ln(1/\theta) \propto A_D L^{5/3}$, with the exponent margin $5/3 = 11/3 - 2$. For U(1), -2	Reading (physics), Cited (Tanizaki–Ünsal)	Dilute regime, one loop. The place where the non-abelian works "within this reduction"	—	26-03

Component	Content in one line	Label	Conditions in the same row	Lean	Chapter
β-monotonicity	The vortex free energy increases in β. A problem of the same weight as the β-monotonicity of the string tension in lattice units	Assumption (unproved for SU(2))	Proofs exist only in limited cases (small two-dimensional lattices; near $\beta \to 0$ and $\beta \to \infty$ on a fixed lattice), and the book does not set them out	—	26-06
Renormalisation comparison	The vortex free energy of $(2N, \beta) \leq K \cdot (N, \beta')$, $\beta' < \beta$. The only place asymptotic freedom enters	Proved here (one-loop level), Reading (the doubling test), Assumption (the non-perturbative comparison)	On the strong-coupling side, crude bookkeeping of constants; on the weak-coupling side, the one-loop level. U(1) goes the other way	—	26-06
Chessboard	Under reflection positivity in z, frame decorrelation is replaced by a single density κ_{cb}. The band is concentrated into the uniform slab tension and the uniform slab gap	Proved here (the chessboard at fixed L), Assumption (the uniform slab tension, the uniform slab gap)	If L is fixed. Making it uniform in L is open	`ReflPos` (the algebra of the kernel)	26-05

What the Lean column guarantees is only the algebra of finite sums and real numbers. That θ is the sector bias, the Haar measure and the positivity of the transfer matrix lie outside Lean. Names and assumptions are written in the classification table of each chapter.

Of the rows of Table 1, **non-abelian dynamics enters only the rows concerned with "the lateral-widening inequality" and "the uniform slab tension"** (the third assumption of the reduction, the place of the non-abelian, the renormalisation comparison). These have a form that fails for U(1) and $\mathbb{Z}_2$. The other rows are structural facts that hold for any group, or arrangements of logic.

05

Confluence at the level of the goal, difference at the level of the wall

The phrase "the two roads reach the same wall" is **correct at the level of the goal, and not accurate at the level of the walls at which they now stop**.

The correct part is as in § 01. The four-dimensional goal of Part 23 (the physical-unit target level of 23-15: the upper-bound assumption and the one-sided bridge with the same constant

as $\beta \to \infty$) and the goal (1) of the road from the direction of space are both statements that must cross the band.

The inaccurate part is the **present wall** of Part 23. The "summit-curvature shortfall wall" of 23-09 § 06 (that the diagonal $15/4$ of one site at the summit does not balance the sum of couplings 4) was the wall of the first step of the staircase, on the double well in 0+1 dimensions. The length at which things cross over there is a **classical** length fixed by the first conjugate point of the Jacobi equation at the summit; it does not depend on $\hbar$ and is finite in lattice units. The length $1/\Lambda$ at which things cross over in the band is born quantum mechanically and diverges in lattice units. Table 2 sets out the differences.

Table 2 The wall of the first step and the band (details in 26-07)

	The wall of the first step (Part 23)	**The crossover band (Part 26)**
Place	0+1 dimensions, double well, the lattice with $\varepsilon = \frac{1}{2}$	Four-dimensional lattice, $SU(2)$
Crossover length	The first conjugate point at the top (classical, independent of $\hbar$)	$1/\Lambda$ (born quantum mechanically through dimensional transmutation)
Size in lattice units	Finite	Diverges as $a \to 0$
Tools at the two ends	Convexity for short chains, mixing for long chains	One loop and ultraviolet analysis at weak coupling, expansion at strong coupling
Type	The same: there are different tools at the two ends, the estimate of the worst configuration stops in the middle, and there is a prospect that uniformity comes out on moving to averaging tools	

So the wall of the first step is **a miniature** of the band, but it is not the band itself. At the first step one can think of a road that crosses with a finite block longer than the crossover length and strong mixing (labelled [Assumption]), but in four dimensions there is no "finite block longer than the crossover length". This difference, and what the two have in common in type, are written from the side of time in 26-07. The reason for the difference — that the four-dimensional coupling is dimensionless and has no classical length — is in 24-07 § 01.

06

The map of Part 26 — thirteen chapters, and the one line of 24-07

Table 3 The chapters of Part 26 (labels are those of the main rows of each chapter; conditions in the same row)

Chapter	Title	One line	Main labels and conditions
26-00	Entrance — confluence	This page	Map (physics figure + the label of each row)
26-01	The correct form of the band	θ moving away from 1 is not enough; $\ln(1/\theta(L,L)) \geq cL^2$ is needed	[Proved here] the logic of the shortfall; [Cited] Tomboulis–Yaffe; [Reading] the U(1) count (physics)

Chapter	Title	One line	Main labels and conditions
26-02	Reduction to a single inequality	The lateral-widening inequality (strong form, weak form) and iterated doubling. The Conditional Reduction Theorem	[Proved here] under the three assumptions of the base point, frame decorrelation and lateral widening
26-03	The only place where non-abelian enters	$5/3 = 11/3 - 2$; the contrast with U(1) and $\mathbb{Z}_2$	[Reading] physics (dilute regime, one loop; "within this reduction")
26-04	Structure alone cannot cross	Reflection positivity, duality and the transfer matrix are the same for $\mathbb{Z}_2$ and U(1). $\theta = Y(1)/Y(0) \geq 0$	[Proved here] "with this set of inputs alone"
26-05	The chessboard and simultaneous flips	Frame decorrelation turned into the density κ_{cb}. The band is concentrated into the uniform slab tension and the uniform slab gap	[Proved here] for fixed L; [Assumption] uniformity
26-06	Renormalisation comparison — a comparison that distinguishes groups	Band = β-monotonicity + renormalisation comparison. Asymptotic freedom enters only through $\beta' < \beta$	[Proved here] one-loop level; [Reading] doubling test; [Assumption] non-perturbative
26-07	The band seen from the side of time	The form of the statement that corresponds to the band on the road of Part 23. The wall of the first step is a miniature of the band but not the band	[Proved here] the computation of the first conjugate point; [Assumption] the statement that crosses the band itself
24-07	Precisely because it is four-dimensional	Kept in Part 24. The dimensionless coupling and topology — why the length of the band is born quantum mechanically	The label of each row as in 24-07
26-08	A map of the roads tried and stopped	The roads tried and why they stopped. Tomboulis's (2007) claim, and the criticism that it does not distinguish U(1)	[Cited] only literature that has been checked; unchecked items are marked
26-09	The measured record	Widening T^2 makes the U(1) vortex costlier, but not the SU(2) one. The slab tension increases and saturates. The test of doubling the lattice	[Reading] lattice size, β, number of seeds and error on each row
26-10	Candidate new viewpoints	Four faces of the middle band. Including viewpoints that the study found negative	[Reading] physics estimate

Chapter	Title	One line	Main labels and conditions
26-11	Ultraviolet control — what it means to close at infinitely many steps	The foundation under both roads. What closing at one step means, three difficulties, the five steps of the series and where it stopped, a proposed statement	[Proved here] Background–Fluctuation Decomposition Lemma, Coupling-Smallness Induction Lemma (the one-step remainder = the net one-step bracket of the coupling is unproved); [Cited] the five steps of Bałaban's series; [Reading] the class of one-link heat kernels (one link, registered range); [Assumption] the ultraviolet statement for a small twisted four-dimensional torus (proposal, unproved)
26-12	A sieve for arguments about the band — seven test models	Six models besides U(1), listing for each type of argument which models it fails. A table sorting this book's theorems and the candidate statements for the band into true, false or undefined for each model	[Cited], [Reading], [Proved here], [Assumption] (only what was checked in the literature is Cited; computations on small lattices are conditional)

We recommend reading in numerical order. In 26-01 and 26-02 we narrow the band down to a single inequality, in 26-03 to 26-05 we see what enters where in that inequality, and in 26-06 and 26-07 we look at the band itself from two directions. 26-08 to 26-10 are records and candidates. 26-11 is the foundation under the two roads (ultraviolet control), the content of the "ultraviolet control" row of the ledger of conditions. 26-12 is a list of models against which to test arguments about the band. The conditional statement as a whole is gathered in Part 27 (27-01).

07

The table of the five grades and the dependency graph

The labels of the rows of Tables 1 and 3 are fixed in the classification table of each chapter. The rows this page sets out itself are the following twelve.

Table 4 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Λ is born through dimensional transmutation and, in lattice units, diverges as $a \to 0$	Cited	4-07	One- and two-loop flow	—
2	A comparison that reshapes the fluctuations gives $\beta - \beta' = 8b_0 \ln 2 = 0.258$	Proved here	25-07	One-loop level, per doubling	—
3	A comparison isometric in the fluctuations gives only the direction $\beta' \geq \beta + 0.023$	Proved here	25-07	One-loop level	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	The doubling test: $\beta - \beta' \approx +0.13$–$+0.25$ for SU(2), the opposite direction for U(1)	Reading	§ 03 (ii), 26-09	$4 \times 8 \to 8 \times 16$, $4^4 \to 8^4$, $\beta = 2.5$–3.0, 1–3 seeds	—
5	The string tension on a small T^2, $\propto \Lambda^2 (\Lambda L)^{5/3}$	Cited	Tanizaki–Ünsal eq. (2.37)	Dilute regime, one loop	—
6	The margin $5/3 = 11/3 - 2$ (the coefficient of asymptotic freedom beats the transverse entropy)	Reading (physics)	26-03	Dilute regime, one loop	—
7	The area law at strong coupling	Cited	5-02	Small β	—
8	The renormalisation comparison at the strong-coupling end	Assumption	26-06 (bookkeeping sketched)	Very small β	—
9	Ultraviolet control (the twisted small-T^2 box; the continuum limit in the periodic cubic box was shown by the published construction, Volume 2)	Assumption	7-07, 27-02	From a to L_0	—
10	The goals of the two roads are both statements of "the same constant at every scale", and cross the band	Reading (physics)	§ 01	The level of the goal	—
11	The crossover length at the wall of the first step is classical and finite in lattice units	Proved here	26-07 § 02	0+1 dimensions, $\varepsilon = \frac{1}{2}$	—
12	The wall of the first step is a miniature of the band, not the band	Reading (physics)	§ 05	A comparison of types	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  dimensional transmutation: Λ is quantum, diverges in lattice units  4-07
 2  PRV  reshaping comparison gives β − β' = 0.258                          one-loop leve
 3  PRV  isometric comparison: direction β' ≥ β + 0.023                     one-loop leve
 4  RDG  doubling test: SU(2) β' < β, U(1) the opposite                     small lattice
 5  CIT  string tension ∝ Λ²(ΛL)^(5/3)                                      dilute, one
 6  RDG  margin 5/3 = 11/3 − 2 ⟸ 5                                          physics
 7  CIT  strong-coupling area law                                           small β
 8  ASM  renormalisation comparison at the strong-coupling end              bookkeeping s
 9  ASM  ultraviolet control                                                a to L₀
10  RDG  the goals of the two roads are of one type and cross the band ⟸ 1   level
11  PRV  the crossover length of the first step is classical, finite        26-07 §02
12  RDG  the first-step wall is a miniature of the band, not the band ⟸ 1, 10, 11
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The rows of Figure 2 are a rearrangement of the rows of Table 4 from the reshaping comparison to ultraviolet control, and the figure itself creates no new node. The conclusion-side nodes (that the goals of the two roads are of the same type, and that the wall of the first step is a miniature of the band) are "Reading (physics)", and this page has no node for the statement that crosses the band itself (the uniform slab tension, the uniformity of the upper-bound assumption). That is placed as an assumption in the ledger of conditions of Part 27.

What this chapter can say

What is said	What is not said
Reading (physics) That the goals of the two roads are of the same type (the same constant at every scale), and that both cross the band	Crossing the band. The goal of either road
Proved here The direction of the comparison in the band at the one-loop level (the difference between the reshaping comparison and the isometric comparison)	The non-perturbative comparison. Estimating the error at the intermediate couplings of the band
Reading That in the doubling test on small lattices the directions for SU(2) and U(1) separate	Determining the value. Behaviour on large lattices
Reading (physics) That the wall of the first step is a miniature of the band, not the band	Getting past the wall of the first step. Whether the tools of the first step cross the band

Sources and reproduction

Item	Kind	Source / tools
The string tension on a small twisted T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (Prog. Theor. Exp. Phys. 2022), eqs. (2.32) and (2.37). Chapter 25-02
Upper bound on the Wilson loop (from the vortex free energy)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442 (generalisation to SU(N)). Chapter 26-01
The strong-coupling area law, dimensional transmutation, the multiscale ultraviolet analysis	Cited	Sources of chapters 5-02, 4-07, 7-07
The coefficient of the one-loop heat kernel and the shift per doubling	Proved here	Chapter 25-07 (one-loop level)
Numbers from the doubling test	Reading	Replica exchange and Bennett's acceptance-ratio method. The table in chapter 26-09 (lattice size, β, seeds, errors)
The goal of Part 23 and the wall of the first step	This book	23-15 § 07, § 08, 23-09 § 06, 24-07 § 01

Terms used in this chapter — the crossover band (the band), the ladder of scales, the uniform slab tension, the lateral-widening inequality, frame decorrelation, the base point at small T^2, β-monotonicity, the renormalisation comparison, the chessboard density κ_{cb}, the diffusive-relaxation-time assumption. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-01 The correct form of the band — why "θ moves away from 1" is not enough.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band the sector bias, the area exponent, Tomboulis–Yaffe · Kiichi, Shiori and Rin

The correct form of the band

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — How the statement that crosses the band should be written in the language of the sector bias θ. The naive form "θ stays away from 1 uniformly in the scale" is not enough: the lower bound on the string tension that the container estimate gives from that form is $\ln(1/\theta_0)/L^2$, which goes to 0 as $L \to \infty$, and weakly coupled four-dimensional U(1) can be estimated to satisfy that form. What is needed is the form in which the bias becomes small with an area exponent, $\ln(1/\theta(L, L)) \geq cL^2$; this is the local form of 't Hooft's criterion for confinement, conditioned on the frame. We also see how the Tomboulis–Yaffe inequality connects decay of the vortex free energy with an area exponent to the area law.

Prerequisite chapters — 26-00 (the band), 25-01 (the twist and 't Hooft flux), 25-03 (containers and the sector bias), 25-04 (slab confinement), 5-06 (U(1) and the non-abelian case).

Said first — The proofs in this chapter are only manipulations of elementary inequalities. The count for the Coulomb phase of U(1) is an estimate by an effective theory ([Reading], physics), not a theorem. Of the conclusion "the naive form is not enough", the part that can be said on paper is "the lower bound that the container estimate gives from that form goes to 0"; the part "from that form, confinement does not follow by any road" stands on the U(1) estimate. This chapter shows neither confinement nor the correct form (7).

The order of this chapter

1. The naive form — θ stays away from 1 uniformly in the scale
2. What follows from the naive form — the lower bound on the tension goes to 0 as L→∞
3. The count for the Coulomb phase of U(1) — it satisfies the naive form and does not confine
4. The correct form — an area exponent and the Tomboulis–Yaffe inequality
5. The touchstone — the behaviour for U(1), $\mathbb{Z}_2$ and SU(2) (Table 1)

6. What this form imposes on the band
7. The table of the five grades and the dependency graph

01

The naive form of the band — θ stays away from 1 uniformly in the scale

We take the notation from 25-03. We write the Wilson action of $SU(2)$ on the lattice $(\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$ as the ensemble summed over Mack–Petkova twists (links U and closed ± 1 variables z_p, with weight $\exp(\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p)$), and fix the background twist $n_{12} = 1$ in the two transverse directions 1 and 2. For a disc or square D in the plane (directions 3 and 4), we call $D \times T^2_L$ the **container**, the $\mathbb{Z}_2$ flux $\sigma_D = \prod_{p \in D \times \{y_0\}} z_p$ through D its **sector**, and define the **sector bias**, conditioned on all the variables outside the container (the **frame**) Φ, as

$$\theta_D(\Phi) \;:=\; \big|\,\mathbb{E}[\sigma_D \mid \Phi]\,\big| \;\in [0,1] \quad (1)$$

We write $\theta(R; T^2_L)$ for its value at a typical frame for a container of size R with transverse T^2_L, and in particular $\theta(L,L) := \theta(L; T^2_L)$. Given the frame, the insides of the containers are conditionally independent, and for disjoint containers D_j inside a loop C in the plane

$$|\langle W(C)\rangle| \;\le\; \mathbb{E}\prod_j \theta_{D_j}(\Phi) \quad (2)$$

holds (the Mack–Petkova container estimate Cited , 25-03).

The line of the road from the direction of space was this: confinement is visible on a small T^2_L (25-04), and carrying it to $L \to \infty$ gives confinement on $\mathbb{R}^4$. So the first statement of the band that comes to mind is the following **naive form of the band** (the naive form, below).

(naive form) $\qquad \sup_{L \ge L_0} \theta(L,L) \;\le\; \theta_0 \;<\; 1 \qquad$ (at a typical frame, uniform in the lattice sp

It is the statement that "even if the container and the transverse torus are enlarged together, the sector bias does not stick to 1" — disorder remains at every scale.

02

What follows from the naive form — the lower bound on the tension goes to 0 as L → ∞

Tension Lower Bound from the Naive Form Proved here Suppose that, for some L, when the plane is tiled with square containers of side L, the product on the right-hand side of (2) splits into the product of the individual factors up to a constant $C_{\mathrm{D}} \geq 1$ (frame decorrelation Assumption), and that each factor is at most θ_0. Then for a square loop C of area A (with side a multiple of L)

$$|\langle W(C)\rangle| \;\leq\; C_{\mathrm{D}}\,\theta_0^{A/L^2} \;=\; C_{\mathrm{D}}\,e^{-\sigma_L A}, \qquad \sigma_L := \frac{\ln(1/\theta_0)}{L^2}. \tag{4}$$

Proof Place A/L^2 square containers of side L inside C and use (2) and the assumption: $|\langle W(C)\rangle| \leq C_{\mathrm{D}} \prod_j \theta_0 = C_{\mathrm{D}}\theta_0^{A/L^2}$. Rewriting the exponent gives (4). ∎

The tension σ_L in (4) is positive for each L. But the naive form (3) gives only *one* constant θ_0, so the lower bound on the tension obtained by this road is $\ln(1/\theta_0)/L^2$, which goes to 0 in the four-dimensional limit ($L \to \infty$, where the slab $T_L^2 \times \mathbb{R}^2$ approaches $\mathbb{R}^4$). In physical units, whatever multiple of $1/\Lambda$ we take L to be, the lower bound is $\Lambda^2 \ln(1/\theta_0)/(L\Lambda)^2$, and it gives no uniform tension.

What is said here is that "the lower bound that follows from (3) **by the container estimate** goes to 0". That a theory satisfying (3) does not confine does not follow from the Tension Lower Bound from the Naive Form. The U(1) estimate in the next section fills that step.

03

The count for the Coulomb phase of U(1) — it satisfies the naive form and does not confine

We apply the same language of containers to compact $U(1)$ in four dimensions (the twist enters in the same way, as a centre flux with values ± 1). What follows is an estimate by an effective theory Reading (physics).

Weakly coupled $U(1)$ is in the Coulomb phase, and on the slab $T_L^2 \times \mathbb{R}^2$, after integrating over the transverse directions at long distance, it becomes two-dimensional $U(1)$ with coupling e^2/A_T ($A_T = L^2$ is the area of T^2). In two-dimensional $U(1)$, the partition function for flux Φ through a region of area A_D can be written as a Poisson sum over the wrapping k:

$$Z(\Phi) \;\propto\; \sum_{k\in\mathbb{Z}} e^{ik\Phi}\, e^{-e^2 A_D k^2/(2A_T)} \tag{5}$$

The sectors $\sigma_D = \pm 1$ correspond to $\Phi = 0$ and $\Phi = \pi$, so when $q := e^{-e^2 A_D/(2A_T)}$ is small

$$\theta(R; T_L^2) \;=\; \frac{Z(0) - Z(\pi)}{Z(0) + Z(\pi)} \;\approx\; 2q \;=\; 2\exp\Big(-\frac{e^2 A_D}{2A_T}\Big). \quad (6)$$

Taking the container size $R = L$, A_D/A_T is a number independent of L (1 for a square), so $\theta(L, L)$ is a constant independent of L and satisfies the naive form (3). On the other hand, that weakly coupled four-dimensional $U(1)$ does not confine is a theorem (Guth 1980, Fröhlich–Spencer 1982, Villain-type action Cited , 5-06). Together, the two give the conclusion that **(3) cannot lead to confinement**. But (6) is an estimate, so this conclusion carries the same label, [Reading] (physics).

(6) is consistent with the string tension $\sigma \sim e^2/(2L^2)$ of U(1) reduced to two dimensions. U(1) on the slab confines, but its tension goes to 0 as $1/L^2$. The $\ln(1/\theta_0)/L^2$ of the Tension Lower Bound from the Naive Form has exactly the same form as this behaviour.

Note The classical energy count we tried first (the energy $(\beta/2)(\pi/L^2)^2 L^4 = \beta\pi^2/2$ of the flux π spread uniformly over a container of radius L does not depend on L) gives the same conclusion, but it drops the sum over the wrapping k, so we put the forms (5) and (6) in the text.

04

The correct form of the band — an area exponent

To get a uniform tension, the bias must become small with an exponent in the area of the container. In physical units,

$$\text{(correct form)} \qquad \ln\frac{1}{\theta(L,L)} \;\geq\; c\,L^2 \qquad (L \geq L_0, \text{ at a typical frame, uniform in } a) \quad (7)$$

This is the correct form of the band. Seen from the side of the container estimate, (7) turns directly into a tension independent of L.

Area Law from the Area Exponent Proved here Under the same frame-decorrelation assumption as the Tension Lower Bound from the Naive Form, if $\theta \leq e^{-cL^2}$ for containers of side L, then for square loops with side a multiple of L, $|\langle W(C)\rangle| \leq C_{\mathrm{D}}\, e^{-cA}$.

Proof Replacing θ_0 by e^{-cL^2} in the proof of the Tension Lower Bound from the Naive Form gives $C_{\mathrm{D}}(e^{-cL^2})^{A/L^2} = C_{\mathrm{D}} e^{-cA}$. ∎

(7) is 't Hooft's criterion for confinement — that the free energy of a vortex (a centre flux) becomes small with an exponent in the area of the cross-section — put in a local

form conditioned on the frame. The global form can be written with partition functions with twisted boundary conditions. On the torus, define the vortex free energy by $e^{-F_v} := Z_-/Z_+$ as the ratio of the partition function Z_- with twist flux in the plane $\mu\nu$ of the loop to the one Z_+ without; then the global version of the sector bias is

$$\theta \;=\; \frac{Z_+ - Z_-}{Z_+ + Z_-} \;=\; \tanh\frac{F_v}{2} \tag{8}$$

(from the definition, Proved here). Since $\tanh x \le x$, and $\operatorname{artanh}\theta \le \theta/(1-\theta^2) \le 2\theta$ for $0 \le \theta \le 1/\sqrt{2}$, give $\theta \le F_v/2 \le 2\theta$, the conditions $\ln(1/\theta) \ge cL^2$ and $F_v \lesssim e^{-cL^2}$ are the same up to constants.

It is the Tomboulis–Yaffe inequality that connects the global form to the area law Cited . In $SU(2)$ lattice gauge theory, for a rectangle C (area A_C) in the plane of the directions with periods L_μ, L_ν,

$$\langle W(C)\rangle \;\le\; 2\Big\{\tfrac{1}{2}\big(1-e^{-F_v}\big)\Big\}^{A_C/(L_\mu L_\nu)} \tag{9}$$

holds. Here F_v is the free energy of the twist flux in the same plane as C; the vortex sheet wraps the remaining two directions and winds once around C (this is a different twist from the background twist n_{12} of the slab). T. Kanazawa extended this inequality to $SU(N)$ and to general classical spin systems, using reflection positivity Cited .

Tension Lower Bound from Tomboulis–Yaffe Proved here Under (9), if $F_v \le e^{-\rho L_\mu L_\nu}$ (with $\rho > 0$ independent of the size of the torus), then $\langle W(C)\rangle \le 2e^{-\rho A_C}$; that is, the string tension is at least ρ.

Proof From $1-e^{-F} \le F$, $\tfrac{1}{2}(1-e^{-F_v}) \le \tfrac{1}{2}e^{-\rho L_\mu L_\nu}$. Raising to the power $A_C/(L_\mu L_\nu)$ gives $2^{-A_C/(L_\mu L_\nu)}e^{-\rho A_C} \le e^{-\rho A_C}$. ∎

ρ is called the 't Hooft string tension. The assumption of the Tension Lower Bound from Tomboulis–Yaffe and (7) are conditions of the same type — the former a ratio of global partition functions, the latter the bias of a container conditioned on the frame. The global form does not need frame decorrelation; in exchange, the base point (the estimate on a small T^2) must also be shown for the global partition functions (the note in 26-02).

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05
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The touchstone — the behaviour for U(1), $\mathbb{Z}_2$ and SU(2)

An argument that crosses the band must fail for a theory that does not confine. The naive form (3) does not pass this touchstone; the correct form (7) is expected to pass. All the behaviour in Table 1 is estimated Reading (physics).

Table 1 The behaviour of $\theta(L, L)$ (estimates; four dimensions, weak coupling)

Theory	$\theta(L, L)$ as L grows	Naive form (3)	Correct form (7)	Confinement
$U(1)$ (Coulomb phase)	Constant ((6))	Satisfied	Not satisfied	No Cited
$\mathbb{Z}_2$ (weak coupling)	Tends to 1 (the flux makes a wall of the area of T^2)	Not satisfied	Not satisfied	No
$SU(2)$	Tends to 0 (expected)	Satisfied (expected)	Satisfied (expected)	Yes (physical expectation)

Satisfying the naive form is not a touchstone for confinement, because U(1) satisfies it. The right way to impose "failing for U(1)" is as "θ does not become small with an area exponent as L grows".

06

What this form imposes on the band

(7) is a statement that joins the two sides of the band. At small L (the regime of a dilute gas of fractional instantons), $\ln(1/\theta)$ is expected to grow as the product of the string tension $\propto \Lambda^2(\Lambda L)^{5/3}$ and the area of the container (26-03); at large L (the confinement regime) it is expected to take the form σL^2 minus the logarithm of the entropy of the position of the vortex sheet Reading (physics). (7) asks that the region in between — $L \sim 1/\Lambda$ — be passed with a single constant c.

Two intermediate goals weaker than (7) can also be named.

- (a) $F_v(L, L) \to 0$: excluding a non-confining phase. Stronger than the naive form, but gives no uniform lower bound on the tension.
- (b) $\ln(1/\theta(L, L)) \geq cL^2/\ln L$: almost area law. In the form of the Tension Lower Bound from the Naive Form, the lower bound on the tension is $c/\ln L$, still not uniform.

Both are unsolved.

The conclusion of this chapter also changes the standing of the tools of Part 25. The flow lemma $\theta \leq 1 - a$ (25-06) gives only an estimate of the form $\theta_0 \sim \frac{1}{2}$ — the naive

form (3) — because the flip probability a of a one-step kernel can be at most of order $\frac{1}{2}$. So that tool cannot be used to test the band; its use is limited to supporting a lower bound on the slab tension at fixed L Proved here .

In the next chapter, 26-02, we reduce (7) to a single inequality that fixes the container and widens only the transverse torus.

07

The table of the five grades and the dependency graph

Table 2 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The container estimate (2)	Cited	Mack–Petkova, 25-03	The ensemble summed over twists	—
2	Frame decorrelation (the product in (2) splits up to a constant C_{D})	Assumption	A-06	—	—
3	Tension Lower Bound from the Naive Form: the lower bound on the tension that the container estimate gives from the naive form is $\ln(1/\theta_0)/L^2 \to 0$	Proved here	Proof in § 02	Under frame decorrelation, loops with side a multiple of L	—
4	Weakly coupled four-dimensional U(1) does not confine	Cited	Guth, Fröhlich–Spencer (5-06)	Villain-type action	—
5	In the Coulomb phase of U(1), $\theta \approx 2\exp(-e^2 A D/(2AT))$, and $\theta(L, L)$ is constant	Reading (physics)	§ 03	Coulomb phase, two-dimensional U(1) at long distance, an estimate	—
6	The naive form cannot lead to confinement	Reading (physics)	§ 03	On the estimate for the Coulomb phase of U(1)	—
7	$\theta = \tanh(F_v/2)$, $\theta \le F_v/2 \le 2\theta$ ($\theta \le 1/\sqrt{2}$)	Proved here	§ 04	Global form	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
8	The Tomboulis–Yaffe inequality (9)	Cited	Tomboulis–Yaffe (the form of the formula checked in Kanazawa's restatement)	$SU(2)$, lattice, torus	—
9	Extension of (9) to $SU(N)$ and classical spin systems	Cited	Kanazawa (abstract)	Under reflection positivity	—
10	Tension Lower Bound from Tomboulis–Yaffe: if $F_v \le e^{-\rho L_\mu L_\nu}$, the tension is $\ge \rho$	Proved here	Proof in § 04	Under the Tomboulis–Yaffe inequality	—
11	Area Law from the Area Exponent: if $\theta \le e^{-cL^2}$, then $\lvert\langle W\rangle\rvert \le C_{\mathrm{D}} e^{-cA}$	Proved here	Proof in § 04	Under frame decorrelation	—
12	The behaviour in Table 1 (U(1) constant, $\mathbb{Z}_2$ to 1, SU(2) to 0)	Reading (physics)	§ 05	Four dimensions, weak coupling, an estimate	—
13	The Flow Lemma gives only estimates of the naive form	Proved here	§ 06, 25-06	One-step kernel	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  container estimate |W| ≤ E∏θ                    ensemble summed over
 2  ASM  frame decorrelation (constant C_D)
 3  PRV  lower bound of the naive form ln(1/θ₀)/L² → 0 ⟸ 1, 2
 4  CIT  weakly coupled 4D U(1) does not confine        Villain type
 5  RDG  θ in the Coulomb phase of U(1) is independent of L  estimate
 6  RDG  the naive form cannot lead to confinement ⟸ 3, 4, 5
 7  PRV  θ = tanh(F_v/2), θ ≤ F_v/2 ≤ 2θ                global form
 8  CIT  the Tomboulis–Yaffe inequality                 SU(2)
 9  CIT  extension to SU(N)                             abstract
10  PRV  if F_v ≤ e^(−ρLL), tension ≥ ρ ⟸ 8
11  PRV  if θ ≤ e^(−cL²), |W| ≤ C_D e^(−cA) ⟸ 1, 2
12  RDG  the behaviour in Table 1 ⟸ 5                    estimate
13  PRV  the Flow Lemma gives only naive-form estimates ⟸ 3
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The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. Of the conclusion nodes of this chapter, the ones that stand on paper are four: the Tension Lower Bound from the Naive Form, the Tension Lower Bound from Tomboulis–Yaffe, the Area Law from the Area Exponent,

and the Flow Lemma's estimate; all are elementary manipulations on an assumption row (frame decorrelation) or a literature row (the Tomboulis–Yaffe inequality). The whole of "the naive form is not enough" has an estimate row (the count for the Coulomb phase of U(1)) as a premise, so it is "Reading (physics)". There is no node for the correct form (7) itself. That is the statement of the band, and this chapter does not show it.

What this chapter can say

What is said	What is not said
Proved here The lower bound on the tension that the container estimate gives from the naive form is $\ln(1/\theta_0)/L^2$ and goes to 0 (under frame decorrelation). With the area-exponent form the tension is uniform (under the same assumption)	Frame decorrelation itself. The correct form (7)
Cited + Proved here From the Tomboulis–Yaffe inequality, a lower bound on the tension from decay of the vortex free energy with an area exponent	The decay of the global vortex free energy itself
Reading (physics) The Coulomb phase of U(1) satisfies the naive form. So the naive form cannot lead to confinement by any road	A proof of (6) as a theorem. An estimate of the errors of the effective theory of U(1)
Reading (physics) Table 1, the touchstone	That $\theta(L, L) \to 0$ for SU(2)

Sources and reproduction

Item	Kind	Source / tools
The Tomboulis–Yaffe inequality (9)	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. The form of the formula and the setting checked in eq. (1) of T. Kanazawa, arXiv:0805.2742 and in the restatement in arXiv:0808.3442 (the text of the original paper not obtained)
Extension to $SU(N)$	Cited	T. Kanazawa, Generalizing the Tomboulis–Yaffe inequality to SU(N) lattice gauge theories and general classical spin systems, arXiv:0808.3442 (checked in the abstract)
The container estimate, the ensemble summed over twists	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442, 125 (1980) 117. Chapter 25-03
't Hooft flux and the criterion for confinement	Cited	G. 't Hooft, Nucl. Phys. B153 (1979) 141. Chapter 25-01

Item	Kind	Source / tools
Non-confinement of four-dimensional U(1)	Cited	A. H. Guth, Phys. Rev. D 21 (1980) 2291. J. Fröhlich, T. Spencer, Commun. Math. Phys. 83 (1982) 411. Chapter 5-06
The count for the Coulomb phase of U(1), (5) and (6)	Reading	Text § 03 (an estimate by an effective theory; Poisson sum)
Tension Lower Bound from the Naive Form, Area Law from the Area Exponent, Tension Lower Bound from Tomboulis–Yaffe, the estimate (8)	Proved here	Proofs in § 02 and § 04 of the text

Terms used in this chapter — container, sector, frame, the sector bias θ, the naive form, the correct form (an area exponent), the vortex free energy F_v, the 't Hooft string tension, frame decorrelation, touchstone. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-02 Reduction to a single inequality — the correct form, as a single inequality that widens the transverse directions.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
the lateral-widening inequality, the doubling iteration, the conditional reduction · Kiichi, Shiori and Rin

Reduction to a single inequality

> This chapter is this book's own investigation. § 08 is a map of open questions.

What this chapter shows — The conditional reduction theorem, which derives the correct form of the band $\ln(1/\theta(L,L)) \geq cL^2$ (26-01) under three assumptions — the small-T² base point, frame decorrelation and the lateral-widening inequality — and its proof. Alternating between nesting, which widens the container in the directions of the plane, and an inequality that fixes the container and widens the transverse T^2, we obtain decay with an area exponent from the fact that the sequence of biases t_k satisfies $t_{k+1} \leq Kt_k^4$ (the doubling iteration). We also look at the strong and weak forms of the lateral-widening inequality, at the fact that the strong form is stronger than confinement (it includes monotonicity of the slab tension), and at why we state things restricted to scales $L \geq L_{\min}(a)$ free of lattice artefacts.

Prerequisite chapters — 26-01 (notation, the container estimate, the Area Law from the Area Exponent), 25-03 (nesting of containers), 25-04 (slab confinement), A-04 (Lean).

Said first — The Conditional Reduction Theorem and the Reduction Theorem in the Weak Form are conditional, and none of the three assumptions has been shown. In particular the lateral-widening inequality is the only place in this reduction where non-abelian dynamics enters, and the difficulty of the band has only moved there. The conclusion is a statement on the dyadic sequence of scales $R_k = 2^k R_0$, and whether the constants are uniform in the lattice spacing a is decided by whether the constants of the assumptions are uniform. What was checked in Lean is only the real-number algebra of the iteration.

The order of this chapter

1. Two moves — nesting in the directions of the plane, and widening the transverse T^2
2. Three assumptions — the base point, frame decorrelation, the lateral-widening inequality (strong form, weak form)
3. The Conditional Reduction Theorem and its proof
4. The Reduction Theorem in the Weak Form, and what replaces the jump step

5. The strong form of the lateral-widening inequality is stronger than confinement — monotonicity of the slab tension
6. Lattice artefacts and $L_{\min}(a)$
7. Reading the Lean statements — Doubling (`Doubling.iterate`, `Doubling.decay`)
8. What remains
9. The table of the five grades and the dependency graph

01

Two moves — nesting in the directions of the plane, and widening the transverse T^2

The notation is as in 26-01 § 01. $\theta(R;T^2_L)$ is the sector bias at a typical frame of the container $D_R \times T^2_L$ of size R in the directions of the plane, and we write $\theta(R,R) := \theta(R;T^2_R)$. The goal is, for some $c>0$ uniform in physical units,

$$\ln \frac{1}{\theta(R,R)} \;\geq\; c\,R^2 \qquad (1)$$

There are two moves that change $\theta(R;T^2_L)$.

Nesting in the directions of the plane. Keeping T^2_L fixed, split a large container R by an interior frame Φ_{int} into small containers $R_1,\dots,R_k$; conditioned on the frame, the small containers are independent, so

$$\big|\mathbb{E}[\sigma_R \mid \Phi_R]\big| \;\leq\; \mathbb{E}\Big[\prod_i \theta_{R_i} \;\Big|\; \Phi_R\Big] \qquad (2)$$

holds (the nested form of the Mack–Petkova container identity Cited , 25-03). If the container is enlarged s times, the right-hand side is a product of s^2 factors. If the factors are close to their typical value and nearly independent of each other, $\theta(sR;T^2_L) \approx \theta(R;T^2_L)^{s^2}$, and the disorder propagates to large scales. But this is a matter inside a slab with T^2 fixed, and it has the same content as slab confinement (25-04).

Widening the transverse T^2. (1) asks that the container and the transverse torus be enlarged *together*. Nesting does not change T^2, so it alone does not reach (1). Replacing T^2_L by $T^2_{L'}$ changes the very surface that the vortex sheet wraps, and a relation connecting containers at the two scales does not come out of structure (26-04). Setting this relation as an assumption is the lateral-widening inequality of the next section.

02

Three assumptions

We call the assumptions by name. All lengths are in physical units and at least $L_{\min}(a)$, where lattice artefacts can be neglected (§ 06).

The base point at small T^2 Assumption For some physical $L_0 \geq L_{\min}(a)$ and $\varepsilon_0 \in (0,1)$,

$$\theta(L_0; T^2_{L_0}) \;\leq\; 1 - \varepsilon_0 . \tag{3}$$

In the estimate of a dilute gas of fractional instantons, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ (25-02). A line of showing it under ultraviolet control and uniform acceptance of insertions has been considered, but it has not been shown.

Frame decorrelation Assumption There is a constant $C_{\mathrm{D}} \geq 1$ such that for all R, L and integers $s \geq 1$,

$$\theta(sR; T^2_L) \;\leq\; C_{\mathrm{D}}\, \theta(R; T^2_L)^{s^2} . \tag{4}$$

The right-hand side of (2) is the average of a product of factors that share a frame, and Hölder's inequality alone loses the power (only the form $\mathbb{E}\theta^4_{2R} \leq \mathbb{E}\theta^4_R$ comes out). (4) is the assumption that this product splits, up to a constant, into the product of typical values. If L is fixed, it can be replaced, under reflection positivity of z, by the chessboard density κ_{cb} (26-05). Read in terms of κ_{cb}, the power in the nesting holds with constant 1 for all L (the Nested Supermultiplicativity Lemma of 26-05, and the Reduction Read in the Chessboard Density in § 03). The form uniform in L with θ as it is remains open.

The lateral-widening inequality (strong form) Assumption There is a constant C such that for $L_{\min}(a) \leq L \leq L' \leq R$,

$$\theta(R; T^2_{L'}) \;\leq\; C\Big(\frac{L'}{L}\Big)^2\, \theta(R; T^2_L). \tag{5}$$

It is the statement that, even if the container is fixed and the transverse T^2 is widened by a factor $s = L'/L$, the bias grows by at most a factor Cs^2. The factor s^2 is the entropy of the transverse position of the vortex sheet. Setting $L' = L$, we need $C \geq 1$ (if $\theta > 0$).

The lateral-widening inequality (weak form) Assumption For the dyadic scales $R_k = 2^k R_0$, there are $0 \leq \varepsilon_k < 1$ with $\sum_k \varepsilon_k < \infty$ and a constant $C' \geq 0$ such that

$$\ln \frac{1}{\theta(R_{k+1}; T^2_{R_{k+1}})} \;\geq\; (1-\varepsilon_k)\, \ln \frac{1}{\theta(R_{k+1}; T^2_{R_k})} \;-\; C' . \tag{6}$$

It is the statement that at each step of doubling the transverse directions one may lose a fraction ε_k of $\ln(1/\theta)$ and a constant C', provided the sum of the fractions lost is finite. The strong form (5) at $s = 2$ loses only the constant $\ln(4C)$ from $\ln(1/\theta)$, so it is the case $\varepsilon_k = 0$ of (6).

03

The conditional reduction theorem

Conditional Reduction Theorem Proved here Assume the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form). Set $K := 4C\,C_{\mathrm{D}}$ (≥ 1), and choose an integer $s \geq 1$ so that

$$t_0 \;:=\; C\,C_{\mathrm{D}}\,s^2(1-\varepsilon_0)^{s^2} \;<\; K^{-1/3} \quad (7)$$

(this is possible because $s^2(1-\varepsilon_0)^{s^2} \to 0$). Setting $R_0 := sL_0$ and $R_k := 2^k R_0$, for all $k \geq 0$

$$\theta(R_k, R_k) \;\leq\; K^{-1/3}\big(K^{1/3}t_0\big)^{4^k}, \qquad \text{in particular} \qquad \ln\frac{1}{\theta(R_k,R_k)} \;\geq\; \gamma\,R_k^2, \quad \gamma := \frac{\ln(1}{}$$

Moreover, under frame decorrelation, by the Area Law from the Area Exponent in 26-01, for square loops on the slab $T^2_{R_k} \times \mathbb{R}^2$ with side a multiple of R_k, $|\langle W(C)\rangle| \leq C_{\mathrm{D}} e^{-\gamma A}$. The lower bound γ on the tension does not depend on k.

Proof In three steps.

Step 1 (jump). Using frame decorrelation with $R = L = L_0$ and s, and putting in the base point at small T^2, gives $\theta(sL_0; T^2_{L_0}) \leq C_{\mathrm{D}}(1-\varepsilon_0)^{s^2}$. Next, using the lateral-widening inequality (strong form) with $L = L_0$ and $L' = R = sL_0$,

$$t_0' \;:=\; \theta(R_0,R_0) \;=\; \theta(sL_0; T^2_{sL_0}) \;\leq\; C\,s^2 \cdot C_{\mathrm{D}}(1-\varepsilon_0)^{s^2} \;=\; t_0.$$

Step 2 (one doubling). Set $t_k := \theta(R_k, R_k)$. Using frame decorrelation with $R = L = R_k$ and $s = 2$ gives $\theta(2R_k; T^2_{R_k}) \leq C_{\mathrm{D}} t_k^4$. Using the lateral-widening inequality (strong form) with $L = R_k$ and $L' = R = 2R_k$,

$$t_{k+1} \;=\; \theta(2R_k; T^2_{2R_k}) \;\leq\; 4C \cdot C_{\mathrm{D}}\,t_k^4 \;=\; K\,t_k^4. \quad (9)$$

Step 3 (iteration). Setting $c := K^{1/3} \geq 1$, (9) becomes $t_{k+1} \leq c^3 t_k^4$. Multiplying both sides by c gives $ct_{k+1} \leq (ct_k)^4$, so by induction $ct_k \leq (ct_0')^{4^k} \leq (ct_0)^{4^k}$ (using $0 \leq ct_0' \leq ct_0$). Hence $t_k \leq c^{-1}(ct_0)^{4^k}$. From $4^k = R_k^2/R_0^2$, $ct_0 < 1$ and $c \geq 1$,

$$\ln \frac{1}{t_k} \;\geq\; 4^k \ln \frac{1}{ct_0} + \ln c \;\geq\; \frac{\ln(1/(ct_0))}{R_0^2}\, R_k^2 \;=\; \gamma R_k^2.$$

For the last claim, take $L = R_k$ in the Area Law from the Area Exponent and replace e^{-cL^2} by $e^{-\gamma R_k^2}$. ∎

The algebra of the iteration in step 3 — from $t_{k+1} \leq c^3 t_k^4$ to $ct_k \leq (ct_0)^{4^k}$ and $t_k \leq (ct_0)^{4^k}/c$ — is checked in Lean (§ 07).

What the Conditional Reduction Theorem does not say.

- The conclusion is a statement on the dyadic sequence R_k. Extending it to the scales in between, $R_k < R < R_{k+1}$, needs an additional property such as monotonicity of $\theta(R, R)$ in R, which we do not assume here.
- The constant γ is uniform in the lattice spacing a only when the constants $\varepsilon_0, C_{\mathrm{D}}, C$ of the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), and L_0, are uniform in a.
- Frame decorrelation is used in two places — the power in the nesting (steps 1 and 2) and the bridge from the bias to the Wilson loop (the Area Law from the Area Exponent).
- The meaning of "typical frame" is put inside frame decorrelation. We have not used a form that treats the tail of the distribution of frames (the rarity of bad frames) separately.

Note (the global version). If the same chain is written with the ratio of twisted partition functions on the torus $T_L^2 \times T_M^2$ ((8) of 26-01) instead of θ conditioned on the frame, the bridge to the area law is carried by the Tomboulis–Yaffe inequality (the Tension Lower Bound from Tomboulis–Yaffe in 26-01), and frame decorrelation is not needed. In exchange, the base point at small T^2 must also be shown for the global partition functions — as a lower bound saying that vortices are *cheap*. This is also open.

Reduction Read in the Chessboard Density (the reduction read in terms of κ_{cb}) Proved here For the $\kappa_{\mathrm{cb}}(R; L)$ of 26-05 (the density with the container taken to be the whole interior of the block; M fixed to a multiple of all the $2R_k$ of the steps considered), assume

$$\begin{aligned} &(\mathrm{A_{cb}}) \;\; \kappa_{\mathrm{cb}}(L_0; L_0) \leq 1 - \varepsilon_0, \\ &(\mathrm{C_{cb}}) \;\; \kappa_{\mathrm{cb}}(R; L') \leq C\,(L'/L)^2\, \kappa_{\mathrm{cb}}(R; L) \quad (L_{\min}(a) \leq L \leq L' \leq R) \end{aligned}$$

The formula label $(\mathrm{A_{cb}})$ is the chessboard-density base point, $(\mathrm{C_{cb}})$ is the lateral comparison of the chessboard density, and $(\mathrm{B_0})$ below is the base-point translation

inequality (names as in A-06). If the Nested Supermultiplicativity Lemma of 26-05 ($\kappa_{\mathrm{cb}}(sR;L) \le \kappa_{\mathrm{cb}}(R;L)^{s^2}$, constant 1, all L) is used in place of frame decorrelation, the proof of the Conditional Reduction Theorem goes through as it is with $C_{\mathrm{D}} = 1$, $K = 4C$, $t_0 = C\,s^2(1-\varepsilon_0)^{s^2} < K^{-1/3}$, and $t_k := \kappa_{\mathrm{cb}}(R_k;R_k) \le K^{-1/3}(K^{1/3}t_0)^{4^k}$. The step down to loops is made directly by (5) of 26-05, so the C_{D} of the Area Law from the Area Exponent is not needed either. Step 3 is the same algebra as Doubling.iterate, and the doubling form is Lean: LatticeGaugeNestedBlocks (doubling_iterate, doubling_bound). In this form frame decorrelation does not remain as an independent assumption, and the content uniform in L goes into $(\mathrm{C_{cb}})$. If the base point at small T^2 is kept with θ as it is, adding the single-scale line

$$(\mathrm{B}_0)\quad 1-\kappa_{\mathrm{cb}}(L_0;L_0) \;\ge\; \eta\,\big(1-\theta(L_0;L_0)\big) \quad (\eta>0 \text{ uniform in } a\ )$$

gives $(\mathrm{A_{cb}})$ with $\varepsilon_0 \to \eta\varepsilon_0$ Proved here . None of $(\mathrm{A_{cb}})$, $(\mathrm{C_{cb}})$, (B_0) has been shown Assumption .

04 Reduction with the weak form

Even with the weak form (6), the iteration keeps the area exponent. But the jump step cannot be made with the weak form, so the base point has to be set sufficiently small at one scale.

Reduction Theorem in the Weak Form Proved here Assume frame decorrelation, and the lateral-widening inequality (weak form) on the dyadic sequence $R_k = 2^k R_0$, and set $P := \prod_k(1-\varepsilon_k) > 0$, $A := \ln C_{\mathrm{D}} + C'$, $u_k := \ln(1/\theta(R_k,R_k))$. If $u_0 > A/(3P)$, then for all k

$$u_k \;\ge\; \Big(P\,u_0 - \frac{A}{3}\Big)4^k \;=\; \gamma'\,R_k^2, \qquad \gamma' := \frac{Pu_0 - A/3}{R_0^2} > 0. \tag{10}$$

Proof Using frame decorrelation with $s=2$ gives $\ln(1/\theta(R_{k+1};T_{R_k}^2)) \ge 4u_k - \ln C_{\mathrm{D}}$. Putting this into (6) and using $0 \le 1-\varepsilon_k \le 1$ and $\ln C_{\mathrm{D}} \ge 0$ gives $u_{k+1} \ge (1-\varepsilon_k)(4u_k - \ln C_{\mathrm{D}}) - C' \ge 4(1-\varepsilon_k)u_k - A$. Setting $v_k := u_k/4^k$, $v_{k+1} \ge (1-\varepsilon_k)v_k - A\,4^{-(k+1)}$. For $P_k := \prod_{i<k}(1-\varepsilon_i)$ and $S_k := \sum_{j=1}^{k} 4^{-j}$ we show $v_k \ge P_k v_0 - AS_k$ by induction: for $k=0$ it is an equality. If it holds for k, then from $AS_k \ge 0$ and $1-\varepsilon_k \le 1$, $v_{k+1} \ge (1-\varepsilon_k)P_k v_0 - AS_k - A4^{-(k+1)} = P_{k+1}v_0 - AS_{k+1}$. From $v_0 = u_0 > 0$, $P_k \ge P$ and $S_k < 1/3$, $v_k \ge Pu_0 - A/3$. ∎

The strong form is the case $\varepsilon_k = 0$ ($P=1$), $C' = \ln(4C)$, and the condition there, $u_0 > (\ln C_{\mathrm{D}} + \ln 4C)/3 = \ln K/3$, is the same condition as $t_0 < K^{-1/3}$ in the Conditional

Reduction Theorem. The condition on w_0 in the Reduction Theorem in the Weak Form moves the role that the jump step played in the Conditional Reduction Theorem onto the assumption of the base point.

05

The strong form of the lateral-widening inequality is stronger than confinement — monotonicity of the slab tension

We define the string tension of the two-dimensional theory obtained by shrinking the transverse directions to T_L^2 (the slab tension) as

$$\sigma_{\mathrm{eff}}(L) \;:=\; \liminf_{M\to\infty} \frac{-\ln\theta(M;T_L^2)}{M^2} \tag{11}$$

The uniform slab tension ((1) of 26-00) is $\inf_L \sigma_{\mathrm{eff}}(L) > 0$.

Slab-Tension Monotonicity Proposition Proved here If the strong form (5) holds for all $R \geq L'$, then $\sigma_{\mathrm{eff}}(L') \geq \sigma_{\mathrm{eff}}(L)$ for $L_{\min}(a) \leq L \leq L'$.

