Yang–Mills and the mass gap

# Volume 2 Existence — reading the published construction

Kiichi, Shiori and Rin · Edition of 2026-11-14 · https://computoergosum.com/en/principia/yang-mills/

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| Chapter | | Page |
|---|---|---|
| **Part 6 What has been proved** | | |
| 6-00 | Entrance — the promise of this volume | 7 |
| 6-01 | The four requirements of the problem statement, and the published construction | 16 |
| 6-02 | Reading the statements — the public theorem, the construction theorem, the committee's combined target | 25 |
| 6-03 | The skeleton of the proof (one picture) | 37 |
| 6-04 | What is not shown — a bridge to Volume 3 | 50 |
| **Part 7 Constructive field theory** | | |
| 7-01 | What constructive field theory is | 59 |
| 7-02 | Solvability in two dimensions and the Yang–Mills measure | 75 |
| 7-03 | Constructing scalar fields | 88 |
| 7-04 | The cluster expansion | 101 |
| 7-05 | The sequence of renormalisation group steps | 114 |

| Chapter | | Page |
|---|---|---|
| 7-06 | Ultraviolet stability of three-dimensional Yang–Mills | 126 |
| 7-07 | Four dimensions | 139 |
| 7-08 | Triviality of $\phi^4$ in four dimensions | 151 |
| 7-09 | Stochastic quantisation and regularity structures | 162 |
| 7-10 | Where four dimensions stops | 175 |
| 7-11 | Obstacles other than divergence | 188 |
| 7-12 | Detour: "if 1/0 is given a symbol" | 200 |
| **Part 8 From the lattice to the continuum — objects** | | |
| 8-01 | The lattice, the Wilson measure, conventions | 212 |
| 8-02 | The set-up — the sequence of spacings, the sequence of volumes, growing sides | 221 |
| 8-03 | Labels of local fields and their lattice representatives | 229 |
| 8-04 | Renormalised insertions | 238 |
| 8-05 | Correlation functions and the order of limits | 246 |
| **Part 9 One step of renormalisation** | | |
| 9-01 | The block kernel — one step is a probability kernel, not a map | 254 |
| 9-02 | Patches and weights — centred blocks and convolutions of the uniform distribution | 263 |
| 9-03 | Two kinds of gauge fixing — symmetric minimisation, and a randomly drawn axial gauge | 271 |
| 9-04 | The predictor and projection to the group — a weighted average of transports, onto the group | 279 |
| 9-05 | Symmetries — invariance under the hypercubic group and coarse translations, gauge covariance | 287 |

| Chapter | | Page |
|---|---|---|
| 9-06 | The density representation — the vacuum term, the main action, regular activities, the gas of forced activities | 295 |
| 9-07 | The small-field threshold and the truncated action — a step-dependent threshold | 304 |
| **Part 10 Matching the coupling, and asymptotic freedom** | | |
| 10-01 | What "the coupling at step m" is — a coefficient in the representation | 312 |
| 10-02 | The one-step map and the one-loop computation — the shell integral and the coefficient $11C_g/(24\pi^2)$ | 320 |
| 10-03 | Rereading — reading the coupling from the log-ratio of the density at two configurations | 327 |
| 10-04 | The shooting method — choosing the bare coupling by nested intervals and the intermediate value theorem | 335 |
| 10-05 | Why the sides grow, and the calendar — dividing the one-step error by log b to remove it | 344 |
| 10-06 | The running theorem and the short-distance effective coupling — inside and outside the dependencies | 352 |
| **Part 11 Multistep estimates and large fields** | | |
| 11-00 | Entrance — the induction hypothesis and the one-step certificate | 361 |
| 11-01 | The form of the density at each step, and the induction hypothesis | 369 |
| 11-02 | The finest step — splitting the bare Wilson weight into the representation | 376 |
| 11-03 | Decomposing the one-step action, and regions | 382 |
| 11-04 | Energy per seed | 389 |
| 11-05 | Allocating prices, and polymer sums | 397 |
| 11-06 | Reorganising activities at the coarse step | 403 |
| 11-07 | Converting between ledgers, and persistence across steps | 409 |

| Chapter | | Page |
|---|---|---|
| 11-08 | The calendar — the order in which constants are chosen | 416 |
| 11-09 | The small-field step | 422 |
| 11-10 | Convergence at a fixed step, and the step that removes the step index | 432 |
| 11-11 | What this part has not yet read | 442 |
| **Part 12 Infinite volume and reconstruction** | | |
| 12-01 | Lattice reflection positivity — two kinds of plane | 451 |
| 12-02 | The chessboard, and the root seminorm | 460 |
| 12-03 | The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm | 468 |
| 12-04 | The transfer operator and equicontinuity | 476 |
| 12-05 | Arzelà–Ascoli, and the properties that pass to the limit | 484 |
| 12-06 | Euclidean invariance | 493 |
| 12-07 | Phase decomposition | 502 |
| 12-08 | Reconstruction and the Wightman axioms | 511 |
| **Part 13 Evidence that it is Yang–Mills** | | |
| 13-01 | The stress tensor lies on a single line | 521 |
| 13-02 | The trace anomaly, and not being free | 529 |
| 13-03 | Free Maxwell fields at short distance | 538 |
| 13-04 | The operator product expansion and contact terms | 548 |
| **Part 14 Comparison — this book's Lean, and ideas this book's study lacked** | | |
| 14-01 | Comparing skeletons — a table stage by stage | 555 |
| 14-02 | Comparing one-step designs — the push-forward map and the probability kernel | 566 |

| Chapter | | Page |
|---|---|---|
| 14-03 | Comparing matchings — a quantity fixed by the measure, and a coefficient in the representation | 576 |
| 14-04 | Large fields — where this book's study got stuck, and how the published construction handles them | 586 |
| 14-05 | Ideas this book's study lacked — a list in seven groups | 596 |
| 14-06 | What this book has and the published construction does not — a bridge to Volume 3 | 608 |
| **Part 15 Record of the attempts at ultraviolet control** | | |
| 15-00 | Entrance — what ultraviolet control is | 620 |
| 15-01 | Why the ultraviolet is hard — dimensionless coupling, group-valued fields, gauge | 633 |
| 15-02 | A map of the sources — Bałaban's programme, the BBS chain, two rows absent from the sources | 645 |
| 15-03 | The decomposition statement — a class that closes with finitely many parameters, and three remainders | 655 |
| 15-04 | The one-link heat-kernel class and the hierarchical model | 676 |
| 15-05 | The flux-network theorem — the transverse contraction and the parameter flow in one projection | 693 |
| 15-06 | Counting the parameters — couplings by loop shape and the attracting class | 707 |
| 15-07 | Background-dependent fluctuations η — absorption step by step | 721 |
| 15-08 | The four rows that do not close, and proposals for them | 762 |
| 15-09 | What cannot be said, the index of questions, the dependency diagram | 807 |
| 15-10 | Systematic attempts — naming the recurring patterns of the record | 829 |
| 15-11 | Entrance — one step and one scale | 857 |

| Chapter | | Page |
|---|---|---|
| 15-12 | One step and many steps of the free field — weights (1/8, 3/4, 1/8), the vacuum background, linearisation | 860 |
| 15-13 | Linearisation of the block map and the covariance of the push-forward | 872 |
| 15-14 | What the second half (15-11 to 15-13) says, and what it does not | 881 |

# Entrance — the promise of this volume

> This chapter is a guide.

**What this chapter shows** — What Volume 2 reads (the published Lean code and the committee's assessment). How sources are cited. The meaning of the Volume 2 labels Source Inferred Assessment , and how they relate to the labels of Volumes 1 and 3. The level of certainty "the committee's assessment is that it is proved in Lean", and where the line falls for what this book has checked itself. For each part, how far the reading has got.

**Prerequisite chapters** — 0-03 Reading the problem statement word by word, 4-09 The axioms of quantum field theory, 4-10 The Osterwalder–Schrader axioms and reconstruction. For the historical background of constructions, 7-07 Four dimensions.

**Said first** — This book has neither built nor run the published Lean code. Whether the proof is correct is quoted at the level of the committee's assessment. What this book has checked itself goes only as far as reading the definitions and the form of the statements in the source. The published construction does not show the mass gap (6-04).

Order of this chapter

1. What this volume reads — two primary sources, and the committee
2. Levels of certainty — three levels, and what this book does not do
3. Key to the labels — Source Inferred Assessment , and how they relate to the labels of Volumes 1 and 3
4. How we cite — how locations are written, not copying, names
5. Which target a statement belongs to — five distinctions, and the meaning of "used"
6. How deep the reading goes — for each part, what has been read and what has not
7. The shape of the chapters of Volume 2, and the chapters of this part — the four-section shape, and the five chapters of Part 6

01

## What this volume reads

On 10 November 2026, code written in Lean 4 for a construction establishing the existence of four-dimensional pure Yang–Mills theory was published, together with the assessment of a committee that examined it Assessment . Volume 2 reads these two as primary sources.

| Source | Contents | Version |
|---|---|---|
| Code `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) | The Lean 4 source submitted by OpenAI on 2026-10-28. 9,477 files, about 990,000 lines Source | 3456e5e |
| Assessment `docs/assessment-1.pdf` in `yang-mills-group/YM` | The committee's assessment (35 pages, 2026-11-10). Part I compares the results of the construction with the problem statement; Part II is the record of the checks | 55ba9fc |
| The committee's statement | The third statement on the page of the Kavli Institute for Theoretical Physics (2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov and Nikita Nekrasov | — |

The text of the assessment does not name the committee members; the names come from the signatures on the statement. The verdict at the opening of the assessment is as follows Assessment . The submitted code proves, for every simple compact group, that a relativistic quantum field theory satisfying the Wightman axioms can be built as the scaling limit of four-dimensional Wilson lattice gauge theory. It does not prove that the infinite-volume limit is unique, or that there is a mass gap.

The assessment also says the following about itself.

- Only the Lean source was used for the assessment; a separate manuscript of a paper proof was not used Assessment .
- To help read the Lean code, OpenAI's model Astra was used Assessment .
- The committee does not claim yet to understand the proof fully ("We do not claim to fully understand the proof yet.") Assessment .

This book takes over this position of the committee as it is. When Volume 2 writes "what has been proved", it means "what the committee has assessed as such".

02

## Levels of certainty

Each statement in Volume 2 sits at one of the following three levels. The upper two are on the committee's side; the third is on this book's side.

| Level | Contents | Who checked | What this level does not guarantee |
|---|---|---|---|
| Accepted by the Lean kernel | Rebuilding, auditing the axioms, re-checking in the kernel, and matching statements and definitions with Comparator all passed (assessment §9.1, §9.2) | The committee Assessment | That the definitions express the intent of the problem |
| The statements express the intent of the problem | A human reading of the definitions (the assessment describes it as "extensive, but not exhaustive") and a comparison with independently written axioms (§9.3, §10, §11) | The committee Assessment | That a human has fully understood the proof |
| The form of the statements and definitions has been read | The source files were opened, and the names, lines, order of quantifiers and hypotheses of the declarations were read | This book Source | That the content of the proof is correct |

We state here, once and plainly, what this book **does not do**.

- It has not built the published Lean code. Nor has it run it.
- It has not repeated the re-checking in the kernel or the Comparator check on its own machine.
- Of the content of the proof (the sequences of tactics and the proofs of lemmas), it has read only part of the outline. Where the reading falls short is written in the last section of each chapter.

Volume 2 therefore writes every result of the published construction at the level "the committee's assessment is that it is proved in Lean". What this book can say goes only as far as: the statement is written in the source, at that location, in that form.

03

## Key to the labels

The labels specific to Volume 2 are the following three. Each sentence carries one label.

| Label | Meaning | What accompanies it | What it does not guarantee |
|---|---|---|---|
| Source | The source of the published construction (the wording of definitions and statements), or the source on the assessment side (the code and examples of the committee's checks), has been read | The relative path of the file, the name of the declaration, the line | That the proof is correct |
| Inferred | This book's inference (from names, explanatory text, the outline of the proof) | What it was inferred from | That the source says so |
| Assessment | Based on the committee's assessment | The section of the assessment | That this book has checked it |

A Source that needs a qualification carries it in brackets, as in "Source (within the scope of a name search)". When we write "not in the source", we add the words searched for and the scope.

We note how these relate to the labels of Volume 1 (Parts 0–5) and Volume 3 (Parts 16–28).

- In the text of Volume 2 too, the labels of the whole book are used, with the same meaning as in each volume: the Volume 3 labels Proved here Cited Computer-assisted Reading Assumption and Lean , and in parts of Parts 7 and 15 the Volume 1 labels ( Lean , Paper , Computed , Known , Physics ). The meanings of the labels are in "Labels" in 0-01.
- Cited is used only for known results checked in the literature (the Osterwalder–Schrader reconstruction, the Arzelà–Ascoli theorem and the like). Sentences about the theorems and definitions of the published construction carry Source , and sentences touching the level "it has been proved" carry Assessment . Cited and Source are not put side by side on one sentence.
- Reading is used only in the sense of "reading or observing numbers" (floating point, Monte Carlo, fits); having read the source is expressed by Source .
- In the "type" column of the source tables, bibliographic items such as the problem statement and the statement of the committee are marked "Cited".
- When the section "What this book's investigation had seen" of Volume 2 quotes this book's Lean theorems, it writes them as Volume 3 does (the theorem's name and its scope on the same line).

04

## How we cite

There are two kinds of source.

- **Code**: the relative path of the file, the name of the declaration, the line. Example: `JointExistence` in `ChallengeSpec.lean` (line 186). Unless stated otherwise, paths are relative to the root of the repository `OpenAI-YM-20261028`. The code of the committee's checks is written relative to `submissions/openai-20261028/` in the repository `YM`.
- **Assessment**: the section number. Example: assessment § 5. The glossary in the appendix is "assessment Appendix A".

We do not copy long passages of the published construction's text or code. We write the mathematics in our own words. Lean statements are shown as they are only where the form itself matters, and then for a few lines at most (such as the public theorem in 6-02).

The conventions for names are as follows.

- Objects of the published construction are called by **names that say what they mean** (the probability kernel, the predictor, the truncated action, forced activities, the induction hypothesis, and so on). The identifiers are added in monospace as sources.
- Many of the locations and declarations in the code have names unconnected with the mathematical content. The assessment also writes that the naming departs from convention ( § 9.1) `Assessment` . Such names are written only as locations, and are not used as names in the running text.
- Line numbers refer to the versions above.

05

## Which target a statement belongs to

The published code contains several "main theorems" of different strength. Which one a statement belongs to changes how strongly it may be quoted. Volume 2 distinguishes the following five.

| Distinction | Contents | Location |
|---|---|---|
| The public theorem | The statement in the form of the published question. Its only hypothesis is "the Lie algebra of the group is simple" | `JointExistence` in `ChallengeSpec.lean`, proved by `jointExistence` in `Solution.lean` |
| The construction theorem | A stronger statement that contains the public theorem as a projection | `constructed` in `Construction.lean` |

| Distinction | Contents | Location |
|---|---|---|
| The committee's combined target | A statement in which the committee bundles, over a single family, its independent checks on top of the construction theorem | `combined_acceptance` in `submissions/openai-20261028/checks/CombinedAcceptance.lean` of `YM` |
| Reached from one of the headline theorems | In the dependencies of one of the 17 headline theorems listed by the verification record (the status in the table is headline for the headline theorems themselves, used for dependencies) | `verification/joint-headlines.txt`, `declaration-usage.tsv` |
| Outside the dependencies | Not reached from any of the 17 headlines | Status outside in the same table |

One caution is needed. The status in the table of the verification record, `verification/declaration-usage.tsv`, takes three values: headline (the 17 headlines themselves), used (100,151) and outside (150,151) Source . "Used" means reached from **one** of the 17 headline theorems (`verification/Usage.lean`, lines 17–40) Source . Besides the public theorem and the construction theorem, the 17 include headlines on other subjects Source . So "used" does not necessarily mean "used by the public theorem".

In Volume 2 we write "used in the proof of the main theorem" only for declarations whose chain of dependencies we have checked link by link up to the public theorem or the construction theorem. What has not been checked is written as "not checked". As an example, take the short-distance effective coupling (Theorem 7.2 of assessment § 7.4). The assessment does not name its location in Lean, and the source contains three things of a similar form Source .

- `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean:145`): a statement in the form of a ratio; status outside (outside the dependencies).
- `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean:50`): status used. The first headline is `continuum_with_position_tests`, the direct user is `crest_ready`, and it is used as an intermediate step in the analysis of the trace. This is the one whose wording is close to the statement of Theorem 7.2 Inferred . Its conclusion does not survive into the conclusion of the public theorem, the construction theorem or the committee's combined target Source .
- `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean:36`): a headline theorem itself (headline). It is the running of the coupling $\beta(N_i, k_i, i)^{-1/2}$ at step $i$ with a cutoff, a different quantity from $gW$.

This is treated in detail in 6-02 and 13-03.

06

## How deep the reading goes

For each part, we list the range read in the source and the main places not yet read. The "What has not been read" section at the end of each chapter gives this in detail.

| Part | What has been read Source | Main places not yet read |
|---|---|---|
| Part 6 What has been proved | The full text of the public theorem, the statement of the construction theorem, the chain of main theorems, the statement of the committee's combined target, the summary of the verification record | The proofs in each file of the committee's independent checks |
| Part 8 From the lattice to the continuum — objects | The definitions of the Wilson family, of correlations and of the order of limits; the definitions of field labels and quotients | The content of the inverse of the renormalisation transport (how it mixes with the identity field) |
| Part 9 One step of renormalisation | The probability kernel, patches and weights, the predictor, the branching of gauge fixing, the statements of symmetry | The details of the proof of the averaging that makes a symmetric gauge fixing |
| Part 10 Matching the coupling, and asymptotic freedom | The calendar, the shooting band, the statement of the difference of couplings in one step | The explicit form of constants satisfying the conditions of the calendar |
| Part 11 Multistep estimates and large fields | The form of the induction hypothesis and of the one-step certificate, the definitions of large-field regions and seeds, the statements of the one-step inequalities | The core of the proof of the one-step theorem, the reorganisation of activities after one step, the lemma on the Haar loss |
| Part 12 Infinite volume and reconstruction | Reflection positivity, the chessboard inequality, the input from the ultraviolet, the Arzelà–Ascoli extraction, the phase decomposition, the statement of reconstruction | The body of the estimates on the transfer operator, the outline of the proof that the algebra of phases is commutative |
| Part 13 Evidence that it is Yang–Mills | The statements of the stress tensor, the Ward identities, the trace law and the operator product expansion | The content of some fields of the strong forms concerning the trace, the location of the second coupling |

On the whole, what this book has been able to read goes as far as "what is stated, where, and in what form" and "in what order it is assembled". Of the multistep estimates (Part 11), which take up about 70% of the code, only the form of the statements and the outline of some lemmas have been read Inferred .

07

## The shape of the chapters of Volume 2, and the chapters of this part

The chapters of Parts 8–13 have the following four sections, in this order.

1. **Statements** — what the published construction defines and states on the subject, with sources.
2. **Outline of the mechanism** — in what order, and using what, it is assembled. With formulas.
3. **What this book's investigation had seen** — how far this book's research and Lean had got on the subject. If nothing, "nothing".
4. **What has not been read** — what has not been read and what has not been checked, with locations.

At the end of each chapter, as in Volumes 1 and 3, there are the tables "What this chapter can claim" and "Sources and reproduction". Part 14 is the part of comparison, and Part 15 is the record of this book's attempts at the ultraviolet.

This part (Part 6) is the part of the entrance and the overview. It does not follow the four-section shape as it is, but each chapter has the sections "What this book's investigation had seen" and "What has not been read".

| Chapter | In one line |
|---|---|
| 6-01 The four requirements of the problem statement, and the published construction | Of the four — axioms, local fields, short distances, mass gap — what has been reached and what has not. |
| 6-02 Reading the statements | The difference in scope between the public theorem, the construction theorem and the committee's combined target. Who chooses what. What the verification record has checked. |
| 6-03 The skeleton of the proof | The ten stages from the calendar to the evidence, in one picture and one table. A guide to the chapters of Volume 2 that treat each stage, and the size of each layer of the code. |
| 6-04 What is not shown | Convergence and uniqueness, a probability measure, the mass gap, Wilson loops, one-particle states and more. A bridge to Volume 3. |

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## What this chapter can claim

| Said | Not said |
|---|---|
| Assessment The committee judged that the submitted code proves the construction of a theory satisfying the Wightman axioms (opening of the assessment) | The committee itself does not say that it fully understands the proof. Uniqueness and the mass gap have not been shown |
| Source The statements Volume 2 quotes are written in the source, at those locations, in that form | The correctness of the content of the proofs (this book has neither built nor run the code) |

| Said | Not said |
|---|---|
| The conventions (labels, levels, citation, distinctions of targets) | — (these are conventions, not mathematical statements) |

## Sources and reproduction

| Item | Type | Source, tool |
|---|---|---|
| The published code | Source | yang-mills-group/OpenAI-YM-20261028 (Apache-2.0, submitted 2026-10-28, version 3456e5e). The numbers of files and lines were counted on this machine |
| The assessment | Assessment | docs/assessment-1.pdf in yang-mills-group/YM (2026-11-10, 35 pages, version 55ba9fc). Opening, § 9, § 12. The committee members are not named in the text |
| The committee's statement | Cited | The third statement on the page of the Kavli Institute for Theoretical Physics (2026-11-09). Signed by Michael R. Douglas, David J. Gross (chair), Arthur Jaffe, Igor R. Klebanov and Nikita Nekrasov |
| The meaning of "used" and the three values of status | Source | `verification/Usage.lean`, lines 17–40; `verification/joint-headlines.txt` (17 lines); `verification/declaration-usage.tsv` |
| The three locations of the short-distance effective coupling | Source | `YM/PertExtension/Running.lean:145`, `YM/TraceExtension/Atlas/Sink.lean:50`, `YM/Pert/Asymptotics/RawAxis.lean:36` |
| The problem statement | Cited | The problem statement of Jaffe–Witten. For how to read it, 0-03 |

**Terms used in this chapter** — the assessment, the committee's combined target, the public theorem, the construction theorem, headline theorems, outside the dependencies, Comparator, re-checking in the kernel. Their meanings are in A-02 Glossary.

Next chapter to read: 6-01 The four requirements of the problem statement, and the published construction — the four things the problem statement asks for, and where the published construction has got to, side by side in one table.

2026-11-13 · chapter Yang–Mills and the mass gap Part 6 What has been proved problem statement and construction · Kiichi, Shiori and Rin

# The four requirements of the problem statement, and the published construction

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's investigation.

**What this chapter shows** — The four requirements that the assessment extracts from the problem statement. How far the published construction reaches on each of them (what is reached, what is not, the location in Lean, the section of the assessment). The standard of "existence" the committee adopted, and the reservations the committee itself writes. Where among the four the results of Volume 3 of this book lie.

