Volume 3 Toward the mass gap — the questions that remain, and this book's results

Part 28 The tree of premises

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Entrance — the statements to be shown, and sufficient conditions through a representation

> This chapter is a guide.

What this chapter shows — Part 28 lays out "what would suffice" as a tree of the root and its sufficient conditions. This entrance shows the distinction between the root and the sufficient conditions, how to read the premise labels, and the order of the chapters in this part.

Prerequisite chapters — 19-07 Statements to be shown. The firm theorems are in Parts 20–22 (20-00, 21-00, 22-00).

Said first — This part is a list of premises and implications, and it shows none of the roots. Of the roots, the four on the ultraviolet side (matching, convergence of the full sequence, non-degeneracy, equicontinuity) and the tree of premises that supports them are statements about this book's design (measures pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design, and the committee's assessment is that it is proved in Lean (Volume 2; the two designs are compared in Part 14, and the record of this book's ultraviolet attempt is in Part 15). So the roots and the tree on the ultraviolet side are kept as a record of the statements to be shown and the sufficient conditions within this book's design. The roots on the infrared side (uniform decay and the mass gap that follows from it, and the band) remain open, and these are the main subject of Volume 3. The premise labels are as of 2026-11-13; they are updated in A-08 A map to the mass gap.

The order of this chapter

1. The root, and sufficient conditions through a representation (Table 1)
2. Key to the premise labels (Table 2)
3. Symbols for why a form is not fixed (Table 3)
4. The chapters of this part

01

The root, and sufficient conditions through a representation

In this part we call the statements to be shown the "roots" of the tree, and their sufficient conditions the "leaves".

The **roots** are the six statements to be shown, written only with the objects defined in Part 19 (the Wilson measure, the push-forward measure, the vortex ratio, the vessel of reconstruction) (19-07).

The **sufficient conditions** are the premises we place in order to derive the root on the present road. Many of them are statements made through quantities used to write down the theory at coarse scales — effective couplings, large-field activities, background fields, classes — and this book calls such a description a **representation**. Many of the quantities of a representation are not yet defined from the lattice.

Table 1 The root, and the sufficient conditions on the present road

Root (19-07)	Language of the root	Sufficient conditions on the present road (names of rows in 28-04)	Language of the sufficient co
Matching	Expectations under the push-forward measure, β_n	One-step bracket of the coupling and recursion of the activity (definition and continuity of the effective coupling, net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin)	Effective coupling u_k, activ supply w_k
Convergence of the full sequence	Push-forward measure $\nu_{n,m}$	Contraction of the one-step map and matching at the endpoint, the estimates of the premise of ultraviolet stability	Coordinates of the one-step class, background field
Non-degeneracy	$\nu_{n,m}$ and the set of small fields	Rarity of two-step events (the plaquette part, the background-invariant part)	The plaquette part in the sa language as the root; the background-invariant part background fields
Equicontinuity	$\nu_{n,m}$ and the family of observables	Equicontinuity of observables	Same language as the root
Uniform decay	Covariances under $\nu_{n,m}$, the vessel	Uniform decay of block observables, of the same form as the root; the seam to the vessel of the mass gap; the rows for fine observables, infinite volume, rotation and non-triviality (from approximation of flowed observables by block variables to the limit is not a free field); band comparison (in the language of the gap) and from the area law to the gap	Flowed observables, rotate lattices, box masses
Band	Vortex ratio θ	Base point, nesting and widening (non-negativity of θ, base point at small T^2, single-scale criterion, question of the cost of a twist-inserting map, frame decorrelation, lateral-widening inequality (strong form)), tension doubling (tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison)	θ, sheet tension

The root is read for the sequence fixed by matching. If it is read in a form where the sequence of couplings can be chosen freely, $\beta_n \equiv 0$ alone makes the mass-gap and band parts of the root stand (19-07 § 01). The "Matching" row of Table 1 is not one row alongside the other five; it is the row that fixes how the other five are quantified.

The root does not depend on the representation. If the representation is changed (a different definition of the background, a different norm, and so on), the sufficient conditions change, but the root does not. The list (28-04) writes the root at the top and the sufficient conditions below it, as those "on the present road".

02

Key to the premise labels

A premise label is written as one column of a table. It is separate from the labels used in the text (Proved here, Cited, Computer-assisted, Reading, Assumption — the system of 23-00). When the text writes out the statement of a premise itself, its label in the text is Assumption .

Table 2 Premise labels

Label	Meaning	What it does not guarantee
Theorem	Machine-checked in Lean, and a check that compares the definitions and the statement with a separate pair of eyes has confirmed the content of the statement	That it is the premise in exactly its stated form (the scope is written in the same column)
Theorem (weak form)	The same, but only up to a form weaker than the premise. We say separately whether it pays the column as it stands or is a similar but different statement	That it reaches the form of the premise

Label	Meaning	What it does not guarantee
On paper	There is an argument on paper	That it is free of errors (the check comparing definitions and statement with a separate pair of eyes has not been done)
Form fixed, unproved	The form of the statement is settled. "Conditional" means: for the sequence fixed by matching, and only when the restrictions on threshold and period are imposed	That it holds
Form not fixed	The form of the statement is not yet settled. The reason is given by a symbol (Table 3)	—
Depends on design	Determined by a design choice (28-05)	That the choice is determined uniquely
Literature	Depends on a result in the literature	That this book has re-derived it

03

Symbols for why a form is not fixed

Table 3　Reason symbols

Symbol	Meaning
(a)	Too light: it stands for a trivial object, or because a quantity can be chosen freely
(b)	Too strong: it is false, or there are examples where it is false
(c)	Object undefined: a quantity in the statement is not defined from the lattice
(d)	There are numerical signs against the present form
(e)	There is a form at one scale, but no form across steps
(f)	The order of quantifiers is not fixed

A list sorted by origin is in 28-06.

04

The chapters of this part

Chapter	Contents
28-01 Subsequential limits, and the properties that pass to the limit	The parts that stand without premises, and the parts where premises do the work
28-02 Convergence of the full sequence	Convergence of the full sequence through matching at the endpoint
28-03 From decay to the mass gap, and the reduction of the band	From uniform decay to the mass gap of the vessel; from base point, nesting and widening to the area exponent
28-04 The list of premises — tree and labels	The root and the tree of 57 rows of premises, tables by stage, counts by label, a dictionary of other names
28-05 What depends on design choices	Nine choices, and which theorems and premises depend on which
28-06 Open places, and places where the object does not yet exist	Where the 24 rows whose form is not fixed come from, and what would fix their form. What is not claimed

The three chapters on implications are theorems that write "what follows if the premises hold", with all the assumptions shown. What follows can be weaker than the root (28-01 reaches only subsequential limits), and there are also implications that go beyond the root (28-03 goes from uniform decay to the mass gap of the vessel).

What this chapter can say

Can say	Cannot say
The distinction between the root and the sufficient conditions; the meaning of the premise labels and the reason symbols	None of the root is shown

Sources and reproduction

Item	Type	Source, tools
The tree of the root and sufficient conditions, the labels	Assumption	The inventory of premises in this book's study (2026-11-13). The full text is in 28-04

Words used in this chapter — root, sufficient condition, representation, premise label, reason symbol. Their meanings are in A-02 Glossary.

Next chapter: 28-01 Subsequential limits, and the properties that pass to the limit — reads what follows from the premises, with all the assumptions shown. To start from the list, read 28-04.

Subsequential limits, and the properties that pass to the limit

> This chapter is this book's study.

What this chapter shows — Let $\nu_{n,m}$ be the measure obtained by pushing the fine measure μ_n on a box of side $2^n L$ forward to the coarse level m by the block map. For it we read, with all the assumptions shown, the conditional theorem "if many assumption fields hold, then a subsequential limit exists, and the limit has reflection positivity, symmetry and a non-degeneracy lower bound" (Lean: `BoxWalk.chain_end_to_end_box`). The content of this theorem splits in two. (a) Properties that hold uniformly on finite lattices pass to subsequential limits. (b) From the one-step structure for the coupling, the matching point can be taken inside an explicit bracket.

Prerequisites — 19-04 The block map and the push-forward measure, 19-01 The Wilson measure, 19-03 Reflection, 19-07 The statements to be shown.

Said first — This is an implication, and none of its assumptions has been shown on the four-dimensional lattice. The existence of a subsequential limit holds, without assumptions, for any family of probability measures. The reading that joins (a) and (b) ("the fine measure is the Wilson measure at the coupling fixed by the matching point") lies outside the theorem. The quantities put into the assumptions (effective coupling, activity, supply, remainders, background invariants, the sequence of observables) are not defined from the lattice. So the fact that this theorem is closed does not mean that the content of the continuum limit has been shown. The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design (Volume 2). The implication of this chapter is a record of what this book's design (measures pushed forward by a fixed block map) placed as premises (§ 08).

The order of this chapter

1. The form of the theorem — what is fixed and what is left free — only the block map and the push-forward are fixed by definition

2. The table of assumption fields — Lean names, mathematical wording, and whether they are paid by concrete objects (Tables 1 and 2)
3. The first conclusion — the matching point lies in a bracket — a statement about the family of functions u alone, not connected to the measure
4. The second conclusion — convergence along a subsequence — from compactness, it holds for any family of measures
5. Properties that pass to the limit — reflection positivity, invariance, lower bounds on the mass of closed sets, the modulus of continuity of observables (Table 3)
6. The version with the Wilson measure as the fine measure — the scope in which two fields disappear, and the fields that remain
7. The reading that joins the two lies outside the theorem — joining the sequence of bare couplings to the effective coupling needs a definition of the effective coupling
8. Relation to the conditional implication assembled earlier — its assumptions are statements written with the push-forward measure

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01

The form of the theorem — what is fixed and what is left free

One road to building a continuum limit goes as follows. On boxes whose lattice spacing is halved step by step (periodic boxes of side $2^n L$, $n = 0, 1, 2, \ldots$) we place fine measures μ_n, and push them forward by the block map to the coarse box of side $2^m L$ (19-04). The limit $n \to \infty$ with the coarse level m fixed is the candidate for "the distribution of the block variables of the continuum limit" on a box of fixed physical size.

We fix notation. $\mathbb{B}$ is the closed unit ball of the quaternions, and E_m is the space of configurations that put a value in $\mathbb{B}$ on each link of the box of side $2^m L$. Since E_m is a finite product of balls, it is a compact metric space. $B_m : E_{m+1} \to E_m$ is the block map, $\mathrm{down}_{n \to m}$ its $n - m$-fold composite, and the push-forward measure is

$$\nu_{n,m} = (\mathrm{down}_{n \to m})_* \, \mu_n \qquad (m \le n) \quad (1)$$

(Lean: `ChainBlockMap.blockMap`, `down`, `vP`). The block map includes a retract that returns the average of path products to the ball with threshold $r_0 > 0$ (19-04). r_0 is a free argument of the theorem (field `hr : 0 < r₀`), and the limit may depend on r_0. In

the box version the family of paths and the weights are also fixed (weights $(1/8, 3/4, 1/8)$, six orderings; Lean: `BoxWalk.boxPaths`, `boxW`).

This much is fixed by definition. Inside the theorem, the fine measure μ_n is an arbitrary family of probability measures. Besides this, the coupling functions u_k, activities z_k, supplies w_k, remainders $r_k^{\S}, r_k^{]}$, plaquette variables $Y_{m,p}$, background invariants $f_{m,x}$, the sequence of observables s_n, the reflection Θ_m, the symmetry actions, the set of small fields C_m and so on all enter as free variables. The assumption fields are conditions on these free quantities.

Assumptions	First conclusion	Second conclusion
The one-step structure for the coupling, rarity, splitting of observables, reflection positivity, symmetry, representation, and others (§ 02) All Assumption . The quantities are not defined from the lattice	For any target value t and step n, a point x with $u_n(x) = t$ lies in an explicit bracket (§ 03) A statement about u alone	Along some subsequence, $\nu_{n,m}$ converges at every level m, and the limit inherits the assumed properties (§ 04, § 05) The convergence itself holds without assumptions

Lean: `ChainBlockMap.chain_end_to_end_blockMap` (version with a general family of paths and weights), `BoxWalk.chain_end_to_end_box` (box version) — machine-checked as an implication (assumptions as in the tables). The two conclusions sit side by side, joined by "and", in a single statement.

02

The table of assumption fields

We list all the assumption fields. The column "Paid?" says whether the field is shown for concrete objects (measures or maps with a definition), or remains an assumption about abstract quantities. In the box version, the continuity of the block map and the consistency of the push-forward ($(B_m)_*\nu_{n,m+1} = \nu_{n,m}$) are paid from the definitions inside the theorem and do not appear as fields (Lean: `ChainBlockMap.continuous_blockMap`, `hcons_blockMap`).

Table 1 Fields on the side of measures and observables ($\nu_{n,m}$ is as in (1). "For large n" means "there is n_0 such that for $n \geq n_0$")

Lean field	Mathematical wording	Paid?

Lean field	Mathematical wording	Paid?
`rarity`	Assumption For each level m, large n and each block i of the region R_m: $\nu_{n,m}$(the plaquette variable $Y_{m,p}$ of block i exceeds the threshold $y_h(m)$, or the background invariant $f_{m,x}$ exceeds the threshold $f_h(m)$ (in Lean this threshold is named `θ`, which has nothing to do with the vortex ratio))$\le n_p(m)P_1(m) + n_{pt}(m)S(m)$. Rarity	Abstract. $Y_{m,p}$ and $f_{m,x}$ are free functions, and the event itself is not defined
`hCc`, `hgood`	Assumption C_m is a closed set and contains the event "in every block, plaquette variable $\le y_h$ and invariant $\le f_h$"	Abstract
`hθc`, `hθp`	Assumption Θ_m is continuous. For large n and F in the class $\mathcal{A}_m$, $\int F(\Theta_m\omega)F(\omega)\,d\nu_{n,m} \ge 0$. Reflection positivity	**There is a version paid with the Wilson measure** (§ 06; the class is a strip of fixed width)
`hsc`, `hsym`	Assumption The symmetry action g is continuous. For large n, $\nu_{n,m}$ is invariant under g	Abstract (remains in the Wilson version too)
`split`, `lattice_vanishing`, `shift_vanishing`	Assumption $\lvert s_n(d) - s_n(d')\rvert \le \tau_n + \Omega(\mathrm{dist}(d,d'))$, $\tau_n \to 0$, $\Omega(r) \to 0$ $(r \downarrow 0)$. Splitting of observables (a split form of equicontinuity)	Abstract. s_n is free
`hcount`, `hdense`, `hbd`	Assumption The set D in the space X of observable placements is countable and dense, and $\lvert s_n(d)\rvert \le K$ (K is one number independent of d and n; its Lean name is `Bd`)	Abstract
`hac`, `ham`, `hbr`, `hbv`	Assumption The symmetry action on X is continuous and preserves D, and the breaking $\lvert s_n(gd) - s_n(d)\rvert \le \varepsilon_b(g,n,d) \to 0$	Abstract
`hpos`	Assumption For large n, $\sum_{a,b} c_a c_b\, s_n(x_{ab}) \ge 0$ (the matrix form of reflection positivity)	Abstract
`HR`	Assumption For each $d \in D$ and large n, $s_n(d) = \int F_d\,d\nu_{n,\mathrm{lev}(d)}$ (F_d is a bounded continuous function at level $\mathrm{lev}(d)$). Representation	Abstract (holds trivially if s_n is *defined* as the expectation under ν)

Table 2 Fields on the side of the coupling (the structure `TransientChainError.CouplingStepsTransientE` and two outside it. The domain is $\{x : u_j(x) \ge u_\star\ (j \le k)\}$)

Lean field	Mathematical wording	Paid?

Lean field	Mathematical wording	Paid?
`flow`	Assumption $u_{k+1} = u_k - b + r^{\mathrm{s}}_k + r^{\mathrm{l}}_k$ (the one-step decrease of the coupling)	Abstract. $u, z, w, r^{\mathrm{s}}, r^{\mathrm{l}}$ are free functions $\mathbb{N} \times \mathbb{R} \to \mathbb{R}$, and $W, \Phi, \mathcal{D}, e$ are free functions $\mathbb{R} \to \mathbb{R}$
`small_net`, `transient`	Assumption $b_{\mathrm{inf}} + \delta_k - e(u_k) \le b - r^{\mathrm{s}}_k \le b_{\mathrm{p}} + e(u_k)$, $\sum_{k<n} \max(-\delta_k, 0) \le D_0$ (the net bracket for small fields, and the sum of transients)	
`large_rem`	Assumption $\lvert r^{\mathrm{l}}_k \rvert \le \mathcal{D}(u_k)\, z_k$ (the large-field remainder)	
`act_rec`, `act_nonneg`, `act_init`, `supply`	Assumption $z_{k+1} \le \rho z_k + A w_k + B z_k^2$, $z_k \ge 0$, $z_0 \le W(u_0)$, $w_k \le \Phi(u_k)$ (the recursion of activities, and the supply)	
`fix_shift`, `DW`, `margin`	Assumption $\rho W(s) + A\Phi(s) + B W(s)^2 \le W(s')$ ($u_\star \le s$, $u_\star \le s' \le s + D_0$), $\mathcal{D}(s) W(s) \le Y$, $e(u_\star) + Y \le b_{\mathrm{inf}}$ (a moving invariant interval, and the margin)	
`ustar_pos`, `he`, `hD`, `hp`, `hA`, `hB`	Assumption $u_\star > 0$, e is decreasing on $[u_\star, \infty)$, $\mathcal{D} \ge 0$, $\rho, A, B \ge 0$	
`h0`, `hCu`	Assumption $u_0(x) = x$. $x \mapsto u_{k+1}(x)$ is continuous on the domain	Abstract

The fields of the two tables can be satisfied simultaneously Proved here . For example, take the region R_m, the class $\mathcal{A}_m$, P and the index sets of the two symmetry actions to be empty, $C_m = E_m$, X a single point, $s_n(d) = \int F_d \, d\nu_{n,\mathrm{lev}(d)}$, $u_k(x) = x - k$, $b = b_{\mathrm{inf}} = b_{\mathrm{p}} = 1$, the remainders, activities and supplies and $W, \Phi, \mathcal{D}, e, \delta, D_0, Y, \rho, A, B$ all 0, and $u_\star = 1$. With this choice, however, the corresponding conclusions are also trivial. No choice that pays the fields with quantities of real content is known.