Proof For $M \geq L'$, taking the logarithm of (5) gives $-\ln\theta(M;T_{L'}^2) \geq -\ln\theta(M;T_L^2) - \ln\bigl(C(L'/L)^2\bigr)$. Dividing by M^2 and taking the lower limit as $M\to\infty$, the constant term vanishes and $\sigma_{\mathrm{eff}}(L') \geq \sigma_{\mathrm{eff}}(L)$. ∎

So the strong form includes that the slab tension does **not decrease** in L. Confinement (the area law) needs only $\inf_L \sigma_{\mathrm{eff}}(L) > 0$, so the strong form is a stronger assumption than confinement. At small L the tension is expected to grow as $\sigma_{\mathrm{eff}} \propto \Lambda^{11/3}L^{5/3}$, and at large L to approach the tension σ_∞ of $\mathbb{R}^4$ Reading (physics); but if at $L \sim 1/\Lambda$, σ_{eff} goes once above σ_∞ and then comes back (overshoots), the strong form with constant C is false, while confinement can still hold. The weak form allows this kind of overshoot, as long as the sum of the fractions lost is finite.

On the numerical side, on the slab with $SU(2)$, $\beta = 2.5$, $M = 16$, the Creutz ratio of the 4×4 loop is $0.059,\ 0.061,\ 0.076,\ 0.071$ for $N_T = 4, 6, 8, 10$ (errors 0.009–0.002, 2000 measurements): it increases and saturates, and no significant overshoot is seen Reading . The Creutz ratio contains the short-distance Coulomb term, so we read only the trend in N_T, not the values. The lattice is at a single β and the range of N_T is narrow, so this is not a reading about the band as a whole. The table of measurements is in 26-09.

The form that widens the transverse directions in one direction only (making T^2 a rectangle $L_1 \times L_2$ and enlarging only L_1) is expected not to be treatable in the same way as the square form, because the small direction of the two behaves like the time of

finite temperature Reading (physics). The chain of this chapter uses only the square form $L_1 = L_2$.

06

Lattice artefacts and $L_{\min}(a)$

We state (5) and (6) restricted to $L \geq L_{\min}(a)$, where lattice artefacts can be neglected. $L_{\min}(a)$ is kept constant in physical units, and in lattice units $L_{\min}/a \to \infty$. The goal of the band (1) is a statement about the continuum limit uniform in a, so this restriction is natural.

That the restriction is needed also shows in the numbers. For $SU(2)$, $\beta = 2.5$, $M = 8$, going $N_T = 2 \to 4$ changes the vortex free energy as $-0.01 \pm 0.12 \to 0.63 \pm 0.18$ and the bias as $\theta \lesssim 0.06 \to 0.31$, a ratio of about 5 Reading . This sits at the boundary, within errors, of the $Cs^2 = 4C$ that the strong form allows at $s = 2$. It can be read as follows: on the twisted 2×2 T^2 with $N_T = 2$, the vortex enters cheaply as a thin vortex on the lattice rather than as a smooth fractional instanton, and the slab tension comes out large. The strong form cannot be tested with values from small lattices.

07

Reading the Lean statements — `Doubling.iterate`, `decay`

The algebra of step 3 of the Conditional Reduction Theorem is two theorems in the Lean 4 file `Doubling.lean` (namespace `Doubling`). We read the statements in the words of the text.

`Doubling.iterate`: for a real number c and a real sequence $t : \mathbb{N} \to \mathbb{R}$, under the assumptions $0 \leq c$, $0 \leq t_k$ for all k, and $t_{k+1} \leq c^3 t_k^4$ for all k, we have $c\,t_k \leq (c\,t_0)^{4^k}$ for all k.

`Doubling.decay`: under the assumptions $0 < c$, $0 \leq t_k$, $t_{k+1} \leq c^3 t_k^4$, we have $t_k \leq (c\,t_0)^{4^k}/c$ for all k.

The K of the text is c^3 in Lean. What machine checking guarantees is only the algebra of these real inequalities. That t_k is the sector bias, and that the recursion (9) holds — that is, the base point at small T^2, frame decorrelation and the lateral-widening inequality — lie outside Lean. The axioms of the two theorems are, in the record, the three standard ones (`propext`, `Classical.choice`, `Quot.sound`), and they have passed this book's re-check (A-04).

08

What remains

The Conditional Reduction Theorem reduced the correct form of the band to three assumptions. The three are different in character.

- **The base point at small T^2** is a statement on the weak-coupling side, and a line of showing it under ultraviolet control and uniform acceptance of insertion has been considered (Part 25, 27-02).
- **Frame decorrelation**, read in terms of κ_{cb}, holds with constant 1 for all L by the Nested Supermultiplicativity Lemma of 26-05, and does not remain as an independent assumption (the Reduction Read in the Chessboard Density in § 03). The content uniform in L moves to $(\mathrm{C}_{\mathrm{cb}})$, which is the lateral-widening inequality (strong form) read in terms of κ_{cb}. In exchange, the base point at small T^2 and the lateral-widening inequality (strong form) must be read in terms of κ_{cb} (or the single line (B_0) added).
- **The lateral-widening inequality** is the only place in this reduction where non-abelian dynamics enters. It fails for U(1) and $\mathbb{Z}_2$, and is expected to hold for SU(2) in the dilute regime with the margin $5/3 = 11/3 - 2$ (26-03, [Reading] physics). It does not follow from structural inputs alone, such as reflection positivity or duality (26-04).

What is open is to show the strong form (or the weak form) of the lateral-widening inequality non-perturbatively through the region between the dilute regime and the confinement regime — $L \sim 1/\Lambda$. The difficulty of the band has gathered into this single inequality.

09

The table of the five grades and the dependency graph

Table 1 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Nesting in the directions of the plane (2)	Cited	Mack–Petkova, 25-03	T^2 fixed	—
2	The base point at small T^2 (3)	Assumption	25-02, 27-02	$L_0 \geq L_{\min}(a)$	—
3	Frame decorrelation (4)	Assumption	26-05, A-06	Constant $C_{\mathrm{D}} \geq 1$	—
4	The lateral-widening inequality (strong form) (5)	Assumption	A-06	$L_{\min}(a) \leq L \leq L' \leq R$, square T^2	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
5	The lateral-widening inequality (weak form) (6)	Assumption	A-06	Dyadic sequence, $\sum \varepsilon_k < \infty$	—
6	The strong form is the case $\varepsilon_k = 0$, $C' = \ln 4C$ of the weak form	Proved here	§ 02	—	—
7	Iteration: if $t_{k+1} \le c^3 t_k^4$, then $ct_k \le (ct_0)^{4^k}$ and $t_k \le (ct_0)^{4^k}/c$	Proved here	§ 03 step 3	Real-number algebra	Doubling (`Doubling.iterate`) (assumptions: $0 \le c$, $0 \le t_k$, $t_{k+1} \le c^3 t_k^4$), Doubling (`Doubling.decay`) (assumptions: $0 < c$ and the same two)
8	Conditional Reduction Theorem: under the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k, R_k)) \ge \gamma R_k^2$, and the area law on the slab	Proved here	Proof in § 03, the Area Law from the Area Exponent in 26-01	Under the three assumptions, dyadic sequence, $L \ge L_{\min}(a)$	— (only the iteration part; the iteration row of the table)
9	Reduction Theorem in the Weak Form: under frame decorrelation, the lateral-widening inequality (weak form) and $u_0 > A/(3P)$, $u_k \ge \gamma' R_k^2$	Proved here	Proof in § 04	Under the two assumptions and the condition on the size of the base point	—
10	Slab-Tension Monotonicity Proposition: under the strong form, σ_{eff} does not decrease in L	Proved here	Proof in § 05	$L \ge L_{\min}(a)$	—
11	σ_{eff} grows as $L^{5/3}$ at small L, and goes to σ_∞ at large L	Reading (physics)	§ 05, 26-03	Dilute regime, one loop / confinement regime	—
12	The Creutz ratio increases and saturates for $N_T = 4 \to 10$, no significant overshoot	Reading	§ 05, 26-09	SU(2), $\beta = 2.5$, $M = 16$, 2000 measurements, trend only	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
13	For $N_T = 2 \to 4$ the ratio of biases is about 5	Reading	§ 06	SU(2), $\beta = 2.5$, $M = 8$, at the boundary within errors	—
14	The lateral-widening inequality must be stated restricted to $L \geq L_{\min}(a)$	Reading (physics)	§ 06	Lattice artefacts on a small T^2	—
15	The lateral-widening inequality (strong form) fails for U(1) and $\mathbb{Z}_2$, with margin $5/3$ for dilute SU(2)	Reading (physics)	26-03	Dilute regime, one loop. "Within this reduction"	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  nesting in the directions of the plane (2)            T² fixed
 2  ASM  base point at small T² θ(L₀;T²_L₀) ≤ 1 − ε₀
 3  ASM  frame decorrelation θ(sR;T²_L) ≤ C_D θ(R;T²_L)^(s²)
 4  ASM  lateral widening, strong form θ(R;T²_L') ≤ C(L'/L)² θ(R;T²_L)
 5  ASM  lateral widening, weak form (finite total loss)
 6  PRV  the strong form is a special case of the weak form ⟸ 4
 7  PRV  t_(k+1) ≤ c³t_k⁴ ⇒ ct_k ≤ (ct₀)^(4^k)                 Lean Doubling.i
 8  PRV  Conditional Reduction Theorem: ln(1/θ(R_k,R_k)) ≥ γR_k² ⟸ 1, 2, 3, 4
 9  PRV  Reduction Theorem in the Weak Form ⟸ 3, 5
10  PRV  Slab-Tension Monotonicity: strong form ⇒ σ_eff does not decrease ⟸ 4
11  RDG  shape of σ_eff (grows at small L, saturates at large L)  physics
12  RDG  the Creutz ratio increases and saturates              β = 2.5, M = 16
13  RDG  N_T = 2 → 4 gives a ratio of about 5                  β = 2.5, M = 8
14  RDG  restriction to L ≥ L_min(a) is needed ⟸ 13
15  RDG  lateral widening fails for U(1), Z₂; margin 5/3 for dilute SU(2)   26
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The premises of the nodes of the two reduction theorems always include assumption rows (the base point at small T^2, frame decorrelation, the two forms of the lateral-widening inequality). The assumption rows do not contain the quantity of the conclusion (the area exponent of $\theta(R_k, R_k)$): the base point at small T^2 is a statement at one small scale, frame decorrelation a statement of nesting with T^2 fixed, and the lateral-widening inequality (strong form) a statement of comparison with the container fixed. The numerical rows (the Creutz ratio on the slab, the ratio of biases) and the estimate rows (the shape of the slab tension, the failure depending on the group) are premises of no theorem.

What this chapter can say

What is said	What is not said
Proved here Under the three of base point, frame decorrelation and the lateral-widening inequality (strong form), $\ln(1/\theta(R_k, R_k)) \geq \gamma R_k^2$ on the dyadic sequence of scales, and a tension on the slab independent of k (the Conditional Reduction Theorem)	Any of the three assumptions. The scales in between. Uniformity of the constants in a (it depends on the constants of the assumptions)
Proved here With the weak form too, the same conclusion if the base point is small enough at one scale (the Reduction Theorem in the Weak Form)	The weak form itself. A jump from the weak form alone
Proved here The strong form includes monotonicity of the slab tension and is stronger than confinement (the Slab-Tension Monotonicity Proposition)	That the slab tension does not overshoot
Reading The Creutz ratio on the slab increases and saturates. On a small T^2, lattice artefacts touch the boundary of the strong form	The behaviour of the slab tension across the whole band. Values on large lattices
Lean: the real-number algebra of the iteration (Doubling (`Doubling.iterate`, `Doubling.decay`))	That the bias satisfies the recursion (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The container identity and nesting	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442, 125 (1980) 117. Chapter 25-03
Conditional Reduction Theorem, Reduction Theorem in the Weak Form, Slab-Tension Monotonicity Proposition	Proved here	Proofs in the text, § 03– § 05
The form of the string tension on a small T^2 (the base point and the small-L side of σ_{eff})	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166, eqs. (2.32) and (2.37). Chapter 25-02
The Creutz ratio on the slab, the bias for $NT = 2 \to 4$	Reading	Monte Carlo for SU(2) (Creutz ratio), and replica exchange and Bennett's acceptance-ratio method (vortex free energy). The table in chapter 26-09
Lean	Lean	Doubling (`Doubling.iterate`, `Doubling.decay`). Guarantees only the real-number algebra. A-04

Terms used in this chapter — nesting in the directions of the plane, the lateral-widening inequality (strong form, weak form), the base point at small T^2, frame decorrelation, the doubling iteration, jump, the slab tension σ_{eff}, overshoot, the scale of lattice artefacts $L_{\min}(a)$. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-03 The only place where non-abelian enters — the $5/3 = 11/3 - 2$ that decides the sign of the lateral-widening inequality.

The only place where non-abelian enters

> This chapter is this book's own investigation. The established knowledge it builds on is in 25-02 and 3-05.

What this chapter shows — Where non-abelian dynamics enters within the reduction of 26-02. Of the three assumptions (the small-T^2 base point, frame decorrelation, the lateral-widening inequality) and the nesting inequality, only the lateral-widening inequality fails for U(1) and $\mathbb{Z}_2$, and we see that what fixes its sign is the exponent margin in the dilute regime $5/3 = 11/3 - 2$ — the margin by which the coefficient of asymptotic freedom $11/3$ beats the 2 of the number of positions of a lump in the plane (the transverse two-dimensional entropy). For contrast we set side by side that abelian flux can only spread out uniformly, so that the vortex free energy of U(1) increases when T^2 is widened, and that a vortex of $SU(2)$ can localise in a scale-invariant self-dual lump.

Prerequisite chapters — 26-01 (the sector bias θ, the U(1) count), 26-02 (the three assumptions and the Conditional Reduction Theorem), 25-02 (vortices and fractional instantons), 3-05 (instantons and the Bogomolny bound), 4-06 (asymptotic freedom), 5-06 (U(1) and the non-abelian).

Said first — The "only place" of the title is a way of speaking **within this reduction**. In another reduction (the renormalisation comparison of 26-06), the non-abelian input enters a place of a different form. The counts of the behaviour of U(1), $\mathbb{Z}_2$ and $SU(2)$ are counts that the original paper itself states to be estimates, and in this chapter too they carry the label [Reading] (physics). We say nothing outside the dilute regime and one loop. The propositions set out on paper are only an elementary lemma that flux spreads out uniformly, and arrangements of logic.

The order of this chapter

1. The question — what fixes the sign of the lateral-widening inequality
2. Abelian flux can only spread out uniformly — the Flux-Equalisation Lemma and the counts for U(1) and $\mathbb{Z}_2$
3. A non-abelian vortex can localise in a scale-invariant lump — the self-dual lump with $Q = \frac{1}{2}$
4. The margin $5/3 = 11/3 - 2$ — the count for a dilute gas and the Tanizaki–Ünsal formula
5. The only place "within this reduction" — the Contrapositive of the Reduction, and its limits
6. Numerical signs — the vortex free energy when T^2 is widened (Table 2)
7. The table of the five grades and the dependency graph

01

The question — what fixes the sign of the lateral-widening inequality

The notation is as in 26-01 § 01 and 26-02 § 01. $\theta(R;T^2_L)$ is the sector bias, in a typical frame, of the container $D\times T^2_L$ of size R in the planar directions (area A_D), with the twist $n_{12}=1$ fixed on the transverse torus T^2_L (area $A_T=L^2$). The Conditional Reduction Theorem of 26-02 derived the correct form of the band, $\ln(1/\theta(R,R))\geq cR^2$, from three assumptions. The third of them was the lateral-widening inequality (strong form)

$$\theta(R;T^2_{L'})\;\leq\;C\Big(\frac{L'}{L}\Big)^2\,\theta(R;T^2_L)\qquad(L_{\min}(a)\leq L\leq L'\leq R)\tag{1}$$

If we widen the transverse T^2 while keeping the container fixed, which way does the sector bias θ move? The smaller θ is, the greater the disorder of the sector (the more vortices), so the question is whether widening the transverse directions makes vortices **costlier** (θ moves towards 1) or **not costlier**. Even if they become costlier, (1) allows it only up to the factor s^2 of the number of positions ($s=L'/L$).

We write the answer of this chapter first. In an abelian theory the flux of a vortex can only spread out uniformly over the whole transverse cross-section, so the wider the transverse directions, the costlier the vortex, and (1) fails. In $SU(2)$ a vortex can localise in a self-dual lump of size about L, and its action does not depend on the size, so classically the cost of widening the transverse directions is 0. The remaining quantum effects lean, in the dilute regime, to the side of asymptotic freedom, and (1) is expected to hold with margin $5/3$.

02

Abelian flux can only spread out uniformly

First we set out one elementary fact about the classical action.

Flux-Equalisation Lemma Proved here Let Ω be a planar region of area $|\Omega|$, and f a square-integrable real function on Ω with $\int_\Omega f=\Phi$. Then

$$\int_\Omega f^2\;\geq\;\frac{\Phi^2}{|\Omega|},\tag{2}$$

with equality only when f is almost everywhere equal to the constant $\Phi/|\Omega|$ on Ω.

Proof (2) follows from the Cauchy–Schwarz inequality $\big(\int_\Omega f\cdot 1\big)^2\leq\int_\Omega f^2\cdot\int_\Omega 1$. Equality holds only when f and 1 are proportional, that is, only when f is constant. ∎

The action of an abelian gauge field, $\frac{1}{2e^2}\int F_{\mu\nu}F_{\mu\nu}$, is the square of the field strength: a linear field theory. Consider a configuration in which the same flux Φ passes through each cross-section $D\times\{y\}$ ($y\in T^2_L$) of the container $D\times T^2_L$. Using the Flux-Equalisation Lemma on each cross-section and integrating over the transverse T^2_L, we get

$$\frac{1}{2e^2}\int_{D\times T^2_L}F_{34}^2\;\geq\;\frac{A_T}{2e^2A_D}\,\Phi^2\tag{3}$$

Proved here (under the condition that the flux through each cross-section is the same Φ; a lower bound on the classical action). Equality holds for the configuration in which the flux spreads out uniformly over the whole container, and **localising the flux only increases the action**. The right-hand side is proportional to the transverse area A_T, so if we fix the container and widen the transverse directions by a factor s, the minimal action of the flux π (corresponding to the sector $\sigma_D = -1$) becomes s^2 times larger. An abelian vortex becomes costlier when the transverse directions are widened: that is the answer at the classical level.

The count at the quantum level is as in 26-01 § 03 Reading (physics). Seen at long distance on the slab $T^2_L \times \mathbb{R}^2$, U(1) in the Coulomb phase is a two-dimensional U(1) (coupling e^2/A_T), and the partition function with flux Φ is the Poisson sum over wrappings k, $Z(\Phi) \propto \sum_k e^{ik\Phi} e^{-e^2 A_D k^2/(2A_T)}$. This is the sum over k of the classical action $\frac{A_T}{2e^2 A_D}(\Phi + 2\pi k)^2$ of (3), rewritten with the Poisson summation formula, and when $q := e^{-e^2 A_D/(2A_T)}$ is small

$$\theta(R; T^2_L) \;\approx\; 2q \;=\; 2\exp\Big(-\frac{e^2 A_D}{2A_T}\Big), \qquad \ln\frac{1}{\theta} \;\approx\; \frac{e^2 A_D}{2A_T} \;\propto\; A_D\, L^{-2}. \tag{4}$$

If the transverse directions are widened by a factor s, A_T becomes s^2 times larger and q becomes q^{1/s^2}, so $\theta_{\text{new}} \approx 2(\theta_{\text{old}}/2)^{1/s^2}$. The ratio is

$$\frac{\theta(R; T^2_{sL})}{\theta(R; T^2_L)} \;\approx\; 2^{\,1-1/s^2}\, \theta(R; T^2_L)^{-(1-1/s^2)} \;\longrightarrow\; \infty \qquad (\theta(R; T^2_L) \to 0,\; s > 1) \tag{5}$$

and it cannot be bounded by any constant Cs^2. **For U(1), the lateral-widening inequality (1) fails** Reading (physics; one line of algebra once (4) is granted). The weak form (the lateral-widening inequality (weak form) of 26-02) also fails, because the slab tension goes to 0 like e^2/L^2 Reading (physics).

The four-dimensional $\mathbb{Z}_2$ theory is, at weak coupling (large β), in a non-confining phase (5-10). There the flux through the container creates a wall of area A_T spanning the whole transverse T^2, and its tension does not depend on A_D. Widening the transverse directions increases the cost of the wall in proportion to A_T, and (1) fails Reading (physics). What the two abelian theories have in common is that **the cost of the flux increases with the transverse area**: in U(1) because of linearity (the Flux-Equalisation Lemma), in $\mathbb{Z}_2$ because of the area of the wall.

03

A non-abelian vortex can localise in a scale-invariant lump

In $SU(2)$ things change. On $T^2_L \times \mathbb{R}^2$ with the twist $n_{12} = 1$, the topological charge Q can take half-integer values ('t Hooft's twist, 25-01 Cited). And the action has the Bogomolny bound

$$S \;\geq\; \frac{8\pi^2}{g^2}\,|Q| \tag{6}$$

(3-05 Cited), with equality for self-dual (or anti-self-dual) configurations. A configuration with $Q = \frac{1}{2}$ — a fractional instanton — has action at least $4\pi^2/g^2$, and plays the role of a vortex that flips the planar sector (25-02).

The right-hand side of (6) depends neither on the size L of T^2 nor on the size R of the container. This is because the classical action of four-dimensional Yang–Mills does not change under a change of scale (the coupling g is dimensionless, 24-07 § 01). The estimate of the original paper is that the minimal-action configuration with $Q = \frac{1}{2}$ is a self-dual lump localised in a region of size about L, whose action equals the bound $4\pi^2/g^2$ and does not depend on the ratio of the transverse and longitudinal sizes Reading (physics). On the lattice, it has been measured that the action of cooled fractional instantons approaches $\pi^2\beta$ from below (with $\beta = 4/g^2$, $4\pi^2/g^2 = \pi^2\beta$; the values obtained by cooling are upper bounds on the minimal action on the lattice) (the table in 25-02).

Here the difference from the Flux-Equalisation Lemma of § 02 becomes clear.

- **Abelian**: the action is a quadratic form, so the minimal action for a given flux is attained by the configuration that spreads the flux out uniformly, and its value is proportional to the transverse area A_T ((3)). Localising does not make it cheaper; one pays the cost over the whole region it is spread over.
- **Non-abelian**: the self-duality equations are non-linear, and a solution of finite action $4\pi^2/g^2$ can be localised to a size of about L. As long as the container is large enough to hold the lump ($R \geq L'$), widening the transverse directions from L to L' only requires re-spreading the lump, and the classical cost is 0.

This is not a difference of local kinematics, but a difference in the existence of solutions of the self-duality equations — in non-linearity. Since at the classical level the cost of lateral widening is 0, the sign of (1) is fixed by quantum effects: the number of positions of the vortices, and the running of the coupling. That is the count of the next section.

04

The margin $5/3 = 11/3 - 2$

The count for a dilute gas. For $L\Lambda \ll 1$ (Λ is the scale of dimensional transmutation, 4-07), the coupling $g(L)$ is small, and the vortices (lumps with $Q = \pm\frac{1}{2}$) can be treated as a dilute gas, far apart from one another. For $SU(N)$, Tanizaki–Ünsal derived the ground-state energy and the string tension from the facts that on a small twisted T^2 the centre vortices behave like points on the plane $\mathbb{R}^2$ and that the weight of each is $Ke^{-SI/N}$ ($S_I = 8\pi^2/g^2$, with K the prefactor per unit area):

$$E_k(\theta) \;=\; -2Ke^{-SI/N}\cos\frac{\theta - 2\pi k}{N} \;\sim\; -\Lambda^2(\Lambda L)^{5/3}\cos\frac{\theta - 2\pi k}{N}, \qquad T_R \;\sim\; \Lambda^2(\Lambda L)^{5/3}\sin^2\frac{\pi|R|}{N} \quad (\theta =$$

(Tanizaki–Ünsal, eqs. (2.32) and (2.37) Cited , dilute regime, one loop). The θ here is the vacuum angle, a symbol different from this book's sector bias θ. The exponent $5/3$ comes out as follows: from the one-loop running of the coupling $8\pi^2/g^2(L) = \frac{11N}{3}\ln\frac{1}{L\Lambda}$ (4-06 Cited) we get $e^{-SI/N} = (L\Lambda)^{11/3}$, and from the size L of a lump the number of positions per unit area is $1/L^2$, so $(L\Lambda)^{11/3}/L^2 = \Lambda^2(\Lambda L)^{5/3}$.

We move on to the sector bias. In $SU(2)$, let ζ be the density per unit area of vortices and antivortices together, and approximate the positions of the vortices by independent Poisson points Reading (physics; the dilute-gas approximation). The sector σ_D of the container is $(-1)^n$ for the number n of vortices inside D. For n in a Poisson distribution with mean m,

$$\mathbb{E}\big[(-1)^n\big] \;=\; \sum_{n\ge 0} e^{-m}\frac{m^n}{n!}(-1)^n \;=\; e^{-2m} \qquad (8)$$

so Proved here (an elementary identity), putting $m = \zeta A D$,

$$\theta(R;T_L^2) \;\approx\; e^{-2\zeta A D}, \qquad \ln\frac{1}{\theta} \;\approx\; 2\zeta A D \;\propto\; A D\,\Lambda^{11/3}L^{5/3} \qquad (\Lambda \text{ fixed}) \qquad (9)$$

Reading (physics; dilute regime, one loop). Set beside (4), the power in L of $\ln(1/\theta)$ with the container fixed is as follows.

Table 1 The power in L of $\ln(1/\theta)$ with the container fixed (estimates; four dimensions, weak-coupling side)

Theory	Form of the vortex cost	Power of $\ln(1/\theta)$	On widening the transverse directions	Lateral-widening inequality (1)
$U(1)$ (Coulomb phase)	The flux spreads out uniformly over the transverse cross-section (the Flux-Equalisation Lemma)	L^{-2}	θ moves towards 1	Fails ((5))
$\mathbb{Z}_2$ (weak coupling)	A wall of area A_T	— (with the wall cost $\propto L^2$, $\theta \to 1$)	θ moves towards 1	Fails
$SU(2)$ (dilute regime)	A self-dual lump of size L, weight $(L\Lambda)^{11/3}$, density of positions $1/L^2$	$L^{+5/3} = L^{11/3-2}$	θ decreases	Holds (the ratio is at most 1)

The power $5/3$ for $SU(2)$ is the difference between the coefficient of asymptotic freedom $11/3$ and the 2 of the number of positions of a lump in the plane (what this part has been calling "the transverse two-dimensional entropy", Figure 2 of 26-00). The power -2 for U(1) can be read as the form in which there is no asymptotic-freedom term and only the term of the number of positions remains. **The margin $5/3 > 0$ by which asymptotic freedom beats the transverse two-dimensional entropy is the sign of the lateral-widening inequality itself** Reading (physics).

From (9), the ratio when the transverse directions are widened by a factor s in $SU(2)$ in the dilute regime is

$$\frac{\theta(R;T_{sL}^2)}{\theta(R;T_L^2)} \;\approx\; \exp\Big(-2A D\big(\zeta(sL)-\zeta(L)\big)\Big) \;\le\; 1 \qquad (10)$$

($\zeta \propto L^{5/3}$ is increasing), and (1) holds with $C = 1$, without using the factor s^2 Reading (physics). But (9) and (10) can be used only in the dilute regime $L\Lambda \ll 1$. In the confining regime ($L\Lambda \gg 1$) one expects $\ln(1/\theta) \approx \sigma A D - \ln(A_T/\xi^2)$ (ξ is the correlation length); the dependence on A_T is then only the logarithm of the entropy of the position of the vortex sheet, and (1) is expected to hold within the factor s^2 Reading (physics). **There is no estimate that passes non-perturbatively between the two regimes, $L \sim 1/\Lambda$ — the band.**

05

The only place "within this reduction"

We set out the components of the chain of the Conditional Reduction Theorem of 26-02 according to how they behave when the group is changed.

- **The nesting in the planar directions** ((2) of 26-02) is a statement that comes from the Markov property — that conditioning on the frame makes the insides of the containers independent — and from the container identity of Mack–Petkova, and it does not use the non-commutativity of the group Cited .
- **The base point at small T²** is an estimate that holds for U(1) too. Putting $R = L$ in (4), $\theta(L, L) \approx 2e^{-e^2/2}$ is a constant less than 1 independent of L (26-01 § 03) Reading (physics).
- **Frame decorrelation**, if L is fixed, is replaced under reflection positivity of z by the chessboard density κ_{cb} (26-05), and its structure is the same for $\mathbb{Z}_2$ and U(1) too (26-04). There is no estimate that U(1) fails in the form uniform in L (the uniform slab gap); U(1) is expected to fail only on the side of the uniform slab tension Reading (physics, 26-05 § 05).
- **The lateral-widening inequality** fails for U(1) and $\mathbb{Z}_2$ (§ 02), and is expected to hold with margin $5/3$ for $SU(2)$ in the dilute regime (§ 04).

Contrapositive of the Reduction Proved here If, in some theory, the conclusion of the Conditional Reduction Theorem of 26-02 ($\ln(1/\theta(R_k, R_k)) \geq \gamma R_k^2$ on a dyadic sequence of scales, and a tension on the slab independent of k) is false, then at least one of the base point at small T², frame decorrelation and the lateral-widening inequality (strong form) is false in that theory.

Proof The contrapositive of the Conditional Reduction Theorem. ∎

For U(1) at weak coupling, the tension on the slab is estimated to go to 0 like e^2/L^2 (26-01 § 03), and the conclusion of the Conditional Reduction Theorem is false. By the Contrapositive of the Reduction one of the three assumptions is false, and in the list above the base point at small T² holds and the lateral-widening inequality (strong form) fails. **Within this reduction, the only assumption that is necessarily used to exclude U(1) and $\mathbb{Z}_2$ is the lateral-widening inequality**, and that is where non-abelian dynamics enters Reading (physics; resting on estimates of the behaviour of U(1), $\mathbb{Z}_2$ and $SU(2)$).

Two qualifications go with this way of speaking.

- **It depends on the reduction.** "The only place" is a way of speaking within the chain of 26-02. In another reduction that splits the band into "β-monotonicity" and "the renormalisation comparison" (26-06), asymptotic freedom enters through the shift of the coupling in the comparison, $\beta' < \beta$. It is natural to read this as the same physics entering a differently shaped inequality in each reduction, but the precise correspondence between the two reductions has not been shown.
- **The estimates of the sign are only at the two ends.** The margin $5/3$ is a count in the dilute regime ($L\Lambda \ll 1$) at one loop, and the "only a logarithm" of the confining regime is also an estimate. There is no tool that shows the lateral-widening inequality in the band $L \sim 1/\Lambda$. The difficulty of the band is concentrated in this one interval of scales, within this one inequality.

06

Numerical signs — the vortex free energy when T^2 is widened

These are values of the vortex free energy $\Delta F = F_v$ (the ratio to the partition function with the twist flux n_{34} inserted, $\theta = \tanh(\Delta F/2)$) when the transverse T^2 is widened to $N_T \times N_T$, in the global form in which the container is the whole of the planar directions (T^2_M with period M) ((8) of 26-01), measured by replica exchange and Bennett's acceptance-ratio method [Reading] . The lattice is $N_T^2 \times M^2$, with twist $n_{12} = 1$ and the Wilson action. The errors are the estimates given by the source, and the source does not state the number of seeds.

Table 2 ΔF when the transverse T^2 is widened (global form; [Reading])

Theory, β, M	$N_T = 2$	4	6	8
$U(1)$, $\beta = 1.5$, $M = 8$	-0.03 ± 0.04	0.83 ± 0.11	2.10 ± 0.47	5.14 ± 0.78
$SU(2)$, $\beta = 2.5$, $M = 8$	-0.01 ± 0.12	0.63 ± 0.18	0.33 ± 0.22	1.51 ± 0.29
$U(1)$, $\beta = 1.5$, $M = 12$	—	0.20 ± 0.21	1.35 ± 0.30	1.72 ± 0.47
$SU(2)$, $\beta = 2.5$, $M = 12$	—	0.57 ± 0.22	0.42 ± 0.24	-0.15 ± 0.36
$SU(2)$, $\beta = 2.4$, $M = 8$	—	0.31 ± 0.10	—	0.11 ± 0.25

The reading is as follows [Reading] .

- For $U(1)$, at both values of M, ΔF increases when the transverse directions are widened (the vortex becomes costlier). At $M = 8$, θ is $0.39,\ 0.78,\ 0.99$ at $N_T = 4, 6, 8$, and $\ln(1/\theta)$ goes $0.93 \to 0.012$ for $N_T = 4 \to 8$ (about a factor of 80 smaller). This is the direction predicted in § 02.
- For $SU(2)$ the increase is much gentler, or there is none. At $M = 8$, $\beta = 2.5$, $\ln(1/\theta)$ goes $1.19 \to 0.45$ for $N_T = 4 \to 8$ (about a factor of 2.6 smaller), and the ratio of θ, $0.64/0.31 \approx 2$, is within the $4C$ that the lateral-widening inequality (strong form) allows at $s = 2$. At $M = 12$ and at $\beta = 2.4$, ΔF decreases for $N_T = 4 \to 8$. The exception is $N_T = 8$ at $\beta = 2.5$, $M = 8$ (the symmetric torus $N_T = M$), where ΔF rises from 0.33 to 1.51.

We write the limits with the same weight. Since this is a regime where θ is of order 1, the ratio of θ for U(1) is also held to about 2.5 because $\theta \leq 1$, and the discriminating power in this geometry is weak. β takes one or two points for each theory, and $\beta = 1.5$ for U(1) and $\beta = 2.5$ for $SU(2)$ are not matched to the same physics. The errors are 0.04–0.8. The values at $N_T = 2$ are in the region of lattice artefacts (26-02 § 06). **These are signs of a direction, not the conclusion of a test of the lateral-widening inequality.** The detailed tables of the measurements are in 25-08 and 26-09.

07

The table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The Bogomolny bound $S \geq 8\pi^2 \lvert Q\rvert /g^2$	Cited	3-05	Continuum classical action	—
2	Under the twist $n_{12} = 1$, Q can take half-integer values	Cited	't Hooft, 25-01	$T^2_L \times \mathbb{R}^2$, $SU(2)$	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
3	One-loop running $8\pi^2/g^2(L) = \frac{11N}{3}\ln\frac{1}{L\Lambda}$	Cited	4-06	One loop, $L\Lambda \ll 1$	—
4	The string tension on a small twisted T^2, $T_R \sim \Lambda^2(\Lambda L)^{5/3}\sin^2(\pi\lvert R\rvert/N)$	Cited	Tanizaki–Ünsal eqs. (2.32), (2.37)	Dilute regime, one loop, $\theta = 0$	—
5	Flux-Equalisation Lemma: $\int f^2 \geq \Phi^2/\lvert\Omega\rvert$, with equality when constant	Proved here	Proof in § 02	Square-integrable	—
6	Lower bound (3) on the abelian action: $\geq \frac{A_T}{2e^2 A_D}\Phi^2$, equality for the uniform configuration	Proved here	The Flux-Equalisation Lemma on each cross-section	The same flux Φ through each cross-section, classical action	—
7	In the Coulomb phase of U(1), $\theta \approx 2e^{-e^2 A_D/(2A_T)}$	Reading (physics)	26-01 § 03	Coulomb phase, two-dimensional U(1) at long distance, an estimate	—
8	For U(1) the lateral-widening inequality fails (the ratio (5) is unbounded)	Reading (physics)	§ 02	On top of the estimate for the Coulomb phase of U(1)	—
9	For $\mathbb{Z}_2$ at weak coupling the lateral-widening inequality fails (a wall of area A_T)	Reading (physics)	§ 02, 5-10	Weak coupling, an estimate	—
10	The minimal action with $Q = \frac{1}{2}$ in $SU(2)$ is a self-dual lump of size about L, and the classical cost of lateral widening is 0	Reading (physics)	§ 03, 25-02	Stated as an estimate by the original paper; the container large enough to hold the lump	—
11	The Poisson identity $\mathbb{E}[(-1)^n] = e^{-2m}$	Proved here	(8) of § 04	Poisson distribution with mean m	—
12	In a dilute gas $\theta \approx e^{-2\zeta A_D}$, $\ln(1/\theta) \propto A_D L^{5/3}$	Reading (physics)	(9) of § 04	Dilute regime, one loop, approximation by independent points	—
13	The margin $5/3 = 11/3 - 2$ fixes the sign of the lateral-widening inequality	Reading (physics)	§ 04	Dilute regime, one loop	—
14	For dilute $SU(2)$ the ratio on lateral widening is at most 1 ((10))	Reading (physics)	§ 04	$L\Lambda \ll 1$	—
15	In the confining regime the dependence on A_T is only logarithmic	Reading (physics)	§ 04	$L\Lambda \gg 1$, an estimate	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
16	Contrapositive of the Reduction: if the conclusion of the Conditional Reduction Theorem is false, one of the base point at small T^2, frame decorrelation and the lateral-widening inequality (strong form) is false	Proved here	§ 05 (contrapositive)	—	—
17	Within this reduction, only the lateral-widening inequality excludes U(1) and $\mathbb{Z}_2$	Reading (physics)	§ 05	Within the chain of 26-02	—
18	On lateral widening, ΔF increases for U(1); for $SU(2)$ it increases gently or not at all	Reading	§ 06 Table 2	$M = 8, 12$, U(1) $\beta = 1.5$, SU(2) $\beta = 2.4$, 2.5, $N_T = 2$–8, errors 0.04–0.8, the exception at $N_T = M = 8$	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  Bogomolny bound S ≥ 8π²|Q|/g²                              continuum, classical
 2  CIT  Q half-integer under the twist                             T²_L × R²
 3  CIT  8π²/g²(L) = (11N/3) ln(1/LΛ)                               one loop
 4  CIT  T_R ~ Λ²(ΛL)^(5/3) sin²(π|R|/N)                            dilute, one loop
 5  PRV  Flux-Equalisation Lemma: ∫f² ≥ Φ²/|Ω|
 6  PRV  abelian action ≥ (A_T/2e²A_D)Φ² ⟸ 5                         same flux through each cr
 7  RDG  U(1): θ ≈ 2exp(−e²A_D/2A_T)                                estimate
 8  RDG  lateral widening fails for U(1) ⟸ 6, 7
 9  RDG  lateral widening fails for Z₂                              weak coupling, wall
10  RDG  SU(2) vortex is a self-dual lump of size L ⟸ 1, 2          classical cost 0
11  PRV  E[(−1)^n] = e^(−2m)
12  RDG  θ ≈ e^(−2ζA_D), ln(1/θ) ∝ A_D L^(5/3) ⟸ 3, 4, 11           dilute, one loop
13  RDG  margin 5/3 = 11/3 − 2 is the sign ⟸ 8, 10, 12
14  RDG  ratio ≤ 1 for dilute SU(2) ⟸ 12
15  RDG  only a logarithm in the confining regime                   estimate
16  PRV  Contrapositive of the Reduction                            26-02
17  RDG  in this reduction only lateral widening separates groups ⟸ 8, 9, 13, 16
18  RDG  direction of Table 2                                       small lattices, one or two
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The nodes that stand on paper are four: the Flux-Equalisation Lemma, the lower bound on the abelian action, the Poisson identity and the Contrapositive of the Reduction, all of them elementary. The claims of the chapter (that the margin 5/3 = 11/3 − 2 fixes the sign, and that within this reduction only the lateral-widening inequality separates the groups) are "Reading (physics)", and their premises include estimate rows (the Coulomb phase of U(1), the self-dual lump of SU(2), the bias of the dilute gas). The numerical row (the direction of Table 2) is not a premise of any conclusion. There is no node for the lateral-widening inequality itself. It is placed as an assumption in the ledger of conditions of 27-02.

What this chapter can say

What is said	What is not said
Proved here For the abelian action, the minimal action for a given flux is attained by the uniformly spread configuration and is proportional to the transverse area (under the condition of the same flux through each cross-section; classical action)	A proof, as a theorem, of the quantum U(1) count (4)
Reading (physics) For U(1) and $\mathbb{Z}_2$ the lateral-widening inequality fails; for dilute $SU(2)$ it holds with the margin $5/3 = 11/3 - 2$	The lateral-widening inequality in the band $L \sim 1/\Lambda$. The sign outside the dilute regime and one loop
Proved here + Reading (physics) Within this reduction, only the lateral-widening inequality excludes U(1) and $\mathbb{Z}_2$	An "only place" independent of the reduction. The precise correspondence with the reduction of 26-06
Reading On small lattices, the direction in which widening the transverse directions makes the U(1) vortex costlier and the $SU(2)$ one not (with one exception)	Behaviour on large lattices and at many values of β. The values of the constants of the inequality

Sources and reproduction

Item	Kind	Source / tools
The gas of centre vortices on a small twisted T^2 and the string tension (7)	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 (Prog. Theor. Exp. Phys. 2022), eqs. (2.32) and (2.37). Chapter 25-02
The Bogomolny bound, half-integer topological charge	Cited	Sources of chapters 3-05 and 25-01 (G. 't Hooft, Nucl. Phys. B153 (1979) 141 and others)
One-loop running of the coupling	Cited	Sources of chapter 4-06
The count for the Coulomb phase of U(1); the estimate that the lateral-widening inequality separates the groups	Reading (physics)	The proposition "the inequality separates the theories" of the paper on the reduction of the band (a paper that came out of this book, A-07) (the paper itself states that it is heuristic). Chapter 26-01
Flux-Equalisation Lemma, (3), (8), Contrapositive of the Reduction	Proved here	The proofs in § 02, § 04, § 05 of the text
The vortex free energies of Table 2	Reading	Replica exchange and Bennett's acceptance-ratio method. The tables of chapters 25-08 and 26-09

Terms used in this chapter — the lateral-widening inequality, the sector bias θ, the self-dual lump (fractional instanton), the dilute gas, the margin $5/3$, the transverse two-dimensional entropy, the Coulomb phase, the crossover band (the band). Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-04 Structure alone cannot cross — why reflection positivity and duality alone do not give the lateral-widening inequality.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band the negative lemma, the electric flux, reflection positivity of z · Kiichi, Shiori and Rin

Structure alone cannot cross

> This chapter is this book's own investigation. § 01 is established knowledge; § 05 is a map of open questions.

What this chapter shows — We set out the structural tools of lattice gauge theory — the positivity of the transfer matrix, the centre and duality identities, reflection positivity, the symmetries of the lattice, the finite lemmas — in the language of the twisted slab, and check that every one of them holds as it stands for $\mathbb{Z}_2$ and U(1). From this follows the negative lemma: **from this set of inputs alone**, the lateral-widening inequality cannot be derived. To cross the band one needs a dynamical input that excludes the non-confining phase. On the way, we set out with proofs two things that can certainly be said from the side of structure: the electric-flux form of the sector bias, $\theta = Y(1)/Y(0) \geq 0$, and the positivity of reflections through lattice points for the z ensemble. The latter is an input to the chessboard estimate of the next chapter.

Prerequisite chapters — 26-01 (the sector bias and the global form (8)), 26-02 (the lateral-widening inequality), 26-03 (that it fails for U(1) and $\mathbb{Z}_2$), 5-03 (reflection positivity and the transfer matrix), 25-01 (the twist and 't Hooft flux), A-04 (Lean).

Said first — The negative lemma is a statement of the form "**from this set of inputs alone**": an arrangement of logic with the list of inputs made explicit. It does not say that none of the inputs is of any use (reflection positivity is used in 26-05). "The lateral-widening inequality fails for U(1) and $\mathbb{Z}_2$" is an estimate of 26-03 ([Reading], physics), so the last step of the conclusion of the negative lemma inherits that label. What was checked in Lean is only the spectral algebra of the electric-flux form and the algebra of the final sum in reflection positivity.

The order of this chapter

1. The list of structural inputs — what we call "structure"
2. The electric-flux form — $\theta = Y(1)/Y(0) \geq 0$ and the tanh comparison
3. Reflection positivity of z — reflections through lattice points, and how transverse reflections break down under the twist
4. Lemma that Structural Inputs Alone Do Not Suffice — from this set of inputs alone

5. The inputs needed — what excludes the non-confining phase
6. Reading the Lean statements — `FluxDuality`, `ReflPos`
7. The table of the five grades and the dependency graph

01

The list of structural inputs — what we call "structure"

Of the five inputs in this section, three (positivity of the transfer matrix, the centre and duality identities, the lattice symmetries) summarise existing knowledge. The introduction is in 5-03 § 05 (the transfer matrix) and 5-03 § 04 (reflection positivity).

In lattice gauge theory with the Wilson action on the lattice $\Lambda = (\mathbb{Z}/L_1) \times (\mathbb{Z}/L_2) \times (\mathbb{Z}/M)^2$, we place the background twist $n_{12} = 1$ in the two transverse directions 1 and 2 (25-01). Directions 3 and 4 are the plane. What this chapter calls the **structural inputs** is the following set $\mathcal{I}$ of five.

1. **Positivity of the transfer matrix**: the transfer matrix T with a planar direction (3 or 4) as time is a positive operator (5-03, Osterwalder–Seiler, Lüscher Cited).
2. **The centre and duality identities**: inserting twist flux is inserting a centre transformation, and it can be decomposed by the projections onto electric flux ('t Hooft Cited). The Electric-Flux Decomposition Lemma of § 02 is one of this form.
3. **Reflection positivity in the planar directions**: the positivity of reflections through lattice points in directions 3 and 4 for the z ensemble with the twist fixed (the reflection positivity of z in § 03).
4. **Symmetries of the lattice**: translations, rotations, reflections, and the fact that they preserve the class of the twist.
5. **The finite lemmas**: the container identity and nesting (25-03), the slab estimate $\sup_\Phi \theta_D \le \tanh(\beta L^2)^{|D|}$ (25-04), the flow lemma $\theta \le 1 - a$ (25-06). All of them use only properties of measures and finite sums.

The question is whether the lateral-widening inequality ((5) of 26-02) follows from $\mathcal{I}$. The answer, in § 04, is "it does not". Before getting there, we set out two things that can certainly be said about the sector bias within $\mathcal{I}$.

02

The electric-flux form — $\theta = Y(1)/Y(0) \geq 0$

Take the lattice to be $T^2_{L1\times L2} \times T^2_{M\times M}$, and write $Z(1, n34)$ for the partition function with, in addition to $n12 = 1$, the twist flux $n34 \in \{0, 1\}$ through the plane (3,4) inserted. The global bias of (8) of 26-01 was

$$\theta \;:=\; \frac{Z(1,0) - Z(1,1)}{Z(1,0) + Z(1,1)} \;=\; \tanh\frac{F_v}{2}, \qquad e^{-Fv} := \frac{Z(1,1)}{Z(1,0)} \quad (1)$$

We choose direction 3 as time. The space at one time is $T^2_{L1\,L2} \times T^1_M$ (directions 1, 2, 4), and in it there is the spatial magnetic flux $m_4 = n_{12} = 1$. The transfer matrix $T_3 \geq 0$ commutes with the centre transformation $\hat{C}_4$ of direction 4 (the operation that multiplies the links going once round direction 4 by the centre element -1), and $\hat{C}_4^2 = 1$. When space is split by the eigenvalues $(-1)^{e4}$ of $\hat{C}_4$, $e_4 \in \{0, 1\}$ is the **electric flux** in direction 4 ('t Hooft Cited). Let the projections be P_e and put

$$Y(e) \;:=\; \mathrm{Tr}\big(P_e\, T_3{}^M\big) \qquad (e = 0, 1) \quad (2)$$

This defines Y(e).

Electric-Flux Decomposition Lemma (the electric-flux form) Proved here In the setting above, with $Y(0), Y(1) \geq 0$,

$$Z(1,0) = Y(0) + Y(1), \qquad Z(1,1) = Y(0) - Y(1). \quad (3)$$

Hence when $Z(1,0) + Z(1,1) > 0$, $\theta = Y(1)/Y(0) \geq 0$. If moreover the positivity assumption for the twisted partition function holds, then $\theta \leq 1$; that is, with $F_v \geq 0$, $0 \leq \theta \leq 1$.

Proof The partition function is the trace of the transfer matrix, $Z(1,0) = \mathrm{Tr}\, T_3{}^M$, and the twist $n34 = 1$ through the plane (3,4) is the same as inserting $\hat{C}_4$ somewhere around the time circle, so $Z(1,1) = \mathrm{Tr}(\hat{C}_4 T_3{}^M)$ ('t Hooft's construction Cited). Substituting $\hat{C}_4 = P_0 - P_1$ and $1 = P_0 + P_1$ gives (3). P_e is an orthogonal projection commuting with T_3, so $P_e T_3{}^M P_e = P_e T_3{}^M$ is a positive operator and its trace is at least 0. When $Z(1,0) + Z(1,1) = 2Y(0) > 0$, putting (3) into (1) gives $\theta = 2Y(1)/2Y(0)$. From $Z(1,1) = Y(0) - Y(1) \geq 0$, $Y(1) \leq Y(0)$. ∎

Two remarks. First, $\theta \geq 0$ and $|Z(1,1)| \leq Z(1,0)$ are **spectral** facts that follow from (3) and $Y(e) \geq 0$ alone, but the positivity assumption for the twisted partition function needed for $\theta \leq 1$ is not. $Z(1,1)$ is at least 0 because it is the integral of a positive function containing the twisted plaquette weight $e^{-\frac{\beta}{2}\mathrm{tr}U_p}$, but this is the positivity of the Boltzmann weight and does not follow from the spectrum of the transfer matrix.

Second, the Electric-Flux Decomposition Lemma is a lattice version of 't Hooft's electric–magnetic duality: "the vortex free energy (magnetic flux) is small $\Longleftrightarrow$ the state with electric flux $e_4 = 1$ is heavy compared with $e_4 = 0$". In the confining regime $Y(1)/Y(0)$ is expected to be small because of the electric string tension Reading (physics, 26-05 § 04).

We add one elementary lemma, used when writing lateral widening in the global form.

tanh Comparison Lemma Proved here Let $\theta = \tanh(F/2)$, $\theta' = \tanh(F'/2)$, $F, F' \geq 0$. If $F' \leq KF$ $(K \geq 1)$, then $\theta' \leq K\theta$.

Proof $\tanh$ is monotonically increasing and concave on $[0, \infty)$ with $\tanh 0 = 0$, so for $K \geq 1$, $\tanh(Kx) \leq K \tanh x$ (for a concave function g with $g(0) = 0$, $g(x) = g(\frac{1}{K} \cdot Kx + (1 - \frac{1}{K}) \cdot 0) \geq \frac{1}{K} g(Kx)$). Hence $\theta' = \tanh(F'/2) \leq \tanh(KF/2) \leq K \tanh(F/2) = K\theta$. ∎

That is, lateral widening in the form of the vortex free energy, $F_v(L') \leq C(L'/L)^2 F_v(L)$, implies lateral widening in the form of the bias.