**Prerequisite chapters** — 6-00 Entrance (labels and levels of certainty), 0-03 Reading the problem statement word by word, 4-09 The axioms of quantum field theory, 4-06 Asymptotic freedom.

**Said first** — The wording of the four requirements is the summary in § 1 of the assessment, not the original text of the problem statement itself. The published construction claims nothing about the fourth, the mass gap. Every "reached" in this chapter is at the level "the committee's assessment is that it is proved in Lean".

Order of this chapter

1. The four requirements of the problem statement — the summary in § 1 of the assessment
2. What the published construction gives — the verdict at the opening of the assessment, and § 2
3. Requirement by requirement — what is reached, what is not, locations (Table 1)
4. The committee's standard of "existence", and its reservations — subsequences, the vacuum, the degree of understanding
5. What this book's investigation had seen — where among the four this book's results lie (Table 2)

6. What has not been read — comparison with the original text, statements whose location is unknown

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Labels: Source = the source of the published construction has been read / Inferred = this book's inference / Assessment = based on the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) have the same meaning as in the rest of the book (for details, "Labels" in 6-00).

01

## The four requirements of the problem statement

The introduction is 0-03. Assessment § 1 summarises what the Jaffe–Witten problem statement asks of a constructed Yang–Mills theory as the following four Assessment .

1. **A non-trivial theory satisfying the axioms.** Positivity, spacetime symmetry, locality, regularity and the representation of the vacuum hold at the strength of the Wightman or Osterwalder–Schrader (OS) framework, with genuine local observables.
2. **The local fields of Yang–Mills.** The space of operators corresponds to the gauge-invariant polynomials in the curvature $F$ and its covariant derivatives.
3. **The predicted short-distance behaviour.** Asymptotic freedom and the theory of perturbative renormalisation, the stress tensor, and an operator product expansion with the predicted singularities.
4. **A positive mass gap.** For the non-negative Hamiltonian $H$, the following holds for some $\Delta > 0$.

$$\sigma(H) \subset \{0\} \cup [\Delta, \infty). \quad (1)$$

For the fourth, the assessment adds "that the supremum of such gaps is finite" Assessment . The $\Delta$ of (1) is the mass gap. It means that the energies of states orthogonal to the zero-energy space are separated by at least $\Delta$. That the zero-energy space consists only of multiples of the vacuum (the uniqueness of the vacuum) is an axiom separate from (1). In Euclidean language it corresponds to the exponential decay of correlations (4-10).

02

## What the published construction gives

We write the verdict at the opening of the assessment once more [Assessment] . The submitted Lean code proves, for every simple compact group, that a relativistic quantum field theory satisfying the Wightman axioms can be built as the scaling limit of four-dimensional Wilson lattice gauge theory. It does not prove that the infinite-volume limit is unique, or that there is a mass gap.

Assessment § 2 divides what the construction gives into four [Assessment] .

- Starting from Wilson's lattice gauge theory, it associates renormalised lattice observables with polynomials in the curvature, and shows that their correlations converge to a limit that defines a Euclidean field theory. From there, by OS reconstruction, it builds a theory satisfying the construction's own version of the Wightman axioms. It also shows that the theory is non-trivial.
- It defines renormalised local operators corresponding to polynomials in the curvature, including a conserved stress tensor. There are several short-distance checks.
- It gives an operator product expansion with error bounds, but does not show agreement with the predictions of the perturbative expansion. It gives the scaling behaviour of the coupling needed for asymptotic freedom. It defines the trace of the stress tensor and shows its properties.
- **On the mass gap, it gives no evidence, either for or against.**

These four correspond almost one to one with the four requirements of § 01. In the next section we list the locations in the source requirement by requirement.

03

## Requirement by requirement

What the verdict at the opening of the assessment says is "proved" is the construction of a theory satisfying the Wightman axioms, which corresponds to requirement 1 [Assessment] . § 7 states the individual short-distance results (the short-distance law of the trace, the effective coupling, the ratio of two couplings, the comparison with the free Maxwell field) as theorems the submission proves ("The submission proves that ...", § 7.2, § 7.4) [Assessment] . The assessment then calls them together "substantial short-distance evidence at the level of the leading terms", and, together with the lattice origin, the local fields, the stress tensor and the operator product expansion, substantial evidence that the construction describes Yang–Mills (end of § 7.4, opening) [Assessment] . For requirements 2 and 3, the assessment lists the definition of the local operators and "several short-distance checks", calls them together

"substantial evidence that it is Yang–Mills", and states that agreement with perturbation theory at all orders has not been proved (§ 2, § 7, § 9.4) Assessment. Please read rows 2 and 3 of Table 1 in this light. The "Location" column of Table 1 gives the fields in the statement of the public theorem (`ChallengeSpec.lean`). The differences between the public theorem, the construction theorem and the committee's combined target are treated in 6-02.

Table 1 The four requirements of the problem statement, and the published construction (labels are per clain row)

| Requirement | What the construction gives | What is not reached | Location, assessme |
|---|---|---|---|
| 1 Axioms and non-triviality | Assessment The infinite-volume theory is decomposed into phases, and each component (phase) satisfies a variant of the OS axioms including clustering. A Wightman theory is reconstructed from each component. In a chosen phase the two-point function of the trace is not 0 | Assessment The volume limit is along a subsequence, and uniqueness has not been shown. The uniqueness of the vacuum of the theory before decomposition has not been shown. Assessment What the committee compared with independently written OS axioms is only the scalar hierarchy of the trace in the chosen phase (§ 10.1) | `Hierarchy.Reconstr` (line 117), `Properties.reconst` `nonfree` (lines 177- Assessment § 5, § ( |
| 2 The local fields of Yang–Mills | Assessment It takes as labels the space $V_d$ of gauge-invariant polynomials in the curvature of weight at most $d$, and takes as the space of fields $F_d = V_d/N_d$, the quotient by the part $N_d$ whose correlations at separated points all vanish. Assessment $F_d$ is linearly isomorphic to the space of polynomials of the free Maxwell theory | Source In the public statement the type of labels `Token` is an abstract type, and its meaning as polynomials in the curvature is not written (it is on the side of `C.Makes` in the construction theorem). Source An example among the committee's checks writes that a linear isomorphism is not an agreement of correlations (lines 84–86 of the same file). Source An example among the committee's checks writes that the labels are polynomials at a point, and that completeness is not shown for non-local observables such as line defects (`submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean`, lines 35–38) | `Hierarchy.Field` (l `Token` in `PublicWilson.lean` 44). Assessment § 3 § 9.4 |

| Requirement | What the construction gives | What is not reached | Location, assessme |
|---|---|---|---|
| 3 Short-distance behaviour | Source The bare coupling approaches a positive slope in $\beta_{N,p}/\log(1/aN)$ (uniformly in the volume). Assessment The stress tensor is a point on a one-dimensional space of conserved symmetric tensors, normalised by the Ward identity. Assessment The short-distance law (2) for the two-point function of the trace. Assessment An operator product expansion with error bounds. Assessment Along one sequence of scales, the correlations of the stress tensor approach those of the free Maxwell field | Assessment Agreement with perturbation theory at all orders has not been shown. Assessment The coefficients of the operator product expansion are not compared with perturbative coefficients. Source In the public statement, the trace field is an arbitrary element of $F_4$, and it is not written that it is the trace of the stress tensor. Source There are three declarations of a form close to the short-distance effective coupling (assessment § 7.4), and none survives into the conclusions of the three targets: `Running.ratio` is outside the dependencies, `wilson_link` is an intermediate step inside the dependencies (its wording is close to Theorem 7.2 Inferred ), and the headline `Asymptotics.comparison` is the coupling at a step with a cutoff, a different quantity from $g_W$ (6-00 § 05) | `WilsonRegularity.r` (lines 13–14), `Trace` (lines 168–175), `OperatorProductExp` Assessment § 7 |
| 4 The mass gap | — | Assessment Not claimed. Source What there is are polynomial bounds uniform in each phase (the spectral weight at low energy is at most $Mu^2$, the heat bound at most $M\tau^{-2}$), not exponential decay | `light_bound` in `YM/EveryPhase/Ligh` (line 221), `inverse_` in `YM/EveryPhase/Deca` (line 173). Assessm § 8 |

The short-distance law in the third row of Table 1 has the following form for the two-point function of the centred trace $\Theta$ in the chosen phase $\xi_*$ Source (`ChallengeSpec.lean`, lines 173–174; formula (16) of assessment § 7.2).

$$\lim_{r \downarrow 0} \; r^{8} \, \log^{2}(1/r) \; S^{\xi_*}_{\Theta,\Theta}\big(-\tfrac{r}{2}e_0,\; \tfrac{r}{2}e_0\big) = \frac{3 \, \dim_{\mathbb{R}} \mathfrak{g}}{\pi^{4}}. \qquad (2)$$

$r^{-8}$ is the naive power of the two-point function of a field of dimension 4. The $\log^{-2}$ is a trace of the fact that the trace has a coefficient that becomes small through the running of the coupling Inferred . Because this logarithm is a form that does not arise from polynomials in free fields of finitely many masses, "not free" is concluded (13-02).

The polynomial bounds in the fourth row are of a general form that follows from local commutativity and the forward light cone Inferred . They are compatible with a massless theory too, so they do not replace (1).

```
04
```

### The committee's standard of "existence", and its reservations

The committee writes for itself at what strength it accepted "existence".

- **Subsequences.** The infinite-volume limit is taken only along a subsequence obtained by compactness (Arzelà–Ascoli) Assessment . Assessment § 5 writes that the Jaffe–Witten problem statement regards "a subsequential limit obtained by compactness as insufficient without a separate argument showing properties of the limit", and then states that the reconstruction and the theorems on local fields give information beyond compactness Assessment .
- **The vacuum.** The assessment writes that the standard of existence adopted here "does not require a precise characterisation of the vacuum state" Assessment .
- **Missing uniqueness.** The committee writes that the fact that uniqueness of the infinite-volume limit and of the vacuum has not been shown does not contradict the general conjecture that they are unique Assessment . A conjecture is a conjecture, not something shown.
- **Degree of understanding.** The committee does not claim yet to understand the proof fully. What was used for the assessment is the Lean source; the manuscript of the paper proof was not used Assessment .

In addition, the public theorem (`JointExistence`) originally did not include part of the connection with the lattice in its statement. Assessment § 9 writes that the committee was informed of this by OpenAI, and that the committee's verdict rests on a stronger combined target Assessment . This difference is read in 6-02 § 05.

In sum, what the opening verdict says is "proved" is the construction of requirement 1, at a strength that includes subsequences and the choice of a phase. Requirements 2 and 3 are present in partial form: labels and quotients of local fields, and short-distance laws at the level of the leading terms. The individual short-distance results are stated as theorems of the submission, and the assessment calls them together evidence "that it is Yang–Mills". Requirement 4 is, before any question of reaching it, outside what is claimed.

05

## What this book's investigation had seen

In Volume 3, this book has statements corresponding to some of the four requirements, **on a single lattice with the lattice spacing held fixed**. It shows nothing about the continuum theory. Table 2 places them in the same rows as Table 1.

Table 2 The four requirements, and where the results of Volume 3 of this book lie (all statements on this

book's side are at fixed lattice spacing)

| Requirement | The published construction | Volume 3 of this book |
|---|---|---|
| 1 Axioms | A Wightman theory for each phase, in the continuum limit | On the lattice inside the strong-coupling window: the infinite-volume limit without subsequences, reflection positivity, reconstruction (20-03, 20-04). No continuum limit |
| 2 Local fields | The space of labels and the quotient $Fd$ | None. This book's observables are plaquette averages, Wilson loops and ratios of twisted partition functions |
| 3 Short distances | Running of the coupling, stress tensor, trace, operator product expansion | Components at the one-loop level, and a theorem at one ultraviolet scale (latter half of Part 15). No control over all scales |
| 4 The mass gap | Not claimed | Only on the lattice inside the strong-coupling window, the gap of the transfer operator (20-04). The vortex floor (21-03) is not a statement of the mass gap |

Of what this book has listed as "statements to be shown" (19-07), the part corresponding to existence on the ultraviolet side is answered by the published construction at the level of the committee's assessment. The side of uniform decay and the mass gap remains open on both sides (6-04).

06

## What has not been read

- The four of § 01 are the assessment's summary. This chapter has not matched them one by one against the sentences of the original Jaffe–Witten text (for a reading of the original, 0-03).
- We have not found the Lean statement corresponding to "the theory of perturbative renormalisation" in requirement 3. Nor have we checked the location of the second coupling of assessment § 7.4 (the one measured by the tensor–scalar correlation). We infer that it is near `YM/Pert/Position/Four.lean` Inferred .
- We have read the fields of the reconstruction statement (`Hierarchy.Reconstructed`) as a list, but have not matched each field one by one against the axioms of 4-09. The committee's independent comparison (assessment § 10) concerns the scalar hierarchy of the trace Assessment .
- We have not read the proofs of the polynomial bounds (`light_bound`, `inverse_square`). That they come from local commutativity is inferred from the names of the lemmas in the proofs Inferred .

## What this chapter can claim

| Said | Not said |
|---|---|
| Assessment The committee's assessment is that requirement 1 (the construction of a theory satisfying the Wightman axioms) is proved in Lean, in a form that includes subsequences and the choice of a phase | Uniqueness of the infinite-volume limit, uniqueness of the vacuum of the theory before decomposition |
| Assessment Requirements 2 and 3 are present as labels and quotients of local fields and as short-distance laws at the level of the leading terms. The individual short-distance results are stated as theorems of the submission, and the assessment calls them together substantial evidence "that it is Yang–Mills" | Agreement with perturbation theory at all orders, a complete identification of the local fields |
| Source Statements of the form in Table 1 are at the locations in Table 1 | The correctness of the proofs (not read) |
| Assessment Requirement 4 (the mass gap) is not claimed | — |
| This book's results in Table 2 are statements at fixed lattice spacing | This book shows nothing about the continuum limit, weak coupling or the substance of the mass gap |

## Sources and reproduction

| Item | Type | Source, tool |
|---|---|---|
| The four requirements | Assessment | Assessment § 1. The original is the Jaffe–Witten problem statement (reference [JW] of the assessment). For how to read it, 0-03 |
| The verdict, and what the construction gives | Assessment | Assessment opening, § 2 |
| The fields of the public statement | Source | OpenAI-YM-20261028: `ChallengeSpec.lean`, lines 12–48 and 117–181; `PublicWilson.lean`, lines 35–46 |
| Polynomial bounds | Source | `YM/EveryPhase/Light.lean`, line 221; `YM/EveryPhase/Decay.lean`, line 173 |
| Subsequences, the vacuum, degree of understanding, the manuscript | Assessment | Assessment opening, § 5, § 9 |
| The three locations of the short-distance effective coupling | Source | `YM/PertExtension/Running.lean:145` (outside), `YM/TraceExtension/Atlas/Sink.lean:50` (used), `YM/Pert/Asymptotics/RawAxis.lean:36` (headline). `verification/declaration-usage.tsv` |
| This book's results | — | The source tables of each chapter of Parts 20 and 21 |

**Terms used in this chapter** — mass gap, Wightman axioms, OS axioms, phase decomposition, field labels, trace anomaly, operator product expansion, subsequential limit. Their meanings are in A-02 Glossary.

Next chapter to read: 6-02 Reading the statements — reading in full the public theorem to which the "Location" column of Table 1 belongs, and setting out how it differs from the construction theorem and the committee's combined target.

# Reading the statements — the public theorem, the construction theorem, the committee's combined target

> This chapter is existing knowledge (a reading of the published construction). § 08 is this book's own study.

**What this chapter shows** — The full text of the public theorem and its logical form. What sits inside the existential quantifiers (what the reader chooses, what the prover chooses, and the conditions the statement imposes). What the construction theorem adds to the public theorem. What the combined target the committee wrote itself adds. How the three scopes differ (Table 3). What the verification record checked and what it did not.

**Prerequisite chapters** — 6-00 Entrance ( § 05, distinguishing the targets), 6-01. For how to read Lean, see A-04.

**Said first** — What this chapter reads is the form of the statements, not the content of the proofs. This book has not repeated the verification on its own machines. All results of the verification are taken from the assessment report and the copies of the logs Assessment .

The order of this chapter

1. The public theorem — the full text and its logical form
2. What sits inside the existential quantifiers — who chooses what (Table 1)
3. The four conclusions of the public theorem — wilson, reconstruction, ope, nonfree
4. The construction theorem — the penalty coefficient, the family certificate, refining the order and the bundle of conclusions
5. The committee's combined target — checks on one family and one witness (Table 2)
6. How the three scopes differ — item by item (Table 3)
7. What the verification record checked, and what it did not — rebuild, kernel, axioms, Comparator (Table 4)

8. What this book's study had seen — a comparison with the form of this book's statements
9. What has not been read

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment report. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (see "Labels" in 6-00). Unless stated otherwise, paths are relative to the root of the repository `OpenAI-YM-20261028`.

01

## The public theorem

The statement in the form of the public question is at lines 186–188 of `ChallengeSpec.lean` Source. Because the form matters, here are the three lines verbatim.

```
def JointExistence : Prop := ∀ g : GaugeGroup, g.IsSimple → ∃ C : WilsonFamily g,
  WilsonRegularity C ∧ ∀ p : ℕ → ℕ, StrictMono p → ∃ q : ℕ → ℕ, StrictMono q ∧
    ∃ T : Hierarchy C, T.Properties (p ∘ q)
```

In mathematical language this reads as follows.

For every closed subgroup of matrices $G \subset U(m)$ whose real Lie algebra $\mathfrak{g}$ is simple, there is a Wilson family $C$ such that the following holds. $C$ satisfies the four conditions of regularity, and however a strictly increasing sequence $p$ of volume indices is given, there are a refinement $q$ of it (strictly increasing) and a package $T$ of continuum theory such that $T$ has four properties along the refined sequence $p \circ q$.

The proof is `jointExistence` in `Solution.lean` (4 lines); it calls the construction theorem with penalty coefficient 1 and merely extracts part of its conclusion Source.

"Simple" is defined by `IsSimple` in `PublicTheory.lean` (line 39): the Lie algebra is not commutative (there is a pair of elements whose commutator is not 0), and it has no ideals other than 0 and the whole Source. This is the only hypothesis of the public theorem Source.

02

## What sits inside the existential quantifiers

The most important thing in reading the public theorem is that the content of each ∃ is **chosen by the prover**. What the statement imposes is only the conditions the chosen objects must satisfy. Table 1 separates the reader's side (∀) from the prover's side (∃).

Table 1 What sits inside the quantifiers of the public theorem (located in `PublicWilson.lean` and `ChallengeSpec.lean`)

| Object | Chosen by | Content | What the statement imposes |
|---|---|---|---|
| Group $G$ | Reader (∀) | Closed subgroup of matrices | Hypothesis: the Lie algebra is simple |
| Wilson family $C$ | Prover (∃) | Side of the volume $L_p$, denominator $D_N$ (spacing $a_N = 1/D_N$), bare coupling $\beta_{N,p}$, measure on the group, type of labels, lattice insertion for each label (`WilsonFamily`, lines 35–46) | $L_p$ and $D_N$ strictly increasing. The measure is a probability measure and is Haar measure. The four conditions of regularity (below) |
| Sequence of volumes $p$ | Reader (∀) | Strictly increasing sequence | — |
| Refinement $q$ | Prover (∃) | Strictly increasing sequence | — |
| Package of continuum theory $T$ | Prover (∃) | Hilbert space and Wightman theory, spaces of fields by degree, map from labels to fields, space of phases and a measure on it, the chosen phase, correlations, analytic continuation, coefficients of the operator product expansion, the centred-trace field (`Hierarchy`, lines 24–48) | Four properties `Properties` (lines 177–181) |

The four conditions of regularity `WilsonRegularity` (lines 12–22) are as follows `Source` .

- **Running**: there is a positive number $s$ (the slope) with $\beta_{N,p}/\log(1/a_N) \to s$. The convergence is uniform in the volume $p$.
- **Continuity**: each insertion is continuous in the link configuration.
- **Gauge invariance**: each insertion is unchanged by lattice gauge transformations.
- **Locality**: for each label there is a finite stencil (a finite set of displacements in lattice units), and the insertion depends only on the links inside that stencil. The stencil does not depend on the spacing, the volume or the position.

Three things can be read from Table 1.

- The statement asks only that a positive slope $s$ exists; no value is in it Source . The value the construction satisfies is $s = 11C_{\mathfrak{g}}/(24\pi^2)$ (assessment report, Appendix A) Assessment .
- The type of labels is an abstract type, and which polynomial in the curvature a label represents is not in the statement Source . The centred-trace field is an arbitrary element of the space of fields of degree 4 (line 48) Source . The only thing that constrains the trace field is the condition in the fourth property of § 03.
- The sequence of spacings $a_N$ is also chosen by the family Source . So universality in the sense "it converges for every sequence of spacings" is outside the statement (assessment report § 4.3) Assessment .

03

## The four conclusions of the public theorem

We read the four fields of `Properties` for their meaning Source .

1. **wilson** (line 178): the link to the lattice. The lattice correlation $C_{N,p}(v;t)$, smeared with a test function $t$ on configurations of mutually distinct points, has a limit as $N \to \infty$ for each volume $p$ in which it fits, and that limit approaches $\int t\, S^{\mathrm{mix}}$ along the refined sequence of volumes ( `Converges` in `PublicWilson.lean`, lines 85–90). The continuum limit runs along the whole sequence of spacings chosen by the family. It is the volume limit that is taken along a subsequence.
2. **reconstruction** (line 179): the Wightman axioms, faithfulness of the fields, the phase measure being a probability measure with full support, permutation symmetry and smoothness of the correlations, continuity in the phase, the average over phases being equal to the mixed correlation, analytic continuation and boundary values ( `Hierarchy.Reconstructed` , lines 117–143). The axioms, including uniqueness of the vacuum, and the accompanying fields (faithfulness, analytic continuation, boundary values) are stated for the theory of the chosen phase. The Wightman axioms in every phase are on the side of the construction theorem ( § 04).
3. **ope** (line 180): the coefficients of the operator product expansion, and a power-law upper bound on the remainder after truncating by degree. The bound on the remainder is uniform in the state (13-04).
4. **nonfree** (line 181): the two-point function of the centred-trace field satisfies, in the chosen phase, the short-distance law of equation (2) in 6-01, and agrees near 0 with no finite sum of Wick forms of finitely many mass shells (lines 168–175). One of these fields, `amplitudePositive` , is an inequality between constants that does not involve the theory $T$.

All correlations are defined only where the points are mutually distinct Source . Where points coincide is outside the public theorem (6-04).