Detour: what the one-step structure for the coupling actually says

May be skipped. The structure in Table 2 has seventeen fields, but the content that matters for the first conclusion is that "the one-step decrease $u_k - u_{k+1}$ of u lies in the bracket $[\, b_{\mathrm{inf}} + \delta_k - e(u_k),\ b_{\mathrm{p}} + e(u_k) \,]$", together with numerical conditions on $u_\star, e, \delta, D_0$ alone Proved here . Indeed, if an arbitrary u satisfies this bracket, then setting $r^{\mathrm{s}}_k := u_{k+1} - u_k + b$, $r^{\mathrm{l}} = z = w = 0$, $W = \Phi = \mathcal{D} = 0$, $Y = \rho = A = B = 0$ makes `flow`, `small_net`, `large_rem`, the four on activities and supply, `fix_shift`, `DW`, `hD`, `hp`, `hA` and `hB` hold. The remaining four — $u_\star > 0$ (`ustar_pos`), e decreasing on $[u_\star, \infty)$ (`he`), the sum of transients $\sum_{k<n} \max(-\delta_k, 0) \le D_0$ (`transient`), and $e(u_\star) \le b_{\mathrm{inf}}$ (the $Y = 0$ form of `margin`) — do not follow from this choice, and remain as conditions on the choice of $u_\star, e, \delta, D_0$. The fields for the large-field remainder, activity and supply give a name to one "way of splitting" that produces the bracket; as long as the quantity that fixes the splitting (the decomposition of the effective action) is not defined from the lattice, they add no content by themselves. Which splitting carries meaning is the subject of the list of premises (28-04) and of the open places (28-06).

03

The first conclusion — the matching point lies in a bracket

The first conclusion has the following form.

$$\forall t \geq u_\star + D_0,\ \forall n,\ \exists x:\quad \begin{cases} t + (b_{\mathrm{inf}} - e(u_\star) - Y)\, n - D_0 \ \leq\ x \ \leq\ t + (b_{\mathrm{p}} + e(u_\star) + Y)\, n, \\ u_n(x) = t, \qquad u_k(x) \geq t - D_0 \ \ (k \leq n). \end{cases}$$

In the intended reading, x is the bare inverse coupling of the fine lattice, $u_k(x)$ the effective inverse coupling after k steps of renormalisation, and t the target value matched at the coarse end. Formula (2) reads: "the bare value x that becomes exactly t after n steps lies in an explicit bracket linear in n, and does not leave the domain at intermediate steps." The bracket closes around the one-loop decrease b because the one-step decrease lies in $[b_{\mathrm{inf}} - e,\ b_{\mathrm{p}} + e]$.

Formula (2) is a statement about the family of functions u and the constants of Table 2 alone. The fine measure μ_n, the push-forward measure $\nu_{n,m}$ and the observables s_n appear neither in (2) nor in the fields used to derive (2) (Table 2). Inside the theorem there is not a single field joining u and μ.

The proof runs in two steps. (i) On the domain, induct on k for the activity and the coupling together, and derive that the one-step decrease lies in the bracket (the large-field remainder is bounded by $\mathcal{D}W \leq Y$ and absorbed into the bracket by `margin`). (ii) From $u_0(x) = x$ and the continuity of u_{k+1}, take x with $u_n(x) = t$ inside the bracket by the intermediate value theorem Proved here . Both are arguments about real functions; the lattice does not appear.

04

The second conclusion — convergence along a subsequence

The second conclusion begins with the following existence statement.

$$\exists\, \varphi \text{ strictly increasing},\ \exists\, (\nu_m)_{m \geq 0} \text{ probability measures}:\qquad \nu_{\varphi(k),m} \xrightarrow[k \to \infty]{} \nu_m \quad (\text{wea}$$

Formula (3) **holds for any family of probability measures μ_n, without using a single assumption field**. The reason is that E_m is a compact metric space, and the space of probability measures on it is compact (and metrisable) in the topology of weak convergence Cited (Prokhorov's theorem, in the case of a compact space). Since the product over countably many levels m is also compact, one subsequence φ common to all m can be taken (Lean: `WeakUltravioletExistence.exists_common_subseq`).

So formula (3) by itself says nothing about what the limit is. The limit may depend on the choice of the subsequence φ. The limit has content reflecting the choice of the fine measure only through the "properties that pass to the limit" of the next section. Convergence of the full sequence (not a subsequence) is not among the conclusions of this theorem. It is the subject of the next chapter, 28-02, and one of the statements to be shown (convergence of the full sequence in 19-07, stated for the sequence of couplings fixed by matching).

The consistency of the push-forward $(B_m)_* \nu_{n,m+1} = \nu_{n,m}$ holds by definition, and since B_m is continuous it passes to the limit: $(B_m)_* \nu_{m+1} = \nu_m$ (Lean: `WeakUltravioletExistence.limit_consistent`). The family of limits is a projective system consistent with the block maps.

05

Properties that pass to the limit

The rest of the second conclusion is that the fields of Table 1 pass to the subsequential limit. All are of one type: "a closed condition that holds uniformly at finite n (for large n) is preserved in the weak limit."

Table 3 Properties for large n, and properties in the limit (the Lean conclusion structures `WeakUltravioletExistence.BlockConclusion` and `ObservableConclusion`)

Assumption (for large n)	Conclusion (in the limit)	Tool that carries it over
`h0c`, `h0p`	`reflection_positive`: for $F \in \mathcal{A}_m$, $\int F(\Theta_m \omega) F(\omega)\, d\nu_m \geq 0$	$F \circ \Theta_m \cdot F$ is bounded and continuous, so the integral converges, and ≥ 0 is a closed condition Proved here
`hsc`, `hsym`	`symmetric`: ν_m is invariant under the symmetry action	The same (an equality of integrals of bounded continuous functions) Proved here
`rarity`, `hgood`, `hCc`	`small_field_mass`: $\nu_m(C_m) \geq c_m$, $c_m = 1 - \lvert R_m \rvert (n_p P_1 + n_{pt} S)$	A union bound gives $\nu_{n,m}(C_m) \geq c_m$ for large n. The mass of a closed set does not drop in the weak limit: $\limsup_n \nu_n(C) \leq \nu(C)$ Cited
`HR`	For $d \in D$, $S(d) = \int F_d \, d\nu_{\mathrm{lev}(d)}$ and $s_{\varphi(k)}(d) \to S(d)$	The definition of weak convergence itself Proved here
`hbd`	`bounded`: $\lvert S(x) \rvert \leq K$ for all $x \in X$	A closed condition Proved here
`split`, `lattice_vanishing`, `shift_vanishing`, `hcount`, `hdense`	`continuous`, `modulus`: S extends uniformly continuously to all of X, with a modulus of continuity coming from Ω	Equicontinuity by an $\varepsilon/2$ argument; the limit on the countable dense D is extended to X Proved here

Assumption (for large n)	Conclusion (in the limit)	Tool that carries it over
`hac`, `ham`, `hbr`, `hbv`	`invariant`: $S(gx) = S(x)$	The breaking goes to 0, so equality on D; extended to X by continuity Proved here
`hpos`	`positive`: $\sum_{a,b} c_a c_b S(x_{ab}) \geq 0$	A closed condition Proved here

In Lean the non-degeneracy lower bound c_m is written as a subtraction of values in $[0, \infty]$, and when $|R_m|(n_p P_1 + n_{pt} S) \geq 1$ it is read as $c_m = 0$ (a lower bound without content). The lower bound has content only when the right-hand side of the rarity field is small compared with the size $|R_m|$ of the region.

This whole section shows that the implication on the side of measures goes only as far as "properties that hold uniformly at finite n pass to the weak limit". The properties themselves (rarity, reflection positivity, symmetry, equicontinuity of observables) all sit on the side of the assumptions. Of these, only the two fields in the next section are shown for concrete objects.

06

The version with the Wilson measure as the fine measure

There is a version in which the fine measure is the normalised SU(2) Wilson measure (coupling β_n, 19-01) on the box $(\mathbb{Z}/2^n L)^4$ of side $2^n L$. Here $\beta : \mathbb{N} \to \mathbb{R}$ is an *arbitrary sequence*. In addition, the reflection Θ_m is fixed to be the time reflection $x_0 \mapsto -x_0$ in a plane of lattice sites, and the class $\mathcal{A}_m$ is fixed to be the following $\mathcal{A}_m^{(c)}$.

$$\mathcal{A}_m^{(c)} = \{F \text{ bounded continuous : } F \text{ depends only on coarse links with base time in } \{1, \dots, 1$$

Then the reflection-positivity field `h0p` and the reflection-continuity field `h0c` are proved inside Lean and disappear (Lean: `BoxWalk.wilson_h0p`, `continuous_reflCfg`; the former is the assumption-free statement "for any sequence of couplings β and large n, the pushed-forward Wilson measure is reflection positive for the class of strip (4)"). The resulting statement is `BoxWalk.chain_end_to_end_box_wilson2` — machine-checked as an implication (the assumptions are those of Tables 1 and 2 minus `h0p` and `h0c`).

We state the scope of this version precisely.

- **Within the scope in which the two fields disappear, it is confirmed.** The class (4) is non-empty where m is large enough (it contains the function $\cos(\mathrm{Re}\, U_{\ell_0})$ of one link; Lean: `BoxWalk.stripSet_witness`), and the reflection positivity of the limit has content.

- **Read as "the chain has been closed with the Wilson measure", it is conditional.** The one-step structure for the coupling, rarity, the symmetry `hsym`, the representation and the observable fields all remain assumptions.
- The class (4) is a narrow class. The strip width c is one and the same number at every level m, does not widen as m grows, and does not include links in the time-0 plane. It is smaller than the positive-time class $\mathcal{A}_m(P)$ (the whole positive half) of the conditional implication assembled earlier (§ 08). That class is defined by reflection in a symmetric plane of the coarse lattice (for example a plane between lattice sites) and by pull-back to supports on the positive side of the fine lattice, and the reflection itself differs from $x_0 \mapsto -x_0$ in a plane of lattice sites. Moreover the conclusion is "some subsequence, and some limit" for each c, so "one limit is reflection positive for the strips of all c" does not follow from this one statement.

Even in the Wilson-measure version, β_n is an arbitrary sequence, and no relation at all is placed between it and u. This is the subject of the next section.

07

The reading that joins the two lies outside the theorem

So far, the content of the theorem has split in two.

(a) The side of measures	(b) The side of the coupling
Properties that hold uniformly on finite lattices (reflection positivity, symmetry, lower bounds on the mass of closed sets, equicontinuity of observables) pass to subsequential limits. The subsequential limit itself exists without premises.	From the one-step structure for the coupling, a matching point x with $u_n(x) = t$ can be taken inside an explicit bracket linear in n.

The intended reading joins the two by one sentence: "the fine measure μ_n is the Wilson measure at the coupling $\beta_n = 4x_n$ fixed by the matching point x_n (the point taken in (2) for the target t)." This sentence is **not in the theorem**. The Wilson version (§ 06) takes an arbitrary sequence β and places no relation between β_n and x_n. So all that can be read from the theorem is that (a) and (b) hold separately.

In the conditional implication assembled earlier (§ 08), this sentence is placed as a definition of the setting (the fine measure is defined to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling). Even in that implication,

however, the effective coupling u_k is defined not from the push-forward measure $\nu_{n,m}$ but as a quantity built from the Gibbs expectations (or ratios of partition functions) of separate finite lattices fixed for each step k, and the quantities on the side of the coupling and those on the side of the measure (for example a threshold on the coupling side, and the threshold $y_h(m)$ of the rarity event) are not joined inside the theorem. In the Lean chain theorem this definition is not placed either, and the fine measure is an arbitrary family.

To join them, the effective coupling u_k has to be defined from the lattice. For example, if u_k is defined as a function of some expectation under the Wilson measure at bare coupling x, then $u_n(x) = t$ in (2) becomes a statement about defined objects only: "match the fine Wilson measure so that some quantity at the coarse level takes a fixed value." That form is in 19-07 as one of the statements to be shown, "Matching" (19-07 states all the statements other than matching for the sequence of couplings fixed by matching). A choice remains as to which quantity to match by, and that belongs to the design choices (28-05).

In addition, the other quantities of Tables 1 and 2 are not defined from the lattice either. The effective coupling and the remainders (quantities that come from a decomposition of the effective action), the large-field activity and supply, the background field and its invariants, and the sequence of observables and its representation are all free variables inside the theorem. The fact that this theorem is "closed" is no substitute for defining these quantities from the lattice and showing the fields.

08

Relation to the conditional implication assembled earlier

This theorem rewrites an implication of the same type as the chain theorem of the conditional implication that this book's study assembled earlier (a form that derives weak ultraviolet existence of four-dimensional SU(2) lattice Yang–Mills theory in a small box by a machine-checked chain from per-step premises; 2026-10-30), with the coupling fields later put in a different form (the width of the bracket written with an error function e and transient terms δ_k) and the block map replaced by a concrete map on the box. The Lean of the earlier implication is a statement over an abstract block map. In the earlier implication, the fine measure is defined to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling (§ 07). Among the assumptions of the earlier implication, the small-field set assumption (at each level there is a closed set C_m with $\nu_{n,m}(C_m) \geq 1 - \eta_m$ for large n), equicontinuity of observables (equicontinuity of the expectations s_n) and the summability-of-increments assumption ($|s_{n+1}(x) - s_n(x)|$ is summable) are, once the fine measure is

given, statements written only with the push-forward measure and its expectations. The remaining assumption (the one-step bracket assumption for the coupling) is a statement about the effective inverse coupling u_k; the definition of u_k is on paper, and in Lean it is a free function. None of these assumptions has been shown in four dimensions, and the main theorem is an implication. The existence of four-dimensional Yang–Mills theory itself was shown by the construction published on 2026-11-10, with a different design (Volume 2; the two designs are compared in Part 14). In this book we first set out the statements that can be written with the push-forward measure alone as "statements to be shown" (19-07), and read the theorem of this chapter as the implication that takes charge of the part about subsequential limits and the properties that pass to the limit.

What this chapter can say

Can say	Cannot say
Implication: if all the fields of Tables 1 and 2 hold, then (2) holds, and the properties of Table 3 hold for the subsequential limit of (3). Lean: `BoxWalk.chain_end_to_end_box` (the box version of `ChainBlockMap.chain_end_to_end_blockMap`) — machine-checked as an implication (assumptions as in the tables)	None of the fields is shown for concrete quantities on the four-dimensional lattice (except the two of § 06). The quantities put into the assumptions are not defined from the lattice
Cited The subsequential limit of (3) exists for any family of probability measures (compactness). Lean: `WeakUltravioletExistence.exists_common_subseq`	That the limit does not depend on the subsequence (convergence of the full sequence; 28-02)
Proved here Formula (2) is a statement about u and the constants of Table 2 alone, and is not connected to the measure	That μ_n is the Wilson measure fixed by the matching point (a reading outside the theorem)
Wilson version: when the fine measure is the Wilson measure (an arbitrary sequence of couplings), `h0p` and `h0c` disappear. Lean: `BoxWalk.chain_end_to_end_box_wilson2`, `wilson_h0p` — confirmed within the scope in which the two fields disappear	That the chain has been closed with the Wilson measure (the remaining fields are assumptions). The class is the fixed-width strip (4), not the whole positive half
Proved here The fields of Tables 1 and 2 can be satisfied simultaneously (by trivial objects)	A choice that satisfies the fields simultaneously with objects of real content

Sources and reproduction

Item	Type	Source, tools
The statement of the implication and the conclusion structures (§ 01–§ 05)	Lean	Lean of this book's study: `ChainBlockMap.chain_end_to_end_blockMap`, `BoxWalk.chain_end_to_end_box`, structures `TransientChainError.CouplingStepsTransientE`, `WeakUltravioletExistence.BlockConclusion`, `ObservableConclusion`, `Representation`. The statements were copied by reading the source text
Wilson version (§ 06)	Lean	`BoxWalk.chain_end_to_end_box_wilson2`, `wilson_h0p`, `stripSet`, `stripSet_witness`, `reflCfg`. Reflection positivity on the fine lattice is in 20-01
Compactness of the space of probability measures (Prokhorov's theorem), upper semicontinuity of the mass of closed sets	Cited	Standard facts of measure theory. Bibliography only (unchecked). The Lean uses Mathlib lemmas
Example in which the fields are satisfied simultaneously (§ 02)	Proved here	The choice given in the text
Assumptions of the conditional implication assembled earlier (small-field set assumption, equicontinuity of observables, summability-of-increments assumption, one-step bracket assumption for the coupling; § 08)	—	Record of this book's study (2026-10-30)

Words used in this chapter — implication, push-forward measure, subsequence, weak convergence, projective system, one-step structure for the coupling, matching point, bracket, rarity, set of small fields, non-degeneracy lower bound, equicontinuity, representation, strip class. Their meanings are in A-02 Glossary.

Next chapter: 28-02 Convergence of the full sequence — reads conditions sufficient to remove the subsequence, in the form of a Lipschitz contraction of the one-step map and matching at the endpoint.

Convergence of the full sequence

> This chapter is this book's study.

What this chapter shows — The theorem of the previous chapter gives only subsequential limits. To remove the subsequence, it suffices that the differences of coarse-scale quantities when the lattice is made one step finer be summable. In this chapter we read, with all the assumptions shown, one sufficient condition that bounds those differences — a Lipschitz contraction of the one-step renormalisation map, and matching of the coupling at the coarse end — as the convergence of skew-product orbits matched at the endpoint (Lean: `SkewOrbit.coarse_tendsto` and others). Lean examples also show that the conditions $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$ are necessary and sufficient for this method, and that matching at the endpoint cannot be removed.

