Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 27 Closing — conditional statements

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Entrance — the promise of this part

> This chapter is a guide (the entrance to this part).

This part is a record of the study. The theorems that have been machine-checked are in Parts 20–22, and the current list of premises is in 28-04.

What this chapter shows — The promise on how Part 27 is written, and the differences that remain even when the ledger of conditions is empty. The figure of the two roads meeting at the band, shown again, and a map of 27-01 to 27-04.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain). You can read this part without having finished Parts 23 to 26.

01

The promise of this part

Part 27 writes conditionally until the conditions disappear.

- The statement is split arrow by arrow, and each arrow carries a label. We call something a "theorem" only for arrows whose paper proof has been placed in this book. That is why the title of 27-01 is "the conditional statement".
- The conditions are placed one row at a time in the ledger of conditions (27-02). The ledger is written with the facts as they stand after the publication of 2026-11-10. The continuum limit in a periodic cubic box was shown by the published construction (at the level of the committee's assessment; Volume 2), but the form this book's road needs (twisted boundary conditions, elongated boxes, a form uniform in the lattice spacing) is not there, so the row of ultraviolet control remains. A row disappears when its paper proof has been placed in the book and, if it has a Lean entry, the re-check of that entry is done. As research advances, rows disappear, but they can also split and increase.
- Even if every row of the ledger disappears, the difference from the problem statement remains: the axioms on $\mathbb{R}^4$ (the infinite-volume limit, the recovery of rotations, non-triviality; the infinite-volume limit (along a subsequence) from a sequence of periodic boxes, and the reconstruction, were shown by the published construction, but the reduction from the family of twisted slabs on this book's road

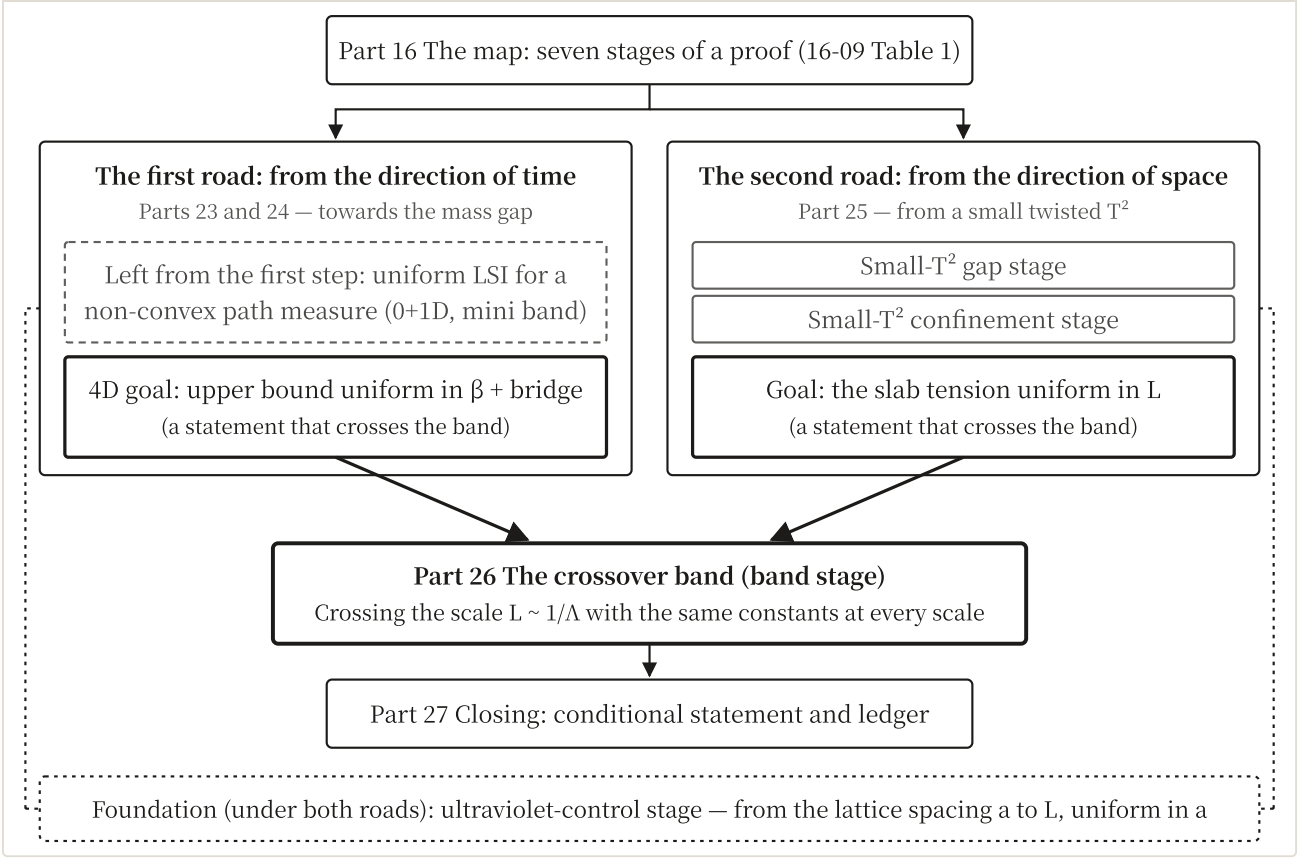
is open), all compact simple groups (existence was shown by the published construction for any compact simple group; the vortex tools cannot be used for groups without a centre), the arrow from the area law to the mass gap (tying the two roads together), and the verification and peer review of the whole. These four (the limit to R^4 [split into four rows in 27-02 Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown) are placed in a separate frame outside the ledger (27-02 § 05). An empty ledger does not mean that the problem is solved.

- This part has no chapter called "solved".

02

The two roads meet at the band (shown again)

Figure 1 16-09 Figure 1, shown again. On both roads, the arrow flowing into the band leaves only from the row of the four-dimensional goal. The ultraviolet-control stage lies under both roads. Part 27 corresponds to the one box below the band



03

A map of the four chapters

Table 1 The chapters of Part 27

Chapter	Contents (one line)

Chapter	Contents (one line)
27-01 The conditional statement	Under assumptions, to the mass gap and confinement. The label on each arrow, and the arrow still left blank (from the area law to the gap)
27-02 The ledger of conditions	The conditions, one row each. The state in the record, the label, what is needed for it to disappear, the dependency diagram, and the differences that remain outside the ledger
27-03 The table of questions	A cross-index that looks up the questions of Parts 23 to 26 by chapter and item number
27-04 What is not claimed	A cross-index that looks up "what is not claimed" in Parts 23 to 26 by chapter and item number

Sources

Figure 1 is the same figure as in 16-09. The grounds for the four items of the promise are in 27-02 § 01, § 05 and § 06.

Next chapter to read: 27-01 The conditional statement.

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the conditional statement, a label for each arrow, the open arrow · Kiichi, Shiori and Rin

The conditional statement — the mass gap and confinement, arrow by arrow

> This chapter is a map of open questions. § 03 is this book's own investigation.

What this chapter shows — Under the assumptions of ultraviolet control, uniform acceptance of insertion, the base point at small T^2, frame decorrelation, the uniform slab tension, the uniform slab gap and tying the two roads together, which arrows the mass gap and confinement pass through. The gap comes out of the small-T^2 gap stage and the transfer matrix, and confinement out of the small-T^2 confinement stage and the band stage; we put a label on every single arrow. The only arrows whose paper proof we could place in the book are the algebra of the doubling iteration (the Doubling-Iteration Theorem) and slab confinement at fixed lattice spacing (the other arrow at fixed lattice spacing, the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3), has a proof with citations in the paper, and its label in the book is Assumption). The arrow from the area law to the mass gap is open, and even the statement of what should be added has not been settled.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain), 27-00 (the promise of this part), 25-04 (slab confinement), 25-05 (the insertion identities), 26-02 (reduction to a single inequality), 26-05 (the chessboard). The names and contents of the assumptions are in A-06 § 06, and the present state of each assumption is in 27-02 Table 2 (the rows of the ledger of conditions) and Table 3 (the differences that remain outside the ledger); this chapter refers to the assumptions by the names of the ledger rows.

Said first — This chapter shows neither the mass gap nor confinement. We do not title it "the conditional theorem" because many of the arrows that make up the statement have no paper proof. Only the arrows whose paper proof we could place in the book are called "theorems"; the rest are listed as labelled "arrows". Even if every row of the ledger of conditions disappears, a difference from what the problem statement requires remains (§ 05).

Order of this chapter

1. The form of the statement — the assumptions, the two conclusions, the two levels (Table 1)
2. The gap side — the small-T^2 gap and the transfer matrix (two arrows)
3. The confinement side — the arrows of small-T^2 confinement and of the band, the Doubling-Iteration Theorem
4. Tying the two roads together — the open arrow tying the two roads together, and the statement of the road from the direction of time
5. What cannot be said, and the differences that remain even when the ledger is empty
6. The label of each arrow, and the dependency graph — Table 2

01

The form of the statement — the assumptions, the two conclusions, the two levels

The statement of this chapter is not a single "if ... then" but a bundle of arrows. There are two conclusions.

- **The mass gap**: the spectrum of the transfer matrix (the Hamiltonian) above the vacuum is $\{0\} \cup [\Delta, \infty)$ with $\Delta > 0$ (4-11).
- **Confinement**: the expectation value of a large Wilson loop decreases exponentially in the area (the area law, 5-08).

The two are different statements. There is no known road that gets the gap from the area law, and "showing confinement also gives the gap" is a misconception (16-08). So this chapter derives the two conclusions each through its own bundle of arrows, and at the end places, still open, the arrow that joins the two on the same theory.

We also split the levels into two. The **theory on a small $\mathbf{T^2}$** is the theory on $\mathbb{R}^2 \times$ a small twisted T^2 (side L, 't Hooft twist $n_{12} = 1$), with the limit of lattice spacing $a \to 0$ taken. It is not what the problem statement requires but an intermediate target. The **theory on $\mathbf{R^4}$** is the theory on $\mathbb{R}^4$, reached from the theory on a small T^2 through the stage of widening the T^2 (the band) and the stage of the infinite-volume limit.

Table 1 lists the assumptions used. Their labels are all Assumption: rows without a proof. The names are taken from the contents.

Table 1 The assumptions placed by the statement of this chapter (the numbers are those of the dependency graph in § 06; the ledger column gives the row of 27-02; the stages are the seven stages a proof must contain, of 16-09)

No.	Ledger	Name	Contents (one line)	Stage	Label
1	Ultraviolet control	Ultraviolet control	From the lattice spacing a up to the scale L, a multiscale analysis uniform in a holds: locality and analyticity of the small-field effective action (the small-field analyticity condition), rarity of large fields (the large-field rarity condition), and the effective coupling stays in $(0, \gamma]$. The continuum limit in a periodic cubic box was shown by the published construction (Volume 2), but a construction with twisted boundary conditions or elongated boxes is not there (within what we searched). Ultraviolet control here is the still open condition that the same construction holds under twisted boundary conditions	Ultraviolet-control stage	Assumption
2	Uniform acceptance of insertion	Uniform acceptance of insertion	Under the small-field frame, the mean of $\tanh(W/2)$ for the dynamically driven vortex insertion is at most $1 - \delta$, uniformly in a (25-06)	Small-T^2 confinement stage	Assumption
3	Base point at small T^2	Base point at small T^2	At a small physical L_0, $\theta(L_0; T^2_{L_0}) \le 1 - \varepsilon_0$ (θ is the sector bias of the container, 25-03). The expected size is $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ (a one-loop estimate). In the outline it follows from ultraviolet control and uniform acceptance of insertion	Small-T^2 confinement stage	Assumption

No.	Ledger	Name	Contents (one line)	Stage	Label
4	Frame decorrelation	Frame decorrelation (form with L fixed)	The correlations between the frames of containers decay, and the Wilson-loop estimate $\mathbb{E}\prod\theta$ can be split into the product of the individual θ at the loss of a constant	Small-T² confinement stage, band stage	Assumption
5	Small-T² gap (form uniform in the lattice spacing)	Decay of block observables (form with L fixed)	The correlations of block observables decay exponentially at a rate $m \geq c/L$ uniform in a (the form placed by the research programme for the theory on a small T²)	Small-T² gap stage	Assumption
6	A form inside positivity of the 't Hooft tension (the place of the input to the reduction is the renormalisation comparison)	Lateral-widening inequality	With the container fixed, widening the transverse T^2 by a factor s increases θ by a factor of at most Cs^2 (strong form), or a form in which the total loss is finite (weak form). The assumption when the band is stated in the form of the paper on the reduction of the band (26-02)	Band stage	Assumption
7	The weak form of positivity of the 't Hooft tension	Uniform slab tension	The string tension $\sigma_{\text{eff}}(L)$ of the slab $T_L^2 \times \mathbb{R}^2$ is positive uniformly in L	Band stage	Assumption
8	Uniform decay of block observables	Uniform slab gap	The correlations of block observables decay exponentially, uniformly in the lattice spacing and in the slab size L (the premise of the arrow of the R⁴ gap)	Band stage	Assumption
9	Limit to R⁴ (outside the ledger)	The limit to $\mathbb{R}^4$	The infinite-volume limit, the OS axioms, the recovery of rotation invariance, uniqueness of the limit, non-triviality	R⁴ stage	Assumption
10	Tying the two roads together (outside the ledger)	Tying the two roads together	That the gap road (carrying the small-T² gap stage through the band) and the confinement road (from the small-T² confinement stage and the band stage to $\mathbb{R}^4$) give both conclusions on the same theory on $\mathbb{R}^4$. The statement of what it suffices to assume has itself not been settled	Small-T² gap stage, small-T² confinement stage, R⁴ stage	Assumption (statement not settled)

We state the correspondence with the ledger first. In the ledger, frame decorrelation (form with L fixed) corresponds to the row of frame decorrelation, and the uniform slab gap to the row of uniform decay of block observables; the lateral-widening inequality and the uniform slab tension are treated as forms inside the row of positivity of the 't Hooft tension. The non-abelian input to the reduction to the band is placed in the ledger in the row of the renormalisation comparison. The row of reflection positivity of z is a closed row (its proof is placed in 26-04 § 03; the algebra of the last step is also closed in Lean (ReflPos)), and in this chapter it is used inside the chessboard arrow. The limit to R⁴ and tying the two roads together are differences outside the ledger; they are not work of removing assumptions but work of carrying the conclusion as far as the problem statement. We put them in the table in this chapter so that they can be named as premises of arrows.

We split the "frame decorrelation" of Table 1 into two forms. Frame decorrelation (form with L fixed) is the form for splitting the container estimate on the confinement side, and decay of block observables (form with L fixed) is the form placed by the research programme on the gap side of the theory on a small T². They are assumptions of the same family, but as sentences they are different, and decay of block observables (form with L fixed) contains the quantity of the conclusion (the rate m) (§ 02). θ is the sector bias of the container $\theta(R; T_L^2)$ (side of the container R, side of the transverse T^2 L), a quantity different from the θ of A-06 Table 6 (the exponent of accumulation of the shortfall).

02

The gap side — the small-T² gap stage and the transfer matrix

The arrow of the small-T² gap (theory on a small T²). Under ultraviolet control and decay of block observables (form with L fixed), the theory on $\mathbb{R}^2\times$ a small twisted T^2 exists (as a subsequential limit of the distributions of block observables), is reflection positive and invariant under translations of $\mathbb{R}^2$, and its transfer matrix satisfies $\|T_t\| \le e^{-mt}$, $m \ge c/L$ on the orthogonal complement of the vacuum Assumption . This is the form put forward as a proposition by the research programme for the theory on a small T², and the proof is at the stage of an outline. The conclusion corresponds to the ledger row of the small-T^2 gap.

To say precisely what this arrow contains: the transfer-matrix step is an abstract lemma of the vessel stage. It is the general fact that, for a self-adjoint contraction on a real inner-product space, if the correlations decay at rate m on a dense set then the operator norm is at most e^{-m}; it is independent of the physical measure. So what carries the content of the mass gap inside the arrow of the small-T² gap is the part "decays at rate $m \ge c/L$" of the assumption decay of block observables (form with L fixed). Decay of block observables (form with L fixed) contains the quantity of the conclusion, and the arrow of the small-T² gap is most accurately read as "an arrow that restates uniform decay of block correlations as a gap of the transfer matrix". That the twist removes the zero modes (the adjoint field has no zero modes, 25-01) is a reason to expect decay of block observables (form with L fixed) to hold, not a proof of it.

The arrow of the R⁴ gap (theory on R⁴). The gap $m \ge c/L$ of the arrow of the small-T² gap approaches 0 as $L \to \infty$, so as it stands it is not a gap on $\mathbb{R}^4$. Carrying it to $\mathbb{R}^4$ needs the mass gap of the slab to be uniform in L (the uniform slab gap), and further uses, inside the infinite-volume limit (the limit to R⁴), that uniform exponential decay passes to a gap of the transfer matrix of the limiting theory Assumption . Here too, the content of the gap lies in the band assumption (the uniform slab gap). The phrase "the gap comes from the small-T² gap stage and the transfer matrix" describes the case in which the gap of the small-T² gap stage could be carried through the band uniformly in L, and the carrying is the problem of the band itself (26-00).

03

The confinement side — the small-T² confinement stage and the band stage

On the confinement side there are two arrows with a paper proof at the stage of fixed lattice spacing; the arrows at the stage of sending the lattice spacing to 0 and at the stage of the band rest on assumptions.

Two arrows at fixed lattice spacing. (a) The arrow of slab confinement: with the lattice spacing and the size N of the T^2 (in lattice units) fixed, the area law holds for every β, and the tension is bounded by $\sigma \ge -\ln\tanh(\beta N^2) \approx 2e^{-2\beta N^2}$ Proved here (25-04). The tension is set by the cost of the surface of a thin $\mathbb{Z}_2$ vortex, and approaches 0 in the continuum limit. (b) The arrow of the fractional-instanton rate (the Confinement Lower Bound by Fractional Instantons): on the lattice $T^2_N \times T^2_M$ with fixed $N \ge 2$, block side $\ell \ge 2$, $\beta \ge 1$ and $M \in 2\ell\mathbb{N}$, the bias θ of the n_{34} flux satisfies $-\ln\theta \ge c(N,\ell)\,\beta^{-a} e^{-\beta(S_*(N,\ell)+\varepsilon(\beta))} M^2$, $\varepsilon(\beta) = O(\sqrt{\ln\beta/\beta})$ Assumption (no proof is placed in this book; a proof with citations is in the paper: it cites the Chessboard Estimate and the Morse–Bott lemma, and the rigidity constants are shown only to exist. The paper is a draft and has not been through outside checking. 25-02 § 07 (e)). S_* is the minimal action of a fractional instanton inside a frame frozen to the vacuum, and the statement says that the rate is set by fractional instantons, not by the $2N^2$ of a thin vortex. In the ledger this is the

confinement-direction row of the upper end of the renormalisation comparison. The Reverse Estimate at a Fixed Plane (Theorem B of Paper 3) of the same paper is the bound with the size M of the plane fixed. Neither is uniform in N, and neither says anything about the continuum limit.