03

Reflection positivity in z — reflections through lattice points, and transverse reflections

Setting. Take the torus $\Lambda = (\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$ with M even. The variables are the links $U_\ell \in SU(2)$ and the plaquette variables $z_p \in \{\pm 1\}$, and we impose the following constraints.

- For every three-dimensional cube c, $\prod_{p \in \partial c} z_p = 1$ (z is a closed $\mathbb{Z}_2$ 2-cochain).
- Fixing the global class, $n_{12}(z) := \prod_{p \in \Pi} z_p = -1$. Π is a (1,2) plane with (x_3, x_4) fixed, and since z is closed this does not depend on the position of Π.

The measure is

$$d\mu \;=\; Z^{-1} \prod_{\ell} dU_\ell \sum_{z} \mathbf{1}[\delta z = 1]\, \mathbf{1}[n_{12}(z) = -1]\, \exp\Big(\frac{\beta}{2} \sum_{p} z_p \operatorname{tr} U_p\Big) \tag{4}$$

(dU is the Haar measure). This is the ensemble of 26-01 § 01 with the twist summed over, with the transverse twist fixed.

Reflection. Let Θ be the reflection through lattice points in direction 3, $x_3 \mapsto -x_3 \pmod M$. The fixed planes are $P_0 = \{x_3 = 0\}$ and $P_1 = \{x_3 = M/2\}$. It acts on links by $U_{(x,\mu)} \mapsto U_{(\Theta x, \mu)}$ for directions $\mu \neq 3$ and by $U_{(x,3)} \mapsto U^{-1}_{(\Theta x - e_3, 3)}$ for direction 3, on plaquette variables by $z_p \mapsto z_{\Theta p}$, and on functions by $(\Theta G)(U, z) := G(\Theta U, \Theta z)$.

Reflection positivity of z Proved here For any bounded function G depending only on the variables of the right half $\Lambda_+ = \{0 \le x_3 \le M/2\}$ (including the variables on the planes $P_0 \cup P_1$), $\langle G \cdot \Theta G \rangle_\mu \ge 0$. The same holds for direction 4.

Proof We show it in five steps.

Step 1 (splitting the variables). Split the variables into three. φ: the variables on the planes $P_0 \cup P_1$ (the links in directions 1, 2, 4 within the planes, and the z of the plaquettes of types (1,2), (1,4), (2,4) within the planes). ψ_+: the right variables (the direction-3 links leaving $0 \le x_3 < M/2$, the links in other directions with $0 < x_3 < M/2$, the plaquettes of type $(3,\nu)$ with $[x_3, x_3+1] \subset [0, M/2]$, and the z of in-plane-type plaquettes with $0 < x_3 < M/2$). $\psi_- := \Theta\psi_+$: the left variables. Θ does not move φ.

Step 2 (decomposing the weight). The plaquette weight $e^{\frac{\beta}{2} z_p \operatorname{tr} U_p}$ belongs to exactly one of right, left and the planes, since p does, and goes into that one. The weights of plaquettes on the planes go into $A(\varphi)$. The cube constraints split in the same way: a cube containing direction 3 spans $[x_3, x_3+1]$ and so depends only on the variables of one side and of the planes; a cube not containing direction 3 depends only on the right variables if $0 < x_3 < M/2$, and only on the plane variables if it lies on a plane. The global constraint n_{12}, measured on $\Pi \subset P_0$, is a function of φ. So the weight has the form $W = A(\varphi)\, F(\psi_+, \varphi)\, F'(\psi_-, \varphi)$, with $A \ge 0$, $F \ge 0$.

Step 3 (the relation between the left and right factors). The action and the constraints are invariant under Θ. Even when the orientation of a plaquette is reversed, in $SU(2)$ we have $\operatorname{tr} U_{\Theta p} = \operatorname{tr} U_p^{-1} = \operatorname{tr} U_p$. Cubes go to cubes, and the plane on which n_{12} is measured is fixed. Hence $F'(\psi_-, \varphi) = F(\Theta\psi_-, \varphi)$.

Step 4 (invariance of the measure). The Haar measure is invariant under taking inverses, and the counting measure of z is invariant under permutations.

Step 5 (positivity). From the above,

$$Z\,\langle G \cdot \Theta G \rangle \;=\; \int d\varphi\, A(\varphi) \Big[\int d\psi_+ F(\psi_+, \varphi) G(\psi_+, \varphi) \Big] \Big[\int d\psi_- F(\Theta\psi_-, \varphi) G(\Theta\psi_-, \varphi) \Big] \;=$$

∎

The algebra from the last equality to the inequality (in the version with finite sums) is checked by ReflPos in Lean (`ReflPos.refl_pos`) (§ 06). The algebra, after splitting into factors, of the Cauchy–Schwarz form that follows from positivity, $\langle G \cdot \Theta H \rangle^2 \le \langle G \cdot \Theta G \rangle \, \langle H \cdot \Theta H \rangle$ (for real G, H), is ReflPos (`ReflPos.refl_cs`). This is an input to the Chessboard Estimate (26-05).

The reflection positivity of z is the Osterwalder–Seiler argument for reflections through lattice points Cited (5-03) with the $\mathbb{Z}_2$ plaquette variables and the closedness constraint added. The key points are that each cube constraint fits on one side plus the planes, and that the global class n_{12} lies in a plane orthogonal to the direction of reflection (3) and can be measured on the plane P_0.

Transverse reflections. If we try the same argument with reflections in directions 1 and 2, the twist gets in the way.

Lemma on the Non-Positivity of Lateral Reflections Proved here In $SU(2)$, write $e^{\frac{\beta}{2}\operatorname{tr}U} = \sum_j c_j \chi_j(U)$ (χ_j is the character of spin j, $c_j = \frac{2(2j+1)}{\beta} I_{2j+1}(\beta) > 0$). Then the twisted plaquette weight is

$$e^{-\frac{\beta}{2}\operatorname{tr}U} = \sum_j (-1)^{2j} c_j \chi_j(U) \quad (5)$$

and the coefficients for half-integer j are negative. Hence, with the twist fixed, the weights of plaquettes straddling the plane of a reflection in the transverse directions 1 and 2 are not of positive type, and the standard argument for reflection positivity does not hold there.

Proof -1 is a centre element of $SU(2)$, and $-\operatorname{tr} U = \operatorname{tr}(-U)$, $\chi_j(-U) = (-1)^{2j}\chi_j(U)$. Hence (5) follows from $e^{-\frac{\beta}{2}\operatorname{tr}U} = e^{\frac{\beta}{2}\operatorname{tr}(-U)} = \sum_j c_j \chi_j(-U)$. The formula for c_j and its positivity are the standard expansion via the modified Bessel functions $I_n(\beta) > 0$ ($\beta > 0$) Cited . Being a weight of positive type (coefficients of the character expansion at least 0) is the property the standard argument needs for the plaquettes straddling the plane of reflection, and (5) does not satisfy it. ∎

The consequence of the Lemma on the Non-Positivity of Lateral Reflections is that **reflection positivity gives no convexity or monotonicity in the transverse size L.** The lateral-widening inequality is a comparison in L, so the road to obtaining it from reflections is closed from the start. What the Lemma on the Non-Positivity of Lateral Reflections says is that the standard argument cannot be used; it does not show that positivity of transverse reflections fails for all functions.

04

The negative lemma — from this set of inputs alone

Lemma that Structural Inputs Alone Do Not Suffice (the negative lemma) Proved here The set of structural inputs $\mathcal{I}$ of § 01 (positivity of the transfer matrix, the centre and duality identities (including the Electric-Flux Decomposition Lemma), reflection positivity in the planar directions (the reflection positivity of z), the symmetries of the

lattice, the finite lemmas) holds, with the same proofs, in the four-dimensional Wilson theory in which the group is changed from $SU(2)$ to $\mathbb{Z}_2$ or U(1) and the twist variables are kept as ± 1. Hence, if there were an argument deriving the lateral-widening inequality using $\mathcal{I}$ alone, the same argument would derive the lateral-widening inequality for $\mathbb{Z}_2$ and U(1) too. Contrapositively, **if the lateral-widening inequality is false for either $\mathbb{Z}_2$ or U(1), it cannot be derived from this set of inputs alone**.

Proof For each input, we check the properties of the group it uses.

- Positivity of the transfer matrix: the Osterwalder–Seiler argument holds for the Wilson action of a compact group, for actions whose character-expansion coefficients are positive. $e^{\beta \operatorname{Re} U_p}$ for $\mathbb{Z}_2$ and U(1) satisfies this Cited .
- The centre and duality identities: all that is used is that -1 is in the centre of the group (the centre of $\mathbb{Z}_2$ and of U(1) is the group itself and contains -1), and that twist flux is the insertion of a centre transformation. The proof of the Electric-Flux Decomposition Lemma goes through as it stands.
- Reflection positivity: the only properties of the group used in the proof of the reflection positivity of z are $\operatorname{tr} U_{\Theta p} = \operatorname{tr} U_p^{-1} = \operatorname{tr} U_p$ in step 3 and the invariance of the Haar measure under inverses in step 4. For U(1), writing the action with $\operatorname{Re} U_p$, $\operatorname{Re} U^{-1} = \operatorname{Re} U = \operatorname{Re} U$; for $\mathbb{Z}_2$, $U^{-1} = U$; both hold.
- Symmetries of the lattice: the definitions of the action and of the class of the twist have a form independent of the group, so they are the same.
- The finite lemmas: the container identity uses independence conditioned on the frame (the Markov property), the slab estimate uses conditioning on the orbits of flips of the plaquette variables, and the flow lemma uses only stationarity; none uses the non-commutativity of the group.

The first half is shown, so the second half is logic: if an argument uses only the properties of $\mathcal{I}$, it gives the same conclusion in every theory that satisfies $\mathcal{I}$. ∎

The premise of the Lemma that Structural Inputs Alone Do Not Suffice, "the lateral-widening inequality is false for $\mathbb{Z}_2$ or U(1)", is the estimate of 26-03 § 02 Reading (physics; the count for the Coulomb phase of U(1) and the count for the wall of $\mathbb{Z}_2$ at weak coupling). Granting it, the conclusion is that **the lateral-widening inequality does not follow from the structural inputs alone**. This last step inherits the label of its premise, [Reading] (physics).

The same argument applies as it stands to the weak form (the lateral-widening inequality (weak form) of 26-02) and to the uniform slab tension. For U(1) and $\mathbb{Z}_2$ the slab tension is estimated to go to 0 as L grows, so neither follows from the same set of inputs alone Reading (physics).

We also write what the Lemma that Structural Inputs Alone Do Not Suffice does not say. First, it does not say that $\mathcal{I}$ is useless for the band. If L is fixed, the reflection positivity of z replaces frame decorrelation by the chessboard density κ_{cb} (26-05). Second, what comes out when one dynamical input is added to $\mathcal{I}$ is outside the lemma. Third, at strong coupling (small β) U(1) also confines, so what the lemma separates is not the groups themselves but **whether there is a non-confining phase**.

The thin-vortex floor — tools of structure alone give only the floor. Even when correlation inequalities and covering comparisons are added to the structural inputs, what follows is a lower bound (a floor) that does not distinguish between groups. For the $SU(2)$ Wilson measure on the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, for every real β, every $N, M \geq 1$ and background $n_{12} \in \{0,1\}$,

$$\theta = \frac{Z_{n34=0} - Z_{n34=1}}{Z_{n34=0} + Z_{n34=1}} \;\leq\; \tanh\!\left(N^2|\beta|\right)^{M^2} \qquad (6)$$

holds Proved here (the thin-vortex floor. The $SU(2)$ system is bounded above by a $\mathbb{Z}_2$ sign system, and the four-dimensional $\mathbb{Z}_2$ system is moved to a two-dimensional system of plaquettes and evaluated in closed form. Checked in Lean as a theorem about the concrete $SU(2)$ Wilson measure). The type of the inequality lies on the line of Mack–Petkova and Fröhlich Cited (not yet checked against the text of the originals). When the coupling is positive, in terms of the tension, $\sigma_N(\beta) \geq -\ln\tanh(N^2\beta) \approx 2e^{-2N^2\beta}$, which goes rapidly to 0 with N and is not uniform in N (inside the strong-coupling window there is a form in which the rate per unit area does not depend on the transverse size; the prefactor remains. Outside the window there is no uniform form. 20-06). The right-hand side is the value of a thin vortex made by stacking N^2 plaquettes along the plane, and the same comparison holds in the same form for the $\mathbb{Z}_2$ system. A comparison of this type becomes an equality for the Gaussian model, which sits at the limit of the class of models it applies to, so the comparison alone does not give that the floor keeps being exceeded at a uniform rate (the tension-doubling inequality of 26-06 § 08) Reading . Where the Lemma that Structural Inputs Alone Do Not Suffice says "does not follow from this set of inputs alone", the same point said from the side of what does follow is "what follows is only the floor". Above the floor is where the non-abelian dynamical input of § 05 is needed. The statement and proof of this floor are in 21-02.

05

The inputs needed — what excludes the non-confining phase

By the Lemma that Structural Inputs Alone Do Not Suffice, an argument that crosses the band must use at least one input outside $\mathcal{I}$. That input must be false for U(1) and $\mathbb{Z}$

2 at weak coupling — the non-confining phase. The candidates that the study of this book can name are all properties of non-abelian dynamics.

Table 1 Candidate inputs needed for the band (none is shown; the labels are estimates for the candidates)

Input	For U(1) and $\mathbb{Z}_2$	Where it enters	Chapter
Asymptotic freedom (the sign of the running of the coupling)	U(1) does not run or runs the other way; $\mathbb{Z}_2$ has no continuous running	The margin $5/3$ in the dilute regime; $\beta' < \beta$ in the renormalisation comparison	26-03 § 04, 26-06
Localisation in a self-dual lump (non-linearity)	Abelian flux can only spread out uniformly	The classical cost of lateral widening is 0	26-03 § 03
Ultraviolet control (multiscale analysis)	For U(1), a different form (conformal at long distance)	The base point at small T^2	the base point at small T^2 in 26-02, 27-02

Each row of Table 1 is an estimate, [Reading] (physics), and none of the inputs is in a form usable in the band $L \sim 1/\Lambda$. On the strong-coupling side, there is a line of argument by a polyhedral expansion in which the lateral-widening inequality holds with the constant $C \approx 1$ (there is a report that the factor $(L'/L)^2$ is consistent with the literature on the leading strong-coupling term (Münster 1981), but checking is not yet done). However, an estimate uniform in the planar size M has not been confirmed, and this book places it under [Assumption]. At strong coupling U(1) also confines, so this row does not contradict the Lemma that Structural Inputs Alone Do Not Suffice.

The Lemma that Structural Inputs Alone Do Not Suffice makes explicit a result that was expected; there is no surprise in it. Its role is to give one **touchstone** for reading arguments about the band: if an argument proceeds using only the properties of $\mathcal{I}$, it gives the same conclusion for U(1), so there is an error somewhere. The record in 26-08 of a road that drew the criticism that it does not distinguish U(1) is a real example of this touchstone.

06

Reading the Lean statements — `FluxDuality`, `ReflPos`

The spectral part of the Electric-Flux Decomposition Lemma and the algebra of step 5 of the reflection positivity of z are five theorems in the Lean 4 files `FluxDuality.lean` and `ReflPos.lean`. We read the statements in the words of the text. All of them are statements about sums of real numbers over a finite set ι (or Φ, Ψ).

`FluxDuality.split`: for families of reals $t, c : \iota \to \mathbb{R}$, under the assumption "for every i, $c_i = 1$ or $c_i = -1$", $\sum_i t_i - \sum_i c_i t_i = 2\sum_{c_i=-1} t_i$ and $\sum_i t_i + \sum_i c_i t_i = 2\sum_{c_i=1} t_i$. In

the text t_i are the eigenvalues of T_3^M and c_i the eigenvalues of $\hat{C}_4$, and the two formulas are $Z(1,0) \mp Z(1,1) = 2Y(1),\ 2Y(0)$ of (3).

`FluxDuality.twisted_le`: under the assumptions "for every i, $0 \le t_i$" and "$c_i = \pm 1$", $\left|\sum_i c_i t_i\right| \le \sum_i t_i$. This is $|Z(1,1)| \le Z(1,0)$ of the text.

`FluxDuality.bias_mem`: under the assumptions "$0 \le t_i$", "$c_i = \pm 1$", "$0 \le \sum_i c_i t_i$", "$0 < \sum_i t_i + \sum_i c_i t_i$", $0 \le \theta \le 1$ for $\theta := (\sum t - \sum ct)/(\sum t + \sum ct)$. **The third assumption $Z_{11} := \sum ct \ge 0$ is used for $\theta \le 1$**; it is not a spectral fact, and corresponds to the positivity of the Boltzmann weight in the text.

`ReflPos.refl_pos`: for finite sets Φ, Ψ and real functions $A : \Phi \to \mathbb{R}$, $F, G : \Psi \to \Phi \to \mathbb{R}$, under the assumption "for every φ, $0 \le A(\varphi)$", $\sum_\varphi \sum_{\psi_1} \sum_{\psi_2} A(\varphi) F(\psi_1, \varphi) F(\psi_2, \varphi) G(\psi_1, \varphi) G(\psi_2, \varphi) \ge 0$. It is the finite-sum version of $\int A \left| \int FG \right|^2 \ge 0$ in step 5 of the reflection positivity of z, and the sign of F does not matter.

`ReflPos.refl_cs`: under the same assumption $0 \le A$, for $g(\varphi) := \sum_\psi F(\psi, \varphi) G(\psi, \varphi)$ and $h(\varphi) := \sum_\psi F(\psi, \varphi) H(\psi, \varphi)$, $\left(\sum_\varphi A\, g\, h\right)^2 \le \left(\sum_\varphi A\, g^2\right)\left(\sum_\varphi A\, h^2\right)$. It writes the Cauchy–Schwarz form for reflections, $\langle G \cdot \Theta H \rangle^2 \le \langle G \cdot \Theta G \rangle \langle H \cdot \Theta H \rangle$, as algebra after the weight has been split into factors.

What machine checking guarantees is only this algebra of finite sums and real numbers. That the transfer matrix is positive, that it commutes with $\hat{C}_4$, that the twist is the insertion of $\hat{C}_4$, that the weight splits into the form $A \cdot F \cdot F$ (steps 1–4 of the reflection positivity of z), and the Haar measure lie outside Lean; the proofs in the text and the literature carry them. The axioms of the five theorems are, in the record, the three standard ones (`propext`, `Classical.choice`, `Quot.sound`), and they pass this book's re-check (A-04).

07

The table of the five grades and the dependency graph

Table 2 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Positivity of the transfer matrix $T \ge 0$	Cited	Osterwalder–Seiler, Lüscher, 5-03	Wilson action of a compact group	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
2	The twist $n34$ is the insertion of the centre transformation $\hat{C}_4$; e_4 is an eigenvalue of $\hat{C}_4$	Cited	't Hooft, 25-01	Groups whose centre contains -1	—
3	Electric-Flux Decomposition Lemma: $Z(1,0) = Y(0) + Y(1)$, $Z(1,1) = Y(0) - Y(1)$, $\theta = Y(1)/Y(0) \geq 0$	Proved here	Proof in § 02 $\Longleftarrow$ 1, 2	$T^2_{L_1 L_2} \times T^2_{M \times M}$, $n_{12} = 1$, $Z(1,0) + Z(1,1) > 0$	FluxDuality (`FluxDuality.split`) (assumption: $c_i = \pm 1$), `twisted_le` (assumptions: $0 \leq t_i$, $c_i = \pm 1$)
4	$\theta \leq 1$ ($F_v \geq 0$)	Proved here	§ 02 $\Longleftarrow$ 3	Uses the positivity assumption for the twisted partition function (positivity of the Boltzmann weight)	FluxDuality (`FluxDuality.bias_mem`) (assumptions: $0 \leq t_i$, $c_i = \pm 1$, $Z_{11} \geq 0$, $Z_{10} + Z_{11} > 0$)
5	tanh Comparison Lemma: if $F' \leq KF$ ($K \geq 1$), then $\tanh(F'/2) \leq K \tanh(F/2)$	Proved here	Proof in § 02 (concavity of $\tanh$)	$F, F' \geq 0$	—
6	Reflection positivity of z: positivity of reflections through lattice points in directions 3 and 4 for the z ensemble	Proved here	Proof in § 03 (the Osterwalder–Seiler argument with z and the constraints added)	$SU(2)$, M even, twist $n_{12} = 1$ fixed	ReflPos (`ReflPos.refl_pos`) (only the algebra of step 5; assumption: $0 \leq A$), `refl_cs` (assumption: $0 \leq A$)
7	Character-expansion coefficients $c_j = \frac{2(2j+1)}{\beta} I_{2j+1}(\beta) > 0$	Cited	Standard expansion (cross-checked numerically)	$SU(2)$, $\beta > 0$	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
8	Lemma on the Non-Positivity of Lateral Reflections: the twisted plaquette weight has negative coefficients for half-integer j; transverse reflections are not of positive type	Proved here	Proof in § 03 ⟸ 7	Twist fixed, reflections in directions 1 and 2	—
9	Lemma that Structural Inputs Alone Do Not Suffice: $\mathcal{I}$ holds for $\mathbb{Z}_2$ and U(1) too, and an argument from $\mathcal{I}$ alone gives the same conclusion there	Proved here	Proof in § 04 ⟸ 1, 3, 6	The form "from this set of inputs alone"; twist ± 1	—
10	For U(1) and $\mathbb{Z}_2$ the lateral-widening inequality is false	Reading (physics)	26-03 § 02	Weak coupling, an estimate	—
11	The lateral-widening inequality (strong and weak forms) and the uniform slab tension do not follow from the structural inputs alone	Reading (physics)	§ 04 ⟸ 9, 10	On top of the estimate that the lateral-widening inequality is false for U(1) and Z_2	—
12	Reflection positivity gives no convexity or monotonicity in the transverse size L	Proved here	§ 03 ⟸ 8	In the sense that the standard argument cannot be used	—
13	At strong coupling the lateral-widening inequality holds with $C \approx 1$	Assumption	Line of argument: polyhedral expansion (an estimate uniform in the planar size is unconfirmed)	Small β	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
14	In the confining regime $Y(1)/Y(0)$ is small because of the electric string tension	Reading (physics)	26-05 § 04	$L\Lambda \gg 1$, an estimate	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  positivity of the transfer matrix                       compact group, 
 2  CIT  twist is a centre insertion, e₄ an eigenvalue           't Hooft
 3  PRV  Electric-Flux Decomposition Lemma: θ = Y(1)/Y(0) ≥ 0 ⟸ 1, 2   Lean s
 4  PRV  θ ≤ 1 ⟸ 3                                               uses Z(1,1) ≥
 5  PRV  tanh Comparison Lemma: F' ≤ KF ⇒ θ' ≤ Kθ                K ≥ 1
 6  PRV  reflection positivity of z (directions 3, 4)            Lean refl_pos,
 7  CIT  character-expansion coefficients c_j > 0                SU(2)
 8  PRV  Lemma on the Non-Positivity of Lateral Reflections ⟸ 7   twist fixed
 9  PRV  Structural Inputs Alone Do Not Suffice ⟸ 1, 3, 6        from this set
10  RDG  lateral widening false for U(1), Z₂                     26-03, estimate
11  RDG  does not follow from structural inputs alone ⟸ 9, 10
12  PRV  no convexity or monotonicity in L from reflections ⟸ 8
13  ASM  C ≈ 1 at strong coupling                                line of argumen
14  RDG  Y(1)/Y(0) small in the confining regime                 estimate
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The nodes that stand on paper are the Electric-Flux Decomposition Lemma, $\theta \leq 1$, the tanh Comparison Lemma, the reflection positivity of z, the Lemma on the Non-Positivity of Lateral Reflections, the Lemma that Structural Inputs Alone Do Not Suffice, and the absence of convexity and monotonicity; they rest on the literature rows (positivity of the transfer matrix, the insertion of the centre transformation, the character-expansion coefficients). Only the claim of the chapter title (that the lateral-widening inequality and the uniform slab tension do not follow from the structural inputs alone) has an estimate row (that the lateral-widening inequality is false for U(1) and Z_2) as a premise, and it is "Reading (physics)". The assumption row (that the lateral-widening inequality holds at strong coupling) is not a premise of any conclusion.

The machine-checked form of reflection positivity is in 20-01, and the sign of θ is in 21-06.

What this chapter can say

What is said	What is not said
Proved here The global bias of the sector is the ratio of electric fluxes $\theta = Y(1)/Y(0) \in [0,1]$ (the upper bound uses the positivity of the weight)	An estimate of the size of $Y(1)/Y(0)$. The electric string tension in the band
Proved here The z ensemble with the twist fixed is positive under reflections through lattice points in directions 3 and 4. Transverse reflections are not of positive type because of the twist	That positivity of transverse reflections fails for all functions. Convexity in L
Proved here The set of structural inputs $\mathcal{I}$ holds for $\mathbb{Z}_2$ and U(1) too, and an argument from $\mathcal{I}$ alone gives the same conclusion there (from this set of inputs alone)	What comes out when a dynamical input is added to $\mathcal{I}$
Reading (physics) The lateral-widening inequality and the uniform slab tension do not follow from the structural inputs alone	A proof, as a theorem, that the lateral-widening inequality is false for U(1) and $\mathbb{Z}_2$
Lean: the spectral algebra (FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`)) and the algebra of the final sum of the reflection (ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`))	The transfer matrix, the centre transformation, the decomposition of the weight, the Haar measure (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
Reflection positivity and positivity of the transfer matrix	Cited	K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. M. Lüscher, Commun. Math. Phys. 54 (1977) 283. Chapter 5-03
Electric and magnetic flux, the twist and centre transformations	Cited	G. 't Hooft, Nucl. Phys. B153 (1979) 141. Chapter 25-01
The ensemble with the twist summed over	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442. Chapter 25-03
Electric-Flux Decomposition Lemma, tanh Comparison Lemma, reflection positivity of z, Lemma on the Non-Positivity of Lateral Reflections, Lemma that Structural Inputs Alone Do Not Suffice	Proved here	The proofs in § 02– § 04 of the text. The same content as the propositions "site-reflection positivity" and "no-go from structure" of the paper on the reduction of the band (a paper that came out of this book, A-07)
The formula for the character-expansion coefficients	Cited	Standard expansion. Checked to agree to 8 decimal places with numerical integration over the Haar measure for $\beta = 0.5, 2.5$, $j \le 3/2$
The line of argument by the strong-coupling polyhedral expansion	Assumption	Line of argument only. The literature for the comparison of the leading term (Münster 1981) is not yet checked

Item	Kind	Source / tools
Lean	Lean	FluxDuality (`FluxDuality.split`, `FluxDuality.twisted_le`, `FluxDuality.bias_mem`), ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`). They guarantee only the algebra of finite sums and real numbers. A-04

Terms used in this chapter — structural inputs, the transfer matrix, the centre transformation, the electric flux e_4, the sector bias θ, the z ensemble, reflection through lattice points, positive type, the negative lemma, the lateral-widening inequality, touchstone. Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-05 The chessboard and simultaneous flips — using the reflection positivity of z to replace frame decorrelation by a single density.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
the chessboard density κ_cb, simultaneous flips, two uniformities · Kiichi, Shiori and Rin

The chessboard and simultaneous flips

> This chapter is this book's own investigation. § 04 and § 05 are a map of open questions; § 06 is guide material.

What this chapter shows — The second assumption of the reduction in 26-02, "frame decorrelation", was the assumption that the product of the biases of containers sharing a frame splits into a product of typical values. If the size L of the transverse torus is fixed, the reflection positivity of z (the Reflection-Positivity Theorem for z in 26-04) and the Chessboard Estimate replace this assumption by **a single density** κ_{cb}. On a slab with L fixed, the Wilson loop is bounded by $\kappa_{\mathrm{cb}}^{n(C)}$, and the only work left is the estimate $\kappa_{\mathrm{cb}} < 1$. As tools for that estimate we set side by side the slab estimate and simultaneous flips, and we also write why simultaneous flips cannot be used to test the band. Finally we see that the band is concentrated into two uniformities, "the uniform slab tension" and "the uniform slab gap".

Prerequisite chapters — 26-02 § 02 (frame decorrelation), 26-04 § 03 (z reflection positivity), 25-03 (containers), 25-04 (slab confinement), 25-06 (the flow lemma and simultaneous flips), 5-03 (reflection positivity).

Said first — The paper results of this chapter (the Chessboard Estimate with the Lateral Size Fixed and the Simultaneous-Flip Estimate) are statements of the form **if L is fixed**. There is not a single estimate uniform in L. The two uniformities of the band remain [Assumption]. The Chessboard Estimate itself is a standard tool (Fröhlich–Israel–Lieb–Simon); what is new is only the construction that applies it to the biases of containers. The subscript of the symbol κ_{cb} is there to distinguish it from the other κ of this book (§ 06).

The order of this chapter

1. What is replaced — frame decorrelation, and the power lost to Hölder
2. The Chessboard Estimate — the chessboard with the lateral size fixed, and the density κ_cb
3. Upper bounds on κ_{cb} — the slab estimate and the Simultaneous-Flip Estimate, and their limits
4. Where the band appears again — the dilute regime, the confining regime, and in between

5. The band concentrated — the uniform slab tension and the uniform slab gap (Table 1)
6. The uses of the symbol κ compared (Table 2)
7. The table of the five grades and the dependency graph

01

What is replaced — frame decorrelation, and the power lost to Hölder

The notation is as in 26-01 § 01. The sector bias $\theta_D(\Phi) = |\mathbb{E}[\sigma_D \mid \Phi]|$ of a container $D \times T_L^2$ is a function of the frame Φ, and if mutually disjoint containers D_j in the plane lie inside the loop C, then

$$|\langle W(C)\rangle| \;\le\; \mathbb{E}\prod_j \theta_{D_j}(\Phi) \qquad (1)$$

(Mack–Petkova Cited). The right-hand side is the mean of a product of factors **sharing the same frame**. If the factors were independent it would split into $\prod_j \mathbb{E}\theta_{D_j}$, but the frame field is connected between containers, so it does not. If we use the Hölder inequality, $\mathbb{E}\prod_{j=1}^{n}\theta_j \le \prod_j (\mathbb{E}\theta_j^n)^{1/n}$, each factor on the right is the n-th root of the mean of the n-th power of θ; this is not a product of typical values θ, and a power proportional to the area is lost (in the nested form only $\mathbb{E}\theta_{2R}^4 \le \mathbb{E}\theta_R^4$ comes out).

So 26-02 § 02 set frame decorrelation,

$$\theta(sR; T_L^2) \;\le\; C_{\mathrm{D}}\,\theta(R; T_L^2)^{s^{\measuredangle}} \qquad (2)$$

as an assumption Assumption . (2) played two roles: the power of the nesting (the jump step and the doubling step of the Conditional Reduction Theorem of 26-02), and the bridge from the bias to the Wilson loop (the Area Law from the Area Exponent in 26-01). This chapter moves the role of splitting the product, **on a slab with L fixed**, to a single translation-invariant quantity.

02

The chessboard estimate — the density κ_{cb}

Setting. Consider the z ensemble with the twist $n_{12} = 1$ fixed ((4) of 26-04) on the lattice $(\mathbb{Z}/L)^2 \times (\mathbb{Z}/M)^2$. Divide the plane (3,4) into square blocks Q_t of side B ($t \in \mathcal{T}$, including the boundary lattice points), and let M be a multiple of $2B$. Inside each block put one container $D_t \times T_L^2$, at positions carried into one another by reflections.

Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) Cited Suppose the measure is reflection positive for reflections in planar directions 3 and 4 through all the lattice planes on the boundaries of the blocks. For each t let f_t be a bounded real function depending only on the variables of the block Q_t (boundary included), and let $f_t^{(s)}$ be the function obtained by carrying f_t to the block Q_s by a composition of reflections. Then

$$\Big|\Big\langle \prod_{t\in\mathcal{T}} f_t \Big\rangle\Big| \;\le\; \prod_{t\in\mathcal{T}} \Big\langle \prod_{s\in\mathcal{T}} f_t^{(s)} \Big\rangle^{1/|\mathcal{T}|}. \tag{3}$$

This is the standard argument of Fröhlich–Israel–Lieb–Simon, obtained by using the reflection Cauchy–Schwarz form repeatedly (26-04 § 03; Lean's ReflPos (`ReflPos.refl_cs`) is the algebra after splitting into those factors). (The original paper is Fröhlich–Israel–Lieb–Simon (1978). In this book we checked from the abstract of M. Biskup's lecture notes, arXiv:math-ph/0610025, that this is the standard form of obtaining the Chessboard Estimate from reflection positivity.)

In the z ensemble, reflection positivity is given by the reflection positivity of z in 26-04. The reflection positivity of z speaks of reflections through $x_3 = 0$ and $x_3 = M/2$, but the measure is invariant under translations in direction 3, so the same holds for reflections through any lattice plane $x_3 = c$. Direction 4 is the same.

Next we check that the bias is a function of the block. Given the frame, the interiors of the containers $D_t \times T_L^2$ are conditionally independent, and the conditional distribution depends only on the variables of the shell just outside the container (the Markov property). Since the container lies inside the block, the shell lies in the block, and $\theta_t := \theta_{D_t}$ is a function of the variables of Q_t. Also, a reflection preserves the measure and carries the sector $\sigma_D = \prod_{p\in D\times\{y_0\}} z_p$ to the sector of the reflected container, so the function obtained by carrying θ_t to the block Q_s by reflection is θ_s.

Chessboard Estimate with the Lateral Size Fixed (the chessboard with L fixed) Proved here Fix L and M. In the setting above, put

$$\kappa_{\mathrm{cb}} \;:=\; \Big(\mathbb{E}\prod_{t\in\mathcal{T}} \theta_t\Big)^{1/|\mathcal{T}|} \tag{4}$$

Then for every subset $S \subset \mathcal{T}$, $\mathbb{E}\prod_{t\in S}\theta_t \le \kappa_{\mathrm{cb}}^{|S|}$. In particular, if $n(C)$ is the number of blocks whose container lies wholly inside the loop C,

$$|\langle W(C)\rangle| \;\le\; \kappa_{\mathrm{cb}}^{\,n(C)}. \tag{5}$$

Proof In (3) put $f_t = \theta_t$ if $t \in S$ and $f_t = 1$ if $t \notin S$. Since $\theta_t \ge 0$, the left-hand side is $\mathbb{E}\prod_{t\in S}\theta_t$. On the right, the factors with $t \in S$ are $\big(\mathbb{E}\prod_{s\in\mathcal{T}}\theta_s\big)^{1/|\mathcal{T}|} = \kappa_{\mathrm{cb}}$, because $\theta_t^{(s)} = \theta_s$

as seen above, and the factors with $t \notin S$ are 1. Hence $\mathbb{E}\prod_{t\in S}\theta_t \le \kappa_{\mathrm{cb}}^{|S|}$. For (5), take D_j in (1) to be the containers inside C and use this estimate. ∎

Since a block of side B occupies the area B^2, (5) gives the lower bound $\ln(1/\kappa_{\mathrm{cb}})/B^2$ on the tension (for square loops whose side is a multiple of B). **With L fixed, z reflection positivity has closed, so the gap to the area law on the slab has become only the estimate $\kappa_{\mathrm{cb}} < 1$** Proved here (a statement with L and M fixed). No assumption about the distribution of the frame, such as frame decorrelation (2), is needed.

κ_{cb} is the geometric mean per block of the product of the biases when containers are placed in all blocks at once. It is not "the bias of one typical container" but "a quantity that looks at all containers at once", so the correlation of the frames is absorbed into κ_{cb}. In exchange, the smallness of κ_{cb} can no longer be read as a property of one container.

Nested Supermultiplicativity Lemma Proved here Take the container to be the whole interior of the block (let D_t be all the (3,4) plaquettes in the interior of Q_t, so that the shell $\partial Q_t \times T_L^2$ lies in the closed block Q_t). Write $\kappa_{\mathrm{cb}}(R;L)$ for (4) for blocks of side R (with M fixed). For an integer $s \ge 1$, if $2sR$ divides M, then for all L

$$\kappa_{\mathrm{cb}}(sR;L) \;\le\; \kappa_{\mathrm{cb}}(R;L)^{s^2}. \quad (5a)$$

Proof A large block Q'_u of side sR is the union of s^2 small blocks of side R, and since the (3,4) plaquettes lie in the interiors of the blocks, the sector is $\sigma'_u = \prod_{t\subset u}\sigma_t$. The union Φ' of the shells of the small blocks contains the union Φ of the shells of the large blocks, and by the Markov property the small containers are conditionally independent given Φ', so $\theta'_u = |\mathbb{E}[\sigma'_u \mid \Phi]| \le \mathbb{E}[\prod_{t\subset u}\theta_t \mid \Phi]$ ((2) of 26-02, the nested form of Mack–Petkova). $X_u := \prod_{t\subset u}\theta_t$ is a function of the variables of the closed large block u, and given Φ the interiors of the large blocks are conditionally independent, so $\prod_u \mathbb{E}[X_u \mid \Phi] = \mathbb{E}[\prod_u X_u \mid \Phi]$. Averaging, $\mathbb{E}\prod_u \theta'_u \le \mathbb{E}\prod_t \theta_t$. The numbers of blocks satisfy $|\mathcal{T}| = s^2|\mathcal{T}'|$, so take the $1/|\mathcal{T}'|$-th power of both sides. ∎

The Nested Supermultiplicativity Lemma does not use reflection positivity. It uses only the Markov property and the nested form, and the constant is 1, independent of L, R and s. The Chessboard Estimate is needed only for the step down from κ_{cb} to a single container ($\mathbb{E}\theta_t \le \kappa_{\mathrm{cb}}$) and to the Wilson loop ((5)). The abstract form on a finite probability space, with the large block as a fibre of s^2 small blocks and the inequality of means placed as an assumption, is Lean: LatticeGaugeNestedBlocks (nested_supermultiplicative). The Markov property is outside Lean.

03

Upper bounds on κ_{cb} — the slab estimate and simultaneous flips, and their limits

(i) The slab estimate. For each face $p \in D$ of the container, the operations that flip the z on the face $\{p\} \times T_L^2$ commute with one another, each flips the sector, and none changes the frame. Conditioning on their orbit gives $\mathbb{E}[\sigma_D \mid \text{orbit}] = \pm \prod_{p \in D} \tanh X_p$, and since $|X_p| \le \beta L^2$, $\sup_\Phi \theta_D(\Phi) \le \tanh(\beta L^2)^{|D|}$ (25-04 Proved here). From $\theta_t \le \sup \theta$,

$$\kappa_{\mathrm{cb}} \;\le\; \tanh(\beta L^2)^{|D|} \;<\; 1. \quad (6)$$

So the area law on a slab with L fixed already follows. But the tension is $-\ln \tanh(\beta L^2) \approx 2e^{-2\beta L^2}$ (per unit area), which for $\beta = 2.5$, $L = 4$ is $2e^{-80}$. **An estimate that goes through the supremum (sup) over frames gets worse superexponentially in L,** so an estimate uniform in L can only come from a mean form — a quantity like κ_{cb}.

(ii) Simultaneous flips. The flow lemma (25-06) says that for any Markov kernel K that preserves the conditional distribution $\mu(\cdot \mid \Phi)$ of the interior of the container, if $a(\Phi)$ is the probability that the sector changes in one step, then $\theta_D(\Phi) \le 1 - a(\Phi)$ Proved here (it uses only stationarity; reversibility is not needed).

Simultaneous-Flip Estimate Proved here With the frame held fixed, run a stationary kernel independently for one step inside each of the mutually disjoint containers D_t ($t \in S$), starting from the conditional distribution. Then

$$\mathbb{E} \prod_{t \in S} \theta_t \;\le\; \mathbb{P}\big(S\text{: no container sector changes}\big). \quad (7)$$

Proof Fix the frame Φ. The interiors of the containers are conditionally independent and the kernels also run independently, so the conditional probability that no container changes is $\prod_{t \in S}(1 - a_t(\Phi))$. From the flow lemma, $\theta_t(\Phi) \le 1 - a_t(\Phi)$, so $\prod_t \theta_t(\Phi) \le \mathbb{P}(\text{none changes} \mid \Phi)$. Averaging over Φ gives (7). ∎

The right-hand side of (7) is a quantity that can be measured by Monte Carlo, so with $S = \mathcal{T}$, $\kappa_{\mathrm{cb}} \le \mathbb{P}(\text{none changes})^{1/|\mathcal{T}|}$ becomes a tool that estimates the upper bound on the Wilson loop **in the correct direction**. The natural kernel is the dynamically driven insertion (25-06, acceptance probability $\min(1, e^{-W})$).

(iii) The limits. The flip probability a of a one-step kernel is, in the range measured, away from 1 (0.6–0.73 for the driven-insertion kernel with acceptance $\min(1, e^{-W})$, 25-08 Computed). So what (7) can guarantee is of the form $\theta_t \le \theta_0$ (with θ_0 a constant less than 1), and κ_{cb} cannot be made small exponentially in the area. Even if this is uniform in L for blocks of size $B \sim L$, the lower bound on the tension that comes out is $\ln(1/\theta_0)/L^2 \to 0$, which is only the naive form of the band treated by the Tension Lower Bound from the

Naive Form in 26-01 — a form that U(1) also satisfies Proved here (logic). **Simultaneous flips cannot be used to test the band.** To check the correct form of the band, $\theta \le e^{-cL^2}$, one needs a kernel with $a \to 1$, and no construction of one is known (25-06 § 07). Treating the measurements of simultaneous flips as "the numerical face of the band" has been withdrawn (25-08); the correct use of this tool is the lower bound on the tension of a slab with L fixed — the estimation of κ_{cb}.

04

Where the band appears again — the dilute regime, the confining regime, and in between

The Chessboard Estimate with the Lateral Size Fixed was a statement with L fixed. When L moves, the block size B needed to show $\kappa_{\mathrm{cb}} < 1$, and the quality of the estimate, change from scale to scale. Everything below is an estimate Reading (physics).

- **The dilute regime ($\Lambda L \ll 1$).** For a container of side ℓ, $\mathbb{E}[-\ln \theta_D] \approx \sigma_{\mathrm{TU}} \ell^2$, with $\sigma_{\mathrm{TU}} \propto \Lambda^2 (\Lambda L)^{5/3}$ (26-03 § 04). The influence of the frame decays over distances of about L/π. The criterion holds, but the blocks needed have $B \gtrsim \sigma_{\mathrm{TU}}^{-1/2}$, and their volume grows exponentially in $1/g^2(L)$.
- **The confining regime ($\Lambda L \gg 1$).** In the form of the electric flux (the Electric-Flux Decomposition Lemma of 26-04), $\theta \approx Y(1)/Y(0) \approx c\,\sigma L^2 e^{-\sigma \ell^2}$ (the factor L^2 is the entropy of the transverse position of the electric string). The block size needed grows only as $\sqrt{\ln L}$.
- **In between ($\Lambda L \sim 1$) — the band.** Two uniformities are needed (§ 05).

As $a \to 0$, both B and L become infinite in lattice units, so checking finitely many boxes is not enough. An inequality that connects the scales is needed, and that is the problem of the band.

Why the usual Dobrushin-type criteria cannot be used. With the frame fixed, the sectors do not interact with one another. All the dependence is in the frame field, and since a continuous field follows the boundary, the influence coefficient measured in total variation is nearly 1. Conditions of the Dobrushin, Dobrushin–Shlosman and van den Berg–Maes type are not satisfied as they stand, and we expect another input to be needed, such as contraction on good blocks (of the type of Bałaban's multiscale analysis) Reading (physics). The original papers of these criteria have not been checked in this book.

05

The band concentrated — the uniform slab tension and the uniform slab gap

In the language of the chessboard, the band is concentrated into the following two statements of uniformity. Neither has been shown, and we set them as assumptions.

The uniform slab tension Assumption The string tension of the two-dimensional theory obtained by shrinking the transverse directions to T^2_L (the slab tension, (11) of 26-02) is positive uniformly in L:

$$\inf_{L \geq L_0} \sigma_{\mathrm{eff}}(L) \;>\; 0 \qquad (\text{physical units, uniform in the lattice spacing } a). \tag{8}$$

In the language of containers it has the form $\inf_L \mathbb{E}[-\ln \theta_D]/(\ell\Lambda)^2 > 0$ and corresponds to the content of the lateral-widening inequality (weak form) of 26-02. The chessboard criterion only carries it to the area law. For U(1) the tension is $\sim e^2/L^2$ for every L, and it is expected to fail **here only** Reading (physics).

The uniform slab gap Assumption That κ_{cb} can be compared across scales: that the estimates of the correlations between biases, of the probability of bad blocks and of the fluctuations of the shells are uniform in L (this corresponds to the mass gap of the slab being uniform in L). This is what frame decorrelation (2) of 26-02 really is when made uniform in L. By the Nested Supermultiplicativity Lemma (§ 02), within a fixed L the density κ_{cb} can be carried from scale R to sR with constant 1, so the content of this statement that is uniform in L moves to the comparison across L

$$(\mathrm{Ccb}) \quad \kappa_{\mathrm{cb}}(R; L') \leq C\,(L'/L)^2\,\kappa_{\mathrm{cb}}(R; L) \qquad (L_{\min}(a) \leq L \leq L' \leq R)$$

— the lateral-widening inequality of 26-02 read in terms of κ_{cb} Paper (the Reduction Read in the Chessboard Density in 26-02 § 03). There is no nesting in the lateral directions (tiling T^2_{2L} by four copies of T^2_L makes the twist $4 \equiv 0 \pmod 2$), so the ensemble with $n_{12} = 1$ cannot be formed).

Table 1 The band concentrated (labels per row; conditions in the same row)

Statement	Label	Conditions in the same row	Correspondence with 26-02	For U(1)
The chessboard with L fixed: $\lvert\langle W\rangle\rvert \leq \kappa_{\mathrm{cb}}^{n(C)}$	Proved here	L and M fixed; under the reflection positivity of z (26-04 § 03)	Takes over, at fixed L, the role of frame decorrelation of "splitting the product"	Holds as well
$\kappa_{\mathrm{cb}} < 1$ (fixed L)	Proved here	From the slab estimate (6). The tension is extremely small, $\approx 2e^{-2\beta L^2}$	—	Holds as well

Statement	Label	Conditions in the same row	Correspondence with 26-02	For U(1)
The uniform slab tension (8)	Assumption	Physical units, uniform in a	The content of the weak form of the lateral-widening inequality	Expected to fail [Reading]
The uniform slab gap	Assumption	Estimates of correlations, bad blocks and shell fluctuations uniform in L. Read in terms of κ_{cb}, within a fixed L constant 1 by the Nested Supermultiplicativity Lemma, and the rest is $(\mathrm{C}_{\mathrm{cb}})$	Frame decorrelation made uniform in L	No estimate that it fails [Reading]

In this way the non-abelian content of the band is placed on the side of the uniform slab tension, and the probabilistic content of crossing scales on the side of the uniform slab gap. Read in terms of κ_{cb}, the content of the uniform slab gap that is uniform in L moves to $(\mathrm{C}_{\mathrm{cb}})$ and becomes the same statement as the lateral-widening inequality Paper (the Nested Supermultiplicativity Lemma). The logical relation to the uniform slab tension (8) (whether one follows from the other) has not been examined.

06

The uses of the symbol κ compared

This book has used κ in several senses. The chessboard density of this chapter is written κ_{cb}, and the subscript is never dropped. Table 2 lists the other quantities written with the same letter. The quantities are not related to one another.

Table 2 Quantities written with the letter κ

Symbol	Meaning	Chapter
κ_{cb}	The chessboard density: the geometric mean per block of the product of the biases of the containers of all blocks (4)	This chapter, 26-00 Table 1
κ (the one-sided bridge)	The comparison constant of the One-Sided Bridge Proposition (of the form $C_P \geq \kappa \xi^2$)	23-01 § 06
κ_{BE}	The lower bound on the Bakry–Émery curvature	24-09
$\hbar\kappa$, κ_f	An upper bound on the strength of the coupling to neighbouring sites; the strength of the Brascamp–Lieb cross term	23-06
$\kappa = 3/1000$	The lower bound outside the defects in Barta's row	23-09
κ (Part 24)	The comparison constant between the Langevin metric and the metric of the kinetic term	24-09
$\kappa_{\mathrm{C7}}(k)$	The convexity constant of a chain of length k (the same quantity as $\kappa(k)$ of 23-11)	26-07
κ_w	Adapted stiffness: the variance of the force along the generator of the twist (width w; one of the candidate quantities for the band)	26-10

The whole correspondence of symbols is in A-01.

07

The table of the five grades and the dependency graph

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The container estimate (1)	Cited	Mack–Petkova, 25-03	The ensemble summed over twists	—
2	Frame decorrelation (2)	Assumption	26-02, A-06	Constant C_{D}	—
3	Reflection positivity of z (directions 3 and 4, any lattice plane)	Proved here	26-04 § 03, translation invariance	$SU(2)$, twist fixed, M even	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (algebra only; assumption: $0 \le A$)
4	The Chessboard Estimate (3)	Cited	Fröhlich–Israel–Lieb–Simon (checked from the abstract of lecture notes)	Under reflection positivity, $2B \mid M$	—
5	θ_t is a function of the variables of the block and is carried to θ_s by reflection	Proved here	§ 02 (Markov property, reflection invariance of the measure)	The container is placed inside the block	—
6	Chessboard Estimate with the Lateral Size Fixed: $\mathbb{E}\prod_S \theta_t \le \kappa_{\mathrm{cb}}^{\lvert S\rvert}$, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$	Proved here	Proof in § 02 $\Longleftarrow$ 1, 3, 4, 5	If L is fixed (M also fixed)	—
7	The slab estimate $\sup_\Phi \theta_D \le \tanh(\beta L^2)^{\lvert D\rvert}$, hence $\kappa_{\mathrm{cb}} < 1$	Proved here	25-04, § 03 (6)	L fixed, tension $\approx 2e^{-2\beta L^2}$	—
8	The Flow Lemma $\theta \le 1 - a$	Proved here	25-06	Stationary kernel (reversibility not needed)	FluxLemma (`FluxLemma.flip_le_one_sub_abs`) (assumptions: $w \ge 0$, $K \ge 0$, row sums 1, w stationary, $\sum w = 1$, $\sigma = \pm 1$)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
9	Simultaneous-Flip Estimate: $\mathbb{E}\prod_S \theta_t \leq \mathbb{P}(\text{none changes})$	Proved here	Proof in § 03 $\Longleftarrow$ 8	Frame fixed, one independent step, starting from the conditional distribution	—
10	The flip probability of a one-step kernel levels off at about $\frac{1}{2}$	Reading	Measurements of 25-08	The driven-insertion kernel	—
11	Simultaneous flips give only an estimate of the naive form and cannot be used to test the band	Proved here	§ 03 (iii) $\Longleftarrow$ 9, the Tension Lower Bound from the Naive Form in 26-01	Logic assuming the levelling off of the one-step flip probability	—
12	Criteria and block sizes in the dilute and confining regimes	Reading (physics)	§ 04	$\Lambda L \ll 1$, $\Lambda L \gg 1$, estimate	—
13	Dobrushin-type criteria cannot be used as they stand	Reading (physics)	§ 04	The estimate that the influence coefficient is nearly 1; original papers not yet checked	—
14	The uniform slab tension (8)	Assumption	§ 05, A-06	Physical units, uniform in a	—
15	The uniform slab gap	Assumption	§ 05, A-06	Uniform in L	—
16	U(1) fails only in the uniform slab tension	Reading (physics)	§ 05	Estimate of the tension $\sim e^2/L^2$	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  container estimate |W| ≤ E∏θ                              ensemble summed ov
 2  ASM  frame decorrelation (what this chapter replaces at fixed L)
 3  PRV  reflection positivity of z (26-04)                        Lean refl_pos, ref
 4  CIT  Chessboard Estimate                                       Fröhlich–Israel–Li
 5  PRV  θ_t is a block function, reflected to θ_s                 Markov property
 6  PRV  Chessboard Estimate with the Lateral Size Fixed: |W| ≤ κ_cb^n(C) ⟸ 1,
 7  PRV  κ_cb ≤ tanh(βL²)^|D| < 1                                  L fixed, tension t
 8  PRV  θ ≤ 1 − a                                                 Lean flip_le_one_s
 9  PRV  Simultaneous-Flip Estimate: E∏θ ≤ P(none changes) ⟸ 8
10  RDG  one-step flip probability levels off near ½               measurement
11  PRV  simultaneous flips cannot test the band ⟸ 9, 10            26-01 Tension Lo
12  RDG  criteria and block sizes in the two end regimes           estimate
13  RDG  Dobrushin type not usable as it stands                    estimate
14  ASM  the uniform slab tension
15  ASM  the uniform slab gap
16  RDG  U(1) fails only in the uniform slab tension ⟸ 14, 15      estimate
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The main conclusion of this chapter (the Chessboard Estimate with the Lateral Size Fixed) stands on paper, and its only premises are literature rows (the container estimate, the Chessboard Estimate) and paper rows (the reflection positivity of z, block functions and reflection). The assumption row (frame decorrelation) is not among the premises of this conclusion — that is the point of this chapter. "Simultaneous flips cannot be used to test the band" is a row of logic, but it has a measurement row (the levelling off of the one-step flip probability) as a premise, so it is a statement within the range where that levelling off holds. The two assumptions of the band (the uniform slab tension, the uniform slab gap) do not contain the quantity of the conclusion (κ_{cb} with L fixed), and are placed in the ledger of conditions of Part 27 (27-02).