04

## The construction theorem

The construction theorem is `constructed` in `Construction.lean` (line 15) Source . Its form is as follows.

For each simple $G$ and each positive **penalty coefficient** $c_{\mathrm{pen}}$ (in the source's wording `kernel.penalty`: the coefficient of the weight in the one-step probability kernel that pulls the coarse links towards the predictor, 9-01), there is a family $C$ such that the following hold. The penalty coefficient of the kernel equals $c_{\mathrm{pen}}$. The family certificate `C.Makes` holds. The four conditions of regularity hold. Moreover, however a pair of subsequences of spacings and volumes (the order `Extraction.Order`) is given, there are a refinement of it, together with a table of correlations, a final witness, a bundle of reconstructions in every phase (`EveryPhase.Almanac`), a reader of the operator product expansion, contact apparatus and a sequence of scales, such that a dozen or so conclusions hold at once Source .

The main things added to the public theorem are these four.

- **The family certificate** `C.Makes` (`YM/Source/Construction.lean` line 44): the renormalisation relation at every step (`path.Selected`), soundness of the local sections, the label stencils fitting in a box, and the insertion coefficients being exactly the label coefficients when read at the reference step (`calibrated`) Source . The meaning of a label as a polynomial in the curvature, and its renormalisation, are fixed here (assessment report § 9.4) Assessment .
- **The strong form of the ultraviolet limit** `FullVolume.PositionsUV C Extraction.Sequence.full`: for each volume there is a kernel, smooth where points do not coincide, and the correlations approach it locally uniformly, with differences, along the whole sequence of spacings Source . The wilson field of the public theorem is only convergence of smeared values; this form is only in the conclusion of the construction theorem Source .
- **Identification of the trace**: the centred-trace field is the trace of the stress tensor minus its vacuum expectation value (`HierarchyBridge.lean` line 42, `identified` at lines 87–91 of `YM/Nonperturbative/HeadlineSpec.lean`) Source .
- **Every phase**: the Wightman axioms not only in the chosen phase but in every phase Source .
- **Comparison with the free Maxwell field**: the correlations of the mixed theory (the average over phases) approach those of the free Maxwell field along one sequence of

scales (`Pert.FirstExtension`). The source's docstring says "only the averaged correlations" (`YM/Pert/FirstTarget.lean` line 4) Source .

The proof of the construction theorem descends the chain of main theorems `Nonperturbative.existence` (`YM/Nonperturbative/Headlines.lean` line 19) → `continuum_with_position_tests` → the choice of one run `constructionRun` (`YM/Independent/ConstructionRun.lean` line 14) Source . It is organised so that a run is chosen only once for each penalty coefficient, and every subsequent object is built on that run Source . The rest of the chain is read in 6-03.

05

## The committee's combined target

According to § 9 of the assessment report, after submission OpenAI informed the committee that the original public target `jointExistence` had dropped part of the link to the lattice construction that the proof sets up Assessment . The committee wrote a target of its own that keeps that link, and based its judgement on it Assessment . That target is `YMReview.Acceptance.combined_acceptance` (line 111 of `submissions/openai-20261028/checks/CombinedAcceptance.lean` in the repository `YM`; the type of the statement is `CombinedAcceptance` at line 172 of `AcceptanceSpec.lean`) Source .

Its form is: "for each penalty coefficient one family is fixed first, and for every order there are one refinement, one table and one final witness, and all the check fields hold on those **same** objects" Source . The proof calls the construction theorem and transfers its conclusions to the committee's fields Source . Table 2 lists the fields.

Table 2 The fields of the committee's combined target (`AcceptanceSpec.lean` lines 34–170)

| Field | Content (field names and types read at the source; meaning per the section on the right) | Asses |
|---|---|---|
| Lattice and scales | The family certificate and regularity. The finite-lattice measure agrees with the Wilson measure of an independently written lattice gauge theory (coupling conversion $\beta_{\mathrm{lgt}} = \beta/m$). The order on the whole sequence of spacings. The public limit. The strong ultraviolet form. Reference spacing $a_0 = 1$. The step-0 coupling has the same positive value for all $N, p$. Normalisation of the read-out of insertions | § 11 ( |
| Phases and OS | The phase measure is a probability measure with full support. The average over phases. The centred-trace hierarchy of the chosen phase satisfies independently written OS axioms and a growth condition (E0′). That trace is the same as the trace of the construction | § 10 ( |
| Translation Ward identity | In a form with fixed signs and coefficients, for every phase and every field. Integrability of the integrand, and independence of the choice of cut-off function | § 11 ( |
| Operator content | Every invariant polynomial is among the labels. A linear isomorphism between the space of fields of each degree and the space of free Maxwell shells (also in each phase). Density of words, and faithfulness | § 3, § |

| Field | Content (field names and types read at the source; meaning per the section on the right) | Asses |
|---|---|---|
| Leading term of the trace | The short-distance law from every direction, for fields of the same chosen phase | § 7.2 ( |
| Positive lattice probe | For a fixed test function, the double limit has a nonzero value | § 11 ( |
| Source conclusions | The bundle of conclusions of the construction theorem, kept as is | § 9.3 |

The docstring of `AcceptanceSpec.lean` itself says that bundling the source conclusions is not a check of the independent meaning of each definition[Source] . The matching of the operator-product-expansion statement (`YMReview.OPE.constructed`) is an endpoint separate from the combined target (assessment report § 11)[Assessment] .

06

## How the three scopes differ

Table 3 The public theorem, the construction theorem, the committee's combined target (○ in the statement, — not. All[Source] )

| Item | Public theorem | Construction theorem | Combined target |
|---|---|---|---|
| Hypothesis | Lie algebra simple | Simple, penalty coefficient $c_{\mathrm{pen}} > 0$ | Same |
| Lattice measure is the Wilson measure | ○ (definition of the family's measure) | ○ | ○ also agrees with an independent definition |
| Labels represent polynomials in the curvature; renormalisation coefficients | — (labels are abstract) | ○ (family certificate) | ○ |
| Continuum limit (smeared values) | ○ whole sequence of the family's spacings | ○ | ○ |
| Continuum limit (smooth kernel, with differences) | — | ○ | ○ |
| Volume limit | Subsequence | Subsequence | Subsequence |
| A unit common to all spacings and volumes ($a_0 = 1$, common step-0 coupling) | — | Inside the family certificate | ○ as a field |
| Wightman axioms | Chosen phase | Every phase | Every phase (source conclusion) |

| Item | Public theorem | Construction theorem | Combined target |
|---|---|---|---|
| Matching against independently written OS axioms | — | — | ○ only the trace scalar of the chosen phase |
| Trace field is the trace of the stress tensor | — | ○ | ○ |
| Translation Ward identity | — | ○ (as the stress law) | ○ in a form with fixed coefficients |
| Free Maxwell field at short distance (correlations of the mixed theory, one sequence of scales) | — | ○ | ○ (source conclusion) |
| Short-distance effective coupling (the theorem of assessment report § 7.4) | — | — (not among the conclusions; a lemma with similar wording is an intermediate step inside the dependencies) | — |
| Mass gap | — | — | — |

The last two rows of Table 3 matter. The body of the assessment report lists the short-distance effective coupling as Theorem 7.2, but does not name where it is in Lean Assessment . The source has three declarations of a similar form Source .

- `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean:145`): a ratio form ($\mu \to \infty$, $a_m \le 1/\mu$). Outside the dependencies (outside) Source .
- `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean:50`): inside the dependencies (used; the first headline is `continuum_with_position_tests`, the direct user is `crest_ready`), an intermediate step of the trace analysis Source . It has `PostT` among its hypotheses, and its conclusion enters an intermediate certificate and then drops out of the conclusions of all three targets Source . This is the one whose wording is close to the statement of Theorem 7.2 ($\log(1/r)\, g_W(r)^2 \to 1/(2b_0)$) Inferred .
- `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean:36`): itself a headline theorem (headline), the running of the step-$i$ coupling with cut-off $\beta(N_i, k_i, i)^{-1/2}$. This is a different quantity from $g_W$ Source .

Volume 2 cites the short-distance effective coupling as "there are declarations of a similar form, but none survives in the conclusions of the public theorem, the construction theorem or the committee's combined target" (10-06, 13-03).

> 07

## What the verification record checked, and what it did not

§ 9 of the assessment report separates formal verification (the proof derives the statement from the definitions and the permitted axioms) from validity (the formal statement correctly represents the intended problem)[Assessment] . Table 4 lists, check by check, what was checked and what was not.

Table 4 The verification record (all [Assessment] ; the lines of the log copies were checked at the [Source] )

| Check | What was checked | What was not checked |
|---|---|---|
| Rebuild ( § 9.1) | With Lean 4.32.0, 9,470 modules rebuilt with no pre-existing artefacts, 0 errors (about 184 minutes). 9,477 source files agree with the submission's manifest | 2,453 warnings remain (mostly unused variables and the like) |
| Inventory ( § 9.1) | Among the 250,319 declarations loaded, no custom axioms, no direct references to `sorryAx`, no `unsafe` marks | — |
| Kernel recheck ( § 9.1) | By the submitter's checker `VerifyJoint.lean`, separate from Comparator. The 100,168 declarations on which the 17 headline theorems depend were rechecked in a fresh kernel environment initialised only with the external libraries. 14,364 external declarations at the boundary. The only axioms are `propext`, `Classical.choice` and `Quot.sound` | Declarations not reachable from the 17 headlines (about 150,000, the rest of the inventory) were not rechecked |
| Comparator ( § 9.2) | Against fixed reference statements and definitions: agreement of statements and definitions, axioms, kernel recheck. The public theorem (10-30), the combined target (11-03) and the operator-product-expansion matching (11-05) passed. Four controls (a correct proof passes; versions with altered statements or definitions, and forbidden axioms, are rejected) | Run on macOS, without a check that isolates potentially malicious build code. No kernel from another implementation was used |
| Human reading ( § 9.3) | The definitions in the appendix glossary were read against the problem statement and the conjecture | Not exhaustive, the assessment report says |
| Matching against independent axioms ( § 10, § 11) | The trace hierarchy of the chosen phase satisfies OS axioms written independently beforehand. The finite Wilson measure agrees with an independent definition. The Ward identity, the common unit, the trace law | The analytic estimates themselves rest on the submitted proof. The matching concerns the hierarchy of the trace scalar, not every field. Test functions flat on the diagonal do not see contact terms |

The log copy `verification/joint-check.log` records the type match of the public theorem ( `TYPE_MATCH YangMills.jointExistence` ) and the numbers of the recheck (project 100,168, external 14,364, endpoints 17)[Source] . The assessment report notes as

a concern about code quality that, compared with the total number of declarations, few enter the dependencies of the main theorem Assessment .

Among the things not checked, two bear on this book's reading. One is that the kernel's acceptance does not decide that the definitions express the intent ( § 9.3) Assessment . The other is that the committee's OS matching is limited to the hierarchy of the trace scalar ( § 10.1) Assessment .

08

## What this book's study had seen

This book's Lean consists of two kinds of statement (19-00): theorems about concrete objects defined on the lattice, and implications with hypotheses placed in fields (chain theorems). The closest forms in the published construction are these three Inferred .

- An implication that takes estimates as hypotheses and concludes a limit ( `positioned` in `YM/FullVolume/Profiles.lean`, line 42) Source .
- A certificate bundling the family certificate, the order, pointwise limits and the table's axioms into one ( `Seed` in `YM/Spec/Closed.lean`, line 20) Source . Reading this as "an assembly that transfers properties to the limiting table under conditions" is this book's inference Inferred .
- An arrangement in which the requested propositions are placed outside as types, closed by proofs elsewhere and passed to the theorem above Inferred .

The difference is the endpoint. The published construction closes every hypothesis at the end and arrives at a statement whose only hypothesis is "simple" Assessment . This book's chain theorems stop with premises such as uniform estimates across steps still in place (28-04). Another difference is that the family is chosen inside the existential quantifier. This book's chain took the sequence of bare couplings as given from outside, not as determined by matching (14-03).

09

## What has not been read

- Of the dozen or so conclusions of the construction theorem, we have not read the contents of `Independent.Core`, `Pert.Position.FourMeters` or the fields of the contact apparatus.
- We have not read the bridge in `HierarchyBridge.lean` (the transfer to the public types), except the lines on the trace field.
- We have not read the proofs in the committee's check files ( `CompleteOS.lean`, `CompletenessAudit.lean`, `LGTWilsonBridge.lean` and others). We read only the types

of the statements (`AcceptanceSpec.lean`).

- We have not computed the hash values in the log copies ourselves to compare them.
- Among the 17 headline theorems, we have not read what those outside the line of the public theorem and the construction theorem (for example `YM.SharpGram.habitat`) say.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The only hypothesis of the public theorem is "the Lie algebra is simple"; the family, the refinement and the package of theory sit inside the existential quantifiers | That the proof is correct (this is at the level of the assessment) |
| Source In the public theorem, the continuum limit is convergence of smeared values along the whole sequence of the family's spacings, and the volume limit is along a subsequence | Universality with respect to the choice of the sequence of spacings; uniqueness of the volume limit |
| Source The meaning of labels, the identification of the trace and the strong ultraviolet form are on the side of the construction theorem | Reading them from the public theorem alone |
| Assessment The judgement rests on the committee's combined target | That the checks of the combined target are a check of the independent meaning of every definition |
| Source There are three declarations of a form close to the short-distance effective coupling, and none survives in the conclusions of the three targets | Which of them is where Theorem 7.2 of the assessment report lives (the report gives no name) |

## Sources and reproduction

| Item | Type | Source / tool |
|---|---|---|
| Public theorem | Source | OpenAI-YM-20261028: `ChallengeSpec.lean` lines 12–48 and 117–188, `PublicWilson.lean` lines 35–90, `PublicTheory.lean` line 39, `Solution.lean` 4 lines |
| Construction theorem | Source | `Construction.lean` lines 15–40, `YM/Source/Construction.lean` lines 44–65, `HierarchyBridge.lean` lines 27–42, `YM/Nonperturbative/HeadlineSpec.lean` lines 84–91, `YM/Nonperturbative/Headlines.lean` lines 19–22, `YM/Independent/ConstructionRun.lean` line 14 |
| Combined target | Source | yang-mills-group/YM (revision 55ba9fc): `submissions/openai-20261028/checks/AcceptanceSpec.lean` lines 34–176, `CombinedAcceptance.lean` lines 111–121 |

| Item | Type | Source / tool |
|---|---|---|
| Why the combined target was written | Assessment | Assessment report § 9 (opening), § 9.3, § 9.4 |
| Verification record | Assessment | Assessment report § 9.1, § 9.2, § 10, § 11, § 12.2. Log copies `verification/joint-check.log`, `build.log` |
| Inside and outside the dependencies | Source | `verification/declaration-usage.tsv`, `verification/Usage.lean`, `verification/joint-headlines.txt` |
| Short-distance effective coupling | Source | `YM/PertExtension/Running.lean:145` (outside), `YM/TraceExtension/Atlas/Sink.lean:50` (used), `YM/Pert/Asymptotics/RawAxis.lean:36` (headline). Assessment report § 7.4 |
| Only the averaged correlations | Source | `YM/Pert/FirstTarget.lean` lines 1–5. Assessment report § 7.4 |

**Words used in this chapter** — existential quantifier, Wilson family, four conditions of regularity, label, penalty coefficient, family certificate, order and refinement, combined target, Comparator, kernel recheck. Their meanings are in the A-02 Glossary.

Next chapter to read: 6-03 The skeleton of the proof — descending the chain of the construction theorem, we see the ten stages from the calendar to the evidence in one figure and one table.

# The skeleton of the proof (one picture)

> This chapter is existing knowledge (a reading of the published construction). § 06 is a guide, and § 07 is this book's own study.

**What this chapter shows** — In what order the published construction builds what, from the Wilson measure on the lattice to the Wightman theory and the "evidence that it is Yang–Mills". What each of the ten stages defines and what it shows (with equations). The interfaces that connect the stages. Where the main declarations of each stage are, and the chapters of Volume 2 that treat them. How the roughly 990,000 lines of code are distributed over the stages.

**Prerequisite chapters** — 6-02 Reading the statements (the construction theorem, and the choice of one run). For the idea of renormalisation steps, see 7-05 The sequence of renormalisation group steps; for reflection positivity and the transfer matrix, 5-03.

**Said first** — This chapter is an overview. The statements of each stage are readings of their form; the content of the proofs is treated in the chapters of each part, only as far as we could read it. The division into stages is this book's arrangement for readability and does not match how the code is laid out. The distribution of line counts is a rough count by unit of location Inferred .

The order of this chapter

1. One picture — ten stages, and one run (Figure 1)
2. Stages 1–5: renormalisation on the lattice — calendar, one step, representation and induction, shooting, calibration
3. Stages 6–10: limits and reconstruction — ultraviolet limit, infinite volume, Euclidean invariance and phases, reconstruction, evidence
4. What connects the stages — the push-forward identity, continuous curves, the root seminorm, the endpoint
5. The size of the code and its distribution over layers — a count by unit of location (Table 2)
6. A guide to the chapters of Volume 2 — chapters by stage (the right-hand column of Table 1)
7. What this book's study had seen — a comparison with this book's set-up

8. What has not been read

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment report. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (see "Labels" in 6-00). Unless stated otherwise, paths are relative to the root of the repository `OpenAI-YM-20261028`.

01

## One picture

For each penalty coefficient $c_{\mathrm{pen}} > 0$, the construction theorem chooses, only once, a "run" that satisfies certain conditions (`constructionRun` in `YM/Independent/ConstructionRun.lean`, line 14) Source . Every later object is built on this one run. Objects whose existence is shown separately might not be compatible when combined afterwards, so the design gathers the existence proof into a single line Inferred . Figure 1 shows the ten stages on that run.

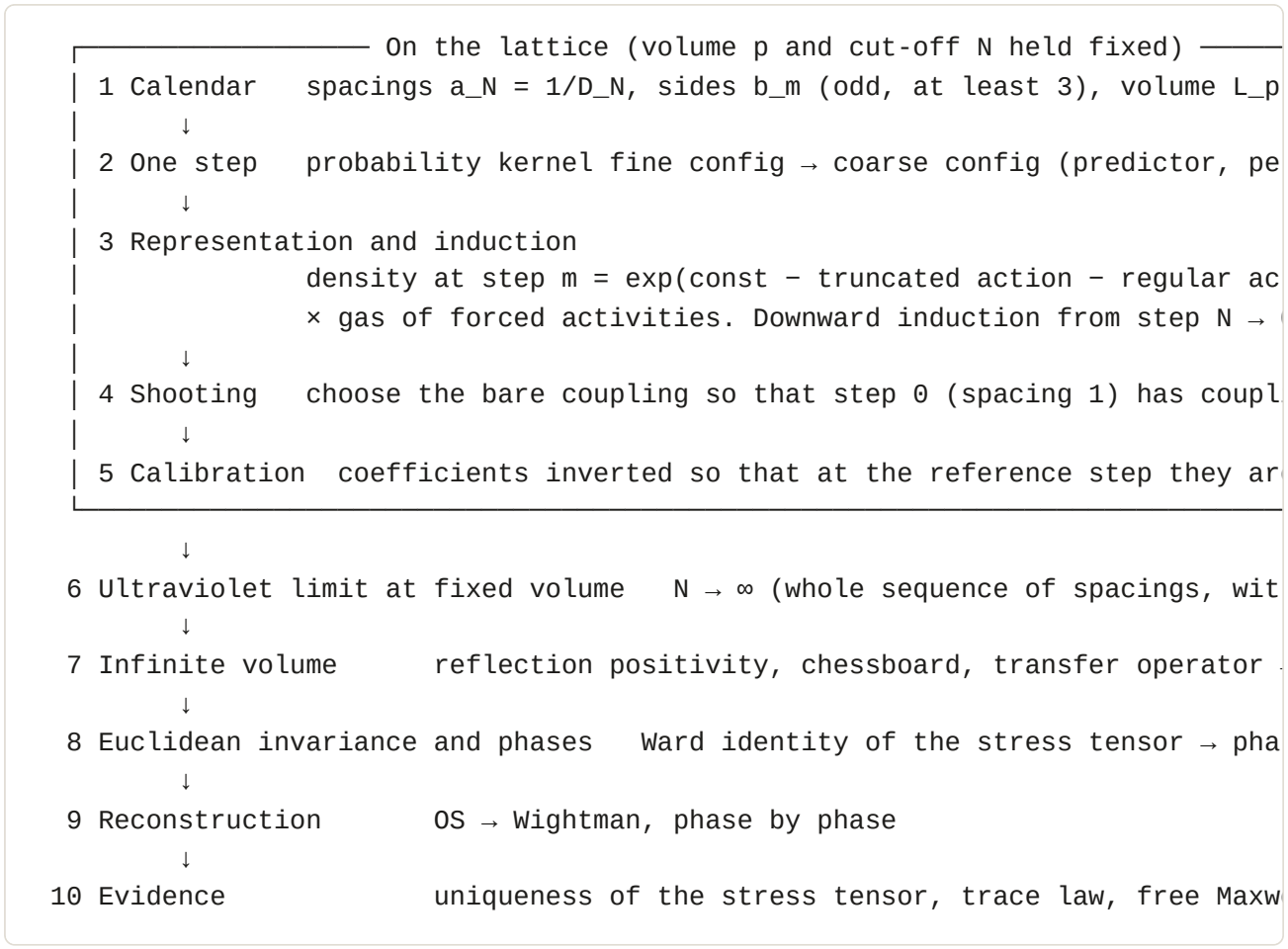



Figure 1 The skeleton of the proof. Stages 1–5 are renormalisation on the lattice; all the steps are built with the volume and the cut-off held fixed. Stage 6 removes the cut-off, and stage 7 removes the volume.

Table 1 summarises, for each stage, its content, its main declarations and the chapters that treat it. The label on the content means that we read the form of the statement in that row at the source.