The arrow of the small-T² area law (theory on a small T²). Under ultraviolet control and uniform acceptance of insertion, Wilson loops in the $\mathbb{R}^2$ plane of the theory on $\mathbb{R}^2\times$ a small twisted T^2 have an area law uniform in a Assumption . The expected form of the tension is $\sigma \geq c' e^{-F/g^2(L)}/L^2$ (F is expected to be the fractional-instanton action $4\pi^2$). Among the parts of the assembly, the insertion identities and the estimate of the mean are closed on paper and in a finite version in Lean (25-05, 25-06), and in the outline the large-deviation step reduces to a chessboard estimate of the same type as the large-field rarity condition of ultraviolet control. The base point at small T² corresponds to the conclusion of this arrow written in the language of the container's bias.

The arrow of the reduction of the band (the form of the reduction of the band). Under the base point at small T², frame decorrelation and the lateral-widening inequality, $\theta(R;T_R^2)\leq e^{-cR^2}$ Assumption (the conditional theorem of the paper on the reduction of the band). The proof in the paper on the reduction of the band is a sketch, and the part "frame decorrelation splits the product at the loss of a constant" is not written as a numerical inequality. In this book we rewrite the assumptions in the form of numerical inequalities, and then put the remaining algebra on paper as the Doubling-Iteration Theorem.

Doubling-Iteration Theorem

Let θ be a function giving a value in $[0,1]$ to each pair of positive lengths (R,L). Let $R_0>0$, $R_k=2^k R_0$, and suppose that for positive constants K_1,K_2, for every $k\geq 0$,

$\theta(2R_k;T_{R_k}^2)\leq K_1\,\theta(R_k;T_{R_k}^2)^4$ (nesting), $\theta(R_{k+1};T_{R_{k+1}}^2)\leq K_2\,\theta(R_{k+1};T_{R_k}^2)$ (lateral widenin

holds. Put $K=K_1K_2$, $t_k=\theta(R_k;T_{R_k}^2)$. If $K^{1/3}t_0<1$, then for every k

$$\theta(R_k;T_{R_k}^2)\;\leq\;K^{-1/3}\exp\bigl(-c\,R_k^2\bigr),\qquad c=\frac{\ln\bigl(1/(K^{1/3}t_0)\bigr)}{R_0^2}>0. \tag{2}$$

Proved here Proof. Since $2R_k=R_{k+1}$, applying the two parts of (1) in turn gives $t_{k+1}\leq K_2\,\theta(2R_k;T_{R_k}^2)\leq K_1K_2\,t_k^4=Kt_k^4$. Putting $u_k=K^{1/3}t_k$ gives $u_{k+1}\leq u_k^4$, and by induction $u_k\leq u_0^{4^k}$. Since $4^k=(R_k/R_0)^2$, we have $u_0^{4^k}=\exp(-(R_k/R_0)^2\ln(1/u_0))$, and (2) follows. ∎ Lean Doubling (`Doubling.iterate`) (assumptions: $c\geq 0$, $t_k\geq 0$, $t_{k+1}\leq c^3t_k^4$; conclusion: $c\,t_k\leq(c\,t_0)^{4^k}$), Doubling (`Doubling.decay`) (the same assumptions and $c>0$; conclusion: $t_k\leq(c\,t_0)^{4^k}/c$). Read $c=K^{1/3}$.

The starting value $K^{1/3}t_0<1$ is made from the base point at small T² with a single big jump. If, in the nesting with $T_{L_0}^2$ fixed, $\theta(sL_0;T_{L_0}^2)\leq C_1(1-\varepsilon_0)^{s^2}$, and, by lateral widening, $\theta(sL_0;T_{sL_0}^2)\leq C_2s^2\,\theta(sL_0;T_{L_0}^2)$ can be said, then the $t_0\leq C_1C_2s^2e^{-\varepsilon_0 s^2}$ at $R_0=sL_0$ becomes smaller than $K^{-1/3}$ for $s^2\gg\varepsilon_0^{-1}\ln(1/\varepsilon_0)$ Proved here (the same algebra).

The Doubling-Iteration Theorem is algebra, and all the content is in (1) and in the two inequalities of the jump. The nesting inequality is exact as a partition inside a container with T^2 fixed (25-03), but the sub-containers share frames and are correlated, so splitting into a product with constant K_1 needs frame decorrelation. The lateral-widening inequality is the assumption of Table 1 itself. That is why the label of the arrow of the reduction of the band is Assumption, and only the part that is the Doubling-Iteration Theorem could be put on paper.

The arrow from the container estimate to the area law (from the form of the reduction to the area law). From $\theta(R;T_R^2) \le e^{-cR^2}$ and frame decorrelation, by the container estimate (the container identity of Mack–Petkova, 25-03 Cited), an area law with tension uniform in L Assumption . The global version that does not use frames uses the ratio of twisted partition functions and the Tomboulis–Yaffe inequality Cited , and in exchange the base point has to be shown globally (26-01).

The chessboard arrow, and the arrow from the two uniformities to the area law (the chessboard form). With the size of the T^2 fixed, from the reflection positivity of z Proved here (a proposition of the paper on the reduction of the band; ReflPos (`ReflPos.refl_pos` , `ReflPos.refl_cs`) is the algebra of the core, 26-04) and the Chessboard Estimate Cited , frame decorrelation is replaced by a single density κ_{cb}, and $|\langle W(C)\rangle| \le \kappa_{\mathrm{cb}}^{n(C)}$ Proved here (a statement with L fixed, 26-05). In this language the band becomes two uniformities, and under the uniform slab tension and the uniform slab gap an area law with tension uniform in L follows Assumption . Reading the uniform slab gap in terms of κ_{cb}, the power in the nesting within a fixed T_L^2 holds with constant 1 for all L (the Nested Supermultiplicativity Lemma of 26-05 Paper), and the content of the uniform slab gap uniform in L moves to the comparison across L, (C $_{\mathrm{cb}}$) — the lateral-widening inequality read in terms of κ_{cb} (the Reduction Read in the Chessboard Density of 26-02).

Confinement in the theory on R^4 follows from the area law of the arrow from the container estimate to the area law or of the arrow from the two uniformities to the area law, together with the limit to $\mathbb{R}^4$ Assumption .

04

Tying the two roads together — the open arrow tying the two roads together, and the statement of the road from the direction of time

The arrow tying the two roads together (open). The gap road of § 02 and the confinement road of § 03 both reach $\mathbb{R}^4$ through assumptions that cross the band (the uniform slab gap, or the lateral-widening inequality and the uniform slab tension). To state both conclusions on the same theory on $\mathbb{R}^4$, we need an arrow saying that the two roads give the same limiting theory and that both hold in that theory. This is tying the two roads together (a difference outside the ledger). Its label is Assumption, but unlike the other assumption rows, there is not yet a sentence saying what it suffices to assume.

Since there is no known road going directly from the area law to the mass gap, the arrow tying the two roads together cannot be written as "an extension of the confinement road". At the level of physical expectation the two are linked (excitations on a string of tension $\sigma > 0$ are heavy, 17-03), but that is not an arrow Reading (a physical estimate). The gap is derived by the road of § 02 (carrying the gap of the small-T^2 gap stage through the band), confinement by the road of § 03, and the arrow tying the two roads together is limited to the role of saying "the two are statements about the same theory".

The statement of the road from the direction of time. The conditional statement of the road of Part 23 has a different form. From the upper side $\xi^2\uparrow$ ($C_P \le K\xi_{\mathrm{RG}}^2$) of the diffusive-relaxation-time assumption and the one-sided bridge (the One-Sided Bridge Proposition, $C_P \ge \kappa\xi^2$), a lower bound on the gap in the physical units of the lattice, $m_{\mathrm{phys}} \ge (\kappa/K)^{1/2}$, follows at each β Proved here (algebra on top of the assumptions, 23-01 § 06). Saying this uniformly as $\beta \to \infty$ is the four-dimensional goal of Part 23 (the ledger row "uniformity of the relaxation-time upper bound and the one-sided bridge"; the physical-unit target level of 23-15 Table 5), and this goal too is a statement that crosses the band. The uniform slab gap of the road from the direction of space and the "$\xi^2\uparrow$ uniform in β, and the bridge" of the road from

the direction of time are uniformity statements of the same type, and neither has been shown inside the band (26-07).

05

What cannot be said, and the differences that remain even when the ledger is empty

None of the arrows of this chapter shows the mass gap or confinement. The only things whose paper proof could be placed in the book are the Doubling-Iteration Theorem (algebra), the arrow of slab confinement at fixed lattice spacing, and the chessboard arrow with T^2 fixed. The arrow of slab confinement says nothing about the continuum limit, and the arrow of the fractional-instanton rate (the Confinement Lower Bound by Fractional Instantons) has no proof in the book (the label is Assumption; a proof with citations is in the paper) and is not uniform in N. The arrow of the small-T^2 gap and the arrow of the small-T^2 area law for the theory on a small T^2 are propositions of the research programme, at the stage of an outline. The band forms, the lateral-widening inequality and the uniform slab tension, contain the quantities of the conclusion (bias, tension) in the form "uniform in L", and the uniform slab gap states decay of correlations in a form uniform in L. The band statement itself carries the content of the conclusion. The statement of this chapter is a conditional statement only insofar as the band is placed among the assumptions.

Even if every row of the ledger disappears, a difference from what the problem statement requires remains: all compact simple groups (for groups without a centre the vortex tools cannot be used), the OS axioms on $\mathbb{R}^4$ with the recovery of rotations and non-triviality, and the arrow from the area law to the mass gap. These are in the separate frame "the differences that remain outside the ledger" of 27-02 § 05 (the limit to R^4 and tying the two roads together of Table 1 also go there). Nor is the ledger a table that only shrinks: as research advances, rows can split and new rows can appear.

06

The label of each arrow, and the dependency graph

We list the arrows in the format of the five grades (23-00 § 01). The label is the label of the arrow, not of its premises. Even for an arrow whose premise is Assumption, the label is Proved here if the inference from the premises to the conclusion is written on paper, and Assumption if it is at the stage of an outline.

Table 2 The label of each arrow ("—" marks rows without machine checking; the Lean column gives only the names of bundles re-checked by the main session)

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Arrow of the small-T^2 gap	$1 \wedge 5 \Longrightarrow$ existence of the theory on a small T^2 and the gap $m \geq c/L$	Assumption	Proposition of the research programme for the theory on a small T^2 (outline). The transfer-matrix step is an abstract lemma of the vessel	Decay of block observables (form with L fixed) contains the quantity m of the conclusion. Uniform in a; L is small	—

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Arrow of the R^4 gap	Arrow of the small-T^2 gap $\wedge$ 8 $\wedge$ 9 $\Longrightarrow$ the gap on $\mathbb{R}^4$	Assumption	Outline	The uniform slab gap (the ledger row of uniform decay of block observables) is uniform decay of block observables; what the arrow of the R^4 gap adds is the transfer to the limiting theory	—
Arrow of slab confinement	Lattice spacing and N fixed $\Longrightarrow$ area law for every β, $\sigma \geq -\ln \tanh(\beta N^2)$	Proved here	25-04	In the continuum limit the tension is 0. Comparison with earlier arguments (Borgs, Tomboulis–Yaffe type) remains	The Lean column of 25-04
Arrow of the fractional-instanton rate	Fixed $N \geq 2$, $\beta \geq 1 \Longrightarrow -\ln \theta \geq c\,\beta^{-a} e^{-\beta(S_* + \varepsilon(\beta))_M 2}$	Assumption	Confinement Lower Bound by Fractional Instantons (no proof placed in the book; a proof with citations is in the paper. 25-02 § 07 (e))	Not uniform in N. Says nothing about the continuum limit. The paper is a draft	—
Arrow of the small-T^2 area law	1 $\wedge$ 2 $\Longrightarrow$ area law for the theory on a small T^2 (uniform in a)	Assumption	Proposition of the research programme (outline). The parts of the insertion are Proved here in 25-05 and 25-06	The form of the tension $e^{-F/g^2(L)/L^2}$ is an expectation	The Lean column of 25-05 (finite version)
Arrow of the reduction of the band	3 $\wedge$ 4 $\wedge$ 6 $\Longrightarrow \theta(R; T_R^2) \leq e^{-cR^2}$	Assumption	Conditional theorem of the paper on the reduction of the band (the proof is a sketch)	Stated at $L \geq L_{\min}(a)$, where there are no lattice artefacts	—
Doubling-Iteration Theorem	(1) $\wedge$ $K^{1/3} t_0 < 1 \Longrightarrow$ (2)	Proved here	The proof in § 03	Algebra independent of the content of θ	Doubling (`Doubling.iterate`, `Doubling.decay`) (assumptions in § 03)
Arrow from the container estimate to the area law	Arrow of the reduction of the band $\wedge$ 4 $\Longrightarrow$ area law with tension uniform in L	Assumption	The Container Identity Cited (25-03) and frame decorrelation	The global version uses Tomboulis–Yaffe Cited and a global base point	—

Arrow	Premises ⟹ conclusion	Label	Grounds	Conditions written in the same row	Machine checking
Chessboard arrow	Reflection positivity of $z \Longrightarrow$ with T^2 fixed, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$	Proved here	The proposition of the paper on the reduction of the band and the Chessboard Estimate Cited (26-05)	A statement with L fixed. Transverse reflections are not positive because of the twist	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (algebra of the core)
Arrow from the two uniformities to the area law	Chessboard arrow $\wedge$ 7 $\wedge$ 8 $\Longrightarrow$ area law with tension uniform in L	Assumption	Outline	The uniform slab tension contains the quantity of the conclusion	—
Arrow of the R^4 area law	(Arrow from the container estimate to the area law, or arrow from the two uniformities to the area law) $\wedge$ 9 $\Longrightarrow$ area law on $\mathbb{R}^4$	Assumption	Outline	—	—
Arrow tying the two roads together	Arrow of the R^4 gap $\wedge$ arrow of the R^4 area law $\wedge$ 10 $\Longrightarrow$ the mass gap and confinement on the same theory on $\mathbb{R}^4$	Assumption	Open arrow. The statement of tying the two roads together is not settled	The area law alone does not give the gap	—
Arrow of the road from the direction of time	$\xi^2\uparrow \wedge$ one-sided bridge $\Longrightarrow m_{\mathrm{phys}} \ge (\kappa/K)^{1/2}$ at each β	Proved here	23-01 § 06 (algebra on top of the assumptions)	A statement at each β. Making it uniform in β is a statement that crosses the band	The Lean column of 23-01

Dependency graph (the numbers of the premises are smaller than the number of the conclusion; 1–10 are the assumptions of Table 1):

```
 1  ASM  ultraviolet control (three conditions)                          ultraviolet-control stag
 2  ASM  uniform acceptance of insertion                                 small-T² confinement sta
 3  ASM  base point at small T²: θ(L₀;T²_L₀) ≤ 1 − ε₀                    small-T² confinement sta
 4  ASM  frame decorrelation (form with L fixed; container form)         small-T² confinement sta
 5  ASM  decay of block observables (L fixed; contains rate m ≥ c/L)     small-T² gap stage
 6  ASM  lateral-widening inequality (strong / weak form)                band stage
 7  ASM  uniform slab tension                                            band stage
 8  ASM  uniform slab gap                                                band stage
 9  ASM  limit to R⁴                                                     R⁴ stage
10  ASM  tying the two roads together (statement not settled)
11  PRV  reflection positivity of z                                      Lean ReflPos.refl_pos, r
12  CIT  Container Identity (Mack–Petkova), Tomboulis–Yaffe inequality
13  PRV  arrow of slab confinement (lattice spacing and N fixed)         tension → 0 in the conti
14  ASM  arrow of the fractional-instanton rate (fixed N, β ≥ 1)         proof with citations in
15  PRV  Doubling-Iteration Theorem                                      Lean Doubling.iterate, d
16  PRV  chessboard arrow: T² fixed, |⟨W(C)⟩| ≤ κ_cb^n(C) ⟸ 11
17  ASM  arrow of the small-T² gap: m ≥ c/L ⟸ 1, 5
18  ASM  arrow of the small-T² area law (uniform in a) ⟸ 1, 2
19  ASM  arrow of the reduction of the band: θ(R;T²_R) ≤ e^(−cR²) ⟸ 3, 4, 6, 15
20  ASM  arrow from the container estimate to the area law (reduction form) ⟸ 4, 12, 19
21  ASM  arrow from the two uniformities to the area law (chessboard form) ⟸ 7, 8, 16
22  ASM  arrow of the R⁴ gap ⟸ 8, 9, 17
23  ASM  arrow of the R⁴ area law ⟸ 9, 20 (or 21)
24  ASM  arrow tying the two roads together: mass gap and confinement on the same R⁴ theory ⟸
```

The cycle check only looks at whether, for every "⟸", the numbers on the right are smaller than the number on the left. Among the assumption rows, those that contain the quantity of the conclusion are decay of block observables (form with L fixed) (the rate m, in a form uniform in a) and the lateral-widening inequality and the uniform slab tension in the form of positivity of the 't Hooft tension (bias, tension, in a form uniform in L). This is not circular, but it shows that what the arrow of the small-T^2 gap, the arrow of the reduction of the band and the arrow from the two uniformities to the area law add is only a restatement of uniformity and algebra, and that the content lies on the side of these assumptions. The ledger (27-02 § 04) writes its rows in forms that do not contain the quantity of the conclusion. Decay of block observables (form with L fixed), the lateral-widening inequality and the uniform slab tension, which do contain it, are "restatements" of ledger rows rewritten to fit the arrows of this chapter, not the ledger rows themselves. The content of the arrow of the R^4 gap also lies on the side of the uniform decay of correlations in the uniform slab gap. The arrow of slab confinement and the arrow of the fractional-instanton rate are nodes that are not among the ancestors of the arrow tying the two roads together; they record what can be said on paper at the stage of fixed lattice spacing. The arrow of the small-T^2 area law writes the same content as the base point at small T^2 in the language of the theory on a small T^2, and in the graph we kept the base point at small T^2 as a leaf.