Prerequisite chapters — 28-01 Implication of subsequential limits, 19-04 The block map, 19-07 Statements to be shown (convergence of the full sequence).

Said first — This is an implication. The theorem is a statement about orbits of points in a metric space, and no measure appears in it. The assumed one-step map (a map on the marginal coordinate u and the irrelevant coordinate v) is not defined from the lattice. So convergence of the full sequence for the continuum limit of the lattice does not follow from this theorem. What follows is only a conclusion of the form "if one step of renormalisation has this form and this contraction". The existence of four-dimensional Yang–Mills theory was shown by the construction published on 2026-11-10, with a different design (Volume 2). The implication of this chapter is a record within this book's design (measures pushed forward by a fixed block map) (28-00).

The order of this chapter

1. From a subsequence to the full sequence — what is missing — if the increments are summable, the full sequence converges
2. The form of the one-step map — the marginal direction and the irrelevant directions — the skew-product map, and families of orbits matched at the endpoint
3. The table of assumption fields — Lipschitz contraction, diameter, matching at the endpoint (Table 1)

4. The statement — the Cauchy property at the coarse end and uniqueness of the limit — difference $\leq D\theta^n$; the limit does not depend on the bare values
5. The route of the proof — a two-point boundary-value problem — differences in u propagate from the coarse side, differences in v from the fine side
6. Conditions and rate — necessary and sufficient for this method — eigenvalues of a linear model, and a rate that cannot be improved
7. Matching at the endpoint cannot be removed — an example where, for the same map, removing the matching gives a non-Cauchy sequence
8. The assumed map is not defined from the lattice — what must be defined for the statement to become a statement about the lattice

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01

From a subsequence to the full sequence — what is missing

The subsequential limit of the previous chapter (28-01 § 04) exists for any family of probability measures. Since the limit may depend on the choice of subsequence, that alone does not allow us to say "one continuum limit has been determined". We need convergence of the full sequence (the sequence without passing to a subsequence).

There is an easy sufficient condition for convergence of the full sequence. For a sequence s_n of one coarse-scale quantity (for example the expectation of an observable at level m), if

$$|s_{n+1} - s_n| \leq \delta_n, \qquad \sum_n \delta_n < \infty \quad (1)$$

then s_n is a Cauchy sequence and converges, and every subsequential limit equals that limit Proved here (Lean: `WeakUltravioletExistence.tendsto_of_summable_increments`, `subseq_limit_eq`; the form for a family of observables is `WeakUltravioletExistence.weak_uv_observable_full`). In the conditional implication that this book's study assembled earlier (28-01 § 08), a condition of the form (1) is placed as an assumption (the summability-of-increments assumption). The summability-of-increments assumption assumes that, on a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. It is not the same statement as weak convergence of the full sequence of push-forward measures (convergence of the full sequence in 19-07, stated for the sequence of couplings fixed by matching): it is stronger by the summability, and the family of functions is different. The step from that family to all bounded continuous functions is an argument on paper.

The question is where to get the δ_n of (1). We push the two fine measures on lattice $n+1$ and lattice n forward to the same coarse level m and compare them. Since renormalisation maps compose, the side of lattice $n+1$ can be seen as "renormalise once, then follow the same road as lattice n". If the effective action after one step differs only a little from the bare action of lattice n, and that difference contracts over the remaining steps, then the difference at the coarse end becomes small. The theorem of this chapter writes the "contracts" part of this line of argument as a statement about orbits of points.

02

The form of the one-step map — the marginal direction and the irrelevant directions

We think of the effective action as described by two coordinates: the marginal coordinate $u \in \mathbb{R}$ (the inverse coupling) and the irrelevant coordinate v (everything else; a point of a metric space X). The one-step map from scale k ($k = n$ is the fine end, $k = 0$ the coarse end) to $k-1$ is

$$\Phi_k(u, v) = \big(u - b_k + r_k(u, v),\ T_k(u, v)\big) \quad (2)$$

(Lean: `SkewOrbit.step`). b_k is the one-step decrease of the coupling (a number near the one-loop value), r_k its correction, and T_k the map on the irrelevant coordinate. The map depends only on the step k, not on the lattice n one started from.

Why a skew product. If a map contracted in every direction, convergence of orbits would follow in the manner of the contraction mapping principle. The renormalisation map does not contract in the direction u of the coupling (under asymptotic freedom, u just decreases by roughly b at each step). Only the irrelevant direction v contracts, and the two directions affect each other through r_k and T_k. A map of this form is called a skew product.

The orbit of lattice n is the sequence of points that starts from the bare value $w_n(n)$ at scale n and goes down to the coarse end $w_n(0)$ by $w_n(k-1) = \Phi_k(w_n(k))$ (Lean: `SkewOrbit.IsOrbit`). For a family of orbits $(w_n)_n$, we call the following three together "admissible" (Lean: `SkewOrbit.Admissible`).

- Each w_n is an orbit from scale n to 0.
- **Matching at the endpoint**: for all n, the marginal coordinate at the coarse end takes the same value $\bar{u}$: $w_n(0)_u = \bar{u}$.
- The irrelevant coordinate stays, at every step, in a set $\mathcal{B} \subset X$ of diameter at most D.

Matching at the endpoint plays the same role as the first conclusion of the previous chapter (28-01 § 03: there is a bare value that becomes exactly the target value after n steps). The two theorems, however, do not refer to each other's quantities.

03

The table of assumption fields

Table 1 Assumptions for the convergence of skew-product orbits matched at the endpoint (fields of the midpoint version `SkewOrbit.coarse_tendsto_mid`; $v, v' \in \mathcal{B}$, u, u' all real numbers)

Lean field	Mathematical wording	Paid?
`hr`	Assumption $\lvert r_k(u,v) - r_k(u',v')\rvert \le \varepsilon \lvert u-u'\rvert + c\, d(v,v')$	Abstract. r_k, T_k, X are free and not defined from the lattice
`hT`	Assumption $d(T_k(u,v), T_k(u',v')) \le c' \lvert u-u'\rvert + \tau\, d(v,v')$	
`hB`	Assumption The diameter of $\mathcal{B}$ is $\le D$	Abstract
`hε`, `hc`, `hc'`, `hτ`, `hD`	Assumption $\varepsilon, c, c', \tau, D \ge 0$	Numerical conditions
`hετ`, `h4`	Assumption $\varepsilon + \tau < 1$, $4cc' < (1-\varepsilon-\tau)^2$	Numerical conditions (necessary and sufficient for this method, § 06)
`hw` (`Admissible`)	Assumption The family of orbits is admissible (being orbits, matching at the endpoint, staying in $\mathcal{B}$)	Abstract
`[CompleteSpace X]`	Assumption X is complete	Abstract

The general version (`SkewOrbit.coarse_step`, `coarse_cauchy` and others) is given one θ and assumes $\tau < \theta < 1 - \varepsilon$ and

$$q := \frac{cc'/(1-\varepsilon-\theta) + \tau}{\theta} < 1 \quad (3)$$

The θ of this chapter is the contraction rate, and has nothing to do with the vortex ratio θ_M (19-02). Taking the midpoint $\theta = (1-\varepsilon+\tau)/2$, (3) follows from $4cc' < (1-\varepsilon-\tau)^2$ (Lean: `SkewOrbit.midpoint_admissible`).

The Lipschitz fields are assumed for all values of u, but this is not a restriction. If the fields are assumed only on a window $u, u' \ge a$, and we add that the orbits stay in the window ($w_n(k)u \ge a$ for all n, k), the same conclusion follows (Lean: `SkewOrbitProbe.coarse_tendsto_window`, `limit_independent_window`; the original theorem is applied to the map cut off by $u \mapsto \max(u, a)$). This suits situations in which only estimates on the weak-coupling window $u \ge u_\star$ are used.

The assumptions can be satisfied simultaneously, and not only in trivial cases. With $X = \mathbb{R}$, $\mathcal{B} = [-1, 1]$ ($D = 2$), $b = 0$, $r(u, v) = \frac{7}{20}v$, $T(u, v) = \frac{7}{10}v + \frac{1}{20}u$ ($\varepsilon = 0$, $c = 7/20$, $c' = 1/20$, $\tau = 7/10$, $4cc' = 0.07 < 0.09$), one can build an admissible family in which the bare irrelevant coordinate oscillates near $\pm\frac{1}{2}$ and does not converge, while the coarse end converges (Lean: `SkewOrbitProbe.Ex.admissible`, `Ex.coarse_converges`, `Ex.bare_not_cauchy`).

04

The statement — the Cauchy property at the coarse end and uniqueness of the limit

Under the assumptions of Table 1, with $\theta = (1 - \varepsilon + \tau)/2$, the following holds.

$$d\big(w_{n+1}(0)_v,\ w_n(0)_v\big) \le D\,\theta^n \qquad (\text{all } n). \ (4)$$

So the irrelevant coordinate $w_n(0)_v$ at the coarse end is a Cauchy sequence; if X is complete it converges **along the full sequence**, and its limit v_∞ satisfies $d(w_n(0)_v, v_\infty) \le D\theta^n/(1-\theta)$. Moreover, the limits of two admissible families matched at the same $\bar{u}$ coincide. The limit does not depend on the choice of the bare irrelevant coordinates $w_n(n)_v$. A continuous function Ψ of the coarse-end point (u, v) converges along the full sequence to $\Psi(\bar{u}, v_\infty)$.

Lean: `SkewOrbit.coarse_step` (formula (4)), `coarse_cauchy`, `coarse_tendsto`, `coarse_rate`, `limit_independent`, `observable_tendsto`, and the midpoint versions `coarse_step_mid`, `coarse_tendsto_mid`, `limit_independent_mid` — machine-checked as an implication (assumptions as in Table 1).

Let us check what (4) compares. Cutting the orbit of lattice $n+1$ to scales n and below gives an orbit that starts at scale n from "the bare value with one step Φ_{n+1} applied" (Lean: `SkewOrbitProbe.one_step_after`). The orbit of lattice n starts from the bare value at scale n. Both are matched to $\bar{u}$ at the coarse end, so this is a comparison of two admissible orbits of the same length n.

Read in the language of the continuum limit, the statement becomes: "if a coarse-scale observable is a continuous function of (u_0, v_0), the sequence of its expectations converges without passing to a subsequence, and the limit does not depend on the choice of the irrelevant part of the bare action." This reading lies outside the theorem (§ 08).

05

The route of the proof — a two-point boundary-value problem

Compare two admissible orbits w, w' of the same length n, and set $x_k = |w(k)_u - w'(k)_u|$, $y_k = d(w(k)_v, w'(k)_v)$.

Step 1: the one-step inequalities. Taking the difference of the first components of (2) and using `hr` of Table 1 gives $x_k \le x_{k-1} + \varepsilon x_k + c\, y_k$, that is,

$$(1-\varepsilon)\, x_k \le x_{k-1} + c\, y_k, \qquad y_{k-1} \le c' x_k + \tau\, y_k \qquad (1 \le k \le n) \ (5)$$

(the latter is `hT`) Proved here . The boundary values are $x_0 = 0$ (matching at the endpoint) and $y_n \le D$ (the diameter of $\mathcal{B}$). x propagates from the coarse side ($k = 0$) towards the fine side, and y from the fine side ($k = n$) towards the coarse side.

Step 2: a weighted maximum. Set $M := \max_{k \le n} y_k\, \theta^{-(n-k)}$. Accumulating the first half of (5) from $x_0 = 0$, induction gives $x_k \le cM\theta^{n-k}/(1-\varepsilon-\theta)$ (it closes exactly because $\theta/(1-\varepsilon-\theta) + 1 = (1-\varepsilon)/(1-\varepsilon-\theta)$) Proved here .

Step 3: tightening. Putting this into the second half of (5) gives, for $k - 1 < n$, $y_{k-1}\theta^{-(n-k+1)} \le qM$ (q as in (3)). At $k = n$, $y_n \le D$. So $M \le \max(D, qM)$, and from $q < 1$ we get $M \le D$, in particular $y_0 \le D\theta^n$ Proved here (Lean: `SkewOrbit.two_point_bound`, `orbit_compare`).

The use of $q < 1$ in Step 3 is where the contraction condition does its work. Since the u direction does not contract, bounding the difference in u by the difference in v requires the difference in u to be 0 at the coarse end (matching at the endpoint).

06

Conditions and rate — necessary and sufficient for this method

Three facts about the choice of θ have been checked in Lean.

- **Necessary and sufficient**: the existence of a θ satisfying $\tau < \theta < 1 - \varepsilon$ and (3) is equivalent to $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$ (Lean: `SkewOrbitProbe.theta_exists_iff`). Taking the midpoint loses nothing in the conditions.
- **The window of rates**: with $s = \sqrt{(1-\varepsilon-\tau)^2 - 4cc'}$ and $\theta_\pm = \big((1-\varepsilon+\tau) \pm s\big)/2$, every θ with $\theta_- < \theta < \theta_+$ can be used (Lean: `SkewOrbitProbe.rate_window`). The rate can be improved to just above θ_-.
- **A rate that cannot be improved**: with $\varepsilon = 0$, $c = 1/6$, $c' = 1/3$, $\tau = 1/2$ ($\theta_- = 2/3$, $\theta_+ = 5/6$), there are sequences satisfying (5) and the boundary values with $y_0 \ge (2/3)^n$

/2. So no decrease faster than θ_- follows from (5) (Lean: `SkewOrbitProbe.Sharp.sharp`).

$\theta_\pm$ are the eigenvalues of the matrix $\begin{pmatrix} 1-\varepsilon & -c \\ c' & \tau \end{pmatrix}$ of the linear model with the sign of the cross term reversed Proved here (the roots of the characteristic polynomial $\lambda^2 - (1 - \varepsilon + \tau)\lambda + (1-\varepsilon)\tau + cc'$). $4cc' \le (1-\varepsilon-\tau)^2$ is the condition for them to be real.

"Necessary and sufficient" here is **relative to this method of proof (the two-point estimate from (5))**. It is not a claim that convergence of the full sequence itself becomes false when the conditions fail. Formula (5) looks only at the sizes of differences and does not use the sign of the cross terms, so for maps where the sign is known there is room for better estimates.

07

Matching at the endpoint cannot be removed

With the same map as in the example of § 03 ($\varepsilon = 0$, $c = 7/20$, $c' = 1/20$, $\tau = 7/10$, $\mathcal{B} = [-1, 1]$), one can build a family in which every member is an orbit and the irrelevant coordinate stays in $\mathcal{B}$, but the marginal coordinates at the coarse end are not aligned. In that family the irrelevant coordinate at the coarse end is not Cauchy (Lean: `SkewOrbitProbe.Ex.matching_needed`).

What lies behind this is that the matrix of the map $\begin{pmatrix} 1 & c \\ c' & \tau \end{pmatrix}$ has a direction with eigenvalue $21/20 > 1$ (a direction close to the marginal direction). In this example, if u at the coarse end is not aligned, the orbits have a component in this direction, and its size alternates from one n to the next. Matching at the endpoint is the operation that fixes the component in this expanding direction afresh for each n.

So convergence of the full sequence does not follow from "contraction" alone. **How the bare coupling is chosen for each n (matching) is part of the statement.** In the language of the lattice this corresponds to "Matching" among the statements to be shown (19-07). The first conclusion of the previous chapter gives an existence of the same type (a bare value that becomes exactly the target value after n steps) for a different quantity, but the u of the two theorems is not set to be the same thing. There too, u is not defined from the lattice (28-01 § 07). Nor has it been shown, for general maps, that an admissible family exists (including that the irrelevant coordinate stays in $\mathcal{B}$).

08

The assumed map is not defined from the lattice

This theorem is a statement about orbits of points in the metric space $\mathbb{R} \times X$. Measures, lattices and block maps appear nowhere in the statement. To read it as convergence of the full sequence for the continuum limit of the lattice, the following must be fixed outside the theorem.

1. **Coordinates**: describe the effective action at step k by (u, v) — in particular the irrelevant coordinate v and its metric. The effective action (the effective measure at step k) itself is not yet defined from the lattice.
2. **The map**: write one step of renormalisation as a map Φ_k on those coordinates (that it takes the form (2), and that b_k is common to the two orbits).
3. **Contraction**: that the map has the Lipschitz estimates of Table 1 with constants satisfying $4cc' < (1 - \varepsilon - \tau)^2$.
4. **Observables**: that expectations of coarse-scale observables are continuous functions of the coarse-end coordinates (u_0, v_0).

All four are at present on the side of the assumptions. In the list of premises (28-04), the row for the Lipschitz contraction of the one-step map and matching at the endpoint carries the label "Form not fixed (object undefined)", and below it hang premises at the level of the representation, such as an induction uniform in the step and one-step renormalisation of small fields. Which coordinates and norm to choose is a design choice (28-05).

We have not checked whether the Lipschitz contraction follows from existing renormalisation-group estimates (estimates of analytic form in the fields and parameters). This chapter makes no claim about that prospect.