Table 1 The ten stages (all content `Source` ; correctness of the proofs is at the level of the assessment)

| Stage | What it defines and what it shows | Main declarations (location) | Chapters |
|---|---|---|---|
| 1 Calendar | Sides $b_m$ (odd, at least 3), volume sides $L_p$ (multiples of 4, strictly increasing), $D_N = \prod_{k<N} b_k$. Constants for each step and the reference coupling $\bar{\beta}$ | `Schedule` (`YM/Lattice/Scale.lean` line 28), `Calendar` (`YM/RG/History.lean` line 64) | 8-02, 10-05, 11-08 |
| 2 One step | A probability kernel from fine configurations to coarse configurations | `StepData.density`, `push` (`YM/RG/Block.lean` lines 71, 77) | 9-01 to 9-05 |

| Stage | What it defines and what it shows | Main declarations (location) | Chapters |
|---|---|---|---|
| 3 Representation and induction | The representation of the density at each step, the induction hypothesis, the one-step relation, the one-step theorem | `Rep`, `History.Conv`, `Transfer`, `Selected` (`YM/RG/History.lean` lines 165, 213, 223, 234), `Directed.Program` (`YM/PCalc/Band.lean` line 65) | 9-06, 9-07, 11-00 to 11-09 |
| 4 Shooting | Choosing the bare coupling so that the step-0 coupling matches $\bar{\beta}$ | `Arc.bought` (`YM/FullVolume/Arc.lean` line 95); on the path of the main theorem, `rodB` (`YM/FullVolume/Stage/Rods.lean` line 26) | 10-04 |
| 5 Calibration | Fixing the insertion coefficients by inverting the renormalisation transport | `calibrated` in `Family.Makes` (`YM/Source/Construction.lean` lines 44–65) | 8-04, 11-10 |
| 6 Ultraviolet limit | For each volume, convergence along the whole sequence of spacings, with kernels and differences | `Campaign.realUV` (`YM/FullVolume/Fields/Filled.lean` line 45), `positioned` (`YM/FullVolume/Profiles.lean` line 42) | 11-10 |
| 7 Infinite volume | Reflection positivity, the chessboard inequality, input from the ultraviolet, equicontinuity, subsequential limits | `reflected_positive` (`YM/Slice/Reflection.lean` line 98), `tiles`, `adds` (`YM/Cells/Chess.lean` lines 80, 108), `Wcmain` (`YM/Extraction/Center.lean` line 38), `first_surface` (`YM/Extraction/Lipschitz.lean` line 166) | 12-01 to 12-05 |
| 8 Euclidean invariance and phases | The stress tensor and the Ward identity, vanishing of the rotational flux, phase decomposition | `YM/Euclid/Stress.lean`, `OS.Selection` (`YM/Phase/Mixture.lean` line 127) | 12-06, 12-07 |
| 9 Reconstruction | OS reconstruction and the Wightman axioms in each phase | `Hierarchy.Reconstructed` (`ChallengeSpec.lean` line 117), `EveryPhase.every` (`YM/EveryPhase/Vacuum.lean` line 146) | 12-08 |
| 10 Evidence | Uniqueness of the stress tensor, the short-distance law of the trace, comparison with the free Maxwell field, the operator product expansion | `YM.Pert.one_line` (`YM/Pert/Weak.lean` line 68), `ChallengeSpec.lean` lines 168–175, `YM/Pert/FirstTarget.lean` | 13-01 to 13-04 |

```
02
```

## Stages 1–5: renormalisation on the lattice

### Stage 1 Calendar

The lattice spacing is $a_N = 1/D_N$, with $D_N = \prod_{k<N} b_k$ Source . The sides $b_m$ are odd numbers at least 3 and grow with the step, but $\log b_m / \log D_m \to 0$ (`Schedule.Gentle`) Source . The volume is a torus of side $L_p$ in physical units, which on the lattice has $L_p D_N$ points per side Source . The spacing at step 0 is 1, and this becomes the common unit of length Assessment .

The calendar fixes reference values for each step Source . With $C_{\mathfrak{g}}$ the Casimir constant of the adjoint representation,

$$\hat{\beta}_m = \bar{\beta} + \frac{11 C_{\mathfrak{g}}}{24\pi^2} \log \frac{1}{a_m}, \qquad \hat{g}_m = \hat{\beta}_m^{-1/2}, \qquad \mathrm{monitor}_m = \exp\big((\log\log \hat{\beta}_m)^2\big). \tag{1}$$

The small-field threshold is $\hat{g}_m \cdot \mathrm{monitor}_m$, the allowance for regular activities is $\hat{g}_m^{11/5}$, and the allowance for forced activities is $\exp(-\mathrm{monitor}_m^{1/5})$ (`YM/RG/History.lean` lines 80–87) Source . The slope in (1) is the one-loop coefficient of asymptotic freedom (4-06).

### Stage 2 One step

One step is a **probability kernel** from a fine configuration $U$ to a coarse configuration $V$ Source . It is not a deterministic map. The kernel has the following form (`YM/RG/Block.lean` lines 53–72) Source .

$$K_\beta(U,V) = \sum_t \ell(U,t) \prod_b \frac{e^{-\beta\, c_{\mathrm{pen}}\, \mathrm{price}(V_b\, Q_b(U,t)^{-1})}}{Z_\beta}. \tag{2}$$

The sum is over choices $t$ (a gauge frame at each coarse lattice point and a predictor label for each coarse link), and $\ell(U,t)$ is the probability of that choice. $Q(U,t)$ is the **predictor** of the coarse links, $c_{\mathrm{pen}}$ is the penalty coefficient (`kernel.penalty` in the source's wording), $\mathrm{price}(u) = 1 - \mathrm{Re}\,\mathrm{tr}\, u/m$ is the distance from the identity, of the same form as the Wilson action (`YM/Gauge/MatrixGroup.lean` line 112), and $Z_\beta$ is the normalisation. The coarse links are scattered in Gaussian fashion around the predictor Inferred . The push-forward is $\mathrm{push}_\beta F(V) = \int K_\beta(U,V) F(U)\, dU$ (line 77) Source . Averaging over the choices with probabilities gives a gauge fixing that keeps the symmetry of the lattice (assessment report § 6.3) Assessment .

### Stage 3 Representation and induction

The density of the theory with cut-off $N$ and volume $p$, pushed forward to step $m \le N$, is held in the following form (`YM/RG/History.lean` lines 165–179) Source .

$$\mathrm{eval}_m(U) = \exp\Big(v\,|\Lambda_m| - A_m(U) - \sum_X E_X(U)\Big)\cdot \mathrm{Gas}_m(U). \quad (3)$$

$A_m$ is the action of the path (the action truncated at the threshold), $E_X$ are the regular activities, and $\mathrm{Gas}_m$ is the gas of **forced activities** that exactly covers the marks of the faces exceeding the threshold Inferred . The **induction hypothesis** ( `History.Conv` , line 213) is that at each step this representation stays within the calendar's allowances and the coupling lies in $[\hat\beta_m/2,\ 2\hat\beta_m]$ Source . The **one-step relation** ( `Transfer` , line 223) consists of the following two Source .

$$\mathrm{eval}_m = \mathrm{push}_{\beta_{m+1}}\,\mathrm{eval}_{m+1}, \qquad \Big|\beta_{m+1} - \beta_m - \frac{11C_{\mathfrak g}}{24\pi^2}\log b_m\Big| \le \mathrm{drift}_{m+1}. \quad (4)$$

The first is an identity, not an approximation Source . So the partition function is preserved at every step ( `History.partition_eq` , line 242) Source . The all-steps statement `History.Selected` (line 234) bundles together: step $N$ is the bare Wilson measure; the induction hypothesis and the one-step relation at every step; the coupling equals $\bar\beta$ at step 0; and the running of the coupling Source . The one-step theorem is written in the form "a curve of representations, continuous on an interval of couplings" ( `YM/PCalc/Band.lean` line 65) Source . Its proof is the subject of Part 11, and this book has read only part of it.

### Stage 4 Shooting

Inducting downward from step $N$ to 0 produces an arc of representations over an interval of bare couplings Source . The ends of the band are

$$\mathrm{low}_m = \tfrac{3}{4}\bar\beta + \tfrac{5}{8}\,s\log D_m, \qquad \mathrm{high}_m = \tfrac{5}{4}\bar\beta + \tfrac{3}{2}\,s\log D_m, \qquad s = \frac{11C_{\mathfrak g}}{24\pi^2} \quad (5)$$

( `YM/FullVolume/Arc.lean` lines 21–22) Source . At step 0 the coupling along the arc has ends $\frac{3}{4}\bar\beta$ and $\frac{5}{4}\bar\beta$, so by the intermediate value theorem a bare coupling can be chosen for which the step-0 coupling is exactly $\bar\beta$ Source . In this way the coupling at spacing 1 takes the same value for all $N, p$ (assessment report § 11) Assessment . The one-step theorem is carried in a form continuous in the coupling in order to use this intermediate value theorem Inferred .

### Stage 5 Calibration

For each label of degree $d$, a reference step $r(d)$ independent of the cut-off is fixed Source . The renormalisation transport induces an invertible linear map on the space of coefficients of local polynomials Assessment . The insertion coefficients at cut-off $N$ are fixed through its inverse, so that "when carried to step $r(d)$ they are exactly the label coefficients" ( `calibrated` at line 62 and `normalized` at line 65 of

`YM/Source/Construction.lean`) Source . There is no explicit formula for the entries of the inverse matrix (assessment report, Appendix A) Assessment .

03

## Stages 6–10: limits and reconstruction

### Stage 6 Ultraviolet limit at fixed volume

For each volume, convergence along the whole sequence of spacings, with kernels and differences (`realUV` at line 45 of `YM/FullVolume/Fields/Filled.lean`) Source . Its form is the implication `positioned` (`YM/FullVolume/Profiles.lean` line 42), which derives locally uniform convergence with differences from uniform boundedness on separated configurations and pointwise Cauchy property at general position Source . The Cauchy property is obtained by combining convergence in the cut-off of the objects at a fixed coarse step $j$ with an estimate that removes step $j$ as $j \to \infty$ Inferred . The latter uses the separation of the points, so the test functions avoid coinciding points Inferred . The source of convergence is not a quantitative comparison estimate but compactness, uniqueness of limits and the subsequence principle Inferred (11-10).

### Stage 7 Infinite volume

Reflection positivity is stated for the Wilson measure of the original lattice, for both planes through lattice points and planes through midpoints of edges, along all four axes ($\beta \geq 0$) Source . From it follow the chessboard inequality and the triangle inequality of its root (the root being a seminorm) Source . The main input from the ultraviolet, `Wcmain`, is that the size of renormalised observables measured by the chessboard root is bounded, uniformly in all volumes, as follows Source .

$$\mathrm{root}(O_d) \leq A\, a_m^{-d-\varepsilon} \qquad (m \geq r(d),\ N \text{ sufficiently large, all } p). \tag{6}$$

From this and the representation by the transfer operator follow local boundedness and equicontinuity, and Arzelà–Ascoli gives limits along subsequences of volumes (`YM/Extraction/Lipschitz.lean` line 166) Source . `Wcmain` is not the only input from the ultraviolet; other fields are also passed along the same trajectory Source . The hypotheses of `first_surface` (`TransferLip`, `Bound`) are upper bounds and rates of continuity, and contain no ingredient for uniqueness of the limit Source .

### Stage 8 Euclidean invariance and phases

The lattice has only finitely many symmetries, so continuum Euclidean invariance is shown separately. § 6.3 of the assessment report lists three parts Assessment : that the gauge fixing keeps the lattice symmetries; a divergence-free stress tensor and the local Ward identity; and that the stress flux vanishes fast enough at infinity so that

invariance is not spontaneously broken. After that, the commutative algebra of operators compressed to the zero-energy space is given a Gelfand representation, yielding a space of phases $X$ and a probability measure $\mu$ (`phaseMeasure` in the source; $\nu$ in equation (12) of the assessment report) Source .

$$S^{\mathrm{mix}} = \int_X S^{\xi}\, d\mu(\xi). \tag{7}$$

The proof that the algebra is commutative uses rotational invariance as a hypothesis, so the order is "Euclidean invariance → phase decomposition" Source .

### Stage 9 Reconstruction

Each phase $S^{\xi}$ satisfies a variant of the OS axioms including clustering, and a Wightman theory is reconstructed from it Assessment . In each phase the zero-energy space is one-dimensional (uniqueness of the vacuum, without a rate) Source . The chosen phase $\xi_*$ gives the Wightman theory of the public theorem (12-08).

### Stage 10 Evidence

The stress tensor lies in a one-dimensional space of conserved symmetric tensors Source . The two-point function of the centred trace, multiplied by $r^8 \log^2(1/r)$, approaches $3 \dim \mathfrak{g}/\pi^4$ Source . Along one sequence of scales, the stress correlations of the mixed theory (the average over phases) approach those of the free Maxwell field Assessment . The coefficients of the operator product expansion do not depend on the state, and the remainder is bounded uniformly in the state Source . The details are in Part 13.

04

## What connects the stages

Here are four interfaces that hold the skeleton together.

- **The push-forward identity** (stage 3): because the first relation in (4) is an identity, the representation at step 0 is exactly the push-forward of the original Wilson measure Source . Approximation errors are carried as activities inside the representation Inferred .
- **Curves continuous in the coupling** (stage 3 → stage 4): because the one-step theorem has a form continuous in the coupling parameter, the intermediate value theorem can be used after all the steps Inferred .
- **The root seminorm** (stage 6 → stage 7): estimates uniform in the volume are handed over in the language of the chessboard root built from reflection positivity of the original lattice Source . On the renormalisation side too, the one-step ledger is

a pointwise sum and does not depend on the volume Inferred . What involves an exponent in the volume are estimates at the level of densities and partition functions Inferred .

- **The endpoint is correlations of local fields** (stages 5 and 6): because the endpoint is correlations of local fields at separated points rather than the coarse lattice measure, calibration of observables (stage 5) and a step that removes the coarse step $j$ (stage 6) are needed Inferred .

05

## The size of the code and its distribution over layers

The source consists of 9,477 files and about 990,000 lines Source . There are 250,319 declarations, including automatically generated ones, of which 100,168 enter the dependencies of the 17 headline theorems Assessment . Assigning the locations (directories) roughly to the stages of Table 1 and counting gives Table 2.

Table 2 Lines by layer, divided by unit of location (the contents of a location are mixed, so the proportions are a guide Inferred ; the numbers are a count over the map of locations)

| Layer | Stage | Locations | Files | Lines | Share |
|---|---|---|---|---|---|
| Public statements and assembly | — | 10 | 97 | 10,071 | 1.0% |
| Foundations (groups, lattice, measure) | 1 | 20 | 133 | 16,449 | 1.6% |
| Definition of one step | 2 | 15 | 88 | 12,028 | 1.2% |
| Multistep induction and estimates | 3, 4 | 22 | 6,223 | 696,999 | 69.9% |
| Ultraviolet at fixed volume | 5, 6 | 3 | 1,184 | 97,171 | 9.7% |
| Infinite volume | 7 | 4 | 39 | 4,607 | 0.5% |
| Reconstruction | 8, 9 | 12 | 249 | 40,167 | 4.0% |
| Local fields, operator product expansion, trace | 10 | 14 | 1,253 | 100,546 | 10.1% |
| Comparison with perturbation theory | 10 | 2 | 148 | 12,726 | 1.3% |
| Unused locations | — | 2 | 63 | 6,299 | 0.6% |

The line counts in Table 2 count one extra line at the end of each file (total 997,063; counting line endings gives 987,586) Source . Three things can be read from it Inferred .

- About 70 per cent of the weight of the proof is in the multistep induction and estimates (stages 3 and 4). Part 11 corresponds to this.
- The definitional layer is shallow and short. `YM/RG/History.lean`, which fixes the representation of the induction, is about 250 lines (251 lines) Source .
- The infinite-volume locations are small and shallow. The heavy input comes from the uniform estimates on the ultraviolet side.

06

## A guide to the chapters of Volume 2

The right-hand column of Table 1, arranged by part in reading order.

- **Part 8 From the lattice to the continuum — objects**: the sequences of spacings and volumes of stage 1, labels and renormalised insertions (the definitional side of stage 5), the order of limits. From 8-01.
- **Part 9 One step of renormalisation**: the probability kernel of stage 2, and the form of the representation of stage 3. From 9-01.
- **Part 10 Matching the coupling, and asymptotic freedom**: the reasons for the calendar of stage 1, one step of the coupling in stage 3, the shooting of stage 4. From 10-01.
- **Part 11 Multistep estimates and large fields**: the content of the induction of stage 3, and the fixed-step convergence of stage 6. From 11-00.
- **Part 12 Infinite volume and reconstruction**: stages 7–9. From 12-01.
- **Part 13 Evidence that it is Yang–Mills**: stage 10. From 13-01.
- **Part 14 Comparison**: a comparison with this book's Lean, stage by stage. From 14-01.

Part 7 (constructive field theory) is the historical background of this skeleton. § 4.3 of the assessment report says that stages 2–4 follow Bałaban's approach, with many improvements Assessment .

07

## What this book's study had seen

This book's study divided the stages a proof must contain into seven (16-09). The set-up "ultraviolet limit at fixed volume → infinite volume by compactness" runs in the same direction as the published construction. The differences stage by stage are as follows (details in 14-01).

- **Stages 1 and 4**: this book took the sequence of bare couplings as given from outside. There was no anchor fixed by matching (fixing the step-0 coupling). Centred blocks with odd sides were a direction this book's study had also tried (the side-3 design, 14-02). Growing the sides with the step was not done.
- **Stage 2**: this book's block map is a deterministic map, and the push-forward measure is its image (19-04). It is not a probability kernel.
- **Stage 3**: this book's ultraviolet Lean goes only as far as theorems at one scale (one step) (the second half of Part 15), and uniform estimates across steps remain

premises (28-04).

- **Stage 6**: this book's chain had the form of subsequential limits. The published construction has convergence of the whole sequence in the spacing.
- **Stage 7**: reflection positivity and the Chessboard Estimate are also in this book (20-01, 22-01). This book has reflection positivity for planes through lattice points and for midplanes. The triangle inequality of the root (the root being a seminorm) is not in this book's catalogue of firm theorems.
- **Stages 8–10**: this book's Lean has no phase decomposition, reconstruction or stress tensor in the continuum limit. What it has is reconstruction and the mass gap on the lattice inside the strong-coupling window (20-04).

08

## What has not been read

- The core of the proof of the one-step theorem (located at `Mallow.faces`, line 41 of `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Mallow/Pediment.lean`). We read only the statement.
- The construction that, after one step, puts the activities back within the allowances of the next step, and the lemma on the loss of Haar measure.
- The mathematical meaning of the conditions for choosing the run (the fields of `Funded`). We infer that they are numerical conditions Inferred .
- The body of the transfer-operator estimates (inside `Time.powering` in `YM/Slice/Calendar.lean`), the line of argument that uses rotations in proving commutativity of the phase algebra, and how uniqueness of the vacuum is extended to every phase.
- For many of the large-field lemmas, following the dependency table leads to a different theorem as the headline. Whether they lie on the path of the main theorem has not been checked, except for those confirmed along the chain (from the shooting's `rodB` to the one-step induction, from `realUV` to the synthesis of the run, and so on).
- The assignment to layers in Table 2 is by unit of location; the distribution within a location was not measured.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The declarations of Table 1 are at those locations in those forms. (1)–(7) are readings of the source's definitions and statements | Correctness of the proofs (at the level of the assessment). The core of the proof of the one-step theorem |
| Source The first one-step relation is an identity, and the partition function is preserved at every step | That, because it is an identity, this book's premises fall away |
| Source Shooting matches the step-0 coupling to $\bar{\beta}$ using the Arc band (5) and the intermediate value theorem | Explicit values of constants that satisfy the calendar's conditions |
| Inferred About 70 per cent of the code is multistep induction and estimates | The exact distribution within locations |

## Sources and reproduction

| Item | Type | Source / tool |
|---|---|---|
| Stages 1–4 | Source | OpenAI-YM-20261028: `YM/Lattice/Scale.lean` lines 28–44 and 128–129, `YM/RG/History.lean` lines 64–87 and 165–250, `YM/RG/Block.lean` lines 53–77, `YM/PCalc/Band.lean` line 65, `YM/FullVolume/Arc.lean` lines 21–22 and 95, `YM/FullVolume/Stage/Rods.lean` line 26 |
| Stages 5, 6 | Source | `YM/Source/Construction.lean` lines 44–65, `YM/FullVolume/Profiles.lean` lines 25, 42, `YM/FullVolume/Fields/Filled.lean` line 45 |
| Stages 7–9 | Source | `YM/Slice/Reflection.lean` line 98, `YM/Cells/Chess.lean` lines 80, 108, `YM/Extraction/Center.lean` lines 38–41, `YM/Extraction/Lipschitz.lean` line 166, `YM/Phase/Mixture.lean` line 127, `YM/EveryPhase/Vacuum.lean` line 146, `ChallengeSpec.lean` line 117 |
| The three parts of stage 8; stages 2–4 and Bałaban's approach | Assessment | Assessment report § 4.3, § 6.3 |
| Common unit, calibration, number of declarations | Assessment | Assessment report § 9.1, § 11, Appendix A |
| Line counts | Source | Lines of `.lean` files counted on our machine (9,477 files, 987,586 lines). The distribution over layers is a count over the map of locations |

**Words used in this chapter** — calendar, probability kernel, predictor, penalty coefficient, induction hypothesis, one-step relation, truncated action, regular activity, forced activity, shooting, calibration, root seminorm, phase decomposition. Their meanings are in the A-02 Glossary.

Next chapter to read: 6-04 What is not shown — where this skeleton does not reach, set out from the assessment report's five open questions and the source's "what it does not say".

# What is not shown — a bridge to Volume 3

> This chapter is existing knowledge (a reading of the published construction). § 08 is this book's own study, and the "not shown" side of § 01– § 07 is a map of the unsolved.

**What this chapter shows** — Nine items the published construction does not reach. For each, what the assessment report says, how far the source goes, and where it stops. What the polynomial estimates that stand in place of a mass gap say and do not say. At which item's opposite end (strong coupling, fixed lattice spacing) the results of this book's Volume 3 lie.

**Prerequisite chapters** — 6-01 (the four requirements), 6-02 (the three targets), 6-03 (the skeleton).

**Said first** — "Not shown" does not mean "false". The committee writes that the fact that uniqueness is not shown does not contradict the general expectation that it holds Assessment . Much of what we describe as "not in the source" is a judgement within the range of our searches. All the results of this book's Volume 3 are statements at a fixed lattice spacing, and this book shows none of the items listed here.

The order of this chapter

1. Overview — nine items and where they stand (Table 1)
2. Convergence and uniqueness — subsequences of volumes, and quantities that may depend on the subsequence
3. A probability measure for the continuum theory — only the table of correlations and the OS space
4. A uniform mass gap — what exists are polynomial estimates
5. Wilson loops and the area law, one-particle states — outside the local labels
6. Sequences of spacings, coinciding points, perturbation theory — universality, contact terms, agreement to all orders
7. Fermions — a construction of pure Yang–Mills
8. A bridge to Volume 3 — where this book's results stand, item by item (Table 2)
9. What has not been read

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment report. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (see "Labels" in 6-00). Unless stated otherwise, paths are relative to the root of the repository `OpenAI-YM-20261028`.

01

## Overview

§ 8 of the assessment report lists the main points the present construction does not address as five open questions Assessment .

1. Show convergence rather than subsequential convergence, and show that the infinite-volume limit is unique.
2. Define a probability measure for the continuum theory on a suitable space, such that it gives the Schwinger functions of the construction.
3. Show a uniform mass gap, including exponential decay of the corresponding correlations and a gap in the spectrum of the OS Hamiltonian.
4. Construct Wilson-loop observables and relate them to the local observables of the construction. Show the area law for Wilson loops.
5. Construct one-particle states and study their energy–momentum spectrum. In many constructive models in two and three dimensions, this analysis has been carried out.