What this chapter can say

What is said	What is not said
The doubling iteration (the Doubling-Iteration Theorem): if nesting and lateral widening are assumed in the form of the numerical inequalities (1), then $\theta(R_k;T^2_{R_k}) \le K^{-1/3}e^{-cR_k^2}$ Proved here	The two inequalities of (1) themselves (frame decorrelation and the lateral-widening inequality)

What is said	What is not said
The area law with the lattice spacing and the T^2 fixed (slab confinement) Proved here	Showing, inside this book, the bound on the fractional-instanton rate at fixed N (the Confinement Lower Bound by Fractional Instantons) (the label is Assumption; a proof with citations is in the paper). A bound uniform in N. The tension in the continuum limit
The chessboard estimate with T^2 fixed, $\lvert\langle W(C)\rangle\rvert \le \kappa_{\mathrm{cb}}^{n(C)}$ Proved here	Comparing κ_{cb} across scales (the uniform slab gap)
That the statement has been split into arrows, with a label on each arrow. That the gap and confinement come out of different bundles of arrows	Any of the two gap arrows, the arrow of the small-T^2 area law, the arrows from the reduction of the band to the R^4 area law, and the arrow tying the two roads together (outlines on top of assumptions, or an open arrow). The mass gap, confinement, the continuum limit
That the arrow of the road from the direction of time (a lower bound on the gap at each β) is algebra on top of the assumptions Proved here	The form uniform in β (a statement that crosses the band)

Sources and reproduction

Item	Kind	Source / tools
Names and contents of the assumptions in Table 1	This book	A-06 § 06, 27-02
The form of the arrow of the small-T^2 gap and the arrow of the small-T^2 area law (the research programme for the theory on a small T^2)	Assumption	The research programme as rewritten in this book's study (SU(2) on $\mathbb{R}^2\times$ a small twisted T^2). The propositions are at the stage of an outline
Arrow of slab confinement	Proved here	25-04
Arrow of the fractional-instanton rate	Assumption	Theorem A (the Confinement Lower Bound by Fractional Instantons) and Theorem B (the Reverse Estimate at a Fixed Plane) of the paper on lattice fractional instantons (draft) "Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus" (A-07). No proof is placed in the book (a proof with citations is in the paper)
Arrow of the reduction of the band, chessboard arrow, the definition of the lateral-widening inequality	Assumption, Proved here	The conditional theorem, the proposition on reflection positivity and the chessboard corollary of the paper on the reduction of the band (draft) "Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics" (A-07)
Doubling-Iteration Theorem	Proved here	The proof in the text. The algebra is the Lean bundle `Doubling`
Container Identity	Cited	G. Mack, V. B. Petkova (1980). Checked in the source table of 25-03
The Tomboulis–Yaffe inequality	Cited	E. T. Tomboulis, L. G. Yaffe, Commun. Math. Phys. 100 (1985). Checked in the source table of 26-01
Arrow of the road from the direction of time	Proved here	23-01 § 06

Terms used in this chapter — the conditional statement, arrow, the theory on a small T^2 and the theory on R^4, ultraviolet control, uniform acceptance of insertion, the base point at small T^2, frame decorrelation, the lateral-widening inequality, the uniform slab tension, the uniform slab gap, tying the two roads together, the doubling iteration, the chessboard density κ_{cb}, the open arrow. Their meanings are in the A-02 Glossary and A-06 § 06.

Next chapter to read: 27-02 The ledger of conditions — the present state of each assumption, and what is needed for it to disappear.

The ledger of conditions

> This chapter is a map of open questions. § 01, § 02 and § 06 are guide material.

What this chapter shows — A ledger that lists, one row at a time, the conditions placed by the conditional statement of Part 27 (27-01). For each row: the mark of its state in the record, its label in the book, what is needed for it to disappear, the rows it presupposes, and the chapters that treat it. A diagram of the dependencies between rows. And, in a frame separate from the ledger, what remains as a difference from the problem statement even when every row of the ledger has disappeared.

Prerequisite chapters — 16-09 (the table of the seven stages a proof must contain, and the figure of the two roads). The rows of the first road are taken from 23-15 Table 1 and 23-15 § 07.

Said first — The ledger is a list of conditions, not a gauge of progress. We do not count its rows and read off "this much left". This chapter adds no new results. The proofs and counterexamples for the contents of the rows are in the chapters in the right-hand column (including chapters still being written).

Order of this chapter

1. The definition of "a row disappears" — the rule placed at the head of the ledger
2. A copy of the table of stages — the marks of 16-09 Table 1, and the rows of the ledger (Table 1)
3. The ledger — twelve rows and their breakdown (Table 2)
4. The diagram of dependencies between rows — the numbers of premises are smaller than those of conclusions
5. The differences that remain outside the ledger — what remains even when every row has disappeared (Table 3)
6. The day rows increase — the ledger is not a table that only shrinks

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01

The definition of "a row disappears" — the rule placed at the head of the ledger

Rule

A row of the ledger disappears when, and only when, the following two are both in place. (1) A paper proof of the statement of the row has been placed in some chapter of this book (a proof existing in the record is not enough). (2) If the row has a Lean entry, the theorems written in that entry have passed a re-check on the side of this book that reads the axioms again.

The point of this rule is to separate the state of the record from the state of the book. Even a row that is closed in the record (mark ■) is not removed from the ledger until its paper proof has been placed in a chapter of the book. For example, reflection positivity of z was a row closed in the record, and its proof has been placed in 26-04 § 03. The algebra of the last step is also closed in Lean (ReflPos). This row is closed.

One more thing. When a row disappears, the rows that use it as a premise do not disappear automatically. For example, even if the base point at small T^2, frame decorrelation and the renormalisation comparison all disappear, positivity of the 't Hooft tension disappears only when the paper for the reduction from the base point at small T^2, frame decorrelation and the renormalisation comparison to positivity of the 't Hooft tension has been placed in the book (26-02). And even if positivity of the 't Hooft tension disappears, not one of the differences outside the ledger (§ 05) moves.

02

A copy of the table of stages — the marks of 16-09 Table 1, and the rows of the ledger

Table 1 copies the column of marks of 16-09 Table 1 and lists which rows of the ledger bear on each stage. The marks mean the same as in 16-09: they mark the state in the record, and are not a gauge of how far things are proved (16-09 § 02). The contents of the stages and the column of what is known are kept only in 16-09 and are not repeated here.

Table 1 The marks of the seven stages a proof must contain (a copy of 16-09 Table 1) and the rows of the ledger (■ closed in the record / ◐ partial / □ open)

Stage	Mark	Ledger rows
Vessel stage	◐	Reflection positivity of z (writing out the assembly of the vessel is not set up as a row. § 06)

Stage	Mark	Ledger rows
Ultraviolet-control stage	◐	Ultraviolet control (the continuum limit in a periodic cubic box was shown by the published construction; the form needed by this book's road is open)
Small-T^2 gap stage	◐	Small-T^2 gap
Small-T^2 confinement stage	◐	Uniform acceptance of insertion, base point at small T^2 (and the form of frame decorrelation with L fixed)
Band stage	□	In the language of confinement: frame decorrelation, renormalisation comparison, positivity of the 't Hooft tension; in the language of the gap: band comparison (in the language of the gap), uniform decay of block observables. In the language of the time side: uniform gap of the double well (in miniature), uniformity of the relaxation-time upper bound and the one-sided bridge
R^4 stage	◐	Outside the ledger: the four parts of the limit to R^4, tying the two roads together (infinite volume from a sequence of periodic boxes, and the reconstruction, were shown by the published construction)
General-group stage	□	Outside the ledger: general groups (existence was shown by the published construction for every compact simple group)

03

The ledger — twelve rows and their breakdown

Table 2 is the ledger. Rows 1–10 are the rows of the second road (from the direction of space), and the uniform gap of the double well and the uniformity of the relaxation-time upper bound and the one-sided bridge are the rows of the first road (from the direction of time). The rows of the band are set up in two languages. Frame decorrelation, the renormalisation comparison and positivity of the 't Hooft tension are rows in the language of confinement (the vortex free energy, the charged sector with electric flux), and the band comparison (in the language of the gap) and uniform decay of block observables are rows in the language of the mass gap (the neutral sector without electric flux). We put the ledgers of the two roads into one because both roads stop at the same band (16-09 Figure 1). The rows are numbered in an order in which premises come before conclusions (§ 04). We write for the case of the group SU(2), the Wilson action on the lattice, and the 't Hooft twist $n_{12} = 1$ put on T^2.

Table 2 The ledger of conditions (to the right of the row name is the mark = the state in the record. The label is the label in the book, for each claim within the row. "What is needed for it to disappear" is what is needed in addition to (1)(2) of the rule in § 01. The six rows under the renormalisation comparison are its breakdown)

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
1 **Ultraviolet control** ◐	A multiscale analysis uniform in a from the lattice spacing a up to a small physical scale L_0 (locality and analyticity of the small-field effective action, rarity of large fields, the effective coupling staying small). On a small T^2	Assessment The continuum limit in a periodic cubic box was shown by the published construction (at the level of the committee's assessment; Volume 2) / Assumption A construction with twisted boundary conditions or in an elongated box, and a statement in a form uniform in the lattice spacing, are not in the published construction (within the range searched). The ultraviolet control of this row is a condition that is still open: that the same construction stands under twisted boundary conditions	Extending the published construction to the form this book's road needs (twisted boundary conditions, elongated boxes, a form uniform in the lattice spacing) (Volume 2, 16-03, the seam in Table 2 of A-08). On the side of this book's road (Part 15), completing the written-out steps of Bałaban (including the steps the original sources call unpublished), and verifying them. An assumption that depends on asymptotic freedom, which does not apply to U(1) or $\mathbb{Z}_2$ Premises: — Chapters: 7-07, 25-00, 26-11 (summary) → Part 15 (15-00–15-09; the decomposition statement is in 15-03, the four rows that do not close in 15-08) Tables that split this row into small questions: 15-09 Table 1 (the table of ultraviolet questions, with the current state of the answers), 15-08 Table 1 (the four rows that do not close) The small-field contribution to the coupling holds, under conditions, in the same form on the group as in the sources for flat fields; what the sources lack is the explicit quadratic term between blocks and the group version of its local cancellation (15-08 § 06)
2 **Small-T^2 gap** ◐	The twist removes the zero modes, and the gap of the transfer matrix $\geq c/L$ is uniform in a	Proved here the twist part (uniqueness of the vacuum, the gap $4\sin^2(\pi/2N)$ of the two-dimensional lattice Laplacian, $S_{\min} > 0$ in the (1,1) sector; N fixed) / Assumption the form uniform in a	Under ultraviolet control, writing the lower bound on the gap uniformly in a Premises: ultraviolet control Chapters: 25-01

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
3 **Uniform acceptance of insertion** ◐	Under the small-field frame, the Barker-type acceptance rate $b_F = (1 - \mathbb{E}[\tanh(W/2)])/2$ of the vortex insertion dynamically driven by a procedure of length $m(a)$ is at least $\delta/2$, uniformly in a. Said in terms of the mean, the mean of $\tanh(W/2)$ is at most $1 - \delta$ (W is the work of the insertion). "Uniform in the lattice spacing" becomes false unless the length of the procedure $m(a) \propto (L/a)^2 (r/a)^2$ is put into the statement (with the length fixed, the dissipation grows like $(L/a)^2/m$)	Paper the sandwich of acceptance rates: for the Metropolis type $e^{-\Delta F} O \le a_F \le \min(e^{-\Delta F}, O)$, for the Barker type $\frac{1}{2} a_F \le b_F \le \min(a_F, 1/(1+e^{\Delta F}))$. O is the overlap of the path measures of the forward and reverse paths, and $O \ge \frac{1}{2} e^{-D}$ (D is the dissipation). As a corollary, uniform acceptance of insertion is, up to constants, the same as boundedness of the insertion free energy (ΔF bounded uniformly in a) $\wedge$ the uniform lower bound on the path overlap (O bounded below uniformly in a). Uniform acceptance of insertion $\Rightarrow$ base point at small T^2 is one line ($b_F \le 1/(1+e^{\Delta F})$), and uniform acceptance of insertion contains the base point at small T^2 (Lean LatticeGaugeAcceptanceSandwich (`barker_le_of_crooks`, `overlap_sandwich`). The algebra of inequalities with a Crooks-type identity placed as an assumption; the Bretagnolle–Huber step is outside Lean) / Cited linear response of the dissipation (Sivak–Crooks 2012) / Assumption an estimate of the friction $\zeta \le \mathrm{Var}/\mathrm{gap}$ uniform in a, and the remainder of linear response / Reading making the procedure longer lowers the work, and $m \cdot D$ approaches a constant for $m = 64$–256 (the computation record (2026-10-22). The acceptance rates in 25-08 Tables 1 and 2 are Barker-type values. Measurement conditions in 25-08)	A lower bound uniform in a for the uniform lower bound on the path overlap: an estimate of the friction (boundedness of the fluctuation per plaquette of the face, and the gap of the heat bath of the container $\gtrsim (a/r)^2$; on the side of ultraviolet control and the small-T^2 gap, not on the side of frame decorrelation and the renormalisation comparison) and an estimate of the remainder of linear response. Boundedness of the insertion free energy is the content of the base point at small T^2, so removing uniform acceptance of insertion also means attaching a measurable certificate to the base point at small T^2 Premises: ultraviolet control, small-T^2 gap Chapters: 25-06, 25-08

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
4 **Base point at small T² ◐**	At a small physical length L_0, in the symmetric container $(D_{L_0} \times T^2_{L_0}$, twist $n_{12} = 1)$, $\theta(L_0) \le 1 - \varepsilon_0$ uniformly in a, $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$	Cited the form of the string tension (Tanizaki–Ünsal, arXiv:2201.06166 eq. (2.37), dilute regime, one loop) and the numerics of fractional instantons on the twisted lattice (Bergner–González-Arroyo–Soler, arXiv:2505.10396) / Paper Lemma on the Upper Bound for the Minimal Action of a Symmetric Box: the minimal action of the symmetric box satisfies $S_{\min}(N,N) \le 2N^4(1-\cos(\pi/N^2)) < \pi^2$ for every N (a lattice version of 't Hooft's constant-curvature solution; an abelian configuration is placed explicitly. Lean LatticeGaugeAbelianTwistAction (`totalAction_eq`, `totalAction_lt_pi_sq`). That the minimal action of sector (1,1) is at most the action of this configuration is outside Lean). With the lattice fixed, the rate as $\beta \to \infty$ is $S_{\min}(N,N)$ (the Laplace principle) / Assumption the form uniform in a. It splits into two conditional rows: ultraviolet control around a vortex background (a background of constant curvature, including the extension to non-flat backgrounds) and insertion acceptance in a symmetric container (uniform acceptance of insertion in a symmetric container of size L_0)	The two rows, ultraviolet control around a vortex background and insertion acceptance in a symmetric container. The one-loop estimate of the work in insertion acceptance in a symmetric container becomes a ratio of determinants of the lattice fluctuation operator (explicitly writable) around the constant-curvature background. A lower bound on the action (the lattice BPS bound) is not needed for the base point at small T² (it is needed on the upper side of the dilute regime, in the converse direction of the upper end of the renormalisation comparison). Finite volume does not appear in the base point at small T² (it is on the side of the upper end of the renormalisation comparison and positivity of the 't Hooft tension). The lattice version is the same semiclassical picture as the upper end of the renormalisation comparison Premises: ultraviolet control, uniform acceptance of insertion Chapters: 25-02
5 **Reflection positivity of z ■**	The ensemble summed over twists is positive under reflections through lattice points in directions 3 and 4. As a corollary, the Chessboard Estimate	Proved here / Lean: ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`) (the algebra of the core on non-negative weights)	— (the proof has been placed in 26-04 § 03, and the same section has the fact that transverse reflections are not positive because of the twist. The algebra of the last step is also closed in Lean (ReflPos). This row is closed) Premises: — Chapters: 26-04, 26-05
6 **Frame decorrelation (the confinement side) ◐**	The correlations between the frames of containers decay uniformly in L. Taking the container to be the whole interior of a block, on a fixed T^2_L the nested inequality $\kappa_{\mathrm{cb}}(sR;L) \le \kappa_{\mathrm{cb}}(R;L)^{s^2}$ holds with constant 1 for all L, R, s	Paper Nested Supermultiplicativity Lemma (nested supermultiplicativity. It uses only the Markov property and the nested form of Mack–Petkova, and does not use reflection positivity. Lean LatticeGaugeNestedBlocks (`nested_supermultiplicative`) is an abstract form on a finite probability space; the Markov property is outside Lean). With L fixed, reflection positivity of z and the chessboard replace it by a single density κ_{cb} (Proved here, L fixed). The chessboard is used only in the step that descends from κ_{cb} to a single container and Wilson loops / Assumption the comparison across L (lateral comparison of the chessboard density)	Writing the reduction of 26-02 with $\kappa_{\mathrm{cb}}(L)$ instead of θ gives $C_D = 1$, and the part uniform in L does not remain as an independent assumption Paper. What remains is the comparison across L (lateral comparison of the chessboard density) $\kappa_{\mathrm{cb}}(R;L') \le C(L'/L)^2\kappa_{\mathrm{cb}}(R;L)$, which is the lateral-widening inequality read in terms of κ_{cb}. There is no nesting sideways (tiling T^2_{2L} with four copies of T^2_L makes the twist even, and transverse reflections are not positive because of the twist), so this comparison sits in the same place as the renormalisation comparison. Split into columns, it becomes the same problem as one of the faces of the middle of the renormalisation comparison (twist cost of a column). If the base point at small T² is kept in terms of θ, a one-scale row $1 - \kappa_{\mathrm{cb}}(L_0;L_0) \ge \eta(1 - \theta(L_0;L_0))$ is needed. Placing the paper (the Nested Supermultiplicativity Lemma, and the reduction written with κ_{cb}) in the book Premises: reflection positivity of z (only the descending step) Chapters: 26-02, 26-05