The machine-checked form of the chessboard estimate is in 22-01.

What this chapter can say

What is said	What is not said
Proved here If L is fixed, z reflection positivity and the chessboard estimate bound the Wilson loop by $\kappa_{\mathrm{cb}}^{n(C)}$, and frame decorrelation is not needed	An estimate of κ_{cb} uniform in L (the uniform slab gap)
Proved here Simultaneous flips are a tool that estimates κ_{cb} in the correct direction, and with a one-step kernel they cannot be used to test the band	A kernel with $a \to 1$. A numerical test of the band

What is said	What is not said
Reading (physics) The band is concentrated into the uniform slab tension and the uniform slab gap, and U(1) fails only in the former	Either of the two uniformities. The logical relation between the two
Lean: the algebra of the final sums of reflection, and a finite version of the flow lemma	The chessboard estimate itself, the Markov property, the Haar measure (outside Lean)

Sources and reproduction

Item	Kind	Source / tools
The chessboard estimate	Cited	The standard argument of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (1978). Checked from the abstract of M. Biskup, Reflection positivity and phase transitions in lattice spin models, arXiv:math-ph/0610025 (the text of the original paper not obtained). Examples of use in lattice gauge theory (Borgs–Seiler) not yet checked
The container estimate, the ensemble summed over twists	Cited	G. Mack, V. B. Petkova, Ann. Phys. 123 (1979) 442. Chapter 25-03
Reflection positivity of z	Proved here	Chapter the reflection positivity of z in 26-04
Chessboard Estimate with the Lateral Size Fixed, Simultaneous-Flip Estimate, (6)	Proved here	Proofs in § 02 and § 03 of the text. The same content as the corollaries "chessboard" and "simultaneous flips" of the paper on the reduction of the band (a paper that came out of this book, A-07)
The levelling off of one-step kernels; the withdrawal of the measurements of simultaneous flips	Reading	The table in chapter 25-08
The criteria of Dobrushin, Dobrushin–Shlosman, van den Berg–Maes	—	Not yet checked (names only)
Lean	Lean	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), FluxLemma (`FluxLemma.flip_le_one_sub_abs`). They guarantee only the algebra of finite sums and real numbers. A-04

Terms used in this chapter — frame decorrelation, the Chessboard Estimate, the chessboard density κ_{cb}, block, simultaneous flips, the Flow Lemma, the uniform slab tension, the uniform slab gap, the crossover band (the band). Their meanings are in A-02 Glossary and A-06 Index of names.

Next chapter: 26-06 Renormalisation comparison — a comparison that distinguishes groups — another reduction, which splits the band into β-monotonicity and the renormalisation comparison.

Renormalisation comparison — a comparison that distinguishes groups

> This chapter is this book's own investigation. § 04, § 07 and § 08 are a map of open questions.

What this chapter shows — We split the crossover band into two statements. One is that, with the lattice geometry held fixed, the vortex free energy increases in β (β-monotonicity). The other is that the lattice obtained by doubling the whole lattice can be compared, up to a constant factor, with the original lattice at a stronger coupling (the renormalisation comparison). Asymptotic freedom enters only through the "direction of strengthening the coupling" $\beta' < \beta$ of the renormalisation comparison. β-monotonicity looks like an easy component, but it turned out to be a problem of the same weight as the monotonicity in β of the string tension in lattice units. The renormalisation comparison holds at both the strong-coupling end and the one-loop end, but on the strong-coupling side it is a line of proof with extremely crude constants, and on the weak-coupling side it is a computation at the one-loop level. In the band between them, comparing $SU(2)$ and $U(1)$ numerically, the direction of the comparison does not change over the whole band for $SU(2)$, and for $U(1)$ it reverses at the position of the known phase transition. Finally, from the fact that the band is a bounded interval in β, we prove an elementary lemma that reduces the renormalisation comparison inside the band to "the two-sided area law at fixed β (with an error of the order of the perimeter)", and say where in the band the difficulty gathers.

Prerequisite chapters — 26-01 (the correct form of the band, $\ln(1/\theta) \geq cL^2$), 26-02 (reduction to a single inequality), 25-01 (the twisted T^2 and 't Hooft flux), 25-07 (why fixed maps fail, and the one-loop table), 4-06 (asymptotic freedom), 5-02 (the strong-coupling expansion), 5-06 ($U(1)$ and the non-abelian).

Said first — The level is [Res]. This chapter does not show the renormalisation comparison at any β of the band. What it shows is the range and conditions in which the comparison holds at the two ends, numerical readings, and one reduction lemma (§ 06). The assumption of the reduction lemma contains the unsolved problem of

confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice. We write the chessboard density as κ_{cb}.

The order of this chapter

1. Splitting the band in two — β-monotonicity and the renormalisation comparison
2. β-monotonicity is not easy — rewritten with the electric flux, it stands beside the monotonicity of the string tension
3. The comparison that holds at the two ends — the strong-coupling end and the one-loop end (Table 1)
4. The map over β — labels interval by interval (Figure 1)
5. Numerics — doubling the lattice — $SU(2)$ and $U(1)$ (Tables 2–4)
6. The band is a bounded interval — the Lemma Reducing the Comparison inside the Band, and the Corollary Crossing the Band in Finitely Many Steps
7. The four faces of the band in the middle — candidate local quantities (Table 5)
8. Conditions an argument closing the band must satisfy — the criterion "distinguish from the U(1) transition"
9. What cannot be said
10. The table of the five grades and the dependency graph — Table 6

01

Splitting the band in two — β-monotonicity and the renormalisation comparison

The object is the slab of 25-01. On the lattice $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ we put the $SU(2)$ Wilson action and insert the twist $n_{12} = 1$ in the $(1,2)$ plane. We take the flux n_{34} of the $(3,4)$ plane as the sector, and write the vortex free energy as

$$\Delta F(N,M,\beta) \;=\; -\ln \frac{Z_{n_{34}=1}(N,M,\beta)}{Z_{n_{34}=0}(N,M,\beta)}, \qquad \theta = \tanh \frac{\Delta F}{2} \tag{1}$$

θ is the quantity used in 't Hooft's criterion, and the correct form of the band was "$\ln(1/\theta(L,L)) \geq cL^2$ uniformly in L" (26-01). The larger the vortex free energy, the smaller θ, which is the confining side.

We split the statement that crosses the band into the following two Assumption (both set as unproved statements).

$$\text{(Mon)}\quad \frac{\partial}{\partial\beta}\Delta F(N,M,\beta)\ \geq\ 0 \qquad \text{(with the lattice geometry fixed)}\ (2)$$

$$\text{(RG-cmp)}\quad \Delta F(2N,2M,\beta)\ \leq\ K\cdot\Delta F(N,M,\beta'),\qquad \beta'<\beta,\qquad K \text{ independent of } N,M$$

> The formula label (Mon) stands for β-monotonicity, and (RG-cmp) for the renormalisation comparison (names as in A-06). The (Mon) of (5) below is the same.

The left-hand side of (2) can be written as an equality, as an average of the difference of actions: $\partial_\beta\Delta F=\langle\Delta S\rangle$ (the difference of the average actions in the sectors with and without a vortex) Proved here (just differentiate the partition function in β). (3) is the comparison that a vortex on the lattice coarsened by a factor 2 costs no more than a vortex on the lattice of the original size at a stronger coupling.

The scope of (2). (2) is a statement about the slab of this chapter — the group $SU(2)$, the background twist $n_{12}=1$ and the slab geometry $(\mathbb{Z}/N)^2\times(\mathbb{Z}/M)^2$. It is not a general property of non-abelian groups: on small lattices for the finite non-abelian groups Q_8 and D_4 there are points where $\partial_\beta\Delta F<0$ even without a background twist, and, with the background, there are points where it fails in geometries that are long in the transverse directions Reading (floating-point values from transfer matrices of finite groups; two independent implementations agree). On the other hand, for the $\mathbb{Z}_2$ sign system, whose only variables are the signs of the plaquettes, monotonicity of the type of (2) ($\mathbb{Z}_2$ monotonicity) holds for non-negative couplings Proved here (a proof of correlation-inequality type; the statement for abstract sign systems was checked in Lean; the application to the lattice of the four-dimensional torus is on paper).

Together, the two give a line that crosses the band. Iterating (3) as $\beta_0>\beta_1>\beta_2>\cdots$, the vortex free energy on the lattice enlarged 2^k times is bounded by K^k times the value on the original lattice at coupling β_k. Once β_k enters the strong-coupling range, the vortex free energy there is small with an area exponent, and it becomes doubly exponentially small with each step, so it beats the loss K^k. (2) plays the role of an adjustment: nothing is lost if, in the course of the iteration, β' is chosen smaller than necessary.

The rate form of (3). The constant K in (3) is bookkeeping; the content is a comparison of slab tensions. Write the slab tension, and that quantity multiplied by the number of plaquettes, as

$$\sigma_N(\beta) := \lim_{M\to\infty} \frac{-\ln \Delta F(N,M,\beta)}{M^2}, \qquad s(N,\beta) := N^2 \sigma_N(\beta) \quad (3')$$

(assuming the limit exists; otherwise the lower limit). Taking the logarithm of both sides of (3), dividing by M^2 and letting $M \to \infty$, K disappears, and what remains is $s(N,\beta') \le s(2N,\beta)$ Proved here (this form follows from (3); the converse needs control of the difference between ΔF and the area term). In this book, this rate form is read as the main form of the renormalisation comparison. If one looks only at symmetric boxes $N = M$ and allows $K > 1$, $\beta' < \beta$ holds even for $U(1)$ Reading. What forces $\beta' \ge \beta$ is the side $M \to \infty$ with N fixed, and the rate form carries that direction.

Asymptotic freedom enters only through $\beta' < \beta$ in (3). (2) is a statement of a form that contains no mechanism that distinguishes groups. For $\mathbb{Z}_2$ and $U(1)$, it follows from Ginibre's inequality when there is no twisted background, but the slab of this chapter has the twist $n_{12} = 1$, and in that case it does not follow from Ginibre's inequality even for $\mathbb{Z}_2$ Assumption (the application of Ginibre's inequality has not been checked against the text of the original source). The direction of (3), on the other hand, changes with the group. For $U(1)$ one loop gives $\beta' \ge \beta$, and the iteration does not head for strong coupling: it fails, correctly. That $\beta' < \beta$ for the non-abelian $SU(2)$ is a restatement of asymptotic freedom (4-06), that is, of the effective coupling becoming stronger when the lattice is coarsened. In this sense (3) has to be "a comparison that distinguishes groups".

Example. At the strong-coupling end, (3) holds with $K \approx 4$ (§ 03). Doubling the lattice makes the area of the vortex sheet 4 times larger, and it balances at the β' that makes the rate $\sigma(\beta)$ of the area exponent 4 times larger, that is, $\sigma(\beta') = 4\sigma(\beta)$. For $\beta = 1.0$, $\beta' \approx 0.013$ (Table 1). At strong coupling the coupling moves a long way in one step.

Tomboulis (2007) tried to build a comparison of type (3) with Migdal–Kadanoff decimation, and Ito–Seiler and Kanazawa pointed out that the flow heads for strong coupling also for $U(1)$ Cited (Ito–Seiler reduce the remaining step to the inequality $A(\alpha) > A^{+}(\alpha)$ of their eq. (3.5), and write that it has not been shown). The history and the checking are in 26-08. What this chapter needs is just one point: a comparison that does not distinguish groups cannot cross the band.

Looking again at the direction of the comparison in the language of effective actions: at tree level the coarse action of the genuine one step is no greater than the Migdal–Kadanoff action (in the coarse directions it is smaller, down to 1/4, and it agrees at long wavelengths); in the language of couplings this is $\beta/4 \le \beta' \le \beta$, the direction favourable to confinement Proved here (lower side) Reading (the upper side is the numerics of the measured boxes). It still cannot be used for the comparison, because the inequality of partition functions $Z \le Z_{\mathrm{MK}}$ does not lift to the form with the coarse

variables fixed. In the one-loop asymptotics the speed of the Migdal–Kadanoff flow (0.2503 per step) is only slightly slower than the genuine speed 0.2575, and the direction is favourable; but in U(1) neither runs, and the difference between the groups shows up on the Migdal–Kadanoff side only in an exponentially small flow Reading . The one-step sandwich $\langle W\rangle_L < \langle W\rangle_R < \langle W\rangle_{\mathrm{MK}}$ of the same observable holds in all 12 rows for SU(2) and U(1), and does not distinguish the groups Reading (Metropolis, $4^4 \to 2^4$, $8^4 \to 4^4$, $\beta \in \{2, 4, 8\}$).

The inequality $A(\alpha) > A^{+}(\alpha)$ to which Ito–Seiler reduced the problem is the same as the monotonicity "the vortex free energy grows with the coupling", and in the numerics of small boxes it holds even for deconfined U(1) Reading (4^4, one direction of the MK image, $\alpha \in \{0.9, 1.0\}$). What has to distinguish the groups is the extension beyond it, and the device that moves the interpolation parameter away from 1.

Measuring "the Migdal–Kadanoff flow is slower than the genuine flow" with the twist free energy, over one step of the orbit (2.47 → 2.19 in the effective β), in the $4^4/2^4$ box the genuine one step looked faster by 0.15 converted to Wilson's β ($R = 1.34(7)$); but in $6^4/3^4$ the difference is 0.02(2) ($R = 1.17(20)$), which is within the resolution of the measurement (statistics, and the systematic floor of 0.1–0.25 of the ΔF method). The speeds of the two flows look nearly equal inside the band, and the 7% discretisation of the 2^4 box may have produced most of the initial difference Reading .

02

β-monotonicity is not easy — rewritten with the electric flux, it stands beside the monotonicity of the string tension

β-monotonicity looks like "a structural component that can probably be shown". In fact it is not easy. Rewriting it with the electric flux makes this clear.

Splitting the partition function of the sector with flux n_{34}, by the Fourier transform on $\mathbb{Z}_2$, into the partition functions Y_0, Y_1 with electric flux $e \in \{0, 1\}$ gives $Z_{n_{34}=0} = Y_0 + Y_1$, $Z_{n_{34}=1} = Y_0 - Y_1$. Setting $E_e = \frac{d}{d\beta} \ln Y_e$,

$$\frac{\partial \Delta F}{\partial \beta} = \frac{2\, Y_0 Y_1\, (E_1 - E_0)}{Y_0^2 - Y_1^2}, \qquad \theta = \frac{Y_1}{Y_0} \quad (4)$$

Proved here Proof. Differentiate $\Delta F = \ln(Y_0 + Y_1) - \ln(Y_0 - Y_1)$, insert $Y_e' = E_e Y_e$ and bring to a common denominator; the numerator is $(E_0 Y_0 + E_1 Y_1)(Y_0 - Y_1) - (E_0 Y_0 - E_1 Y_1)(Y_0 + Y_1) = 2 Y_0 Y_1 (E_1 - E_0)$. The expression for θ is $\tanh(\Delta F/2) = (e^{\Delta F} - 1)/(e^{\Delta F} + 1)$ with $e^{\Delta F} = (Y_0 + Y_1)/(Y_0 - Y_1)$ inserted. ∎

If $Y_1 > 0$ (the positive case of $\theta = Y(1)/Y(0) \geq 0$ in 26-04), then from (4)

$$(\mathrm{Mon}) \iff E_1 \geq E_0 \iff \theta \text{ increases in } \beta \iff \text{the electric-flux free energy } -\ln(Y_1/Y_0)$$

Proved here . In the confining phase, the free energy of the electric flux is of the order of the product of the string tension and the length the flux passes through. So it can be read that, if β-monotonicity holds on every lattice, it follows that the string tension in lattice units σa^2 decreases in β Assumption (basis: the step connecting the electric-flux free energy with the string tension has not been written out). **The monotonicity in β of the string tension in lattice units has not been proved for $SU(2)$.** Even the monotonicity in β of the expectation value of the Wilson loop has not been shown, as far as we searched. β-monotonicity is a problem at least of the same weight as this Assumption . The initial estimate that "β-monotonicity is structural and can probably be shown" is corrected here.

The partial results to hand are the following Assumption (the proofs of none of them are set out in this book). In two dimensions with 1 or 2 sheets (by the monotonicity of the conditional energy of the $O(4)$ chain); near $\beta \to 0$ on any fixed lattice; near $\beta \to \infty$ in the twisted n_{12} background ($\partial_\beta \Delta F$ approaches a positive constant because $SU(2)$ has no flat connection carrying both n_{12} and n_{34}). No counterexample has been found in any of the closed two-dimensional formulas ($N \leq 100$, $\beta \leq 20$), the exhaustive enumeration of $2 \times 2 \times 2 \times 1$ for $\mathbb{Z}_2$, or the four-dimensional Monte Carlo on 1^4 and 2^4 Reading .

Two reasons why it gets stuck are recorded. In the Mack–Petkova representation, when the $SO(3)$ configuration carries a twist class, the conditional Ginibre inequality goes the wrong way, and $\partial_\beta \Delta F$ becomes exactly "the difference of the two contributions we want to measure". In the character expansion, in three or more dimensions the sign of the spin-network integrals is not fixed Reading (physical estimate).

Consequence. The chain across the band does not need β-monotonicity on every lattice. If (3) is shown directly in the form $\Delta F(2N, 2M, \beta) \leq K \Delta F(N, M, \beta')$, the freedom in choosing β' absorbs the role of β-monotonicity. So below we concentrate on the renormalisation comparison.

03

The comparison that holds at the two ends — the strong-coupling end and the one-loop end

The strong-coupling end

We write the coefficient of the character expansion of the Wilson action in the fundamental representation as $u(\beta) = I_2(\beta)/I_1(\beta)$ ($u \approx \beta/4$ as $\beta \to 0$). Expanding the

vortex free energy of the slab by the strong-coupling expansion (5-02), the leading terms are the N^2 flat sheets wrapping the $(3,4)$ plane (area M^2, weight u^{M^2}). The record contains a proposition and a line of proof of the following form.

Strong-Coupling-End Expansion Proposition. There are absolute constants u_0, K_0, C such that, for $u(\beta) \le u_0$ and all $N, M \ge 2$,

$$\Delta F(N,M,\beta) = 2N^2 e^{-\sigma_N(\beta) M^2}(1+\epsilon), \qquad |\epsilon| \le C M^2 (K_0 u)^{\max(1, M/100)} \le C' u \tag{6}$$

holds. σ_N depends on β, N and n_{12} but not on M, and $\sigma_\infty = -\ln u - 4u^4 + O(u^5)$. Assumption (line: in the polymer expansion the sign of the twist is fixed by the $\mathbb{Z}_2$ homology class; integrating the flat parts of the sheet by gluing two-dimensional characters gives exactly u^{M^2}; a Peierls estimate for contours on the sheet. The bookkeeping of the interaction between volume polymers and sheet contours (step 5 of the proof) is sketched, and the proof is not set out in this book. The form of the expansion is of the Münster (1981) type, but the text of the original source has not been checked)

Comparison at the Strong-Coupling End. Defining β' by $\sigma_N(\beta') = 4\sigma_{2N}(\beta)$, for all N, $M \ge 2$, $\Delta F(2N, 2M, \beta)/\Delta F(N, M, \beta') = 4(1 + O(u))$, $\beta' < \beta$, $u' = u^4(1 + O(u^4))$ Assumption (algebra on top of the Strong-Coupling-End Expansion Proposition).

The constants are extremely crude: $u_0 \approx 10^{-80}$, which in β is about 4×10^{-80}. On paper, the strong-coupling end is closed only in the range of extremely strong coupling. This is a matter of constants, and there is a prospect of large improvement by tightening the counting, but the expansion in flat sheets stops at roughening ($\beta_R \approx 1.9$), so the strong-coupling side alone cannot cover the band Reading (physical estimate). $U(1)$ also confines at strong coupling and satisfies (3). The difference between the groups appears on the weak-coupling side.

Table 1 The root β' of the leading-order strong-coupling equation $u(\beta') = u(\beta)^4$ (the movement of the coupling in one step) Reading

β	$u(\beta)$	β'	$\beta - \beta'$
0.5	0.1237	0.0009	0.499
1.0	0.2402	0.0133	0.987
1.5	0.3441	0.0561	1.444
2.0	0.4331	0.1409	1.859

The β of Table 1 lie outside the range of the Strong-Coupling-End Expansion Proposition ($u \le u_0$). The table is a rough guide to the direction and size of the comparison from the leading terms only, not a claim that the comparison holds.

The one-loop end

On the weak-coupling side we build (3) as a variational inequality. For a measurable bijection T carrying the measure μ_0 of the sector $n_{34}=0$ to the sector 1, the Gibbs variational principle gives

$$\Delta F \;=\; \mathbb{E}_{\mu_0}\big[S\circ T - S - \ln J_T\big] - \mathrm{KL}\big(T_{\#}\mu_0 \,\|\, \mu_1\big) \;\le\; \mathbb{E}_{\mu_0}\big[S\circ T - S - \ln J_T\big] \quad (7)$$

Cited (the standard form of the Gibbs variational principle; J_T is the Jacobian of T). The right-hand side is written only with averages in μ_0, so the estimate needs only one side. We consider the following construction: block the configuration of the fine lattice $(2N,\beta)$ and map it to the coarse lattice (N,β'), use a nearly optimal insertion map on the coarse lattice, and carry the fluctuations when going back to the fine lattice. Depending on how the fluctuations are carried, the one-loop answer splits in two.

The one-loop count is done with the trace of the heat kernel of the $SU(2)$ fluctuations (Feynman gauge) in a constant background magnetic field B. Per volume, apart from $1/(16\pi^2 s^2)$, the vector (4 polarisations) gives $\frac{Bs}{\sinh Bs}(2+2\cosh 2Bs)$ and the ghosts give $-2\frac{Bs}{\sinh Bs}$. Inserting $\frac{x}{\sinh x} = 1-\frac{x^2}{6}+O(x^4)$ and $\cosh 2x = 1+2x^2+O(x^4)$ for $x = Bs$,

$$\underbrace{\frac{x}{\sinh x}(2+2\cosh 2x) - \frac{2x}{\sinh x}}_{\text{total}} = 2+\tfrac{11}{3}x^2+O(x^4), \qquad \underbrace{\frac{4x}{\sinh x} - \frac{2x}{\sinh x}}_{\text{orbital only}} = 2-\tfrac{1}{3}x^2+O(x^4)$$

and the $+4$ in $\frac{11}{3} = -\frac{1}{3}+4$ comes from the spin term $\cosh 2Bs$ Proved here (one-loop level; the expansion is a two-line computation). $\frac{11}{3}$ is the coefficient of asymptotic freedom. The table and the details of the derivation are in 25-07; here we use only the conclusion.

- In **an isometric comparison** (a map isometric in the fluctuations: the identity map, translation in the group, gauge transformation, Migdal–Kadanoff-type decimation), the estimate (7) is the first-order trace $\frac{1}{2}\operatorname{tr}((M_B - M_0)M_0^{-1})$, and the contribution of the spin term is exactly 0. The estimate is worse than the orbital value alone, so what it can give is at best the $-\frac{1}{3}$ side, that is, $\beta' \ge \beta + 0.023$ (per doubling): the direction against asymptotic freedom, the same direction as $U(1)$ Proved here (one-loop level).
- In **a comparison that reshapes the fluctuations** (changing the shape of the fluctuations by an optimal linear map of determinant 1), the exact $\frac{1}{2}\ln\det$ is reproduced with error $\sim \Delta F^2/n$ (n is the number of degrees of freedom), giving $\beta' = \beta - 0.258$ (per doubling) Proved here (one-loop level).

Where the numbers come from: with $\beta = 4/g^2$ and $b_0 = 22/(48\pi^2)$, the one-loop movement of the coupling per doubling is $\beta - \beta' = 4 \cdot 2b_0 \ln 2 = 0.2575$. For the orbital part alone, replacing $\frac{11}{3}$ by $-\frac{1}{3}$ gives $0.2575 \times (-\frac{1}{11}) = -0.0234$ Proved here (arithmetic).

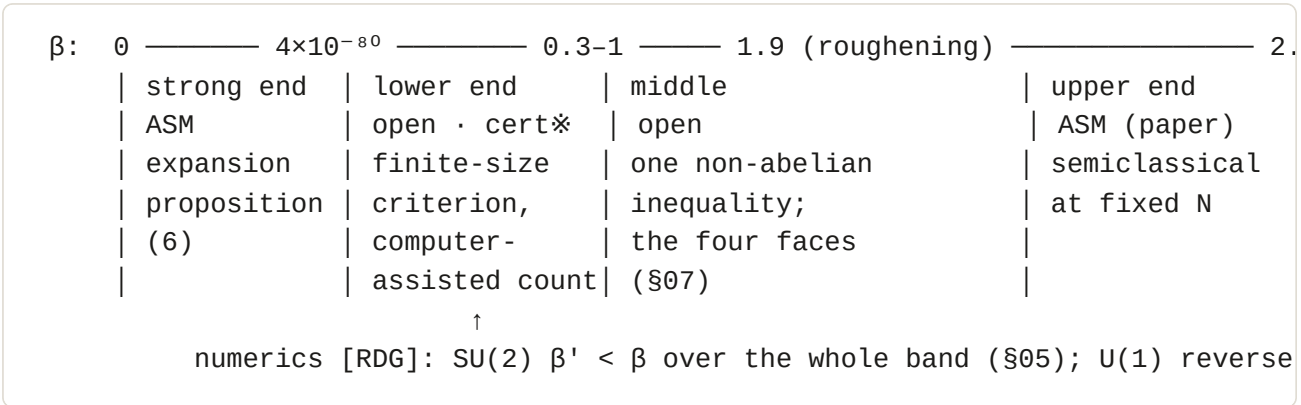
This explains as a number, at the one-loop level, why the method of Tomboulis (decimation = isometric) could not distinguish groups. The mechanism that sees asymptotic freedom lies in the reshaping of the fluctuations. What remains is to control the error in the estimate of the reshaped transport (the local KL near the vortex sheet and the higher loops) at intermediate coupling and in large fields, and this is open.

Always read the statements at the two ends with their conditions. One cannot say "the renormalisation comparison holds at both ends, so only the width of the band remains". The strong-coupling side is a line, part of whose bookkeeping is sketched, in the range $u \le u_0 \approx 10^{-80}$, and the weak-coupling side is a computation at the one-loop level. A non-perturbative weak-coupling comparison has not been shown at any β.

04

The map over β — labels interval by interval

Figure 1 sets out, over the β of the Wilson action, the status of the renormalisation comparison and of confinement interval by interval. The boundaries of the intervals are rough guides, and none of them is a threshold of a theorem.



- **The strong-coupling end** Assumption : the Strong-Coupling-End Expansion Proposition. With the constant $u_0 \approx 10^{-80}$ as it stands, the proof cannot be set out in the book, so it is "Assumption". At the same strong-coupling end there are results of this book checked in Lean (20-06): the area law, and a vortex floor whose rate per unit area does not depend on the transverse size (the prefactor remains). Both are statements at fixed lattice spacing, only inside the strong-coupling window (β below about 10^{-3}), and are not the renormalisation comparison itself. Outside the window there is no form uniform in the transverse size.

- **The lower end** ($\beta \approx 0.3$–1): open. However, the upper side of the two-sided area law at fixed β at the strong-coupling end has been extended (※ in Figure 1). Splitting an element of $SU(2)$ into a centre sign and a section of $SO(3)$ ($U = \sigma V$), the twist free energy is sandwiched by means of ratios for a $\mathbb{Z}_2$ gauge glass with the $SO(3)$ background frozen (couplings $|K_p| \le \beta$, signs random) Paper . Checking the Kotecký–Preiss criterion Cited for the polymer expansion of its closed sheets (the form, uniform in the background, of the mixing of box magnetic flux δ_C (26-10 § 03)) by full enumeration up to 10 plaquettes and a tree bound on the tail, $|\Delta F(N, M, \beta)| \le 8aN^2 e^{-bM^2}$ ($a = 0.0454$, $b = 0.01$; lowering β makes b larger) holds for $N, M \ge 11$ and all $\beta \le 0.1039$ Computer-assisted (interval arithmetic; for all $N, M \ge 2$, $\beta \le 0.0680$. The final arithmetic of the rational version $\beta \le 0.1029$ is Lean: CenterFluxPolymerCertificate (tree_k, kp_bound, kp_lt_one, rate_rational, final_real). $\tanh \beta \le \beta$, the lemma of the induction on trees and the KP theorem are outside Lean). This is of the form of the upper side of the two-sided area law at fixed β; the rate b is much smaller than the true σ, and there is no lower side (an upper bound on the string tension). The same splitting holds for $U(1)$ and for $\mathbb{Z}_2$, so this estimate does not distinguish groups and cannot cross the transition $\beta \approx 0.44$ of pure $\mathbb{Z}_2$ Reading . $\beta \ge 0.12$ cannot be shown at the threshold of the enumeration. It is likely an estimate of a known type, and no novelty is claimed. The candidates above that are a Dobrushin–Shlosman-type finite-size criterion using the influence $\delta(L_0, \beta)$ on the inside when the centre magnetic flux on the boundary of the box is changed, and a road that shows the growth constant of the admissible sheets by computer-assisted enumeration and raises u_0 to 0.04–0.12 (the range of the prospect is a physical estimate). Neither is expected to reach above $\beta \approx 1$.
- **The middle** ($\beta \approx 1$–2.7): open. One inequality that uses non-commutativity is needed, and there are four candidate local quantities to carry it (§ 07).
- **The upper end** (large β at fixed N): for the direction of confinement there is a paper proof in the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3, which came out of this book's study; $-\ln \theta \ge c\beta^{-a} e^{-\beta(S_* + \varepsilon)} M^2$ at fixed N). However, it cites the general chessboard theorem and a Morse–Bott lemma, and the proof is not set out in this book, so we label it "Assumption" Assumption . The reverse estimate uniform in M, and whether the rate S_* equals the minimal action $S_{\min}$, are open. Increasing N is the same problem as the middle.
- **One loop** Proved here : (8) of § 03 and the two comparisons. This is the perturbative level.

How the width of the band is counted depends on where the lower end is put. The strong-coupling expansion is considered numerically effective up to about $\beta \approx 0.3$–0.5, and the record writes this band as "roughly 0.5 to 2.7 in the β of the Wilson action"

Reading (physical estimate). On paper the band is much wider, $[4 \times 10^{-80},\ 2.7]$, because u_0 is so small.

05

Numerics — doubling the lattice

We test the renormalisation comparison directly with numerics. The procedure is to measure the vortex free energy on the fine lattice and read off, from the ΔF curve of the coarse lattice, the β' of the coarse lattice that gives the same value. Every measurement makes samples that cross sectors by replica exchange and obtains the free-energy difference by Bennett's acceptance ratio method (BAR). The errors are statistical; at some points the difference between two runs with different seeds (initial values of the random numbers) is 1–2 times the displayed error, so the errors may come out on the small side. Every number is a reading, not a proof Reading .

Table 2 Doubling the whole lattice of the slab: $(N, M) = (4, 8) \to (8, 16)$ $(n_{12} = 1)$ Reading

Group	β	$\Delta F(4, 8, \beta)$	$\Delta F(8, 16, \beta)$	Seeds	Matching β'	$\beta - \beta'$
$SU(2)$	2.7	2.97 ± 0.33	0.92 ± 0.30 (weighted mean of 0.65 ± 0.39 and 1.31 ± 0.48)	2	≈ 2.57 (2.45–2.65 including the error of the curve)	$\approx +0.13$ ($+0.05$–$+0.25$)
$SU(2)$	2.6	1.43 ± 0.19	-0.50 ± 0.34 (consistent with 0)	2	$\lesssim 2.45$	$\gtrsim +0.15$
$SU(2)$	2.5	0.63 ± 0.18	0.41 ± 0.34	1	2.2–2.5	0–$+0.3$
$U(1)$	1.5	0.83 ± 0.11	2.32 ± 0.45 (48 replicas, minimum exchange acceptance rate 0.37)	1	> 1.7 (outside the curve)	< -0.2

The curves on the coarse side are, for $SU(2)$, $\Delta F(4, 8, \beta') = 0.05, 0.00, 0.31, 0.63, 1.43, 2.97$ at $\beta' = 2.2, 2.3, 2.4, 2.5, 2.6, 2.7$, and for $U(1)$, $0.42, 0.55, 0.83, 0.92, 1.46$ at $\beta' = 1.3$–1.7. For $SU(2)$, doubling lowers the vortex free energy ($2.97 \to 0.92$ at $\beta = 2.7$), and the matching β' is smaller than β. For $U(1)$ it rises ($0.83 \to 2.32$, about 3σ), and $\beta' > \beta$. The direction of the comparison separated for the two groups. The first $U(1)$ run (32 replicas) had a minimum exchange acceptance rate of 0.03 and mixed badly, and is not used.

Table 3 ΔF on the symmetric torus N^4 $(n_{12} = 1)$, and β' for one step $2^4 \to 4^4$. One seed each Reading

Group	β	$\Delta F(2^4)$	$\Delta F(4^4)$	$\beta - \beta'$
$SU(2)$	1.75, 2.0	0.573, 1.39	0.01, -0.03	> 0.75 ($\beta' < 1.0$ = outside the 2^4 curve)
$SU(2)$	2.25	2.71	0.50	0.53
$SU(2)$	2.5	4.29	2.46	0.30

Group	β	$\Delta F(2^4)$	$\Delta F(4^4)$	$\beta - \beta'$
$SU(2)$	2.75	5.95	5.10	0.13
$SU(2)$	3.0	7.88	7.33	0.07
$U(1)$	0.9	1.15	0.05	> 0.3
$U(1)$	1.0	1.80	1.07	$+0.12$
$U(1)$	1.1	2.46	2.71	-0.04
$U(1)$	1.2	3.10	3.68	-0.11
$U(1)$	1.4	4.18	4.46	-0.05

The errors are 0.01–0.08 on 2^4 and 0.03–0.26 on 4^4. For $SU(2)$, $\beta - \beta' > 0$ at every β that can be read (1.75–3.0), that is, the step function of one step has no fixed point. The closer to strong coupling, the larger the movement in one step, in a form that connects to the strong-coupling values of Table 1. The decrease to 0.07 on the weak-coupling side is likely a lattice artefact, because 2^4 is too small for the one-loop regime. For $U(1)$, **the sign reverses between** $\beta = 1.0$ **and** 1.1. This is at the position of the known phase transition $\beta_c \approx 1.01$: in the confining phase $\beta' < \beta$, in the Coulomb phase $\beta' > \beta$.

Table 4 $4^4 \to 8^4$ for $SU(2)$ ($n_{12} = 1$). ΔF is the simple mean of the per-seed values, and the error is half the difference between seeds Reading

β	$\Delta F(4^4)$	$\Delta F(8^4)$	Seeds (8^4)	$\beta - \beta'$
2.3	—	-0.04 ± 0.10 (≈ 0)	1	$\gtrsim 0.1$
2.5	2.46 (1 seed)	0.48 ± 0.48	1	$\approx +0.25$ ($+0.19$–$+0.5$)
2.7	4.72 ± 0.25	3.11 ± 0.46	2	$+0.16$ ($+0.13$–$+0.19$)
2.9	6.56 ± 0.33	4.68 ± 0.63	3	$+0.20$ ($+0.12$–$+0.29$)
3.0	7.58 ± 0.25	5.78 ± 0.40	2	$+0.18$ ($+0.14$–$+0.25$)

For the 4^4 curve we also used the points $\beta = 2.6$ (3.96 ± 0.13, 1 seed), 2.75 (5.18 ± 0.10) and 2.8 (5.60 ± 0.27). For $4^4 \to 8^4$ too, $\beta - \beta' > 0$ at every β measured, with a mean of about $+0.2$. This is close to the one-loop asymptotic value 0.258 and to the value of about 0.23 expected from the known values of $a\sqrt{\sigma}$ Reading .

How to read this. The test of the sign of the step function distinguishes whether there is a phase transition. $U(1)$ reverses at the position of the transition, and $SU(2)$ does not reverse in the range measured. This is the numerical face of the criterion "distinguish from the U(1) transition" of § 08. However, the lattices are small, up to 8^4, the β points are few, and there are only 1–3 seeds. "For $SU(2)$ the sign does not change over the whole band" is a reading on these lattices and points.

06

The band is a bounded interval — reduction to the area law at fixed β

The band is a bounded closed interval $I = [\beta_a, \beta_b]$ in β. A comparison that crosses infinitely many scales shrinks to finitely many steps on a bounded interval. We write this in the form of the following lemma.

Assumption of two-sided area laws inside the band (the two-sided area law at fixed β on the band I). There are constants $N_1, M_0 \geq 2$, $c_-, c_+ > 0$, $p \geq 0$ and a function $\sigma_N(\beta) > 0$ such that

- Two-sided area law at fixed β: for all $\beta \in I$, $N \geq N_1$, $M \geq M_0$, $c_- N^2 e^{-\sigma_N(\beta) M^2 - pM} \leq \Delta F(N, M, \beta) \leq c_+ N^2 e^{-\sigma_N(\beta) M^2 + pM}$.
- Uniform positivity and continuity of the rate: with $\sigma_* := \inf_{N \geq N_1,\, \beta \in I} \sigma_N(\beta) > 0$, for some $\delta > 0$: if $\beta, \beta' \in I$ and $|\beta - \beta'| \leq \delta$, then for all $N \geq N_1$, $\sigma_N(\beta') \leq \sigma_{2N}(\beta) + \sigma_*$.

The upper side of the two-sided area law at fixed β is the area law (with an error pM of the order of the perimeter), and the lower side is that "vortices are not too cheap", that is, an upper bound on the string tension. The uniform positivity and continuity of the rate says that the rate is uniformly positive on the band and does not change much when β is moved a little.

Lemma Reducing the Comparison inside the Band. Under the assumption of two-sided area laws inside the band, for $\beta \in I$, $\beta' \in [\beta - \delta, \beta) \cap I$, $N \geq N_1$, $M \geq M_0$,

$$\Delta F(2N, 2M, \beta) \leq K \cdot \Delta F(N, M, \beta'), \qquad K = 4 \frac{c_+}{c_-} \exp\Big(\frac{9p^2}{8\sigma_*}\Big) \tag{9}$$

holds. K does not depend on N, M or β.

Proved here Proof. Since $2N \geq N_1$ and $2M \geq M_0$, the upper side of the two-sided area law at fixed β gives $\Delta F(2N, 2M, \beta) \leq c_+ (2N)^2 e^{-\sigma_{2N}(\beta)(2M)^2 + p(2M)} = 4c_+ N^2 e^{-4\sigma_{2N}(\beta) M^2 + 2pM}$. The lower side gives $\Delta F(N, M, \beta') \geq c_- N^2 e^{-\sigma_N(\beta') M^2 - pM}$. Setting $\eta := 4\sigma_{2N}(\beta) - \sigma_N(\beta')$, the uniform positivity and continuity of the rate gives $\eta \geq 3\sigma_{2N}(\beta) - \sigma_* \geq 2\sigma_*$ (using $\sigma_{2N}(\beta) \geq \sigma_*$). The ratio is

$$\frac{\Delta F(2N, 2M, \beta)}{\Delta F(N, M, \beta')} \leq 4 \frac{c_+}{c_-} e^{-\eta M^2 + 3pM} \leq 4 \frac{c_+}{c_-} e^{9p^2/(4\eta)} \leq 4 \frac{c_+}{c_-} e^{9p^2/(8\sigma_*)}$$

The second inequality is the maximum $9p^2/(4\eta)$ of the quadratic $-\eta M^2 + 3pM$, and the third is $\eta \geq 2\sigma_*$. ∎

Corollary Crossing the Band in Finitely Many Steps. The band can be crossed in at most $\lceil(\beta_b - \beta_a)/\delta\rceil$ steps, and the loss over them is K to the power of the number of steps, a constant independent of N and M Proved here . The same proof goes through if β' is lowered further, as long as it satisfies $\sigma_N(\beta') \le 4\sigma_{2N}(\beta) - 2\sigma_*$.

Calibration. The Strong-Coupling-End Expansion Proposition ((6)) gives the two-sided area law at fixed β with $p = 0$ and $c_\pm = 2(1 \pm C'u)$. Then (9) holds with $K = 4(1 + C'u)/(1 - C'u)$, which agrees with $4(1 + O(u))$ of the Comparison at the Strong-Coupling End in § 03 Proved here (the Strong-Coupling-End Expansion Proposition is an assumption, so the agreement is an agreement of form).

What the Lemma Reducing the Comparison inside the Band says. The renormalisation comparison inside the band reduces to "confinement at fixed β (two-sided asymptotics with an error of the order of the perimeter)" at each β of the band. The comparison across scales itself is not needed. The band is entered when the physical box is large, so the two-sided area law at fixed β is needed only in the range where N, M is sufficiently larger than the correlation length $\xi(\beta)$.

What the Lemma Reducing the Comparison inside the Band does not say. The assumption of two-sided area laws inside the band is unproved at every β of the band. The upper side is itself the unsolved problem of confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice. As far as we searched, there is no proof of confinement on the four-dimensional $SU(2)$ lattice at $\beta \approx 1$. The lower side (an upper bound on the string tension to the accuracy of the order of the perimeter) has not been written out either. The lemma does not depend on the group; all the non-commutativity is in the assumption of two-sided area laws inside the band. For $U(1)$ ($\beta \in (1.01, 2.7)$), the assumption of two-sided area laws inside the band is false, so no conclusion follows. This lemma is a reduction, not a tool, and we do not say it has made the band easier.

Where the difficulty gathers Reading (physical estimate). After the Lemma Reducing the Comparison inside the Band, what remains inside the band are problems at fixed β (estimates uniform in the volume, non-convex large fields, the absence of a phase transition). The difficulty of crossing scales does not vanish from inside the band; it gathers at **the upper end of the band** and at **the seam with ultraviolet control**. At the upper end of the band, $\xi(\beta)$ exceeds 10 in lattice units, and the road of checking the two-sided area law at fixed β by computations in finite boxes stops on size. And beyond the upper end of the band ($\beta \to \infty$) is the range taken on by ultraviolet control (7-07), and the two must be joined with the same constant. In terms of the three roots of 24-07 § 05, inside the band it is the source "no global gauge fixing" (the non-convexity of large fields) and excluding a phase transition, and the seam at the upper end is the

source "no expansion linking scales" (there is no expansion that joins the ultraviolet and the infrared).

Example. On the strong-coupling side ($u \le u_0$), a statement of exactly the form of the base point at small T^2 exists as the Strong-Coupling-End Expansion Proposition (at the level of a line of proof). The work of pushing the lower end of the band up to $\beta \approx 0.3$–1 (the lower end of § 04) is, in the words of the Lemma Reducing the Comparison inside the Band, "widening the interval on which the two-sided area law at fixed β holds", and needs no comparison across scales.

07

The four faces of the band in the middle

It has been narrowed down to this: crossing the band in the middle needs one inequality that uses non-commutativity. On **which local quantity to write it in**, there are four candidates. We call them "faces" in the sense that it is enough to show any one of them uniformly in the scale, in a form that fails for $U(1)$. Table 5 lists only the names and the expected behaviour; the detailed examination is in 26-10. Every row is an estimate Reading (physical estimate).

Table 5 The four faces of the band in the middle (expectation rows are physical estimates; numbers are readings)

Face	Local quantity	Expectation for $SU(2)$	Where it fails for $U(1)$
κ_w	Softness in the twist direction: the variance of the force along the generator of the twist (in a strip of width w)	Decreases with w (the soft direction of Nielsen–Olesen; a non-perturbative version of the one-loop $+4$ spin term)	No spin term, $\kappa_w \ge c\beta$
$\delta(L_0)$	The total variation distance on the inner cube when the centre magnetic flux on the boundary of the box is changed	Goes to 0 exponentially in L_0	Remains $O(\beta)$ in the Coulomb phase
$f(\ell)$	The twist free energy of an $\ell \times \ell$ column with its outer frame held fixed	Of the type $e^{-\sigma \ell^2}$	Of the type $\pi^2\beta/\ell^2$ ($\approx 2\beta$ for $\mathbb{Z}_2$)
Pressure margin	The pressure difference $p_\infty - p_{G,L}$ between the system with and without the twist pattern (the exponent of the chessboard density κ_{cb})	A positive margin remains and gives the area law at fixed β	Does not produce a small κ_{cb}

A numerical example. Measuring the column face $f(\ell)$ by the twist bias $\mathbb{E}|\theta_B|$ on $T_4^2 \times \ell \times \ell$ with the outer frame held fixed, $SU(2)$ ($\beta = 2.3$) gives $0.23, 0.07, 0.09$ at $\ell = 4, 5, 6$, and the Coulomb phase of $U(1)$ ($\beta = 1.3$) gives $0.61, 0.49, 0.41$ (0.25 at $\ell = 8$); for $\ell \ge 5$,

$SU(2)$ is 1/5 to 1/7 of $U(1)$ Reading . However, the $SU(2)$ values are close to the statistical floor, and at this size $e^{-\sigma \ell^2}$ cannot be told apart from a power.

For each of the four faces, the comparison across scales (for example $f(2\ell, 2N, \beta) \leq Kf(\ell, N, \beta')$) is a local version with the same difficulty as the global renormalisation comparison, and has not become easier. What changing the face gains is that the object of the proof becomes "a statement about a finite region with its boundary held fixed".

08

Conditions an argument closing the band must satisfy — the criterion "distinguish from the U(1) transition"

The criterion "distinguish from the U(1) transition". $U(1)$ has, inside this band ($\beta \approx 1.01$), a phase transition to a non-confining phase. An argument that closes the band must show that $SU(2)$ has no phase transition, distinguishing it from $U(1)$. This is of the same weight as the unsolved problem that "four-dimensional $SU(2)$ has no transition to a non-confining phase at zero temperature", and proposals that do not satisfy this condition are discarded from the start.

The sign of the step function in § 05 was the numerical face of this criterion. Put another way, **a proof that closes the band is a proof that the direction $\beta' < \beta$ of the comparison does not change over the whole band for $SU(2)$**. Purely structural arguments (reflection positivity, duality, the transfer matrix) hold in the same form for $\mathbb{Z}_2$ and $U(1)$, so they alone do not satisfy the criterion (26-04). Renormalisation that does not distinguish groups (the isometric comparison, the Migdal–Kadanoff flow) does not satisfy it either (§ 03).

Even without closing the whole band at once, there are intermediate goals that satisfy the criterion "distinguish from the U(1) transition".

1. $\Delta F(L, L) \to 0$ (at any speed). In the Coulomb phase of $U(1)$, $\Delta F(L, L)$ tends to $\pi^2 \beta_R /2 - \ln 2$ (the classical half-flux value $\pi^2 \beta_R/2$ minus $\ln 2$ for the two orientations $\pm\pi$ of the half flux; β_R is the renormalised coupling), and in the non-confining phase of $\mathbb{Z}_2$ it increases, so showing this would exclude the known non-confining phases Assumption (the behaviour of the phases is an estimate). It does not give the mass gap. Numerically, for $U(1)$ ($\beta = 1.2$), $2^4 \to 4^4$ is flat at $3.10 \to 3.68$, and for $SU(2)$ ($\beta = 2.75$) it is $5.95 \to 5.10$ Reading .
2. The tension-doubling inequality: there are $\delta_0 > 0$ and N_1 such that $4\sigma_{2N}(\beta) \geq e^{\delta_0} \sigma_N(\beta)$ for all $N \geq N_1$, that is, $s(2N, \beta) \geq e^{\delta_0} s(N, \beta)$ (notation as in the rate form of the renormalisation comparison in § 01). In the Coulomb phase of $U(1)$, s is constant

across scales and sits exactly on the boundary $\delta_0 = 0$, and in the non-confining phase of $\mathbb{Z}_2$, $s \to 0$ and it is on the negative side, so it satisfies the criterion "distinguish from the U(1) transition" Reading (the values for the three phases come from closed-form models and small lattices). What follows is $s(2^k N_1) \geq e^{k\delta_0} s(N_1) \to \infty$, which excludes the Coulomb phase and the non-confining phases of discrete groups Proved here (the arithmetic of iteration). To pass to the first statement above, $\Delta F(L, L) \to 0$, control of the difference between the values on symmetric boxes and the slab tension is needed separately. Neither the mass gap nor the area law follows. Its row in the list of premises is in 28-04.

3. $\ln(1/\theta(L, L)) \geq cL^2/\ln L$ (nearly the area law). Through the Tomboulis–Yaffe inequality Cited, the Wilson loop decreases faster than the perimeter law (of the type $\langle W \rangle \leq 2\exp(-cA/\ln A)$).
4. The band itself: $\ln(1/\theta(L, L)) \geq cL^2$ (26-01).