Table 1 adds to these five four more items, from the body of the assessment report and from the source's "what it does not say".

Table 1 Nine items not shown (the right-hand column lists related chapters of Volume 3)

| Item | Assessment report | In the source | Not in the source | Volume 3 |
|---|---|---|---|---|
| Convergence and uniqueness | § 8 item 1, § 5, § 6.2 | Limits along subsequences of volumes; uniqueness of the limit along the same subsequence | Convergence in the volume; independence of the subsequence | 16-03, 20-03 |
| Probability measure | § 8 item 2, § 3 | Table of correlations and OS space | Continuum measure | 16-02 |
| Uniform mass gap | § 8 item 3, § 2 | Polynomial estimates uniform in each phase | Exponential decay, spectral gap | 20-04, 16-04 |
| Wilson loops and the area law | § 8 item 4 | Parallel transport along lattice paths | Continuum loop observables, area law | 20-06, 21-03 |

| Item | Assessment report | In the source | Not in the source | Volume 3 |
|---|---|---|---|---|
| One-particle states | § 8 item 5 | — | One-particle states, mass shell | 16-02 |
| Universality in the sequence of spacings | § 4.3, § 9.4 | Convergence along the whole of the one sequence of spacings chosen by the family | Convergence along other sequences of spacings | — |
| Where points coincide | § 7.5, § 10 | Correlations at separated points; existence of an extension with contact terms | Uniqueness of the extension, OS positivity, convergence from the lattice | — |
| Agreement with perturbation theory to all orders | § 7.4, § 7.5 | One-loop running of the coupling; coefficient of the leading term of the trace | Agreement at every order | — |
| Fermions | (not treated) | — | Matter fields | 20-05 |

02

## Convergence and uniqueness

The volume limit is taken only along subsequences, by Arzelà–Ascoli compactness (assessment report § 5)[Assessment] . It is shown that one subsequence works for all correlations simultaneously[Assessment] . Whether the limiting correlations depend on the chosen subsequence is open[Assessment] .

On the source side, what goes into the theorem that takes the volume limit, `first_surface` (`YM/Extraction/Lipschitz.lean` line 166), is only upper bounds for local boundedness and equicontinuity[Source] . Statements that could serve as material for uniqueness — such as the difference when the volume is changed, decay of correlations, or the vanishing of dependence on boundary conditions — are not among them[Source] . The uniqueness stated by `Order.unique` (`YM/Extraction/Orders.lean` line 70) is uniqueness of the limit along the same subsequence, that is, uniqueness of limits in a Hausdorff space[Source] . This lemma is outside the dependencies (outside) and is not reached from any headline[Source] .

There are other quantities that may depend on the subsequence[Assessment] .

- Whether the components of the phases are equal to one another. Each component has clustering, but clustering of the mixed theory $S^{\mathrm{mix}}$ is not shown ( § 6.2). The assessment report writes that if the mixture had clustering all components would be equal, but that this would still not suffice for uniqueness of the infinite-volume limit.

- That the coefficients of the operator product expansion do not depend on the choice of subsequence of volumes ( § 7.5).
- That the short-distance effective coupling $g_W(r)$ does not depend on the subsequence of volumes ( § 7.4).

The spacing side is different. The ultraviolet limit at fixed volume converges along the whole sequence of spacings chosen by the family Source . Subsequences remain only on the volume side (6-02 Table 3).

03

## A probability measure for the continuum theory

§ 3 of the assessment report writes that defining, on a suitable space, a probability measure corresponding to the formal expression $Z^{-1}\exp\!\big(-\frac{1}{2g^2}\int\sum_{\mu<\nu}\|F_{\mu\nu}\|^2\big)\,\mathcal{D}A$ is not something the submission claims Assessment . The continuum objects constructed are the correlation functions of local observables and the quantum fields reconstructed from them Assessment .

On the source side too, the continuum objects are the table of correlations and the OS Hilbert space, not a measure Inferred . For the intermediate objects of renormalisation as well, the effective density at a fixed step $m$ is Cauchy uniformly in the cut-off, but its limiting measure is not defined as a named object Inferred . Reflection positivity of the push-forward measure is not stated either Inferred , because the patches of the one-step kernel straddle the reflection plane Inferred .

04

## A uniform mass gap

The construction gives no evidence, for or against, about the mass gap (assessment report § 2) Assessment . What § 8 item 3 asks for is exponential decay of correlations and a gap in the spectrum of the OS Hamiltonian $H$ Assessment .

What the source has are the following two polynomial estimates, uniform in each phase Source .

$$\sigma_A\big((0,u]\big)\le M\,u^{2}\quad(0<u\le 1),\qquad 0\le\big\langle A,\,(e^{-\tau H}-P_0)\,A\big\rangle\le M\,\tau^{-2}\quad(\tau\ge 1).\tag{1}$$

The left is the weight at energies $(0,u]$ of the spectral measure $\sigma_A$ (named `vletter` in the source) of a vector $A$ built from finitely many words ( `light_bound` in `YM/EveryPhase/Light.lean`, line 221) Source . The right is a heat estimate, where $\tau$ is Euclidean time and $P_0$ is the projection onto the zero-energy space ( `inverse_square` in

`YM/EveryPhase/Decay.lean`, line 173) Source . Both are used in the dependency table, and the first headline is `YM.SharpGram.habitat` Source . Whether they reach the public theorem or the construction theorem has not been checked.

If there were a mass gap $\Delta$, the weight on the left would be 0 for $u < \Delta$, and the right would decrease like $e^{-\Delta\tau}$. (1) is no substitute for that, because its form is compatible with theories that have massless particles Inferred . We read the origin of (1) as local commutativity and the forward light cone, not an input specific to Yang–Mills Inferred . Put differently, the mass gap question is the question "make the weight in (1) vanish below some $\Delta$" Inferred .

One more point about the edge of the construction's scope. The renormalisation steps stop at step 0, at spacing 1, where the coupling reaches the reference value $\bar{\beta}$ Source . What can be read from the source about $\bar{\beta}$ goes only as far as the calendar's conditions giving $\hat{\beta}_m > 1$ (`fits.delayed`, used at line 142 of `YM/PertExtension/Running.lean`) Source . How large it is taken is inference Inferred . The long distances beyond that are handled not by renormalisation but by reflection positivity, the chessboard and Arzelà–Ascoli Source . The mass gap question can probably be restated as the question of following this lattice theory at spacing 1 out to long distances Inferred . To restate it that way, three things are needed first: the limiting object, local estimates uniform in the volume, and the image of the observables (16-03).

05

## Wilson loops and the area law, one-particle states

Parallel transport along lattice paths is in the source's foundations (`travel` in `YM/Lattice/Path.lean`, line 105) Source . However, we have not found Wilson-loop observables in the continuum limit or a statement of their area law Source (within the range of name searches). Across all `.lean` files, the words area law, area_law, Wilson loop, wilson_loop and WilsonLoop occur 0 times. The word loop occurs often, but as in one-loop coefficients or the calendar's `loopCap` Inferred . An example among the committee's checks writes that the labels are polynomials in the curvature at a point, and that completeness is not shown for non-local observables such as line defects (`submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` lines 35–38) Source . § 8 item 4 asks not only to construct Wilson loops but to relate them to the local observables Assessment .

We have also not found in the source one-particle states of the interacting theory or an analysis of their spectrum Source (within the range of name searches: one particle, single particle, oneParticle). The words occur in only one place, the docstring "One particle test vectors" at line 4 of `YM/TraceExtension/Fock/Weyl.lean` Source . Since it is in the Fock location, we infer that it belongs to the Fock space of the free field Inferred .

Separately, the docstring at line 5 of `YM/TraceExtension/Damping.lean` has the heading "states of a one-point insertion and their spectral representation", and the same docstring says it does not assume a mass gap Source . § 8 item 5 writes that this analysis has been carried out in many constructive models in two and three dimensions Assessment (7-03).

06

## Sequences of spacings, coinciding points, perturbation theory

### Universality in the sequence of spacings

The sequence of spacings $a_N$ is chosen by the family. The convergence is along the whole of that chosen sequence Assessment . § 4.3 of the assessment report writes that there is no universality result saying that every kind of sequence of spacings gives the same limit Assessment . The construction's sequence is a special one, whose sides $b_m$ are odd numbers at least 3 that grow with the step Source . We read growing the sides as a choice made to remove the one-step error by dividing it by $\log b_m$ Inferred (10-05).

### Where points coincide

Correlations are defined only on test functions on configurations of mutually distinct points Assessment . Using the operator product expansion, the correlations can be extended to test functions that include coinciding points Assessment . However, uniqueness of that extension, OS positivity on unrestricted Euclidean test functions, and convergence from the lattice are not shown ( § 7.5) Assessment . The committee's independent OS matching uses test functions all of whose derivatives of every order vanish on every diagonal, so contact terms supported on the diagonal are not seen ( § 10) Assessment .

### Agreement with perturbation theory to all orders

Agreement with perturbation theory at every order is not shown ( § 7.5) Assessment . The truncation degree $D$ of the operator product expansion is the degree of the labels, not the loop order Assessment . The committee's matching neither computes the expansion coefficients independently nor compares them with perturbative coefficients ( § 11) Assessment . The agreement that is stated is at the level of leading terms: the one-loop running of the coupling (6-01 Table 1) and the coefficient of the short-distance law of the trace Assessment . There are three declarations of a form close to the short-distance effective coupling ( § 7.4): `Running.ratio` is outside the dependencies, `wilson_link` is an intermediate step inside the dependencies, and the headline `Asymptotics.comparison` is the step coupling with cut-off, a different quantity from $g_W$ ; none survives in the conclusions of the three targets (6-02) Source .

07

## Fermions

The construction is of pure Yang–Mills theory. The lattice measure of the public theorem is Haar measure on the link variables weighted by the Wilson action, and has no matter fields (`wilsonLaw` in `PublicWilson.lean`, lines 29–33) Source . Searching the source's `.lean` files for the words fermion and Grassmann found only one occurrence, in a docstring in the free-field Wick location: "no signs from a fermionic grading" Source . A construction of a theory with quarks (quantum chromodynamics) is outside the scope of this construction Inferred .

08

## A bridge to Volume 3

Volume 3 treats the infrared side of the items listed here (uniqueness, the mass gap, the area law). All of this book's results are statements at a fixed lattice spacing, and show nothing about the continuum theory. They are statements at the opposite end (strong coupling, fixed lattice spacing) from the questions the published construction left, and statements of a kind the published construction does not have. Table 2 lists the position item by item.

Table 2 Items not shown, and the results of this book's Volume 3 (this book's side is all at a fixed lattice spacing)

| Item | This book's result | What it does not say |
|---|---|---|
| Convergence and uniqueness | Inside the strong-coupling window, an infinite-volume limit without subsequences (convergence of the whole sequence, 20-03) | Weak coupling, the continuum limit |
| Mass gap | On the lattice inside the strong-coupling window, a gap for the transfer operator (20-04) | Outside the window; a uniform gap in the continuum limit |
| Area law, confinement | On the lattice inside the strong-coupling window, the area law for Wilson loops and an upper bound on the vortex free energy (20-06). Inside the window there is a form whose rate does not depend on the transverse size (the prefactor and the transverse area remain). The vortex floor in a twisted box (21-02, 21-03) | A form uniform in the transverse size outside the window; the area law in the continuum limit |
| Dependence on boundary conditions | Ratios of partition functions with twisted boundary conditions, and Polyakov loops (21-06) | Implications for uniqueness of the infinite-volume limit |
| Rarity of large fields | For the Wilson measure at one scale, rarity uniform in the volume (22-01) | Measures at coarse scales; a form across steps |
| Fermions | A vessel with the Wilson–Dirac weight (20-05) | The continuum limit, chiral symmetry |

The published construction treats only lattices with periodic boundaries and has no statements about twisted boundary conditions or the vortex free energy Inferred . This book's vortex results are quantities that measure dependence on boundary conditions. The chain that would carry them to the continuum theory or to a uniform gap is still left as premises (28-04). The open questions are organised in 16-02, and this book's position in 16-04.

09

## What has not been read

- The items described as "not in the source" (continuum Wilson loops, one-particle states, the continuum measure) are judgements within the range of the word searches described in § 05. We have not excluded the possibility of declarations that treat the same content under other names.
- We have not read the proof of (1). That its origin is local commutativity is inferred from the names of lemmas in the proof.
- We have not checked explicit conditions on how large the construction takes $\bar{\beta}$.
- Whether there really is no named statement about the limiting measure at a fixed step has been searched only in part of the locations (under `YM/FullVolume/`).
- We have not read how the "One particle test vectors" of `Weyl.lean` and the "states of a one-point insertion" of `Damping.lean` relate to one-particle states of the interacting theory.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Assessment The assessment report lists five open questions and writes that the construction gives no evidence about the mass gap | That they are false (they are merely not shown) |
| Source Only upper bounds go into the volume-limit theorem, and uniqueness holds only along the same subsequence | That the limit does not depend on the subsequence |
| Source There are polynomial estimates (1), uniform in each phase | Exponential decay, a spectral gap |
| Source The lattice measure has no matter fields | — |
| This book's results in Table 2 are statements at a fixed lattice spacing | This book shows none of the items of Table 1 for the continuum theory |

## Sources and reproduction

| Item | Type | Source / tool |
|---|---|---|
| Five open questions | Assessment | Assessment report § 8 |
| Dependence of subsequences, phases, coefficients, couplings | Assessment | Assessment report § 5, § 6.2, § 7.4, § 7.5 |
| Inputs to the volume limit, and uniqueness | Source | OpenAI-YM-20261028: `YM/Extraction/Lipschitz.lean` line 166, `YM/Extraction/Orders.lean` line 70 |
| Polynomial estimates | Source | `YM/EveryPhase/Light.lean` line 221, `YM/EveryPhase/Decay.lean` line 173 |
| Parallel transport | Source | `YM/Lattice/Path.lean` line 105 |
| Completeness for line observables | Source | yang-mills-group/YM: `submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` lines 35–38 |
| Probability measure, universality, contact terms, perturbation theory | Assessment | Assessment report § 3, § 4.3, § 7.5, § 10, § 11 |
| Being pure Yang–Mills | Source | `PublicWilson.lean` lines 29–33. Word search (fermion, Grassmann): only `YM/Maxwell/Wick.lean` line 10 |
| This book's results | — | The source tables of each chapter of Parts 20–22 |

**Words used in this chapter** — subsequential limit, uniqueness, clustering, mass gap, spectral measure, heat estimate, Wilson loop, area law, one-particle state, universality, contact term, agreement with perturbation theory. Their meanings are in the A-02 Glossary.

Next chapter to read: 7-01 What constructive field theory is — we return to the tools and history of constructive field theory, the background of the published construction. To go on to Volume 3, read 16-02 Five open questions.

# What constructive field theory is — splitting the "making" of quantum field theory into six stages

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — The idea that to "make" a quantum field theory is to exhibit a non-Gaussian probability measure satisfying the axioms as the limit of a sequence of approximations. The six stages along that road and the difficulty of each. Which stages each chapter of Part 7 deals with.

**Prerequisite chapters** — 4-10 The Osterwalder–Schrader axioms and reconstruction. See also 4-02, 4-11 and 5-01. Within Volume 2, read this after 6-04 What is not shown.

**Said first** — The six steps are an arrangement this book makes following the account of the problem statement (Jaffe–Witten), not standard terminology in the field. This chapter explains a known framework and shows nothing about four-dimensional Yang–Mills. As the preface to Volume 2, Part 7 sets out the tools of constructive field theory and what the literature before 2026 reached, before the published construction (Part 6) is read in detail. How far the published construction got through the steps is treated in 6-01 and 6-04.

The order of this chapter

1. What is it that one makes — a non-Gaussian measure
2. Where the formal expression is broken — three places (a table of calculations)
3. The six stages — a map and how far each theory has got (moving figure)
4. The regularisation and finite-volume steps — what is kept, what is broken
5. The ultraviolet-limit step — renormalisation and uniform estimates
6. The infinite-volume-limit step — clustering
7. The axiom step — reflection positivity
8. The uniform-gap step — "uniformly" is the heart of it
9. Detour: taking a one-dimensional chain through all six stages — moving figure
10. How to read Part 7 — stages and chapters

01

## What is it that one makes

Physics textbooks compute assuming the theory exists. The mathematical question comes before that: whether there is an object satisfying the axioms (4-09, 4-10). The problem statement writes that not a single non-trivial relativistic field theory satisfying the axioms is known in four dimensions (JW § 6, 2000).

The problem statement divides the ways of showing existence into two (JW § 6.1). (i) Find an exact solution in closed form. (ii) Solve a sequence of approximate problems and show that the solutions converge to a limit. **Constructive field theory** (constructive quantum field theory) refers to (ii), and at its centre are inequalities independent of the approximation: uniform a priori estimates. Known

In the Euclidean form (4-02), the object to be made is a probability measure $\mu$ on the space of distributions $\mathcal{S}'(\mathbb{R}^d)$. If its moments

$$S_n(f_1,\dots,f_n) = \int \phi(f_1)\cdots\phi(f_n)\,\mathrm{d}\mu(\phi) \quad (1)$$

satisfy the OS axioms, the reconstruction theorem gives a Hilbert space and a Hamiltonian ($\phi(f)$ is the field smeared with a test function $f$). In physics the same measure is written

$$\mathrm{d}\mu(\phi) = \frac{1}{Z}\,e^{-S(\phi)}\,\mathcal{D}\phi \quad (2)$$

This is the target form, but none of the symbols on the right is a mathematical object as it stands.

**Example:** the free field of mass $m$ is the Gaussian measure with mean 0 and covariance $C = (-\Delta + m^2)^{-1}$ ($\Delta$ is the Laplacian, different from the $\Delta$ of the mass gap). It satisfies the OS axioms, but has no interaction and is trivial (JW § 6.2). The work is to make a **non-Gaussian** measure. According to the problem statement, the first constructions of the form satisfying the axioms (as non-Gaussian measures) came in the 1970s, for fields in two dimensions with small polynomial interactions (JW § 6.2).

A figure lining up, in eleven rows, what switches with the dimension is 1-12 §10 Figure 4 (opens at d = 4; the bottom row is the YM row of Table 2).

02

## Where the formal expression is broken

Expression (2) is broken in three places. (i) $\mathcal{D}\phi$: on an infinite-dimensional space, the only measure that is unchanged by translation and gives every ball a finite weight is 0 (1-09 § 02). (ii) Values at points: in two or more dimensions samples of the free field are distributions, and $\phi(x)^4$ has no meaning (1-11). (iii) Volume: on the whole of $\mathbb{R}^d$ the action is infinite. The action of a scalar field is

$$S(\phi) = \int_{\mathbb{R}^d} \left( \tfrac{1}{2} |\nabla \phi|^2 + \tfrac{1}{2} m^2 \phi^2 + \lambda \, \phi^4 \right) \mathrm{d}x \qquad (3)$$

and for Yang–Mills the field is a connection $A$ and the action is the integral of the square of the curvature $F$ (3-03).

(ii) can be seen in numbers. The variance at a point of the free field on the lattice $a\mathbb{Z}^d$ is

$$G_a(0) = \langle \phi(x)^2 \rangle_a = \int_{[-\pi/a,\, \pi/a]^d} \frac{\mathrm{d}^d k}{(2\pi)^d} \, \frac{1}{m^2 + \frac{4}{a^2} \sum_j \sin^2 \frac{k_j a}{2}} \qquad (4)$$

where the denominator is the lattice version of $m^2 + |k|^2$, and the momenta are cut off at $|k_j| \le \pi/a$.

Computed Table 1 $G_a(0)$ ($m = 1$). Numerical integration of the heat-kernel representation $G_a(0) = a^{2-d} \int_0^\infty e^{-m^2 a^2 s} \, [e^{-2s} I_0(2s)]^d \, \mathrm{d}s$ ($I_0$ is a modified Bessel function). For $d = 1$ it agrees to 5 digits with the closed form $1/(m\sqrt{m^2 a^2 + 4})$.

| $a$ | $d = 1$ | $d = 2$ | $d = 3$ | $d = 4$ |
|---|---|---|---|---|
| 1 | 0.44721 | 0.25405 | 0.1705 | 0.127 |
| 1/4 | 0.49614 | 0.49320 | 0.9289 | 2.431 |
| 1/16 | 0.49976 | 0.71675 | 3.9634 | 39.598 |
| 1/64 | 0.49998 | 0.93768 | 16.0950 | 634.524 |
| $a \to 0$ | $\to 0.5$ | $\frac{1}{2\pi} \log \frac{1}{a} + 0.2758$ | $0.2527/a$ | $0.1549/a^2$ |

In two or more dimensions the variance goes to infinity, and values at points are not random variables. $\phi^4$ has to be redefined with a subtraction depending on $a$, and this is the entrance to the renormalisation of the ultraviolet-limit step.

03

## The six stages

Replace the broken places with finite things (regularisation and finite volume), and remove them at the end (the two limits). Once the limits are taken, check the axioms and the spectral gap. The problem statement too writes, for the lattice, that one should

establish the limits of expectation values as the lattice spacing goes to 0 and the volume to infinity (JW § 6.5).