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
7 **Renormalisation comparison** (the core of the band) □	The vortex free energy at side $2N$ and coupling β is at most a constant times that at side N and coupling β', with $\beta' < \beta$. Non-perturbatively, for every β	Six rows of breakdown	An estimate with the error kept within a constant factor, for every β. An argument that closes the band must be able to distinguish U(1) (U(1) has a transition inside the band, at $\beta_c \approx 1.01$). The construction published on 2026-11-10 gives no handle inside the band (it carries every step at weak coupling and stops at the lattice of spacing 1). At the weak end there is a non-perturbative input that "the finer the step, the weaker the coupling", but it is a statement about the coefficients of the effective action, and it works only where the coefficients are very large (coefficients of about 9×10^5 or more, at very weak coupling). It is not the "input that distinguishes U(1)" needed in the middle of the band (Volume 2) Premises: reflection positivity of z, frame decorrelation Chapters: 26-06
Strong-coupling end of the renormalisation comparison ■	The range $\beta \lesssim 10^{-79}$	Assumption (there is a proof in the record. The constants are crude and the bookkeeping is a sketch, so it stays Assumption until the paper is placed in the book)	Writing out the steps, and improving the constants. In the strong-coupling window (β about 10^{-3} or less), this book's Lean has the area law and a vortex floor whose rate per area does not depend on the transverse size (20-06; the lattice spacing is fixed and the prefactor remains; it is not the renormalisation comparison itself) Chapters: 26-06
Lower end of the renormalisation comparison □	Pushing confinement and the comparison up to $\beta \approx 0.3$–1	Assumption	A finite-size criterion by mixing of box magnetic flux, and computer assistance (a certificate with interval arithmetic) The computer-assisted chain for the hierarchical model is a theorem about the Migdal–Kadanoff image itself, and Tomboulis's representative (the image shrunk uniformly) is pushed up in the transverse coordinate by at least (5/3)(1−α) in one step and leaves the tube, so it cannot be used as it stands as a tool for the lower end of the renormalisation comparison either Proved here (the note in 15-04 § 05) Chapters: 26-06, 15-04

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
Middle of the renormalisation comparison □	One inequality that uses non-commutativity. Any one of the four faces (softness in the twist direction, mixing of box magnetic flux, twist cost of a column, pressure margin), uniformly in the scale	Assumption (none of the faces is proved)	A proof in a form that fails for U(1) What is proved in Tomboulis's sandwich (arXiv:0707.2179) is an upper and lower inequality for a single number, the partition function; the step needed for the form of the renormalisation comparison, "representing the partition functions with and without a twist by the same representative (a common interpolation parameter)", is not proved (K. R. Ito and E. Seiler). This step must become false for U(1), and sits in the same place as the middle of the renormalisation comparison Cited Proved here Reading A candidate for the fifth face (the proposal of a one-step inequality with a fixed partner of comparison): the string tension of the one-step Migdal–Kadanoff image does not exceed the string tension of the genuine coarse theory, $\sigma[T_{\mathrm{MK}}w] \le 4\sigma[w]$. In the leading strong-coupling term and in the one-loop asymptotics the directions agree, and it has a form that fails for U(1) at the step where the MK image enters confinement first. Inside the band it is unknown, and it has not been shown at any β Assumption (an idea) On finite boxes the threshold of the proposal of a one-step inequality with a fixed partner of comparison is 1 ($R := \Delta F(\mathrm{coarse}; T_{\mathrm{MK}}w)/\Delta F(\mathrm{fine}; w) \ge 1$. ΔF is dual to σ, and the genuine one step preserves ΔF) Proved here . Over one step of the orbit, $4^4/2^4$ is $R = 1.34(7)$ and $6^4/3^4$ is $R = 1.17(20)$, within the resolution, so no verdict is given. There is neither evidence of failure nor numerical support for the side where it holds. It is a weak test that even the do-nothing map passes, and the failure for U(1) does not appear in a one-step box Reading (15-04 § 05) Chapters: 26-06, 26-10, 15-04
Upper end of the renormalisation comparison ◐	Semiclassics at fixed N and large β	Assumption the confinement direction (the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) of the draft paper: at fixed N, $-\ln\theta \ge c\beta - a_e - \beta(S_*+\varepsilon)M^2$. No proof is placed in this book. A proof with citations is in the paper: the general chessboard theorem and the Morse–Bott argument are cited) / Assumption the converse uniform in M, and $S_* = S_{\min}$	A uniform bound in the converse direction (already reduced to a single positivity lemma), and agreement of the minimal actions Chapters: 25-02, 26-06
One-loop end of the renormalisation comparison ■	With the comparison that reshapes the fluctuations, $\beta - \beta' = 0.258$. With the comparison that keeps the shape of the fluctuations, $\beta' \ge \beta + 0.023$ (the anti-asymptotically-free direction)	Proved here (at the one-loop level; not a non-perturbative claim) / Reading the direction of the numbers (SU(2) has $\beta' < \beta$ throughout the band; for U(1) the sign flips at $\beta_c \approx 1.01$. Lattices, seeds and errors in 26-09)	Making it non-perturbative is ultraviolet control and the middle of the renormalisation comparison Chapters: 25-07, 26-09

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
Band in the language of the coupling (mixed derivative) □	In a box of side L with a uniform background holonomy (angles α, γ) at the two ends in time (the Schrödinger functional box), the mixed derivative $M = \partial^2 \ln Z/\partial\alpha\,\partial\gamma$, normalised to β at tree level as M_R, satisfies $M_R(2L) \le (M_R(L) - \delta)_+$ on the interval of the band, or $-\Delta \ln M_R/\Delta L^2 \ge \sigma_0 > 0$ (δ, σ_0 independent of the lattice spacing). M drops exactly the terms that depend on one boundary only, and measures the free energy of the electric flux running through from bottom to top (the dual of the vortex of the middle of the renormalisation comparison) Reading . What is measured is the difference D_R over two values of γ. The difference $h := \beta_R - D_R$ from the coupling of the same box, $\beta_R = 4/\bar{g}^2$ (the derivative at the single point $\eta = \pi/4$), is the marker of the one-sided terms Ultraviolet: M_R has the same one-loop running $-8b_0 \ln(L/a)$ as β_R (at one loop on the lattice the coefficient is 1.009 times), and its doubling step is 0.2575, the same as for β_R. Only the constant differs, $M_R - \beta_R = +0.21$; this difference comes entirely from the charged fluctuations and is 0 for $U(1)$ Cited Computed Band: h has the form of a function of $\bar{g}^2$; it turns positive at $\bar{g}^2 \approx 4.5$, has its maximum 0.14–0.18 at 5–10, falls to 0 at 60–140, and is slightly negative (-0.01–-0.03) on the strong-coupling side. All 36 points with $L/a = 3$–8 lie on one curve, pointwise within 2σ Computed Strong coupling: for fixed $L \ge 4$, below the convergence region of the expansion ($\beta < \beta_0(L)$), $h < 0$. The leading term is the one-sided band term $-(8/s)u^{3L} \approx -(16/\pi)L^2u^{3L}$ ($u = I_2(\beta)/I_1(\beta)$, $s = \sin(\pi/2L^2)$); it does not enter D_R), and the term of the tube joining top and bottom is also negative Paper Lean (StrongCouplingOneSidedTerm, 9 theorems. Assumptions: the remainder $\lvert R\rvert \le C_L u^{3L+1}$, $\sin x > x - x^3/4$, the tree normalisation $\hat{D} \le 3$ of D_R, the coefficients on paper). Convergence of the expansion is Cited (Osterwalder–Seiler). A β_0 uniform in L is not yet done	Assumption (the band inequality is not shown at any β) / ultraviolet Cited Computed / band Computed / strong coupling Paper Lean (convergence Cited)	A non-perturbative proof of the inequality on the interval of the band. Distinguishing $U(1)$ takes the shape of $h(\bar{g}^2)$, not the size of h: for $U(1)$, $h \approx 0$ deep in the Coulomb phase and in the confined phase ($\lvert h\rvert \le 0.023$ at 11 of 12 points), with a peak only at one point inside the width of the transition of the finite box (0.35 at $L/a = 4$, still 0.29(4) at $L/a = 8$); for $SU(2)$ the peak lies on a band of width $\bar{g}^2 \approx 5$–10 Computed Reading . The one-loop ultraviolet difference +0.21 is also 0 for $U(1)$ Computed (the computation record (2026-10-23 to 10-28)) The Schrödinger functional box and the coupling $\bar{g}^2$ are as defined by Lüscher–Narayanan–Weisz–Wolff (hep-lat/9207009) Cited Premises: ultraviolet control (the coupling at the ultraviolet end) Chapters: 26-06, 26-09
8 **Band comparison (in the language of the gap)** □	For the dimensionless quantity $x(L) = L\,m_0(L)$, the lowest mass $m_0(L)$ of the neutral sector of the box measured by the size of the box, the doubling map has no fixed point on the interval of the band: $x(2L) \ge x(L) + \delta$, with δ independent of the lattice spacing	Assumption / Paper In the Coulomb phase of U(1), x stays at the photon value $x^* \approx 2\pi$, so the band comparison (in the language of the gap) fails. The ratio-of-rates form $m_0(2L) \ge (\frac{1}{2} - \varepsilon)m_0(L)$ ends up holding on periodic boxes for U(1) ($m_0(2L)/m_0(L) = 1/(2\cos(\pi/2L)) > \frac{1}{2}$), so it is not a form that distinguishes the groups	A proof in a form that fails for U(1). Of the four faces of the middle of the renormalisation comparison, the two that also work in the language of the gap are κ_W (softness in the twist direction) and $\delta(L_0)$ (the total-variation distance inside when the centre flux on the boundary of the box is changed), extended to all boundary conditions; as a new face, one step function of x is added. The phases to be excluded are a massless phase and first-order coexistence; the non-confining phase of $\mathbb{Z}_2$ (which also has a gap) need not be excluded. The Tomboulis–Yaffe inequality holds for U(1) by the same proof and is a tool that descends to the charged sector, so it is not a tool for the band comparison (in the language of the gap) Reading . Since it can be written in a language that does not use the centre, it can be defined even for the groups without a centre among general groups Premises: ultraviolet control (the coupling at the ultraviolet end is placed at the finite value that ultraviolet control hands over) Chapters: 26-06

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
9 **Uniform decay of block observables (the gap side)** ☐	For block observables F, G with vacuum expectation 0 (their supports a distance t apart), $\lvert\langle FG\rangle_{a,L}\rvert \le C_F C_G\, e^{-mt}$, with $m > 0$ independent of a and L. The form of the gap of the small-T^2 gap uniform in L (the task of a uniform gap in 16-07)	Assumption / Paper Uniform decay of block observables gives the mass gap under the limit to R^4 (the content that the arrow of the R^4 gap in 27-01 asks of the uniform slab gap has this form, and frame decorrelation is not enough) / Cited uniform exponential decay under arbitrary boundary conditions gives the area law (Chatterjee 2021, CMP 385, Theorem from Uniform Decay to the Area Law (Theorem 2.4 of Chatterjee 2021); on the lattice)	The band comparison (in the language of the gap), and carrying the small-T^2 gap uniformly in L. Positivity of the 't Hooft tension is the band in the language of confinement, and even if the base point at small T^2, frame decorrelation and the renormalisation comparison disappear, uniform decay of block observables does not move. On the side of time, uniformity of the relaxation-time upper bound and the one-sided bridge is a row of the same type Premises: ultraviolet control, small-T^2 gap, band comparison (in the language of the gap) Chapters: 27-01, 16-07
10 **Positivity of the 't Hooft tension** (the band in the language of confinement) ☐	$\ln(1/\theta(L,L)) \ge cL^2$, with c independent of L (the 't Hooft string tension is positive)	Proved here the reduction from the base point at small T^2, frame decorrelation and the renormalisation comparison (the algebra of the iteration in Lean: Doubling (`Doubling.iterate`)). In the form written with κ_{cb}, the chessboard-density base point and the lateral comparison of the chessboard density, the same iteration goes through with $K = 4C$ (Lean: LatticeGaugeNestedBlocks (`doubling_iterate`)) / Assumption the band itself	The base point at small T^2, frame decorrelation and the renormalisation comparison disappearing, and the paper for the reduction being placed in the book. Positivity of the 't Hooft tension is the band in the language of confinement; the mass gap needs uniform decay of block observables separately. Intermediate milestones: (a) exclusion of a deconfined phase, (b) $\ln(1/\theta) \ge cL^2/\ln L$ Premises: base point at small T^2, frame decorrelation, renormalisation comparison Chapters: 26-01, 26-02
11 **Uniform gap of the double well** (what remains of the first step) ☐	For the path measure of the 0+1-dimensional double well, a gap uniform in k and $\hbar$ for chain length $k \ge 8$ (a uniform logarithmic Sobolev inequality for a measure with non-convex regions). The band in miniature	Assumption . The convex side (length ≤ 7) and the uniqueness of the minimum are closed in the record (the section that places them in the book is a supplement to 23-15)	Closing the rows of 23-15 Table 1 in a form uniform as $\hbar \to 0$, or stitching finite blocks together with strong mixing The side $\hbar \to 0$ (the small-ħ remaining item) reduces to showing, as $\hbar \to 0$, one margin among the inputs of the Six-Node In-Phase Poincaré Proposition, $\hbar\rho - 2\hbar\kappa \ge 0.067$ Paper . The two limiting conditions (the bridge landscape: the global minimum is unique and interior, with $\lambda_{\min} \ge 0.2$ and $\kappa_0 \le 0.1282$; the coarse landscape: $U_0''(t^*) \ge 1.943$) are checked with interval arithmetic, and the limit of the margin is at least 0.758 Computer-assisted (the action gap is in floating point Reading). To overlap with the range of the record ($\hbar \ge 0.005$), the quantitative uniform Laplace method and a lemma on one-dimensional Muckenhoupt asymptotics on the side of $\hbar\rho$ are newly needed Assumption . The value of $\hbar_0$ expected from the quantitative uniform Laplace method is 4×10^{-4}–8×10^{-4} Reading (the computation record (2026-10-22)) For $\hbar \in [0.001949, 0.005]$, the margin of the lattice target level $\hbar\rho - 2\hbar\kappa \ge 0.067$ holds by interval arithmetic on six subintervals Computer-assisted (the arithmetic on the subintervals and the logic of the covering are Lean : LatticeGaugeH1HbarInterval, 15 theorems). The gap to the $\hbar_0$ of the quantitative uniform Laplace method is not yet done; from the next subinterval on, the estimate of the corner tail does not close with this computing machinery, and the machinery has to be replaced (keeping the weights as logarithms, or revisiting the estimate of the corner tail) Reading (the computation record (2026-10-24 to 10-28)) Premises: — Chapters: 23-09 § 06, 23-15, 26-07

Row	What is to be shown (with its conditions in the same row)	Label	What is needed for it to disappear, premises, chapters
12 **Uniformity of the relaxation-time upper bound and the one-sided bridge** (the four-dimensional goal, side of time) □	The upper side of the diffusive-relaxation-time assumption (the relaxation-time upper-bound assumption, $CP \le K\xi^2$) and the one-sided bridge, uniform as $\beta \to \infty$ (the physical-unit target level of 23-15 Table 5)	Assumption	Crossing the band in the language of time. Even if the uniform gap of the double well disappears, the uniformity of the relaxation-time upper bound and the one-sided bridge does not move (the uniform gap of the double well is a miniature, not an implication) Premises: ultraviolet control Chapters: 23-01, 23-15 § 07, 26-07

What to read from Table 2. (a) The rows marked ■ in the record are only reflection positivity of z and two rows of the breakdown of the renormalisation comparison (the strong-coupling end and one loop). For reflection positivity of z the proof has been placed in 26-04 § 03, and the algebra of the last step is also closed in Lean (ReflPos), so this row is closed. The two rows of the breakdown have not yet disappeared by the rule of § 01. (b) Non-abelian dynamics enters at the middle of the renormalisation comparison (and at frame decorrelation and positivity of the 't Hooft tension, which gather into it), and this row has a form that correctly fails for U(1) and $\mathbb{Z}_2$. The other rows are structural facts, or rows that depend on ultraviolet control. (c) The weak form of positivity of the 't Hooft tension (the weak form of the lateral-widening inequality, the uniform slab tension) has almost the same content as the band. We read the reduction not as having made the problem easier but as having pinned down the middle of the renormalisation comparison as the place where the non-abelian input is needed. (d) Uniform acceptance of insertion, the base point at small T^2 and frame decorrelation are conditional (◐) rows, but their content has not disappeared; it has moved to other rows. Uniform acceptance of insertion is the same as the conjunction of boundedness of the insertion free energy and the uniform lower bound on the path overlap, and contains the base point at small T^2; the base point at small T^2 splits into the two rows ultraviolet control around a vortex background and insertion acceptance in a symmetric container; and the part of frame decorrelation uniform in L moves, as the lateral comparison of the chessboard density, to the same place as the renormalisation comparison. (e) Even if positivity of the 't Hooft tension, the band in the language of confinement, disappears, uniform decay of block observables remains for the mass gap. That the two roads stop at the same band is right as a type, but they arrive at different sectors.

04

The diagram of dependencies between rows — the numbers of premises are smaller than those of conclusions

Figure 1 writes the dependencies between the rows of the ledger and the rows outside it, split into layers. Arrows point from premise to conclusion, and every row has a larger number than its premise rows. The rows of the ledger are written in forms whose statements do not contain the quantities of the conclusion (the gap on $\mathbb{R}^4$, the area law on $\mathbb{R}^4$) (the assumption rows ultraviolet control, uniform acceptance of insertion, frame decorrelation, the middle of the renormalisation comparison, the band comparison (in the language of the gap), uniform decay of block observables, and uniformity of the relaxation-time upper bound and the one-sided bridge). The assumptions in forms that contain the quantities of the conclusion (the assumption of the arrow of the small-T^2 gap and the forms of the band) are marked explicitly as "restatements" in 27-01 (assumptions of arrows that restate uniform decay of correlations as a gap, and uniform bias and tension as an area law). Positivity of the 't Hooft tension is the band itself and contains a quantity close to the conclusion of the conditional statement. We put positivity of the 't Hooft tension in the ledger to show, in one row, the road to removing it once the reduction (from the base point at small T^2, frame decorrelation and the renormalisation comparison to positivity of the 't Hooft tension) is on paper. There are no exceptions to the numbering rule.

```
Layer 0   1 ultraviolet control ◂                5 reflection positivity of z ■
Layer 1   2 small-T² gap ◂ (← 1)
          6 frame decorrelation ◂ (← 5; only the descending step)
Layer 2   3 uniform acceptance of insertion ◂ (← 1, 2)
          7 renormalisation comparison □ (← 5, 6)
          8 band comparison (in the language of the gap) □ (← 1)
Layer 3   4 base point at small T² ◂ (← 1, 3)
          9 uniform decay of block observables □ (← 1, 2, 8)
Layer 4  10 positivity of the 't Hooft tension □ (← 4, 6, 7)
———————— below this line: outside the ledger (§05) ————————
Layer 5   limiting measure on R⁴ and its positivity ◂ (← ultraviolet control in arbitrary volume)
          rotation invariance □ (← ultraviolet control in arbitrary volume + a new lemma)
          local fields and OS regularity □ (← ultraviolet control in arbitrary volume + the task of local fields)
          uniqueness and non-triviality of the limit □ (← 10, 9)
Layer 6   tying the two roads together □ (← 9, 10, uniqueness and non-triviality of the limit)
          general groups □ (← limit to R⁴)
Layer 7   verification and peer review □ (← all rows)

ultraviolet control in arbitrary volume = ultraviolet control + extensive extension to arbitrary volume + continuity in
(the rows outside the ledger include parts that the published construction showed, in the form from a sequence of period

First road   12 uniformity of the relaxation-time upper bound and the one-sided bridge □ (← 1). The row of the same type
             11 uniform gap of the double well □ is 12 in miniature (not an arrow of implication)
```

Figure 1 Dependencies between the rows (to the right of ← are the premises). The base point at small T^2 and the upper end of the renormalisation comparison stand on the same semiclassical picture, but this is not an arrow of implication, so it is not in the figure. Uniform acceptance of insertion ⇒ base point at small T^2 follows in one line, but the relation is that uniform acceptance of insertion contains the base point at small T^2, so the figure only puts uniform acceptance of insertion among the premises of the base point at small T^2. The premises of uniqueness and non-triviality of the limit are, as written in 16-07, positivity of the 't Hooft tension and tying the two roads together; since the gap side of tying the two roads together has been moved to uniform decay of block observables, we write positivity of the 't Hooft tension and uniform decay of block observables Reading . "Area law ⇒ mass gap" is not set up as an arrow (§ 05).