Reading the band in the language of the gap — the band comparison (in the language of the gap). The vortex free energy is a quantity on the side of electric flux $e \neq 0$ ((4)(5) of § 02), while the mass gap needs the side $e = 0$. A comparison crossing the band on the neutral side can be written as the absence of a fixed point of the doubling map of $x(L) = L\, m_0(L)$, the gap of the box measured in units of the box size; that is, in the form $x(2L) \geq x(L) + \delta$ on the interval of the band (with δ independent of the lattice spacing) Assumption. In the Coulomb phase of $U(1)$, x stays at the photon value 2π (in a periodic box $x(2L)/x(L) = 1/\cos(\pi/2L) \to 1$), so this form satisfies the criterion "distinguish from the U(1) transition" Paper. The form of a ratio of rates, $m_0(2L) \geq (\frac{1}{2} - \varepsilon) m_0(L)$, is satisfied trivially by $U(1)$, with $m_0(2L)/m_0(L) = 1/(2\cos(\pi/2L)) > \frac{1}{2}$, so it cannot be used to distinguish Paper. The content of the distinction lies in the step δ being uniform. Of the four faces (§ 07), κ_W and $\delta(L_0)$ (extended to all boundary conditions) also act on the neutral side, while $f(\ell)$ and the pressure margin are quantities on top of the centre twist and belong only to the side of confinement Reading. The Tomboulis–Yaffe inequality is a structural inequality that holds with the same proof for $U(1)$, and its output is in the charged sector, so it is a tool for stepping down to the side of confinement, not a tool for the band comparison (in the language of the gap) Reading. The criterion on the gap side is narrower than the one on the confinement side, in that it need not exclude the non-confining phase of $\mathbb{Z}_2$ (which also has a gap). None of these rows has been shown.

The test models other than U(1) (Z_2, finite non-abelian groups, the adjoint Higgs limit, mixed actions, SU(N), G_2) are listed in 26-12.

09

What cannot be said

(a) We show neither the renormalisation comparison nor β-monotonicity at any β of the band.

(b) The strong-coupling end is a line, part of whose bookkeeping is sketched, in the range $u \le u_0 \approx 10^{-80}$. The weak-coupling end is at the one-loop level, not a non-perturbative claim. Do not read "it holds at both ends" with these two conditions removed.

(c) The Lemma Reducing the Comparison inside the Band is a reduction, and the assumption of two-sided area laws inside the band contains an unsolved problem. We do not say that the lemma made the band easier.

(d) The numbers of § 05 are readings on small lattices up to 8^4, with few β and 1–3 seeds, and are not evidence that there is no phase transition.

(e) The four faces of Table 5 are candidates, and none of them has an estimate uniform in the scale.

(f) The relation between β-monotonicity and the monotonicity of the string tension includes a step that is a reading (§ 02). Nor do we say that β-monotonicity is false.

10

The table of the five grades and the dependency graph

Table 6 The rows of this chapter ("—" marks a row with no machine checking; the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
1	$\partial_\beta \Delta F = \langle \Delta S \rangle$	Proved here	§ 01	With the lattice geometry held fixed
2	The Gibbs variational principle (7)	Cited	§ 03	Measurable bijection T
3	(4)(5): β-monotonicity $\Leftrightarrow E_1 \ge E_0$ $\Leftrightarrow \theta$ increases in β	Proved here	§ 02	$Y_1 > 0$
4	β-monotonicity has the same weight as the β-monotonicity of the string tension in lattice units	Assumption	§ 02 (the step connecting the electric flux with the tension has not been written out)	On every lattice
5	Partial results on β-monotonicity (1 and 2 sheets in two dimensions, $\beta \to 0$, $\beta \to \infty$ in the twisted background)	Assumption	§ 02 (the proofs are not set out in the book)	The conditions of each case

Row	Statement (one line)	Grade	Basis	Scope
6	Strong-Coupling-End Expansion Proposition: the form (6) of the vortex free energy at the strong-coupling end	Assumption	Line: the bookkeeping of step 5 of the proof is sketched	$u \le u_0 \approx 10^{-80}$, N, $M \ge 2$
7	The Comparison at the Strong-Coupling End $K = 4(1 + O(u))$, $\beta' < \beta$	Assumption	Algebra on top of the Strong-Coupling-End Expansion Proposition	As above
8	The expansion (8) of the heat-kernel trace: $\frac{11}{3} = -\frac{1}{3} + 4$	Proved here	§ 03	One loop, constant background
9	The isometric comparison gives $\beta' \ge \beta + 0.023$ (against asymptotic freedom)	Proved here	§ 03, 25-07	One-loop level, per doubling
10	The comparison that reshapes the fluctuations gives $\beta' = \beta - 0.258$	Proved here	§ 03, 25-07	One-loop level, per doubling
11	Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3; the direction of confinement at fixed N and large β)	Assumption	A paper proof in the paper (with citations). Not set out in the book	Fixed N
12	Table 2: for $(4, 8) \to (8, 16)$, $SU(2)$ gives $\beta' < \beta$, $U(1)$ gives $\beta' > \beta$	Reading	§ 05	Lattices, β and seeds as in the table
13	Tables 3 and 4: for $SU(2)$ the sign does not change; for $U(1)$ it reverses at $\beta = 1.0$–1.1	Reading	§ 05	Up to 8^4, 1–3 seeds
14	Lemma Reducing the Comparison inside the Band: the comparison (9) inside the band under the assumption of two-sided area laws inside the band	Proved here	§ 06	Two-sided area law at fixed β, uniform positivity and continuity of the rate, $N \ge N_1$, $M \ge M_0$
15	Corollary Crossing the Band in Finitely Many Steps: the band can be crossed in finitely many steps, with a loss independent of N, M	Proved here	§ 06	As above
16	Calibration: $K = 4(1 + C'u)/(1 - C'u)$ in the form of the Strong-Coupling-End Expansion Proposition	Proved here	§ 06	Agreement of form (the Strong-Coupling-End Expansion Proposition is an assumption)

Row	Statement (one line)	Grade	Basis	Scope
17	The difficulty gathers at the upper end of the band and at the seam with ultraviolet control	Reading	§ 06 (physical estimate)	—
18	The four faces of the band in the middle and their expectations	Reading	§ 07 (physical estimate)	—
19	The criterion "distinguish from the U(1) transition": the transition of $U(1)$ inside the band, and the condition on an argument closing the band	Reading	§ 08 (the position of the transition is a known numerical value; physical estimate)	—
20	$\Delta F(L, L) \to 0$ excludes the known non-confining phases	Assumption	§ 08 (the behaviour of the phases is an estimate)	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  ∂_β ΔF = ⟨ΔS⟩
 2  CIT  Gibbs variational principle (7)
 3  PRV  β-monotonicity ⇔ E₁ ≥ E₀ ⇔ θ increases ⟸ 1                Y₁ > 0
 4  ASM  β-monotonicity weighs as much as β-monotonicity of the tension ⟸ 3
 5  ASM  partial results on β-monotonicity
 6  ASM  Strong-Coupling-End Expansion Proposition                u ≤ u₀ ≈ 10⁻
 7  ASM  Comparison at the Strong-Coupling End, K ≈ 4 ⟸ 6
 8  PRV  11/3 = −1/3 + 4                                        one loop
 9  PRV  isometric comparison gives β' ≥ β + 0.023 ⟸ 2, 8        one loop
10  PRV  reshaping comparison gives β' = β − 0.258 ⟸ 2, 8        one loop
11  ASM  Confinement Lower Bound by Fractional Instantons        fixed N
12  RDG  direction of (4,8)→(8,16) separates by group
13  RDG  sign of the step function: SU(2) unchanged, U(1) reverses
14  PRV  Lemma Reducing the Comparison inside the Band
15  PRV  the band takes finitely many steps ⟸ 14
16  PRV  calibration ⟸ 6, 14                                     agreement of
17  RDG  difficulty at the upper end and the UV seam ⟸ 14
18  RDG  the four faces ⟸ 9, 10, 12, 13
19  RDG  criterion "distinguish from the U(1) transition" ⟸ 13
20  ASM  ΔF(L,L) → 0 excludes non-confining phases ⟸ 19
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The two-sided area law at fixed β and the uniform positivity and continuity of the rate are written as conditions in the row of the Lemma Reducing the Comparison inside the Band and are not made nodes, because the assumption rows do not contain the quantity of the conclusion (the comparison that crosses the band). Among the nodes of this chapter there is no node for the comparison at any β of the band, none for confinement, and none for the mass gap.

What this chapter can say

What is said	What is not said
Proved here β-monotonicity and its restatement (4)(5) in terms of the electric flux. That, at the one-loop level, the isometric comparison gives only the direction against asymptotic freedom ($\beta' \geq \beta + 0.023$), and the comparison that reshapes the fluctuations gives $\beta - \beta' = 0.258$	β-monotonicity itself. The non-perturbative weak-coupling comparison
Proved here Lemma Reducing the Comparison inside the Band: that the comparison inside the band reduces to the two-sided area law at fixed β (the assumption of two-sided area laws inside the band)	A proof of the assumption of two-sided area laws inside the band at any β. Having made the band easier
Assumption The Comparison at the Strong-Coupling End ($u \leq u_0 \approx 10^{-80}$, part of the bookkeeping sketched). The upper end at fixed N (the Confinement Lower Bound by Fractional Instantons; Theorem A of Paper 3)	A proof set out in the book. Raising u_0 to a practical range
Reading That on lattices up to 8^4 the direction of the comparison does not change for $SU(2)$ and reverses at the position of the transition for $U(1)$	Evidence that there is no phase transition. The same reading on large lattices

Sources and reproduction

Item	Kind	Source / tools
The split into β-monotonicity and the renormalisation comparison, (4)(5), the partial results, the expansion of the heat-kernel trace and the two comparisons	This book's study	The record of this book's study (the idea of splitting the band into β-monotonicity and the renormalisation comparison, the examination of β-monotonicity, the one-loop check). The computations of (4) and (8) were checked in this chapter
Strong-Coupling-End Expansion Proposition and the Comparison at the Strong-Coupling End	This book's study (Assumption)	The record of this book's study (the strict form of the comparison on the strong-coupling side, and the section correcting it). The type of the polymer expansion is that of G. Münster's (1981) strong-coupling expansion of the vortex free energy (the bibliographic data and the text of the original source have not been checked)
Table 1	Reading	Numerical values of the ratio of modified Bessel functions I_2/I_1, and numerical roots of $u(\beta') = u(\beta)$ [4]. The numbers were checked in this chapter

Item	Kind	Source / tools
Tables 2–4, the column bias	Reading	The numerical record of this book's study (the test of the renormalisation comparison, the scan of the step function on the symmetric torus, the $U(1)$ control, the addition of $4^4 \to 8^4$ and the merging of seeds, the twist cost of a column). Replica exchange and Bennett's acceptance ratio method
Lemma Reducing the Comparison inside the Band, Corollary Crossing the Band in Finitely Many Steps, calibration	Proved here	This book's study (the reduction lemma from the examination of the band candidates). The proof was checked line by line in this chapter
The four faces, the criterion "distinguish from the U(1) transition", the intermediate goals	This book's study	The record of this book's study (the arrangement of roads for closing the crossover band). Details in 26-10
The claim of Tomboulis and its criticism	Cited	E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930 (eq. (3.5)), arXiv:0803.3019 (detailed study). T. Kanazawa, arXiv:0805.2742 (Appendix A). For the history, see 26-08
Tomboulis's short paper of the 1980s	Cited	E. T. Tomboulis, Phys. Rev. Lett. 50 (1983) 885 (title not confirmed; the text has not been read)
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) (the table of sources of 26-01)

Terms used in this chapter — the crossover band, β-monotonicity, the renormalisation comparison, the isometric comparison, the comparison that reshapes the fluctuations, the step function, the four faces, the criterion "distinguish from the U(1) transition", the reduction lemma. Their meanings are in A-02 Glossary.

Next chapter: 26-07 The band seen from the side of time — the statement corresponding to the band on the road of Part 23, and why the first-step wall, the "summit-curvature shortfall wall", is a miniature of the band and not the band.

The band seen from the side of time

> This chapter is this book's own investigation. § 06 is a map of open questions.

What this chapter shows — Part 23 was a road that approaches the mass gap from the side of Langevin time (relaxation of a stochastic dynamics). This chapter states exactly which statement on that road corresponds to the crossover band, and settles the relation between the band and the wall, the "summit-curvature shortfall wall", at which Part 23 actually stops. There are two answers. (1) The four-dimensional goal of Part 23 (the physical-unit target level of 23-15) is a statement uniform over all scales — that the upper side of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption) and the one-sided bridge hold with the same constant as $\beta \to \infty$ — and it must cross the band. (2) The first-step wall, the "summit-curvature shortfall wall", is a miniature of the band, but it is not the band. This is because its crossover length is fixed classically by the first conjugate point of the Jacobi equation at the summit (about 7.84 sites), does not depend on $\hbar$, and is finite in lattice units. The length $1/\Lambda$ of the four-dimensional band is born by quantisation and diverges in lattice units. The two walls **are of the same type and differ in content**. We put this into a comparison table of six columns.

Prerequisite chapters — 23-15 § 07 (the three levels of the goal), 23-01 § 06 (the one-sided bridge), 23-09 § 06 (the curvature $\frac{15}{4}$ of one site at the summit, and 4), the short-core step of 23-11 § 05 (convexity up to length 7), 24-06 (paying with rarity), 24-07 (precisely because it is four-dimensional; the three sources of difficulty), 26-06 (the renormalisation comparison and the four faces), 4-07 (dimensional transmutation).

Said first — The level is [Res]. The only proofs set out are the computation of the Jacobi equation at the summit and of the first conjugate point in § 02, the proof of the Hidden-Convexity Proposition in § 03, and the one line of algebra in § 01. The contrast between the two walls is a physical estimate, and the statement that crosses the band is itself an assumption on either road. A note on words: 23-01 § 05 in Part 23 uses the word "band statement" for the statement that C_P lies in a range of width a constant times the square of a length (the range of constant-multiple width = the band). The "band" of this chapter is the "crossover band" of Part 26 (the crossover at $L \sim 1/\Lambda$), a different word. The list of places where "band" is used in other senses in Parts 23 and 24 is in 26-00 Table 0. It is also different from the "window" of Parts 23 and 24 (the site window, the $\hbar$ window, the time window). The convexity constant up to length 7 is written $\kappa_{C7}(k)$, to distinguish it from the chessboard density κ_{cb} (the same quantity as $\kappa(k)$ of 23-11).

The order of this chapter

1. The statement on the road of Part 23 that corresponds to the band — the physical-unit target level, and uniformity with the same constant
2. The wall of the first step is fixed by a classical length — the Jacobi equation at the top and the first conjugate point
3. The type is the same — different tools at the two ends, the worst-case estimate stops in the middle, and on to averaging tools
4. The content differs — the origin of the crossover length, rare and typical, phase transitions
5. The comparison table of the two walls — six columns (Table 1)
6. A supplement to Part 23 — the road of blocks and mixing — towards the third branch of 23-15
7. What cannot be said
8. The table of the five grades and the dependency graph — Table 2

01

The statement on the road of Part 23 that corresponds to the band — the physical-unit target level, and uniformity with the same constant

Part 23 split the goal of the mass gap on the four-dimensional lattice into three levels (23-15 Table 5): the lattice target level, that the correlation length is finite at each β on the lattice; the physical-unit target level, that the lattice mass measured in physical units, $m_{\mathrm{lat}}(\beta)/a(\beta)$, tends to a positive finite value Δ_∞; and the continuum target level, that the gap is positive in the continuum theory. On the road from the side of time, the physical-unit target level is what corresponds to the band.

The lower half of the physical-unit target level (that the mass does not collapse to 0) follows by one line of algebra from two statements: the upper-bound assumption, the upper side of the diffusive-relaxation-time assumption ((14) of 23-01 § 05), and the one-sided bridge (in the form of the One-Sided Bridge Proposition of 23-01 § 06).

$$\exists\,\beta_0,\ C,\ \kappa>0\ \ \forall \beta\ge\beta_0:\qquad \underbrace{a(\beta)^2\,C_P(\beta)\ \le\ C}_{\xi^2\uparrow},\qquad \underbrace{C_P(\beta)\ \ge\ \kappa\,\xi(\beta)^2}_{\text{one-sided bridge}}\ \ (1)$$

$$\Longrightarrow\qquad \frac{m_{\mathrm{lat}}(\beta)}{a(\beta)}\ =\ \frac{1}{a(\beta)\,\xi(\beta)}\ \ge\ \sqrt{\kappa/C}\qquad (\beta\ge\beta_0)\ \ (2)$$

Proved here Proof. From the two parts of (1), $\kappa\xi^2 a^2 \le a^2 C_P \le C$; taking the reciprocal of the square root of both sides gives (2). ∎ (The same algebra as (16) of 23-01 § 06. C_P is the Langevin Poincaré constant of the Wilson measure, $\xi = 1/m_{\mathrm{lat}}$ the correlation length in lattice units, and $a(\beta)$ the lattice spacing.)

The content of (1) lies in the order of the two quantifiers: C **and** κ **do not depend on** β, that is, it holds with the same constant on all lattices as $\beta\to\infty$. The one-sided bridge can be shown, for continuous time and a finite spatial lattice, as a proposition with named assumptions, but its lifting to the four-dimensional lattice is only a prospect of form Assumption (the row of the fourth step in 24-07 Table 2). The upper-bound assumption is an assumption in four dimensions Assumption .

Why this statement crosses the band Reading (physical estimate). As β grows, the lattice spacing $a(\beta)$ goes to 0, and the correlation length in lattice units grows like $\xi(\beta)\propto e^{3\pi^2\beta/11}$ (the one-loop bookkeeping, 23-03 Table 2). The Wilson measure at one fixed β contains all scales from one lattice step up to $\xi(\beta)$, and the effective coupling at the scale of length L is small for $L\ll 1/\Lambda$, becomes $O(1)$ at $L\sim 1/\Lambda$, and beyond that enters strong-coupling behaviour. $C_P(\beta)$ is a supremum over all test functions, so it includes the slowest relaxation, that is, the behaviour at the longest scales. So showing (1) with a C independent of β includes passing, with the same constant, at every β, through the scales at which the effective coupling becomes $O(1)$ — the crossover band. The goal of the road from the side of space (the uniform slab tension: a tension positive uniformly in L, 26-00) is a statement of the same type. That the two roads arrive at a statement of one type, "the same constant at every scale", is what the confluence of Part 26 means.

This uniformity takes up again something Part 23 already said. The end of 23-15 § 07 wrote that what is missing is not the number of steps but uniformity, holding at every scale with the same constant, and that it appears at the first step in a form uniform as $\hbar\to 0$ and on the four-dimensional lattice in a form uniform as $\beta\to\infty$. The question of this chapter is whether that first-step version is **the same thing** as the four-dimensional band.

02

The wall of the first step is fixed by a classical length — the Jacobi equation at the top and the first conjugate point

First, the place. The first step is the path measure $\mu\propto e^{-S/\hbar}$ of the double well in 0+1 dimensions, with

$$S(z)=\sum_i\Big[\frac{(z_{i+1}-z_i)^2}{2\varepsilon}+\varepsilon V(z_i)\Big],\qquad V(z)=\frac{(z^2-1)^2}{8},\qquad \varepsilon=\tfrac12\ \ (3)$$

(23-04). The wall appears in the conditional measure on the interval (the core) $Q_k=[-x_c,0]^k$ ($x_c=0.89$, Neumann faces) in which the path stays between the summit $z=0$ and the well, and the goal is that its Poincaré constant is uniform in the length k of the core, the end values and $\hbar$ (the core Poincaré uniformity assumption, 23-15 § 02).

The Hessian at the top configuration $z\equiv 0$ is the tridiagonal $\mathrm{tridiag}(-2,\frac{15}{4},-2)$, and the diagonal $\frac{15}{4}$ of one row falls short of the sum of couplings 4 by $\frac14$ (23-09 § 06). We reread this shortfall in the language of the calculus of variations.

The Jacobi equation at the top. Linearising the Euler–Lagrange equation of S at the top $z\equiv 0$ gives, for the displacement u_n,

$$-2u_{n-1}+\tfrac{15}{4}\,u_n-2u_{n+1}=0\ \ (4)$$

For general ε the diagonal is $2/\varepsilon+\varepsilon V''(0)=2/\varepsilon-\varepsilon/2$ and the coupling to neighbours is $1/\varepsilon$, and (4) can be written $u_{n-1}+u_{n+1}=(2-\varepsilon^2/2)u_n$. Substituting $u_n=e^{\pm in\theta}$, the characteristic equation is $2\cos\theta=2-\varepsilon^2/2$, that is,

$$\cos\theta=1-\frac{\varepsilon^2}{4}\qquad (\varepsilon=\tfrac12\ \text{gives}\ \cos\theta=\tfrac{15}{16},\ \ \theta=0.35542\ldots)\ \ (5)$$

and since $|\cos\theta|<1$, the solution is a rotation $u_n=\sin(n\theta)$ ($u_0=0$). Seen through the linearisation $\ddot u=V''(0)u=-\frac12 u$ of the Euclidean equation of motion $\ddot z=V'(z)$ too, the top is not a point of exponential departure but a point of rotation.

The first conjugate point. The position $n=\pi/\theta$ at which the solution starting from $u_0=0$ next becomes 0 is the first conjugate point. At $\varepsilon=\frac12$,

$$\frac{\pi}{\theta}=8.8391\ldots,\qquad \frac{\pi}{\theta}-1=7.8391\ldots\ \approx\ 7.84\ \ (6)$$

Proved here (the numerical value of $\theta=\arccos(15/16)$. As an independent check, running the recursion (4) from $u_0=0,u_1=1$ gives $u_8=0.844>0$, $u_9=-0.164<0$, and the zero of the linear interpolation is 8.837).

The threshold of convexity = the conjugate point. The smallest eigenvalue of the Hessian $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$ at the top of a lump of length k with both ends fixed is

$$\kappa_{\mathrm{C7}}(k) \;=\; \tfrac{15}{4} - 4\cos\frac{\pi}{k+1} \;=\; 8\sin^2\frac{\pi}{2(k+1)} - \frac{1}{4} \quad (7)$$

(the closed formula for the eigenvalues of a tridiagonal matrix), and it is positive only when $\pi/(k+1) > \theta$, that is, $k+1 < \pi/\theta$. From (6), it is positive for $k \le 7$ and negative for $k \ge 8$ Proved here ($\kappa_{\mathrm{C7}}(7) = 0.0545$, $\kappa_{\mathrm{C7}}(8) = -0.0088$. The difference between numerical diagonalisation and (7) is of order 10^{-15} for $k = 5$–10). "Convex up to length 7, broken from 8" (the Short-Core Convexity Lemma of 23-11 § 05) is exactly the discrete version of the Jacobi criterion of the calculus of variations: whether a conjugate point falls in the interval $[0, k+1]$.

This length is classical and does not depend on $\hbar$. $\hbar$ never appears in (4)–(7). What appears is only the form of the action S (the coefficient $1/\varepsilon$ of the kinetic term and the curvature at the top $V''(0) = -\frac{1}{2}$). The limit taken at the first step is $\hbar \to 0$ with $\varepsilon = \frac{1}{2}$ fixed, so the crossover length stays at about 7.84 sites throughout the limit and is finite in lattice units. In units of time it is $\varepsilon \cdot \pi/\theta \approx 4.42$, close to the value $\pi/\sqrt{|V''(0)|} = \pi\sqrt{2} \approx 4.44$ (the half period of rotation at the top) in the limit where time is made continuous. Sending ε to 0 makes the number of sites grow as $1/\varepsilon$, but that is not the limit the problem of the first step takes.

Example. For $\varepsilon = \frac{1}{4}$ it is $\pi/\theta - 1 \approx 16.75$ sites, which in units of time is ≈ 4.44. As a number of sites the length changes with the step; as a physical time it is fixed by the number $V''(0)$ in the action.

A numerical agreement with the mixing length (the mechanism is different). The number of sites over which the influence of the worst end value (the top) drops below 10^{-2} in total variation hardly moves for $\hbar = 10^{-2}$–10^{-5} and is about 9; changing the step to $\varepsilon = 0.35, 0.5, 0.7, 0.9$ gives $12, 9, 7, 5$, numerically close to $\pi/\theta = 12.7, 8.8, 6.3, 4.9$ Reading . But the mechanism is not "staying near the top for half a period". The optimal path as $\hbar \to 0$ starts going down from the top at once and reaches the corner $-x_c$ of the box in a finite number of steps (7 at $\varepsilon = \frac{1}{2}$), and the mixing length is roughly that number of steps plus 2 Reading . Why the two values are close is not known. What this chapter claims about the conjugate point is only the convexity threshold of (7).

03

The type is the same — different tools at the two ends, the worst-case estimate stops in the middle, and on to averaging tools

Setting the two walls side by side shows the same type Reading (physical estimate; a comparison of types, implying no statement on either side).

There are different tools at the two ends. At the first step, short lumps ($k \le 7$) are bounded uniformly in the end values and $\hbar$ by the convexity of (7) (the Short-Core Convexity Lemma Proved here), and inside the wells local Poincaré within the phase (the in-phase local Poincaré condition, 23-06) works. In the four-dimensional band, the expansion works at the strong-coupling end, and the one-loop comparison at the weak-coupling end (26-06 § 03; on the strong-coupling side the line of argument for $u \le u_0 \approx 10^{-80}$, on the weak-coupling side the one-loop level).

In the middle, the estimate of the worst configuration stops. At the first step, every tool that takes the worst configuration as its entrance comes back to the same one-line shortfall $\frac{15}{4} < 4$ Proved here (each row is a two-line computation).

- The one-site Dobrushin influence coefficient: the conditional one-site measure is $\frac{15}{4}$-convex, and with the coupling $1/\varepsilon = 2$ to neighbours the influence from one neighbour is $\le 2/\frac{15}{4} = \frac{8}{15}$, and from two $\frac{16}{15} > 1$.
- The influence of a block of length 7: by Brascamp–Lieb and an entrywise comparison of M-matrices, the influence from the left end to the right end is $2\,(M_7^{-1})_{71} = 2 \cdot 2^6/\det M_7 \approx 1.18 > 1$ (7×7 for $M_7 = \mathrm{tridiag}(-2, \frac{15}{4}, -2)$, $\det M_7 = 1770735/16384 \approx 108.08$).
- The Otto–Reznikoff criterion: it needs the positivity of $\mathrm{tridiag}(-2, \frac{15}{4}, -2)$, which fails on the infinite chain with lower end $-\frac{1}{4}$.

In the four-dimensional band, the road that estimates with the worst frame plays the same role. The tension that comes out of the slab-confinement estimate $\sup_\Phi \theta \le \tanh(\beta L^2)^{\ell^2}$ (25-04) is $\sigma \ge -\ln\tanh(\beta L^2) \approx 2e^{-2\beta L^2}$, which gets worse superexponentially in L ($2e^{-80}$ at $\beta = 2.5$, $L = 4$). Structural inputs alone (reflection positivity, duality, the transfer matrix) hold in the same form for $\mathbb{Z}_2$ and $U(1)$, and cannot cross the band (26-04). Each side has a "worst single point": at the first step the face $x \equiv 0$ where every site is at the top, in four dimensions the place where the plaquettes are at the centre element -1.

Moving to averaging tools, there is a prospect that uniformity comes out. At the first step, the correlation length of the transfer operator gets shorter as $\hbar$ is made smaller (from 1.68 to 0.48 as $\hbar = 0.1$ goes to 3×10^{-4}) Reading , and the influence of the worst end value also vanishes over a length almost independent of $\hbar$ (§ 02). This is because the rarity of configurations near the top ($P(z > -0.5) \approx 10^{-252}$ at $\hbar = 3 \times 10^{-4}$) Reading is paid automatically inside the transfer operator ("paying with rarity" of 24-06 § 06). On the four-dimensional side, if L is fixed, frame decorrelation is replaced by the single chessboard density κ_{cb} (an averaged quantity) (26-05). Estimates across scales are entrusted to one of the four faces (26-06 § 07) Assumption .

A first-step version of the averaging tool has been proposed down to a concrete form (§ 06). We set out, with proof, the uniqueness proposition at its entrance.

Hidden-Convexity Proposition. For any $\varepsilon > 0$, $c > 0$, $l \geq 1$ and end values $a, b \in [-c, 0]$, the action S (with the V of (3) and both ends fixed at a, b) has exactly one local minimum on the box $[-c, 0]^l$, and it is the global minimum.

Proved here Proof. Put $z_i = -\sqrt{\rho_i}$ ($\rho_i \in [0, c^2]$). (i) $(z_{i+1} - z_i)^2 = \rho_{i+1} + \rho_i - 2\sqrt{\rho_i \rho_{i+1}}$, and the geometric mean $\sqrt{xy}$ is concave on $[0, \infty)^2$, so $-\sqrt{\rho_i \rho_{i+1}}$ is convex. (ii) The end term $(z_1 - a)^2 = \rho_1 + a^2 - 2|a|\sqrt{\rho_1}$ is also convex, since $-\sqrt{\rho_1}$ is convex (likewise at the right end). (iii) $V(z) = (z^2 - 1)^2/8 = (\rho - 1)^2/8$ is strictly convex in ρ with second derivative $\frac{1}{4}$. Hence $\tilde{S}(\rho) := S(-\sqrt{\rho})$ is strictly convex on the convex set $[0, c^2]^l$ with $\nabla^2 \tilde{S} \geq (\varepsilon/4) I$, and its local minimum is unique and global. $\rho \mapsto -\sqrt{\rho}$ is a homeomorphism from $[0, c^2]^l$ to $[-c, 0]^l$, so the local minima correspond to each other. ∎

What the Hidden-Convexity Proposition says is the uniqueness of the minimum, not a Poincaré inequality. The Dirichlet form is written in the flat metric of z, so convexity in the coordinates ρ does not give Bakry–Émery as it stands. Writing the measure in ρ attaches $\prod_i (2\sqrt{\rho_i})^{-1}$ to the density, and the concavity of its logarithm breaks convexity only for $\rho_i \lesssim \sqrt{2\hbar/\varepsilon}$ (a thin layer very close to the summit) (comparing $\frac{\varepsilon}{4\hbar}$ with $\frac{1}{2\rho^2}$). One can read the non-convexity of the "summit-curvature shortfall wall" as moving from what looked, in the coordinate z, like lumps at the summit of length 8 or more, to rare thin layers site by site in the coordinate ρ Assumption (the argument that pays for this layer with rarity has not been built).

04

The content differs — the origin of the crossover length, rare and typical, phase transitions

Even with the same type, three points that decide the weight of the wall differ Reading (physical estimate).

(i) The origin of the crossover length. The crossover length of the first step is the first conjugate point (about 7.84 sites), a classical length fixed only by numbers written in the action S (the coefficient of the kinetic term and the curvature at the top). Throughout the limit $\hbar \to 0$ it stays finite in lattice units (§ 02). The length of the four-dimensional band is $1/\Lambda$, and the action has no ruler of length (in four dimensions the coupling is dimensionless, 24-07 § 01). The length is born by quantisation, in the one-loop bookkeeping in the form $\Lambda = \mu \exp(-1/(2b_0 g^2(\mu)))$ (dimensional transmutation, 4-07) Cited , and in lattice units $1/(a\Lambda)$ diverges like $e^{3\pi^2 \beta/11}$ as $\beta \to \infty$. All derivatives of $\exp(-1/g^2)$ vanish at $g = 0$, so this length appears at no order of the expansion in the coupling.

(ii) Whether bad configurations are rare or typical. At the first step, configurations near the summit are rare (the 10^{-252} of § 03), and the currency of "paying with rarity" of 24-06 works. In the four-dimensional band, the bad configurations (thick centre vortices, fractional instantons) are the mechanism of confinement itself, and are typical. The rate of rarity runs out in the band. This is the same point as the row of the source "no expansion linking scales" in 24-07 § 05 (in the infrared, configurations for which the Gaussian approximation fails become typical, and there is no currency of rarity).

(iii) Whether a phase transition is possible. The core of the first step is a one-dimensional nearest-neighbour chain over a compact one-site space. The transfer operator is a positive integral kernel, so there is no phase transition that would prevent estimates uniform in the length of the chain Assumption (a fact about the principal eigenvalue of positive integral kernels; the same bibliography as [001] of 23-15 Table 1, the original still to be obtained). At fixed β in four dimensions, an estimate uniform in the volume has the same weight as "there is no phase transition", and $U(1)$ actually has a transition inside the band ($\beta \approx 1.01$). An argument that closes the band must distinguish itself from $U(1)$: this is the criterion "distinguish from the U(1) transition" (26-06 § 08).

One decisive difference follows from (i). At the first step, if one chooses a **finite** block (length 12) longer than the crossover length, a two-stage road stands: analysis around a single minimum inside the block, and strong mixing between blocks (§ 06). In four dimensions the crossover length $1/\Lambda$ diverges in lattice units, so there is no finite block longer than it. **So we cannot expect the tools of the first step to cross the band as they are.** The "dimensionless coupling = no classical length" of 24-07 § 01 is the reason for this difference.

A supplement. Through the reduction lemma of 26-06 § 06, the inside of the band becomes "an estimate uniform in the volume at fixed β", a statement of the same form as the wall of the first step (an estimate uniform in the length k at fixed $\hbar$). Three things are common: a fixed parameter, uniformity in the volume, and passing through a non-convex region. That the weights still differ is because of (ii) and (iii). And after the reduction, the difficulty of crossing scales gathers at the seam between the upper end of the band and ultraviolet control (ibid. § 06). At the first step there is no flow of the coupling (among the sources of difficulty, the source "dimensionless coupling" is absent), so there is nothing corresponding to this seam.

05

The comparison table of the two walls

Table 1 gathers § 02– § 04 into six columns. The label of each cell is written inside the cell. Comparisons of type (readings across rows) are all estimates Reading .

Table 1 The wall of the first step and the four-dimensional band (columns (1)–(6))

	(1) Place and object	(2) Crossover length and its origin	(3) Tools that work at the two ends	(4) Tools that stop in
The first-step wall, the "summit-curvature shortfall wall"	The path measure of the double well in 0+1 dimensions ($\varepsilon = \frac{1}{2}$, $\hbar \to 0$). The Poincaré constant of the conditional measure on the core $[-x_c, 0]^k$, uniform in k, the end values and $\hbar$ (the core Poincaré uniformity assumption)	The first conjugate point at the summit, $\pi/\theta - 1 \approx 7.84$ sites, $\cos\theta = 1 - \varepsilon^2/4$. A classical length fixed by the action alone, independent of $\hbar$, finite in lattice units Proved here	Short lumps: convexity for $k \le 7$ (the Short-Core Convexity Lemma, $\kappa_{C7}(7) = 0.0545$) Proved here . Inside the wells: local Poincaré within the phase (the in-phase local Poincaré condition; labels per row in 23-06)	Curvature at the wc the rows of Dobrusl of length 7 (≈ 1.18),), Brascamp–Lieb a back to $\frac{15}{4}$ against 4 worst point is the fa
The four-dimensional band	The vortex free energy of the slab $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$ ($n_{12} = 1$) of the four-dimensional $SU(2)$ lattice. $\ln(1/\theta(L,L)) \ge cL^2$ uniformly in L (26-01)	$1/\Lambda$. The action has no length (the coupling is dimensionless); it is born by quantisation (dimensional transmutation) Cited . Diverges in lattice units as $\beta \to \infty$	The strong-coupling expansion ($u \le u_0 \approx 10^{-80}$, part of the bookkeeping sketched) Assumption . The one-loop comparison (with reshaped fluctuations, $\beta - \beta' = 0.258$) Proved here (one-loop level)	The worst frame: su gets worse superexp 04). Structural inpu for $\mathbb{Z}_2$ and $U(1)$ (26- comparison goes ag freedom (26-06). Th centre element -1

How to read Table 1. Reading columns (3)–(5) across, the type is the same (different tools at the two ends, the worst-case estimate stops in the middle, then averaging tools); comparing columns (2) and (6) down, the content differs (whether the origin of the length is classical or quantum, whether a phase transition is possible). The answer to the question "can the summit-curvature shortfall wall be written in the same words as the band?" is: **the type can be written in the same words, and the content (the origin of the crossover length) differs**.

06

A supplement to Part 23 — the road of blocks and mixing

The following road has been proposed as a candidate for supporting the core Poincaré uniformity assumption without going through the LSI constant A_W of the site window (23-09). As the third branch, "the road of blocks and mixing", of the fork of Part 23 (23-15 § 02), its table of rows and dependency graph are in the subsection on the road of blocks and mixing in 23-15. Here we write only the skeleton needed for the relation to the band.

1. **Block LSI** Assumption : on blocks of length $\ell_B \ge 12$ (avoiding the conjugate point ≈ 7.84 and the length 8 at which convexity breaks), uniform in the end values and $\hbar$. The entrance is the Hidden-Convexity Proposition (§ 03) Proved here ; the curvature at the minimiser for the worst end values $\lambda_*(12) \ge 0.255$ Reading (interval arithmetic for the minimisers at the four corners remains); and Witten's Morse theory together with the Menz–Schlichting decomposition, which derive from a single non-degenerate minimum that there are no exponentially small eigenvalues (the form at the corners of a Neumann box not yet checked).
2. **Strong mixing** Assumption : in three stages — the optimal path as $\hbar \to 0$ reaches the corner of the box within 7 steps from every starting point (the record is a 50-digit KKT certificate, the Hidden-Convexity Proposition and Topkis monotonicity; Reading until it is checked whether this is interval arithmetic with outward rounding), concentrates within a width $\approx 0.8\sqrt{\hbar}$, and has a Doeblin overlap in the boundary layer at the corner — one gets $\mathrm{TV}(n) \le (1-\delta)^{\lfloor n/10 \rfloor}$ with δ independent of $\hbar$. The second and third stages are along the lines of the constrained Laplace method.
3. **The standard theorem for one-dimensional chains** Assumption : from block LSI and strong mixing, an LSI uniform in the length of the chain (of the Zegarlinski and Martinelli–Olivieri type; the version for compact continuous spins not yet checked). Poincaré follows from LSI and reaches the core Poincaré uniformity assumption.

Where it stands. Even if the road of blocks and mixing closes, the first step does not close. The rows common to the two roads (the tube, the reduced kink gas, the cross term, the extension of the in-phase local Poincaré condition and others: the 13 common rows of 23-15 Table 1) remain on the road of blocks and mixing too. If a mild dependence on $\ln(1/\hbar)$ remains in the mixing length, the power statement ($\ln CP / \ln \xi \to 2$) is expected to be unaffected, but rows that need the constant to be uniform in $\hbar$ are affected (details in the subsection on the road of blocks and mixing in 23-15).

The relation to the band. The road of blocks and mixing is the first-step version of the stage "move to averaging tools" in the type of § 03. And as in § 04 (i), its key was that there is "a finite block longer than the crossover length". The four-dimensional band has no such finite block. Even if the road of blocks and mixing succeeds at the first step, we cannot say it gives a tool that crosses the band. In the form carried over to four dimensions (a box criterion with fixed boundary + strong mixing, the face $\delta(L_0)$), that "strong mixing" is precisely where $U(1)$ breaks Reading (physical estimate).

07

What cannot be said

(a) Neither half of the statement (1) that crosses the band has been shown in four dimensions. (2) is one line of algebra on top of (1).

(b) The comparison of the types of the two walls (§ 03, § 04, readings across the rows of Table 1) is an estimate, and we do not say that results at the first step imply anything in four dimensions. Only the form of the statement lifts (23-03 § 06).

(c) All we claim about the conjugate point is that the convexity threshold of (7) is fixed by the first conjugate point. The reason for the numerical agreement with the mixing length is not known.

(d) The rows of the road of blocks and mixing, apart from the Hidden-Convexity Proposition, are "Assumption" and "Reading". We do not say that the road of blocks and mixing closes the first step.

(e) That a one-dimensional chain has no phase transition is a statement resting on bibliography whose original has not been obtained.

(f) That four dimensions "has no finite block" is not a claim that the road of blocks is impossible in four dimensions. It means that the form of the first-step tools does not carry over as it is.

08

The table of the five grades and the dependency graph

Table 2 The rows of this chapter (the rows of this chapter have no Lean column)

Row	Statement (one line)	Grade	Basis	Scope
1	(1) $\Longrightarrow$ (2): $m_{\mathrm{lat}}/a \geq \sqrt{\kappa/C}$	Proved here	§ 01 (the same algebra as (16) of 23-01)	$\beta \geq \beta_0$
2	The one-sided bridge $CP \geq \kappa\xi^2$	Proved here	23-01 § 06	Continuous time, finite spatial lattice, the ground-state assumption, the stationary path-measure assumption, the metric comparison assumption
3	Lifting the one-sided bridge to the four-dimensional lattice	Assumption	24-07 Table 2	—
4	The upper-bound assumption: $a^2CP \leq C$, with C independent of β	Assumption	23-01 § 05	Four dimensions, $\beta \geq \beta_0$
5	Showing (1) with the same constant includes crossing the band	Reading	§ 01 (physical estimate)	—
6	The Jacobi equation at the summit (4) and $\cos\theta = 1 - \varepsilon^2/4$	Proved here	§ 02	$V = (z^2-1)^2/8$
7	The first conjugate point $\pi/\theta - 1 = 7.839\ldots$	Proved here	§ 02 (two independent counts)	$\varepsilon = \frac{1}{2}$
8	(7): $\kappa_{C7}(k) > 0 \iff k+1 < \pi/\theta \iff k \leq 7$	Proved here	§ 02, the short-core step of 23-11	$\varepsilon = \frac{1}{2}$, summit configuration
9	The crossover length does not depend on $\hbar$ and is finite in lattice units as $\hbar \to 0$	Proved here	§ 02 (no $\hbar$ in (4)–(7))	$\hbar \to 0$ with ε fixed
10	The numerical closeness of the mixing length ≈ 9 and π/θ (the mechanism is the time to reach the corner)	Reading	§ 02	$\hbar = 10^{-2}$–10^{-5}, four values of ε
11	The worst-configuration tools (Dobrushin, blocks, Otto–Reznikoff) come back to $\frac{15}{4}$ against 4	Proved here	§ 03	$\varepsilon = \frac{1}{2}$
12	The worst-frame estimate gets worse superexponentially in L	Proved here	25-04	With N and the lattice spacing fixed
13	The Hidden-Convexity Proposition (uniqueness of the minimum)	Proved here	§ 03	End values $a, b \in [-c, 0]$, box $[-c, 0]^l$
14	The non-convexity moves, in the coordinate ρ, to a thin layer $\rho \lesssim \sqrt{2\hbar/\varepsilon}$ (a reading)	Assumption	§ 03	—

Row	Statement (one line)	Grade	Basis	Scope
15	$1/\Lambda$ is born through dimensional transmutation and diverges in lattice units	Cited	4-07, 24-07 § 01	One-loop bookkeeping
16	The one-dimensional chain of the core has no phase transition	Assumption	§ 04 (positive integral kernel; the original still to be obtained)	Compact one-site space
17	The two walls are of the same type and differ in content (Table 1)	Reading	§ 03– § 05 (physical estimate)	—
18	The road of blocks and mixing: block LSI + strong mixing $\Longrightarrow$ the core Poincaré uniformity assumption	Assumption	§ 06, the third branch of 23-15	$\varepsilon = \frac{1}{2}$, core $[-x_c, 0]^k$
19	Four dimensions has no finite block longer than the crossover length, and the tools of the first step do not carry over as they are	Reading	§ 04, § 06 (physical estimate)	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  (1) ⟹ m_lat/a ≥ √(κ/C)
 2  PRV  One-Sided Bridge Proposition                    continuous time, finite spatial lattice
 3  ASM  lifting the bridge to four dimensions ⟸ 2
 4  ASM  upper-bound assumption (C independent of β)     four dimensions
 5  RDG  showing (1) with one constant crosses the band ⟸ 1, 3, 4
 6  PRV  Jacobi equation at the summit, cos θ = 1 − ε²/4
 7  PRV  first conjugate point π/θ − 1 ≈ 7.84 ⟸ 6          ε = ½
 8  PRV  convexity threshold k ≤ 7 = conjugate point ⟸ 6, 7
 9  PRV  crossover length independent of ħ, finite ⟸ 6, 7  ħ → 0 with ε fixed
10  RDG  numerical closeness of mixing length ≈ 9 and π/θ
11  PRV  worst-configuration tools come back to 15/4 vs 4 ⟸ 8
12  PRV  worst frame gets superexponentially worse in L (25-04)
13  PRV  Hidden-Convexity Proposition
14  ASM  non-convexity moves to thin layers in ρ ⟸ 13
15  CIT  1/Λ from dimensional transmutation, diverges in lattice units
16  ASM  the one-dimensional chain of the core has no phase transition
17  RDG  same type, different content ⟸ 9, 11, 12, 15, 16
18  ASM  road of blocks and mixing ⟹ core Poincaré uniformity ⟸ 13
19  RDG  no finite block in four dimensions ⟸ 9, 15, 18
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The "Proved here" nodes of this chapter are the one line of algebra, the linear computations at the summit (the Jacobi equation, the first conjugate point, the convexity threshold, the classical length, the worst-configuration tools), and the Hidden-Convexity Proposition; the statements that cross the band (the upper-bound assumption in four dimensions, and showing it with the same constant) are "Assumption" and "Reading". There is no node for the mass gap.

What this chapter can say

What is said	What is not said
Proved here That the lower half of the physical-unit target level follows in one line from the upper-bound assumption and the one-sided bridge with the same pair of constants	The upper-bound assumption and the bridge in four dimensions. The physical-unit target level itself
Proved here That the convexity threshold of the first step (length 7) is fixed by the first conjugate point $\pi/\theta - 1 \approx 7.84$ of the Jacobi equation at the summit and does not depend on $\hbar$. That the worst-configuration tools come back to $\frac{15}{4}$ against 4. The Hidden-Convexity Proposition	The core Poincaré uniformity assumption itself. Why the mixing length is close to the conjugate point
Reading That the two walls are of the same type, and differ in the origin of the crossover length (classical vs quantum), rare vs typical, and whether there is a phase transition (Table 1)	Implications of first-step results for four dimensions. A tool that crosses the band
Assumption That the road of blocks and mixing (block LSI + strong mixing) is a candidate for supporting the core Poincaré uniformity assumption	Proofs of each stage of the road of blocks and mixing (apart from the Hidden-Convexity Proposition)

Sources and reproduction

Item	Kind	Source / tools
The three levels of the goal, the upper-bound assumption, the one-sided bridge	This book	23-15 Table 5, 23-01 § 05, 23-01 § 06

Item	Kind	Source / tools
The Hessian at the summit and $\frac{15}{4}$ against 4, the Short-Core Convexity Lemma	This book	23-09 § 06, 23-11 § 05
The Jacobi equation, conjugate points (4)–(7)	Proved here	This book's study (the section on linearisation at the top in the record of proposals for the wall of Part 23). $\theta = \arccos(15/16)$ and π/θ are numerical, and were checked in this chapter by two independent checks, the sign change of the recursion and diagonalisation of the tridiagonal matrix (a computation of a few seconds)
The values of the three worst-configuration tools	Proved here	The same record (the section on worst-case analysis). $2(M_7^{-1})_{71}$ was recomputed in rationals in this chapter (1.184, against the rounded value 1.19 of the record)
Hidden-Convexity Proposition	Proved here	The same record (the section on the proof of the Hidden-Convexity Proposition). The proof was checked line by line in this chapter. Arguments of the Benguria–Brezis–Lieb and Brezis–Oswald type (originals not checked)
The correlation length of the transfer operator, the mixing length, the optimal path, $\lambda_*(12)$	Reading	The same record (the numerical section), and the arranged version of the road to a uniform LSI
The standard theorems of the road of blocks and mixing	Assumption (candidate for Cited)	The LSI for one-dimensional chains of B. Zegarlinski and F. Martinelli–E. Olivieri, Witten's Morse theory of B. Helffer–F. Nier and D. Le Peutrec, the decomposition of G. Menz–A. Schlichting (none of the originals checked yet)
Dimensional transmutation, the dimensionless coupling, the three sources of difficulty	Cited	The source tables of 4-07 and 24-07
Each cell on the four-dimensional side	This book	25-04, 26-04, 26-05, 26-06

Terms used in this chapter — the crossover band, the diffusive-relaxation-time assumption (the upper-bound assumption), the one-sided bridge, the Jacobi equation, conjugate point, hidden convexity, block LSI, strong mixing, the road of blocks and mixing, the criterion "distinguish from the U(1) transition". Their meanings are in A-02 Glossary.

Next chapter: 26-08 A map of the roads tried and stopped — a list of the roads tried and stopped, and why they stopped. The two points that work in four dimensions are in 24-07 Because it is four dimensions.

A map of the roads tried and stopped

> This chapter is a map of open questions. § 03 is established knowledge; § 04 is this book's own investigation.

What this chapter shows — We gather into one ledger the roads that were tried in order to cross the crossover band, and that stopped. The ledger has three layers: thirty roads (§ 02); roads discarded, or limited in scope, in the study of the band chapters (§ 05); and three ideas that turned out to be no more than restatements of roads in the ledger (§ 06). Each row carries "why it stopped" and its origin, and the label of the reason is set per row. For rows that rest on literature, only literature checked against the original carries [Cited]; literature not checked is marked "not yet checked". Three roads are looked at in detail: that the average over local containers stays at $R/\ln R$ in four dimensions (§ 03); the claim of Tomboulis (2007) by Migdal–Kadanoff interpolation, and the criticism that it does not distinguish U(1) (§ 03); and that the comparison principle with the twisted T^2 fails numerically (§ 04). Finally we set out the selection criteria that a new candidate should pass first (§ 07).

Prerequisite chapters — 26-00 (the breakdown of the band), 26-01 (the correct form of the band), 26-02 (the lateral-widening inequality), 25-03 (containers), 25-04 (slab confinement), 26-06 (the renormalisation comparison), 23-10 § 04 (the seven types of breakage).

Said first — This page is a map of the record and sets out no new theorem. "Stopped" means "it was found that this set of inputs cannot cross the band"; it does not mean that the parts of that road are of no use elsewhere (slab confinement, for example, is a theorem that holds for every β once the lattice spacing and T^2 are fixed). Most of the reasons for stopping are judgements at the level of physical estimates; negations with proofs are few. The labels are set per row to keep that distinction.

The order of this chapter

1. How to read the ledger — what "stopped" means and how labels are set
2. Thirty roads — why they stopped, and their origins (Table 1)
3. Two roads that had stopped in the literature — the average over local containers, and Migdal–Kadanoff interpolation

4. The comparison principle with the twisted T^2 — true at one loop, false numerically
5. Roads discarded, or limited in scope, in the band chapters (Table 2)
6. Three that were only restatements (Table 3), and the questions of type (Table 4)
7. The selection criteria and calibration against the literature (Table 6)
8. The table of the five grades and the dependency graph

01

How to read the ledger — what "stopped" means and how labels are set

The study in this book tried many roads in search of a tool that crosses the band. Every road stops in the end at the same single point: **there is no non-perturbative principle showing that, at intermediate coupling, disorder becomes stronger as the scale is enlarged**. Setting the ways of stopping side by side, three lessons come up again and again.

- Tools that work only in a perturbative regime (weak or strong coupling) cannot cross the band.
- Local operations cannot change topological quantities such as the twist sector.
- The sources of monotonicity (correlation inequalities, reflection positivity) work in the opposite direction to the mechanism of confinement (disorder).