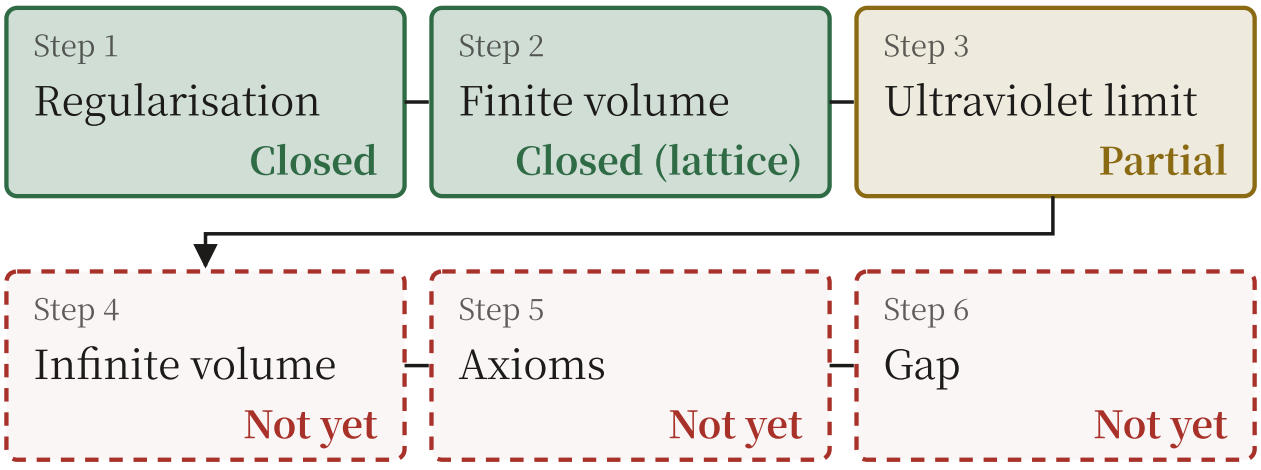
| Stage | What it does | What it needs |
|---|---|---|
| 1 Regularisation | Replace the field by finitely many degrees of freedom | Lattice spacing $a$ |
| 2 Finite volume | Make the measure on a box of side $L$ | Measure $\mu_{a,L}$ |
| 3 Ultraviolet limit | $a \to 0$ | Renormalisation, uniform estimates |
| 4 Infinite volume | $L \to \infty$ | Clustering |
| 5 Axioms | The limit satisfies the OS axioms | Reflection positivity |
| 6 Gap | Gap of the Hamiltonian | Uniform exponential decay |

The four steps from the regularisation step to the infinite-volume step make the double limit

$$\mu = \lim_{L\to\infty} \lim_{a\to 0} \mu_{a,L} \quad (5)$$

and the axiom step and the uniform-gap step are properties of $\mu$. There is also the road, as in strong-coupling results on the lattice, of holding $a$ fixed and taking $L \to \infty$ first, in which case the ultraviolet-limit step is left for later. However, the strong-coupling region (where the lattice coupling $g$ is large and $\beta = 2N/g^2$ small) is far from the continuum limit ($\beta \to \infty$), so this road does not as it stands connect to the ultraviolet-limit step. Known

Figure 1 Choosing a theory (with the buttons below or a row of Table 2) changes how far each step has got




YM four dimensions: Ultraviolet stability on the lattice in finite volume (Bałaban 1987–89; with the assumption that the effective coupling enters a sufficiently small range; 7-07). MRS 1993 is an outline of a construction with an infrared cutoff. The continuum finite-volume measure has not been constructed

Known Table 2 How far each theory had got before 2026 (Closed = there is a theorem / Partial / Not yet / — = the question is hard to pose / Not taken = that road does not pass through the step). From the problem statement and primary sources. Years are the publication years of representative papers (the timeline in 16-01 counts by first public appearance, so Sengupta is 1992–93 and Lévy 2001). Selecting a row changes Figure 1.

| Theory | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Free field | Closed | Closed | Closed | Closed | Closed |
| $\phi^4_2$ weak coupling | Closed | Closed | Closed | Closed | Closed |
| $\phi^4_3$ weak coupling | Closed | Closed | Closed | Closed | Closed |
| YM two dimensions | Closed | Closed | Closed | Closed | Partial |

| Theory | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| YM three dimensions | Closed | Closed (lattice) | Partial | Not yet | Not yet |
| YM four dimensions | Closed | Closed (lattice) | Partial | Not yet | Not yet |

| Theory | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Lattice YM four dimensions, strong coupling | Closed | Closed | Not taken | Closed | Partial |

The current marks, including the published construction (2026), are in 1-12 Table 4 and 7-07 Table 4. In four dimensions, of the six steps of construction, the regularisation, finite-volume (lattice), ultraviolet-limit and axiom steps are closed, the infinite-volume step is partial (up to subsequences of volumes), and the uniform-gap step is not yet done (as marked in 1-12).

The "—" in Table 2 means that the question is hard to pose in its plain form. Two-dimensional Yang–Mills has no propagating degrees of freedom (waves that travel far, the counterpart of light in electromagnetism), so the question of a spectral gap cannot be posed in the same form as in four dimensions (7-02). Known "Not taken" means that the road does not pass through the step. At strong coupling on the lattice $a$ stays fixed, so the road does not pass through the ultraviolet-limit step. For three- and four-dimensional Yang–Mills, the finite-volume step closes by formula (6) on the lattice (§ 04). Building the finite-volume measure directly in the continuum has not yet been done, and that part is recorded as "Partial" for the ultraviolet-limit step and in the summary column.

The four-dimensional YM row of Table 2 is what the literature before 2026 reached. On 2026-11-10, a Lean formalisation that a theory satisfying the Wightman axioms can be built as a scaling limit of four-dimensional Wilson lattice gauge theory was published, together with the committee's assessment report. The committee's assessment is that Lean proves the following: the ultraviolet limit in the form of correlation functions of local observables, the infinite-volume limit along subsequences of volumes, and from there the verification of the axioms and reconstruction (6-01; for reconstruction, 12-08). Convergence and uniqueness of the infinite-volume limit, a probability measure for the continuum theory, and a uniform mass gap (exponential decay) are not shown (6-04).

> 04

## The regularisation and finite-volume measure steps

Regularisation can be done with a **lattice** (putting the field on lattice points or edges) or a **momentum cutoff** (restricting Fourier components to $|k| \le \kappa$), and both break something.

| Property | Lattice (Wilson action) | Momentum cutoff |
|---|---|---|
| Gauge invariance | Kept exactly | Broken. Counterterms are needed to restore it (MRS 1993 § I) |
| Reflection positivity | Kept (Osterwalder–Seiler 1978) | Easily lost. Ways to recover reflection positivity lost through regularisation are generally scarce (JW § 6.5). MRS 1993 § I says that gauge choices lacking positivity (Feynman type) cannot be used directly from the start, and that OS positivity could probably be shown by cutting off first with a lattice and using a momentum cutoff afterwards |
| Rotational symmetry | Broken | The cutoff itself can be made rotationally symmetric. However, gauge fixing (choosing one representative from fields related by gauge transformations) and a finite box can break rotations. MRS 1993 uses a gauge that treats a particular direction specially (axial gauge) and an infrared cutoff |

On the lattice the finite-volume measure has meaning at once. If the gauge group $G$ is compact,

$$\mathrm{d}\mu_{a,L}(U) = \frac{1}{Z_{a,L}}\, e^{-S_W(U)} \prod_{\ell} \mathrm{d}U_\ell \qquad (6)$$

where $\ell$ are the edges of the lattice, $U_\ell \in G$, $\mathrm{d}U_\ell$ is the Haar measure (the invariant measure of total weight 1), and $S_W$ is the Wilson action

$$S_W(U) = \beta \sum_P \left(1 - \tfrac{1}{N} \operatorname{Re}\operatorname{tr} U_P\right), \qquad \beta = \frac{2N}{g^2} \quad (G = SU(N))$$

The sum is over the smallest squares of the lattice (plaquettes) $P$, and $U_P$ is the product of the $U_\ell$ on the four edges of $P$ taken in order around it (5-01). The integrand is bounded and continuous and the domain compact, so $Z_{a,L}$ is finite and positive (JW § 6.5). Known

**Example:** a four-dimensional periodic lattice with 16 points on a side has $4 \times 16^4 = 262144$ edges, and `SU(3)` is 8-dimensional over the reals, so formula (6) is an integral in 2,097,152 dimensions. Computed

Making it directly in the continuum is hard, and even three-dimensional Yang–Mills has not yet been constructed on the torus $T^3$ (Chevyrev 2022). On the lattice, the finite-

volume step is closed by formula (6) ("Closed (lattice)" in Table 2), and instead the recovery of rotational symmetry is passed on to the axiom step.

05

## The ultraviolet-limit step

As $a \to 0$, unless the coefficients of the action (the bare couplings) are moved according to $a$, the limit diverges or collapses to something trivial. This moving is renormalisation (4-04). According to the problem statement, for $\phi^4$, if the cutoff-dependent constants are chosen in the form indicated by perturbation theory, expectation values converge even though the constants diverge. In two dimensions a few constants suffice; in three dimensions one needs all the coefficients plus wave-function renormalisation, that is, renormalising the state vectors (vectors of the Fock space of 2-06) by a constant that diverges with the cutoff (JW § 6.2). Known

What is needed can be guessed by counting dimensions. Known Since the action has no units, in units of mass

$$[\phi] = \frac{d-2}{2}, \qquad [\lambda] = 4 - d, \qquad [g^2] = 4 - d \tag{7}$$

($g$ is the Yang–Mills coupling). For $d < 4$ the coupling has positive mass dimension (**super-renormalisable**), and there are finitely many types of superficially divergent diagrams (pictures of the terms of perturbation theory; 4-03). "Superficially" means that divergence is judged only by counting the powers of momentum in the integral attached to the diagram (4-04; as a textbook, Rivasseau 1991). $d = 4$ is the boundary where the coupling has no units. Table 1 can be read by the same counting. Since $[\phi^2] = d - 2$, $G_a(0)$ grows as $a^{2-d}$, and for $d = 2$ logarithmically (1-12).

**Example:** in two dimensions it suffices to redefine the product as $:\phi^4:_a = \phi^4 - 6G_a(0)\phi^2 + 3G_a(0)^2$ (the Wick-ordered product), together with a constant for the vacuum energy (the $\alpha, \gamma$ of JW § 6.2).

What one wants to show is that expectation values are bounded by constants independent of $a$,

$$\sup_{a>0} \left|\langle \Phi \rangle_{a,L}\right| \le C_\Phi \tag{8}$$

With a uniform estimate (8), one can take a subsequence of the measures that converges in a weak sense (the measure version of taking a convergent subsequence from a bounded sequence of numbers). The limit, however, need not be unique, and the axioms and the gap are separate questions. The problem statement does not accept

weak existence by compactness alone as a solution unless the properties of the limit are shown separately (JW footnote 2). Four dimensions is treated in 7-07.

06

## The infinite-volume-limit step

$L \to \infty$ needs estimates independent of $L$, and the key is in particular that quantities far apart become nearly independent (**clustering**). The problem statement lists as ways to set up estimates (a) correlation inequalities, (b) symmetries, and (c) convergent expansions (cluster expansions), and says that (a) tends to be limited to scalar fields and abelian gauge theories, and (c) became the central tool (JW § 6.1). The form one wants to show is

$$\left|\langle A\, B_x\rangle_L - \langle A\rangle_L \langle B_x\rangle_L\right| \le C\, e^{-m|x|} \quad (C,\ m \text{ independent of } L\ ) \tag{9}$$

($B_x$ is $B$ shifted by $x$). Given a decay rate $m$, the influence of the edge of the box reaches only about $e^{-mL}$.

**Example:** for the one-dimensional chain, the difference between the variance at a point of a ring of length $L$ and the value for the infinite chain is, for large $L$, almost $e^{-mL}/m$. For $m = 1$ it is $1.1 \times 10^{-7}$ at $L = 16$; for $m = 0.1$ it is still 2.5. The rate of approach is set by $m$. Computed

In four-dimensional Yang–Mills the coupling is thought to become strong at long distances. Physics This view comes from the renormalisation group of perturbation theory (4-04). MRS 1993 § I writes, as the reason for not removing the infrared cutoff, 'this would lead to large values of the coupling constant', and further 'there is no easy solvable model of these phenomenons around which to expand'.

As a metaphor, the scaffolding for the expansion is gone. Precisely, there is neither a solvable model to serve as the starting point of an expansion nor a small parameter to expand in. The problem statement too says that present methods do not suggest how to obtain $T^4 \to \mathbb{R}^4$ (JW § 6).

07

## The axiom step

Axioms that hold with the regularisation tend to pass to the limit, and what was broken has to be recovered. The key is **reflection positivity**: for time reversal $\Theta : t \mapsto -t$, for $F$ depending only on the field at $t > 0$,

$$\left\langle\, \Theta F \; F \,\right\rangle \;\ge\; 0 \tag{10}$$

holds. This is the condition for extracting a Hilbert space and a Hamiltonian (5-03).

Formula (10) is an inequality, so it passes to the limit. Known There are three conditions. The $F$ for which the limit is taken can be approximated by functions depending only on the lattice field on the positive-time side. Their expectation values converge. The reflection plane is a lattice plane (a plane through lattice points, or a plane between neighbouring lattice points). Then it is a limit of non-negative values in each $\mu_{a,L}$, so it is non-negative in the limit too. For lattice gauge theory, reflection positivity with respect to lattice planes has been shown (Osterwalder–Seiler 1978; MRS 1993 § I also cites it).

Conversely, ways to recover lost reflection positivity are scarce, and the problem statement writes that setting up the Hilbert space is itself part of the problem (JW § 6.5). Rotational symmetry broken by the lattice has to be shown to recover separately. Known How the published construction satisfies these is read in Part 12 (12-01 Lattice reflection positivity, 12-06 Euclidean invariance, 12-08 Reconstruction and the Wightman axioms).

**Example:** for a Gaussian measure, reflection positivity is the same as positivity of $\Theta C$ restricted to the positive-time side (JW § 6.2). In one dimension $C(t) = e^{-m|t|}/(2m)$, so for $t_i, t_j > 0$

$$\sum_{i,j} f_i \, f_j \, C(t_i + t_j) = \frac{1}{2m} \Big( \sum_i f_i \, e^{-m t_i} \Big)^2 \;\geq\; 0 \tag{11}$$

becomes a square. On a lattice chain with 40 points, trying 20000 random $f$ gives a minimum of $1.4 \times 10^{-8}$, never negative. Being a square, this is to be expected; it is a check of the formula. As a control, with the sign-reversed covariance $-C(t_i + t_j)$ the smallest eigenvalue of the matrix is $-2.49$, and reflection positivity fails. Computed

Stochastic quantisation (7-09) makes the measure as the invariant measure of an equation, so reflection positivity is not part of the output, and has to be shown as a separate task.

08

## The uniform-gap step

Next one asks about the gap in the spectrum of the Hamiltonian (4-11). On the Euclidean side it is the rate of decay of correlations. On the lattice the decay rate in the time direction is the gap in lattice units $m_{\mathrm{lat}}$, and the gap in physical units is

$$\Delta = \lim_{a \to 0} \frac{m_{\mathrm{lat}}(a)}{a} \tag{12}$$

One wants to show that this is positive and finite, and moreover uniform in the volume.

On a lattice in a finite box the gap is there automatically. Known The **transfer matrix** $T$ is the integral operator that carries weight from the slice at time $t$ (the set $u$ of values of the lattice field at that time) to the slice $v$ at time $t + a$

$$(Tf)(u) = \int K(u, v)\, f(v)\, \mathrm{d}v$$

The kernel $K(u, v)$ is made as $e^{-S_{t,t+a}(u,v)}$ from the part $S_{t,t+a}(u, v)$ of the action connecting the two times. Reflection positivity makes $T$ a positive operator, and the Hamiltonian is fixed by $T = e^{-aH}$. If the range of field values is bounded and closed (lattice gauge theory with a compact gauge group is such a case) and the kernel is continuous and positive, the largest eigenvalue $\lambda_0$ is not degenerate. This is the infinite-dimensional version (a generalisation of Jentzsch's theorem) of the Perron–Frobenius theorem for matrices with all entries positive Known (the generalisation is Schaefer 1974; sources in 5-03). So the ratio to the next eigenvalue $\lambda_1$ is $\lambda_1/\lambda_0 < 1$, and there is a gap $-\log(\lambda_1/\lambda_0)/a > 0$ (5-03; the positive self-adjoint transfer matrix of lattice gauge theory is Osterwalder–Seiler 1978). What is asked is uniformity under $a \to 0$ and $L \to \infty$, and the problem statement writes that a gap uniform in the volume needs new ideas (JW § 6.5).

**Example:** for the chain with $m = 1$, $m_{\mathrm{lat}}/a$ approaches 1 as 0.9624 ($a = 1$), 0.9974 ($a = 0.25$), 0.99990 ($a = 0.05$), while $m_{\mathrm{lat}}$ in lattice units itself goes towards 0, about 0.05 at $a = 0.05$. In the continuum limit **the gap in lattice units closing is the correct picture**, and what should be looked at is the ratio (12) (5-09). The ratio of eigenvalues of the transfer matrix, $-\log(\lambda_1/\lambda_0)/a$, agrees with the decay rate of correlations, the difference being at most $5 \times 10^{-14}$. Computed

09

## Detour: taking a one-dimensional chain through all six stages

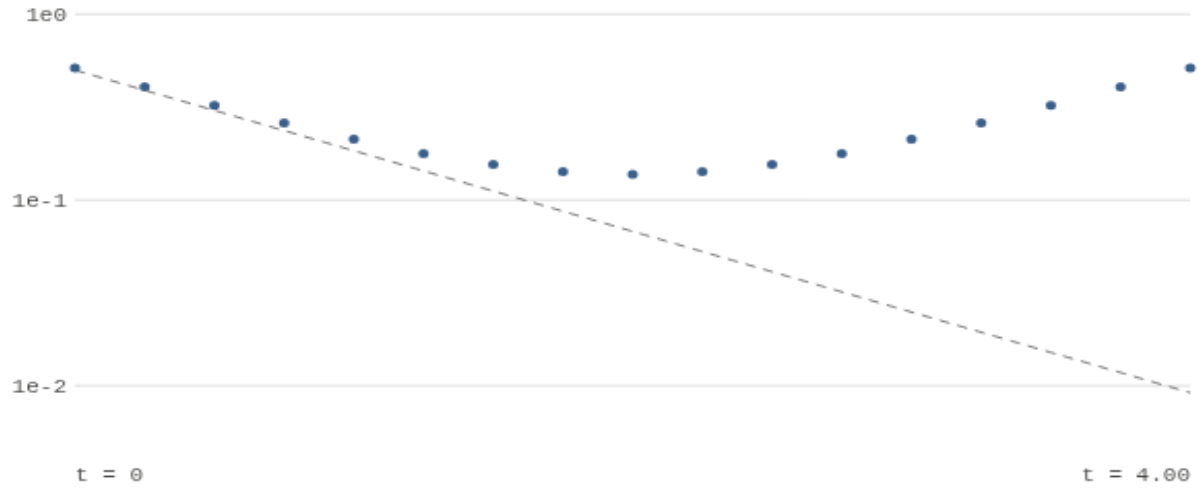
A section you may skip.

Euclidean field theory with $d = 1$ is quantum mechanics: the field $\phi(t)$ is the position of a particle, and the action $\int (\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2)\,\mathrm{d}t$ is the imaginary-time action of the harmonic oscillator (2-06, 2-07). In the regularisation step time is cut into steps of $a$, in the finite-volume step it is made a ring of length $L$, and in the ultraviolet-limit and infinite-volume steps $a \to 0$ and $L \to \infty$. The axiom step is formula (11), and the gap of the uniform-gap step is the level spacing $m$. The correlation on a ring of $N = L/a$ points can be written in closed form.

$$\cosh m_{\mathrm{lat}} = 1 + \tfrac{1}{2}m^2a^2, \qquad \langle \phi_0\, \phi_n \rangle_L = \frac{a\, \cosh\bigl(m_{\mathrm{lat}}(N/2 - n)\bigr)}{2 \sinh m_{\mathrm{lat}}\, \sinh(m_{\mathrm{lat}} N/2)} \quad (13)$$

It agrees, at the level of $10^{-14}$, with the values obtained by inverting the matrix of the action directly. Computed

Figure 2 Moving $a$, $L$ and $m$ changes the correlation on the ring (vertical axis logarithmic) and the lattice gap $m_{\mathrm{lat}}/a$




a = 0.25, L = 4.00 (16 points), m = 1.00, m_lat/a = 0.9974, <φ²>_L = 0.5148 (continuum, infinite 0.5000)

Computed The numbers in this figure were computed on this machine (formula (13)). The points are lattice values, the dashed line the infinite, continuum $e^{-mt}/(2m)$.

Stretching $L$ makes the contribution that goes around the ring (the rise on the right) disappear, and the points fall onto the dashed line. Making $m$ smaller makes the needed $L$ longer in proportion to $1/m$. This is the link between the infinite-volume step and the uniform-gap step, that the gap sets the rate of the infinite-volume limit, and the problem statement too writes that a uniform gap may play a basic role in proving the infinite-volume limit (JW § 5). We look at the two limits in separate tables (both with $m = 1$).

Computed Table 3 Making the lattice spacing $a$ finer (the ultraviolet-limit and uniform-gap steps; infinite chain).

| $a$ | $m_{\mathrm{lat}}/a$ | $\langle\phi^2\rangle$ **(infinite chain)** |
|---:|---:|---:|
| 1 | 0.962424 | 0.447214 |
| 0.25 | 0.997414 | 0.496139 |
| 0.05 | 0.999896 | 0.499844 |
| 0.01 | 0.999996 | 0.499994 |
| $a \to 0$ | $\to 1$ | $\to 1/(2m) = 0.5$ |

Computed Table 4 Stretching the length $L$ of the ring (the infinite-volume step; $a = 0.05$). The value for $L = \infty$ is 0.499844 of the infinite chain (the $a = 0.05$ row of Table 3).

| $L$ | $\langle\phi^2\rangle_L$ | **Difference from the infinite chain** |
|---:|---:|---:|
| 1 | 1.081735 | $5.8 \times 10^{-1}$ |
| 2 | 0.656350 | $1.6 \times 10^{-1}$ |
| 4 | 0.518503 | $1.9 \times 10^{-2}$ |

| $L$ | $\langle\phi^2\rangle_L$ | Difference from the infinite chain |
|---|---|---|
| 8 | 0.500180 | $3.4 \times 10^{-4}$ |
| 16 | 0.499844 | $1.1 \times 10^{-7}$ |

Everything goes through in one dimension because values at points are random variables and the ultraviolet-limit step has no divergence (Table 1). Adding an anharmonic term $\lambda\phi^4$ still makes sense as quantum mechanics. Known

10

## How to read Part 7

Each row of Table 2 shows the steps being closed in order from the regularisation step. The chapters of Part 7 follow, step by step, which tools gave each "Closed", and where the "Partial" and "Not yet" stop. Reading two-dimensional Yang–Mills and scalar fields as examples where the steps went through, the cluster expansion and the renormalisation group as tools for the steps where they work, and the chapters on three and four dimensions as records of the steps that are stuck, shows where each chapter stands. One chapter may touch several steps, so the correspondence below gives the main steps treated.