What this chapter can say

Can say	Cannot say
Proved here If the increments are summable, the full sequence converges, and subsequential limits equal its limit. Lean: `WeakUltravioletExistence.tendsto_of_summable_increments`, `weak_uv_observable_full`	That the increments of lattice expectations are summable (the assumption of the conditional implication assembled earlier, the summability-of-increments assumption)

Can say	Cannot say
Implication: under the assumptions of Table 1, the irrelevant coordinate at the coarse end converges along the full sequence, the differences are at most $D\theta^n$, and the limit does not depend on the bare values. Lean: `SkewOrbit.coarse_step`, `coarse_tendsto`, `limit_independent`, `observable_tendsto` — machine-checked as an implication (assumptions as in the table)	That the assumed maps r_k, T_k and coordinates X are defined from the lattice. Convergence of the full sequence for the continuum limit of the lattice
The existence of a usable θ is equivalent to $\varepsilon + \tau < 1$ and $4cc' < (1-\varepsilon-\tau)^2$. Lean: `SkewOrbitProbe.theta_exists_iff`, `rate_window`, `Sharp.sharp`	That convergence of the full sequence is false when the conditions fail (this is a limit of the method, not a negation of the statement)
Removing matching at the endpoint, there is a non-Cauchy family for the same map. Lean: `SkewOrbitProbe.Ex.matching_needed`	How to match on the lattice (which quantity, at which step)
Proved here The u of this chapter and the coupling function u of the previous chapter are separate variables of separate theorems	Setting the two to be the same. That an admissible family (orbits with matching that stay in $\mathcal{B}$) exists for the actual maps
The assumptions can be satisfied simultaneously by a non-trivial example. Lean: `SkewOrbitProbe.Ex.admissible`, `Ex.bare_not_cauchy`	That the actual renormalisation maps satisfy the assumptions

Sources and reproduction

Item	Type	Source, tools
The theorem on skew-product orbits (§ 02– § 05)	Lean	Lean of this book's study: 13 declarations in the namespace `SkewOrbit` (`two_point_bound`, `midpoint_admissible`, `orbit_compare`, `coarse_step`, `coarse_cauchy`, `coarse_tendsto`, `coarse_rate`, `limit_independent`, `observable_tendsto` and the midpoint versions). Machine-checked in Lean 4 (13 declarations). The axioms are the standard three. Kernel re-check done. The statements were copied by reading the source text
Equivalence of the conditions, window of rates, a rate that cannot be improved, necessity of matching, the window version (§ 03, § 06, § 07)	Lean	Namespace `SkewOrbitProbe` (auxiliary declarations added when the definitions and statements were compared with a separate pair of eyes: `theta_exists_iff`, `rate_window`, `Sharp.sharp`, `Ex.*`, `coarse_tendsto_window` and others). The axioms are the standard three (record of the check)

Item	Type	Source, tools
Summable increments give the full sequence (§ 01)	Lean	`WeakUltravioletExistence.tendsto_of_summable_increments`, `subseq_limit_eq`, `weak_uv_observable_full`
Eigenvalues of the linear model (§ 06)	Proved here	Computation in the text (characteristic polynomial of a 2×2 matrix)
Assumption of the conditional implication assembled earlier (the summability-of-increments assumption)	—	Record of this book's study (2026-10-30). 28-01 § 08

Words used in this chapter — convergence of the full sequence, summability of increments, skew product, marginal coordinate, irrelevant coordinate, one-step map, orbit, matching at the endpoint, admissible family, Lipschitz contraction, two-point boundary-value problem, window of rates. Their meanings are in A-02 Glossary.

Next chapter: 28-03 From decay to the mass gap, and the reduction of the band — reads the two implications on the infrared side, and the fact that there is no arrow between them.

From decay to the mass gap, and the reduction of the band

> This chapter is this book's study.

What this chapter shows — We read two implications on the infrared side. (i) If a sequence of measures has uniform exponential decay, the theory reconstructed from the limit measure has a mass gap (Lean: `OSGap.massGap_of_measures_sub`). Inside the strong-coupling window, these assumptions are all paid by concrete Wilson measures. (ii) If base point, nesting and widening hold for the vortex ratio θ, then $\ln(1/\theta)$ grows in proportion to the area (Lean: `BandReduction.band_reduction` and others). Finally we write down that there is no arrow from the conclusion of (ii) to the assumptions of (i).

Prerequisites — 19-03 The vessel of reconstruction and the definition of the mass gap, 20-04 Reconstruction and the mass gap, 19-02 The vortex ratio θ, 26-02 Reduction to one inequality.

Said first — Both are implications. Outside the window, (i) remains an assumption, and there is no concrete sequence of measures (a sequence in which the lattice spacing moves) that pays the assumptions. Also, the conclusion alone cannot exclude the trivial case in which the Hilbert space contains only the vacuum. (ii) is a statement of inequalities about a real function $\theta(R, L)$; neither lattice nor measure appears in it. That the actual vortex ratio satisfies the three assumptions has not been shown. There is no arrow, either in Lean or on paper, that derives the mass gap from the area exponent. Exponential decay uniform in the lattice spacing is not in the construction published on 2026-11-10 either; what it has goes only as far as estimates of polynomial speed at long distances (16-03). The two implications of this chapter are shaped towards the open question on the infrared side.

The order of this chapter

1. The two implications, and what lies between them — decay ⇒ gap; the three assumptions ⇒ area exponent; area ⇒ gap is absent
2. From decay to the mass gap — the statement — reconstruction over the limit measure, and ‖ T − P_Ω ‖ ≤ e^{−m}

3. From decay to the gap — the table of assumption fields — reflection positivity, symmetry, translation, decay (Table 1)
4. Inside the strong-coupling window, everything is paid — relation to the theorem of 20-04, and outside the window
5. The reduction of the band — the statement — at scales doubling each time, the bias shrinks by a fourth power
6. The reduction of the band — the table of assumption fields — base point, nesting, widening (Table 2)
7. There is no arrow from the area exponent to the mass gap — the two roads exist separately

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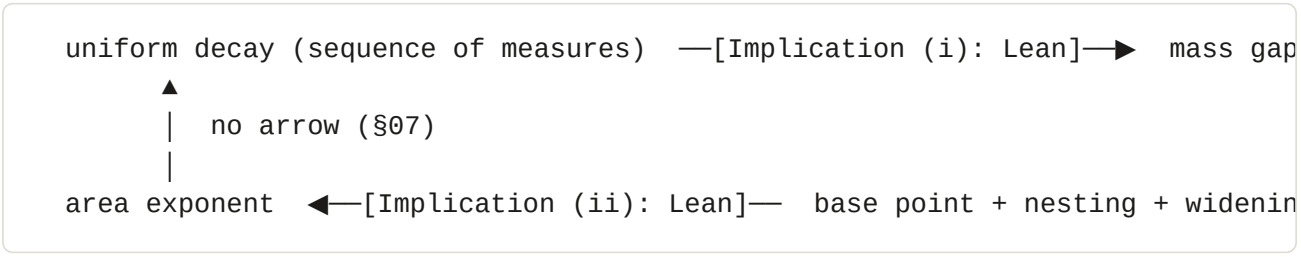
01

The two implications, and what lies between them

On the infrared side (the side of long distances) there are two goals. One is the mass gap (contraction of the transfer operator away from the vacuum); the other is confinement (the vortex ratio and loop expectations decreasing in proportion to the area). The two implications of this chapter take charge of the last step towards each goal.



Both arrows are machine-checked in Lean in the form "if ..., then ...". The assumption sides (uniform decay, the three inequalities) correspond to "mass gap (uniform decay)" and "band" among the statements to be shown (uniform decay and band in 19-07, both stated for the sequence of couplings fixed by matching), and neither has been shown in a form in which the lattice spacing moves. The vertical arrow of the figure, as § 07 explains, does not exist.

02

From decay to the mass gap — the statement

We fix the setting. X is the space of configurations and Y the space of configurations on the positive side. The continuous maps $r, \Theta_r : X \to Y$ are the map that restricts a

configuration to the positive side, and the map that reflects in the mirror and then restricts. $\mathrm{sh} : Y \to Y$ is the translation by one time unit. Observables are bounded continuous functions on Y; among them we use a subspace W that contains the constant 1 and is closed under translation (for example functions that depend only on finitely many time slices). For a measure ν, the pairing and the translation are

$$\langle F, G\rangle_\nu = \int F(\Theta_r x)\, G(rx)\, d\nu(x), \qquad (\tau F)(y) = F(\mathrm{sh}\, y) \quad (1)$$

If the pairing is positive semidefinite and symmetric, and τ is symmetric with respect to the pairing, one can build the Hilbert space obtained by quotienting W by the pairing and completing, the transfer operator T coming from τ, and the vacuum Ω coming from the constant 1 (the vessel of reconstruction; 19-03). A mass gap m means $\|T - P_\Omega\| \le e^{-m}$ (P_Ω is the projection onto the vacuum; 19-03).

Implication (i). Suppose a sequence of probability measures μ_n converges weakly to ν, and the fields of Table 1 hold. Then there is a vessel of reconstruction D over W whose pairing is the pairing (1) of ν, whose translation is τ and whose constant is 1, and D has mass gap m.

Lean: `OSGap.massGap_of_measures_sub` — machine-checked as an implication (assumptions as in Table 1). The conclusion ties the pairing, translation and constant of the vessel to the quantities of ν; it is not a form in which the vessel can be chosen at will.

The route of the proof has three steps. (1) Reflection positivity, symmetry and the symmetry of translation pass to the weak limit because $F \circ \Theta_r \cdot G \circ r$ is bounded and continuous (the same type as 28-01 § 05) Proved here . (2) With these, the vessel over ν can be built. (3) The decay field also passes to the limit, and $\langle \tau^k F, \tau^k F\rangle_\nu - \langle 1, F\rangle_\nu^2 \le C_F e^{-2mk}$ holds for all $F \in W$. Translated into the language of Hilbert space, this says that T^k contracts by e^{-mk} on vectors orthogonal to the vacuum, which gives $\|T - P_\Omega\| \le e^{-m}$ (Lean: `OSGap.Datum.hasMassGap_of_decay`).

03

From decay to the gap — the table of assumption fields

Table 1 Assumptions of `OSGap.massGap_of_measures_sub` ("for large n" means "there is n_0 such that for $n \ge n_0$"; $F, G \in W$)

Lean field	Mathematical wording	Inside the strong-coupling window	Outside the window

Lean field	Mathematical wording	Inside the strong-coupling window	Outside the window
`h`	Assumption $\mu_n \to \nu$ (weak convergence)	All paid by the Wilson measure (§ 04, 20-04)	Abstract (convergence of a sequence in which the lattice spacing moves has not been shown)
`h1`, `hW`	Assumption $1 \in W$, and W is closed under translation		A choice of the class of observables
`hRP`	Assumption For large n, $\int F(\Theta_r x)F(rx)\,d\mu_n \geq 0$ (reflection positivity)		A theorem for measures on the fine lattice (20-01). For pushed-forward measures it depends on the class
`hsym`, `hshift`	Assumption For large n, the pairing is symmetric, and translation is symmetric with respect to the pairing		Abstract
`hdec`	Assumption For each F there is a constant C_F such that for each k and large n, $\int F(\mathrm{sh}^k\Theta_r x)F(\mathrm{sh}^k rx)\,d\mu_n - \left(\int F(rx)\,d\mu_n\right)^2 \leq C_F e^{-2mk}$ (uniform decay)		Form fixed, not shown (needs an m uniform in the lattice spacing)

In the decay field `hdec`, m and C_F do not depend on n. The mass m of the limit theory is a number in units of one time step of the lattice on which the measures μ_n live. To speak of a physical mass in the continuum limit, block observables at coarse scales must decay at a rate independent of the lattice spacing and the box size. That is "mass gap" among the statements to be shown (uniform decay in 19-07, stated for the sequence of couplings fixed by matching), and paying the assumption `hdec` of this implication in that form is open outside the window. The field `h` also assumes weak convergence of the sequence μ_n. When there is only a subsequential limit, as in the previous chapter (28-01 § 04), applying it to that subsequence gives a conclusion "about that limit point", and the limit itself is not determined uniquely.

The conclusion alone cannot exclude the trivial case. If the space obtained by quotienting W by the pairing is only the vacuum (one-dimensional), then $T - P\Omega = 0$ and a mass gap holds for any m. The assumption fields can also be satisfied simultaneously on a one-point space Proved here . A mass gap with content requires saying separately that there are states outside the vacuum (non-degeneracy). The theorem for the strong-coupling window includes non-degeneracy in its statement (20-04).

> 04

Inside the strong-coupling window, everything is paid

In the strong-coupling window $18\tanh|\beta|<1$ ($|\beta|<0.0556$ Reading), the sequence obtained by lifting the normalised Wilson measures on the periodic boxes $(\mathbb{Z}/2(n+1))^4$ periodically to $\mathbb{Z}^4$ converges weakly, without a subsequence, to a single measure ν_β. On top of that, the properties corresponding to the fields of Table 1 (reflection positivity, symmetry, translation, decay) are all shown for concrete measures, the vessel of reconstruction satisfies $\|T-P_\Omega\|\le(18\tanh\beta)^{1/2}$, and there are states outside the vacuum ($0<\beta$; Lean: `InfMain2.wilson_infvol_main2`; the statement gathered into one in 20-04). This is a formalisation of a result of a known type from the strong-coupling expansion Cited (of the type of Osterwalder–Seiler, Ann. Phys. 110 (1978) 440).

Here is what the theorem inside the window does not say. The lattice spacing is fixed at 1. The window is only $|\beta|<0.0556$, and does not reach the side of the continuum limit ($\beta\to\infty$). The mass figure $-\frac{1}{2}\log(18\tanh\beta)$ is only a lower bound.

So the position of implication (i) is as follows. **Inside the window, all the assumptions of the implication are paid, and the conclusion stands as a theorem about concrete Wilson measures. Outside the window, they remain assumptions.** To use it for the mass gap of the continuum limit, one needs to show the fields of Table 1 for a sequence of measures in which the lattice spacing moves (the sequence of fine-lattice measures pushed forward to a coarse level, 28-01), and none of them has been shown.

05

The reduction of the band — the statement

The goal on the side of confinement is that the vortex ratio becomes small in proportion to the area. In the intended reading, $\theta(R,L)$ is the sector bias at a typical frame, for a container of size R in the directions of the plane and a transverse torus of side L, with lengths measured in physical units (26-02 § 01; the definition of the vortex ratio on the lattice is in 19-02). The goal is $\ln(1/\theta(R,R))\ge cR^2$ for a $c>0$ uniform in physical units.

Implication (ii). Let $\theta:\mathbb{N}\times\mathbb{N}\to\mathbb{R}$ be an arbitrary function, and assume the structure `Hyp` of Table 2 (non-negativity, base point, nesting, widening). The assumptions outside the structure are $L_0>0$, $s\ge1$, $C\ge0$, $C_D\ge0$, $c\ge0$, $c^3=4CC_D$. Taking doubling scales $R_k=2^k sL_0$, for all k

$$c\,\theta(R_k,R_k)\;\le\;\big(c\,t_0\big)^{4^k},\qquad t_0:=Cs^2\,C_D\,(1-\varepsilon_0)^{s^2}\quad(2)$$

holds. Moreover, if $c \geq 1$ and $0 < c\,t_0 < 1$, then when $\theta(R_k, R_k) > 0$ (the assumptions of `area_exponent` are these three and (2))

$$4^k \ln \frac{1}{c\,t_0} \;\leq\; \ln \frac{1}{\theta(R_k, R_k)} \quad (3)$$

and since $4^k = R_k^2/(sL_0)^2$, this is the form of the area exponent, $\ln(1/\theta(R_k, R_k)) \geq c' R_k^2$.

Lean: `BandReduction.band_reduction` (formula (2)), `area_exponent` (formula (3)), and the parts `BandReduction.jump` (from the base point, one nesting and one widening give $\theta(sL_0, sL_0) \leq t_0$) and `step` (one doubling step, $\theta(2R, 2R) \leq 4CC_D\,\theta(R, R)^4$) — machine-checked as an implication (assumptions as in Table 2).

The core of the proof is `step`. Using nesting with $n = 2$ gives $\theta(2R, R) \leq C_D \theta(R, R)^4$, and using widening with $L = R$, $L' = 2R$ gives $\theta(2R, 2R) \leq 4C\,\theta(2R, R)$; together, $t_{k+1} \leq 4CC_D\,t_k^4$. Setting $c^3 = 4CC_D$ gives $c\,t_{k+1} \leq (c\,t_k)^4$, and induction gives (2) Proved here .

There is also a version that, instead of the base point, puts in a number at one size, $\theta(R_0, R_0) \leq t_0$, and uses widening only "at doubling steps where the bias is already small" (Lean: `BandFiniteSize.doubling`; the assumptions are three in the structure — non-negativity, the $n = 2$ case of nesting, and widening when $\theta(2L, L) \leq \delta$ — and $R_0 > 0$, $C \geq 0$, $C_D \geq 0$, $c \geq 1$, $c^3 = 4CC_D$, $\theta(R_0, R_0) \leq t_0$, $c\,t_0 \leq 1$, $C_D t_0^4 \leq \delta$). The conclusion has the same form as (2).

06

The reduction of the band — the table of assumption fields

Table 2 Fields of the structure `BandReduction.Hyp` (θ is an arbitrary function $\mathbb{N} \times \mathbb{N} \to \mathbb{R}$)

Lean field	Mathematical wording	For the intended θ
`nonneg`	Assumption $\theta(R, L) \geq 0$ for all R, L	For the vortex ratio at fixed lattice spacing, a theorem for all real β when the longitudinal size is even (21-06). The correspondence with the container bias is on paper
`base`	Assumption $\theta(L_0, L_0) \leq 1 - \varepsilon_0$ (base point)	In the form with fixed lattice spacing, the thin-vortex floor $\lvert\theta_M\rvert \leq \tanh(N^2\lvert\beta\rvert)^{M^2}$ is a theorem (21-02). A form uniform in the lattice spacing has not been shown
`nest`	Assumption $\theta(nR, L) \leq C_D\,\theta(R, L)^{n^2}$ for all R, L, n (nesting)	Input on paper (the nesting argument of 26-02). The correspondence with the Lean field is also on paper
`widen`	Assumption For $0 < L \leq L' \leq R$, $\theta(R, L') \leq C\,(L'/L)^2\,\theta(R, L)$ (widening)	Not shown. There is one sufficient condition (direction-wise subadditivity of the vortex free energy) (28-04), but whether it holds is open

This implication is a statement about real inequalities. θ is an arbitrary function, and lattice, measure and lattice spacing do not appear in the statement. That the intended θ (the container bias) satisfies the three assumptions is placed as input on paper and has not been shown. In particular, in the intended reading all lengths are in physical units, and the assumptions must be uniform in the lattice spacing. The floor theorems at fixed lattice spacing (21-02, 21-03) do not give this uniformity.

The assumptions can be satisfied simultaneously Proved here . With $\theta(R, L) = e^{-R^2}$, $C_D = C = 1$ and ε_0 small, nesting holds with equality and widening holds because $(L'/L)^2 \geq 1$. In this example the conclusion is of a form known from the start, and the example only shows that the assumptions are not contradictory. Also, using nesting with $n = 1$ forces $C_D \geq 1$ when $\theta > 0$.