The purpose of the ledger is not to repeat the same road. When a new candidate is thought of, we first check whether it is a restatement of something in this ledger (§ 06 gives examples).

The columns of each row are as follows.

- **Road**: a one-line summary of the idea tried.
- **Why it stopped**: where the road became unable to cross the band.
- **Label**: the label of the reason for stopping. Proved here means the reason has a proof on paper (set out in a chapter of this book); Cited means it rests on literature checked against the original; Computer-assisted is computer assistance; Reading is a numerical reading; "Reading (physics)" is a judgement at the level of a physical estimate. Even when the reason is an estimate, the judgement not to re-examine the road does not change. But a road stopped for a reason that is an estimate can be reopened if there is a new input.

- **Origin**: the record of this book's study, or the literature. Literature checked against the original is marked [Cited]; literature not checked is marked "not yet checked".

A note on words. In this part, following the convention of 26-00 § 02, we write "band", and distinguish it from the "window" of Parts 23 and 24 (the site window, the ħ window, the time window).

02

Thirty roads — why they stopped, and their origins

Numbers 1–28 in Table 1 are roads that this book's study tried in search of a principle that crosses the band; 29 and 30 are roads found to have already stopped in the literature when that study was re-checked by another hand. The labels are labels of "why it stopped".

Table 1 Thirty roads (labels are labels of why it stopped; "this book's study" is the record of the study in this book)

#	Road	Why it stopped	Label	Origin
1	Reflection positivity, chessboard estimates	The inequality points the wrong way, and only one side (the lower bound) of the flux free energy comes out	Reading (physics)	This book's study. The chessboard itself is in 26-05
2	Continuous monotonicity in the box size L (the Feynman–Hellmann derivative)	There is no correlation inequality for $SU(2)$ that fixes the sign of the difference	Reading (physics)	This book's study
3	Checking finitely many L by interval arithmetic	In four dimensions the size of the computation is out of reach. In the hierarchical model an interval-arithmetic certificate could be made (551 sectors, passed on re-running all sectors; a result for the hierarchical model only)	Reading (physics: a judgement of scale), Computer-assisted (hierarchical model only)	This book's study
4	Projection to $\mathbb{Z}_2$ and GKS (the splitting $SO(3) \times \mathbb{Z}_2$)	The premise of the splitting was rejected by four-dimensional data	Reading	This book's study
5	The infrared bound (Fröhlich–Simon–Spencer)	The wrong direction. It is the inequality on the side that shows a massless phase	Reading (physics)	Fröhlich–Simon–Spencer (not yet checked)

#	Road	Why it stopped	Label	Origin
6	Proof by contradiction extending the result for a small T^2 continuously in L	There is no quantity that excludes a fixed point, and it comes back to the band itself	Reading (physics)	This book's study
7	Kramers–Wannier duality	It only restates the band as a one-variable inequality (the magnetic flux free energy is smaller at scale $2L$ than at L)	Reading (physics)	This book's study (the duality itself not yet checked)
8	Local entropy and finite-energy arguments	Topological conservation: a local closed surface in $\mathbb{R}^4$ meets a plane an even number of times, so local operations do not change the twist sector	Reading (physics)	This book's study
9	Slab confinement	With the lattice spacing and T^2 fixed, the area law holds for every β. A bound uniform in N is the band itself, and the tension goes to 0 in the continuum limit	Proved here (the theorem with N fixed)	25-04
10	GKS for the $\mathbb{Z}_2$ effective model of centre vortices	Removing the frustration also removes the mechanism of confinement	Reading (physics)	This book's study
11	The Nishimori structure	**Withdrawn** (annealed and quenched were confused; negated numerically)	—	This book's study
12	The data-processing inequality	No channel independent of the sector can be built. It becomes a detector that tells sectors apart, but not a driving force that produces disorder	Reading (physics)	This book's study
13	Group averaging over thick vortices	It works on a small T^2, but on $\mathbb{R}^4$ it comes back to the problem of the tension of the vortex sheet	Reading (physics)	This book's study
14	Renormalising the worst flip cost	It becomes monotonically non-increasing, but decrease is not guaranteed	Reading (physics)	This book's study
15	Percolation of vortex sheets	Stochastic domination needs a worst-case lower bound, and reduces to strong coupling	Reading (physics)	This book's study

#	Road	Why it stopped	Label	Origin
16	Stability of the gap, Lieb–Robinson bounds	Works only inside the perturbative radius, and reduces to strong coupling	Reading (physics)	This book's study
17	Convexity in the coupling of the sheet	Only the cost of thin sheets is visible	Reading (physics)	This book's study
18	Adaptive averaging over the shape of the sheet	A rewriting of a partition of unity; a tautology	Reading (physics)	This book's study
19	Volume independence at large N_c	The band comes back, changed into the effective volume $N_c L$	Reading (physics)	This book's study
20	Positivity of the coefficients of the character expansion	It follows automatically from reflection positivity, and carries no new information	Reading (physics)	This book's study
21	Extracting monotonicity from the hierarchical model (ratio ordering, TP_2 and the like)	False in general	Reading	This book's study
22	Single-site Bakry–Émery curvature	The single-site gap is always positive; the wall lies in the influence between plaquettes	Reading (physics)	This book's study
23	Monotone quantities from strong subadditivity, a lattice version of the a-theorem	The lattice-only version does not distinguish phases. The version that does needs rotation invariance, which is available only after crossing the band (circular)	Reading (physics)	This book's study (Casini–Huerta-type arguments not yet checked)
24	Curvature of the space of gauge orbits, renormalisation of a membrane gas, tools for sharp phase transitions	The curvature correction is $O(1)$ and cannot cancel the negative $O(\beta)$ of the Hessian of the action, reducing to strong coupling / four-dimensional membranes have no screening mechanism / the surface representation of $SU(2)$ has weights that change sign, and no FKG	Reading (physics)	This book's study

#	Road	Why it stopped	Label	Origin
25	Spin and the index theorem (counting soft modes)	At one loop it is a restatement of asymptotic freedom, and the non-perturbative part comes back to the band	Reading (physics)	This book's study
26	Renormalisation of curvature (Bakry–Émery curvature and O'Neill's formula, the curvature-dimension condition of the quotient)	Negative curvature is not preserved by coarse-graining. Following the effective one-link weight on a one-dimensional chain by convolution, the minimum of the curvature gets worse at the first step ($SU(2)$: $-6 \to -14$; $U(1)$: $-4 \to -8$)	Reading	This book's study
27	The comparison principle with the twisted T^2 (for all loops)	True at one loop, false non-perturbatively by numerics (§ 04)	Reading	This book's study (§ 04)
28	The multiscale Bakry–Émery criterion (the Polchinski equation)	It can pass non-convex regions, but scale-by-scale control of the renormalised potential is the band itself. The version for compact groups is also not in place	Reading (physics)	Bauerschmidt–Bodineau (not yet checked)
29	Averaging over local vortex containers	The identity is known, and with the container length fixed it reaches only $R/\ln R$ in four dimensions (§ 03)	Cited	Mack–Petkova (1980), Kovács–Tomboulis hep-lat/0108017. 25-03 § 06
30	Monotonicity of the vortex free energy by Migdal–Kadanoff interpolation	The key inequality ((3.5) of Ito–Seiler; the interpolation-parameter commonality inequality) has not been shown outside strong coupling. The argument does not distinguish $U(1)$ (§ 03)	Cited	Tomboulis arXiv:0707.2179, Ito–Seiler arXiv:0711.4930, Kanazawa arXiv:0805.2742

N_c in Table 1 is the number of colours, a different quantity from the size N of T^2_N. Number 11 is kept here as a single line only. Its content was withdrawn, so it is not used elsewhere in this book.

Note that most labels in Table 1 are "Reading (physics)". Most of the reasons for stopping are not counterexamples or negative theorems but estimates of the form "this road comes back to the band itself at this point". The rows with proofs are number 9 (slab confinement is a theorem with N fixed, and is not uniform in N) and, in Table 2,

the stopped road "isometric comparison", the stopped road "structural inputs only", the stopped road "flip certificate" and the stopped road "supremum estimate over frames" (the scope is in the brackets of each row's label). Numbers 29 and 30 of Table 1 are the "Cited" rows that had stopped in the literature. The "Computer-assisted" of number 3 is a result on the hierarchical model and says nothing about the four-dimensional lattice (the hierarchical model is a different thing from four dimensions in that Migdal–Kadanoff decimation is exact there).

03

Two roads that had stopped in the literature — the average over local containers, and Migdal–Kadanoff interpolation

This section summarises existing knowledge (two roads that had stopped in the literature). The introduction is in 17-02 § 05.

Numbers 29 and 30 in Table 1 are roads that, after this book's study had tried them itself, were found to have already stopped in the literature. Both were checked against the text of the original [Cited].

The average over local vortex containers — only up to R/ln R in four dimensions

The identity that, with the frame (the outside of the container) fixed, the signs of the sectors inside the containers become independent and the Wilson loop is bounded by the product of the biases of the containers, was the Container Identity of 25-03 § 03. This is the "vortex container" identity of Mack–Petkova (1980), written in eqs. (18)–(22) of Kovács–Tomboulis (hep-lat/0108017). The only differences are that we wrote the sectors with the sum ensemble and took the average over the frame rather than the maximum; this book claims no novelty.

§ 3 of the same paper states the limit of this road in four dimensions. The length of a container passing through a point inside a rectangular loop with sides $T \times R$ ($T > R$) is of order R^2, and to keep the cost of the container constant the width of its cross-section must be fattened to order $\sqrt{\ln R}$. Only about $RT/\ln R$ independent containers fit inside the loop, and what comes out is

$$V(R) \;\geq\; \text{const} \cdot \frac{R}{\ln R} \qquad \text{(Kovács–Tomboulis, eq. (25))} \quad (1)$$

and no more [Cited]. A linear potential (the area law) needs a single fat container the size of the loop. This is exactly the global form of Tomboulis–Yaffe (26-01 § 04), the same condition as crossing the band.

In one line, the reason it stopped is: **the road of laying out containers of fixed length independently does not reach the area law on** $\mathbb{R}^4$. On a slab with a small T^2, the

length of the container (the area of the vortex sheet) stops at the area of T^2, so this obstacle does not appear (25-03 § 06). The obstacle comes back when T^2 is widened — that is, in the band.

Monotonicity of the vortex free energy by Migdal–Kadanoff interpolation

Tomboulis (arXiv:0707.2179) claimed confinement at every coupling for four-dimensional $SU(2)$. The flow of the argument is as follows [Cited].

1. By Migdal–Kadanoff-type decimation (lattice spacing $a \to ba$), build upper and lower bounds on the partition function ((3.4) and (3.7) of the original).
2. Interpolate their coefficients with a parameter t, and obtain a representation in which decimation becomes an equality at some value t^* ((3.12), (3.16), (3.35)). Build the same representation in the twisted sector ((4.19), (4.26)).
3. **The key step**: that the same t^* can be taken in the untwisted and the twisted sectors ((5.14)). This is to be derived from the inequality $A \geq A^+$ ((5.15)), and it is stated that showing it at strong coupling is enough.
4. In conclusion, the true vortex free energy is at most the vortex free energy of the decimated system ((5.22)). The decimated system flows to strong coupling, so the strong-coupling expansion gives $F \sim L\, e^{-\hat{\sigma} A}$, and the Tomboulis–Yaffe inequality gives the area law.

Against this there are two criticisms [Cited].

- **Ito–Seiler (arXiv:0711.4930)**: that t^* can be taken in common for the two sectors reduces to their inequality (3.5), $A(\alpha) > A^+(\alpha)$, which has not been shown outside strong coupling. Further, the Migdal–Kadanoff flow goes to strong coupling **regardless of whether the group is abelian or non-abelian**, so if the argument went through as it stands, $U(1)$ would also confine at every coupling. Four-dimensional $U(1)$ has a non-confining phase, so an input that distinguishes groups is needed somewhere.
- **Kanazawa (arXiv:0805.2742, Appendix A)**: the continuity argument in the interpolation parameter does not hold (a counterexample is given). It is not shown which step of the argument fails for $U(1)$. For any compact connected group the Migdal–Kadanoff flow goes to strong coupling, so the renormalisation side cannot distinguish $U(1)$ from $SU(2)$, and **all the non-abelian content goes into the existence of t^***.

Footnote 10 of the original itself writes that "for abelian systems, inequalities of type (5.15) follow from the Griffiths inequalities". Together with the remaining steps this

would show confinement for $U(1)$, a sign that the argument does not distinguish groups.

In one line, the reason it stopped is: **a renormalisation that does not distinguish groups cannot be an argument that closes the band**. This is an example in the literature of the criterion "distinguish from the U(1) transition" of 26-06 § 08 (show that $SU(2)$ has no transition, distinguishing it from the transition that $U(1)$ has inside the band, at $\beta \approx 1.01$). The lateral-widening inequality of this book (26-02 § 02) restricts the comparison to the form "change only the geometry, within one theory and one coupling", and has a form that fails for $U(1)$ (26-01 § 03, 26-09 § 02). But the difficulty of showing it is of the same type, and it is still needed across the band.

Relatedly, Theorem 1 of Kanazawa (arXiv:0805.2742) is an identity relating the ratio of twisted partition functions on lattices of different sizes to a ratio at shifted couplings, but it also holds for $U(1)$, so it says nothing about confinement [Cited]. Another paper of Ito–Seiler (arXiv:0803.3019) is not yet checked.

04

The comparison principle with the twisted T² — true at one loop, false numerically

This is number 27 of Table 1. **The comparison principle** is the claim that for every planar loop C

$$\langle W(C)\rangle_{\mathbb{Z}^4} \;\le\; \langle W(C)\rangle_{\mathbb{Z}^2\times T^2_N,\; n_{12}=1} \tag{2}$$

holds (the right-hand side is the twisted slab, 25-04 § 02). If it held, then together with slab confinement (the area law for every β with N fixed) it would give the area law for every β on $\mathbb{Z}^4$. That is why it was worth trying.

True at one loop [Reading]. Comparing with the heat kernel of fluctuations around the background, the inequality holds at each time. The colour weight of the sum over images is $c(n) = 4\cdot\mathbf{1}[n\in(2\mathbb{Z})^2] - 1$, and we checked this numerically to machine precision (a numerical check at the one-loop level; no proof on paper is set out).

False non-perturbatively [Reading]. By Monte Carlo we compared the twisted slab on the lattice $16\times16\times N_T\times N_T$ (the $\mathbb{R}^2$ directions periodic with 16, T^2 with N_T) with 16^4 ($SU(2)$, Wilson action).

Test of the comparison principle (plaquette expectation values; parentheses give the statistical error in the last digit)

β	N_T	**Twisted slab**	16^4	**Reading**

β	N_T	Twisted slab	16^4	Reading
2.1	2	0.52952(67)	0.53494(40)	The slab is lower: **7σ** taking errors from both sides, **5.5σ** in an independent rerun. $W(2,2)$ is also lower
1.8	2	0.43966(56)	0.44129(18)	The slab is lower (about 2.7σ)
2.4	2	0.6275(10)	0.6300(2)	The slab is lower (about 2.4σ)
2.6	6, 8	Signs that the twisted slab overshoots 16^4 downwards (2.5σ and 1.9σ in the plaquette)		Undetermined

Taking C in (2) to be a single plaquette, the right-hand side is the plaquette of the slab and the left-hand side the plaquette of 16^4. At $\beta = 2.1$, $N_T = 2$ the slab is 7σ lower, so (2) fails for small loops. This can be read as the frustration of the twist (the cost of forcing a twist into a small T^2) lowering the plaquette [Reading] (physics). The overshoot on the weak-coupling side ($\beta = 2.6$) lacks statistics and is not settled.

Further, at $N = 1$ it contradicts the scaling behaviour of asymptotic freedom [Reading] (physics). The slab theorem gives $\sigma_{\mathrm{tw}}(1) \geq -\ln\tanh\beta \approx 2e^{-2\beta}$, but if the scaling behaviour of asymptotic freedom is accepted, $\sigma_\infty a^2 \propto e^{-(6\pi^2/11)\beta} = e^{-5.38\beta}$, and at large β (2) points the other way.

In one line, the reason it stopped is: **an ordering that holds at one loop breaks down non-perturbatively, starting from small loops**. Two criteria come out of this (§ 07): the criterion "not a rewording of one loop" (it is a non-perturbative estimate) and the criterion "not broken by small loops" (it does not fail at short scales, or it has a form that claims only large loops). The asymptotic form for large loops only ($\sigma_{\mathrm{tw}}(L) \leq \sigma_\infty$) has not been refuted, but there is no mechanism to show it, and it hits the same wall as number 2 of Table 1 (there is no inequality for $SU(2)$ that fixes the sign of the difference).

One positive observation also remains [Reading] (physics). At weak coupling the twisted T^2 is close to infinite volume already from small N_T, and approaches it from the opposite side to the untwisted case. This fits the picture of adiabatic continuity, but there are signs that the approach is not monotone, a fact unfavourable to tools based on monotonicity.

05

Roads discarded, or limited in scope, in the band chapters

In the study of Parts 25 and 26 there are, apart from Table 1, roads that were discarded and roads whose usable scope was limited. Table 2 gathers them. The first five (1–5) are "roads discarded", the rest "roads limited in scope".

Table 2 Roads discarded, or limited in scope, in the band chapters (labels are labels of why it stopped)

#	Road	Why it stopped	Label	Chapter
1	**Isometric comparison**: isometric comparisons (the identity map, group translation, gauge transformation, decimation)	In a comparison that does not change the shape of the fluctuations, the contribution of the one-loop spin term $+4$ is exactly 0, and only $\beta' \geq \beta + 0.023$ (the anti-asymptotic-freedom direction) comes out	Proved here (one-loop level)	25-07, 26-06 § 03
2	**Structural inputs only**: structural inputs only (reflection positivity, duality, the transfer matrix, the centre identities)	They hold as they are for $\mathbb{Z}_2$ and $U(1)$ too, so they cannot distinguish groups (the negative lemma)	Proved here (in the form "from this set of inputs alone")	26-04
3	**Deterministic squeeze**: getting the area law by a squeeze between deterministic maps	The remainder (relative entropy) of the Transport Identity (the Gibbs variational principle) grows with the area of the vortex sheet, so the form $e^{-\sigma A}$ does not come out. It can be used for the qualitative goal $\Delta F(L, L) \to 0$	Reading (physics)	26-10 § 02
4	**Glassy squeeze**: a squeeze by the $\mathbb{Z}_2$ glass in the upper half of the band	For $\beta \gtrsim 2$ the mean effective $\mathbb{Z}_2$ coupling is 1.1–1.9, far above the pure $\mathbb{Z}_2$ transition point 0.44, and the glass is on the side of the ordered phase. The squeeze is loose	Reading	26-09 § 08, 26-10 § 03
5	**Flip certificate**: using flip certificates (flipping containers) and simultaneous flips to test the band	The flip probability a of a one-step kernel levels off at about 1/2, and $\theta \leq 1 - a$ gives only $\sigma \geq c/L^2$ (a weak form that $U(1)$ also satisfies)	Proved here ($\theta \leq 1 - a$), Reading (the levelling off of a)	25-06, 26-01 § 02
6	**Supremum estimate over frames**: bounds using the supremum over frames $\sup_\Phi$	Of the form $\tanh(\beta L^2)^{\ell^2}$, getting worse superexponentially in L. For uniformity only forms averaged over the frame can be used	Proved here	25-04 § 02
7	**Dobrushin-type criterion**: using the usual Dobrushin, Dobrushin–Shlosman and van den Berg–Maes criteria with frames	The continuous field follows the boundary, and the influence coefficients are close to 1	Reading (physics)	(literature not yet checked)

#	Road	Why it stopped	Label	Chapter
8	**β-monotonicity from easy parts**: taking β-monotonicity as the easy component, band = β-monotonicity + the renormalisation comparison	β-monotonicity turned out to be a problem of the same weight as the monotonicity in β of the string tension in lattice units (a correction)	Reading (physics)	26-06 § 02
9	**One-directional lateral widening**: restricting lateral widening to one direction only	Doubtful in view of the behaviour of the spatial string tension at high temperature (lattice measurements by Bali et al., not yet checked); restricted to the square shape only	Reading (physics)	26-02 § 02
10	**Transport by Wilson flow**: using the Wilson flow to transport vortex insertions	Negative numerically. The form recast as forward transport (Moser-type) is open	Reading	25-06
11	**Conformal frame**: the conformal frame $H^2 \times S^2$ (turning scale into position)	Negative on examination: hyperbolic nesting does not increase the number of containers, and the core stays in the same place	Reading (physics)	26-10 § 04
12	**Strong-coupling expansion only**: covering the band with the strong-coupling expansion alone	The expansion about flat surfaces stops at roughening ($\beta_R \approx 1.9$, not yet checked)	Reading (physics)	26-06 § 04

06

Three that were only restatements, and the questions of type

When thinking of a new candidate, we first check that it is not a restatement of something in the ledger. Table 3 lists three ideas that, when the band candidates were examined, were not made candidates but judged to be restatements of the ledger.

Table 3 Three ideas that were only restatements

#	Idea	What it was a restatement of	Label
1	**Approach of the holonomy to Haar**: that the law of the holonomy of large loops approaches the Haar measure (the boundary holonomy centralises under the Weyl measure)	The decay of the part odd under the centre (half-integer representations) is the Wilson area law itself, a restatement of the ring made by the column decomposition and the Tomboulis–Yaffe inequality. The even part (adjoint representations) has a perimeter law and does not help to distinguish groups	Reading (physics)

#	Idea	What it was a restatement of	Label
2	**Anomaly of the centre and the θ term**: consistency of the mixed anomaly between the centre and the θ term	A kinematic constraint (the criterion "dynamical" of § 07). At $\theta = 0$ it forces nothing	Reading (physics)
3	**Success in the hierarchical model**: taking success in the hierarchical model (a model in which Migdal–Kadanoff decimation is exact) as evidence for the band	In the hierarchical model $U(N)$ also confines, so it is no test of the criterion "uses non-commutativity essentially"	Reading (physics)

The study of detours in Part 23 (23-10 § 04) extracted seven types from failures on the double well in 0+1 dimensions. The individual detour plans do not overlap directly with four-dimensional candidates, but the questions of type apply to four-dimensional candidates as they are. Table 4 lists those questions.

Table 4 The seven types of breakage, and the questions to put to a new candidate

Type	One line	Question to put to the candidate
1	There is one source (the curvature at one site at the top)	On the side of the worst configuration, what is being paid?
2	Structural factors that enter as ratios have a ceiling	Does it change the defect set, or does it only move the factor of the ratio?
3	Paying for the same place twice	Have the places paid been drawn in the same figure?
4	Confusing currencies (the speed of exit and rarity)	Does the step that pays for rarity go through entropy?
5	The address of the difficulty	Which quantity does the difficulty rest on?
6	The mismatch between the direction of fluctuation and the soft direction	Is a vector being collapsed to a scalar?
7	Transfer	Is the sum an invariant (does another column become heavier by as much as the lightened column)?

The candidate rules of Part 24 (Table 3 of 24-08 § 02) can also be used here, not as exclusions but as a record of "which root they do not work on". No candidate has "works" written in the column of the root that connects the ultraviolet and the infrared (source: no expansion linking scales, of 24-07), and the "coupling window" (a word of Part 24, 24-06) that appears in the rarity class closed by one step of coarse-graining (rule of axiomatising scale + rule of the rarity class) was a rewording of that root. Anyone who says a band candidate works on the lack of an expansion linking scales must name an input different from every candidate.

> 07

The selection criteria and calibration against the literature

Turning the reasons for stopping inside out gives candidate conditions that a tool crossing the band should satisfy. Table 6 sets those conditions side by side with the ways of stopping they came from. All are candidates for necessary conditions, not sufficient conditions that guarantee crossing the band if satisfied [Reading] (physics). Especially heavy are the four "uses non-commutativity essentially", "not a rewording of one loop", "crosses the smallest band" and "distinguish from the U(1) transition".

Table 6 The selection criteria (with the ways of stopping they came from)

Criterion	Content	Way of stopping it came from
Stands on the lattice alone	Holds on the lattice alone (does not rely on rotation invariance)	Number 23 of Table 1
Distinguishes phases	Distinguishes phases (fails in a massless phase, or the limiting values differ)	Number 23 of Table 1
Does not rely on positivity	Does not rely on positivity or FKG	Number 24 of Table 1 (the surface representation of $SU(2)$ has weights that change sign)
Uses non-commutativity essentially	**Uses non-commutativity essentially**. A principle that gives the same conclusion in models with a massless phase, such as four-dimensional $U(1)$ or two-dimensional $O(2)$, is wrong	Number 30 of Table 1, "structural inputs only" of Table 2, "success in the hierarchical model" of Table 3
Compatible with asymptotic freedom	Compatible with asymptotic freedom	The contradiction at $N = 1$ in § 04
Handles topological sectors	Can handle topological sectors (twists)	Number 8 of Table 1
Not a rewording of one loop	Is **a non-perturbative estimate, not a rewording of one loop**	Numbers 25 and 27 of Table 1
Dynamical	Is dynamical (uses the action and the flow of the coupling). Kinematic constraints alone (duality, reflection positivity, anomalies, positivity of the partition function) cannot reach, since a non-confining phase can also exist without contradiction	"Structural inputs only" of Table 2, "anomaly of the centre and the θ term" of Table 3
Passes through non-convex regions	Can pass through non-convex (negatively curved) regions. Tools based only on convexity, log-concavity or lower bounds on curvature cannot be used in the band	Number 26 of Table 1

Criterion	Content	Way of stopping it came from
Crosses the smallest band	**Can cross the smallest band**: in two-dimensional asymptotically free models (the $O(N)$ model at large N, the Gross–Neveu model), with the ultraviolet cutoff removed. A monotone quantity that cannot cross this cannot cross four dimensions either	Calibration against the literature (below)
Not broken by small loops	Does not fail for small loops (short scales). Or has a form that claims only large loops	§ 04
Non-isometric linearisation	The linearisation of the comparison map is not isometric in the fluctuations	"Isometric comparison" of Table 2
Distinguish from the U(1) transition	**Shows that $SU(2)$ has no transition, distinguishing it from the transition that $U(1)$ has inside the band** ($\beta \approx 1.01$)	Number 30 of Table 1, 26-06 § 08

Behind the criterion "crosses the smallest band" is the fact that even for the two-dimensional $O(3)$ model, exponential decay at all temperatures at the lattice level (Polyakov's conjecture) is unsolved (as stated in the lecture notes of Peled–Spinka, arXiv:1708.00058; not yet checked). A tool that shows a uniform gap in four dimensions must at least have the power to show that.

Calibration against the literature. Where is the current frontier on each side of the band? Only literature whose checking is complete is marked [Cited].

- The strong-coupling side: Cao–Nissim–Sheffield (arXiv:2505.16585) showed, for $U(N)$ lattice Yang–Mills, the area law for all N at β below a $\beta_0(d)$ depending only on the dimension d [Cited]. The same authors, in arXiv:2509.04688, check the mass-gap condition of Durhuus–Fröhlich (1980) by methods of stochastic dynamics and obtain the area law in the 't Hooft regime ($\beta \lesssim c_d N$) [Cited]. Both are results on the strong-coupling side and do not reach β inside the band. This book's Lean also contains an area law only inside the strong-coupling window, and a vortex floor whose rate per unit area does not depend on the transverse size (20-06; the lattice spacing is fixed, and the prefactor remains). This does not reach the band either.
- The universality of the band: there is a paper that states explicitly that, for every version of the Gross–Neveu model, there is no proof of chiral symmetry breaking in the limit with the ultraviolet cutoff removed (Fabbri–Goller arXiv:2606.13075, not yet checked). The infrared analysis of two-dimensional models at large N (Kopper–Magnen–Rivasseau 1995, Kopper 1999) is with an ultraviolet cutoff (not yet checked).
- Computer-assisted renormalisation: a result showing with computer assistance the flow near the high-temperature fixed point in two-dimensional tensor renormalisation (Ebel–Kennedy–Rychkov arXiv:2506.03247, not yet checked) can be

seen as one of the current frontiers. It is far from the band of four-dimensional $SU(2)$.

In summary, **within what we have examined, there is no proof of mass generation in an asymptotically free theory in the limit with the ultraviolet cutoff removed**. The existence of the continuum limit of four-dimensional Yang–Mills was shown by the construction published on 2026-11-10 (at the level of the committee's assessment; Volume 2), but it does not show the mass gap. The band is not a hole peculiar to this book's road, but the unsolved core common to asymptotically free theories. This sentence, however, is a statement about what we have examined, and includes literature not yet checked.

A table that rearranges the reasons for stopping by type of argument is Table 2 of 16-11.

08

The table of the five grades and the dependency graph

The labels of each row of Tables 1–3 are placed in the tables. The rows this page sets out itself are the following fourteen.

Table 7 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The Container Identity with the frame fixed is the container identity of Mack–Petkova	Cited	Kovács–Tomboulis eqs. (18)–(22)	Differing only in the sum ensemble and the averaged form	—
2	Independent containers of fixed length give only up to $V(R) \gtrsim R/\ln R$ in four dimensions	Cited	Kovács–Tomboulis eqs. (23)–(25)	$\mathbb{R}^4$, independent containers	—
3	The key step of Migdal–Kadanoff interpolation reduces to the interpolation-parameter commonality inequality and has not been shown outside strong coupling	Cited	Ito–Seiler arXiv:0711.4930	The argument of Tomboulis arXiv:0707.2179	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	The Migdal–Kadanoff flow goes to strong coupling regardless of the group, and the renormalisation side does not distinguish $U(1)$ from $SU(2)$	Cited	Kanazawa arXiv:0805.2742, Appendix A	Any compact connected group	—
5	The comparison principle holds at one loop	Reading	§ 04 (the heat-kernel inequality at each time, numerically)	One-loop level	—
6	The comparison principle fails at $\beta = 2.1$, $N_T = 2$	Reading	The table in § 04	$16^2 \times 2^2$ against 16^4, 7σ (independent rerun 5.5σ)	—
7	The comparison principle at $N = 1$ contradicts the scaling behaviour of asymptotic freedom	Reading (physics)	§ 04	As long as the scaling behaviour of asymptotic freedom is accepted	—
8	Slab confinement holds for every β with N and the lattice spacing fixed, and is not uniform in N	Proved here	25-04	N fixed, lattice spacing fixed	The Lean column of 25-04
9	An isometric comparison gives only the direction $\beta' \geq \beta + 0.023$	Proved here	25-07	One-loop level	—
10	Structural inputs alone cannot distinguish groups	Proved here	26-04	"From this set of inputs alone"	—
11	The reasons for stopping of the remaining rows of Tables 1, 2 and 3	Reading (physics)	This book's study	The conditions of each row	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
12	In the hierarchical model a certificate can be made by interval arithmetic over finitely many sectors	Computer-assisted	Number 3 of Table 1	Hierarchical model only. Says nothing about the four-dimensional lattice	—
13	The frontier on the strong-coupling side is the area law at small β for all N	Cited	Cao–Nissim–Sheffield arXiv:2505.16585, 2509.04688	$\beta \le \beta_0(d)$, $\beta \lesssim c\,dN$	—
14	The thirteen selection criteria (Table 6) are candidates for necessary conditions on a tool that crosses the band	Reading (physics)	§ 07	Candidates for necessary conditions, not sufficient conditions	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  CIT  the Container Identity is known                          Kovács–Tombouli
 2  CIT  independent containers reach only R/ln R ⟸ 1              R⁴
 3  CIT  the key step of Tomboulis reduces to (3.5), unproved     outside strong
 4  CIT  the Migdal–Kadanoff flow does not distinguish groups     any compact con
 5  RDG  the comparison principle holds at one loop               one-loop level
 6  RDG  the comparison principle fails at β = 2.1, N_T = 2       7σ, rerun 5.5σ
 7  RDG  comparison principle at N = 1 contradicts asymptotic freedom   physic
 8  PRV  slab confinement (N fixed)                               25-04
 9  PRV  isometric comparison gives β' ≥ β + 0.023                25-07, one loop
10  PRV  structure alone does not distinguish groups              26-04
11  RDG  why the remaining rows stopped                           physical estima
12  CMP  certificate for the hierarchical model                   hierarchical mo
13  CIT  the frontier on the strong-coupling side                 small β
14  RDG  the selection criteria ⟸ 2, 4, 5, 6, 7, 9, 10, 11
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The row of the selection criteria is the only node on the conclusion side, and its label is "Reading (physics)". No row of this page shows either that the band can be crossed or that it cannot. A stopped road is only recorded as "cannot cross with this set of inputs"; a road that adds something to those inputs lies outside the ledger.

What this chapter can say

What is said	What is not said
Cited The average over local containers reaches only $R/\ln R$ in four dimensions. The argument by Migdal–Kadanoff interpolation has an unproved key step and does not distinguish groups	That the claim of Tomboulis (2007) is false (what can be said is that an unproved step remains)
Reading That the comparison principle fails for small loops ($\beta = 2.1$, $N_T = 2$, 7σ, rerun 5.5σ)	That the asymptotic form for large loops only is false
Proved here Slab confinement, the isometric comparison, the negative lemma for structure alone (under the conditions of each chapter)	That roads adding something to these cannot cross
Reading (physics) The reasons for stopping of the remaining roads, and the selection criteria	Proofs of most of the reasons for stopping. That a tool meeting the criteria crosses the band

Sources and reproduction

Item	Kind	Source / tools
The vortex container identity and its four-dimensional limit	Cited	G. Mack, V. B. Petkova, Ann. Phys. 125 (1980) 117. T. G. Kovács, E. T. Tomboulis, hep-lat/0108017 (Vortex waistlines and long range fluctuations), eqs. (18)–(25). Chapter 25-03
The claim by Migdal–Kadanoff interpolation	Cited	E. T. Tomboulis, arXiv:0707.2179 ((3.4), (3.7), (3.12), (3.16), (3.35), (4.19), (4.26), (5.14), (5.15), (5.22) and footnote 10 of the original)
Its criticism	Cited	K. R. Ito, E. Seiler, arXiv:0711.4930 ((3.5)). T. Kanazawa, arXiv:0805.2742 (Appendix A, Theorem 1). Ito–Seiler arXiv:0803.3019 not yet checked
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. T. Kanazawa, arXiv:0808.3442. Chapter 26-01
The frontier on the strong-coupling side	Cited	S. Cao, R. Nissim, S. Sheffield, arXiv:2505.16585, arXiv:2509.04688
The rest of the calibration against the literature	Not yet checked	Fabbri–Goller arXiv:2606.13075, Ebel–Kennedy–Rychkov arXiv:2506.03247, Bauerschmidt–Bodineau (Comm. Pure Appl. Math. 2021), Kopper–Magnen–Rivasseau (1995), Kopper (1999), Peled–Spinka arXiv:1708.00058, Fröhlich–Simon–Spencer, the measurements of the spatial string tension by Bali et al.
Test of the comparison principle	Reading	Numbers from this book's study ($SU(2)$, Wilson action, the twisted slab on $16 \times 16 \times N_T \times N_T$ and 16^4, plaquettes and small Wilson loops). The errors are values re-taken in a re-check (7σ) and an independent rerun (5.5σ)

Item	Kind	Source / tools
The thirty roads, roads discarded in the band chapters, restatements, the selection criteria	This book's study	The record of this book's study (the ledger of exclusions, and the examination of band candidates)

Terms used in this chapter — the ledger of exclusions, why it stopped, vortex container, the comparison principle, Migdal–Kadanoff decimation, the isometric comparison, the selection criteria, the crossover band (the band), β-monotonicity, the renormalisation comparison. Their meanings are in A-02 Glossary.

Next chapter: 26-09 The measured record — the numbers this part cites, on one page.

2026-10-15 · chapter Yang–Mills and the mass gap Part 26 The crossover band
vortex free energy, slab tension, the step function, the column bias · Kiichi, Shiori and Rin

The measured record

This chapter is this book's own investigation. The established knowledge it builds on is in 5-07 and 5-01.

What this chapter shows — We gather on one page the numbers that Part 26 cites. When the small twisted T^2 is widened, the vortex free energy of $U(1)$ rises and that of $SU(2)$ does not (§ 02). The string tension of the slab increases with T^2 and saturates, with no significant overshoot visible (§ 03). The β-dependence of the vortex free energy (§ 04). When the whole lattice is doubled, the $SU(2)$ vortex becomes cheaper and the $U(1)$ vortex costlier (§ 05). The sign of the step function of one step flips for $U(1)$ at the known phase transition $\beta_c \approx 1.01$, and does not flip for $SU(2)$ in the range measured (§ 06). The width needed to keep the column bias small with the outer frame held fixed hardly grows with N for $SU(2)$, and grows for $U(1)$ (§ 07). A diagnosis of the $\mathbb{Z}_2$ glass representation (§ 08).

Prerequisite chapters — 25-01 (the twisted T^2 and flux), 26-01 (the sector bias θ), 26-02 (the lateral-widening inequality), 26-06 (the renormalisation comparison), 5-07 (the Monte Carlo method).

Said first — Every number on this page is a numerical reading [Reading]. The lattices are small, up to 8^4; the points in β and the numbers of seeds (initial values of the random numbers) and frames are few; the errors are statistical only and may come out on the small side. No row is evidence of crossing the band; each is a record of a tendency, for choosing the direction of candidates. Each row states the lattice size, β, the number of seeds or frames, and the error. Rows for which the record does not state the number of seeds are marked "not recorded".

The order of this chapter

1. How we measured, and symbols — the vortex free energy, the sector bias, notes on artefacts
2. Widening the T^2 — $U(1)$ rises, $SU(2)$ does not (Tables 1–3)
3. The string tension of the slab — it increases and saturates (Tables 4 and 5)
4. The β-dependence of the vortex free energy (Table 6)

5. Doubling the whole lattice (Tables 7–10)
6. The sign of the step function — the $U(1)$ contrast (Table 11)
7. The column bias — the $\ell \times N$ scan (Tables 12–14)
8. A diagnosis of the $\mathbb{Z}_2$ glass (Table 15)
9. Artefacts of small lattices — a list of cautions for reading
10. The table of the five grades and the dependency graph

01

How we measured, and symbols

The lattice. The slab lattice is $(\mathbb{Z}/N_T)^2 \times (\mathbb{Z}/M)^2$; the first two directions (size N_T) are the small T^2, and the remaining two (size M) stand in for the plane $\mathbb{R}^2$. The T^2 carries the 't Hooft twist $n_{12} = 1$ (25-01). A symmetric torus is written N^4. The action is the Wilson action $-\frac{\beta}{2}\sum_p z_p \operatorname{tr} U_p$ ($SU(2)$) and its $U(1)$ version.

The vortex free energy. We write the free-energy difference with the sector in which the flux n_{34} in the planar directions is changed as

$$\Delta F = F(n_{34}=1) - F(n_{34}=0) = -\ln \frac{Z(n_{34}=1)}{Z(n_{34}=0)} \quad (1)$$

and set the sector bias to $\theta = \tanh(\Delta F/2)$ (the same quantity as F_v of 26-01 § 04). The larger ΔF, the costlier the vortex (the harder it is to enter), and θ approaches 1. In the confinement regime one expects $\Delta F \to 0$.

How we measured. ΔF is obtained by replica exchange (exchanging configurations between replicas with interpolated couplings to produce samples that span the sectors) and by Bennett's acceptance-ratio method (BAR). The errors are statistical. At some points the difference between two runs with different seeds is 1–2 times the stated error, so the errors may come out on the small side. The string tension of the slab is read from the Creutz ratio

$$\chi(R) = -\ln \frac{W(R,R)\,W(R-1,R-1)}{W(R,R-1)\,W(R-1,R)} \quad (2)$$

($W(R,T)$ is the $R \times T$ Wilson loop). The Creutz ratio at small R includes the Coulomb term and overestimates the tension, so we read only the tendency, not the absolute value.

Notes on artefacts (in advance). Cautions common to all the tables are listed in § 09. In particular, $N_T = 2$ is the regime of lattice artefacts, 2^4 is too small to see one-loop behaviour, and column-bias values of about 0.1 are at the statistical floor.

02

Widening the T² — U(1) rises, SU(2) does not

We hold the size M in the planar directions fixed and widen only the size N_T of the T^2. This is a test of the sign of the lateral-widening inequality (26-02 § 02).

Table 1 $\Delta F(N_T)$, $M = 8$ ($n_{12} = 1$, replica exchange + Bennett's acceptance ratio method; number of seeds not recorded) [Reading] — the numbers are the same measurement given in 25-08 Table 3. We do not copy them here.

For $U(1)$, widening the T^2 makes ΔF increase sharply, and θ goes $0 \to 0.39 \to 0.78 \to 0.99$. $SU(2)$ goes $0 \to 0.63 \to 0.33 \to 1.51$, much more gently ($\theta$ goes $0 \to 0.31 \to 0.16 \to 0.64$). Under $N_T = 4 \to 8$, $\ln(1/\theta)$ goes $0.93 \to 0.012$ for $U(1)$ (about 1/80), and $1.18 \to 0.45$ for $SU(2)$ (about 1/2.6). In this geometry, however, θ is of order 1 and the ceiling $\theta \leq 1$ weakens the power to discriminate; and $N_T = 8 = M$ for $SU(2)$ is the symmetric torus 8^4 (physical size $L\sqrt{\sigma} \approx 1.5$). $\beta = 1.5$ for $U(1)$ and $\beta = 2.5$ for $SU(2)$ are not matched to the same physical size.

Table 2 $\Delta F(N_T)$, $M = 12$ (remeasured with a larger container) [Reading] — the numbers are the same measurement given in 25-08 Table 4. We do not copy them here.

For $U(1)$ at $N_T = 8$, 24 replicas gave 0.54 ± 0.53 with poor mixing; we give the value with 40. The number of seeds is not recorded.

Table 3 $SU(2)$, $\beta = 2.4$, $M = 8$ [Reading] — the numbers are the same measurement given in 25-08 Table 5. We do not copy them here.

Reading [Reading]. In three settings ($M = 8$ and 12 at $\beta = 2.5$, $M = 8$ at $\beta = 2.4$), the vortex free energy of $SU(2)$ does not increase when the transverse T^2 is widened (except for the symmetric point $N_T = M = 8$ at $\beta = 2.5$, $M = 8$). For $U(1)$ it increases at both values of M. This is a numerical sign of the sign of the lateral-widening inequality. The errors are large, 0.1–0.5, and there are only two points in β. That $N_T = 8$ at $\beta = 2.4$ is smaller than the same point at $\beta = 2.5$ fits with the same lattice being physically larger at $\beta = 2.4$ ($L\sqrt{\sigma} \approx 2.2$ against 1.5).

03

The string tension of the slab — it increases and saturates

One of the necessary conditions for the uniform slab tension (26-02) is that the string tension of the slab $\sigma_{\mathrm{eff}}(N_T)$ does not overshoot much in N_T. We measured the Creutz ratio (2) for $SU(2)$, $\beta = 2.5$, $M = 16$. The four-dimensional value is $\sigma a^2 \approx 0.036$.

Table 4 Creutz ratios of the slab, 400 measurements ($SU(2)$, $\beta = 2.5$, $N_T \times N_T \times 16 \times 16$; number of seeds not recorded) [Reading] — the numbers are the same measurement given in 25-08 Table 6. We do not copy them here.

Table 5 The same setting with five times the statistics (2000 measurements) and $N_T = 10$ [Reading] — the numbers are the same measurement given in 25-08 Table 7. We do not copy them here.

Reading [Reading]. The peak in $\chi(4)$ at $N_T = 6$ seen with 400 measurements (0.089 ± 0.013, about 2.3σ from $N_T = 8$) became 0.061 ± 0.004 with 2000 measurements; it was a statistical fluctuation. $\chi(4)$ increases over $N_T = 4 \to 8$ and nearly saturates at $N_T = 10$ with 0.071 (1.7σ from $N_T = 8$). No significant overshoot is visible. $\chi(4)$ includes the Coulomb term, so it comes out larger than the four-dimensional $\sigma a^2 \approx 0.036$. We read only the tendency in N_T, and β is a single point.

04

The β-dependence of the vortex free energy

Table 6 $SU(2)$, $N_T = 4$, $M = 8$ ($4 \times 4 \times 8 \times 8$, $n_{12} = 1$, replica exchange + Bennett's acceptance ratio method, number of seeds not recorded) [Reading]

β	2.2	2.3	2.4	2.45	2.5	2.55	2.6	2.7
ΔF	0.05 ± 0.05	0.00 ± 0.05	0.31 ± 0.10	0.30 ± 0.13	0.63 ± 0.18	0.66 ± 0.13	1.43 ± 0.19	2.97 ± 0.33

Reading [Reading]. Within the errors, it increases monotonically in β. The reading is that this does not contradict β-monotonicity (26-06 § 02) (β-monotonicity has not been shown for $SU(2)$). For $\beta \le 2.3$ the vortex is nearly free ($\theta \approx 0$), and at $\beta = 2.6$–2.7 it becomes costly sharply. This can be read as the fixed lattice $4 \times 4 \times 8 \times 8$ becoming physically small and entering the dilute regime ($\theta \to 1$). The points at 2.45 and 2.55 are interpolation points added for the fit in § 05.

05

Doubling the whole lattice

We test the renormalisation comparison (26-06 § 05) directly. We measure ΔF on the fine lattice (all directions doubled) and read off, from the curve of the coarse lattice

(Table 6 and others), the β' of the coarse lattice that gives the same value. The prediction of asymptotic freedom is $\beta' < \beta$ for $SU(2)$, and $\beta' \geq \beta$ for $U(1)$, which does not run.

Table 7 Slab $(N_T, M) = (4, 8) \to (8, 16)$ $(n12 = 1)$ [Reading]

Group	β	$\Delta F(4, 8, \beta)$	$\Delta F(8, 16, \beta)$: per seed	Combined	Seeds	Matching β'	$\beta - \beta'$
$SU(2)$	2.7	2.97 ± 0.33	0.65 ± 0.39, 1.31 ± 0.48 (the second run has a minimum exchange acceptance rate of 0.10)	0.92 ± 0.30 (error-weighted mean)	2	≈ 2.57 (2.45–2.65 including the error of the curve)	$\approx +0.13$ ($+0.05$–$+0.25$)
$SU(2)$	2.6	1.43 ± 0.19	-0.57 ± 0.46, -0.41 ± 0.51	-0.50 ± 0.34 (consistent with 0)	2	$\lesssim 2.45$	$\gtrsim +0.15$
$SU(2)$	2.5	0.63 ± 0.18	0.41 ± 0.34	—	1	2.2–2.5	0–$+0.3$
$U(1)$	1.5	0.83 ± 0.11	2.32 ± 0.45 (48 replicas, minimum exchange acceptance rate 0.37)	—	1	> 1.7 (outside the curve)	< -0.2

The coarse-lattice curve for $U(1)$ is $\Delta F(4, 8, \beta') = 0.42, 0.55, 0.83, 0.92, 1.46$ ($\beta' = 1.3, 1.4, 1.5, 1.6, 1.7$). The first run for $U(1)$ (32 replicas) gave -0.18 ± 0.12, but the minimum exchange acceptance rate was 0.03 and the mixing poor, so it is not used.

Reading [Reading]. For $SU(2)$, doubling lowers the vortex free energy ($2.97 \to 0.92$ at $\beta = 2.7$), and the matching β' is smaller than β. For $U(1)$ it rises ($0.83 \to 2.32$, about 3σ), and $\beta' > \beta$. The direction of the comparison separates between the two groups. The $+0.13$ at $\beta = 2.7$ is somewhat smaller (about 1.5σ) than the one-loop 0.258 and than the roughly 0.23 expected from the known values of $a\sqrt{\sigma}$. The record notes that the reversed direction for $U(1)$ may also include artefacts of the small lattice.

Next, on the symmetric torus N^4, we look at the two steps $2^4 \to 4^4$ and $4^4 \to 8^4$.

Table 8 $SU(2)$, $\Delta F(N^4, \beta)$ on the symmetric torus ($n12 = 1$, one seed each) [Reading]

β	1.0	1.25	1.5	1.75	2.0	2.25	2.5	2.75	3.0
2^4	0.034	0.094	0.213	0.573	1.39	2.71	4.29	5.95	7.88
4^4	0.01	−0.01	0.01	0.01	−0.03	0.50	2.46	5.10	7.33

The errors are 0.01–0.08 at 2^4 and 0.03–0.26 at 4^4. The β' of the single step $2^4 \to 4^4$ (fitting the 4^4 values to the 2^4 curve) is as in Table 9.

Table 9 The single step $2^4 \to 4^4$ for $SU(2)$ [Reading]

β	≤ 2.0	2.25	2.5	2.75	3.0
β'	< 1.0 (outside the curve)	1.72	2.20	2.62	2.93

β	≤ 2.0	**2.25**	**2.5**	**2.75**	**3.0**
$\beta - \beta'$	> 0.75–1	0.53	0.30	0.13	0.07

At every readable β (1.75–3.0), $\beta - \beta' > 0$; that is, the step function of one step has no fixed point. The motion per step grows towards strong coupling; that it falls to 0.07 on the weak-coupling side is seen as an artefact of 2^4 being too small for the one-loop regime.

Table 10 $4^4 \to 8^4$ for $SU(2)$. ΔF is given per seed and as their simple mean (the error is half the difference between seeds; for three runs, the mean and the spread) [Reading]

β	$\Delta F(4^4)$: per seed → combined	$\Delta F(8^4)$: per seed → combined	Seeds (8^4)	$\beta - \beta'$
2.1	—	-0.04 ± 0.17	1	— (≈ 0)
2.3	—	-0.04 ± 0.10	1	$\gtrsim 0.1$
2.5	2.46 (1 run)	0.48 ± 0.48	1	$\approx +0.25$ ($+0.19$–$+0.5$)
2.6	3.96 ± 0.13 (1 run)	—	—	—
2.7	4.96, 4.47 → 4.72 ± 0.25	3.57, 2.65 → 3.11 ± 0.46	2	$+0.16$ ($+0.13$–$+0.19$)
2.75	5.10, 5.26 → 5.18 ± 0.10	—	—	—
2.8	5.33, 5.87 → 5.60 ± 0.27	—	—	—
2.9	6.88, 6.23 → 6.56 ± 0.33	5.68, 3.52, 4.84 → 4.68 ± 0.63	3	$+0.20$ ($+0.12$–$+0.29$)
3.0	7.33, 7.83 → 7.58 ± 0.25	5.38, 6.17 → 5.78 ± 0.40	2	$+0.18$ ($+0.14$–$+0.25$)

Reading [Reading]. For $4^4 \to 8^4$ too, $\beta - \beta' > 0$ at every β measured, with a mean of about $+0.2$. This is close to the one-loop asymptotic value 0.258 and to the roughly 0.23 expected from the known values of $a\sqrt{\sigma}$. At some points the difference between seeds is 1–2 times the stated error (at $\beta = 2.9$, 8^4 gives 5.68 and 3.52), and we read the errors as quite likely on the small side. The reading with added seeds came closer to the asymptotic behaviour than the single-seed reading ($+0.08$–$+0.13$). It also fits the reading of the 8×16 slab at $\beta = 2.7$ (Table 7, $+0.13$).