- From regularisation to infinite volume: 7-02 two-dimensional Yang–Mills
- All six steps: 7-03 $\phi^4_2$, $\phi^4_3$ (examples where all six steps went through)
- The infinite-volume step and the uniform-gap step: 7-04 the cluster expansion
- The ultraviolet-limit step: 7-05 the renormalisation group, 7-06 three dimensions, 7-07 four dimensions, 7-08 triviality, 7-12 Detour: 1/0
- The finite-volume and ultraviolet-limit steps by another road: 7-09 stochastic quantisation
- All: 7-10 where four dimensions stops, 7-11 obstacles other than divergence, 16-01 the timeline

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## What this chapter can say

| | Content |
|---|---|
| **Said** | Constructive field theory is a method of exhibiting non-Gaussian measures satisfying the axioms as limits of sequences of approximations, with uniform estimates at its centre (JW § 6.1) Known |
| **Said** | The variance at a point of the lattice free field is finite for $d = 1$, and grows as $\log(1/a)$, $1/a$, $1/a^2$ for $d = 2, 3, 4$ ($a \geq 1/64$) Computed |

| | Content |
|---|---|
| **Said** | For the one-dimensional chain the six stages go through in closed form, and the decay rate of correlations agrees with the eigenvalue ratio of the transfer matrix Computed |
| **Not said** | The division into six stages is this book's arrangement, not a standard division in the field |
| **Not said** | The content of the "Closed" entries in Table 2 (the proofs of the original papers were not followed. $\phi^4_2$ and $\phi^4_3$ are from the problem statement. The gap for $\phi^4_3$ is from the title of the paper) |
| **Not said** | A way to advance uniqueness of the infinite-volume limit and the uniform-gap step for four-dimensional Yang–Mills (the problem statement [2000] says there are no ideas pointing the direction; the construction published on 2026-11-10 does not show these two either; 6-04) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Accounts and quotations of JW (§5, §6, §6.1, §6.2, §6.5, footnote 2) | Primary source (checked in the text) | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000) |
| The OS axioms, textbook | Bibliographic data only | K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83–112; 42 (1975) 281–305. J. Glimm, A. Jaffe, *Quantum Physics*, 2nd ed., Springer (1987) |
| $\phi^4_2$, $\phi^4_3$ | Bibliographic data only (the reference list of the problem statement) | Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585–632; Glimm–Jaffe, Fortschr. Phys. 21 (1973) 327–376; J. Feldman, K. Osterwalder, "The Wightman axioms and the mass gap for weakly coupled $\phi^4_3$ quantum field theories", Ann. Phys. 97 (1976) 80–135 |
| Textbook on perturbation theory and renormalisation | Bibliographic data only (the reference list of the problem statement) | V. Rivasseau, *From Perturbative to Constructive Renormalization*, Princeton Univ. Press (1991) |
| Reflection positivity on the lattice, the positive self-adjoint transfer matrix, infinite volume at strong coupling | Bibliographic data and abstract only | K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 |

| Item | Kind | Source / tools |
|---|---|---|
| The Perron–Frobenius and Jentzsch theorems | Known theorems (Perron 1907 and Jentzsch 1912: bibliographic data only. The generalisation to kernels on compact spaces is Schaefer 1974 [bibliographic data only, claim checked in secondary sources] and Krein–Rutman 1948 [bibliographic data only]) | See 5-03 |
| Exponential decay at strong coupling | Checked in the abstract | H. Shen, R. Zhu, X. Zhu, Comm. Math. Phys. 400 (2023) 805–851, arXiv:2204.12737 |
| Continuum measure in two dimensions (the plane) | Checked in the abstract and introduction | B. K. Driver, Comm. Math. Phys. 123 (1989) 575–616 |
| Continuum measure in two dimensions (compact surfaces) | Bibliographic data only (Lévy 2003 checked as far as the abstract) | A. Sengupta, Mem. Amer. Math. Soc. 126 no. 600 (1997); T. Lévy, Mem. Amer. Math. Soc. 166 no. 790 (2003), arXiv:math/0101239 |
| Ultraviolet stability in three and four dimensions | 1985 and 1989 (CMP 122, 355–392): the opening and the main theorem (the Conditional Ultraviolet Stability Theorem; Theorem 1) checked. 1987 and the rest of the series: bibliographic data only | T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275; 109 (1987) 249–301; 122 (1989) 355–392 |
| Construction with an infrared cutoff | Abstract and § I checked | J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383 |
| The three-dimensional measure has not been constructed | Text § 1.1 checked | I. Chevyrev, arXiv:2202.13359 (2022) |
| Table 1, dimension counts, the one-dimensional chain | Computed | Python on this machine (numpy, scipy): `point_variance.py`, `counts.py`, `chain.py` (the transfer matrix diagonalised with positions on 1601 points) |
| Figures 1 and 2 | Figures | SVG, canvas and plain JavaScript. Figure 1 reads Table 2 (if JavaScript is disabled, a static figure for YM in four dimensions). Figure 2 computes formula (13) |

Next chapter to read: 7-02 Solvability in two dimensions and the Yang–Mills measure (an example in which the four steps from regularisation to infinite volume close).

# Solvability in two dimensions and the Yang–Mills measure — on the plane the plaquettes become independent, and the measure can be made

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — On a two-dimensional lattice, fixing the gauge on a tree makes the plaquettes independent, and the area law holds exactly at every coupling. In the continuum limit the heat kernel of the group appears, and continuum measures have been constructed on the plane and on closed surfaces. On the torus, stochastic quantisation also arrived at the same measure.

**Prerequisite chapters** — 5-01 Wilson's lattice. Characters are supplied on the spot, within the range of 1-04. Expectation, independence and densities are in 1-08, white noise and Brownian motion in 1-09, the six stages in 7-01.

**Said first** — Pure Yang–Mills in two dimensions is not a scaled-down version of four dimensions. It has no propagating degrees of freedom, and the measure sees only area. The core of the solvability cannot be carried over to four dimensions ( § 09). This chapter presents known results.

Order of this chapter

1. Why start with two dimensions — two dimensions is the only case where a continuum measure has been constructed for a non-abelian group
2. Fixing a tree makes the plaquettes independent — the number of edges matches the number of faces
3. The area law is exact at every β — peeling off faces one at a time with a lemma on characters
4. The continuum limit: the heat kernel of the group — area becomes the time of the heat equation
    - Detour: a theory that sees only area — area-preserving deformations leave expectation values unchanged

5. Driver and Gross–King–Sengupta — the field strength becomes white noise, and parallel transport becomes a stochastic differential equation
6. On closed surfaces — the faces are no longer independent, but the measure has been constructed
   - Detour: letting the area go to 0 — the volume of the space of flat connections comes out
7. Seen with the transfer matrix — the lowest excitation is proportional to the length of the circle, and no particles propagate
8. Reconstruction by stochastic quantisation — on the torus, the same measure obtained as an invariant measure
9. What can be carried out, and what cannot — the methods extend, but the independence of faces does not

---

01

## Why start with two dimensions

Two dimensions is, so far, the only case in which a continuum Yang–Mills measure has been constructed for a non-abelian gauge group. Chatterjee (2018) writes that no continuum limit in which Wilson loops behave non-trivially has been constructed in any dimension higher than two. Known The four-dimensional construction published on 2026-11-10 also builds the continuum theory in the form of correlation functions of local observables, and does not build a probability measure or Wilson-loop observables (6-04). In two dimensions the field strength has one component.

$$F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2], \qquad S(A) = \frac{1}{2g^2} \int_{\mathbb{R}^2} |F_{12}|^2 \, d^2x \quad (1.1)$$

$A$ is a Lie-algebra-valued connection (3-02), and $|\cdot|$ a conjugation-invariant length. $F$ has the dimension of an inverse length squared, so for the action to be dimensionless $g^2$ must have the dimension of an **inverse area**, and $g$ itself provides the scale. In four dimensions $g$ is dimensionless, and the scale is thought to be generated by quantum effects Physics (4-07). The protagonist of this chapter is a property found only in two dimensions: **fixing the gauge suitably makes the degrees of freedom of the faces independent**. In terms of the six steps (7-01), the steps from the regularisation step to the infinite-volume step close, the axiom step is partial (Euclidean invariance; § 05), and the gap of the uniform-gap step is not a question with the same meaning as in four dimensions ( § 07).

02

## Fixing a tree makes the plaquettes independent

The set-up is as in 5-01: $U_\ell \in G$ on an edge $\ell$, $U_p$ for the product around a face $p$, and the weight $\prod_p e^{\beta \operatorname{Re}\operatorname{tr} U_p}$ ($dU$ is the Haar measure). The β of this chapter is half of the $\beta_W$ customary for $SU(2)$ (Wilson action $\beta_W \sum_p \left(1 - \tfrac{1}{2}\operatorname{Re}\operatorname{tr} U_p\right)$; in four dimensions $\beta_W = 4/g^2$): $\beta = \beta_W/2$. β = 1 in § 03 corresponds to $\beta_W = 2$. A gauge transformation $U_{xy} \mapsto g_x U_{xy} g_y^{-1}$ does not change $\operatorname{tr} U_p$. The variables on a tree (a set of edges with no cycles) can be set to 1 by choosing $g_x$ in order from the root, without changing expectation values of gauge-invariant quantities. Known Counting on an $L \times L$ square lattice (free boundary),

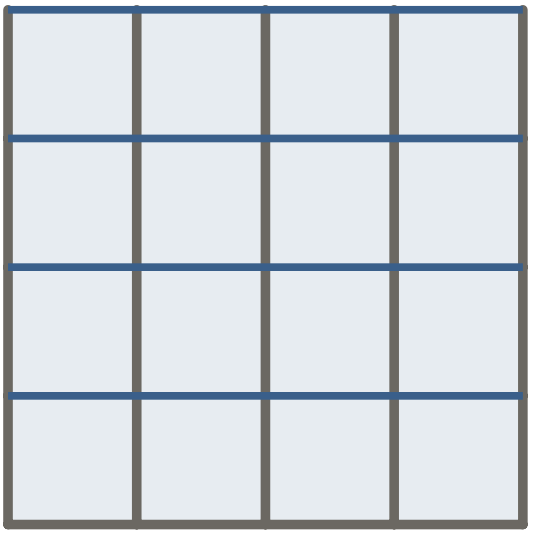
$$V = (L+1)^2, \quad E = 2L(L+1), \quad P = L^2, \qquad E - (V-1) = P. \tag{2.1}$$

the tree has $V - 1$ edges, so the remaining edges are as many as the faces (Euler's formula for the disc). Taking the tree to be the vertical edges and the bottom edges (as in Driver 1989), what remains are the horizontal edges $h_{i,j}$ ($h_{i,0} = 1$), and

$$U_{p(i,j)} = h_{i,j}\, h_{i,j+1}^{-1} \iff h_{i,j+1} = U_{p(i,j)}^{-1} h_{i,j}. \tag{2.2}$$

can be solved from the bottom up in one-to-one correspondence, and since the Haar measure is invariant under multiplication the measure carries over. **The face variables are mutually independent, each with density $\rho_\beta(U) = e^{\beta \operatorname{Re}\operatorname{tr} U}/z(\beta)$.** Known Extending to the whole plane, the infinite-volume limit exists independently of the boundary condition and, on a finite region, agrees with the measure with free boundary (Theorem 7.2 of Driver, shown by integrating out the outer edges using the invariance of the Haar measure; Driver writes that this theorem is essentially contained in Dosch–Müller 1979). Known

**Figure 1 The tree gauge** — The knob $L$ changes the size of the lattice. "Play" fixes the tree (grey) to 1 and fills the faces from the bottom. Each face filled fixes one edge above it.



L = 4: 25 vertices, 40 edges, 16 faces. Fixing the 24 edges of the tree to 1 leaves 16 edges, matching the number of faces.

Computed The numbers in this figure (numbers of vertices, edges and faces) were computed on this machine. The numbers are formula (2.1).

03

## The area law holds exactly at every β

The tool is the **character** $\chi_\lambda(U) = \operatorname{tr} \lambda(U)$ ($\lambda$ an irreducible unitary representation, of dimension $d_\lambda$). It is conjugation-invariant, and $\int_G \chi_\lambda \chi_\mu \, dU = \delta_{\lambda\mu}$. For $SU(2)$ the representation of spin $j$ has dimension $2j+1$, and for an element with eigenvalues $e^{\pm i\theta}$, $\chi_j = \sin((2j+1)\theta)/\sin\theta$ (1-04). The key is a lemma: if the density of $X$ is conjugation-invariant and $X$ is independent of $A$, then

$$\mathbb{E}[\chi_\lambda(XA)] = w_\lambda \, \chi_\lambda(A), \qquad w_\lambda = \mathbb{E}[\chi_\lambda(X)]/d_\lambda . \tag{3.1}$$

This is because $\mathbb{E}[\lambda(X)]$ commutes with every $\lambda(V)$, so by Schur's lemma it is $w_\lambda I$. When a simple closed curve $C$ encloses $n$ faces, the variables of the enclosed faces appear once each in the tree-gauge $U_C$, accompanied by conjugations by other face variables. The densities are conjugation-invariant, so (3.1) applies as it is, and moving faces one at a time to the front by cyclicity of the trace and peeling them off gives

$$\frac{\mathbb{E}[\chi_\lambda(U_C)]}{d_\lambda} = w_\lambda(\beta)^n, \qquad \langle W_C \rangle = e^{-\sigma_{\text{lat}} n}, \quad \sigma_{\text{lat}} = -\ln w_{\text{fund}} . \tag{3.2}$$

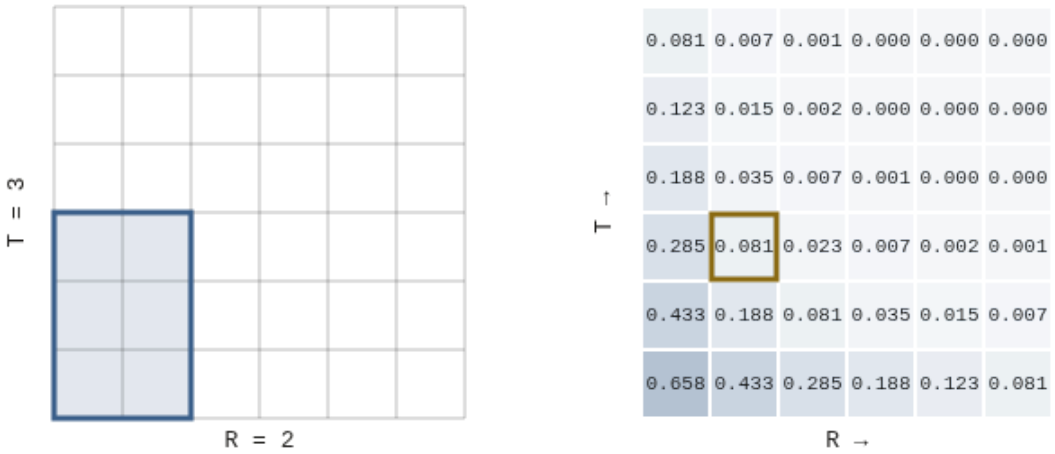
Here $W_C = \chi_{\text{fund}}(U_C)/d_{\text{fund}}$ (the value in the fundamental representation; $\frac{1}{2}\operatorname{tr} U_C$ for $SU(2)$). Migdal (1975) showed that in two dimensions the transformation rule (recursion) for the weights under coarsening the lattice becomes exact, wrote the formula in which the expectation of a character is exponential in the enclosed area $A$,

$\langle \chi_q \rangle = d_q e^{-g_q^2 A}$ ($q$ a representation, $g_q^2$ a constant for each representation), and stated that it is "nothing but Wilson's area law". Known The value is fixed by the area alone, and holds **at every β**. This contrasts with four dimensions, where the area law has been shown only on the strong-coupling side (5-02). The coefficient can be written by the following formula in modified Bessel functions, and the difference from numerical integration was at most $2 \times 10^{-16}$. Computed

$$SU(2):\ w_j = \frac{I_{2j+1}(2\beta)}{I_1(2\beta)}, \qquad U(1):\ w_n = \frac{I_n(\beta)}{I_0(\beta)} \quad (3.3)$$

$I_n$ is the modified Bessel function, and for integer $n$, $I_n(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta\, d\theta$. When $x$ is large the ratio $I_n(x)/I_1(x)$ approaches 1, and § 04 uses how it approaches.

**Figure 2 The area law (SU(2))** — The knob β changes the value $w$ of a single face, and R and T the shape of the loop. On the right is $w^{RT}$ for the 6×6 rectangles; the same area gives the same value.




β = 2.0 : w = 0.6580、σ = 0.4185。2×3 (area 6, perimeter 10) : ⟨W⟩ = 0.0812。

Computed The numbers in this figure were computed on this machine. The $w$ in the figure is the Bessel series (3.3) summed on the spot, agreeing with values computed with scipy to the fifth decimal place. If JavaScript is disabled, see Table 1.

**Table 1** $SU(2)$. "Random" is heat-bath Monte Carlo on a 6×6 lattice (free boundary) **without gauge fixing**. Brackets give the error in the last digit. Computed

| β | $w$ | $\sigma_{\text{lat}}$ | 2×2 formula | 2×2 random | 3×3 formula | 3×3 random |
|---|---|---|---|---|---|---|
| 1 | 0.4331 | 0.8367 | 0.0352 | 0.0366(11) | 0.0005 | −0.0002(15) |
| 2 | 0.6580 | 0.4185 | 0.1875 | 0.1893(12) | 0.0231 | 0.0223(16) |
| 4 | 0.8192 | 0.1994 | 0.4505 | 0.4514(12) | 0.1662 | 0.1649(29) |

The calculation without gauge fixing agrees with the tree-gauge formula within errors.

04

## The continuum limit: the heat kernel of the group appears

At lattice spacing $a$, a loop of area $A$ encloses $n = A/a^2$ faces. Taking the length in (1.1) as $|X|^2 = -2\,\mathrm{tr}\,X^2$, the expansion $U_p \approx e^{a^2 F_{12}}$ gives, for $SU(2)$, $\beta = 2/(g^2 a^2)$. From $I_n(x)/I_1(x) = 1 - \frac{n^2-1}{2x} + O(x^{-2})$, $w_j = 1 - j(j+1)/\beta + O(\beta^{-2})$, so

$$w_j(\beta)^{A/a^2} \longrightarrow e^{-g^2 A\, j(j+1)/2} \qquad (a \to 0). \tag{4.1}$$

The right-hand side is a coefficient of the **heat kernel** $Q_t$ of the group. $\Delta$ is the Laplacian on the group (different from the $\Delta$ of the mass gap), the operator that has the characters as eigenfunctions, $\Delta\chi_\lambda = -c_\lambda \chi_\lambda$. $c_\lambda \geq 0$ is the Casimir value determined by each representation, $c_j = j(j+1)$ for $SU(2)$. The heat kernel is the solution of the heat equation $\partial_t Q_t = \frac{1}{2}\Delta Q_t$ concentrated at the identity at time 0, and expanded in characters

$$Q_t(U) = \sum_\lambda d_\lambda\, e^{-c_\lambda t/2} \chi_\lambda(U), \qquad Q_s * Q_t = Q_{s+t} \tag{4.2}$$

The right-hand side of (4.1) is the coefficient $e^{-c_j t/2}$ at $t = g^2 A$. That is, **the holonomy of a simple closed curve of area $A$ follows the heat kernel $Q_{g^2 A}$.** Known The semigroup property is "adding areas adds times". In Table 2 each of $j = \frac{1}{2}, 1, \frac{3}{2}$ approaches the heat kernel, and the error becomes 1/4 each time $a$ is halved. Computed

**Table 2** $SU(2)$, $g = 1$, area 1, $\beta = 2/a^2$. Values are $w_j^n$. Computed

| $a$ | $n$ | $j = \frac{1}{2}$ | $j = 1$ | $j = \frac{3}{2}$ |
|---|---|---|---|---|
| 1/2 | 4 | 0.67919 | 0.35713 | 0.14579 |
| 1/4 | 16 | 0.68527 | 0.36504 | 0.15119 |
| 1/32 | 1024 | 0.68726 | 0.36783 | 0.15332 |
| Heat kernel | ∞ | 0.68729 | 0.36788 | 0.15335 |

Taking the face weights to be heat kernels from the start gives, by the semigroup property, a measure independent of how the faces are divided (Migdal's observation). Driver showed that lattices with heat-kernel weights (the Villain action) and with the Wilson action converge to the same continuum measure, using for the latter a group-valued central limit theorem (Borgs–Seiler). Known

### Detour: a theory that sees only area

A subsection you may skip.

According to Corollary 6.7 of Driver, expectation values of gauge-invariant quantities do not change under **area-preserving diffeomorphisms** (as long as the arrangement of curves is carried to one of admissible form). As a metaphor, it is a theory on a rubber sheet: stretching the sheet while preserving area changes nothing. Precisely, expectation values can be written as finite-dimensional integrals of products of heat kernels, into which only the areas of the faces and the way the curves cross enter (Theorem 6.4).

05

## The continuum measure: Driver and Gross–King–Sengupta

The continuum version of the tree gauge is the **complete axial gauge** ($A_2 = 0$, and $A_1 = 0$ on the $x$ axis). Then $[A_1, A_2] = 0$, so the field strength becomes linear, and the weight $e^{-S}$ becomes Gaussian in $F$.

$$F_{12} = -\partial_2 A_1, \qquad \mathbb{E}\big[F^a(R)F^b(R')\big] = g^2\delta^{ab}|R \cap R'| \quad (5.1)$$

$F$ is **Lie-algebra-valued white noise** ($F^a(R)$ is the integral over a region $R$, $|R|$ its area; 1-09), and its values on disjoint regions are independent. It is the continuum version of "independent for each face" on the lattice. Known $F$ is not defined pointwise, so parallel transport becomes a **stochastic differential equation**: an ordinary differential equation whose driving term contains white noise (increments of Brownian motion), solved after fixing one way of taking stochastic integrals. The driving term of parallel transport along a horizontal curve is the integral of $F$ between the curve and the $x$ axis, and each component is a Brownian motion whose time is that area (1-09) (Proposition 3.2 of Driver). This method goes back to Gross–King–Sengupta (1989).

Driver's results are (i) that expectation values of gauge-invariant functions of holonomies can be written as finite-dimensional integrals with products of heat kernels as density (times are the face areas, arguments the way the curves cross; Theorem 6.4), (ii) invariance under area-preserving maps (Corollary 6.7), and (iii) convergence of lattices. Known From Corollary 6.7 follows Euclidean invariance of gauge-invariant expectation values (part of the axioms). Driver does not claim verification of the whole of the Osterwalder–Schrader axioms (4-10), including reflection positivity. The observables are quantities along curves, and Driver writes that Wilson-loop variables are "quite singular for $d = 3$, and especially for $d = 4$".

06

## On closed surfaces: Sengupta and Lévy

On a closed surface of genus $\gamma$ Euler's formula becomes $V - E + P = 2 - 2\gamma$, and after fixing a tree $P - 1 + 2\gamma$ edges remain. The face variables are subject to one overall

constraint, and $2\gamma$ variables going around the holes are added, so the faces are not independent. Even so, the measure has been constructed. Sengupta (1992–97) constructed it on compact surfaces (original papers: bibliographic data only), and Lévy (2003) treated the discrete case and the continuum limit in a unified way and showed a **Markov property** with respect to cutting and gluing surfaces (when cut along a curve, fixing the values on the cut makes the two sides not influence each other). Ways of gluing gauge fields on a surface come in topological kinds (1-07); the non-trivial kinds (non-trivial principal bundles) are Lévy (2006), and the general theory of holonomy fields is Lévy (2010). Known The discrete version puts a heat kernel on each face $F$.

$$\mathbb{E}[f] = \frac{1}{Z}\int f(h)\prod_F Q_{g^2|F|}(h(\partial F))\prod_e dh_e, \qquad Z_\gamma(T) = \sum_\lambda d_\lambda^{2-2\gamma} e^{-c\lambda g^2 T/2} \tag{6.1}$$

$h(\partial F)$ is the product along the boundary of the face, $T$ the total area, and $Z$ is up to a constant factor. Known The $Z_1$ of the torus has the same form as the $\operatorname{Tr} e^{-\tau H}$ of the Hamiltonian of § 07.