The forms of other statements about the band — the doubling inequality for the tension, monotonicity, one-scale criteria and so on — are laid out with labels in the list of premises (28-04). This chapter does not give them chapters of their own.

07

There is no arrow from the area exponent to the mass gap

There is no arrow, either in Lean or on paper, from the conclusion of implication (ii) (the area exponent) to the assumption of implication (i) (uniform decay of block observables).

"Area law ⇒ mass gap" in its naive form is false in general Proved here . For example, on a two-dimensional lattice, take a gauge theory with an area law and add to it, as a direct product, a decoupled model without a mass gap (for example the critical two-dimensional Ising model; that the gap of the transfer matrix closes at the critical point is in 2-09 § 07). The expectation of a Wilson loop is determined by the gauge part alone, so the area law remains, but the correlations of the Ising part do not decay exponentially, so the theory as a whole has no mass gap (the gap of the transfer matrix of a direct product is the smaller of the two gaps). So deriving decay from the area exponent requires a separate argument that uses the absence of such decoupled parts.

In the opposite direction there is the following result Cited (Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229; 17-02). If centre symmetry is not broken, Wilson loops satisfy $|\langle W(R \times T)\rangle| \leq e^{-V(R)T}$ with $V(R) \to \infty$ (this stops short of saying that the string tension is positive), and if moreover one assumes a strong form of gap — uniform exponential decay under arbitrary boundary conditions — the area law follows. This strong form of decay is not the same as the assumption of implication (i) of this chapter, nor as the mass gap in the Clay sense. The two implications of this book have no arrow in that direction in Lean either.

So on the present map the two roads exist separately.

- **The road of the mass gap**: uniform decay ⇒ mass gap (implication (i)). The assumed uniform decay is shown inside the window and open outside it.
- **The road of confinement**: base point, nesting, widening ⇒ area exponent (implication (ii)). The three assumptions have not been shown in a form uniform in the lattice spacing.

By which road the last arrow to the mass gap is to be drawn (whether, for example, to use the comparison rows that write band quantities in the language of the mass gap) is the subject of the list of premises (28-04) and of the open places (28-06). This chapter does not claim which road goes through.

What this chapter can say

Can say	Cannot say
Implication (i): if a sequence of measures converges weakly and the fields of Table 1 hold, the vessel of reconstruction over the limit measure has mass gap m. Lean: `OSGap.massGap_of_measures_sub` — machine-checked as an implication (assumptions as in the table)	Showing the fields outside the window, for sequences in which the lattice spacing moves. The conclusion alone does not give non-degeneracy (states outside the vacuum)
In the strong-coupling window $18 \tanh \lvert\beta\rvert < 1$, $0 < \beta$, the fields are all paid by concrete Wilson measures, and a mass gap with non-degeneracy holds. Lean: `InfMain2.wilson_infvol_main2` (20-04)	Moving the lattice spacing, outside the window, an upper bound on the mass (only a lower bound)
Implication (ii): from base point, nesting and widening, $\ln(1/\theta)$ grows in proportion to the area at doubling scales. Lean: `BandReduction.band_reduction`, `area_exponent`, `jump`, `step`, `BandFiniteSize.doubling` — machine-checked as an implication (assumptions as in the table)	That the intended θ satisfies the three assumptions (input on paper). Base point and widening uniform in the lattice spacing
Proved here The naive "area law ⇒ mass gap" is false when there is a decoupled part without a mass gap	An arrow deriving uniform decay from the area exponent (neither in Lean nor on paper)

Sources and reproduction

Item	Type	Source, tools
The implication from decay to the mass gap (§ 02, § 03)	Lean	Lean of this book's study: `OSGap.massGap_of_measures_sub`, the vessel `OSGap.Datum`, `HasMassGap`, `OSGap.Datum.hasMassGap_of_decay`. The statements were copied by reading the source text

Item	Type	Source, tools
The theorem for the strong-coupling window (§ 04)	Lean, Cited	`InfMain2.wilson_infvol_main2` (20-04). Source of the type: K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440
The reduction of the band (§ 05, § 06)	Lean	`BandReduction.Hyp`, `band_reduction`, `area_exponent`, `jump`, `step`, `iterate`, `BandFiniteSize.Hyp`, `doubling`. The statements were copied by reading the source text. The reduction on paper is in 26-02
Examples in which the assumptions are satisfied simultaneously (§ 03, § 06)	Proved here	The choices given in the text
A result in the direction from the gap to confinement (§ 07)	Cited	S. Chatterjee, Comm. Math. Phys. 385 (2021) 1007, arXiv:2006.16229

Words used in this chapter — implication, vessel of reconstruction, transfer operator, projection onto the vacuum, mass gap, uniform decay, non-degeneracy, strong-coupling window, vortex ratio, sector bias, base point, nesting, widening, area exponent. Their meanings are in A-02 Glossary.

Next chapter: 28-04 The list of premises — tree and labels — lays out, with labels, the tree whose root is the statements to be shown and whose leaves are the sufficient conditions on the present road, including the assumptions of the three implications.

The list of premises — tree and labels

This chapter is this book's study. The dictionary of § 09 is a guide.

What this chapter shows — A tree with the six statements of 19-07 (the roots) at the top and the 57 rows of sufficient conditions that support them on the present road below, and a table of the 57 rows. For each row we give the name of its content, a one-line statement, the object the statement is about, the label, and, if the form is not fixed, the reason symbol. We also give counts by label, and a table matching these names to the names in the conditional implication assembled earlier and to the names in the ledger of Part 27.

Prerequisite chapters — 28-00 Entrance (key to the labels), 19-07 The statements to be shown. The definitions of the objects are in Part 19 (19-01–19-04).

Said first — This list is as of 2026-11-13, and the labels move. Updates are made in A-08 A map to the mass gap. Many statements in the tables use quantities not yet defined from the lattice (effective coupling, activity, background field and so on). Those rows are marked "Form not fixed" or "Object: no definition", and do not claim that the statement has a meaning. The rows supporting the roots on the ultraviolet side (most of Tables 2 and 3) are a record of sufficient conditions in this book's design (the measure pushed forward by a fixed block map). The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10 (Volume 2; the two designs are compared in Part 14). The rows on the infrared side (uniform decay in Table 4, and the band in Table 5) are aimed at questions that remain open.

The order of this chapter

1. How to read the stages and labels — the order of the tables, and seven labels
2. The tree — the root, and sufficient conditions through a representation (Figure 1)
3. Counts by label, and what they tell us (Table 1)
4. Premise of the box continuum limit — the continuum limit in a fixed box (Table 2, 17 rows)
5. Premise of ultraviolet stability — estimates of ultraviolet stability (Table 3, 21 rows)

6. Premises of full-sequence convergence, observables, infinite volume, rotation and non-triviality (Table 4, 7 rows)
7. The band — rows on the vortex free energy (Table 5, 12 rows)
8. Five rows added outside the 57 (Table 6)
9. A dictionary of other names (short version, Table 7)

01

How to read the stages and labels — the order of the tables, and seven labels

The order of the tables. The rows are arranged in four tables, in the order of the stages into which the road to a proof is divided. The number in the left column of each table is a running number within that table. In the text, rows are referred to by the name of their content.

Stage	Table	Rows
Premise of the box continuum limit: the continuum limit in a box of fixed size (premises of the chain of block maps)	Table 2	17
Premise of ultraviolet stability: estimates of ultraviolet stability (step-by-step estimates; the kinds of estimate, in order, are free-field input, constrained Green's functions, background field, small-field expansion, large fields, and induction across steps)	Table 3	21
Premises of full-sequence convergence, fine observables, infinite volume, rotation and non-triviality	Table 4	7
The band (crossover of the vortex free energy; the rows around the band in the ledger of conditions, 27-02)	Table 5	12

The free-field premise (absolute conditions on the free field) is a reference to rows of the premise of ultraviolet stability (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm), large-field estimate), so it has no rows of its own. The fermion premise is outside the scope of this list.

Labels. A premise label is one column of a table, and is separate from the labels of the text (Proved here, Cited, Computer-assisted, Reading, Assumption) (key in 28-00).

Label	Meaning
Theorem	Machine-checked in Lean, and a check that compares the definitions and the statement with a separate pair of eyes has confirmed the content of the statement

Label	Meaning
Theorem (weak form)	The same, but only up to a form weaker than the premise. The same column says whether it **pays the premise's column as it stands** or is a **similar but different statement** (a different object, a subfamily, a linearisation and so on)
On paper	There is an argument on paper. The check comparing definitions and statement with a separate pair of eyes has not been done
Form fixed, unproved	The form of the statement is settled. There is no proof. "Conditional" means: only when, for the sequence fixed by matching, the restrictions on threshold and period are put into the statement
Form not fixed	The form of the statement is not yet settled. The reason is given by a symbol
Depends on design	A row determined by a design choice (28-05)
Literature	Depends on a result in the literature. Not re-derived in this book

Reason symbols (for rows whose form is not fixed): (a) too light (stands for trivial objects), (b) too strong (false, or there are examples where it is false), (c) object undefined, (d) there are numerical signs against the present form, (e) there is a form at one scale but no form across steps, (f) the order of quantifiers is not fixed.

The "Object" column of the tables says which object the statement is about. Objects with a definition in Part 19 are linked by anchor; those without are marked "no definition".

02

The tree — the root, and sufficient conditions through a representation

The root of the tree is the six statements of 19-07. The root is written with defined objects only. Below it hang the premises (leaves) we place in order to derive the root on the present road. Many of the leaves are statements made through a **representation** (quantities used to write down the theory at coarse scales): effective couplings, activities, background fields, classes.