05

The differences that remain outside the ledger — what remains even when every row has disappeared

Separate frame

Even if all twelve rows of the ledger disappear, four differences from the requirements of the problem statement (Jaffe–Witten 2000, 0-03) remain (the limit to R^4 [split into four rows in Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown). An empty ledger does not mean that the problem is solved.

Table 3 The differences that remain outside the ledger (the correspondence with the tasks of 16-07 is this book's arrangement)

Row	Difference	Mark	Present state	16-07
Limiting measure on R^4 and its positivity	Of the limit to $\mathbb{R}^4$, the limiting measure, translations, preservation of reflection positivity, the Hilbert space and $H \geq 0$	◐	Assessment Infinite volume from a sequence of periodic boxes (along a subsequence) and the reconstruction were shown by the published construction (the Wightman axioms phase by phase; there is no convergence or uniqueness of the limit; Volume 2) / Assumption The reduction from the family of twisted slabs of this book's road is open. Its conditions are ultraviolet control in arbitrary volume (ultraviolet control extended to arbitrary volume with extensive constants) and continuity in the sources uniform in the lattice spacing and the volume. The rest is known argument (compactness, preservation of inequalities under weak limits, a symmetric contraction semigroup) Cited . The statement of ultraviolet control alone is not enough: no source derives, as a theorem, boundedness uniform in a of the expectations of gauge-invariant observables Cited	Task toward infinite volume
Rotation invariance	Recovery of rotation invariance	☐	It is present within the reconstruction of the published construction (Volume 2). From the family of twisted slabs of this book: it does not follow from invariance under the hypercubic group and continuity. What is needed is a lemma that the limits of regularisations with rotated lattice orientations agree, or a direct estimate that the breaking vanishes as $a \to 0$; neither ultraviolet control nor positivity of the 't Hooft tension promises it. The reflections of reflection positivity of z are positive only in directions 3 and 4, and reflection positivity in directions 1 and 2 comes only from rotation invariance Reading	Task of the remaining axioms
Local fields and OS regularity	Local fields and OS regularity, the linear-growth condition (Osterwalder–Schrader axiom E0′) (Wightman fields)	☐	It is present within the reconstruction of the published construction (Volume 2). From the family of twisted slabs of this book: renormalisation of composite operators and their uniform estimates	Task of the remaining axioms, task of local fields and short distances
Uniqueness and non-triviality of the limit	Uniqueness of the limit, non-triviality, clustering	☐	The published construction has no convergence or uniqueness of the limit. Clustering and a unique vacuum are on the side of decay uniform in the volume (uniform decay of block observables) and of positivity of the 't Hooft tension. Uniqueness is not required on our reading of the problem statement, but it is needed on the road that derives the recovery of rotation invariance from uniqueness. That the two roads give the same theory is also included here	Task of non-triviality
Tying the two roads together	Tying the two roads together: stating both the mass gap and confinement on the same theory on $\mathbb{R}^4$	☐	The area law alone does not give the gap (16-09 § 03). The gap side (the form "carry the gap of the small-T^2 gap through the band to $\mathbb{R}^4$ with the same constants") has been moved to uniform decay of block observables in the ledger, and "the same theory" is included in uniqueness and non-triviality of the limit. What substantively remains is the consistency of scales $0 < \inf m/\sqrt{\sigma} \leq \sup m/\sqrt{\sigma} < \infty$ (along the same sequence $\beta \to \infty$); this is a condition for stating confinement in physical units, and lies outside the requirements of the problem statement (what the problem statement asks for is the mass gap Cited). In three-dimensional U(1) there is an example in which both hold on the lattice and yet the ratio goes to 0, so the condition is not empty Cited (summary)	Task of one scale

Row	Difference	Mark	Present state	16-07
General groups	All compact simple groups	☐	Existence was shown by the published construction for every compact simple group (Volume 2). The vortex tools cannot be used for groups without a centre. On the mass gap: for SU(N), it is expected that the rows can be turned into conditional ones Reading . The places specific to SU(2) are of four kinds, of which two need redoing: the properties of SU(2) = S^3 (the conditional law of one link being one-parameter, the explicit formula of the heat kernel, the Heat-Kernel Fluctuation Rarity Lemma), and -1 lying in the Weyl group (a cubic treatment replacing the evenness of class functions). On top of these comes redoing the numerics of 26-09 for SU(3) (on top of ultraviolet control and the renormalisation comparison, which are open for SU(2)). What gets filled: the twist by the centre $\mathbb{Z}_N$ (for primitive twists the clock-and-shift pair is unique up to conjugation Cited (González-Arroyo, arXiv:hep-th/9807108 § 4.2); the adjoint gap is $4\sin^2(\pi/(NL))$, agreeing with the numerics for SU(3) at $L = 3$–6 Reading (the computation record (2026-10-22)); one loop gives $\beta - \beta' = 11N^2 \ln 2/(12\pi^2)$ Paper), the n-colour version of the colour lemma (Lean ColorAverageN (`colorAverage` , `design_identity`); no assumptions, including the existence of the weighted quartic design; the application to su(N) is on paper), and the SU(N) version of the Tomboulis–Yaffe inequality Cited (Kanazawa, arXiv:0808.3442 Theorem 2; its application to the slabs of this book is not done). All compact simple groups remain open: for groups without a centre (G_2, F_4, E_8), neither the vortex tools nor the arrows that use the centre can be used Cited (16-07 § 07), and for Sp(n) ($n \geq 2$) the centraliser of the pair that absorbs the twist is continuous, so a single twist on T^2 does not remove the zero modes Reading	16-07 § 07
Verification and peer review	Verification	☐	Writing out the whole, Lean (as far as it reaches), expert refereeing, publication. The rules of the prize require refereed publication followed by a period of acceptance	—

The differences outside the ledger are not put in the ledger because they are not conditions of the conditional statement (27-01) but the distance between the conclusion of the statement and the requirements of the problem statement. The rows of the ledger are "work of removing assumptions"; the rows outside are "work of carrying the conclusion as far as the problem statement". If the two were mixed in one table, then on the day the ledger became empty it would read as if the problem had been solved. The gap side of tying the two roads together (crossing the band in the language of the gap) is a condition of the same type as the assumption of the arrow of the R^4 gap in 27-01, so it has been placed inside the ledger as uniform decay of block observables.

What is not set up as an arrow. "Area law ⇒ mass gap" is kept as a physical expectation, read as Reading , and is not made an arrow of Figure 1. From reflection positivity and the area law alone it is false (the direct product of a lattice measure with an area law and U(1) in the Coulomb phase keeps reflection positivity and the area law and has no gap Paper ; no counterexample within simple groups is known). The theorem known on the lattice goes the other way: uniform exponential decay under arbitrary boundary conditions gives the area law Cited (Chatterjee 2021 Theorem from Uniform Decay to the Area Law, Durhuus–Fröhlich 1980). In the language of the transfer matrix, the area law is the gap of the charged sector with electric flux, and the mass gap is the gap of the neutral sector without electric flux.

06

The day rows increase — the ledger is not a table that only shrinks

As research advances, rows disappear, but they can also increase. For example, when the reduction of the band was worked out, the lateral-widening inequality split into two rows, a strong form and a weak form, and it turned out that the strong form is stronger than confinement (it contains monotonicity of the slab tension). When the renormalisation comparison was worked out, two new rows appeared, the strong-coupling end and the one-loop end, and the middle row split into four faces. When tying the two roads together was reviewed, two rows that cross the band in the language of the gap, the band comparison (in the language of the gap) and uniform decay of block observables, appeared inside the ledger, the row of frame decorrelation became the row on the confinement side, frame decorrelation, and the limit to R^4 split into four rows. Writing out the assembly of the vessel (the vessel stage) has no row for now, but a row may appear at the stage of writing it out.

So the number of rows of this ledger, laid out over time, is not a gauge of progress. What to read is the content of each row, and what is needed to remove it.

The current list of premises is in 28-04.

What this chapter can say

What is said	What is not said
The list of conditions of the conditional statement, and for each row its state in the record, its label, and what is needed for it to disappear (Table 2)	The proof of any row. This chapter adds no new results

What is said	What is not said
Proved here the reduction to positivity of the 't Hooft tension, reflection positivity of z, the twist part of the small-T^2 gap, the confinement direction of the upper end of the renormalisation comparison, the one-loop level of the one-loop end of the renormalisation comparison (within the conditions written in the rows)	Positivity of the 't Hooft tension itself, the non-perturbative form of the renormalisation comparison, ultraviolet control, the uniform lower bound on the path overlap, the lateral comparison of the chessboard density, the band comparison (in the language of the gap), uniform decay of block observables
Paper the sandwich of acceptance rates in uniform acceptance of insertion, the Lemma on the Upper Bound for the Minimal Action of a Symmetric Box in the base point at small T^2, the Nested Supermultiplicativity Lemma in frame decorrelation, the reduction of the small-ħ remaining item in the uniform gap of the double well (in the computation record; the paper proofs are not yet placed in the chapters of the book)	That these rows disappear by the rule of § 01
That there are four differences that remain even when every row of the ledger has disappeared (the limit to R^4 [split into four rows in Table 3], tying the two roads together, general groups, verification and peer review; seven rows with the breakdown) (Table 3)	That an empty ledger would mean the problem is solved (this cannot be said; the differences outside the ledger remain)
That, of the limit to R^4, only the limiting measure and its positivity are connected by known arguments, on the condition of ultraviolet control in arbitrary volume	That ultraviolet control in arbitrary volume holds; rotation invariance; local fields and OS regularity; uniqueness and non-triviality of the limit
The arithmetic of the theorem names in the Lean column (the algebra of finite sums and the inequality of the iteration, the inequality of the sandwich of acceptance rates, the action of the abelian twisted configuration, the inequality of nested averages, the identity of the quartic design)	That the Lean column checks the statement of the row itself

Sources and reproduction

Item	Kind	Source / tools
The rows of the ledger, the breakdown, the dependencies	This book	The material of Parts 25 and 26 (the list of tasks up to a proof with its dependency diagram, and the list of assumptions), renumbered so that premises come first. The marks are copies of the state in the record
Rows of the first road, uniform gap of the double well and uniformity of the relaxation-time upper bound and the one-sided bridge	This book	23-15 Table 1, Table 5, 23-09 § 06
The form of the string tension in the base point at small T^2	Cited	Y. Tanizaki, M. Ünsal, arXiv:2201.06166 eq. (2.37) (bibliographic data and the equation in the source table of 25-02)
The sandwich in uniform acceptance of insertion, the Lemma on the Upper Bound for the Minimal Action of a Symmetric Box in the base point at small T^2, the Nested Supermultiplicativity Lemma in frame decorrelation, the small-ħ remaining item of the uniform gap of the double well, the adjoint gap for general groups	Paper, Computer-assisted, Reading	The computation record (2026-10-22). Interval arithmetic only for the two limiting conditions of the uniform gap of the double well
The dissipation in uniform acceptance of insertion	Cited	C. Jarzynski, Phys. Rev. Lett. 78 (1997); G. E. Crooks, Phys. Rev. E 60 (1999); D. A. Sivak, G. E. Crooks, Phys. Rev. Lett. 108 (2012)
The lattice numerics in the base point at small T^2	Cited	G. Bergner, A. González-Arroyo, I. Soler, arXiv:2505.10396
The converse theorem in tying the two roads together	Cited	S. Chatterjee, Commun. Math. Phys. 385 (2021) Definition 2.3, Theorem 2.4 (Theorem from Uniform Decay to the Area Law); B. Durhuus, J. Fröhlich (1980)
The twist and TY for general groups	Cited	A. González-Arroyo, arXiv:hep-th/9807108 § 4.2; T. Kanazawa, arXiv:0808.3442 Theorem 2
The Confinement Lower Bound by Fractional Instantons in the upper end of the renormalisation comparison	Assumption	Draft paper 'Fractional instantons confine on the lattice: a rigorous semiclassical bound for SU(2) gauge theory on a twisted small torus' (2026-10, A-07). The general chessboard theorem and the Morse–Bott argument are cited. A proof with citations is in the paper, and no proof is placed in the book, so the label is Assumption
The reduction to positivity of the 't Hooft tension	Proved here	The conditional theorem of the draft paper 'Where the twisted-slab route to confinement stops: a corrected window target, a reduction to transverse widening, and SU(2) versus U(1) numerics' (2026-10, A-07), and 26-02
The tasks of the differences outside the ledger	This book	16-07 Table 1 and § 07, 16-09
The problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000)
Lean	Lean	ReflPos (`ReflPos.refl_pos`, `ReflPos.refl_cs`), Doubling (`Doubling.iterate`), LatticeGaugeAcceptanceSandwich, LatticeGaugeAbelianTwistAction, LatticeGaugeNestedBlocks, ColorAverageN. What the bundles guarantee is only the algebra of finite sums, the inequality of a real iteration, and the elementary inequalities and identities written in the rows (A-04)

Terms used in this chapter — the ledger, a row disappears, the marks of the state in the record (■◐□), the differences that remain outside the ledger, the renormalisation comparison, the four faces, the band in miniature, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter to read: 27-03 The table of questions. The conditional statement itself is in 27-01.

The table of questions (a cross-index)

> This chapter is an index; it points to the relevant sections of each chapter.

What this chapter shows — The questions that the parts of this book left open, gathered into one index. The seventeen questions of Part 23, and the nineteen questions and six candidate rules of Part 24, are looked up by chapter and item number, without moving the items. The questions left by the text of Parts 25 and 26 are given running numbers per table on this page, listed with the section of the chapter they come from and the rows of the ledger of conditions they concern (the rows of 27-02 Table 2, and the differences outside the ledger). The two tables of "the next move" listed by the record of the study are also looked up in the language of the ledger rows.

Prerequisite chapters — 27-02 (the ledger of conditions; the numbers and names of the rows are fixed there). The content of the questions is in the chapters in the right-hand column of each row.

Said first — This chapter adds no new results. Questions are questions, not claims. The status of a question (done, partly done, not yet, new) and "what does not move even if settled" are kept only in the tables of the chapters they come from, and are not copied here. Whichever question is settled, the problem of the mass gap does not move. What moves is one row of the ledger, or one label in one chapter.

Order of this chapter

1. How to read the index — the five sources, the numbering convention, and the correspondence with the ledger rows (Table 1)
2. The questions of Part 23 (Table 2)
3. The questions and candidate rules of Part 24 (Tables 3 and 4)
4. Questions from Part 25 (Table 5)
5. Questions from Part 26 — the assumptions of the reduction of the band, criteria, β-monotonicity and more (Table 6)
6. The next moves listed by the record of the study — two tables looked up by ledger row (Table 7)
7. Looking up from the ledger rows — for each row, the questions that concern it (Table 8)

01

How to read the index — the five sources and the numbering convention

The questions have five sources. Table 1 gives, for each source, how the numbers are assigned and the table that the numbers point to. The numbers are those of the source chapter, used as they are and not renumbered on this page. Parts 25 and 26 do not yet have a chapter gathering their questions, so we pick up sentences in the form of questions from the sections "what cannot be said", "what remains" and "what is lacking" of the text of those parts, and number them on this page. The numbers are plain running numbers per table, without letters, and in the text questions are referred to by short names.

Table 1 The five sources of the questions

Source	Numbers	Count	Table to look in	How the size of a question is written
Part 23 (the first road, first half)	Running numbers 1–17	17	23-17 Table 3	Tools (SAT, paper, python3, interval arithmetic, Lean, literature) and the form of settlement
Part 24 (the first road, second half)	Running numbers 1–19, candidate rules 1–6 and three combinations	19 + 9	24-08 Table 1, Table 2	Kind of work (paper, reading, computation, Lean) and status
Parts 25 and 26 (the second road and the band)	Running numbers 1–16 of Tables 5 and 6	16 + 16	Tables 5 and 6 of this page	As in the sentences of the source sections. No estimate of size is written
Part 15 (ultraviolet control)	Running numbers 1–31	31	15-09 Table 1	As in the sentences of the source chapter (the conditions are in the tables of the source chapter)
"The next move" of the record of the study	Order 1–7 of the record's table, and order of work 1–4	7 + 4	Table 7 of this page	Looked up after replacing them with the language of the ledger rows

The names of the ledger rows are as in 27-02 Table 2 (the twelve rows from ultraviolet control to uniformity of the relaxation-time upper bound and the one-sided bridge, and the six rows of the breakdown of the renormalisation comparison). The differences outside the ledger are the limit to R^4, tying the two roads together, general groups, and verification and peer review (27-02 Table 3). The "rows concerned" column of the tables is written with row names.

The column "rows concerned" of the tables gives, as this book's arrangement, which ledger rows' content a question would touch if it were settled. Settling a question does not necessarily make a row disappear. A row disappears only when its paper proof has been placed in the book and, if it has a Lean entry, the re-check is done (27-02 § 01).

02

The questions of Part 23

The questions of Part 23 are "small questions" expected to be settled at the size of an undergraduate thesis. Table 2 lists only a one-line summary and where to look. The form of settlement and "what does not move when settled" are in each row of 23-17 Table 3. The assumption around which the questions of Part 23 turned is the diffusive-relaxation-time assumption (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption).