06

The sign of the step function — the U(1) contrast

We carried out the same $2^4 \to 4^4$ scan for $U(1)$. Four-dimensional $U(1)$ has a phase transition at $\beta_c \approx 1.01$ from the confining phase to the Coulomb phase.

Table 11 $U(1)$, $\Delta F(N^4, \beta)$ on the symmetric torus and the single step $2^4 \to 4^4$ ($n_{12} = 1$, one seed each, errors not recorded) [Reading]

β	**0.6**	**0.8**	**0.9**	**1.0**	**1.1**	**1.2**	**1.4**	**1.6**
2^4	0.14	0.62	1.15	1.80	2.46	3.10	4.18	5.23

β	0.6	0.8	0.9	1.0	1.1	1.2	1.4	1.6
4^4	0.00	0.00	0.05	1.07	2.71	3.68	4.46	5.85
$\beta - \beta'$	> 0.3	> 0.3	> 0.3	$+0.12$	-0.04	-0.11	-0.05	< 0

Reading [Reading]. The sign of $\beta - \beta'$ for $U(1)$ flips between $\beta = 1.0$ and 1.1. This is the position of the known phase transition $\beta_c \approx 1.01$. In the confining phase $\beta' < \beta$, and in the Coulomb phase $\beta' > \beta$. For $SU(2)$, the same scan gave $\beta - \beta' > 0$ at every β measured (Tables 9 and 10). The sign of the step function of one step works as a diagnostic that tells whether there is a phase transition. This is the numerical face of the criterion "distinguish from the U(1) transition" (26-06 § 08), which an argument that closes the band must satisfy.

What this sign test can say is only that, on lattices up to 8^4 and at the β points of the table, the sign for $SU(2)$ did not change. It is not evidence that $SU(2)$ has no phase transition.

Relatedly, there are two readings [Reading] concerning the intermediate goal of seeing $\Delta F(L, L) \to 0$ at any rate (26-06 § 08). For $U(1)$ ($\beta = 1.2$), $2^4 \to 4^4$ gives $3.10 \to 3.68$, flat, which does not contradict the picture of heading towards the classical half-flux value in the Coulomb phase. For $SU(2)$ ($\beta = 2.75$), it falls, $5.95 \to 5.10$.

07

The column bias — the ℓ × N scan

We measure the column face $f(\ell)$ (26-10 § 02). The outer frame of the column $T^2_N \times \ell \times \ell$ is taken from an equilibrium configuration and held fixed; for each frame, the conditional free energy $\Delta F_B = \ln Z_B(+)/Z_B(-)$ when a sheet of n_{34} is put inside the column, and the bias $\theta_B = \tanh(\Delta F_B/2)$, are obtained by replica exchange + Bennett's acceptance ratio method, and we take $\mathbb{E}|\theta_B|$ over the frames. It is this $\mathbb{E}|\theta_B|$ that enters the Chessboard Estimate (26-05). $SU(2)$ is at $\beta = 2.3$, and $U(1)$ at $\beta = 1.3$ in the Coulomb phase.

Table 12 Varying ℓ at $N = 4$ (numbers of frames in parentheses) [Reading]

ℓ	$SU(2)$, $\beta = 2.3$: $\mathbb{E}\lvert\theta_B\rvert$	$U(1)$, $\beta = 1.3$: $\mathbb{E}\lvert\theta_B\rvert$
2	0.93 ± 0.06 (5)	—
3	0.58 ± 0.17 (4)	0.95 ± 0.05 (4)
4	0.23 ± 0.04 (12)	0.61 ± 0.21 (4)
5	0.07 ± 0.03 (12)	0.49 ± 0.09 (12)
6	0.09 ± 0.04 (4)	0.41 ± 0.06 (12)

ℓ	$SU(2)$, $\beta = 2.3$: $\mathbb{E}\lvert\theta_B\rvert$	$U(1)$, $\beta = 1.3$: $\mathbb{E}\lvert\theta_B\rvert$
8	—	0.25 ± 0.05 (8)

For $\ell \geq 5$, the column bias of $SU(2)$ is 1/5 to 1/7 of that of $U(1)$. $U(1)$ decreases slowly (about $\ell^{-1.4}$ over $\ell = 5 \to 8$), while $SU(2)$ drops to 1/3 over $\ell = 4 \to 5$. The values of $SU(2)$ for $\ell \geq 5$, however, are close to the floor made by the per-frame statistical errors (which bias $|\theta_B|$ positive), and at these sizes one cannot decide whether it is $e^{-\sigma\ell^2}$ or a power. The sign can change from frame to frame (for $SU(2)$ at $\ell = 4$, the mean of ΔF_B is -0.05). In frames where the holonomy around the block is close to -1, the sector with the sheet is favoured.

Table 13 Varying N at $\ell = 5$ (numbers of frames in parentheses) [Reading]

N	$SU(2)$, $\beta = 2.3$	$U(1)$, $\beta = 1.3$
4	0.07 ± 0.03 (12)	0.49 ± 0.09 (12)
6	0.20 ± 0.05 (6)	0.97 ± 0.01 (6)
8	0.39 ± 0.05 (5; the sixth frame stopped partway at the time limit)	—

With ℓ fixed, the bias approaches 1 with N for both groups. While the bias is small, it grows roughly in proportion to N^2 ($\Delta F_B \propto N^2 f(\ell)$), and the proportionality coefficient for $SU(2)$ is about 1/7 of that for $U(1)$. The difference between the groups should appear in how the ℓ needed to keep θ small grows, so we varied both ℓ and N.

Table 14 The $\ell \times N$ scan: $\mathbb{E}\lvert\theta_B\rvert$ (numbers of frames in parentheses) [Reading]

	$\ell = 4$	**5**	**6**	**7**	**8**	**10**
$SU(2)$, $\beta = 2.3$, $N = 4$	0.23 (12)	0.07 (12)	0.09 (4)	0.11 (4)		
$SU(2)$, $N = 6$	0.43 (4)	0.20 (6)	0.15 (4)	0.08 (4)		
$SU(2)$, $N = 8$	0.43 (4)	0.38 (5)	0.13 (4)	0.29 ± 0.06 (3)		
$U(1)$, $\beta = 1.3$, $N = 4$	0.61 (4)	0.49 (12)	0.41 (12)		0.25 (8)	0.27 (4)
$U(1)$, $N = 6$		0.97 (6)	0.98 (4)		0.51 (4)	0.45 (4)

Reading [Reading]. For $SU(2)$, at every one of $N = 4, 6, 8$, the bias falls to the statistical floor (about 0.1) at $\ell \approx 6$–7, and the needed ℓ hardly grows with N. For $U(1)$, even at $N = 4$ it stays at 0.27 at $\ell = 10$, and at $N = 6$ it is 0.45 even at $\ell = 10$; the needed ℓ grows with N, and the fall is slower. The 0.29 at $N = 8$, $\ell = 7$ comes from only 3 frames (cut off at the time limit); that it is larger than the 0.13 at $\ell = 6$ we take as a fluctuation due to the small number of frames (all 3 frames have $\Delta F_B > 0$, 0.43–0.85). This does not contradict the expectation ($\ell \sim \sqrt{\ln N}$ for $SU(2)$, $\ell \propto N$ for $U(1)$), but the lattices are small and the frames few, so the form of the growth cannot be determined.

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08
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A diagnosis of the Z_2 glass

Writing $U = \sigma V$ ($\sigma \in \mathbb{Z}_2$, V a section of $SO(3)$), the twisted sector becomes "a $\mathbb{Z}_2$ random-plaquette gauge model on an $SO(3)$ background" (26-10 § 03). The effective $\mathbb{Z}_2$ coupling per plaquette is $K_p = (\beta/2)|\mathrm{tr}\, U_p|$, and the frustration (a $\mathbb{Z}_2$ monopole) is a cube c for which $\prod_{p \in \partial c} \mathrm{sign}\, \mathrm{tr}\, U_p = -1$.

Table 15 $SU(2)$, periodic 8^4, heat bath with 150 thermalisation sweeps and 100 measurements (number of seeds not recorded, errors not recorded) [Reading]

β	$\langle P \rangle$	Fraction with $\mathrm{tr}\, U_p < 0$	$\langle K \rangle$	Fraction with $K < K_c = 0.4407$	Frustrated cubes	p_{eff}
1.0	0.243	0.297	0.455	0.494	0.481	0.209
1.5	0.363	0.204	0.741	0.282	0.452	0.162
2.0	0.501	0.109	1.120	0.149	0.349	0.091
2.3	0.602	0.051	1.437	0.081	0.216	0.045
2.5	0.653	0.029	1.660	0.051	0.140	0.027
2.7	0.686	0.018	1.870	0.034	0.093	0.017

p_{eff} is the frustration density ρ of the cubes, converted to a probability of independent sign flips by $(1 - 2p_{\mathrm{eff}})^6 = 1 - 2\rho$. K_c is the transition point of pure $\mathbb{Z}_2$ gauge theory, and the value to compare with is the Nishimori multicritical point $p_c \approx 0.11$ of the four-dimensional random-plaquette gauge model (checking not yet done).

Reading [Reading]. For $\beta \gtrsim 2.0$, the mean coupling $\langle K \rangle$ is 3–4 times K_c, and $p_{\mathrm{eff}} < 0.11$. The $\mathbb{Z}_2$ glass is likely on the side of the ordered phase. For $\beta \lesssim 1.5$, $p_{\mathrm{eff}} \geq 0.16$, and soft plaquettes ($K < K_c$) are 28–49%: the disordered side. The conversion to independent signs is crude (soft plaquettes and frustration are correlated), so the boundary is a rough guide. From this diagnosis, the glass road is limited to the lower half of the band (the stopped road "glassy squeeze" of 26-08 § 05).

09

Artefacts of small lattices — a list of cautions for reading

- **$N_T = 2$ (and 2^4) is the regime of lattice artefacts.** At $N_T = 2$ one side of the T^2 is two lattice sites, the frustration of the twist is large, and it is far from the continuum picture (a gas of fractional instantons). That the $\beta - \beta'$ of the single step $2^4 \to 4^4$ falls to 0.07 on the weak-coupling side (Table 9) is also seen as due to 2^4 being too small for the one-loop regime. Values at $N_T = 2$ are kept as "a reference for the direction".

- **The ceiling of θ.** In geometries where θ is of order 1 ($M = 8$ of Table 1), $\theta \leq 1$ holds the ratio down and weakens the power to discriminate between the groups.
- **The statistical floor.** The column bias $\mathbb{E}|\theta_B|$ is biased positive by the per-frame statistical errors. Values of about 0.1 are close to the floor, and one cannot decide between an area exponent and a power (Tables 12 and 14).
- **The Coulomb term.** The Creutz ratio at small R includes the Coulomb term and overestimates the tension (Tables 4 and 5).
- **Poor mixing.** Runs with a small minimum replica-exchange acceptance rate (replicas 24 and 32 for $U(1)$, the 0.10 of the second $SU(2)$ run) fluctuate strongly in value. Runs we did not use are noted under the tables.
- **Number of seeds and errors.** There are 1–3 seeds. At some points the difference between two runs with different seeds is 1–2 times the stated error, so the errors are quite likely on the small side.
- **How β is matched.** The β of $U(1)$ and of $SU(2)$ are not matched to the same physical size. What we compare is only the direction (sign) within each group.
- **A withdrawn measurement.** We have withdrawn reading the measurement that flips containers simultaneously as "the numerical face of the band". The flip probability of the one-step kernel levels off at about 1/2, and what comes out is only the weak form, which $U(1)$ also satisfies (the stopped road "flip certificate" of 26-08 § 05). The measurement itself is in 25-08.

10

The table of the five grades and the dependency graph

All the rows of this page are numerical readings. The label is "Reading", with " (physics)" added for readings that include a physical estimate.

Table 16 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	Widening the T^2 increases ΔF for $U(1)$ and not for $SU(2)$	Reading	Tables 1–3	$M = 8, 12$, $\beta = 2.4$, 2.5 ($SU(2)$), $\beta = 1.5$ ($U(1)$), $N_T \leq 8$	—
2	The slab $\chi(4)$ increases with N_T and saturates, with no significant overshoot	Reading	Tables 4 and 5	$\beta = 2.5$, $M = 16$, 2000 measurements, $N_T \leq 10$	—
3	ΔF increases in β	Reading	Table 6	$4 \times 4 \times 8 \times 8$, $\beta = 2.2$–2.7	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
4	Doubling the whole lattice: $\beta' < \beta$ for $SU(2)$, $\beta' > \beta$ for $U(1)$	Reading	Tables 7–10	$(4,8) \to (8,16)$, $2^4 \to 4^4$, $4^4 \to 8^4$, 1–3 seeds	—
5	$\beta - \beta'$ for $4^4 \to 8^4$ averages about $+0.2$, close to the one-loop 0.258	Reading	Table 10	$\beta = 2.5$–3.0, 1–3 seeds	—
6	The sign of the single step for $U(1)$ flips between $\beta = 1.0$ and 1.1; for $SU(2)$ it does not flip in the range measured	Reading	Tables 9 and 11	$2^4 \to 4^4$, one seed	—
7	The ℓ that keeps the column bias small hardly grows with N for $SU(2)$, and grows for $U(1)$	Reading	Tables 12–14	$N \le 8$, $\ell \le 10$, 3–12 frames	—
8	The $\mathbb{Z}_2$ glass is on the ordered side for $\beta \gtrsim 2$ and on the disordered side for $\beta \lesssim 1.5$	Reading (physics)	Table 15	8^4; the conversion to independent signs is crude	—
9	The measurements 1–7 do not contradict the directions of the lateral-widening inequality, the renormalisation comparison and the criterion "distinguish from the U(1) transition"	Reading (physics)	§ 02–§ 07	Tendencies on small lattices. Not evidence	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
1  RDG  widening T²: U(1) rises, SU(2) does not                M = 8, 12
2  RDG  slab tension increases and saturates                   β = 2.5, 2000 m
3  RDG  ΔF increases in β                                      4×4×8×8
4  RDG  doubling test: SU(2) β' < β, U(1) the opposite ⟸ 3     1–3 seeds
5  RDG  β − β' ≈ +0.2 for 4⁴ → 8⁴ ⟸ 4                         1–3 seeds
6  RDG  sign of the step function: U(1) flips at β_c ⟸ 4       2⁴ → 4⁴
7  RDG  ℓ × N of the column bias                               N ≤ 8, ℓ ≤ 10
8  RDG  diagnosis of the Z₂ glass                              8⁴
9  RDG  consistency of directions ⟸ 1, 2, 4, 6, 7              tendency
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. This page has no "Proved here", "Cited" or "Assumption" rows. Nor does it have nodes for the statements that cross the band (the uniform slab tension, the lateral-widening inequality, the renormalisation comparison); those are placed as assumptions in the ledger of conditions of Part 27 (27-02).

What this chapter can say

What is said	What is not said
Reading That on small lattices the directions of $SU(2)$ and $U(1)$ separate in every test: widening the T^2, doubling the lattice, widening the column	Behaviour on large lattices. Determining the values. The lateral-widening inequality and the renormalisation comparison themselves
Reading That the sign of the step function for $U(1)$ flips at the position of the known transition, and that for $SU(2)$ it does not flip in the range measured	That $SU(2)$ has no phase transition
Reading (physics) That the $\mathbb{Z}_2$ glass is on the ordered side in the upper half of the band	The position of the phase boundary of the glass (the conversion is crude)

Sources and reproduction

Item	Kind	Source / tools
All the tables	Reading	The record of numbers from this book's study (numerical computations of vortex insertion: the global flux free energy, the string tension of the slab, the β-dependence, tests of the renormalisation comparison, scans of the step function on the symmetric torus, the $U(1)$ contrast, the added $4^4 \to 8^4$ runs and the combining of seeds, the twist cost of a column, the diagnosis of the $\mathbb{Z}_2$ glass), and the numerical section of the survey of roads that would close the crossover band. Replica exchange and Bennett's acceptance ratio method, heat bath, Creutz ratios
The phase transition $\beta_c \approx 1.01$ of $U(1)$	Reading	Known numerical lattice values (5-06)
The Nishimori multicritical point $p_c \approx 0.11$	Checking not yet done	Numerical literature on the four-dimensional random-plaquette gauge model
The one-loop 0.258	Proved here	Chapter 25-07 (one-loop level)

Terms used in this chapter — the vortex free energy ΔF, the sector bias θ, the Creutz ratio, replica exchange, Bennett's acceptance ratio method, the step function, the column bias $\mathbb{E}|\theta_B|$, the $\mathbb{Z}_2$ glass, frustration. Their meanings are in A-02 Glossary.

Next chapter: 26-10 Candidate new viewpoints — into inequalities for which local quantities these tendencies might be grown.

Candidate new viewpoints

> This chapter is a map of open questions. It marks the boundary between the parts that close and the parts that do not.

What this chapter shows — We have narrowed things down to this point: crossing the middle of the crossover band needs one inequality that uses non-commutativity (26-06 § 07). This chapter sets out candidates for the local quantity in which to write that one inequality. For the four faces — the softness of the twist direction κ_W, the flux mixing of the box $\delta(L_0)$, the twist cost of the column $f(\ell)$, the pressure margin — we write the local quantity that carries each, why we expect it to hold for $SU(2)$, where it breaks for $U(1)$, and whether there are numbers (§ 02). We then give the $\mathbb{Z}_2$ glass representation, restricted to the lower half of the band (§ 03), the conformal frame $H^2 \times S^2$, which turned out negative on examination (§ 04), and three new candidates — reduction by the compactness of the band, the log-concavity of the Weyl measure and the height–azimuth decomposition, and the local step function of the column (§ 05).

Prerequisite chapters — 26-06 (the renormalisation comparison, the list of the four faces, the reduction lemma), 26-08 (stopped roads and the selection criteria), 26-09 (the measured record), 25-03 (containers), 26-01 (the correct form of the band).

Said first — The candidates on this page are all **at the level of physical estimates**. None of them closes the band, and none gives a theorem at any β in the band. Paper proofs are set out only for the elementary facts that serve as footholds for the candidates (the transport identity, the Jensen sandwich, the density of a projection, and so on), each labelled "Proved here". We do not give numbers for the chances that a candidate succeeds (such numbers are only an aid to judgement, not numbers with a basis).

The order of this chapter

1. How to read the candidates — the checkpoints a candidate must pass
2. The four faces of the band in the middle (Table 1)
3. The $\mathbb{Z}_2$ glass — restricted to the lower half of the band
4. The conformal frame $H^2 \times S^2$ — negative on examination
5. Three new candidates — reduction by compactness, the Weyl measure, the step function of the column
6. Five other candidates (Table 2)
7. What cannot be said

8. The table of the five grades and the dependency graph

01

How to read the candidates — the checkpoints a candidate must pass

The candidates on this page are set out after passing the selection criteria of 26-08 § 07 as checkpoints. In particular, we name the following four for each candidate.

- **Does it use non-commutativity essentially?** A candidate that gives the same conclusion for $U(1)$ is wrong. This is why we write "where it breaks for $U(1)$" for each candidate. A table that widens the opponents of the test beyond U(1) is in 26-12.
- **Is it not a rewording of one loop?** A quantity that at one loop becomes asymptotic freedom itself is not enough on its own.
- **Does it not fail at small scales?** Claims are put in a form restricted to outside the lattice artefacts, such as $\ell \geq \ell_{\min}$.
- **Does it distinguish from the transition of $U(1)$ within the band ($\beta \approx 1.01$)?**

For each candidate we write (i) the local quantity that carries it, (ii) why we expect it to hold for $SU(2)$ (the place where non-commutativity enters), (iii) where it breaks for $U(1)$ and $\mathbb{Z}_2$, (iv) the check against the ledger (26-08), (v) whether there are numbers. The labels are "Proved here" for elementary facts, "Reading" for numbers, and "Reading (physics)" for estimates.

02

The four faces of the band in the middle

There are four candidates for the local quantity in which to write the non-abelian inequality that crosses the band. We call them "faces" in the sense that it suffices to show any one of them, uniformly in the scale, in a form that fails for $U(1)$. For every face, a comparison across scales (for example $f(2\ell, 2N, \beta) \leq K f(\ell, N, \beta')$) is a local version of the same difficulty as the global renormalisation comparison, and does not become easier. What changing the face gains is that the object of the proof becomes "a statement about a finite region with its boundary held fixed".

Table 1 The four faces (expectations and reasons are physical estimates; numbers are readings)

Face	Local quantity that carries it	Why we expect it to hold for $SU(2)$	Where it breaks for $U(1)$, $\mathbb{Z}_2$	Numbers
Softness of the twist direction κ_w	The variance of the force along the generator of the twist, within a band of width w (adapted stiffness)	Decreases with w. The soft direction of Nielsen–Olesen; a non-perturbative version of the one-loop spin term $+4$	$U(1)$ has no spin term, so $\kappa_w \geq c\beta$. $\mathbb{Z}_2$ has no continuous path and can use only thin sheets (2β per unit area)	None

Face	Local quantity that carries it	Why we expect it to hold for $SU(2)$	Where it breaks for $U(1)$, $\mathbb{Z}_2$	Numbers
Flux mixing of the box $\delta(L_0$)	The total variation distance of the law of the inner cube when the centre flux at the boundary of the box is changed	Goes to 0 exponentially with L_0	In the Coulomb phase of $U(1)$ it stays at $O(\beta)$	None
Twist cost of the column $f(\ell)$	The conditional free energy of the twist and the bias $\mathbb{E}\lvert\theta_B\rvert$ of the column $T^2_N \times \ell \times \ell$ with its outer frame held fixed	Of type $e^{-\sigma\ell^2}$	$U(1)$ is of type $\pi^2\beta/\ell^2$, $\mathbb{Z}_2$ is $\approx 2\beta$	Yes (26-09 § 07)
Pressure margin	The pressure difference p $\infty - pG,L$ between the system with the twist pattern and the system without it (the exponent of the chessboard density κ cb)	A positive margin remains at intermediate β	$U(1)$ does not produce a small κ_{cb}	None

The softness in the twist direction $\kappa_{\mathcal{W}}$ — joining sectors by transport

The foothold is an identity for the free energy given by a map that carries one sector to the other. Take a measurable bijection T (Jacobian J_T) from the configuration space of sector 0 to that of sector 1; let μ_0, μ_1 be the Gibbs measures of the two sectors and $\Delta F = -\ln(Z_1/Z_0$). Then

$$\Delta F \;=\; \mathbb{E}_{\mu_0}\left[S\circ T - S - \ln J_T\right] \;-\; \mathrm{KL}\big(T_{\#}\mu_0 \,\big\|\, \mu_1\big) \quad (1)$$

holds [Proved here].

Proof. By the change of variables, $Z_1 = \int e^{-S(Tx)} J_T(x)\,dx$, so setting $G := S\circ T - S - \ln J_T$ we have $Z_1/Z_0 = \mathbb{E}_{\mu_0}[e^{-G}]$. The measure $\tilde{\mu} \propto e^{-G}\mu_0$ is μ_1 pulled back by T^{-1}, and $\mathrm{KL}(\mu_0\|$ $\tilde{\mu}) = \mathbb{E}_{\mu_0}[G] + \ln \mathbb{E}_{\mu_0}[e^{-G}]$. Since T is a bijection, $\mathrm{KL}(\mu_0\|\tilde{\mu}) = \mathrm{KL}(T_{\#}\mu_0\|\mu_1)$. Rearranging gives (1). ∎

The first term of (1) is "the action cost of the map", and the second is the remainder. If the map T is deterministic, the remainder (the relative entropy) grows with the area of the vortex sheet, so an area law of type $e^{-\sigma A}$ does not come out [Reading] (physics) (the stopped road "deterministic squeeze" of 26-08 § 05). The goal of the transport road is therefore not the area law but the qualitative $\Delta F(L, L) \to 0$ (the intermediate goal of 26-06 § 08).

(ii) Why we expect it to hold for $SU(2)$ [Reading] (physics). The "adapted" map that carries sectors along the generator of the twist is a triangular change of variables with $J_T = 1$: a map that preserves volume while changing the shape of the fluctuations. It is the non-perturbative version of the principle of 25-07 — that what hides the antiscreening $+4$ is not

preserving volume but not changing the shape of the fluctuations. κ_w is the size of the fluctuations of the action cost of that map, and we expect it to decrease with w if the soft direction of Nielsen–Olesen works within a band of width w.

(iii) Where it breaks. $U(1)$ has no spin term, and $\kappa_w \geq c\beta$ remains. $\mathbb{Z}_2$ has no continuous path joining the sectors and can use only thin sheets (2β per unit area). **(iv) The ledger**: there is a risk of hitting item 25 of Table 1 (spin and the index theorem are a restatement of asymptotic freedom at one loop); the dividing line is whether the non-perturbative part can be cut out as an estimate of the variance κ_w. **(v) Numbers**: none.

The mixing of box magnetic flux $\delta(L_0)$ — a finite-box criterion

A candidate for a gauge version of a Dobrushin–Shlosman-type finite-size criterion. The form of the goal is "if $\delta(L_0, \beta) \leq \varepsilon_d$, then the area law"; the criterion itself has not been written down [Reading] (physics). It does not use correlation inequalities (GKS), so it does not hit the obstacle that $SU(2)$ has no GKS (items 2 and 4 of Table 1).

For $SU(2)$ we expect $\delta(L_0)$ to go to 0 exponentially with L_0. In the Coulomb phase of $U(1)$, in the Villain model in the Gaussian approximation, the relative entropy stays at $O(\beta)$ independent of L_0 (a count at the level of estimates) [Reading] (physics). Caution: if the ordinary Dobrushin criterion is used with a frame, the continuous field follows the boundary and the influence coefficient becomes nearly 1 (the stopped road "Dobrushin-type criterion" of 26-08). $\delta(L_0)$ is a quantity that changes only the centre flux precisely to avoid this following. No numbers.

The twist cost of a column $f(\ell)$ — the same quantity as the Wilson loop of the box

Given the skeleton Σ (the outer frame of the columns), the interiors of the columns are conditionally independent, so

$$|\theta| \;\leq\; \mathbb{E} \prod_B |\theta_B(\Sigma)| \qquad (2)$$

holds [Proved here] (the same framework as the Container Identity of 25-03 § 03, with no new content). What is new is how $f(\ell)$ is read. The electric flux in a box with its outside held fixed is **the same quantity as the area law of the Wilson loop of the box** [Reading] (physics). This closes a loop: the Tomboulis–Yaffe inequality (from θ to the Wilson area law, 26-01 § 04) and (2) (from the Wilson area law of the box to θ).

Two things are missing to go round the loop (open): the comparison between periodic and fixed boundary conditions, and the loss of $\ln N$. The estimate is that even if the loss remains, $\ln(1/\theta(L, L)) \geq cL^2/\ln L$ comes out, and through Tomboulis–Yaffe this gives $\langle W \rangle \leq 2 \exp(-cA/\ln A)$ (decay faster than the perimeter law) [Reading] (physics).

There are numbers (26-09 § 07) [Reading]. On $T_4^2 \times \ell \times \ell$ with the outer frame held fixed, $\mathbb{E}|\theta_B|$ for $SU(2)$ ($\beta = 2.3$) at $\ell \geq 5$ is 1/5 to 1/7 of that for $U(1)$ ($\beta = 1.3$), and the ℓ needed to keep

θ small hardly grows for $SU(2)$, staying at 6–7 for $N \le 8$, while it grows for $U(1)$. At these sizes, the difference between fixed and periodic boundary conditions does not seem to matter. The values for $SU(2)$, however, are close to the statistical floor, and $e^{-\sigma \ell^2}$ cannot be told apart from a power.

The pressure margin — a computer-assisted squeeze

We write the chessboard density as $\kappa_\Lambda = \exp(|B|(p_{G,\Lambda} - p_\Lambda))$ (p_G is the pressure of the system with the twist pattern, $|B|$ the volume of a tile). From above: if the pattern is reflection-symmetric, the periodic transfer matrix is positive, and the monotonicity of Schatten norms gives $p_{G,\Lambda} \le p_{G,L}$ (Λ a multiple of L). From below: $p_\Lambda \ge p_\infty$, together with a lower bound p_{lb} on the infinite-volume pressure taken from a dual certificate of the loop-equation bootstrap (the method of Kazakov–Zheng; checking not yet done). Combined, $\kappa_\Lambda \le \exp(|B|(p_{G,L} - p_{\mathrm{lb}}))$ becomes uniform in Λ, and an explicit area law for the tension comes out at fixed β — that is the estimate [Reading] (physics).

This face does not cross the band (uniformly in the lattice spacing). What it could give is confinement at fixed β for four-dimensional $SU(2)$ at intermediate coupling (for example $\beta = 1.0$). In the range we searched, no proof of that was found. Combined with the reduction of § 05 (the candidate "reduction by compactness of the band"), a result of this kind at fixed β becomes a component of the band as it stands. $\beta = 1.2$ for $U(1)$ should not produce a small κ, and can be used as a control. No numbers.

03

The Z_2 glass — restricted to the lower half of the band

Writing an element of $SU(2)$ as $U = \sigma V$ ($\sigma \in \mathbb{Z}_2$, V a section of $SO(3)$), the partition function of a twisted sector can be written $Z_n = \int d\omega\, Z_n(\omega)$. $Z_n(\omega)$ is the partition function of a $\mathbb{Z}_2$ gauge theory with couplings $\beta J_p(\omega)$ fixed by the $SO(3)$ background ω ($J_p = \frac{1}{2}\operatorname{tr} V_p$, with random signs) and twist n. Setting $X(\omega) = \ln Z_0(\omega)/Z_1(\omega)$ and $P_n(d\omega) \propto Z_n(\omega) d\omega$, we have

$$\mathbb{E}_{P_1}[X] \;\le\; \Delta F \;\le\; \mathbb{E}_{P_0}[X] \qquad (\Delta F = \ln Z_0/Z_1) \tag{3}$$

[Proved here].

Proof. The left side follows from $Z_0/Z_1 = \int Z_1(\omega) e^{X(\omega)} d\omega / Z_1 = \mathbb{E}_{P_1}[e^X] \ge e^{\mathbb{E}_{P_1} X}$ (Jensen). Likewise the right side from $Z_1/Z_0 = \mathbb{E}_{P_0}[e^{-X}] \ge e^{-\mathbb{E}_{P_0} X}$. ∎

The right-hand side of (3) is the twist free energy of a $\mathbb{Z}_2$ random-plaquette gauge model quenched in the $SO(3)$ background. This representation alone exists in the same form for $U(1)$, so it does not satisfy the criterion "distinguish from the U(1) transition". The distinction between groups enters **dynamical quantities**: the frustration of the background ($\mathbb{Z}_2$ monopoles) and the density of soft plaquettes [Reading] (physics).

Result of the diagnosis (26-09 § 08) [Reading]. On 8^4, for $\beta \gtrsim 2.0$ the mean of the effective $\mathbb{Z}_2$ coupling is 1.1–1.9, well above the transition point 0.44 of pure $\mathbb{Z}_2$, and p_{eff}, the frustration converted to independent signs, is 0.017–0.09, below the Nishimori multicritical point 0.11. **In the upper half of the band, the glass is on the side of the ordered phase**. The $SO(3)$ field already distinguishes the sectors ($|X|$ is large), and the sandwich (3) becomes very loose. Confinement there can be read as carried by the $SO(3)$ part (thick centre vortices). This fits the physical picture that centre vortices are thick at weak coupling [Reading] (physics).

For $\beta \lesssim 1.5$, $p_{\text{eff}} \geq 0.16$, soft plaquettes are 28–49%, and the glass is on the disordered side. **The glass road is therefore restricted to the lower half of the band**. We have discarded it for the upper half (the stopped road "glassy squeeze" of 26-08 § 05). A candidate use in the lower half is a variant of the mixing of box magnetic flux: condition the outside on the $SO(3)$ part only, flip only the centre flux at the boundary (the $\mathbb{Z}_2$ sign), and measure the total variation distance $\delta_c(L_0, \beta)$ of the inner law. The $SO(3)$ frame does not move, so the obstacle of the continuous field following the boundary (Dobrushin-type criterion) is avoided. The only coupling left is the $\mathbb{Z}_2$ glass. In the upper half of the band δ_c should also be close to 1, and this variant belongs to the work of pushing up the lower end of the band (26-06 § 04) [Reading] (physics).

04

The conformal frame H² × S² — negative on examination

The space $\mathbb{R}^4$ with one straight line removed is conformal to $H^2 \times S^2$ [Proved here]: with r the distance from the line and t the coordinate along the line,

$$dt^2 + dr^2 + r^2 d\Omega^2 \;=\; r^2\Big[\frac{dt^2 + dr^2}{r^2} + d\Omega^2\Big] \quad (4)$$

and the bracket is the product of the hyperbolic plane H^2 (the upper half-plane model) and the unit sphere. If a 't Hooft line is placed on the line, every sphere surrounding the line carries $\mathbb{Z}_2$ flux 1 (an exact analogue of $n_{12} = 1$). And the sphere $\{t\} \times S^2_r$ corresponding to a point $(t, r) \in H^2$ winds an odd number of times with a loop C in the half-plane if and only if (t, r) lies inside the region enclosed by C [Proved here] (the intersection of C with the three-dimensional ball spanned by the sphere is the intersection of the segment $\{t\} \times [0, r)$ with C). This avoids the topological obstacle that stopped the road using local closed surfaces in $\mathbb{R}^4$ (item 8 of Table 1: a closed surface meets a plane an even number of times) by using the line as "the outside". Near the line ($r \to 0$) is the region of small spheres (weak coupling), far away ($r \to \infty$) is the infrared: **scale turns into position**.

The promise went only this far; on examination the result was negative [Reading] (physics).

- A container at height r with hyperbolic size 1 has Euclidean size of order r and carries a sphere S^2_r. This is the band quantity $\theta(r, r)$ itself (with S^2 in place of T^2).

- Hyperbolic nesting: the hyperbolic area of a unit container at height $2r$ (about $[-r, r] \times [r, 3r]$) is $2r\,(1/r - 1/(3r)) = 4/3$, and only 1–2 unit containers at height r fit inside it [Proved here] (computation of an integral). Hyperbolic area does not change under shape-preserving dilation, so nesting does not increase the number of sub-containers; what comes out is about $\theta(2r) \le \theta(r)^{1.3}$, not the exponent 4 of planar nesting.
- The hyperbolic area of a loop in the half-plane (from height d to $d+\ell$, length T) is $T(1/d - 1/(d+\ell)) \approx T/d$, and a uniform θ_0 independent of height gives only the perimeter law. The area law needs $\theta \le e^{-c(r\Lambda)^2}$ for unit containers at height r: the same condition as the band.

Conclusion: the conformal frame puts renormalisation into the language of space as "change of scale = translation in H^2", but kinematics alone does not give the area law, and the core (the growth of disorder through asymptotic freedom) remains in the same place. Its value lies in the clarity of a framework that states the $\mathbb{R}^4$ stage (16-09) without going through slabs. The relation between loops in the measure with a 't Hooft line and loops in $\mathbb{R}^4$ without the line is open, and there is a precedent of the comparison principle failing (26-08 § 04).

05

Three new candidates

We examined eight candidates for the band, and put three of them here (the rest are in § 06). The reasons for the choice: the reduction by compactness of the band is a reduction that can be written reliably on paper, and it moves "where the band is really hard"; the candidate "log-concavity of the Weyl measure" is a two-dimensional checkpoint that can be tried more cheaply than four dimensions, and is one of the few forms that does not fall into the type in which the source of monotonicity and the mechanism of disorder point in opposite directions (the lesson of 26-08 § 01); the candidate "local step function of a column" can be tried directly with the existing numerical tools for the column, and lets us look at the criterion "distinguish from the U(1) transition" directly.

Reduction by compactness of the band

(i) In one line. The band is a bounded closed interval $I = [\beta_a, \beta_b]$ in β, so the renormalisation comparison within the band holds by lowering β' only a little from β, and the band can be crossed in finitely many steps. All that is needed is "confinement at fixed β (two-sided asymptotics with errors of perimeter order)" at each β in the band.

This is set out with a proof as the Lemma Reducing the Comparison inside the Band of 26-06 § 06 [Proved here] (under the two-sided area law at fixed β, the uniform positivity and continuity of the rate, $N \ge N_1$, $M \ge M_0$). Here we write only its meaning as "a viewpoint".

(ii) The local quantity that carries it. Two-sided estimates of the vortex free energy in large volume at fixed β. As local quantities, the area rate $\sigma_N(\beta)$ of the column bias at $\ell \gg \xi(\beta)$,

and the error p of perimeter order.

(iii) Where non-commutativity enters [Reading] (physics). The lemma itself does not depend on the group. Non-commutativity enters entirely through the assumption (confinement for $SU(2)$ within the band). This candidate is a reduction, not a tool, and serves to move the place of the difficulty. After the reduction, what remains within the band is a problem at fixed β (estimates uniform in the volume, non-convex large fields, absence of a phase transition), and the difficulty across scales gathers at the upper end of the band (where the correlation length $\xi(\beta)$ grows sharply in lattice units) and at the seam with ultraviolet control.

(iv) Where it breaks. For $U(1)$ ($\beta \in (1.01, 2.7)$) and $\mathbb{Z}_2$ (on the weak-coupling side of the transition) the assumption is false, so no conclusion follows either. The criterion "distinguish from the U(1) transition" is carried by the assumption.

(v) Check against the ledger. Unlike item 6 of Table 1 (a proof by contradiction extending continuously from a small T^2), the only continuity used is the continuity in β of finite-volume quantities. It is in the same spirit as the idea of considering the step function of the continuum-limit coupling on a bounded interval, but here we stop at the lattice β and do not pass through the continuum limit.

(vi) Relation to known results. On the strong-coupling side, a statement of exactly this form exists at the level of a strand (the Strong-Coupling-End Expansion Proposition of 26-06 § 03). A proof of confinement for the four-dimensional $SU(2)$ lattice at $\beta \approx 1$ within the band was not found in the range we searched (the front line on the strong-coupling side is at small β, 26-08 § 07). Chatterjee (arXiv:2006.16229; abstract checked, text not yet checked) gives a result of the form: if the centre is non-trivial and correlations decay exponentially under arbitrary boundary conditions, then centre symmetry is not broken, and if it is not broken there is confinement. This could be an entrance to qualitative confinement at fixed β, but it does not give the perimeter-order precision of the two-sided area law at fixed β.

(vii) What is needed next. A road that gives the upper side of the two-sided area law at fixed β for each β by finite-box criteria (the mixing of box magnetic flux, the pressure margin), and a road that gives the lower side (that vortices are not too cheap = an upper bound on the string tension) by a variational trial vortex. Both are problems at fixed β, and boxes of size about $\xi(\beta)$ suffice, but at the upper end of the band $\xi(\beta)$ exceeds 10 in lattice units, and finite-box computations in four dimensions stop on scale (item 3 of Table 1). No numbers.

The log-concavity of the Weyl measure, and the height–azimuth decomposition

(i) An elementary fact [Proved here]. The density of the uniform measure on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ projected onto one coordinate t is

$$\rho_n(t) \;\propto\; (1-t^2)^{(n-3)/2}, \qquad t \in [-1,1] \quad (5)$$

Proof. The slice at height t is a sphere S^{n-2} of radius $\sqrt{1-t^2}$, whose area is proportional to $(1-t^2)^{(n-2)/2}$, and the width (length along the sphere) of the band of height dt is $dt/\sqrt{1-t^2}$. Multiplying gives (5). ∎ Hence for $n \geq 3$, ρ_n is log-concave (flat at $n=3$), and at $n=2$ it is log-convex (the arcsine distribution). For $SU(2) \cong S^3$ ($n=4$), the density of $t = \frac{1}{2}\operatorname{tr} U$ is $\propto \sqrt{1-t^2}$ (the Weyl measure); for $U(1) = S^1$ it is $\propto 1/\sqrt{1-t^2}$.

In the two-dimensional $O(3)$ model, writing the spin as $s = (\sqrt{1-t^2}\cos\varphi,\ \sqrt{1-t^2}\sin\varphi,\ t)$, the uniform measure on S^2 is $dt\,d\varphi/(4\pi)$ ($n=3$ of (5)), and $s_i \cdot s_j = \sqrt{(1-t_i^2)(1-t_j^2)}\cos(\varphi_i - \varphi_j) + t_i t_j$, so the model of the azimuths φ conditioned on the heights t is an XY model with couplings $J_{ij} = \beta\sqrt{(1-t_i^2)(1-t_j^2)} \geq 0$ [Proved here]. There is no frustration.

(ii) The local quantity that carries it. The effective azimuthal coupling $J_{\text{eff}}(b)$, measured in the coarse variables of a $b \times b$ block, and the fluctuations of the height.

(iii) Why we expect it to hold [Reading] (physics). The conditioned azimuth model is ferromagnetic, so the Ginibre inequality for XY (checking not yet done) can be used in the right direction: lowering the couplings lowers the correlations. What exponential decay needs is also an upper bound on correlations, so **the source of monotonicity and the mechanism of disorder point the same way**. The disorder comes from the fluctuations of the height, that is, from the measure (5) with $n \geq 3$. $O(2)$ has no height.

(iv) Where it breaks. $O(2)$: there is no height, and the azimuths remain XY with coupling β (the KT phase). Four-dimensional $U(1)$: there is no degree of freedom for the direction of the class (S^2). $\mathbb{Z}_2$: there is no continuous azimuth.

(v) Check against the ledger, and risks. Unlike items 22 (one-point Bakry–Émery) and 26 (renormalisation of curvature) of Table 1, it uses not curvature but the log-concavity of a projection and a conditional Ginibre inequality. Unlike item 10 of Table 1 (GKS for $\mathbb{Z}_2$), the conditioned model has no frustration. There are two risks. (a) There is an argument of Patrascioiu–Seiler (not yet checked) for the opposite conclusion (that $O(3)$ may have a KT-type phase) through embeddings of the equatorial direction, and this candidate stands at the site of that dispute. (b) At the level of a single site, the typical values of J_{ij} exceed the KT threshold, and the comparison says nothing. If it works, it works only at the block level, where there is no direct Ginibre comparison, and there is the risk of the transfer type of detour.

(vi) To four dimensions. One needs a representation in which the remaining $U(1)$, conditioned on n in $U = t + i\sqrt{1-t^2}\,n\cdot\sigma$ ($n \in S^2$ the direction of the class), is ferromagnetic, but this has not been constructed. Showing exponential decay at all temperatures for the two-dimensional lattice $O(n)$ ($n \geq 3$) (Polyakov's conjecture) is unsolved (the criterion "crosses the smallest band" of 26-08 § 07).

(vii) Numbers. None. The form of the first test is fixed: measure $J_{\text{eff}}(b)$ for the two-dimensional $O(3)$ ($\beta = 1.5, 1.7$), the control $O(2)$ (the KT phase and the disordered phase),

and the auxiliary $O(4)$ (whose height density $\sqrt{1-t^2}$ is the same as for $SU(2)$); if J_{eff} stays above the KT guide even at sizes at which correlations of $O(3)$ have already decayed, discard this candidate. If J_{eff} falls with b for $O(2)$, the measurement is lowering the coupling mechanically, and the result is not used. A block-level Ginibre comparison does not exist as a theorem, so even if it passes it remains a watch on a necessary condition.

The local step function of a column

(i) In one line. Replace the renormalisation comparison of the global ΔF by a comparison of column quantities with the outer frame held fixed:

$$f(2\ell, 2N, \beta) \;\le\; K \cdot f(\ell, N, \beta')^{c}, \qquad \beta' < \beta, \;\; \ell \ge \ell_{\min} \quad (6)$$

(K and c independent of ℓ and N). **(ii) The local quantity that carries it.** The conditional twist free energy ΔF_B and the bias $\theta_B = \tanh(\Delta F_B/2)$ of the column $T^2_N \times \ell \times \ell$ (the outer frame held fixed at an equilibrium configuration). Two quantities are compared: the median over frames of ΔF_B (with sign) and $\mathbb{E}|\theta_B|$.

(iii) Why we expect it to hold [Reading] (physics). In the existing numbers (26-09 § 07), the ℓ needed to keep θ small hardly grows with N for $SU(2)$, and grows for $U(1)$. Non-commutativity enters where the vortex can localise in a lump of the size of T^2 (a fractional instanton), because its action, by the Bogomolny bound, does not depend on the size of T^2 (25-02).

(iv) Where it breaks. In the Coulomb phase of $U(1)$ the flux spreads uniformly and the column bias decreases only as a power (about $\ell^{-1.4}$ over $\ell = 5 \to 8$), so the local β' should come out at or above β. For $\mathbb{Z}_2$ the sheet tension remains, likewise.

(v) Check against the ledger. Unlike item 27 of Table 1 (comparison of boundary conditions and geometries), it compares the same object at two scales. Nor does it use the correlation inequalities of item 2 of Table 1. Because of the criterion "not broken by small loops", small ℓ (3 or less) may fail through lattice artefacts, so the claim takes the form $\ell \ge \ell_{\min}$. It is a local version of the step function of the coupling defined with twisted boundary conditions (the method of de Divitiis et al.; not yet checked).

(vi) Numbers. There are as yet no numbers measuring (6) itself. The form of the test is fixed: fine side $(N, \ell) = (4, 4)$, coarse side $(2, 2)$ (and, if time allows, $(4, 6)$ and $(2, 3)$), with $SU(2)$ at $\beta = 2.3$, and $U(1)$ in the confining phase at $\beta = 0.9$ and in the Coulomb phase at $\beta = 1.3$. The prediction: $\beta' < \beta$ for $SU(2)$; for $U(1)$, $\beta' < \beta$ at $\beta = 0.9$ and $\beta' \ge \beta$ at 1.3 (the sign flips with the phase). If the sign does not separate between the two $U(1)$ points, this observable does not distinguish phases, and the candidate is lowered a grade. $(2, 2)$ has large artefacts, so it is treated as a sign until confirmed with $(4, 6)/(2, 3)$.

(vii) Difficulty. Showing (6) is a local version of the same difficulty as the global renormalisation comparison (the opening of § 02). What is gained is that the object of the

proof becomes "a statement about a finite region with its boundary held fixed", and that the assumption of the reduction by compactness of the band (the two-sided area law at fixed β) can be written in the language of columns.

06

Five other candidates

Of the eight candidates examined, we give the remaining five in one line each, with the reason none was given priority over the three of § 05. All are labelled "Reading (physics)"; those that include an elementary identity have "Proved here" added in their row.

Table 2 Five other candidates

Candidate	One line	Why it was not given priority	Label
Confinement as sign cancellation	Under the twist-summed measure P (weight $Z_0(\omega) + Z_1(\omega)$), $\theta = \mathbb{E}_P[\tanh(X/2)]$. Even if the glass is in the ordered phase in the upper half of the band ($\lvert X\rvert$ large), θ is small if the signs of X cancel	The distinction between groups appears in the growth as N is made large, so it is close to a restatement of the same quantity as the column face $f(\ell)$. Numerically it needs the ratio of partition functions of a disordered $\mathbb{Z}_2$ gauge theory for each ω, which is large in scale	Proved here (the identity only rewrites the definition $(Z_0 - Z_1)/(Z_0 + Z_1)$ of θ in terms of ω), Reading (physics)
Centre-only influence coefficients	The $\delta_c(L_0, \beta)$ of § 03. Condition the outside on the $SO(3)$ part only, and flip only the centre flux at the boundary	In the upper half of the band δ_c should also be close to 1: for the lower half of the band only (on the side of the work of pushing up the lower end of the band)	Reading (physics)
Diffusion of the sign of the flux by the flow of $\mathbb{Z}_2$ monopoles	The sign flux through each plane changes only by the parity of the number of closed $\mathbb{Z}_2$ monopole curves crossing between planes. The flux mixes through the winding density of monopoles	A $\mathbb{Z}_2$ version of the $U(1)$ monopole picture; the criterion "uses non-commutativity essentially" is weak. There is a dispute over whether confinement survives with monopole-suppressed actions (not yet checked)	Reading (physics)
Determinant in self-dual coordinates	In each sector, the difference between sectors of the Jacobian when the measure is viewed as a push-forward to the anti-self-dual part	A large risk of hitting item 25 of Table 1 (spin and the index theorem are a restatement of asymptotic freedom at one loop). The lattice has no self-dual decomposition	Reading (physics)

Candidate	One line	Why it was not given priority	Label
A quantitative version, uniform in N, of "discrete 1-form symmetry is not broken" for the slab reduced to two dimensions	Read slab confinement as a lattice version of a statement of the type that discrete 1-form symmetry is not spontaneously broken in two dimensions, and make its quantitative version (tension $\geq$ the density of walls of the cheapest sector) uniform in N	As a road, this is exactly the road of increasing N in the semiclassics of a small T^2 (of the same weight as the adiabatic continuity conjecture of Tanizaki–Ünsal); only the reading is new. The dimension conditions in the literature on generalised symmetries are not yet checked	Reading (physics)

07

What cannot be said

- (a) None of the candidates closes the band. For none of them have we shown a theorem at any β in the band.
- (b) The reduction by compactness of the band is a reduction, and we do not say it has made the band easier. The assumption of two-sided area laws inside the band contains confinement at intermediate coupling on the four-dimensional $SU(2)$ lattice (unsolved).
- (c) Even if the two-dimensional test of the log-concavity of the Weyl measure passes, it implies nothing for Polyakov's conjecture or for four dimensions. A block-level Ginibre comparison does not exist as a theorem.
- (d) We do not say that two-dimensional results lift to four dimensions. Only the form of the statement lifts.
- (e) The $SU(2)$ expectations of the four faces are all estimates. Only the column face $f(\ell)$ has numbers, and those are tendencies on small lattices.
- (f) The negative verdict on the conformal frame is a judgement at the level of estimates, that "kinematics alone does not give the area law"; it does not rule out roads that add other inputs in that frame.
- (g) We do not say that the candidates on this page are not combinations of known facts. The lemma of the reduction by compactness of the band and the log-concavity of the projection in the Weyl-measure candidate may be written somewhere.

08

The table of the five grades and the dependency graph

The candidates on this page are all "Reading (physics)". Paper proofs are set out only for the elementary facts that serve as footholds for the candidates.