Figure 1 The tree of the root and sufficient conditions (as of 2026-11-13; labels in parentheses)

```
Matching (19-07 §02)
  └─ sufficient conditions through the effective coupling
       │  · definition and continuity of the effective coupling (design)
       │  · net one-step bracket of the coupling (not fixed)
       │  · large-field contribution to the coupling (not fixed)
       │  · recursion of the activity (not fixed)
       │  · rarity of supply (not fixed)
       │  · moving invariant interval and margin (not fixed)
       ├─ representation level of the premise of ultraviolet stability
       │     · coefficient of the one-step decrease of the coupling (thm, weak
       │     · value of the one-loop coefficient (lit.)
       │     · small-field one-step renormalisation theorem (not fixed)
       │     · induction uniform in the step (not fixed)
       │     · large-field estimate (not fixed)
       │     · large-field gas (not fixed)
       └─ linking the thresholds (not fixed)
── the roots below are for the sequence β_n fixed by matching ──
Convergence of the full sequence (19-07 §03)
  └─ contraction of the one-step map and matching at the endpoint (not fixed)
       └─ · induction uniform in the step (not fixed)
          · small-field one-step renormalisation theorem (not fixed)
          · contraction of irrelevant terms (thm, weak)
          · Constrained Decay Theorem (fixed, vacuum version)
          · zero-order transport of the background (not fixed)
          · one-step background-field lemma (paper) · multistep background fie
          · single-scale expansion tools (paper)
          · gauge fixing step by step (paper) · class and localisation (paper)
Non-degeneracy (19-07 §04)
  ├─ rarity of two-step events: plaquette part (fixed, conditional)
  │     ← itself a statement about defined objects only
  └─ rarity of two-step events: background-invariant part (not fixed)
        · choice of the small-field set and extension of the background invari
        · positivity with background (not fixed)
        · constant of the quadratic form of the background (paper)
        · one-step background-field lemma (paper)
Equicontinuity (19-07 §05)
  └─ equicontinuity of observables (fixed, conditional)
       │  ← itself a statement about defined objects only
       └─ Lipschitz property of the background map
            · positivity with background (not fixed)
            · constant of the quadratic form of the background (paper)
            · contraction of the background map (thm, weak)
Structure (needed by every root)
     · the seven rows from continuity, composition and push-forward consistenc
       of the block map to family of observables and small-field set (thm / th
Free-field foundation
     · bracketing of the propagator (thm, weak)
     · geometry of many-body sums (correlation norm) (thm, weak)
     · geometry of many-body sums (reach constant) (fixed)
     · constrained inverse identity (thm)
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      · single-scale rarity on the fine lattice (thm, weak)
Uniform decay (19-07 §06) = uniform decay of block observables (fixed, conditi
  ├─ seam to the vessel of the mass gap (fixed)
  ├─ infinite volume: ultraviolet estimate uniform in the box size (not fixed)
  ├─ rotation: one-step ultraviolet estimate for the bridge, independence of t
  ├─ fine observables: approximation of flowed observables by block variables
  ├─ non-triviality: the limit is not a free field (not fixed)
  └─ infrared road
        · band comparison (in the language of the gap) (not fixed)
        · from the area law to the gap (not fixed)
Band (19-07 §07)
  ├─ non-negativity of θ (thm, weak)
  ├─ base point at a small box (fixed, conditional)
  ├─ single-scale criterion (fixed, conditional)
  │     ← question of the cost of a twist-inserting map (not fixed)
  ├─ nesting (frame decorrelation) (paper)
  ├─ lateral widening for doubling (fixed, conditional)
  └─ another road
        · tension-doubling inequality (not fixed)
        · monotonicity of the vortex free energy and renormalisation compariso
        · gap of a small twisted box and uniform acceptance of insertion (not
        · from the vortex area exponent to the Wilson-loop area law (lit.)
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Three things can be read from the tree.

1. **At the level of the root, the forms of the statements are settled** (except for the choice of matching). The root, however, has content only when read for the sequence fixed by matching (if the sequence is taken freely, $\beta_n \equiv 0$ makes the uniform-decay and band parts of the root stand; 19-07 § 01). Of the 24 rows whose form is not fixed, 13 sit at the level of the representation below the root, 8 are rows whose object does not yet exist or which are settled after something upstream, and 3 concern the quantification and truth of band statements (28-06). Changing the representation changes the leaves, not the root.
2. **The plaquette part of non-degeneracy (rarity of two-step events: plaquette part) and equicontinuity (equicontinuity of observables) are premises whose form is, as it stands, a statement about defined objects only**; they do not go through a representation.
3. **The open forms on the side of the band come not from the representation but from the order of quantifiers and from truth** (tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison, from the area law to the gap). Their origin differs from the ultraviolet side.

Inside the chain theorem of the present road (28-01), the sequence of bare couplings β_n and the effective coupling are not connected. The Lean theorem takes the sequence of bare couplings arbitrarily. The conditional implication assembled earlier (28-01

§ 08) joins the two by defining the fine measure to be "the Wilson measure at the β_n chosen by the matching bracket for the bare coupling", but the proof itself splits into the coupling side and the measure side. The candidate place in the chain where the two could be joined is linking the thresholds.

03

Counts by label, and what they tell us

Table 1 Counts by label (as of 2026-11-13, 57 rows; the 5 rows added in § 08 are not counted)

Label	Count	Rows
Theorem	3	Continuity, composition and push-forward consistency of the block map, reflection positivity of the fine lattice, constrained inverse identity
Theorem (weak form)	10	Gauge covariance and locality, commuting with the lattice symmetries, reflection positivity at the coarse level, single-scale rarity on the fine lattice, bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm), non-negativity of θ
On paper	8	Symmetries of the measure, family of observables and small-field set, one-step background-field lemma, multistep background field, single-scale expansion tools, gauge fixing step by step, class and localisation, frame decorrelation
Form fixed, unproved	7	Rarity of two-step events: plaquette part, equicontinuity of observables, Constrained Decay Theorem, uniform decay of block observables, base point at small T^2, single-scale criterion, lateral-widening inequality (strong form) (the Constrained Decay Theorem in the version around the vacuum; the other 6 rows conditional)
Form not fixed	24	Net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin, rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, small-field one-step renormalisation theorem, large-field estimate, large-field gas, induction uniform in the step, contraction of the one-step map and matching at the endpoint, approximation of flowed observables by block variables, ultraviolet estimate uniform in the box size, one-step ultraviolet estimate for the bridge, independence of the infinite-volume limit from the box shape, the limit is not a free field, question of the cost of a twist-inserting map, tension-doubling inequality, monotonicity and renormalisation comparison, gap of a small twisted box and uniform acceptance of insertion, band comparison (in the language of the gap), from the area law to the gap
Depends on design	2	Definition and continuity of the effective coupling, row sums of the lift uniform in the step
Literature	3	Value of the one-loop coefficient, from the vortex area exponent to the Wilson-loop area law, forest interpolation formula

The objects that rows with a theorem label speak about. The 13 rows labelled "Theorem" or "Theorem (weak form)" are statements about one of the following: the Wilson measure on the fine lattice (reflection positivity of the fine lattice, single-scale rarity on the fine lattice, non-negativity of θ), the block map itself (continuity, composition and push-forward consistency of the block map, gauge covariance and locality, commuting with the lattice symmetries), structural properties of the push-forward measure (reflection positivity at the coarse level, for the strip class), the free field, the vacuum background and the linearised blocking (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm)), or an abstract identity (constrained inverse identity). **There is no theorem giving quantitative estimates for measures at renormalised scales (the push-forward measure $\nu_{n,m}$ or the effective measure at step $k \geq 1$).** For the push-forward measure, what exists is only structural properties carried over from the fine measure by the push-forward.

Rows that pay a column as it stands. Of the 10 rows labelled "Theorem (weak form)", only reflection positivity at the coarse level (for the strip class) pays a premise column of the present road as it stands. The others are similar but different statements — a different object, a subfamily, a linearisation, a one-step form, and so on (distinguished in the label column of the tables).

Where the rows whose form is not fixed come from. Of the 24 rows, 20 include (c) (object undefined) or (e) (no form across steps) among their reasons. The 4 that do not (zero-order transport of the background, tension-doubling inequality, monotonicity and renormalisation comparison, from the area law to the gap) are on the side of "too strong, or there are false examples"; for zero-order transport of the background there is an argument on paper that its present form is false (not yet checked by another pair of eyes).

Rows whose form is fixed but unproved, and choices that make them stand trivially. Of the 7 rows, 6 (rarity of two-step events: plaquette part, equicontinuity of observables, uniform decay of block observables, base point at small T^2, single-scale criterion, lateral-widening inequality (strong form)) have a "fixed form" only for the sequence fixed by matching, with the restrictions on threshold and period put into the statement. If the sequence is taken freely, uniform decay of block observables, base point at small T^2, single-scale criterion and lateral-widening inequality (strong form) stand trivially with $\beta_n \equiv 0$ ($\theta = 0$, and variables separated in time are independent) Proved here . The theorem of uniform decay of block observables in the strong-coupling window (20-04) lies in this direction: it is an example in which the statement can be satisfied, not a check of its content. Non-degenerate small examples exist for the version of the Constrained Decay Theorem around the vacuum (one-dimensional,

scalar, for two of the inputs) and for the form of base point at small T^2 with fixed lattice spacing (the thin-vortex floor, 21-02).

04

Premise of the box continuum limit — the continuum limit in a fixed box

These are the premises for the chain of block maps, in a box of fixed size, to give subsequential limits and the properties that pass to the limit. The implication that uses them is in 28-01. Notation: B is the block map, $\nu_{n,m}$ the push-forward measure, u_k the effective coupling at step k (its definition from the lattice is not fixed), z_k the large-field activity, w_k the supply.

Table 2　Premise of the box continuum limit (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Continuity, composition and push-forward consistency of the block map	B is continuous. The map from step n to m is the composite of single steps. $(B_m)_*\nu_{n,m+1} = \nu_{n,m}$	19-04, push-forward	Theorem: for every family of paths, weights and family of measures	—
2	Gauge covariance and locality	$B(U^g) = B(U)$ $g\|_{\text{coarse sites}}$. $B(U)_b$ depends only on fine links within a finite distance of b	19-04	Theorem (weak form, a similar but different statement): the form for the one-step map. The extension to composites and the reach distance are on paper	—
3	Commuting with the lattice symmetries	Translations, reflections and the hypercubic group commute with B	19-04	Theorem (weak form, groundwork): translations checked; reflection in the time direction in Lean (content not yet checked); reflections in the other axes and the hypercubic group on paper	—
4	Reflection positivity of the fine lattice	For bounded measurable f depending only on links on the positive side, $\int f(\Theta U)\, f(U)\, d\mu_W \geq 0$	19-01, reflection	Theorem: even side length. For the plane of lattice sites, all real β; for the plane through the middle of links, $\beta \geq 0$ and gauge-invariant f (20-01)	—

Number	Name	Statement	Object	Label	Reason
5	Reflection positivity at the coarse level	For F in the positive-time class, $\int (F \circ \Theta)\, F\, d\nu_{n,m} \geq 0$; matrix form $\sum c_a c_b\, s_n(x_{ab}) \geq 0$	19-04, positive side	Theorem (weak form, pays the column as it stands): for the class whose positive side is taken as a strip. The matrix form is on paper. The form taking the whole coarse half as the positive side is not placed (28-05)	—
6	Symmetries of the measure	$\nu_{n,m}$ is invariant under lattice symmetries. The breaking of symmetry of observables $\to 0$	19-04	On paper: the symmetry fields for the pure gauge theory are not in Lean. What Lean has is a version for families of measures with fermions (global gauge transformations and unit translations in the four directions)	—
7	Family of observables and small-field set	D is countable and dense, $\lvert s_n \rvert \leq B_d$, $s_n(x) = \int G_x\, d\nu_{n,\mathrm{lev}(x)}$, C_m is closed and contains the two-step event	19-07	On paper: the inclusion (two-step event $\subseteq$ small-field set) depends on how the background invariants are extended (choice of the small-field set and extension of the background invariants)	—
8	Definition and continuity of the effective coupling	$u_0(x) = x$. $x \mapsto u_{k+1}(x)$ is continuous on $\{u_j \geq u_\star\ (j \leq k)\}$	No definition (u_k)	Depends on design	—
9	Net one-step bracket of the coupling	$u_{k+1} = u_k - b + r_k^s + r_k^l$, $b_{\inf} + \delta_k - e(u_k) \leq b - r_k^s \leq b_p + e(u_k)$, $\sum_k \max(-\delta_k, 0) \leq D_0$	No definition (u_k, small- and large-field remainders r_k^s, r_k^l)	Form not fixed	(c)(e)
10	Large-field contribution to the coupling	$\lvert r_k^l \rvert \leq D(u_k)\, z_k$	No definition (r_k^l, z_k)	Form not fixed	(a)(c)
11	Recursion of the activity	$z_{k+1} \leq \rho z_k + A w_k + B z_k^2$, $z_k \geq 0$, $z_0 \leq W(u_0)$	No definition (z_k, w_k)	Form not fixed	(a)(c)(e)

Number	Name	Statement	Object	Label	Reason
12	Rarity of supply	$w_k \le \Phi(u_k)$ (from an exponential moment of the plaquette variable uniform in the step, $\mathbb{E}\, e^{\sigma Y} \le m_1(\sigma)$)	No definition (w_k, plaquette variables at step k)	Form not fixed	(c)(e)
13	Single-scale rarity on the fine lattice	For $\beta \ge 81$, the probability that all plaquettes of X have $\operatorname{Re} U_p < 0$ is $\le q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, with $q_1 < 1$ independent of the volume	19-01	Theorem (weak form, a similar but different statement, not connected to any column): one scale, $\beta \ge 81$, a ratio to the full partition function (22-01)	—
14	Moving invariant interval and margin	$\rho W(s) + A\Phi(s) + BW(s)^2 \le W(s')$ $(u_\star \le s,\ s' \le s + D_0)$, $D(s)W(s) \le Y$, $e(u_\star) + Y \le b$ inf	Real functions W, Φ, D, e	Form not fixed: W, Φ, D, e are free, and taking them all 0 makes it hold	(a)(c)
15	Rarity of two-step events: plaquette part	For large n, $\nu_{n,m}\big(\beta_m(1 - \operatorname{Re} U_p) > y_h(m)\big) \le P_1(m)$	19-04	Form fixed, unproved (conditional: for the sequence fixed by matching, with the threshold floor and $\eta_m < 1$ put into the statement)	—
16	Rarity of two-step events: background-invariant part	For large n, $\nu_{n,m}\big(f_{m,x} > f_h(m)\big) \le S(m)$	No definition (the background-field invariant f)	Form not fixed	(c)
17	Equicontinuity of observables	$\lvert s_n(d) - s_n(d')\rvert \le \tau_n + \Omega(\operatorname{dist}(d, d'))$, $\tau_n \to 0$, $\Omega(r) \to 0$	19-04 (the family of observables is the one in family of observables and small-field set)	Form fixed, unproved (conditional: for the sequence fixed by matching)	—

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05
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Premise of ultraviolet stability

These are the premises for deriving the representation-level premises of the premise of the box continuum limit (net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply) from step-by-step estimates. Notation: G_k is the family of free-field propagators at step k, P/ω the free propagator (19-04 § 07), $\mathcal{M}$ the one-step map.

Table 3 Premise of ultraviolet stability (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Bracketing of the propagator	For every step k and momentum, $c_0\,P/\omega \preceq G_k \preceq P/\omega$, $c_0 = 3/100$	19-04	Theorem (weak form, a similar but different statement): both the upper and lower sides checked. The object is the symbol recursion (a family of matrices for each momentum): free field, vacuum background, linearisation, weights $(1/8, 3/4, 1/8)$ (15-12)	—
2	Positivity with background	$\lambda_{\min}(H_c(B)) \geq \lambda_0(1 - f_1/f_\star)$	Definition on paper (background field B, constrained Hessian H_c)	Form not fixed: the background field and constrained Hessian are definitions on paper, and three forms stand side by side	(c)
3	Contraction of irrelevant terms	$\mathcal{M}^{17}(\omega G_k) \preceq \tau\,\omega G_{k+17}$, $\tau < 1/170$, uniform in k	19-04	Theorem (weak form, a similar but different statement): the τ of the quadratic form, different from the τ_k of the induction. The one-step estimate at step 0 is conditional (15-12)	—

Number	Name	Statement	Object	Label	Reason
4	Coefficient of the one-step decrease of the coupling	The free-field coefficient b_k converges at rate $(3/5)^k$, and the sum of transients is finite	19-04	Theorem (weak form, a similar but different statement): quadratic functionals of arbitrary data on the constrained covariance of the free field converge at rate $(3/5)^k$. That b_k is such a functional is on paper	—
5	Value of the one-loop coefficient	$b_\infty = 2b_0 \ln 2$	—	Literature: not re-derived for this map	—
6	Contraction of the background map	The sum of one-unit displacements $\sum_j \Delta_j < \infty$ (the free-field part is of the type $\|\psi_l\|^2 \le 0.608^l$)	19-04	Theorem (weak form, a similar but different statement): the free-field part	—
7	Geometry of many-body sums (correlation norm)	Correlation norm of the plaquette field $\Lambda \le 640/9$	19-04	Theorem (weak form, a similar but different statement): free field, linearisation, torus (15-12). The reach constant is in geometry of many-body sums (reach constant)	—
8	Constrained inverse identity	$C = G - GT^*(TGT^*)^{-1}TG$, with upper and lower estimates	Abstract linear maps	Theorem: a finite-dimensional abstract identity and estimates	—
9	Constrained Decay Theorem	The one-step constrained Green's function at step k satisfies, for an exponential weight W, $\|Wx\| \le \frac{11}{2}\|Ww\|/r^2$, with constants uniform in k	Abstract (Lean). The concrete objects are on paper	Form fixed, unproved (the version around the vacuum). The version for a general background has form not fixed (c)	—

Number	Name	Statement	Object	Label	Reason
10	Zero-order transport of the background	$\sup_n 4^n\theta_n < \infty$, $\sum_n 4^n\theta_n^{3/2} < \infty$ (θ_n: the ratio of the sup of background plaquettes in units of step j, divided by coarse units)	Definition on paper (the background as minimiser of the action)	Form not fixed: in its present form, it diverges numerically for the background that minimises the action Reading (the argument on paper that it is false has not yet been checked)	(b)(d)
11	One-step background-field lemma	For small fields, the minimum of the constrained action is unique, analytic, exponentially local, and commutes with coarse gauge transformations	Definition on paper (background field)	On paper	—
12	Multistep background field	Unique and regular on nested small-field regions. The inverse of the linearisation is at most 1 in the action norm, independent of the number of steps	Definition on paper (background field)	On paper	—
13	Row sums of the lift uniform in the step	$\sup \lvert Eg\rvert \le R \sup \lvert g\rvert$, with R uniform in k	Definition on paper (the linear lift)	Depends on design: the choice of gauge and norm. The one-dimensional scalar case is a theorem	—
14	Single-scale expansion tools	Cluster expansion on product white-noise measures, and its convergence	Abstract	On paper	—
15	Forest interpolation formula	—	—	Literature: the formula itself is not in Lean	—
16	Small-field one-step renormalisation theorem	Coarse action = β_{k+1} S_W(background) + local expansion + many-body terms. The many-body terms are exponentially small in distance	No definition (the effective action at step k)	Form not fixed	(c)(e)

Number	Name	Statement	Object	Label	Reason
17	Gauge fixing step by step	Landau gauge at each step. The determinant is block lower-triangular. Estimate of the Haar Jacobian	Definition on paper	On paper	—
18	Large-field estimate	For each step t and each hole γ, $\lvert z_t(\gamma)\rvert \le \exp(-\gamma_E \beta S^\flat(\gamma))$, $S^\flat(\gamma) \ge s_* \lvert\gamma\rvert$, with $s_* > 0$ uniform in step and volume	No definition (activities, coarse class)	Form not fixed	(c)(d)(e)
19	Large-field gas	$\sum_{X \ni x} \lvert z(X)\rvert\, e^{2\lvert X\rvert} \le 1$, and that the partition function of the gas equals the corresponding ratio	No definition (activities)	Form not fixed	(a)(b)(e)
20	Class and localisation	The only local gauge-invariant terms of dimension at most 4 are the constant and $\sum \operatorname{tr} F^2$	Definition on paper (class)	On paper: the algebra of localisation is in Lean; identification with the class is on paper	—
21	Induction uniform in the step	$r_{k+1} \le \tau_k r_k + a_k w_k + s_k$, $w_{k+1} \le b_k r_k + \sigma_k w_k + s'_k$; if the product of the coefficients is below $1/4$, it stays in the ball	No definition (r_k, w_k are real sequences with no definition as coordinates of the class)	Form not fixed	(c)(e)

06

Premises of full-sequence convergence, observables, infinite volume, rotation and non-triviality

These rows have their forms fixed after the forms of the premise of the box continuum limit and the premise of ultraviolet stability are settled. For the rows on fine observables and rotation, the objects needed to write the statements are not yet defined (28-06).

Table 4　Premises of full-sequence convergence, fine observables, infinite volume, rotation and non-triviality (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Contraction of the one-step map and matching at the endpoint	$\lvert r_k(u,v)-r_k(u',v')\rvert \le \varepsilon\lvert u-u'\rvert + c\lVert v-v'\rVert$, $\lVert T_k(u,v)-T_k(u',v')\rVert \le c'\lvert u-u'\rvert + \tau\lVert v-v'\rVert$, $4cc' < (1-\varepsilon-\tau)^2$	No definition (the coordinates u, v of the one-step map)	Form not fixed	(c)
2	Approximation of flowed observables by block variables	Approximate an observable at flow time t by a Lipschitz function of the block variables at step m, with an error $\varepsilon_m(t) \to 0$ uniform in the lattice	No object (flowed observables)	Form not fixed: flowed observables are undefined	(c)
3	Ultraviolet estimate uniform in the box size	The premise of the box continuum limit and the premise of ultraviolet stability hold with constants independent of the box size L	All of those two premises	Form not fixed	(e)
4	Uniform decay of block observables	$\lvert\langle F\cdot G\circ\tau_t\rangle - \langle F\rangle\langle G\rangle\rvert \le C_F C_G\, e^{-mt}$, with m independent of the lattice spacing and the box (exact form in 19-07 § 06)	19-04, 19-03	Form fixed, unproved (conditional: for the sequence fixed by matching, with time differences up to half the period). A theorem in the strong-coupling window (20-04), but in the direction where it stands trivially	—
5	One-step ultraviolet estimate for the bridge	The one-step image from a rotated fine lattice to the standard coarse lattice lies in the ball of the chain	No object (the block map from a rotated lattice)	Form not fixed	(c)
6	Independence of the infinite-volume limit from the box shape	The limits of sequences of tori of different shapes coincide	No definition (the infinite-volume limit)	Form not fixed	(c)
7	The limit is not a free field	(by which quantity to say it is not fixed)	—	Form not fixed	(c)

07

The band — rows on the vortex free energy

In this stage many rows can be written with the vortex ratio θ (19-02) alone. The statements are written in the notation $\theta(R;T^2_L)$ of 26-02 (longitudinal size R, size L of the transverse torus). $m_0(L)$ is the box mass of the box of size L (the gap of the transfer operator), and $\sigma_N(\beta) = \liminf_M(-\ln\theta_{N,M})/M^2$ is the slab tension.

Table 5 Premises of the band (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Non-negativity of θ	If M is even, $0 \le \theta_{N,M}(\beta)$ for all real β	19-02	Theorem (weak form, a similar but different statement): the subfamily with M even. The column needs all sizes (21-06)	—
2	Base point at a small box	$\theta(L_0;T^2_{L_0}) \le 1-\varepsilon_0$, uniform in the lattice spacing	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching). The form with fixed lattice spacing is the thin-vortex floor (21-02)	—
3	Single-scale criterion	At a physical size L_*, $\theta(L_*;T^2_{L_*}) < (s^2)^{-1/(s^2-1)}$ (0.63 for $s=2$, 0.7598 for $s=3$)	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching)	—
4	Question of the cost of a twist-inserting map	For a gauge-covariant smooth map T, $\mathbb{E}_+[S\circ T - S - \ln J_T] \le 2\,\mathrm{artanh}(0.7598)$, uniform in the lattice spacing	No object (the map T)	Form not fixed	(c)
5	Nesting (frame decorrelation)	$\theta(nR;T^2_L) \le C_D\,\theta(R;T^2_L)^{n^2}$	19-02	On paper	—
6	Lateral widening for doubling	$\theta(R;T^2_{L'}) \le C\,(L'/L)^2\,\theta(R;T^2_L)$ $(L \le L' \le R)$	19-02	Form fixed, unproved (conditional: for the sequence fixed by matching)	—
7	Tension-doubling inequality	For some $\delta_0 > 0$ and N_1, $4\sigma_{2N}(\beta) \ge e^{\delta_0}\sigma_N(\beta)$ for all $N \ge N_1$	Definition on paper (slab tension σ_N)	Form not fixed: (d) only for the form uniform in β	(b)(f)

Number	Name	Statement	Object	Label	Reason
8	Monotonicity of the vortex free energy and renormalisation comparison	The free energy of a twist insertion is monotone in β. $s(N,\beta') \le s(2N,\beta)$	19-02	Form not fixed	(b)
9	Gap of a small twisted box and uniform acceptance of insertion	Gap $\ge c/L$ uniform in the lattice spacing. The acceptance rate of driven vortex insertions uniform in the lattice spacing	19-02, 19-03	Form not fixed: the gap operator of small twisted boxes and the insertion map are undefined	(c)
10	Band comparison (in the language of the gap)	For $x(L) := L\,m_0(L)$, $x(2L) \ge x(L) + \delta$, together with stability above the threshold	19-03 (box mass)	Form not fixed: the box mass $m_0(L)$ of the box used (with spatial planes) is undefined	(c)
11	From the area law to the gap	From the band (area exponent) to uniform decay of block observables	—	Form not fixed: the naive form is false in two dimensions	(b)
12	From the vortex area exponent to the Wilson-loop area law	—	—	Literature: the typical-frame version has not been re-derived	—

Among the band rows, the tension-doubling inequality and monotonicity and renormalisation comparison still have a choice of how to quantify the form of the statement (uniformly in β, or at fixed physical size). From the area law to the gap is the arrow joining the side of the mass gap and the side of confinement, and this book does not have it (28-03).

08

Five rows added outside the 57

There are five premises that do not fall under any of the 57 rows. So as not to change the counting, they are kept outside the counts of Table 1. The labels of these 5 rows have not yet passed a check at the same level as the 57.

Table 6　Added premises (as of 2026-11-13)

Number	Name	Statement	Object	Label	Reason
1	Linking the thresholds	Tie the scale factor β_{m} of the plaquette variable and the threshold $y_{h}(m)$ of the two-step event to u_{m} and the threshold $y_{s}(u_{m})$ on the coupling side (for example under $\beta_{m} = 4u_{m}$, $y_{h}(m) = y_{s}(u_{m})$, or the analytic floor $y_{h}(m) \le \beta_{m}(1 - \cos\varphi_{c})$ (φ_{c} is a fixed angle that sets the small-field threshold))	No definition (u_{m})	Form not fixed: in the conditional implication assembled earlier, too, the theorem does not tie these two together	(c)
2	Choice of the small-field set and extension of the background invariants	That the two-step event is contained in the small-field set. It holds when the background invariants are extended outside $SU(2)$ by values larger than the threshold f_{h}, and is false when they are extended by 0	Definition on paper (background field)	Depends on design	—
3	Constant of the quadratic form of the background	The three constants of the premise "background estimate" of the implication assembled earlier (constants of the quadratic forms that make up the Lipschitz constant of the background map)	Definition on paper (background field)	On paper	—
4	Geometry of many-body sums (reach constant)	The row sum R of correlations of the plaquette field is finite	19-04	Form fixed, unproved	—
5	Seam to the vessel of the mass gap	The vessel implication (28-03) assumes weak convergence of the full sequence of measures, and that the subspace of observables is closed under time translation. When used along a subsequence, it takes the form "at every limit point"	19-03	Form fixed, unproved (conditional: for the sequence fixed by matching; the same form as the root of full-sequence convergence)	—