Table 2 The questions of Part 23 (summaries; the full text of each row is in 23-17 Table 3)

Q.	Summary	Tools	Rows concerned	Where to look
1	Does a complete family of $k = 3$ cells exist in $d = 9$?	SAT, Lean	— (the combinatorics of one-link integration)	23-17 Table 3, 23-02 § 02
2	The largest dimension in which a complete family exists, $d_{\max}(k)$	Exhaustive search, paper	— (same as above)	23-17 Table 3
3	The constants of $1 - \lambda_k$ for the Hermite family	Paper, python3	— (worked example)	23-17 § 05
4	The $SU(N)$ version, and the Lean statement, of the saturation of the ratio of functions on the forest	Paper, Lean	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
5	$CP(\beta)$ in the link variables of two-dimensional lattice Yang–Mills	Paper, python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (control)	23-17 Table 3

Q.	Summary	Tools	Rows concerned	Where to look
6	The best constant of the curvature after one-link integration, $c^{*} \in [21, 27]$	Global optimisation, interval arithmetic	— (the strong-coupling side)	23-17 Table 3, 5-11
7	The coefficient of N^{-3} in the column model, analytically	Paper, python3	Uniform gap of the double well	23-17 Table 3
8	In the column model, the sign of $1/P$ at finite u	python3	Uniform gap of the double well	23-17 Table 3
9	How much does the shortfall c_{2} change with the choice of gauge in the block transformation?	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
10	C_{P} and ξ^{2} of the two-dimensional $O(3)$ sigma model	python3, literature	Uniformity of the relaxation-time upper bound and the one-sided bridge (an asymptotically free control)	23-17 Table 3
11	C_{P}/ξ^{2} of the three-dimensional $\mathbb{Z}_{2}$ gauge theory	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (control)	23-17 Table 3
12	Monotonicity in volume of $C_{P}(V)$, and boundary conditions	python3	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
13	Collect the primary sources on the dynamical exponent of local algorithms for four-dimensional $SU(2)$	Literature	Uniformity of the relaxation-time upper bound and the one-sided bridge	23-17 Table 3
14	Bound from above, by interval arithmetic, the two-point maximal correlation of the medium-core step ($8 \le k \le 14$)	Interval arithmetic, paper	Uniform gap of the double well	23-17 Table 3, 23-15 Table 1
15	Read the power of the prefactor $\hbar^{-p}$ of C_{P} for the core by the capacity of the degenerate point	python3, paper	Uniform gap of the double well	23-17 Table 3
16	A comparison theorem showing the alternating maximal correlation between boxes $c_{\mathrm{alt}} < 1$	Paper	Uniform gap of the double well	23-17 Table 3
17	When a small term that breaks the conservation is added to a dynamics with conservation, the length at which the gap departs from the square of the side	Paper, python3	Uniformity of the relaxation-time upper bound and the one-sided bridge (the smallest model of Part 24)	23-17 Table 3

Reading Table 2. The questions that concern the row of the uniform gap of the double well (what remains of the first step) are the question "the N^{−3} coefficient of the column model", the question "sign of 1/P in the column model", the question "interval arithmetic for the maximal correlation of the core", the question "power of the prefactor for the core and window" and the question "comparison for the alternating maximal correlation of boxes"; the questions opened by "the road of blocks and mixing" in the supplement to Part 23 (the road of blocks and mixing in 23-15) are gathered in Table 6 of Part 26 as the question "two-sided area laws inside the band". The questions concerning uniformity of the relaxation-time upper bound and the one-sided bridge are questions of how to set up the statement of the diffusive-relaxation-time assumption on the side of the four-dimensional lattice; whichever of them is settled, uniformity of the relaxation-time upper bound and the one-sided bridge itself (the upper-bound assumption and the bridge, uniform as $\beta \to \infty$) does not move.

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03
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The questions and candidate rules of Part 24

The questions of Part 24 came out of the study that repairs the form of the statement of the relaxation-time assumption (the length at which conservation breaks, the sector and local versions, the form the one-sided bridge needs, the vacuum measure). Table 3 gives only summaries and where to look; the status (done, partly done, not yet, new) and "what moves and what does not move if settled" are in 24-08 Table 1.

Table 3 The questions of Part 24 (summaries; the full text of each row and its status are in 24-08 Table 1)

Q.	Summary	Size	Where to look (the section the question came from)
1	For the free field, compare the excitation energies of observables supported in the box	Paper	24-04 § 05
2	Monotonicity of the factor that cuts off times, $\varphi(x)/x^2$	Elementary, Lean	24-04 § 03
3	Whether the static ratio catches the slowness, for the sign of the Polyakov loop on a small periodic box	Small computation	24-01 § 02
4	For $U(1)$ on a two-dimensional layer, how the gap decreases as the strip-shaped region is lengthened	Computation	24-02 § 03
5	Primary sources on the autocorrelation of the topological charge in pure $SU(2)$ gauge theory	Reading, computation	24-07 § 07
6	A many-family version of the Two-Projection Lemma, and its condition on maximal correlation	Paper	24-03 § 06
7	$\inf \operatorname{Hess} V_t$ of the one-link Wilson weight convolved with the heat kernel, and the flow of the weight of the bad set	Paper + small computation	24-06 § 07
8	For Wilson's transfer operator, show the time-step comparison assumption ((D) in the paper "one-sided bridge") restricted to functions cut off at low energy	Paper	24-04 § 08
9	For finite-state reversible Markov chains, close in Lean $\tau_{\mathrm{int}} \le CP$ and the identity of the ground-state transformation	Lean	24-05 § 01
10	Reread the autocorrelations of local modes and topological modes in two-dimensional CP^{N-1} through the eyes of the local version	Reading, computation	24-02 § 07
11	For two-dimensional $SU(2)$, find λ_W and Γ of families of loops at finite β	Computation	24-04 § 06
12	On a four-dimensional box, the values for families of loops the size of the span	Computation	24-04 § 06
13	For the 0+1 double well, sandwich the local ratio by a reduction to one dimension with the position of the kink as coordinate	Paper + small computation	24-04 § 07
14	Add one point $\beta = 7$ for the two-dimensional strip-shaped region, and see whether the slope settles	Computation	24-02 § 05
15	For fields whose ratio lies between the threshold and 1, another argument that sees the mass while staying in the ratio form	Paper	24-04 § 06
16	On Wilson's lattice with discretised time, write the version of the bridge that cuts off times (the In-Window Supremum Proposition and the lower bound in finite volume) under the time-step comparison assumption	Paper	24-04 § 03

Q.	Summary	Size	Where to look (the section the question came from)
17	Whether the Kogut–Susskind Hamiltonian becomes frustration-free when it is split with terms grouped by sector	Paper	24-05 § 04
18	On the CP^{N-1} test bed, add seeds and pairs of N, and read the exponent of the slowness of the topological mode	Computation	24-02 § 07
19	Measure the definition (2) of the length at which conservation breaks with the charge of a four-dimensional subregion	Computation	24-02 § 01

The summaries of Table 3 are shortened from the sentences of the source table. The terms of Part 24 "window factor" and "window version of the bridge" are restated here as "the factor that cuts off times" and "the version of the bridge that cuts off times", so that they are not confused with the band of Part 26 (note on terms: 26-00 § 02). The "strip-shaped region" of the question "U(1) gap on two-dimensional layers" and the question "the single point β=7 of the band" is a width-2 region of a two-dimensional lattice, a different thing from the crossover band. In terms of ledger rows, only the question "reduction to kink positions" touches the uniform gap of the double well; the others are questions that repair the form of the statement of uniformity of the relaxation-time upper bound and the one-sided bridge.

The candidate rules are candidates for the question "if we were to make new mathematics, what would we put as rules?", and each has one question attached (24-08 Table 2). Table 4 lists only the symbol, the name of the rule and where to look. The relation to the three roots of the third step is in 24-08 Table 3.

Table 4 The candidate rules (the one-line candidate and its question are in 24-08 Table 2)

Symbol	Name of the rule	Section where the candidate came from
1	Axiomatising scale	24-07 § 06
2	The algebra of loops	24-07 § 06
3	Conserved quantities and sectors	24-03 § 04
4	The class of measures that pay with rarity	24-06 § 07
5	An axiom of asymptotic freedom	24-07 § 06
6	Curvature rule (Bakry–Émery on the space of orbits)	24-07 § 06
1+4, 3+4, 2+3	Three combinations	The three rows at the bottom of 24-08 Table 2

Seen from the study of Parts 25 and 26, among the candidates the axiom of asymptotic freedom and axiomatising scale sit closest to the problem of the band. In the statements that cross the band, the place where asymptotic freedom enters is narrowed to one (the $\beta' < \beta$ of the renormalisation comparison, 26-06 § 01), and the question of the axiom of asymptotic freedom (what follows from the rule of approaching a free field in the ultraviolet) becomes a question of restating that one place in the language of rules. This is an estimate of position; none of the candidates promises a prospect of crossing the band Reading (a physical estimate).

04

Questions from Part 25

These are the questions that the text of Part 25 (the second road: from the direction of space) left open. The sentence in each row is the sentence of the source section shortened into the form of a question, with its conditions kept in the same row. The group is $SU(2)$, and the small T² carries the 't Hooft twist $n_{12} = 1$. The questions from the three chapters 25-06 to 25-08 (driven insertion, fixed maps, the measured record)

are the five rows at the bottom of the table (from the question "kernels with flip probability near 1" to the question "extending the measurements and their errors").

Table 5 Questions from Part 25 (the rows concerned are rows of the ledger of 27-02; this book's arrangement)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
1 Reverse direction of the fractional-instanton lower bound	The converse of the Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3): at fixed N, an upper bound by a dilute gas uniform in M (a bound on the string tension from above). In the draft it is reduced to a single positivity lemma	Upper end of the renormalisation comparison	A bound uniform in N, the continuum limit	25-02 § 06, A-07 § 06
2 Agreement of the minimal actions	Agreement of the minimal actions $S_*(N,\ell) \to S_{\min}(N,\infty)$ (the minimal action of a column with its frame frozen to the vacuum, and the minimal action of a free fractional instanton). The numbers agree to five digits (a reading)	Upper end of the renormalisation comparison	Same as above	25-02 § 06
3 Existence of fractional instantons	On the twisted $T^2 \times \mathbb{R}^2$, existence of a solution of topological charge $\frac{1}{2}$ attaining the lower bound $4\pi^2/g^2$ of the action (at present an outline with assumptions)	Base point at small T^2	The rate on the lattice, the band	25-02 § 02, § 07 (c)
4 SU(N) versions of the three facts	The $SU(N)$ version of uniqueness of the vacuum that absorbs the twist, the gap of the adjoint field, and the absence of flat configurations in the (1,1) sector	General groups	Groups without a centre	25-01 § 09 (c)
5 Bias of the outer container	The second term when moving to a fixed sector: that the bias of the outer container is typically small (0 as $M \to \infty$)	Frame decorrelation	A uniform estimate of the container free energy	25-03 § 04, § 08 (b)
6 Estimating the container free energy	Estimate the container free energy ΔF_j uniformly over good frames and uniformly in the lattice spacing (needs a multiscale analysis of the conditional measure with the frame held fixed)	Ultraviolet control, frame decorrelation	The band (the stage of widening sideways)	25-03 § 08 (e)
7 Cost of inserting a thick vortex	With the map of a thick vortex, bound the cost of the insertion independently of N (an improvement of the Slab Confinement Theorem towards the continuum limit; at present an outline, Assumption)	Ultraviolet control, uniform acceptance of insertion	The band	25-04 § 04, § 07 (c)
8 Gap from insertion to the area law	The three things lacking between the insertion estimates and the area law: (i) that the event that insertions are refused simultaneously in all cells is exponentially rare in the number of cells, (ii) that the lower bound on the acceptance rate is uniform in the lattice spacing, (iii) an upper bound on the variance of the insertion ratio	Uniform acceptance of insertion	The band	25-05 § 07
9 U(1) negative control	A negative control that applies the insertion estimates to four-dimensional $U(1)$ (not confining at weak coupling) and checks where they stop working	Middle of the renormalisation comparison (the condition that distinguishes groups)	The estimate for $SU(2)$	25-05 § 07, § 08 (f)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
10 Oddness of composed insertions	That the map composing insertions in separate cells satisfies the identity "odd under insertion" (at present an assumption of the Many-Cell Second-Moment Estimate, and an assumption in Lean too)	Uniform acceptance of insertion	The identity for one cell	25-05 § 08 (c)
11 Literature check for slab confinement	Comparing the Slab Confinement Theorem with arguments of Tomboulis–Yaffe type and with the area law for spatial Wilson loops at high temperature (Borgs 1985)	— (a question about the label)	The content of the theorem	25-04 § 06
12 Kernels with flip probability near 1	Constructing a kernel whose flip probability a is close to 1. The Flow Lemma $\theta \le 1 - a$ gives only a naive form, because for a one-step kernel a saturates around $\frac{1}{2}$ (a judgement from measurement and the record)	Positivity of the 't Hooft tension (a form usable for testing the band)	Ultraviolet control	25-06 § 06, § 08 (d)
13 Quasi-static limit	The quasi-static limit: that the work W approaches the conditional free energy of the container as the number of driving steps $m \to \infty$ (a proof including an estimate of how the kernel mixes)	Uniform acceptance of insertion	The band	25-06 § 04, § 08 (b)
14 Error of the shaping map	Bound the error of the map that reshapes the fluctuations (the local relative entropy near the vortex surface, and higher loops) at intermediate coupling and for large fields. At one loop it reproduces $\beta - \beta' = 0.258$	Renormalisation comparison (from one loop to non-perturbative)	The statement that crosses the band itself (the middle of the renormalisation comparison)	25-07 § 08 (a)
15 Ultraviolet cost of a fixed map	Whether the extra ultraviolet cost of fixed maps grows in the same way in four-dimensional gauge theory (whether cancellation by Ward identities occurs). The present numbers are three points in a two-dimensional model	Ultraviolet control	The band	25-07 § 06, § 08 (d)
16 Extending the measurements and their errors	Measurements that widen the T^2 beyond the sizes measured (up to about $L\sqrt{\sigma} \approx 2$), and an estimate of the errors of driven insertion and replica exchange (the treatment of autocorrelation)	— (a question about a reading)	Any ledger row (numbers are not proofs)	(b)(c) of 25-08 § 07

Reading Table 5. The reverse direction of the fractional-instanton lower bound and the agreement of the minimal actions are exactly what remains of the upper end of the renormalisation comparison in the ledger; if they are settled, both sides of the rate match on the lattice at fixed N, but the direction of increasing N is the same problem as the middle of the renormalisation comparison. The question "estimating the container free energy" and the question "cost of inserting a thick vortex" are questions that touch the content of ultraviolet control, a wall separate from the band (25-00 § 02). (ii) of the question "gap from insertion to the area law" is a restatement of the row uniform acceptance of insertion.

05

Questions from Part 26 — the assumptions of the reduction of the band, criteria, β-monotonicity and more

These are the questions that the text of Part 26 (the crossover band) left open. The first three rows put into question form the three assumptions of the reduction of the band (26-02 § 02), and the difficulty of the band is concentrated in the third row, the lateral-widening inequality (26-02 § 08). An argument that

closes the band must be able to distinguish $U(1)$ (the criterion "distinguish from the U(1) transition", 26-06 § 08). This condition applies to every row of Table 6.

Table 6 Questions from Part 26 (the rows concerned are rows of the ledger of 27-02; this book's arrangement)

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
1 Non-perturbative proof of the lateral-widening inequality	Show the lateral-widening inequality (strong form $\theta(R;T_{L'}^2) \le C(L'/L)^2\theta(R;T_L^2)$, or the weak form in which the total loss is finite) non-perturbatively across $L \sim 1/\Lambda$, between the dilute regime and the confining regime. It does not follow from structural inputs alone (reflection positivity, duality, the transfer matrix)	Positivity of the 't Hooft tension (the place of the input to the reduction is the renormalisation comparison)	From the limit to R^4 to verification and peer review	26-02 § 02, § 08
2 Uniformity of frame decorrelation	Show frame decorrelation $\theta(sR;T_L^2) \le C_{\mathrm{D}}\,\theta(R;T_L^2)^{s^2}$ uniformly in L. With L fixed it is replaced by the chessboard density κ_{cb}, so the question becomes a comparison of $\kappa_{\mathrm{cb}}(L)$ across scales	Frame decorrelation	The non-abelian input to the band (non-perturbative proof of the lateral-widening inequality)	26-02 § 02, 27-02 Table 2, frame decorrelation
3 Proof of the base point at small T^2	Show the base point at small T^2, $\theta(L_0;T_{L_0}^2) \le 1-\varepsilon_0$, under ultraviolet control and uniform acceptance of insertion (the expected size $\varepsilon_0 \propto (L_0\Lambda)^{11/3}$ is a one-loop estimate)	Base point at small T^2 (premises: ultraviolet control, uniform acceptance of insertion)	The band	26-02 § 02, 27-01 Table 1, base point at small T^2
4 Excluding a deconfined phase	Exclusion of a deconfined phase: $\Delta F(L,L) \to 0$ (at any speed). It rules out the known non-confining phases, but gives no uniform lower bound on the tension	Partway to positivity of the 't Hooft tension	The mass gap, a uniform tension for the area law	26-01 § 06, 26-06 § 08
5 Almost-area-law question	Almost area law: $\ln(1/\theta(L,L)) \ge cL^2/\ln L$ (almost area law; by the Tomboulis–Yaffe inequality, decay faster than the perimeter law)	Partway to positivity of the 't Hooft tension	A uniform tension	26-01 § 06
6 β-monotonicity question	β-monotonicity: $\partial_\beta \Delta F \ge 0$ with the lattice geometry held fixed. Rewritten in terms of electric flux, it is a problem of the same weight as β-monotonicity of the string tension in lattice units, and as far as we searched, not even β-monotonicity of Wilson-loop expectation values has been shown. In the chain of the band it is not needed if the renormalisation comparison is shown directly	Renormalisation comparison (an auxiliary role)	The renormalisation comparison itself	26-06 § 02
7 Z_2 and U(1) on twisted slabs	Checking that, on a slab with twist n_{12}, β-monotonicity for $\mathbb{Z}_2$ and $U(1)$ does not follow from Ginibre's inequality (without the twist it does)	Renormalisation comparison	β-monotonicity for $SU(2)$	26-06 § 01