Table 3 The rows of this chapter ("—" marks a row with no machine checking)

Row	Statement (one line)	Grade	Basis	Scope	Machine checking

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
1	The free-energy identity (1) for a bijection carrying one sector to the other	Proved here	§ 02 softness in the twist direction	A measurable bijection with a well-defined Jacobian	—
2	For a deterministic map the remainder grows with the area of the sheet, and the area law does not come out	Reading (physics)	Softness in the twist direction in § 02, "deterministic squeeze" of 26-08	—	—
3	The column decomposition $\lvert\theta\rvert \le \mathbb{E}\prod_B \lvert\theta_B(\Sigma)\rvert$	Proved here	Twist cost of a column in § 02, the framework of the Container Identity of 25-03	Conditional independence with the skeleton held fixed	—
4	Column bias: at $\ell \ge 5$, $SU(2)$ is 1/5 to 1/7 of $U(1)$, and the needed ℓ does not grow for $SU(2)$	Reading	26-09 § 07	$\beta = 2.3$ ($SU(2)$), 1.3 ($U(1)$), $N \le 8$, 3–12 frames	—
5	The $SU(2)$ expectations of the four faces, and where they break for $U(1)$ and $\mathbb{Z}_2$	Reading (physics)	§ 02 Table 1	Estimate	—
6	The Jensen sandwich (3) of the $\mathbb{Z}_2$ glass	Proved here	§ 03	The $U = \sigma V$ representation	—
7	The glass is on the ordered side in the upper half of the band; the glass road is restricted to the lower half	Reading (physics)	§ 03, 26-09 § 08	8^4; the conversion is crude	—
8	$\mathbb{R}^4 \setminus$ line $\cong H^2 \times S^2$ (conformal), and the parity criterion for spheres and loops	Proved here	§ 04 (4)	—	—
9	The hyperbolic nesting area $4/3$, the hyperbolic area of a half-plane loop $\approx T/d$	Proved here	§ 04	Computation of an integral	—
10	In the conformal frame kinematics alone does not give the area law, and the core remains in the same place	Reading (physics)	§ 04	Negative on examination	—

Row	Statement (one line)	Grade	Basis	Scope	Machine checking
11	Reduction by compactness of the band: the comparison within the band reduces to two-sided area laws at fixed β	Proved here	The Lemma Reducing the Comparison inside the Band of 26-06 § 06	Two-sided area law at fixed β, uniform positivity and continuity of the rate, $N \geq N_1$, $M \geq M_0$	—
12	The projected density (5), and the height–azimuth decomposition of $O(3)$ (XY with couplings $J_{ij} \geq 0$)	Proved here	§ 05 log-concavity of the Weyl measure	—	—
13	The mechanism of the log-concavity of the Weyl measure: the source of monotonicity and the mechanism of disorder point the same way	Reading (physics)	§ 05 log-concavity of the Weyl measure	No block-level comparison exists as a theorem	—
14	The prediction (6) of the local step function of a column: $\beta' < \beta$ for $SU(2)$; for $U(1)$ the sign flips with the phase	Reading (physics)	§ 05 local step function of a column	$\ell \geq \ell_{\min}$. No numbers for (6)	—
15	The identity $\theta = \mathbb{E}_P[\tanh(X/2)]$ of the candidate "confinement as sign cancellation"	Proved here	§ 06 Table 2	A rewriting of the definition of θ	—

The dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
 1  PRV  the Transport Identity (1)
 2  RDG  remainder of a deterministic map grows with area ⟸ 1   physics
 3  PRV  the column decomposition                              framework of 25-03
 4  RDG  numbers for the column bias                           26-09 §07
 5  RDG  expectations of the four faces ⟸ 1, 2, 3, 4            physics
 6  PRV  the Jensen sandwich (3) of the glass
 7  RDG  the glass road is restricted to the lower half ⟸ 6     8⁴, physics
 8  PRV  conformal equivalence and the parity criterion (4)
 9  PRV  the hyperbolic nesting area
10  RDG  the conformal frame is negative ⟸ 8, 9                 physics
11  PRV  reduction by compactness of the band                  26-06 reduction lem
12  PRV  projected density (5), height–azimuth decomposition
13  RDG  mechanism of the log-concavity of the Weyl measure ⟸ 12   physics
14  RDG  prediction (6) of the local step function of a column ⟸ 3, 4, 11   physic
15  PRV  θ = E_P[tanh(X/2)]
```

The check is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. The conclusion-side nodes (the expectations of the

four faces, the scope of the glass road, the negative result for the conformal frame, and the mechanism and prediction of two of the three new candidates) are all "Reading (physics)", and the "Proved here" nodes are either elementary facts serving as footholds for the candidates, or a reduction whose proof is in another chapter. This page has no node for a statement that crosses the band.

What this chapter can say

What is said	What is not said
Proved here The elementary facts that serve as footholds for the candidates: the transport identity, the column decomposition, the Jensen sandwich of the glass, the conformal equivalence and the parity criterion, the projected density and the height–azimuth decomposition, $\theta = \mathbb{E}_P[\tanh(X/2)]$	That estimates crossing the band follow from them
Proved here That the comparison within the band reduces to two-sided area laws at fixed β (the Lemma Reducing the Comparison inside the Band of 26-06, under the assumption of two-sided area laws inside the band)	A proof of the assumption of two-sided area laws inside the band at any β. That the band has been made easier
Reading (physics) For the four faces and the three candidates, why we expect them to hold for $SU(2)$ and where they break for $U(1)$	An estimate uniform in the scale for any face or candidate
Reading (physics) That the $\mathbb{Z}_2$ glass road is restricted to the lower half of the band; that the conformal frame does not give the area law by kinematics alone	That the glass road pushes up the lower end of the band in the lower half

Sources and reproduction

Item	Kind	Source / tools
The four faces, the glass representation, the examination of the conformal frame	This book's study	The record of this book's study (the survey of roads that would close the crossover band, proposals of mechanisms from the physical point of view)
The three new candidates and the five others	This book's study	This book's study (the examination of candidates for the band). The Lemma of the reduction by compactness of the band is in 26-06 § 06
Numbers for the column bias and the glass diagnosis	Reading	26-09
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985) 313. Chapter 26-01
The action of fractional instantons	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166. Chapter 25-02
An entrance to confinement at fixed β	Abstract checked	S. Chatterjee, arXiv:2006.16229 (text not yet checked)

Item	Kind	Source / tools
Literature not yet checked	Checking not yet done	Kazakov–Zheng (bootstrap for lattice Yang–Mills), Patrascioiu–Seiler (the argument for a KT-type phase of $O(3)$), the Ginibre inequality for XY, de Divitiis et al. (the step function with twisted boundary conditions), the dispute over monopole-suppressed actions, literature of Coleman–Mermin–Wagner type for generalised symmetries, the Dobrushin–Shlosman criterion

Terms used in this chapter — the four faces, the adapted stiffness κ_w, the flux mixing of the box $\delta(L_0)$, the twist cost of the column $f(\ell)$, the pressure margin, the $\mathbb{Z}_2$ glass, frustration, the conformal frame, the compactness of the band, the Weyl measure, height and azimuth, the local step function of the column, the crossover band (the band). Their meanings are in A-02 Glossary.

Next chapter: 26-11 Ultraviolet control — what it means to close at infinitely many steps — the foundation under the two roads. What closing in one step means, and where the series stopped.

Ultraviolet control — what it means to close at infinitely many steps

> This chapter is a guide. § 01 is a map of open questions.

What this chapter shows — When the lattice spacing a is sent to 0 and the number of renormalisation steps grows without bound, what does it mean to show, at every step and without the constants degrading, that "one step returns to the same form"? The body of this chapter has been expanded into the first half (15-00 to 15-10) of Part 15 "A record of attempts at ultraviolet control". This page keeps only the five lines of what this chapter shows, the statement of the decomposition of the ultraviolet wall, and a guide for each section.

Prerequisite chapters — 7-05 (the sequence of renormalisation group steps), 7-07 (four dimensions: Bałaban and MRS), 26-00 (confluence), 27-02 (the ledger of conditions).

Said first — This chapter deals with the "ultraviolet control" row of the ledger of conditions, a different task from the crossover band (the band). It deals only with the side of scales finer than the band. The continuum limit in a periodic cubic box was shown by the construction published on 2026-11-10 (at the level of the committee's assessment; Volume 2). What this chapter and Part 15 deal with is ultraviolet control on a different road set up by this book's study (small twisted boxes, and a form uniform in the lattice spacing), and that has not been shown. A construction with twisted boundary conditions and elongated boxes, and a statement in a form uniform in the lattice spacing, are not in the published construction either. What Part 15 gives is a statement of what has to be shown, line-by-line calculations, elementary lemmas, and numbers measured in the smallest systems; each of them only restates a part of ultraviolet control, or splits ultraviolet control into other conditions.

The order of this chapter

1. What this chapter shows — five lines
2. The decomposition of the ultraviolet wall — a class that closes with finitely many parameters, and three remainders (the statement only)
3. Guide to Part 15 — the order of reading

00

What this chapter shows

- Ultraviolet control means showing, at infinitely many steps and without the constants degrading, that the effective density "returns to the same form in one step". The number of steps is $K = \log_L(L_0/a) \to \infty$, and if the sum $\sum_k \delta_k$ of the one-step losses δ_k is finite uniformly in K, the constants survive Proved here (15-00).
- In four dimensions the coupling is dimensionless, so the sum of the losses of size g_k^2 per step diverges like $\ln K$ Proved here (arithmetic under the one-loop flow). That part must be absorbed exactly into the flow of the coupling, and the rest confined to a sum of finite size or to a form that contracts transversely (15-01, 15-03).
- For a group-valued field the measure of the fluctuations depends on the background, and that dependence η_k is the hard point. In the type where the law of the fluctuations does not depend on the background (a product of heat kernels, one per edge), the chain of one-step estimates for flat fields goes through unchanged on the group, giving $\eta = 0$ Proved here (one step only, without the Wilson action and without large fields; 15-07).
- In the smallest system, with the angle of the single coarse plaquette fixed, the slope $\sum f^2 = \beta'/\beta$ of the invariant curve and the transverse contraction $\lambda = \sum f^4 / \sum f^2$ are determined by a single projection, and $\lambda \le \max f^2$ Proved here (at tree level restricted to commuting configurations, one step seen through the marginal distribution of one coarse plaquette; 15-05). The source (the coefficient of θ^4 poured in at each step) is a number measured at one loop Reading .
- Parameters are best counted as "a one-parameter attracting class plus contracting shapes"; as a class that keeps the ratio of the quadratic couplings at 2, it does not close in one step Proved here , Reading ($SU(2)$, four in a row, leading term; 15-06). Bałaban's four-dimensional theorem assumes the smallness of the effective coupling, the theorem that removes this assumption is unpublished, and the one-step remainder including large fields (the net one-step bracket of the coupling) is not in the original sources Cited (15-02).

01

The decomposition of the ultraviolet wall — a class that closes with finitely many parameters, and three remainders

Take a family of weights $\{w_\theta\}$ indexed by finitely many parameters $\theta \in \Theta \subset \mathbb{R}^m$, and suppose one blocking step T

$$T\big(w_\theta\,(1+\rho)\big) = w_{\Phi(\theta)}\,\big(1 + r_\theta[\rho]\big) \quad (1)$$

takes the form (Φ is the flow of the parameters, $r_\theta[\rho]$ the remainder). We put a statement of the form that ultraviolet control follows, uniformly in the number of steps, from the following four Assumption (the form of the statement only; for details see 15-03 § 02).

- Class-closure condition: the number of parameters is finite independently of the step, and the class consists of the one-parameter attracting class of the one-step map and shape coordinates that contract at each step (the evidence in the smallest systems is in 15-06).
- Coupling-flow remainder condition: $\beta_{k+1} = \beta_k - 8b_0 \ln L + O(1/\beta_k)$.
- Shape remainder condition: of the type in which, with transverse contraction $\lambda < 1$ and one-step remainder $\varepsilon(\beta)$, the deviation is bounded by $\sup_j \varepsilon(\beta_j)/(1-\lambda)$ (15-03 § 04).
- Definition and continuity of the effective coupling: continuity within the domain of definition.

That the condition that the coupling stays small follows from the net one-step bracket of the coupling and the definition and continuity of the effective coupling is the Coupling-Smallness Induction Lemma Proved here (15-03 § 03; the net one-step bracket of the coupling written in the variable β is the coupling-flow remainder condition), and the upper bound on the deviation is the arithmetic of a geometric series Proved here. What has not been shown is that the class-closure condition and the coupling-flow remainder condition including large fields hold for the real four-dimensional step; these are the four open rows (15-08).

02

Guide to Part 15 — the order of reading

There are three reading orders (15-00 § 04): the shortest is 15-00 → 15-03 → 15-09, the line of calculation is 15-04 → 15-05 → 15-06, and the line of the wall is 15-02 → 15-07 → 15-08.

What this chapter can say

Can say	Cannot say
This page is a summary and a guide. What can be said is in the tables of each chapter of Part 15	Ultraviolet control in the form of this book's road (no chapter of Part 15 shows it; the continuum limit in a periodic cubic box was shown by the published construction)
Assumption The decomposition of the ultraviolet wall (the form of the statement)	That the class-closure condition, the coupling-flow remainder condition and the shape remainder condition hold for the real four-dimensional step

Sources and reproduction

The sources are in the source table of each chapter of Part 15 (the bibliography of the original sources is in the source table of 15-02).

Words used in this chapter — ultraviolet control, the number of steps, the same form, the three conditions of ultraviolet control, the decomposition of the ultraviolet wall, the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, the continuity of the coupling, the attracting class, the transverse contraction. Their meanings are in A-02 Glossary.

Chapter to read next: 26-12 A sieve for arguments about the band — seven test models. To read alongside: 15-00 Entrance — what ultraviolet control is.

A sieve for arguments about the band — seven test models

This chapter is a map of what is unsolved. § 05 is this book's study (computations on small lattices).

What this chapter shows — Arguments that would close the band have so far been tested against the criterion "distinguish from the U(1) transition" (26-06 § 08). This chapter widens the test models to seven: U(1) at weak coupling, Z_2 at weak coupling, finite non-abelian groups (the quaternion group, the binary icosahedral group and others), the adjoint-Higgs limit (which becomes the U(1) Wilson action), the first-order transition line of the SU(2) mixed fundamental + adjoint action and its endpoint, the bulk transition of SU(N), and the centreless group G_2. For each model we write which types of argument it rules out, and we give a table that sorts this book's Lean theorems and the candidate statements for the band into "true, false, not defined" for the seven models. In each cell we keep apart what was checked by this book's computations or Lean, what was checked in the literature, and what has not been checked. Finally we list, with their conditions, what was seen in computations on small lattices (counting flat connections of finite groups, the power law on a one-point lattice, the hysteresis of the mixed action, the U(1) constant with the aspect ratio fixed, a trial of the sign term by term). The overall map of ideas is in 16-11.

Prerequisite chapters — 26-04 Structure alone cannot cross, 26-06 Renormalisation comparison, 21-05 The minimal action of a vortex, 5-06 How U(1) differs from the non-abelian case, 5-10 Lattice gauge theory with finite groups.

Said first — This chapter gives no road across the band. There are no proofs in it. The seven models are tests for ruling out arguments, and no new theorem is put forward about any of them. Of the properties of these models, only those this book checked at the relevant place in an original source or a review are marked Cited ; the rest are written as "stated in the literature (not checked by this book)" and are not used as grounds in the text. **The discussion of the endpoint of the mixed action is at a fixed lattice spacing. It does not mean that the continuum SU(2) theory has no mass gap.** The view of widening the test models may itself be known to experts. This book writes it not as a new result but as a sieve used in this book's study.

The order of this chapter

1. How the sieve works — tests in two directions — quantities of confinement and quantities of the gap
2. The seven models — the properties of each, and the arguments it rules out
3. Types of argument, and the models that rule them out (Table 1)
4. Lean theorems and candidate statements for the band × the seven models (Table 2)
5. What was seen in computations on small lattices — five items
6. What this chapter does not say

Labels — Proved here = a hand count whose derivation is in this chapter / Cited = a fact from the literature checked at the relevant place in an original source or a review / Reading = a reading of this book's numbers on small lattices (with lattice, sweeps and seeds) / Assumption = a conjecture or expectation / Lean = a Lean theorem of this book (only inside the strong-coupling window, with the lattice spacing fixed). For details, see "Labels" in 6-00.

Notation. The SU(2) Wilson action is $\beta \sum_p (1 - \frac{1}{2} \operatorname{tr} U_p)$; the couplings of the mixed action are β_F for the fundamental representation and β_A for the adjoint representation; the hopping coupling of the adjoint Higgs is J. ΔF is the vortex free energy of a twisted box, and θ is the vortex ratio (19-02). The slab-tension quantity $s(N) = N^2 \sigma_N$ is the notation of 26-06, and is a different quantity from the one-loop slope s.

Words. The **maximal torus** is the abelian subgroup $\{\mathrm{diag}(e^{i\phi}, e^{-i\phi})\}$ of SU(2) (of the same form as U(1)). The **mixed action** is the Wilson action (the fundamental-representation term) with an adjoint-representation term added. A **first-order transition** is one in which the plaquette average and the like jump as functions of the coupling; a **critical endpoint** is the point where a line of first-order transitions ends. A **bulk transition** is a transition that comes from the lattice action and does not depend on temperature (a zero-temperature transition). It is a phenomenon on the lattice, not physics of the continuum limit. A **flat connection** (a flat configuration, in the wording of 21-07) is a configuration in which the holonomy of every plaquette equals the identity (or the sign of the twist, when there is a twist). The **frozen phase** is the phase of a finite-group gauge theory on the large-β side that sticks close to the flat connections.

01

How the sieve works — tests in two directions

If an argument that would close the band goes through in the same form in another model, and the conclusion is false in that model, then the argument is wrong somewhere or is not enough for the conclusion. This is the sieve. Until now the sieve was U(1) alone: U(1) has a transition inside the band ($\beta \approx 1.01$) to a non-confining phase, so an argument that closes the band must distinguish U(1) (26-06 § 08).

The sieve has two directions Assumption .

- An argument whose core is a **quantity of confinement** (the tension, the vortex ratio, the tension-doubling inequality and so on) must become false in a model that has a non-confining phase.
- An argument whose core is a **quantity of the gap** (the correlation length, mixing in a finite box) must be true in a model that does not confine but does have a gap. A quantity of confinement may be false there. The weak-coupling phase of Z_2 and the centreless group are candidates for this, but this book has not checked in the literature that they have a gap (model 2, model 7).

The mass gap and confinement are different statements (16-09 § 03), and the direction in which the sieve is used is decided by which kind of quantity the core statement is.

02

The seven models

Model 1 — U(1) at weak coupling

Properties. Four-dimensional U(1) lattice gauge theory has a non-confining phase at weak coupling. For the Villain form of the action, it has been shown that the expectation of the Wilson loop has a Coulomb-type lower bound and charges are not confined Cited (Guth 1980, checked in the abstract). Fröhlich–Spencer showed by a different method that four-dimensional U(1) lattice gauge theory has a deconfining transition to a massless (QED) phase; they write mainly in the Villain approximation and state in the text that all the results extend to the Wilson-action model (the extension uses techniques from another paper) Cited (1982, checked in the abstract and § 1 of the preprint). The position of the transition, $\beta \approx 1.01$, is a value from this book's record (26-06 § 04).

Its place on the SU(2) axis. Restricting elements of SU(2) to the maximal torus gives $\frac{1}{2}\operatorname{tr}\operatorname{diag}(e^{i\phi}, e^{-i\phi}) = \cos\phi$, so the SU(2) Wilson action $\beta(1 - \frac{1}{2}\operatorname{tr} U_p)$ is, on the maximal torus, the same expression as the U(1) Wilson action $\beta(1 - \cos\phi_p)$ (with the same β) Proved here . What can be said is only that the expressions for the action coincide; the SU(2) measure is not restricted to the maximal torus. The U(1) model at the same β is in the Coulomb phase for $\beta \gtrsim 1.01$. This book's estimate is that this section of the SU(2) axis is the section that needs an input distinguishing U(1) Assumption .

Arguments it rules out. Arguments using only reflection positivity, the centre and duality identities, and positivity of the transfer matrix (26-04). The renormalisation comparison with the aspect ratio fixed holds on the weak side with a uniform constant for U(1) too, so a comparison statement on the weak side does not distinguish U(1) (§ 05 (d)). What stops U(1) is where the trajectory crosses the section of the transition. However, with a quantification that stretches the aspect ratio, U(1) already fails on the weak side. When the core statement is written, the quantification must be stated explicitly.

Model 2 — Z_2 at weak coupling

Properties. Four-dimensional Z_2 lattice gauge theory has a first-order transition Cited (Creutz–Jacobs–Rebbi 1979, checked in the abstract; the position is the self-dual point $\beta \approx 0.44$, as stated in 26-06 § 04). The weak-coupling side is a non-confining phase; that it has a gap there is stated in the literature (not checked by this book).

Arguments it rules out. Arguments that claim to put a quantity of the gap at the core but use a quantity that becomes false at weak coupling in Z_2. The tension-doubling inequality is false in the non-confining phase of Z_2, since there the slab-tension quantity $s(N)$ goes to 0 (26-06 § 08) Reading . If the statement in the literature that the weak-coupling phase of Z_2 has a gap is correct, then the tension-doubling inequality is not a quantity needed for the gap. Under the same condition, a global logarithmic Sobolev inequality on the torus is expected to fail because mixing between holonomy sectors is slow, so it would not pass as a quantity of the gap Assumption .

Model 3 — finite non-abelian groups

The models. Finite subgroups of SU(2): the quaternion group Q_8 (order 8), the binary tetrahedral group (24), the binary octahedral group (48), the binary icosahedral group (120). All are non-abelian and contain -1 in the centre. A finite-group gauge theory is expected to enter a frozen phase at large β Assumption . The value of β at which it freezes is stated in the literature (not checked by this book).

Arguments it rules out. Arguments whose only input is "non-abelian as an algebra". Finite non-abelian groups are non-abelian, but at weak coupling they do not have the continuous fluctuations of SU(2) (asymptotic freedom). In this book's record there are points

where the covering inequality and monotonicity in β fail for Q_8 (21-01: monotonicity in β in three dimensions without background) Reading . The level of the values in the frozen phase is § 05 (a).

Model 4 — the adjoint-Higgs limit (which becomes the U(1) Wilson action)

The model. To the SU(2) Wilson action we add the term $-J\sum_{x,\mu} n(x)\cdot R(U_{x,\mu})\,n(x+\hat{\mu})$ of a unit vector $n(x)\in S^2$ at each site (adjoint representation) (R is the map from SU(2) to SO(3)).

The form at $J=\infty$ Proved here . As $J\to\infty$ the weight concentrates on configurations with $R(U_{x,\mu})n(x+\hat{\mu})=n(x)$. SO(3) acts transitively on S^2, so a gauge transformation can make $n(x)=e_3$ at every site. The condition then becomes $R(U)e_3=e_3$, that is, U commutes with σ_3, that is, U is an element of the maximal torus. The plaquette products then also lie in the maximal torus, and by the coincidence of the action expressions seen in model 1, the action becomes the U(1) Wilson action (with the same β). The remaining gauge freedom is U(1). That the limiting measure actually concentrates on this set is treated as a formal limit.

As a family. For every value of J, the group is continuous and non-abelian, the centre $\mathbb{Z}_2$ acts trivially on the adjoint field so centre symmetry remains, and the weights are positive Proved here . Reflection positivity is also expected to be kept in this family Assumption .

Arguments it rules out. Arguments that take only "the group is continuous and non-abelian, centre symmetry, positivity of the weights" as input and conclude confinement uniformly for all J in the family. These inputs are the same across the whole family, and $J=\infty$ is the U(1) Wilson action, which has the non-confining phase of model 1 (Fröhlich–Spencer, model 1) Cited . Adding the sign of asymptotic freedom to the inputs is not expected to change this Assumption . The one-loop coefficient of the continuum theory with one real adjoint scalar added (the theory of an adjoint scalar whose length is not fixed), in the normalisation of equation (3) of 4-06 ($16\pi^2 b_0$), is the $22/3$ of pure SU(2) minus $1/3$ for one real scalar (half of a complex scalar), which is 7 and remains positive Proved here . But the limiting theory at $J=\infty$ is itself U(1), and is not asymptotically free. What this model rules out are arguments that conclude confinement uniformly in J from inputs alone that hold for every J in the family.

Model 5 — the first-order transition line of the SU(2) mixed action and its endpoint

The model. The action $\sum_p\left[\beta_F\left(1-\frac{1}{2}\operatorname{tr}U_p\right)+\beta_A\left(1-\frac{1}{3}((\operatorname{tr}U_p)^2-1)\right)\right]$. $\beta_A=0$ is the Wilson action. The adjoint term does not see the sign of the centre.

Properties. In this generalisation of the SU(2) Wilson action (which includes the SU(2), SO(3) and Z_2 models as limits), four-dimensional Monte Carlo has reported that the first-order transition lines of SO(3) and Z_2 meet at a triple point and continue to a critical endpoint Cited (Bhanot–Creutz 1981, checked in the abstract). The position of the endpoint varies in the literature. The apparent endpoint that Bhanot–Creutz saw on small lattices is at about $(\beta_F,\beta_A)\approx(1.5,\,0.9)$ (as quoted in the text of Gavai). Gavai found no first-order bulk transition at $\beta_A=1.25$ on 6^4 to 16^4 lattices, and expects the endpoint to be at $\beta_A\geq 1.25$ Cited (Gavai 1996, checked in the abstract and the text). For a mixed action of the same type for SU(3), it has been reported that approaching the endpoint lowers the mass of the scalar (0^{++}) glueball while the masses of the other glueballs do not change, and that the endpoint is consistent with a critical point at which only the 0^{++} correlation length diverges Cited (Heller 1995, checked in the abstract; the ratio of the tension to the transition temperature is nearly constant). That the scalar mass goes to 0 at the SU(2) endpoint is stated in the literature (not checked by this book). That both sides of the line are confining phases is an inference from the line ending at the endpoint and connecting smoothly to the confining phase on the Wilson axis Assumption .

This book's computation. At $\beta_A=1.5$ and 2.0, strong hysteresis is seen even on a 4^4 lattice (§ 05 (c)) Reading . We do not read the position of the endpoint from this computation.

Arguments it rules out. Arguments that use "there is no transition on the β axis / the step map has no fixed point" over the class of local, reflection-positive, pure-gauge actions. Inside this class there are a line of first-order transitions and a critical endpoint. Those are facts specific to the Wilson axis, and do not extend to the class of actions. Since step 0 of the published construction is not the Wilson action but an effective action with a representation (16-03 § 05), this is not a decorative caution Assumption . Arguments that use "the rate is uniformly positive" over the class of actions cannot be decided by this model: across the first-order line the tension may only jump and remain positive, and what would break it is the gap closing at the endpoint. For the same reason, if at the endpoint the scalar mass goes to 0 in lattice units while the tension remains, then "if there is a tension, a soft argument uniform over the class of actions gives the gap" is false; but since this book has not checked this part for SU(2), we place it as cannot be decided (16-11 Table 3).

Caution. The endpoint is a phenomenon at a fixed lattice spacing. It does not mean that the continuum SU(2) theory has no mass gap. The lesson of this model is that what an argument about the band may use is "being on the renormalised trajectory", and that "being in the class of local, reflection-positive, pure-gauge actions" alone is not enough Assumption .

Model 6 — the bulk transition of SU(N)

Properties. For the plaquette action (Wilson action) of SU(N), it has been reported that the crossover between strong and weak coupling looks like a smooth crossover for SU(4) and is a strong first-order bulk transition for SU(6) Cited (Lucini–Teper–Wenger 2002, checked in the text). That it becomes first-order from SU(5) on, and that both sides of the transition are confining, are stated in the literature (not checked by this book).

Arguments it rules out. That arguments of the same type as for model 5 (arguments relying on the absence of a transition on the Wilson axis) do not extend to general SU(N). Anyone aiming at the mass gap for general SU(N) (A-08 § 06) must pass this sieve too.

Model 7 — the centreless group G_2

Properties. G_2 has a trivial centre, so a "quark" in the fundamental representation is screened by gluons, and the string between static quarks can break. Even so, G_2 gauge theory has been reported to confine (in the sense of the paper) Cited (Holland–Minkowski–Pepe–Wiese 2003, checked in the abstract). That it has a mass gap is stated in the literature (not checked by this book).

Arguments it rules out. Quantities written in terms of centre twists, vortices and the vortex ratio θ are not even defined for G_2 Proved here (there are no centre elements, so there is no twist). Arguments whose core is the tension-doubling inequality or the four faces of inequalities that use non-commutativity do not reach the mass gap for all groups. G_2 needs other quantities (the response to boundary fields and so on) (16-11 § 06).

03

Types of argument, and the models that rule them out

Table 1 Types of argument, and the models that rule them out ("rules out" means that the same inputs are present in that model too, and the conclusion is false or not defined there)

Type of argument (inputs)	Models that rule it out	Reason and label
Only reflection positivity, the centre and duality identities, positivity of the transfer matrix	U(1), Z_2	The same structure holds there too (26-04) Proved here
Only non-abelian as an algebra	Finite non-abelian groups	Non-abelian, but without continuous fluctuations. Failure of monotonicity (21-01, three dimensions, no background) Reading
Concluding confinement uniformly in the family from only: the group is continuous and non-abelian, centre symmetry, positivity of the weights	The adjoint-Higgs family ($J = \infty$ is U(1))	These are the same across the family (model 4) Proved here . The non-confining phase of the U(1) Wilson action Cited (Fröhlich–Spencer)
Being in the class of local, reflection-positive, pure-gauge actions, and "no transition on the axis / no fixed point of the step map"	The first-order line of the mixed action	Inside the class, a line of first-order transitions and a critical endpoint Cited Reading
"The rate is uniformly positive" over the class of actions, or "tension ⇒ gap" by a soft argument uniform over the class of actions	The endpoint of the mixed action (cannot be decided)	Fails only if the gap closes at the endpoint. The SU(3) analogue is Cited ; for SU(2) not checked by this book
No transition on the Wilson axis, for general SU(N)	The bulk transition of SU(N)	Strong first order for SU(6) Cited
Only quantities written with centre twists and vortices	G_2	The quantities are not defined Proved here
Claiming a quantity of the gap while using global mixing	Z_2 at weak coupling (if it has a gap)	Slow mixing between sectors Assumption . The gap of the weak-coupling phase of Z_2 is stated in the literature (not checked by this book)
A lossy renormalisation comparison with the aspect ratio fixed (weak side)	U(1) (does not rule it out on the weak side)	On the weak side it holds with a uniform constant for U(1) too (§ 05 (d)) Proved here Reading . The content of the distinction is pushed into the section of the transition

04

Lean theorems and candidate statements for the band × the seven models

Table 2 has as rows this book's Lean theorems (whether a theorem of the same type holds in the other model) and the candidate statements for the band, and as columns the SU(2) Wilson action (the reference) and the seven models. Labels in the cells: Lean this book's Lean, Proved here a derivation in this chapter, Reading numbers from this book's record, Cited checked in the literature. "—" is a cell this book has not checked, and is not filled in by conjecture. "Not defined" means that the model lacks something needed to define the quantity (a centre twist, a continuous generator).

Table 2 True, false, not defined

Row	SU(2) Wilson (reference)	U(1) weak coupling	Z_2 weak coupling	Finite non-abelian groups	Adjoint Higgs $J = \infty$	Endpoint of the mixed action	SU
Infinite-volume limit, mass gap and area law in the strong-coupling window (20-03, 20-04, 20-06)	True Lean (inside the window)	—	—	—	—	Out of scope (the endpoint is outside the window)	—
Reflection positivity, chessboard, $\theta \geq 0$ (20-01, 21-06)	True Lean	—	—	—	—	—	—
Thin-vortex floor (21-02)	True Lean	—	—	—	—	—	—
Thick-vortex floor (21-03; uses the twisted vacuum)	True Lean	— (the definition does not carry over: there is no twisted vacuum of action 0)	— (the same)	—	— (the definition does not carry over)	—	—
The minimal action of a vortex is positive, $S_N > 0$ (21-05; uses the twisted vacuum)	True Lean	— (the definition does not carry over: there is no twisted vacuum of action 0)	— (the same)	True Proved here (subgroups of SU(2) containing $\mathbf{i}, \mathbf{j}$)	— (the definition does not carry over)	—	—
Tension-doubling inequality	Open	False Reading (26-06 § 08)	False Reading (26-06 § 08)	—	False Proved here Reading ($J = \infty$ is U(1) Proved here, the U(1) values are Reading)	—	—
Softness in the twist direction	Open	—	Not defined Proved here	Not defined Proved here	—	—	—
Mixing of box magnetic flux	Open	—	—	—	—	—	—
Twist cost of a column	Open	—	—	—	—	—	—
Pressure margin	Open	—	—	—	—	—	—
Mixing statement for boxes of physical size	Open (a restatement of the mass gap in a strong form)	—	—	—	—	—	—

Grounds for the cells.

- **The row $S_N > 0$** Proved here. The definition of S_N (21-05) fixes the frame of the block to the twisted vacuum (the configuration of action 0 with the quaternions $\mathbf{i}, \mathbf{j}$ on the links). In finite subgroups of SU(2) containing the quaternions $\mathbf{i}, \mathbf{j}$ (an anticommuting pair) — the four groups of this chapter — the twisted vacuum of the frame is made of $\mathbf{i}$ and $\mathbf{j}$ only, so it can be built as a configuration of the subgroup. Configurations whose interior links are restricted to elements of the subgroup are a subset of the SU(2) configurations, so the lower bound on the action is not smaller than the SU(2) lower bound. The SU(2) lower bound is positive uniformly in the size of the block (the theorem `VortexPos.Svortex_ge_unif` of 21-05). Hence the minimal action of a vortex is positive for the subgroups too (a one-line consequence of the Lean theorem). For abelian groups such as U(1) and Z_2, a single twist cannot be satisfied with action 0 (multiplying the plaquette products over a closed surface leaves only the sign of the twist), so there is no twisted vacuum of action 0 and the definition does not carry over.
- **Expectation under a transferred definition** Assumption. If the definition is transferred to a form without the background twist, the action of the configuration that spreads the measured twist flux π evenly over m^2 plaquettes, multiplied by the number N^2 of transverse positions, is $N^2 m^2 (1 - \cos(\pi/m^2)) \leq N^2 \pi^2 / (2m^2)$, which goes to 0 as the block is made large (Proved here: on the

transferred definition). The same spreading can be done in SU(2) too, inside the maximal torus, once the background twist is removed. The minimal action of a vortex is positive in SU(2) because only the central ± 1 commute with the background twisted vacuum (the quaternions $\mathbf{i}$, $\mathbf{j}$), so the flux cannot be spread continuously (the detour in 21-05).

- **The row of the tension-doubling inequality** Reading . In the Coulomb phase of U(1), $s(N)$ is constant independent of scale, exactly on the borderline; in the non-confining phase of Z_2, $s(N) \to 0$ (26-06 § 08; values from closed-form models and small lattices).
- **The "not defined" cells** Proved here . G_2 has a trivial centre, so quantities written with centre twists (θ, the vortex floors, the tension-doubling inequality, and all of the four faces that use twists or centre magnetic flux) do not exist. Softness in the twist direction uses a derivative along the generator of the twist, so it is not defined for discrete groups (Z_2, finite groups).
- **On the "—" cells.** As far as we have read the statements of this book's Lean theorems (A-04 Table 4g and the pages 21-05, 21-07, 20-06), the proofs of the strong-coupling window, reflection positivity, the chessboard, $\theta \geq 0$ and the thin-vortex floor rely not on the non-commutativity of SU(2) but on other structure of the group (the Dobrushin condition, reflection, the central element -1, compactness), and are expected to carry over to U(1), Z_2 and finite groups Assumption . Theorems that use the twisted vacuum (the thick-vortex floor, the minimal action of a vortex) do not carry over to U(1) and Z_2, definitions included. We have not transferred the proof of each theorem to the other models and checked it, so in the table these are "—". The estimate of the behaviour of the four faces for U(1) is in 26-06 § 07 Table 5 (a physical estimate).

What can be read from Table 2 Assumption . So far the only row whose value differs between the reference column and a model column is the tension-doubling inequality (false for U(1) and Z_2, Reading). The row $S_N > 0$ is true for finite non-abelian groups (Proved here), and its definition does not carry over to U(1). That the facts specific to SU(2) in this book are all facts about the action of configurations is in 16-11 § 03. If we set the criterion that a core candidate is chosen from statements that satisfy both "false for U(1) at weak coupling" and "false at a point of the type of the mixed-action endpoint (if the gap closes at the endpoint; for SU(2) not checked by this book)", then filling in the "—" of Table 2 becomes the next piece of work as it stands.

05

What was seen in computations on small lattices

All are floating-point computations on small lattices, and the values from one run go only as far as "seen" or "did not fail in the trial". The scripts are from this book's study, and were run again for this chapter (the results are written below).

(a) Counting flat connections of finite groups (the 1^4 lattice)

Setting. The 1^4 lattice (one site, four links, six plaquettes). In the frozen limit ($\beta \to \infty$) the partition function is proportional to the number of flat configurations. The untwisted side Z_0 is the number of quadruples of link elements that all commute with one another. The twisted side Z_1 is the number of quadruples with $[U_1, U_2] = -1$ (commutator -1) and the other five pairs commuting.

Hand count Proved here (Z_1 of Q_8). In $Q_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, the anticommuting pairs (U_1, U_2) are two non-central elements on different axes: $6 \times 4 = 24$ ways. U_3, U_4 commute with both U_1 and U_2, so they are the central ± 1: $2 \times 2 = 4$ ways. So $Z_1 = 96$.

Table 3 Numbers of flat connections (full enumeration; no seeds) Reading

Group	Order	Z_0	Z_1	$\ln(Z_0/Z_1)$
Q_8	8	736	96	2.037
Binary tetrahedral group	24	5856	96	4.111
Binary octahedral group	48	18816	384	3.892
Binary icosahedral group	120	76320	480	5.069

On running it again, Z_0 and Z_1 for the four groups agreed with the values above. Reading: in the frozen phase, the free energy of one twist is of the type of a constant independent of L (values seen on 1^4 only).

(b) The power law on a one-point lattice (SU(2), 1^4)

Hand count Proved here . The action is $S = \sum_{\mu<\nu}(1 - \frac{1}{2} z_{\mu\nu} \operatorname{tr} P_{\mu\nu})$, with twist $z_{12} = -1$. If the set of minimal-action configurations is a manifold of dimension m with non-degenerate transverse Hessian, then $Z(\beta) = \int e^{-\beta S}$ behaves as $Z \sim C\beta^{-(12-m)/2}$ (12 dimensions for four links), and $\beta\langle S\rangle = -\beta\, d\ln Z/d\beta \to (12-m)/2$. Twisted side: the minimum has U_1, U_2 anticommuting ($U_1 = i n_1 \cdot \sigma$, $U_2 = i n_2 \cdot \sigma$, $n_1 \perp n_2$: 3 dimensions) and $U_3, U_4 = \pm 1$. Since $m = 3$, $\beta\langle S\rangle \to 9/2$. Untwisted side: configurations in which the four links lie in one maximal torus (2 dimensions for the choice of torus + 4 angles: 6 dimensions), giving $\beta\langle S\rangle \to 3$. This family jumps in dimension at the central points, so a logarithmic correction is expected to enter from there Assumption . The ratio is $Z_1/Z_0 \sim c\beta^{-3/2}$ (with a logarithmic correction).

Computation Reading (Metropolis, 4000 chains × 3000 sweeps, the first 1000 sweeps discarded, one seed, errors only from the statistics of block averages).

β	Twisted $\beta\langle S\rangle$	Untwisted $\beta\langle S\rangle$
30	4.545(3)	2.862(2)
300	4.498(3)	2.897(2)
3000	4.497(2)	2.860(3)

Reading: the twisted side matches $9/2$ at $\beta = 300$ and 3000 ($\beta = 30$ gives 4.545), and the untwisted side is 0.10–0.14 below 3. The difference of power $-3/2$ was seen. The coefficient of the logarithm is not determined by this computation (the value at $\beta = 3000$ is lower than at 300, and there is a suspicion that thermalisation in the flat directions is slow). Rerun: $\beta = 300$ was run with the same seed, and agreed at 4.4977(33) and 2.8970(19).

(c) Hysteresis of the mixed action (SU(2), 4^4)

Setting Reading . The action of model 5, 4^4 periodic, Metropolis (simultaneous updates by sublattice). β_F is raised $0 \to 2.5$ and then lowered, 80 sweeps at each point, from a cold initial configuration. One seed for each β_A (two only for $\beta_A = 2.0$). We look at the maximum of "down − up" for the plaquette average $\frac{1}{2}\operatorname{tr} P$.

β_A	Maximum of down − up (at that β_F)	Reading
0	0.023 or less (everywhere)	No hysteresis. 0.49–0.52 at $\beta_F = 2.0$, 0.65 at 2.5
1.0	0.079 (1.25), 0.063 (1.5)	A size hard to distinguish from fluctuations
1.5	0.317 (1.0), 0.165 (1.25)	Strong
2.0	0.413 (0.5, 0.75), 0.205 (1.0). With another seed 0.516 (0.5), 0.430 (0.75), 0.311 (1.0)	Strong

Differences of the opposite sign appear in the same runs with sizes up to 0.08 ($+0.076$ at $\beta_A = 2.0$, $\beta_F = 0$, and so on), so differences of 0.08 or less are hard to distinguish from fluctuations in this computation. Reading: at $\beta_A = 1.5$ and 2.0, strong hysteresis was seen even on 4^4. The weak difference at $\beta_A = 1.0$ may be a finite-size effect of 4^4 (of the type of Gavai's re-examination, which reported the first-order transition disappearing on larger lattices). We do not read the position of the endpoint from this computation. Hysteresis also appears with slow dynamics, so neither the mass nor the tension at the endpoint comes out of this computation. Rerun: the four values of β_A were run with the same seeds, and all the values agreed.

(d) The U(1) constant with the aspect ratio fixed

Hand count Proved here . Using the slab approximation for the Coulomb phase of U(1) (equation (4) of 26-03, read with $e^2 = 1/\beta$, vortex area M^2 and transverse area N^2), $\theta(N, M, \beta) \approx 2\exp(-M^2/(2\beta N^2))$, and fixing the aspect ratio $r = M/N$, we get

$$\frac{\theta(2N, 2M, \beta)}{\theta(N, M, \beta')} = \exp\Big(\frac{r^2}{2}\Big(\frac{1}{\beta'} - \frac{1}{\beta}\Big)\Big)$$

which does not depend on N. Putting $\beta' = \beta - c$, $1/(\beta - c) - 1/\beta = c/(\beta(\beta - c))$ decreases in β, so uniformly over all $\beta \geq \beta_1$

$$K = \exp\Big(\frac{r^2 c}{2\beta_1(\beta_1 - c)}\Big) \quad (1)$$

gives the form of the renormalisation comparison $\theta(2N, 2M, \beta) \leq K\theta(N, M, \beta - c)$ (on the approximate formula).

Computation Reading (a numerical scan of the approximate formula: 4000 points in $\beta \in [\beta_1, 10^4]$, $N = 2, 4, 8, 16, 64$, $r = 1, 2, 4$; no seeds). The upper bound of the scan agreed with equation (1). Examples: with $c = 0.258$, $\beta_1 = 3$, $K = 1.016$, 1.065, 1.285 for $r = 1, 2, 4$. With $c = 0.258$, $\beta_1 = 10$, 1.0013, 1.0053, 1.0214. Running it again gave the same values.

Reading: on the weak side, the comparison with the aspect ratio fixed holds uniformly for U(1) too, in the form "runs to strong coupling at any speed c you like", and uses no non-commutativity at all. $K(r)$ grows without bound with r, so with a quantification that fixes the transverse size and stretches the longitudinal one, U(1) already fails on the weak side. The approximate formula drops the effect of monopoles (of order $e^{-c\beta}$ at finite β), so we have not checked whether the same constant holds for true U(1).

(e) A trial of the sign term by term (SU(2), 1^4)

Question. When each plaquette is given its own coupling β_p, does ΔF (on the background $z_{12} = -1$, with and without the measured twist z_{34}) increase in each β_p? We look at the sign of $\partial\Delta F/\partial\beta_p = \langle E_p\rangle_{z_{34}=-1} - \langle E_p\rangle_{z_{34}=+1}$ ($E_p = 1 - \frac{1}{2} z_p \operatorname{tr} P$) Proved here (the derivative of $\Delta F = \ln Z_{z_{34}=+1} - \ln Z_{z_{34}=-1}$).

Computation Reading (Metropolis, 3000 chains × 800 sweeps, the first 200 sweeps discarded, β_p uniform random in $[0.2, 3.0]$ for the 6 plaquettes; 6 sets without the adjoint term and 2 sets with adjoint $\beta_A = 1.0$, 48 differences in all). All 48 are positive, and the smallest difference is 0.029 (a set with one β_p at 0.2). The statistical ratios are 58–1700 (errors from block averages, which may come out small because of autocorrelation, but the signs do not move). Rerun: the first 2 sets (seed 1) were run, and all the values agreed.

Reading: 1^4 is a special lattice in which the six plaquettes are joined at one point, and β_p goes only up to 3, so we cannot say that "monotonicity term by term holds". The contrast with the failure of a similar monotonicity for the finite group Q_8 (21-01: monotonicity in β in three dimensions without background, under conditions different from this trial) is worth keeping on record. The next lattice to try is an elongated box (16-11 § 05).

06

What this chapter does not say

- It gives no road across the band. There are no proofs in it.
- The discussion of the endpoint of the mixed action is at a fixed lattice spacing, and does not mean that the continuum SU(2) theory has no mass gap. The position of the SU(2) endpoint varies in the literature, and this book has not checked that the scalar mass goes to 0 at the endpoint.
- We say neither true nor false for the "—" cells of Table 2.
- Of the properties of the seven models, those written as "stated in the literature (not checked by this book)" are not used as grounds for claims in the text.
- We draw no conclusions about large lattices or the continuum theory from values on small lattices.

What this chapter can say

Can say	Cannot say
Lean The reference column of Table 2 (this book's theorems, inside the strong-coupling window, with the lattice spacing fixed)	Theorems of the same type in the other models (the "—" cells)
Proved here That the expressions of the SU(2) and U(1) Wilson actions coincide on the maximal torus. That the adjoint Higgs at $J = \infty$ becomes the U(1) Wilson action (a formal limit). That with one real adjoint scalar the one-loop coefficient is 7 (positive). The row S_N (positive for finite subgroups; the definition does not carry over to abelian groups). The cells not defined for G_2 and discrete groups. $Z_1 = 96$ for Q_8. The count of powers on 1^4. The U(1) constant (1) with the aspect ratio fixed (on the approximate formula)	A check of the concentration of the limiting measure. A uniform constant for true U(1). $S_N = 0$ under the transferred definition
Cited The non-confining phase of U(1) (Villain action; the extension to the Wilson action in the text of Fröhlich–Spencer). The first-order transition of Z_2. The first-order line and critical endpoint of the mixed action, and the re-examination of the endpoint position (expected at $\beta_A \geq 1.25$). That the scalar mass drops near the SU(3) endpoint. The strong first-order bulk transition of SU(6). That G_2 has a trivial centre and its string can break	The mass at the SU(2) endpoint, the bulk transition of SU(5), the gap of Z_2 at weak coupling and of G_2, the freezing β of finite groups
Reading The numbers of flat connections of finite groups (full enumeration). $\beta\langle S\rangle$ of SU(2) on 1^4 (4000 × 3000, one seed). The hysteresis of the mixed action on 4^4 (80 sweeps per point, one seed for each β_A, two for $\beta_A = 2.0$). The 48 signs term by term (3000 × 800)	The position of the endpoint. That monotonicity term by term holds

Sources and reproduction

Item	Kind	Source or tool
The non-confining phase of U(1) (Villain action)	Cited	A. H. Guth, "Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory", Phys. Rev. D 21 (1980) 2291 (checked in the abstract)
The transition of U(1) to a massless phase, and the extension to the Wilson action	Cited	J. Fröhlich, T. Spencer, "Massless phases and symmetry restoration in abelian gauge theories and spin systems", Commun. Math. Phys. 83 (1982) 411 (checked in the abstract and § 1 of the preprint IHES/P/81/40; written mainly in the Villain approximation, with the statement in the text that all the results extend to the Wilson-action model; the extension uses techniques from another paper)
The first-order transition of four-dimensional Z_2	Cited	M. Creutz, L. Jacobs, C. Rebbi, "Experiments with a gauge-invariant Ising system", Phys. Rev. Lett. 42 (1979) 1390 (checked in the abstract)
The first-order lines, triple point and critical endpoint of the mixed action	Cited	G. Bhanot, M. Creutz, "Variant actions and phase structure in lattice gauge theory", Phys. Rev. D 24 (1981) 3212 (checked in the abstract)
The quotation of the apparent endpoint $(1.5, 0.9)$, and the re-examination on larger lattices (no first-order transition at $\beta_A = 1.25$; the endpoint expected at $\beta_A \geq 1.25$)	Cited	R. V. Gavai, "A study of the bulk phase transitions of the SU(2) lattice gauge theory with mixed action", Nucl. Phys. B 474 (1996) 446, arXiv:hep-lat/9603003 (checked in the abstract and the text)
The scalar mass approaching the endpoint of the SU(3) mixed action	Cited	U. M. Heller, "SU(3) lattice gauge theory in the fundamental–adjoint plane and scaling along the Wilson axis", Phys. Lett. B 362 (1995) 123, arXiv:hep-lat/9508009 (checked in the abstract)
The bulk transition of SU(N) (smooth for SU(4), strong first order for SU(6))	Cited	B. Lucini, M. Teper, U. Wenger, "The deconfinement transition in SU(N) gauge theories", Phys. Lett. B 545 (2002) 197, arXiv:hep-lat/0206029 (checked in the text)
That G_2 has a trivial centre and its string can break; confinement	Cited	K. Holland, P. Minkowski, M. Pepe, U.-J. Wiese, "Exceptional confinement in G(2) gauge theory", Nucl. Phys. B 668 (2003) 207, arXiv:hep-lat/0302023 (checked in the abstract)

Item	Kind	Source or tool
The formula for the one-loop coefficient	Proved here	One real adjoint scalar put into equation (3) of 4-06
Statements in the literature that this book has not checked: the gap of the weak-coupling phase of Z_2, the freezing β of finite groups (Petcher–Weingarten 1980), the scalar mass at the SU(2) endpoint, the bulk transition of SU(5) and the phases on both sides, the gap of G_2	—	Stated in the literature (not checked by this book). Not used as grounds for claims in the text
This book's Lean theorems	Lean	The declarations of 20-06, 21-02, 21-03, 21-05, 21-06, 21-07 (`VortexPos.SN_pos`, `TwistBlocks.thick_vortex_floor`, `TwistedWilson.thin_vortex_floor`, `TwistBlocks.theta_nonneg`, `TwistFlat.no_flat_twist11` and others). The statements were checked in A-04 Table 4g and on the pages 21-05, 21-07 and 20-06
The position of the U(1) transition, the position of the Z_2 transition, the values of the tension-doubling inequality in the three phases, the estimates for the four faces	Reading	26-06 § 04, § 07, § 08
(a)–(e) of § 05	Reading	Scripts from this book's study (numpy floating point, full enumeration and Metropolis). Run again for this chapter: (a) all, (b) $\beta = \mathbf{300}$, (c) all plus one other seed, (d) all, (e) the first 2 sets

Words used in this chapter — sieve, quantity of confinement, quantity of the gap, maximal torus, adjoint Higgs, mixed action, first-order transition, critical endpoint, bulk transition, centreless group, flat connection, frozen phase, twisted vacuum, hysteresis, aspect ratio, sign term by term. Their meanings are in A-02 Glossary.

Chapter to read next: 27-00 Entrance — the promise of this part. The overall map of ideas is in 16-11.