```
09
```

A dictionary of other names (short version)

The same premises go by other names in the conditional implication that this book's study assembled earlier (weak ultraviolet existence in small boxes; 28-01 § 08) and in the ledger of Part 27. We include only what is needed to match them up.

Table 7 A dictionary of other names (short version)

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
The implication of weak ultraviolet existence in small boxes (the main theorem of the implication assembled earlier)	Main theorem. Its assumptions are the one-step bracket assumption for the coupling, equicontinuity of observables and the small-field set assumption; its conclusions run from the matching bracket for the bare coupling to non-degeneracy (mass of the small-field set)	Ultraviolet control (premise of the box continuum limit; 27-02)	`WeakUltravi`
The chain theorem from per-step premises	The chain theorem (six per-step assumptions). Its set of assumptions differs from that of the Lean box version	—	`BoxWalk.cha` `ChainBlockM`
One step of the coupling (definition and continuity of the effective coupling, net one-step bracket of the coupling)	One-step bracket assumption for the coupling, net one-step bracket of the coupling, quadratic form of the one-step remainder, window condition for the class, definition and continuity of the effective coupling	—	Chain fields
Large-field contribution to the coupling	Large-field contribution to the coupling	—	Chain field
Recursion of the activity	Recursion of the activity, existence of limiting expectations and continuous extension, large-field weight	—	Chain field
Rarity of supply, single-scale rarity on the fine lattice	Rarity of supply, moments and rarity of a single plaquette	—	`FoldCap.rar`
Moving invariant interval and margin	Moving invariant interval and margin, logarithmic threshold and moving interval	—	Chain field
Rarity of two-step events (plaquette part, background-invariant part)	Small-field set assumption, small-field event uniform in the lattice	—	Chain field
Equicontinuity of observables	Equicontinuity of observables, small-field event uniform in the lattice, background estimate	—	Chain field

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
Reflection positivity (reflection positivity of the fine lattice, reflection positivity at the coarse level)	Support of reflection positivity	Reflection positivity of z (positivity of the ensemble summed over twists; a different thing)	`TorusRP.wil`
Positivity with background	Positivity with background, background estimate	—	—
Large-field gas	Plaquette-level large-field gas	—	—
Induction uniform in the step	Quadratic form of the one-step remainder, window condition for the class	—	`F3Condition`
Constrained Decay Theorem	—	—	`RealSpaceTh`
Contraction of the one-step map and matching at the endpoint	Related to the summability-of-increments assumption (not the same thing; see the note below)	—	—
Uniform decay of block observables	—	Uniform decay of block observables in the ledger	—
Base point at a small box (base point at small T^2)	—	Base point at small T^2 in the ledger	—
Nesting (frame decorrelation)	—	Frame decorrelation in the ledger	—
Lateral widening for doubling (lateral-widening inequality (strong form))	—	Lateral-widening inequality (strong form) in 26-02, and the same row in 27-01	—
Monotonicity of the vortex free energy and renormalisation comparison	—	β-monotonicity, renormalisation comparison (26-06)	—

Name of the content	Name in the conditional implication assembled earlier	Ledger (27-02)	Lean
Gap of a small twisted box and acceptance (small-box gap and acceptance)	—	Small-T^2 gap, uniform acceptance of insertion	—
Band comparison (in the language of the gap)	—	Band comparison (in the language of the gap) in the ledger	—

The summability-of-increments assumption is the assumption that, for a countable dense family of gauge-invariant functions, the step-by-step increments of the values of observables are summable. It is not the same as full-sequence convergence in 19-07 § 03: it is stronger by the summability, and the family of functions is different. Also, the implication assembled earlier defines the fine measure to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling, whereas the Lean theorem takes the sequence of bare couplings arbitrarily.

Among the ten premises of the implication assembled earlier, support of reflection positivity is reflection positivity. The other nine are the premises other than reflection positivity. The existence of four-dimensional Yang–Mills theory itself was shown, with a different design, by the construction published on 2026-11-10 (Volume 2).

What this chapter can say

Can say	Cannot say
The six roots and the 57 rows of premises are laid out as a tree and in tables by stage, with labels and reason symbols (as of 2026-11-13)	None of the premises is shown in this chapter. The labels follow the record of the checks of that date
The 13 rows with theorem labels are statements about the Wilson measure on the fine lattice, the block map itself, structural properties of the push-forward measure, the free field and linearisation, or an abstract identity	There is no theorem giving quantitative estimates for measures at renormalised scales
Of the 24 rows whose form is not fixed, 20 owe this to an undefined object or to the absence of a form across steps	Answers to the questions that would fix the forms of the open rows (only the questions are in 28-06)
Of the 7 rows whose form is fixed but unproved, 6 are conditional on the sequence fixed by matching	If the sequence is taken freely, 4 of them stand trivially with $\beta_n \equiv 0$. Non-degenerate small examples exist for only 2 rows

Sources and reproduction

Item	Type	Source, tools
The 57 rows of premises, labels, reason symbols, the tree	Assumption	The inventory of premises in this book's study (2026-11-13). The labels come from the Lean sources and from the record of the check that compared definitions and statements with a separate pair of eyes
The assumptions and the ten premises of the conditional implication assembled earlier (Table 7)	—	Record of this book's study (2026-10-30). 28-01 § 08
Rows with theorem labels	Lean	The source tables of each chapter (20-01, 21-06, 15-12, 22-01)
Forest interpolation formula	Cited	Bibliography only (unchecked)
From the vortex area exponent to the Wilson-loop area law	Cited	Bibliography only (unchecked)
Rows of the ledger of conditions, and the rows of uniform decay of block observables and band comparison (in the language of the gap)	—	27-02 The ledger of conditions

Words used in this chapter — premise, root, sufficient condition, representation, effective coupling, activity, supply, background field, class, push-forward measure, two-step event, vortex ratio, slab tension, ledger of conditions. Their meanings are in A-02 Glossary.

Next chapter: 28-05 What depends on design choices — tabulates the rows labelled "Depends on design", and which choices the constants of the other rows depend on.

What depends on design choices

> This chapter is this book's study.

What this chapter shows — One step of renormalisation and the statements of the premises contain places that are not decided by mathematics but chosen and fixed by us. For nine such choices we tabulate the present values and the theorems and premises that would have to be redone if the choice changed. We also confirm here that the theorems of Parts 20 and 21 and the rarity of large plaquettes (22-01) do not depend on the choice of block map.

Prerequisites — 19-04 § 08 Weights as a design choice, 28-04 The list of premises.

Said first — This chapter does not claim which choice makes the premises easier to satisfy. The table of constants for other weights, and the computation of whether overlapping block averages preserve reflection positivity of the push-forward measure, have not yet passed a check and are not included. We only write that there is an issue there. The choices here are choices within this book's design (the measure pushed forward by a fixed block map), and are separate from the design of the published construction that showed the existence of four-dimensional Yang–Mills theory (the two designs are compared in Part 14).

The order of this chapter

1. What a design choice is — fixing things so that a definition is unique
2. The nine choices and their present values (Table 1)
3. Transverse weights, scale factor, pull-back radius — choices in the block map itself
4. Overlap of block averages — the issue with reflection positivity
5. Norm, gauge, definition of the background — choices of representation
6. The way of matching, and the definition of compatibility in the gas
7. Which theorems and premises depend on which choices (Table 2)
8. Other fixings (Table 3)

01

What a design choice is — fixing things so that a definition is unique

To define one step of renormalisation (19-04), one has to settle on one choice of the weights of the family of paths, the scale factor of the blocks, the procedure that returns the average to the group, and so on. The statements of the premises (28-04) also involve decisions such as which norm to measure with and around which background to expand.

These are not determined uniquely by mathematics; we choose and fix them. If a choice changes, the constants and certificates of the theorems that depend on it must be redone, and the statements of the premises themselves change. Among the statements to be shown (19-07), the band can be written with the vortex ratio θ alone and does not depend on the choice of block map. The other five depend, through the push-forward measure $\nu_{n,m}$, on the choice of block map (weights, scale factor, radius), and matching, non-degeneracy and equicontinuity depend further on the choice of observables and sets.

This chapter is a table that makes visible, in one place, which choice affects what.

02

The nine choices and their present values

Table 1　Design choices (as of 2026-11-13)

Choice	Present value	Premises that depend on it (names of rows in 28-04)
Transverse weights of blocking $(\alpha, 1-2\alpha, \alpha)$	$\alpha = 1/8$	The free-field inputs (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm)), the Constrained Decay Theorem, zero-order transport of the background, row sums of the lift uniform in the step, large-field estimate
Block scale factor	2	The matching bracket, rarity of two-step events: plaquette part, the free-field inputs (from bracketing of the propagator to geometry of many-body sums), zero-order transport of the background, band doubling (single-scale criterion, lateral-widening inequality (strong form), tension-doubling inequality)
Radius r_0 of the radial pull-back	A free parameter $(0 < r_0 \le 1)$	The range of linearisation, large-field estimate

Choice	Present value	Premises that depend on it (names of rows in 28-04)
Overlap of block averages	Overlapping (an average over paths shifted transversely with $(1/8, 3/4, 1/8)$)	Reflection positivity at the coarse level, recursion of the activity, rarity of supply, uniform decay of block observables
Norm	The sup norm and the sum-of-squares norm are mixed from place to place	The Constrained Decay Theorem, zero-order transport of the background, multistep background field, row sums of the lift uniform in the step, induction uniform in the step
Gauge	Landau slice at each step / axial gauge / tree gauge	The Constrained Decay Theorem, row sums of the lift uniform in the step, gauge fixing step by step
Definition of the background	The minimiser of the constrained action	Rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, one-step background-field lemma, multistep background field
Way of matching (definition of the effective coupling)	Not fixed (observable, step, target value, choice of solution; two proposed definitions of u_k on paper)	Definition and continuity of the effective coupling, net one-step bracket of the coupling, the domain of every row of the premise of the box continuum limit
Definition of compatibility in the gas	The conditional implication assembled earlier: not adjacent / Lean: not overlapping	Large-field contribution to the coupling, large-field gas

03

Transverse weights, scale factor, pull-back radius — choices in the block map itself

Transverse weights. The paths that build a coarse link are shifted by $-1, 0, +1$ units in a transverse direction, with weights $(\alpha, 1-2\alpha, \alpha)$ (19-04 § 03). At present $\alpha = 1/8$. The constants of the one-step free-field theorem (15-12) were all computed for this value and cannot be used as they stand for other weights: the lower constant $c_0 = 3/100$ of the sandwich on the propagator, the 17-step contraction $\tau < 1/170$, the correlation norm $\Lambda \le 640/9$ of the plaquette field, and so on.

There is an issue as to what becomes better and what worse when the weights change (whether the plaquette field measured in coarse units grows with the step, and how the properties of the constraint change with the weights).

Scale factor. One step halves the lattice spacing. The matching bracket (19-07 § 02) is written in terms of the step number $n = \log_2(1/a_n)$, and the threshold of the single-scale criterion on the band side differs for each scale factor s (0.63 for $s = 2$, 0.7598 for $s = 3$).

Pull-back radius. This is the threshold r_0 of the procedure that returns the average of path products to the unit ball. The linearisation theorem for the block map (15-13) has $0 < r_0 \le 1$ among its assumptions.

Theorems that do not depend on the choices. The theorems of Part 20 (reflection positivity, strong coupling) and Part 21 (vortex floors, Polyakov loops), and the rarity of large plaquettes (22-01), use only the Wilson measure on the fine lattice and twisted partition functions, and do not use the block map. So they depend on none of these three choices (nor on the overlap of § 04) Proved here .

04

Overlap of block averages — the issue with reflection positivity

In the present family of paths, the paths that build a coarse link are shifted transversely, so neighbouring blocks share fine links (an overlapping average).

Reflection positivity (20-01) is a theorem for the Wilson measure on the fine lattice. Whether reflection positivity is preserved for the pushed-forward coarse measure $\nu_{n,m}$ is expected to depend on whether blocks straddle the reflection plane, that is, on whether there is overlap, and **there is an issue here.** The computation has not yet passed a check and is not included.

In the present list of premises, reflection positivity at the coarse level (the row on reflection positivity at the coarse level) is placed in the form of the class whose positive side is taken as a strip (coarse observables depending only on links in the positive half of the fine lattice). In this form it is paid as a statement about the Wilson measure on the fine lattice. The form that takes the whole coarse half as the positive side is not placed among the premises.

Besides reflection positivity at the coarse level, what depends on this choice is rarity across steps (recursion of the activity and rarity of supply; the road that uses the Chessboard Estimate from reflection positivity) and the vessel used to go from uniform decay to the mass gap (uniform decay of block observables).

05

Norm, gauge, definition of the background — choices of representation

These three are choices made not in the block map but in writing down (representing) the theory at coarse scales. They do not appear in the root (19-07), only on the side of the sufficient conditions.

- **Norm.** Whether to measure the plaquette field in the sup norm or in the action (sum-of-squares) norm. The present form of zero-order transport of the background (the row in 28-04) is written in the sup norm, and the reasons (b)(d) for its form not being fixed are tied to this choice. The Constrained Decay Theorem and the multistep background field are written in the sum-of-squares or action norm.
- **Gauge.** Landau slice at each step, axial gauge, tree gauge. Whether the row sums of the lift are uniform in the step (row sums of the lift uniform in the step) changes with the choice of gauge. For the one-dimensional scalar lift, the row sums are a theorem.
- **Definition of the background.** We use the background that minimises the action under the block constraint. The rarity of the background invariants (rarity of two-step events: background-invariant part), positivity with background, and the background-field lemmas (one-step background-field lemma, multistep background field) are written on top of this definition. There is also the road of switching to a different background that can be written by an explicit formula.

06

The way of matching, and the definition of compatibility in the gas

The way of matching. The sequence of bare couplings β_n is fixed by matching the expectation of a coarse observable Y_0 at step m_0 to a target value κ_0 (19-07 § 02). What is chosen is Y_0, m_0, κ_0, and how to pick a solution when there are several. In the language of the premises, one possible road is to make this correspond to the definition of the effective coupling u_k (definition and continuity of the effective coupling). There are two proposed definitions of u_k on paper (read as a renormalised coupling, or as a coefficient of the blocked action). Which is taken changes the meaning of the domain $u_m \geq u_\star$ of every row of the premise of the box continuum limit. The conditional implication that this book's study assembled earlier (28-01 § 08) defines the fine measure to be the Wilson measure at the β_n chosen by the matching bracket for the bare coupling, and the Lean theorem takes the sequence of bare couplings arbitrarily. The premise that ties the threshold on the measure side to the threshold on the coupling side (linking the thresholds) is also settled together with this choice.

The definition of compatibility in the gas. This is the definition of what it means, in the large-field gas, for two large-field regions to be "compatible". The conditional implication that this book's study assembled earlier (28-01 § 08) uses "not adjacent", the Lean form uses "not overlapping", and the two are different. What depends on it is the large-field contribution to the coupling and the large-field gas, both of which have forms not fixed.

07

Which theorems and premises depend on which choices

Table 2 Table of dependencies (● means depends). The top 5 rows are theorems, the rest premises (as of 2026-11-13)

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Part 20 (reflection positivity, strong coupling)								
Part 21 (vortex floors, Polyakov loops)								
15-12 One step of the free field	●	●						
15-13 Linearisation and the covariance of the push-forward	●	●	●					
22-01 Rarity of large plaquettes								
Reflection positivity at the coarse level				●				
Definition and continuity of the effective coupling, net one-step bracket of the coupling		●						●
Large-field contribution to the coupling								
Recursion of the activity, rarity of supply				●				

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Rarity of two-step events: plaquette part		●						
Rarity of two-step events: background-invariant part							●	
Free-field inputs (bracketing of the propagator, contraction of irrelevant terms, coefficient of the one-step decrease of the coupling, contraction of the background map, geometry of many-body sums (correlation norm))	●	●						
Positivity with background							●	
Constrained Decay Theorem	●				●	●		
Zero-order transport of the background	●	●			●		●	
Background field (one-step background-field lemma, multistep background field)					● (C2)		●	
Row sums of the lift uniform in the step	●				●	●		
Gauge fixing step by step						●		
Large-field estimate	●		●					
Large-field gas								
Induction uniform in the step					●			

Theorem or premise	Weights	Scale factor	Radius	Overlap	Norm	Gauge	Background	Matchin
Uniform decay of block observables (vessel)				●				
Band doubling (single-scale criterion, lateral-widening inequality (strong form), tension-doubling inequality)		●						

Of the top 5 rows of Table 2, only the two chapters on the free field and linearisation depend on the choices. On the side of the premises, rows that depend on the weights gather in the premise of ultraviolet stability, and rows that depend on the background, norm and gauge gather around the background field and the constrained estimates.

08

Other fixings

These are not choices in the block map or the representation, but fixings that determine the scope of theorems and premises.

Table 3 Other fixings

Fixing	Present value	What depends on it
Threshold for large plaquettes	The rarity theorem uses $\operatorname{Re} U_p < 0$. The premise side uses a threshold $y_h(m)$ for each step	22-01, rarity of supply, single-scale rarity on the fine lattice, rarity of two-step events: plaquette part