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
8 Criterion at the lower end	The criterion at the lower end: push confinement and the comparison up to $\beta \approx 0.3$–1, by a finite-size criterion of Dobrushin–Shlosman type using the influence $\delta(L_0, \beta)$ on the inside when the centre flux on the boundary of the box is changed, and by a computer-assisted count of the growth constant of admissible surfaces	Lower end of the renormalisation comparison	Above $\beta \approx 1$ (the middle)	26-06 § 04
9 Bookkeeping at the strong-coupling end	Write out the bookkeeping of the Strong-Coupling-End Expansion Proposition (the interaction of volume polyhedra and surface contours), and make the constant $u_0 \approx 10^{-80}$ larger	Strong-coupling end of the renormalisation comparison	Beyond roughening ($\beta_R \approx 1.9$)	26-06 § 03
10 Four faces of the middle	Show any one of the four faces of the middle (softness in the twist direction κ_w, mixing of box magnetic flux $\delta(L_0)$, twist cost of a column $f(\ell)$, pressure margin) uniformly in the scale, in a form that fails for $U(1)$. As a preliminary step, test κ_w and $\delta(L_0)$ numerically with $U(1)$ as the control	Middle of the renormalisation comparison	Ultraviolet control	26-06 § 07
11 Two-sided area laws inside the band	At each β in the band, the area law on both sides at fixed β (with errors of the order of the perimeter; the assumption of two-sided area laws inside the band). If this can be said, the comparison inside the band shrinks to finitely many steps (the Lemma Reducing the Comparison inside the Band)	Renormalisation comparison	The seam between the upper end of the band and ultraviolet control	26-06 § 06
12 What remains of the road of blocks and mixing	What remains of the road of blocks and mixing for the first step: the second and third stages of strong mixing (the Laplace method with constraints), checking against the original sources the Varadhan-type limit, Witten's Morse theory (corners of the Neumann box) and the standard theorem for one-dimensional chains, and interval arithmetic for the curvature at the minimum $\lambda_*(12)$ and the reach of the optimal path	Uniform gap of the double well	Uniformity of the relaxation-time upper bound and the one-sided bridge (the uniform gap of the double well is a miniature, not an implication)	the road of blocks and mixing in 23-15, 26-07 § 06
13 Convexity threshold and mixing length	Why, in the first step, the convexity threshold (the first conjugate point at the top, ≈ 7.84) and the mixing length (about 9) are numerically close	Uniform gap of the double well	Uniformity of the relaxation-time upper bound and the one-sided bridge	26-07 § 07 (c)
14 Touchstone of the smallest band	Whether the smallest band can be crossed: test a tool claimed to cross the band first on two-dimensional asymptotically free models (the $O(N)$ model at large N, the Gross–Neveu model), with the ultraviolet cutoff still removed (the screening criterion "crosses the smallest band". Exponential decay at all temperatures for the two-dimensional $O(3)$ model is also said to be unsolved; not yet checked)	Positivity of the 't Hooft tension (a touchstone)	The four-dimensional band	26-08 § 07, the question "C_P and ξ^2 of the O(3) σ model"

Q.	Question (with its conditions in the same row)	Rows concerned	What does not move even if settled	Source
15 Comparison principle for large loops	The asymptotic form of the comparison principle for large loops only, $\sigma_{\mathrm{tw}}(L) \le \sigma_{\infty}$. The form for small loops was broken numerically, but this form has not been refuted (there is no mechanism to show it)	Positivity of the 't Hooft tension	The comparison principle for small loops (broken numerically)	26-08 § 04
16 Testing the three new candidates	Put each of the three new candidates (reduction by compactness of the band, log-concavity of the Weyl measure and a height/azimuth decomposition, local step function of a column) through the tests of the screening criteria	Middle of the renormalisation comparison	None of the candidates closes the band	26-10 § 05, § 07

Reading Table 6. Items 1–3 of Table 6 are the three assumptions of the reduction theorem (26-02 § 03); if all three are in place, the correct form of the band follows. But the three differ in character, and non-abelian dynamics enters only in the question "non-perturbative proof of the lateral-widening inequality" (uniformity of frame decorrelation is replaced by a structural fact once L is fixed, and the question "proof of the base point at small T^{2}" is a statement on the weak-coupling side). From the question "β-monotonicity" to the question "two-sided area laws inside the band" is the breakdown of the renormalisation comparison: the question "criterion at the lower end" is the lower end, the question "four faces of the middle" the middle, and the question "bookkeeping at the strong-coupling end" the strong-coupling end. From the question "touchstone of the smallest band" to the question "testing the three new candidates" are questions coming from the chapters on the roads tried and stopped and on new candidates (26-08, 26-10). The question "what remains of the road of blocks and mixing" and the question "convexity threshold and mixing length" are questions about the first step on the side of time (the first road); even if they are settled, the band does not move (26-07 § 04). The questions of ultraviolet control (the ultraviolet questions) are in Part 15, and 15-09 Table 1 is their index (the fifth source in Table 1).

06

The next moves listed by the record of the study — two tables looked up by ledger row

Among the material of Parts 25 and 26 there are two tables of "the next move", kept as a record of the study. One lists seven rows in decreasing order of the prospect of closing; the other lists four rows in the order of work. Both are tables in the record, not chapters of the book. Table 7 looks up each of their rows in the language of the ledger rows and the chapters of the book. The "amount of work" scale of the record's tables is a number of prospects, not a number with grounds, so it is not copied.

Table 7 The rows of the record's two tables, with the ledger rows and the chapters of the book (the sentences for the content of each row are on the side of the chapters of the book)

Record's table	Order	Content (in the language of the book)	Ledger row	Chapter of the book, question on this page

Record's table	Order	Content (in the language of the book)	Ledger row	Chapter of the book, question on this page
Order of the prospect of closing	1	What remains of strong mixing for the first step (checking the Varadhan-type limit, a uniform form of the Laplace method with constraints)	Uniform gap of the double well	the road of blocks and mixing in 23-15, what remains of the road of blocks and mixing
	2	The logarithmic Sobolev inequality for blocks in the first step (checking Witten's Morse theory and the decomposition against the original sources, interval arithmetic for λ_*)	Uniform gap of the double well	Same as above, what remains of the road of blocks and mixing
	3	Stitching 1 and 2 together (the standard theorem for one-dimensional chains)	Uniform gap of the double well	Same as above, what remains of the road of blocks and mixing
	4	The positivity lemma needed for the converse bound of the paper on lattice fractional instantons (by a low-temperature cluster expansion at one scale)	Upper end of the renormalisation comparison	Reverse direction of the fractional-instanton lower bound, A-07 § 06
	5	Agreement of the minimal actions in the paper on lattice fractional instantons (exponential decay of the minimisers, and cutting off)	Upper end of the renormalisation comparison	The question "agreement of the minimal actions"
	6	Writing out the bookkeeping of the strong-coupling end, and improving the constant u_0	Strong-coupling end of the renormalisation comparison	The question "bookkeeping at the strong-coupling end"
	7	The middle: show the comparison between scales by one of the four faces	Middle of the renormalisation comparison	The question "four faces of the middle"
Order of work	1	Of the faces of the middle, test κ_W and $\delta(L_0)$ numerically with $U(1)$ as the control	Middle of the renormalisation comparison	The question "four faces of the middle"
	2	The upper end: writing out the semiclassical theorem at fixed N	Upper end of the renormalisation comparison	Reverse direction of the fractional-instanton lower bound, agreement of the minimal actions, 25-02 § 06
	3	The lower end: a computer-assisted estimate of the growth constant of surfaces, and formulating the criterion by mixing of box magnetic flux	Lower end of the renormalisation comparison	The question "criterion at the lower end"
	4	Add to the paper on the reduction of the band the scan with step functions and the $U(1)$ control	— (a revision of the paper)	A-07 § 05

None of the rows of the two tables touches the differences outside the ledger. Questions aimed at the differences outside the ledger are in the table of tasks of Part 16 (16-07) and in the unknown column of the table of the seven stages a proof must contain (16-09 Table 1).

07

Looking up from the ledger rows — for each row, the questions that concern it

Table 8 looks up Tables 2–7 again from the side of the ledger rows. The content of each row, its label and what is needed for it to disappear are in 27-02 Table 2; here we list only the numbers of the questions

concerned. A row without questions is a row in which no step small enough to be shrunk into the form of a question is yet in sight.

Table 8 The ledger rows and the questions concerned (this book's arrangement)

Row	Questions concerned
Ultraviolet control	Estimating the container free energy, cost of inserting a thick vortex, the question "ultraviolet cost of a fixed map" (in part), the ultraviolet questions (15-09 Table 1, with the current state of the answers). 15-08 Table 1 (the four rows that do not close) and 15-09 Table 1 are the tables that split this row
Small-T^2 gap	— (the twist part is on paper in 25-01; the form uniform in a depends on ultraviolet control)
Uniform acceptance of insertion	Cost of inserting a thick vortex, gap from insertion to the area law, the question "oddness of composed insertions", the question "quasi-static limit"
Base point at small T^2	The question "existence of fractional instantons", proof of the base point at small T^2
Reflection positivity of z	— (the proof has been placed in 26-04 § 03; this row is closed)
Frame decorrelation	The question "bias of the outer container", estimating the container free energy, uniformity of frame decorrelation
Renormalisation comparison	Strong-coupling end: bookkeeping at the strong-coupling end / lower end of the renormalisation comparison: criterion at the lower end / upper end of the renormalisation comparison: reverse direction of the fractional-instanton lower bound, agreement of the minimal actions / middle of the renormalisation comparison: the question "U(1) negative control", four faces of the middle / testing candidates for the middle of the renormalisation comparison: the question "testing the three new candidates" / outside the breakdown: the question "error of the shaping map", the question "β-monotonicity", the question "Z_2 and U(1) on twisted slabs", two-sided area laws inside the band
Positivity of the 't Hooft tension (the band)	Non-perturbative proof of the lateral-widening inequality (the content), the question "excluding a deconfined phase", the question "almost area law" (intermediate milestones), the question "kernels with flip probability near 1", the question "comparison principle for large loops" (forms usable for testing the band), the question "touchstone of the smallest band" (a touchstone)
Uniform gap of the double well (what remains of the first step)	The question "the N^{−3} coefficient of the column model", the question "sign of 1/P in the column model", the question "interval arithmetic for the maximal correlation of the core", the question "power of the prefactor for the core and window", the question "comparison for the alternating maximal correlation of boxes", the question "reduction to kink positions", what remains of the road of blocks and mixing, convexity threshold and mixing length
Uniformity of the relaxation-time upper bound and the one-sided bridge (the four-dimensional goal, side of time)	Items 4, 5, 9–13 and 17 of Table 2, and items 1–12 and 14–19 of Table 3 (all are questions that repair the form of the statement of this row, not this row itself)
Limit to R^4	—
Tying the two roads together	— (even the statement of what it suffices to assume is not settled, 27-01 § 04)
General groups	The question "SU(N) versions of the three facts" (only the $SU(N)$ part)
Verification and peer review	—

What to read from Table 8. The rows with many questions are the renormalisation comparison, and the uniform gap of the double well and uniformity of the relaxation-time upper bound and the one-sided bridge; the questions on the band itself (positivity of the 't Hooft tension) gather in the non-perturbative proof of the lateral-widening inequality. The questions on excluding a deconfined phase and the almost area law are intermediate milestones weaker than positivity of the 't Hooft tension, and settling them does not make positivity of the 't Hooft tension disappear. The differences outside the ledger have almost no steps that can be shrunk into the form of a question. Tying the two roads together starts, before any question is set, from making a statement of what needs to be shown.

What this chapter can say

What is said	What is not said
Which section of which chapter the questions and candidate rules of Parts 23 and 24 came from (Tables 2–4)	The answer to any question. The status and the full text of "what does not move even if settled" are in the source tables
A list of the questions that the text of Parts 25 and 26 left open, with their source sections (Tables 5 and 6)	Estimates of the size of the questions. Of the sentences of chapters 26-03 to 26-05 and 26-09, only those that overlap with rows of Table 6 have been picked up
The correspondence between questions and ledger rows (Table 8, this book's arrangement)	That settling a question makes a row disappear (the rule for rows disappearing is 27-02 § 01)

Sources and reproduction

Item	Kind	Source / tools
The questions of Table 2	This book	23-17 Table 3
The questions and candidate rules of Tables 3 and 4	This book	24-08 Table 1, Table 2
The questions of Tables 5 and 6	This book	The sections of the chapters in the right-hand column ("what cannot be said", "what remains", "what is lacking", "map")
The record's two tables in Table 7	This book	From the material of Parts 25 and 26, the table of next moves (seven rows in order of the prospect of closing) and the "order of the next work" in the list of tasks up to a proof (four rows). The content of the rows has been replaced with the language of the chapters of the book
Numbers and names of the ledger rows	This book	27-02 Table 2, Table 3

Terms used in this chapter — ledger row, the differences outside the ledger, what does not move even if settled, intermediate milestone, the lateral-widening inequality, frame decorrelation, the base point at small T^2, β-monotonicity, the renormalisation comparison, the four faces, candidate rules, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter to read: 27-04 What is not claimed (a cross-index) — the reverse side of the questions: what this book does not show, does not read strongly, and does not call false either.

2026-10-15 · chapter Yang–Mills and the mass gap Part 27 Closing — conditional statements not shown, not read strongly, not called false · Kiichi, Shiori and Rin

What is not claimed (a cross-index)

> This chapter is an index; it points to the relevant sections of each chapter.

What this chapter shows — What this book does not claim, gathered across the parts into one index. The fourteen items of Part 23 (23-16 Table 2) and the sixteen items of Part 24 (24-08 Table 4) are looked up by chapter and item number, without moving the items. From the sections "what cannot be said" of the text of Parts 25, 26 and 27 we pick up items, give them running numbers per table on this page (items 1 onwards of Tables 4 to 6), and list them with the places in the text that support them. Finally, a table of examples in which sentences of Parts 25 to 27 lose their conditions in a one-line summary.

Prerequisite chapters — None. You can start reading anywhere. The three meanings (not shown, not read strongly, not called false) are used as in 23-16 § 01. The numbers of the ledger rows are those of 27-02 Table 2.

Said first — This book does not solve the problem of Yang–Mills existence and the mass gap. This book's study does not construct the continuum limit (the continuum limit in a periodic box, and the reconstruction phase by phase, were shown by the construction published on 2026-11-10; at the level of the committee's assessment; Volume 2). There is no proof of the mass gap, and no proof of confinement in the continuum limit. This chapter only writes that fact out, split into items, and adds no new results.

The order of this chapter

1. How to read the index — the three meanings, and the number of items per part (Table 1)
2. The fourteen items of Part 23 — looking up 23-16 Table 2 (Table 2)
3. The sixteen items of Part 24 — looking up 24-08 Table 4 (Table 3)
4. The items of Part 25 — on the tools of the second road (Table 4)
5. The items of Part 26 — on the band (Table 5)
6. The items of Part 27 — on the conditional statement and the ledger (Table 6)

7. Conditions drop out in summaries — sentences of Parts 25 to 27 and one-line summaries (Table 7)

01

How to read the index — the three meanings, and the number of items per part

"Not claimed" has three meanings (23-16 § 01). Not shown: on that proposition this book shows nothing. Not read strongly: what has been shown carries conditions or assumptions, and read in a form with the conditions removed it becomes a sentence that has not been shown. Not called false: that a method did not reach something is a property of that method, not a claim that the proposition it did not reach is false. Besides these, there are items of "not judged, not counted": rederivations of known results are not counted as results, and the correctness of other documents is not judged.

The third becomes clearer when written in logical form. Writing "P is claimed" as $\vdash P$, $\neg(\vdash P)$ and $\vdash \neg P$ are different sentences, and the "not claimed" of this chapter always has the meaning on the left. The items of Part 26 have one more direction: cases in which a numerical reading points to the opposite of a proposition (the measurement of the comparison principle, item 4 of Table 5). There too, the book does not claim the opposite proposition as a theorem, but places it as a "reading".

Table 1 The items per part (how they are numbered, and how many)

Part	Numbers	Count	Table or section to look in
Part 23 (the first road, first half)	Items 1–14	14	23-16 Table 2
Part 24 (the first road, second half)	Items 1–16	16	24-08 Table 4 (the five items on the candidate rules are at the end of 24-08 § 02)
Part 25 (the second road)	Running numbers 1–13	13	Table 4 of this page (the "what cannot be said" of each chapter)
Part 26 (the crossover band)	Running numbers 1–12	12	Table 5 of this page
Part 27 (closing)	Running numbers 1–6	6	Table 6 of this page

The items of Part 23 apply as they are to Part 24 (24-08 § 04), and in the same way the items of Parts 23 and 24 apply as they are to Parts 25 to 27. In particular, "not reaching something is not evidence that it is false" (items 8 and 12 of 23-16) and "conditions drop out in summaries" (23-16 § 06) are common to all parts.

02

The fourteen items of Part 23 — looking up 23-16 Table 2

Table 2 lists only the gist of each item, its kind, and the main places in the text that support it. The full text of the items and all the supporting text are in 23-16 Table 2. The assumption named by the items of Part 23 is the diffusive-relaxation-time assumption (the upper side is the relaxation-time upper-bound assumption, the lower side the relaxation-time lower-bound assumption), and the gists in Table 2 use this name.

Table 2 The items of Part 23 (gist; full text in 23-16 Table 2)

Item	Gist	Kind	Supporting text (main places)
1	The solution of the problem of existence and the mass gap; a construction of the continuum limit	Not shown	23-01 § 08
2	Even for the first step, the double-well relaxation-time power statement has not been shown. The First-Step Upper-Bound Proposition is algebra on top of four conditions	Not shown	23-05 § 08
3	There is not yet a single $\hbar$ at which all rows of the four conditions hold at once	Not shown	23-12 § 06
4	Computer-assisted results are inequalities inside the range that was computed	Not read strongly	23-08 § 10
5	Readings are used neither as upper bounds nor as proofs of lower bounds	Not read strongly	23-00 Table 1
6	A Lean theorem checks the algebra from assumptions to conclusion	Not read strongly	23-13 § 05
7	From the first step to the second step on, only the prospect that the form of the sentences and the form of the tools lift	Not read strongly	23-03 § 06
8	That a tool did not reach something is a property of that tool	Not called false	the table of scope in 23-10
9	Rederivations of known results are not counted as results. The correctness of other documents is not judged	Not judged	23-14 § 03
10	The record on the side of the four-dimensional lattice also remains assumption	Not shown	23-01 § 05
11	Under the assumption of the one-sided bridge, the upper-bound assumption implies a lower bound on the gap, but the converse is false in general. The diffusive-relaxation-time assumption is not the mass gap itself	Not read strongly	23-01 § 06

Item	Gist	Kind	Supporting text (main places)
12	The "death" of a detour plan does not mean that the remaining proposition is false	Not called false	23-10 § 03
13	The reading that the upper-bound assumption fails in the periodic box is a conditional reading, resting on numbers from the literature and on a conjecture	Not called false	23-01 § 05
14	A proposition with assumption rows left is not put out as a result	Not counted	23-12 § 08

03

The sixteen items of Part 24 — looking up 24-08 Table 4

Table 3 gives the gist of the items of Part 24, their kind, and the main places in the text that support them. The full text is in 24-08 Table 4. What is not claimed about the candidate rules (neither consistency nor a prospect towards the third step is shown, the Boolean-algebra comparison is a metaphor, and five others) is at the end of 24-08 § 02; here we only point to it.