Range of β	Strong-coupling windows $18 \tanh \lvert\beta\rvert < 1$ and $24(1 - e^{-4\lvert\beta\rvert}) < 1$; rarity for $\beta \geq 81$	Part 20, 22-01, single-scale rarity on the fine lattice
Reflection plane and positive side	Plane of lattice sites (all real β) / plane through the middle of links ($\beta \geq 0$). At the coarse level the positive side is taken as a strip	20-01, reflection positivity of the fine lattice, reflection positivity at the coarse level
Extension of the background invariants	How values outside $SU(2)$ are set (whether to extend by values larger than the threshold)	The inclusion in the family of observables and small-field set, rarity of two-step events: background-invariant part, choice of the small-field set and extension of the background invariants
Class of observables	Flowed observables (no definition yet)	Approximation of flowed observables by block variables, the rotation premise

Fixing	Present value	What depends on it
Geometry of the twist	Background twist $n_{12} = 1$, transverse size N held fixed	Part 21, every row of the band

What this chapter can say

Can say	Cannot say
The present values of the nine choices and the premises that depend on them are tabulated	Which choice makes the premises easier to satisfy
Proved here The theorems of Parts 20 and 21 and 22-01 do not use the block map, so they do not depend on the choice of block map	—
The constants of the one-step free-field theorem in Part 15 were computed with weight 1/8 and scale factor 2	The constants for other weights (the table has not been checked and is not included)
There is an issue between the overlap of block averages and reflection positivity of the push-forward measure	The answer to that issue (the computation has not been checked and is not included)

Sources and reproduction

Item	Type	Source, tools
The list of choices and the dependencies	Assumption	The inventory of premises in this book's study (2026-11-13)
Definition of the block map and the weights	Lean	Source table of 19-04
Constants of the one-step free-field theorem	Lean	Source table of 15-12
The two definitions of compatibility in the gas	—	The conditional implication that this book's study assembled earlier (28-01 § 08) and this book's Lean bundle

Words used in this chapter — design choice, transverse weights, scale factor, pull-back radius, overlap of block averages, norm, gauge, background field, matching, compatibility in the gas. Their meanings are in A-02 Glossary.

Next chapter: 28-06 Open places, and places where the object does not yet exist — lists the premises whose form is not fixed, sorted by origin.

Open places, and places where the object does not yet exist

> This chapter is a map of open questions.

What this chapter shows — A list that divides into three the 24 rows labelled "Form not fixed" in 28-04, according to where the openness comes from. For each row we write whether what would fix its form is an argument on paper, a small numerical experiment, or a design choice. At the end we place the stages for which the objects needed to write the statements do not yet exist (fine observables, rotation), and what this book does not claim.

Prerequisite chapters — 28-04 The list of premises (row names and labels), 19-07 The statements to be shown.

Said first — This chapter is a list and answers none of the questions. The column "What would fix it" records the kind of question, and does not claim that the question will be settled soon. The content moves, so updates are made in A-08 A map to the mass gap.

The order of this chapter

1. Three origins — Table 1
2. Objects of the representation undefined — 13 rows on the ultraviolet side (Table 2)
3. No object at all, or downstream — 8 rows (Table 3)
4. Order of quantifiers and truth — 3 rows of the band (Table 4)
5. The order of the questions that would fix the forms
6. Stages whose objects are not yet defined
7. What this book does not claim

01

Three origins

There are 24 premises whose form is not fixed (28-04 Table 1). Looking only at the reason symbols ((a) too light, (b) too strong, (c) object undefined, (d) numerical signs against the present form, (e) no form across steps, (f) order of quantifiers not fixed) they look scattered, but divided by origin there are three groups.

Table 1 Origins of the 24 rows whose form is not fixed (as of 2026-11-13)

Origin	Rows	Which rows	Relation to the root
The object of the expression is undefined	13	Net one-step bracket of the coupling, large-field contribution to the coupling, recursion of the activity, rarity of supply, moving invariant interval and margin, rarity of two-step events: background-invariant part, positivity with background, zero-order transport of the background, small-field one-step renormalisation theorem, large-field estimate, large-field gas, induction uniform in the step, contraction of the one-step map and matching at the endpoint	The root (19-07) can be written. Below it, the form of the sufficient conditions passing through the effective coupling, the activity, the background field and the class is not fixed
No object at all, or downstream	8	Approximation of flowed observables by block variables, ultraviolet estimate uniform in the box size, one-step ultraviolet estimate for the bridge, independence of the infinite-volume limit from the box shape, the limit is not a free field, question of the cost of a twist-inserting map, gap of a small twisted box and uniform acceptance of insertion, band comparison (in the language of the gap)	Rows for which an object needed in the statement (flowed observables, the map from a rotated lattice, a map that inserts a twist, the vessel of a small twisted box, and so on) does not exist, or whose form is fixed after the upstream forms are settled
Order of quantifiers and truth	3	Tension-doubling inequality, monotonicity of the vortex free energy and renormalisation comparison, from the area law to the gap	The root for the band can be written. Candidate sufficient conditions have false examples, or the order of quantification is not fixed

The 13 rows of the representation sit at the level of the representation below the root. Changing the representation (a different definition of the background, a different norm, and so on) changes the leaves, but not the root. The openness on the side of the band comes not from the representation but from the quantification and truth of the statements.

02

Objects of the representation undefined — 13 rows on the ultraviolet side

Inside the chain theorem of the present road, the effective coupling u, activity z, supply w, remainders, plaquette variables, background invariants and observables are all free variables. Only the block map and the push-forward measure are defined from the lattice (19-04). So the following rows need definitions of their objects before any proof.

Table 2 The 13 rows at the level of the representation, and what would fix their form

Row	Reason	What would fix it	The smallest question
Net one-step bracket of the coupling	(c)(e)	Paper (design)	Define the effective coupling uniquely. If restated in the language of matching (19-07 § 02): which coarse observable to match at which step. Whether the domain $u_m \geq u_\star$ of the other rows can then be written in the same language
Large-field contribution to the coupling	(a)(c)	Small numerical experiment, and design	The present form holds if the activity is chosen freely (light). When a form uniform in volume and step is put into the statement, does that uniformity hold numerically on small lattices for the plain Wilson measure?
Recursion of the activity	(a)(c) (e)	Paper and small numerical experiment	For the measure $\nu_{n,n-1}$ pushed forward by one step only, does pulling back the event "a coarse plaquette is large" to an event on the fine lattice give a form to which the Chessboard Estimate (22-01) applies?
Rarity of supply	(c)(e)	Design	Can the supply be defined as the probability of an event under the push-forward measure (the fine-side image is small, the coarse plaquette is large)? If so, it merges with rarity of two-step events: plaquette part
Moving invariant interval and margin	(a)(c)	Paper, after the definitions of the large-field contribution to the coupling and of rarity of supply	The real functions W, Φ, D, e are free, and it holds if they are all taken to be 0. Do the actual gas coefficients take this form, with the margin numbers holding?
Rarity of two-step events: background-invariant part	(c)	Design (definition of the background)	Whether to make it an event in plaquette variables alone, without background invariants, or to switch to a background that can be written by an explicit formula
Positivity with background	(c)	Design (definition of the background), and paper	Fix the definitions of the background field and the constrained Hessian, and decide which of the three forms standing side by side to take as the premise

Row	Reason	What would fix it	The smallest question
Zero-order transport of the background	(b)(d)	Paper (design)	The present form (measured by the sup over plaquettes) diverges numerically for the background that minimises the action Reading (the argument on paper that it is false has not yet been checked). If the zeroth-order term is written in the action norm, does it close under the same assumptions as the multistep background field?
Small-field one-step renormalisation theorem	(c)(e)	Paper	Is the background field the only undefined object needed to write down the step-0 statement whose input is the Wilson action itself?
Large-field estimate	(c)(d) (e)	Small numerical experiment	Does the lower bound over steps of the action of each hole stay at a positive value proportional to the number of plaquettes as the steps go deeper?
Large-field gas	(a)(b) (e)	Small numerical experiment, and design	The literal form always holds on a fixed lattice (light), while the form with equality on all regions does not hold numerically on small lattices Reading (strong). Fix a form in between that includes uniformity. The choice of the definition of compatibility in the gas (not adjacent / not overlapping) also remains (28-05)
Induction uniform in the step	(c)(e)	Design (the norm of the induction), and paper	When the induction coordinate r_k is defined by coordinates of the class, does the free-field contraction (contraction of irrelevant terms) become a coefficient of the recursion for r_k?
Contraction of the one-step map and matching at the endpoint	(c)	Paper, after the forms of the small-field one-step renormalisation theorem and of induction uniform in the step are settled	In the language of the coordinates of induction uniform in the step, does the contraction condition become a statement in the same coordinates?

Of the 13 rows, 5 (large-field contribution to the coupling, recursion of the activity, rarity of supply, large-field estimate, large-field gas) gather around the treatment of large fields (activities and the gas). Their content (the form of the estimates and the premises on the gas) will not be given chapters in this book until the forms of the statements are settled.

> 03

No object at all, or downstream — 8 rows

Table 3 Rows with no object, or whose form is fixed after the upstream is settled

Row	Reason	What would fix it
Approximation of flowed observables by block variables	(c)	Definitions of the lattice gradient flow and of flowed observables (§ 06)
Ultraviolet estimate uniform in the box size	(e)	Fixed as the uniform version of the premise of the box continuum limit and the premise of ultraviolet stability, after their forms are settled
One-step ultraviolet estimate for the bridge	(c)	Definition of the block map from a rotated fine lattice to the standard coarse lattice (only a sketch on paper)
Independence of the infinite-volume limit from the box shape	(c)	After the definition of the infinite-volume limit (the limit of large boxes)
The limit is not a free field	(c)	The choice of the quantity by which to say it (for example, the limiting value of the same observable at another step m_1, when matched at step m_0)
Question of the cost of a twist-inserting map	(c)	Definition of the map T (the method is not fixed). This row is a sufficient condition for the single-scale criterion, so there is also the road of checking the single-scale criterion directly
Gap of a small twisted box and uniform acceptance of insertion	(c)	Definitions of the operator that measures the gap of small twisted boxes, and of the map for driven vortex insertion
Band comparison (in the language of the gap)	(c)	Definitions of the vessel for boxes with spatial planes, and of the box mass $m_0(L)$

04

Order of quantifiers and truth — 3 rows of the band

Table 4 Rows on the band side whose form is not fixed

Row	Reason	What would fix it
Tension-doubling inequality	(b)(f). (d) only for the form uniform in β	Small numerical experiment. With the transverse size N held fixed and $\beta \to \infty$, there are numbers in which the minimal action of a vortex (21-05) increases with N, a sign pointing against the form uniform in β Reading . Either state it on a sequence at fixed physical size ($N = L_*/a$, $\beta = \beta(a)$), or fix the order of quantifiers

Row	Reason	What would fix it
Monotonicity of the vortex free energy and renormalisation comparison	(b)	On lattices with finite non-abelian groups there are examples where monotonicity fails Reading . Decide what to place as an assumption for $SU(2)$ (the form in 26-06)
From the area law to the gap	(b)	Paper (choice of road). The naive form is false in two dimensions. Whether to fix the last arrow on the side of the mass gap as the single route from band comparison (in the language of the gap) and stability above the threshold, via a uniform lower bound on the box mass, to uniform decay

05

The order of the questions that would fix the forms

Putting first what is settled cheaply (on paper or with small numerics), and, at equal cost, what has more downstream, gives the following order. Items 1–3 are independent of each other.

1. Fix the matching observable Y_0 and step m_0, and define the effective coupling uniquely (definition and continuity of the effective coupling, net one-step bracket of the coupling; paper).
2. Put numbers on the plaquette part of the rarity of two-step events (rarity of two-step events: plaquette part) under the push-forward measure (small numerical experiment). It is a premise written with defined objects only, yet there is no check on small examples.
3. Decide with which background and which norm to state zero-order transport of the background (paper). The definition of the background for rarity of two-step events: background-invariant part is settled at the same time.
4. Whether the recursion of the activity can be stated for the measure pushed forward by one step only (paper and small numerics; after 2).
5. Run the Constrained Decay Theorem through in one dimension from input to output, and check whether the assumptions can be satisfied simultaneously (Lean).
6. Look at the uniformity of the large-field gas in numbers (small numerical experiment).
7. Look at how the large-field estimate depends on the depth of steps (numerics).
8. Run the quadratic part of the induction uniform in the step through in one line (paper; after 1).
9. Look at the shape of the region where the tension-doubling inequality can hold (numerics).

10. Fix the last arrow on the side of the mass gap as a single route (from the area law to the gap; paper).

06

Stages whose objects are not yet defined

For two of the requirements of the problem statement, objects to write the statements about are needed before the forms of the premises can be discussed (19-07 § 08).

Stage	Statement we want to write	Missing object
Fine observables	Convergence in the continuum limit of expectations of point fields and flowed observables	The lattice gradient flow and the definition of flowed observables (not in this book's Lean). The approximation row (approximation of flowed observables by block variables) also has its form not fixed because this definition is missing
Rotation	Agreement of expectations of rotated observables (invariance under the Euclidean group)	Observables on which rotations finer than the hypercubic group act (the fine-observables stage), or one step from a rotated lattice to the standard lattice (the object of the one-step ultraviolet estimate for the bridge)
Infinite volume	Convergence, on windows, of push-forward measures as the box grows	Can be written by adding one definition (restriction to a window in box coordinates, or periodic extension). In the strong-coupling window it can be written by periodic lifting and is a theorem (20-03)
Non-triviality	That the limit differs from the limit of the free field	Can be written, but by which observable to say it is not fixed (the limit is not a free field)

07

What this book does not claim

- **The mass gap itself.** This book shows nothing about the existence of a mass gap for four-dimensional $SU(2)$ Yang–Mills theory (or for any compact simple group). The lower bound on the mass in the strong-coupling window (20-04) is a theorem at one scale with the lattice spacing fixed.
- **Construction of the continuum limit.** This book does not construct the limiting theory as the lattice spacing goes to 0. The existence of four-dimensional Yang–Mills theory was shown, with a different design, by the construction published on 2026-11-10, and the committee's assessment is that it is proved in Lean (Volume 2). The conclusion of the conditional implication that this book's study assembled earlier (weak ultraviolet existence in small boxes in this book's design) goes only as far as

subsequential limits under its premises and the passage of properties of finite lattices to the limit, and many of those premises have forms not fixed (28-01).

- **The band.** This book shows nothing about the statement that spans the strong-coupling side and the weak-coupling side with one constant (19-07 § 07). The vortex floors (21-02, 21-03) are in a form with the lattice spacing and transverse size fixed. The upper bound on the vortex free energy inside the strong-coupling window (20-06) has a rate that does not depend on the transverse size (the prefactor remains), but it does not extend outside the window.
- **Confinement.** Neither a four-dimensional area law outside the strong-coupling window nor a re-derivation of the passage from the vortex area exponent to the area law of Wilson loops (from the vortex area exponent to the Wilson-loop area law) is in this book. The lattice area law inside the strong-coupling window is in 20-06 (a formalisation of a known type; continuum Wilson loops are in neither this book nor the published construction).
- **Non-triviality and rotation invariance.** For these two items of the problem statement, this book has not even fixed the forms of the statements (for rotation, the object does not yet exist; 19-07 § 08).
- **The position of the theorem in the strong-coupling window.** The lower bound on the mass in the strong-coupling window is an example on the side where the statements to be shown stand trivially, not a check of the content of the statements (19-07 § 01).

An index of "what is not claimed" for the earlier parts is in 27-04.

What this chapter can say

Can say	Cannot say
The 24 rows whose form is not fixed are divided into three by origin (representation level 13; no object or downstream 8; quantification and truth 3)	The form of any row is not fixed here
For each row, whether paper, numerics or design would fix its form, and the smallest question, are written down	Answers to the questions. The order of the questions is by cost and amount downstream, not by the prospect of an answer
For the stages of fine observables and rotation, the objects to write the statements about do not yet exist	Definitions of those objects
This book shows nothing about the mass gap itself, the construction of the continuum limit, or the band	—

Sources and reproduction

Item	Type	Source, tools
The division of the 24 rows, the smallest questions, the order of the questions	Assumption	The inventory of premises in this book's study (2026-11-13)
Examples with finite non-abelian groups where monotonicity fails	Reading	Numerical record of this book's study (direct sums on lattices with finite groups). The description in 26-06
That the naive form of "area law to gap" is false in two dimensions	Proved here	The counterexample of 28-03 § 07 (the direct product of a two-dimensional gauge theory with an area law and the critical two-dimensional Ising model; that the gap closes at the critical point is in 2-09 § 07)
Index of what is not claimed	—	27-04

Words used in this chapter — form not fixed, representation, effective coupling, activity, gas, background field, class, order of quantifiers, gradient flow, flowed observable, what is not claimed. Their meanings are in A-02 Glossary.

Next chapter to read: A-01 Table of notation. To read alongside: A-08 A map to the mass gap — updates to this list and progress on the questions are followed there.