Table 3 The items of Part 24 (gist; full text in 24-08 Table 4)

Item	Gist	Kind	Supporting text (main places)
1	The solution of the problem of existence and the mass gap. The continuum limit, recovery of rotations, the OS axioms and non-triviality are not touched	Not shown	24-00 § 05
2	That the relaxation-time assumption is false (the reading for the periodic box extends the numbers of the originals and a conjecture)	Not called false	24-01 § 06
3	The sector and local versions are candidate definitions, not shown at any β	Not shown	24-03 § 07
4	The one-sided bridge is an inequality on a lattice with continuous time. On Wilson's lattice with discretised time, the time-step comparison assumption ((D) in the paper "one-sided bridge") remains	Not read strongly	24-04 § 08
5	"The gap = the Poincaré constant of the vacuum measure" is an identity, and we do not say it has made the problem easier	Not read strongly	24-05 § 07
6	The definition of the length at which conservation breaks, and the restatement of what to compare with, are candidates	Not read strongly	24-02 § 08
7	The results for hierarchical models and two-dimensional models imply nothing for four dimensions	Not read strongly	24-06 § 09

Item	Gist	Kind	Supporting text (main places)
8	The non-perturbative mechanisms of physics (confinement, instantons, centre vortices, duality) are not used as tools of proof in Part 24	Not shown	24-07 § 08
9	The two points of "because it is four dimensions" are estimates as properties of the critical dimension	Not read strongly	24-07 § 01
10	The "what is not claimed" of Part 23 applies to Part 24 as it is	—	23-16 § 02
11	The local version goes only as far as the definition and the ratio form. The local diffusive-relaxation-time assumption has not been shown for the target gauge theory	Not shown	24-03 § 03
12	The thresholds of the local bridge depend on the choice of reference and on the family of test observables	Not read strongly	24-04 § 06
13	For the 0+1 double well, the local ratio is not 1. Taking it to approach 1 at short distances in four dimensions is an assumption that depends on asymptotic freedom	Not read strongly	24-04 § 07
14	The length at which conservation breaks was measured only on the width-2 region of two-dimensional $U(1)$	Not read strongly	24-02 § 03
15	For $SU(2)$, it is undecided whether the freezing of the topological charge breaks the relaxation-time assumption	Not called false	24-07 § 07
16	The results of the two-dimensional CP^{N-1} test bed imply nothing for four dimensions	Not read strongly	24-02 § 07

Item 8 of Part 24 needs care in reading when set beside Parts 25 and 26. Part 24 did not use centre vortices or instantons as tools of proof, but Part 25 puts vortex surfaces and fractional instantons at the centre of its tools. What it uses, however, are only finite identities on the lattice, estimates at fixed lattice spacing and conditional sentences; it does not bring the physical mechanisms into proofs as they are (items 2, 4 and 5 of Table 4).

04

The items of Part 25 — on the tools of the second road

These items are picked up from the sections "what cannot be said" of the chapters of Part 25 (from the direction of space). The group is $SU(2)$, and the small T^2 carries the twist $n_{12} = 1$. The kind column gives the three meanings (and "not judged, not counted").

Table 4 What Part 25 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	"Weak coupling persists up to the size of the small T^2" is not a theorem. It is a physical estimate made on the assumption of ultraviolet control, and an estimate uniform in the lattice spacing is nowhere in this book. That the small T^2 and $\mathbb{R}^4$ are connected without a phase transition (adiabatic continuity) is a conjecture and is not used	Not shown	(a)(b) of 25-00 § 08
2	The Slab Confinement Theorem is a statement with the lattice spacing and the size N of the T^2 fixed. The lower bound $-\ln\tanh(\beta N^2) \approx 2e^{-2\beta N^2}$ goes to 0 in the continuum limit, and also vanishes as $N \to \infty$. It says nothing about a tension uniform in the lattice spacing, nor about four-dimensional confinement	Not read strongly	(a)(b) of 25-04 § 07, § 04
3	The Container Identity is a rewriting of the Mack–Petkova identity, and is known. The rewriting in the summed ensemble, the way of moving to a fixed sector, nesting, the insertion identities and the Slab Confinement Theorem are also elementary, and have not been checked against the earlier literature. No novelty is claimed for the results of this part	Not counted	25-03 § 08 (a), 25-04 § 06, 25-05 § 08 (d)
4	None of the insertion identities and inequalities derives confinement. The large-deviation step, the ultraviolet step and the upper bound on the second moment remain	Not shown	25-05 § 08 (a), § 07
5	The count for a dilute gas of fractional instantons (string tension $\propto \Lambda^2(\Lambda L)^{5/3}$, margin $5/3 = 11/3 - 2$) is a physical estimate in the dilute regime $\Lambda L \ll 1$, at the one-loop level. The action values obtained by cooling are upper bounds on the minimal action, not lower bounds	Not read strongly	(a)(b) of 25-02 § 07
6	The Confinement Lower Bound by Fractional Instantons (Theorem A of Paper 3) is a statement of a draft, and its label in this book is Assumption. It is not uniform in N, and says nothing about the continuum limit or about the mass gap. The converse bound (uniform in M) is open	Not read strongly	25-02 § 06, § 07 (e)

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
7	The count that, in four dimensions, independent containers give only up to $R/\ln R$ is a count on top of the assumption of the original source (that the free energy of thick containers takes a form uniform in the boundary values), not a four-dimensional theorem. Neither that assumption nor its negation has been shown	Not read strongly	25-03 § 08 (d)
8	The three facts about the twisted T^2 (uniqueness of the vacuum, the gap of the adjoint field $4\sin^2(\pi/2N)$, the absence of flat configurations in the (1,1) sector) say nothing about the lattice spacing $a \to 0$. The gap of the adjoint field is a lower bound on classical fluctuations, not a gap of the quantum effective action	Not read strongly	25-01 § 09 (a)
9	The Lean column covers only finite sums and real algebra. The Haar measure of $SU(2)$, the co-closed sign field, the topology and projections of sectors, and that θ is the bias of a sector are outside Lean	Not read strongly	25-03 § 08 (f), 25-04 § 07 (e), 25-05 § 08 (b)
10	No sentence of this part says anything about widening the T^2 beyond $1/\Lambda$. That stage is the crossover band, and the second road stops there	Not shown	25-02 § 07 (d), 25-09
11	The Driven Insertion Identity and the Flow Lemma do not derive confinement. "For a one-step kernel the flip probability saturates around $\frac{1}{2}$" is a judgement from measurement and the record, not a theorem. The Crooks Relation and the Jarzynski Equality are known	Not read strongly	(a)(d)(e) of 25-06 § 08
12	The comparison of isometric maps and reshaping maps is a statement at the one-loop (Gaussian) level. "Upper bounds come only from variation and explicit construction" is a judgement about this book's list of tools, not a theorem. That the leftover relative entropy is proportional to the area is an estimate for general maps	Not read strongly	(a)(c)(e) of 25-07 § 08
13	The measured numbers of Part 25 are not proofs. The trend from widening the T^2 does not show the band numerically (it goes only up to about $L\sqrt{\sigma} \approx 2$), and reading the measurement of simultaneous flips as a numerical picture of the band has been withdrawn	Not read strongly	(a)(b)(f) of 25-08 § 07

05

The items of Part 26 — on the band

These items are picked up from the chapters of Part 26 (the crossover band). The band is the interval around the scale $L \sim 1/\Lambda$ at which weak coupling turns into strong coupling; it is a different thing from the "window" of Parts 23 and 24 (26-00 § 02).

Table 5 What Part 26 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	The band is unsolved. This book does not show, in any form, a statement that crosses the band ($\ln(1/\theta(L,L)) \geq cL^2$ uniformly in L, or an intermediate milestone towards it). Neither the renormalisation comparison nor β-monotonicity has been shown at any β of the band	Not shown	26-01 § 06, 26-02 § 08, 26-06 § 09 (a)
2	The reduction did not make the problem easier. The reduction theorem is a statement on top of three assumptions, and the strong form of the lateral-widening inequality is stronger than confinement (it contains monotonicity of the slab tension). The lemma that shrinks the comparison inside the band to the area law at fixed β is also a reduction, and its assumption contains an unsolved problem	Not read strongly	26-02 § 05, 26-06 § 09 (c), 27-02 § 03
3	Do not read "the renormalisation comparison holds at both ends" with the conditions removed. The strong-coupling end is an outline in the range $u \leq u_0 \approx 10^{-80}$ with part of the bookkeeping only sketched, and the weak-coupling end is a computation at the one-loop level. A non-perturbative weak-coupling comparison has not been shown at any β	Not read strongly	26-06 § 09 (b), § 03
4	The comparison principle (in the form that Wilson loops on $\mathbb{R}^4$ are bounded by loops on the twisted slab) is a statement at the one-loop level and is not used as a non-perturbative tool. With $SU(2)$, the Wilson action and $\beta = 2.1$, the plaquette on the twisted slab $16 \times 16 \times 2 \times 2$ is lower than on 16^4 (7σ with errors taken from both sides, 5.5σ in an independent rerun), so it is broken for small loops. The book places this as a reading that refutes it Reading (not a theorem)	Not read strongly	26-08 § 04

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
5	The numbers of the lattice-doubling test (small lattices up to 8^4, few values of β, 1–3 seeds) are a reading of direction, not evidence that $SU(2)$ has no phase transition to a non-confining phase	Not read strongly	26-06 § 09 (d), 26-00 § 03 (ii)
6	We do not say that β-monotonicity is false (no counterexample has been found; a reading). The four faces of the middle are candidates, and none has an estimate uniform in the scale	Not called false	26-06 § 09 (e)(f)
7	The wall of the first step, the "summit-curvature shortfall wall", is the band in miniature, not the band. We do not say that the results of the first step imply anything in four dimensions. That four dimensions have no "finite block beyond the crossover length" is not a claim that the road of blocks is impossible in four dimensions	Not read strongly / Not called false	(b)(f) of 26-07 § 07, 26-00 § 05
8	We do not say that the road of blocks and mixing closes the first step. Apart from the Hidden-Convexity Proposition, its rows are Assumption and Reading	Not shown	26-07 § 07 (d), the road of blocks and mixing in 23-15
9	"The only place where non-abelian enters" is a phrase with the restriction "within this reduction". The margin $5/3 = 11/3 - 2$ is a physical estimate in the dilute regime at the one-loop level	Not read strongly	26-02 § 08, 26-00 Table 1
10	The negative lemma, that the lateral-widening inequality does not follow from structural inputs alone (reflection positivity, the identities of centre and duality, positivity of the transfer matrix), is a statement of the form "from this set of inputs alone", not a claim that the band cannot be crossed	Not called false	26-00 Table 1, 26-04 § 04
11	None of the candidate new viewpoints closes the band. The reduction by compactness of the band is a reduction, and we do not say it has made the band easier. Nor do we say that the candidates are not combinations of known facts	Not shown	(a)(b)(g) of 26-10 § 07
12	"Stopped" for a road tried and stopped means that its tool could not cross the band, not that the proposition aimed at by the road is false. Roads stopped for reasons of estimate can be reopened if a new input appears	Not called false	26-08 § 01

06

The items of Part 27 — on the conditional statement and the ledger

Table 6 What Part 27 does not claim, and the sentences of the text that support it

Item	What is not claimed	Kind	Supporting sentence of the text (gist)
1	Confinement (the area law) is not the mass gap. There is no known road that gets the mass gap from the area law, and the arrow joining the two on the same theory is open (even the statement of what it suffices to assume is not settled)	Not shown	16-09 § 03, 27-01 § 04, 25-00 § 08 (c)
2	The conditional statement is a statement only insofar as the band is placed among the assumptions. Only the arrows whose paper proof has been placed in the book are called "theorems", and the assumptions in the form of the band (the lateral-widening inequality, the uniform slab tension, the uniform slab gap) carry the content of the conclusion	Not read strongly	27-01 § 05
3	The propositions of the research programme on $\mathbb{R}^2\times$ a small twisted T^2 (the arrow of the small-T^2 gap on the gap side, the arrow of the small-T^2 area law on the confinement side) are conditional and at the stage of an outline. The assumption of the arrow of the small-T^2 gap (block correlations decay at rate $m \geq c/L$) contains the quantity of the conclusion	Not read strongly	27-01 § 02, § 03
4	Even if every row of the ledger disappears, that does not mean the problem is solved. Four things remain: the limit to $\mathbb{R}^4$ (split into four rows in Table 3 of 27-02), tying the two roads together, general groups, and verification and peer review (seven rows with the breakdown)	Not shown	27-02 § 05, 27-00 § 01
5	A row closed in the record (mark ■) does not mean it has disappeared from the book's ledger. A row disappears only when its paper proof has been placed in the book and, if it has a Lean entry, the re-check is done	Not read strongly	27-02 § 01
6	The number of rows of the ledger is not a gauge of progress. Rows can also split and increase	Not counted	27-02 § 06

07

Conditions drop out in summaries — sentences of Parts 25 to 27 and one-line summaries

The error of reading a claim strongly arises, in most cases, at the stage of shrinking a sentence of the text into one line (23-16 § 06). Table 7 sets the conditional sentences of the text of Parts 25 to 27 beside one-liners from which the conditions have been dropped entirely. Every sentence in the right-hand column is a sentence this book has not shown.

Table 7 Sentences of the text (with conditions) and one-liners with the conditions dropped (everything on the right is wrong, or not shown)

Topic	Sentence of the text	One-liner with the conditions dropped
Slab confinement	With the lattice spacing and the side N of the T^2 fixed, $\sigma \geq -\ln\tanh(\beta N^2)$ for every β Proved here (25-04 § 02)	Confinement in lattice gauge theory has been shown
Confinement Lower Bound by Fractional Instantons	At fixed $N \geq 2$, $\ell \geq 2$, $\beta \geq 1$, $M \in 2\ell\mathbb{N}$, a lower bound on the area law with rate $S_*(N,\ell)$ (a statement of the draft; the book's label is Assumption, 25-02 § 06)	It has been shown that fractional instantons cause confinement
The one-loop comparison	With the comparison that reshapes the fluctuations, $\beta - \beta' = 0.258$ (per doubling) Proved here (at the one-loop level, 26-06 § 03)	The renormalisation comparison holds
The doubling test	For $SU(2)$, the $\beta - \beta'$ read off is $+0.13$–$+0.25$ ($4^4 \to 8^4$ and others, $\beta = 2.5$–3.0, 1–3 seeds) Reading	$SU(2)$ has no phase transition inside the band
The reduction theorem	Under the three assumptions of the base point, frame decorrelation and the lateral-widening inequality, the correct form of the band follows Proved here (conditional, 26-02 § 03)	The band has been reduced to a single inequality and become easier
The comparison inside the band	Under the area law on both sides at each β of the band (the assumption of two-sided area laws inside the band), the comparison inside the band shrinks to finitely many steps Proved here (26-06 § 06)	The renormalisation comparison inside the band has been shown
The iteration in Lean	`Doubling.iterate` (assumptions: $c \geq 0$, $t_k \geq 0$, $t_{k+1} \leq c^3 t_k^4$; conclusion: $c\,t_k \leq (c\,t_0)^{4^k}$) is algebra of a sequence of real numbers	The reduction of the band has been checked in Lean

Topic	Sentence of the text	One-liner with the conditions dropped
Reflection positivity of z	The proof is placed in 26-04 § 03. The algebra of the last step is also closed in Lean (ReflPos). This row is closed (27-02 § 01)	The whole proof of reflection positivity of z has been checked in Lean

The error types in Table 7 are the same two as in Table 5 of Part 23. One drops the scope (fixed lattice spacing, fixed N, the one-loop level, small lattices); the other drops the grade (Reading, Assumption, the reach of Lean). In Parts 25 to 27 one more type is added: reading "reduced" as "made easier". A reduction gathers the difficulty into one place; it does not reduce the difficulty.

What this chapter can say

What is said	What is not said
That this book shows nothing about the existence and the mass gap of the problem statement (0-03). Which sentences of which chapters support the items of Parts 23 and 24 (Tables 2 and 3)	The truth or falsity of the claims of the problem statement. Existence of the continuum limit (on the side of this book's study; the published construction showed it in a periodic box). A gap uniform in the volume
The items of Parts 25, 26 and 27, and the places in the text that support them (Tables 4–6)	The truth or falsity of the proposition of each item. Of the sentences of chapters 26-03 to 26-05 and 26-09, only those that overlap with rows of Table 5 have been picked up
Pairs of conditional sentences of Parts 25 to 27 and one-liners with the conditions dropped (Table 7)	That the sentences in the right-hand column of Table 7 are false (only that they have not been shown)

Sources and reproduction

Item	Kind	Source / tools
The three meanings	This book	23-16 § 01
Each row of Table 2	This book	23-16 Table 2 and the chapters in its right-hand column
Each row of Table 3	This book	24-08 Table 4 and the chapters in its right-hand column
Each row of Tables 4–6	This book	The sections "what cannot be said", "what remains" and "the differences that remain outside the ledger" of the chapters in the right-hand column
Table 7	This book	The statements and labels of the chapters in the left-hand column. The numerical conditions are in 26-00 § 03 (ii) and 26-06 § 05

Item	Kind	Source / tools
The problem statement	Cited	A. Jaffe, E. Witten, Quantum Yang–Mills Theory, Clay Mathematics Institute (2000) (the source table of 0-03)

Terms used in this chapter — not shown, not read strongly, not called false, not judged / not counted, the conditions that drop out in a summary, the crossover band, reduction, the differences that remain outside the ledger, the diffusive-relaxation-time assumption. Their meanings are in the A-02 Glossary and the A-06 Index of names.

Next chapter: 28-00 Entrance — the statements to be shown, and sufficient conditions through a representation. To read alongside it, A-07 Papers that came out of this book — the main theorem of each paper, with its label and conditions. To go back to the entrance of Part 27, 27-00.