### Detour: letting the area go to 0

A subsection you may skip.

If $\gamma \geq 2$, then $Z_\gamma \to \sum_\lambda d_\lambda^{2-2\gamma}$ as $T \to 0$. The measure concentrates on connections of zero curvature (flat connections), and from this limit comes the volume of the **moduli space of flat connections** (the space of flat connections identified under gauge transformations). This is Witten's volume formula, and the exact form of the limit was given by Sengupta (2003) (Theorem 4.1 of the survey). Theorem 4.1 assumes that $G$ is simply connected and $\gamma \geq 2$, and $SU(2)$ with $\gamma \geq 2$ satisfies this. Known For $SU(2)$ the dimensions are $1, 2, 3, \ldots$, so the limit is $\zeta(2\gamma - 2)$. Computed

| $\gamma$ | $T = 1$ | $T = 0.1$ | $T = 0.01$ | $T \to 0$ |
|---|---|---|---|---|
| 1 | 2.2738 | 7.5201 | 24.597 | Divergent |
| 2 | 1.2247 | 1.4713 | 1.5849 | ζ(2) = 1.6449 |
| 3 | 1.0482 | 1.0767 | 1.0817 | ζ(4) = 1.0823 |

07

## Seen with the transfer matrix: the lowest excitation is proportional to the length of the circle

The mass gap is the gap in the spectrum of the Hamiltonian (4-11). On the lattice we use the **transfer matrix** $\mathsf{T}$: the operator whose kernel is the weight for advancing time by one step, defining the Hamiltonian $H$ by $\mathsf{T} = e^{-aH}$ (5-03). Take space to be a circle

of $L$ edges, and do not make time periodic (an open cylinder). Then edges in the time direction form no cycles and can be fixed to 1, and gauge-invariant states are functions of the conjugacy class of the holonomy once around, $V_1 \cdots V_L$, that is, superpositions of characters. One step of time evolution multiplies each edge by one independent face variable, so using (3.1) $L$ times gives

$$\mathsf{T}\chi\lambda = w\lambda(\beta)^{L}\chi\lambda, \qquad E\lambda = -L\ln w\lambda(\beta) \quad (\text{lattice units},\ E_0 = 0). \tag{7.1}$$

The lowest excitation is $E = \sigma_{\text{lat}} L$ of the fundamental representation, the energy of a "ring of lines of force" going once around the circle. The tension is the same number as in the area law of § 03. For $SU(2)$ at β = 1, $E_{1/2}$ is 3.3469 at $L = 4$ and 6.6938 at $L = 8$. The ratio $E_1/E_{1/2}$ at β = 1, 32, 512 is 2.4044, 2.6664, 2.6667, approaching the Casimir ratio $8/3$. Lemma (3.1) was also checked with random numbers ($2 \times 10^6$ samples). Computed

The continuum limit can be taken in the same way as in § 04. Holding the physical length $L_{\text{phys}} = La$ fixed and putting $\beta = 2/(g^2 a^2)$ and $-\ln w\lambda = c\lambda/\beta + O(\beta^{-2})$ into (7.1) gives

$$\frac{E\lambda}{a} = -\frac{L_{\text{phys}}}{a^2}\ln w\lambda(\beta) = \tfrac{1}{2}g^2 L_{\text{phys}}\, c\lambda + O(a^2). \tag{7.2}$$

Combined with $\Delta\chi\lambda = -c\lambda\chi\lambda$ (§ 04), the continuum Hamiltonian is a constant multiple of the Laplacian on the group, $-\tfrac{1}{2}g^2 L_{\text{phys}}\Delta$ (acting on class functions). Paper The derivation is this chapter's own; for $SU(2)$ with $g = 1$ and $L_{\text{phys}} = 1$, the value for $j = \tfrac{1}{2}$ at $a = 1/64$ was 0.375011 (limit 0.375), and the difference became 1/4 each time $a$ was halved. Computed

Making the circle longer ($L \to \infty$) sends every eigenvalue other than the vacuum off to infinity, and there are no excitations (particles) propagating with finite energy. We also define the correlation length here. The correlation length is the length $\xi_{\text{lat}}$ (in lattice units) when the connected correlation $\langle P_x P_y\rangle - \langle P_x\rangle\langle P_y\rangle$ of the quantity $P_x = \operatorname{Re}\operatorname{tr} U_{p_x}$ of a plaquette $p_x$ decreases like $e^{-|x-y|/\xi_{\text{lat}}}$. On the plane different plaquettes are independent, so the connected correlation is exactly 0, and $\xi_{\text{lat}} = 0$ at every β (what to hold fixed when taking the limit is in 5-09). The four-dimensional question (whether the correlation length diverges with a finite mass remaining in physical units) does not arise in two dimensions.

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08
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### Reconstruction by stochastic quantisation

**Stochastic quantisation** is the method of not writing the measure directly, but solving a random time evolution $\partial_t A = -\frac{1}{2}\nabla S(A) + \xi$ ($\xi$ is spacetime white noise) that has it as invariant measure. This equation is called the **Langevin equation**. In finite dimensions the invariant measure of this equation is proportional to $e^{-S}$. Even in two dimensions $A$ is a distribution, so renormalisation and consistency with gauge are needed. The tools for this (regularity structures, the DeTurck method) are treated in 7-09; here we look only at the results. Known

Chevyrev (2019) realised the measure on $T^2$ as a distributional 1-form, and Chandra–Chevyrev–Hairer–Shen (2022) built a space of distributional connections and gauge orbits (a Polish space, a good metric space on which probability theory can be used as usual) and defined a **Markov process** (a random process whose future distribution is determined by the present state alone) consistent with gauge in the sense of laws. Chevyrev–Shen (arXiv 2023; Comm. Pure Appl. Math. in 2026) showed that the Yang–Mills measure of the trivial principal bundle on $T^2$ is the **unique invariant probability measure** of this process (Theorem 2.16). As corollaries come long-time existence of the process, that the limits of the Wilson, Villain and Manton lattices are the same measure (the paper writes that for the Villain action this was already known from Lévy 2006), and a decomposition into a Gaussian free field (1-09) and an almost Lipschitz remainder. Known

This road too is not separate from solvability. Chevyrev–Shen use gauge fixing on the lattice and obtain the moment estimates from two-dimensional solvability. The paper calls extending these estimates to higher dimensions or to the two-dimensional Yang–Mills–Higgs model open, and writes that the construction of non-abelian Higgs models is open even in two dimensions. Known In July 2026 two preprints appeared (not refereed; from the abstracts). Chevyrev–Klose–Mohamed give a gauge-fixed representation of optimal regularity for the existing measure, and Chhaibi–Dang–Guedes Bonthonneau–Rivière–Tô say that they used Morse theory (a method of decomposing a surface by the critical points of a function on it) to make the measure on surfaces without passing through lattices, reproducing the formulas of Migdal, Witten and Lévy. Known

09

## What can be carried out of two dimensions, and what cannot

The core of the solvability was the counting of § 02. On a $d$-dimensional lattice, per lattice point, the edges left after removing gauge number $d-1$ and the faces $\binom{d}{2}$: for $d = 2, 3, 4$, 1 and 1, 2 and 3, 3 and 6. In three or more dimensions the face variables are

tied by constraints on each cube (the lattice Bianchi identity) and cannot be taken independently. Known

| Property | Two dimensions | Four dimensions |
|---|---|---|
| Area law | Every β | Only on the strong-coupling side (5-02, 5-06) |
| Coupling constant | Dimension of an inverse area | Dimensionless. How β runs is, with some doubts, mathematically open (Chatterjee 2018). The committee's assessment is that the construction published on 2026-11-10 gives the scaling behaviour of the coupling needed for asymptotic freedom (6-01) |
| Continuum measure | Constructed on the plane and on closed surfaces | Not constructed (the published construction does not build a measure either; 6-04). Not constructed on $T^3$ even in three dimensions (7-06, 7-07) |
| Correlation length | 0 | The heart of the mass gap |

What can be carried out is on the side of methods: taking holonomies along curves as observables, expansion in characters (the tool of the strong-coupling expansion), the pattern of proofs of universality of lattice actions, and the framework of stochastic quantisation. What cannot be carried out are the independence of faces, the dependence on area alone, and the absence of propagating degrees of freedom. What the problem statement of Jaffe–Witten cites as "the only complete example" of an interacting gauge theory satisfying the axioms is likewise the two-dimensional abelian Higgs model (Brydges–Fröhlich–Seiler). Known

## What this chapter can say

| | Content | Label |
|---|---|---|
| Said | On a planar lattice the face variables are independent, and the Wilson loop is $w(\beta)^{\text{area}}$ (at every β). Agrees with random numbers without gauge fixing | Known Computed |
| Said | In the continuum limit holonomies follow the heat kernel $Q_g{}^2{}_A$, with errors proportional to $a^2$. The lowest excitation on the circle is proportional to the length of the circle, and the continuum Hamiltonian is $-\frac{1}{2}g^2 L_{\text{phys}}\Delta$ (formula (7.2) is this chapter's derivation) | Known Paper Computed |
| Said | Construction of the continuum measure on the plane and convergence of lattices (Driver, Gross–King–Sengupta). Construction on closed surfaces (Sengupta, Lévy). On $T^2$, the unique invariant measure of the process of the Langevin equation (Chevyrev–Shen) | Known |
| Said | Euclidean invariance of gauge-invariant expectation values (from Corollary 6.7 of Driver; part of the axiom step) | Known |

| | Content | Label |
|---|---|---|
| Not said | That this method extends to three or more dimensions. Verification of the whole of the OS axioms including reflection positivity (not claimed by Driver, Sengupta's survey or Chevyrev–Shen). Non-abelian Higgs models (not constructed even in two dimensions) | |
| Not said | The content of the 2026 preprints (abstracts only). The texts of works cited from bibliographic data only | |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The recursion is exact in two dimensions, $\langle \chi_q \rangle = d_q e^{-g_q^2 A}$ (area $A$), "nothing but Wilson's area law" | Text § 2 | A. A. Migdal, Zh. Eksp. Teor. Fiz. 69 (1975) 810–822 [English translation Sov. Phys. JETP 42 (1976) 413–418] |
| White noise, Proposition 3.2, Theorem 6.4, Corollary 6.7 (and Euclidean invariance), Theorem 7.2 and the mention of Dosch–Müller, 8.5, 8.8, "quite singular" | Text | B. K. Driver, Comm. Math. Phys. 123 (1989) 575–616 |
| The law of holonomies, the discrete measure, invariance under subdivision, the partition function, Theorem 4.1 ($G$ simply connected, $\gamma \geq 2$) and the attribution of the volume formula | Text of the survey | A. N. Sengupta, arXiv:0707.4136 (2007) |
| Unification and the Markov property / non-trivial bundles / holonomy fields | Abstracts | T. Lévy, Mem. AMS 166 no. 790 (2003); Probab. Theory Relat. Fields 136 (2006) 171–202; Astérisque 329 (2010) |
| The stochastic differential equation for parallel transport | Bibliographic data only | L. Gross, C. King, A. Sengupta, Ann. Phys. 194 (1989) 65–112 |
| The infinite-volume limit (in which Theorem 7.2 is essentially contained) | Bibliographic data only (from reference [23] of Driver) | H. G. Dosch, V. F. Müller, Fortschr. Phys. 27 (1979) 547–559 |
| The group-valued central limit theorem | Bibliographic data only (from Theorem 8.8 of Driver) | C. Borgs, E. Seiler, Comm. Math. Phys. 91 (1983) 329–380 |

| Item | Kind | Source / tools |
|---|---|---|
| The measure on compact surfaces | Bibliographic data only, not checked | A. N. Sengupta, J. Funct. Anal. 108 (1992) (title and pages not checked); Quantum gauge theory on compact surfaces, Ann. Phys. 221 (1993) 17–52; Gauge theory on compact surfaces, Mem. AMS 126 no. 600 (1997) (titles and pages of 1993 and 1997 from the reference list of the survey) |
| The volume formula and its exact limit | Bibliographic data only | E. Witten, J. Geom. Phys. 9 (1992) 303–368 (formula (4.72); attribution from the survey); A. N. Sengupta, J. Geom. Phys. 47 (2003) 398–426 (the source of Theorem 4.1 of the survey). Related: E. Witten, Comm. Math. Phys. 141 (1991) 153–209 |
| Abelian Higgs model | Bibliographic data only | D. Brydges, J. Fröhlich, E. Seiler, Ann. Phys. 121 (1979) 227–284; Comm. Math. Phys. 71 (1980) 159–205; Comm. Math. Phys. 79 (1981) 353–399 |
| Stochastic quantisation | Abstracts | Chevyrev, Comm. Math. Phys. 372 (2019); Chandra–Chevyrev–Hairer–Shen, Publ. Math. IHÉS 136 (2022) |
| Theorem 2.16, the corollaries, the open problems of § 1.2, the statement that the Villain action was known (Lévy 2006) | Text § 1, § 2 | Chevyrev–Shen, arXiv:2302.12160v3 (2023); Comm. Pure Appl. Math. (2026) (volume and pages not checked) |
| The 2026 preprints | Abstracts (not refereed) | Chevyrev–Klose–Mohamed, arXiv:2607.22236; Chhaibi et al., arXiv:2607.24640 |
| No non-trivial continuum limit in higher dimensions, the running of β | Text of the survey | S. Chatterjee, arXiv:1803.01950 (2018) |
| "The only complete example" | Problem statement § 6.4 | Jaffe–Witten, Quantum Yang–Mills theory (Clay) |
| Checks of the tables, figures, lemma and formula (7.2) | Computed | python3 (numpy, scipy). The random numbers are a heat-bath method with SU(2) represented by quaternions (6×6, free boundary, no gauge fixing, 7000 sweeps, 35 blocks, fixed seed). The figures are plain JavaScript |

Next chapter to read: 7-03 Constructing scalar fields.

# Constructing scalar fields — until φ⁴ satisfies the axioms in two and three dimensions

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — The outline of the construction of $\phi^4_2$ and $\phi^4_3$, the first interacting examples to pass the six stages. That in two dimensions Wick products are enough for the ultraviolet, and the wall was "not bounded below". The road of the field equations and the road of statistical mechanics. How to count the one further renormalisation needed in three dimensions.

**Prerequisite chapters** — 7-01 What constructive field theory is. We also refer to 4-10, 1-11 and 4-04.

**Said first** — This introduces known constructions from 1965–1976 and later alternative proofs, and follows the proofs only in outline. The numbers in the tables are computations on finite lattices and with samples, not proofs that the limits exist. Nothing here shows anything about four-dimensional Yang–Mills.

The order of this chapter

1. Why start from scalar fields — the target statement
2. Wick products: which powers become random variables — a table of computations
3. The wall in two dimensions: not bounded below — a moving figure
4. Two roads: field equations and statistical mechanics
5. Weak coupling: the cluster expansion, the mass gap, scattering
6. Three dimensions: one more renormalisation — a moving figure
7. Once more by other roads
8. Detour: strongly coupled $\phi^4_2$ and the phase transition
9. What can be carried to four dimensions, and what cannot

01

## Why start from scalar fields

On the map of 7-01 (the six stages), the examples of interacting theories in which all six stages closed were $\phi^4$ in two and three dimensions. The field carries only one real number at each point, with no gauge symmetry and no constraints, and the difficulty is narrowed to products at a point (the ultraviolet) and volume (the infrared). The Euclidean action is the action of the free field of mass $m_0$ plus the polynomial of formula (3) of the problem statement.

$$S(\phi) = \int \left( \tfrac{1}{2} |\nabla \phi|^2 + \tfrac{1}{2} m_0^2 \phi^2 + \lambda \phi^4 + \tfrac{1}{2} \sigma \phi^2 + c \right) \mathrm{d}x \quad (1)$$

$\lambda > 0$ is the coupling constant, and $\sigma = \alpha\lambda + \beta\lambda^2$ and $c = \gamma\lambda + \delta\lambda^2 + \varepsilon\lambda^3$ are counterterms made to depend on the cutoff (JW § 6.2). The $d$ in $\phi^4_d$ is the dimension of space-time, and in two dimensions $P(\phi)_2$, extended to a polynomial $P$ bounded below, was handled by the same methods. The target is one sentence of the problem statement: 'For $0 < \lambda \ll 1$, one can choose $\sigma(\lambda)$, $c(\lambda)$ so the field theory described by (3) exists, is unique, and has a mass equal to m such that $|m - m_0|$ is small' (JW § 6.2). In four dimensions the lattice limit collapses to a Gaussian (7-08). Known

02

## Wick products: which powers become random variables

In Table 1 of 7-01 (where it breaks) we saw that in two or more dimensions the variance at one lattice point, $c_a = G_a(0)$, becomes infinite as $a \to 0$. A Wick product is a product with lower powers subtracted so that its expectation is 0, and it is fixed step by step by the following conditions (Summers 2016 § 3.1).

$$:\!\phi^0\!: = 1, \qquad \frac{\partial}{\partial \phi} :\!\phi^n\!: = n :\!\phi^{n-1}\!:, \qquad \int :\!\phi^n\!: \mathrm{d}\mu_C = 0 \quad (2)$$

For a single variable with variance $c$, $:\!x^2\!: = x^2 - c$ and $:\!x^4\!: = x^4 - 6cx^2 + 3c^2$: Hermite polynomials with the scale adjusted. The point is that for a Gaussian field with mean 0 and covariance $C$

$$\big\langle :\!\phi(x)^n\!: \, :\!\phi(y)^k\!: \big\rangle = \delta_{nk} \, n! \, C(x-y)^n \quad (3)$$

holds. The variance of the smeared $:\!\phi^n\!:(f)$ is $n! \iint f(x) f(y) C(x-y)^n$, and at short distance $C$ behaves like $\log(1/|x|)$ if $d = 2$ and like $|x|^{-(d-2)}$ if $d \geq 3$, so

$$\int_{|x|<1} C(x)^n \, \mathrm{d}^d x < \infty \iff n(d-2) < d \quad (4)$$

In two dimensions every $n$ is finite; in three dimensions only $n \le 2$ is, $n = 3$ is the logarithmic borderline, and $n = 4$ diverges like $1/a$. Known

Computed Table 1 The lattice $\int C^n \approx a^d \sum_x G_a(x)^n$ (m = 1. The side of the periodic box is 8 in two dimensions and 4 in three). The bottom row for two dimensions is the continuum, infinite-volume $\int_0^\infty 2\pi r\,(K_0(r)/2\pi)^n\,dr$.

| a | d=2, n=2 | n=3 | n=4 | d=3, n=2 | n=3 | n=4 |
|---|---|---|---|---|---|---|
| 1/8 | 0.08086 | 0.01571 | 0.00487 | 0.04771 | 0.02128 | 0.03050 |
| 1/16 | 0.08026 | 0.01518 | 0.00453 | 0.04712 | 0.02589 | 0.06776 |
| 1/32 | 0.08007 | 0.01498 | 0.00436 | 0.04678 | 0.03038 | 0.14358 |
| 1/64 | 0.08001 | 0.01491 | 0.00429 | 0.04661 | 0.03482 | 0.29652 |
| 1/256 | 0.07999 | 0.01487 | 0.00425 | – | – | – |
| $\to 0$ | 0.07958 | 0.01484 | 0.00424 | converges | log | $\propto$ 1/a |

Every two-dimensional column approaches the continuum value. In three dimensions $\int C^2$ converges, the increment of $\int C^3$ each time $a$ is halved approaches about 0.0044 (the continuum prediction is $\log 2/(16\pi^2) = 0.00439$), and $\int C^4$ roughly doubles each time. The two-dimensional $:\phi^4:(f)$ is a random variable, but the three-dimensional one is not. These two columns reappear in § 06 as diagrams with two vertices.

03

## The wall in two dimensions: not bounded below

Even if $V = \int_\Lambda :\phi^4: \mathrm{d}x$ ($\Lambda$ is a finite box) is a random variable, to make the measure $e^{-\lambda V}\,\mathrm{d}\mu C/Z$ we need $e^{-\lambda V}$ to be integrable. The classical $\phi^4$ is non-negative, but the Wick product is not. Since $:x^4: = (x^2 - 3c)^2 - 6c^2$,

$$:x^4: \;\ge\; -6c^2, \qquad V_a \;\ge\; -6\,c_a^2\,|\Lambda| \quad (5)$$

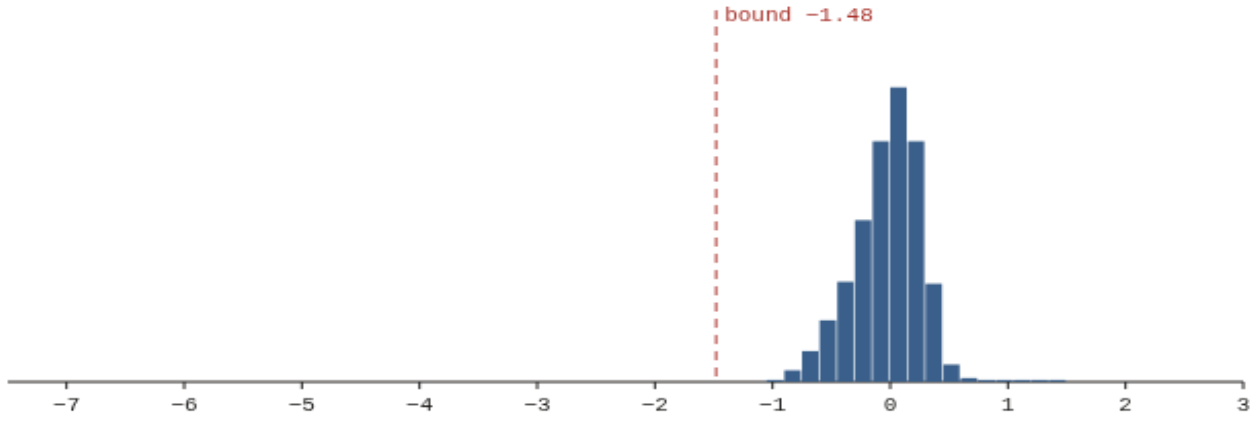
and in two dimensions $c_a = \frac{1}{2\pi}\log\frac{1}{a} + O(1)$, so the lower bound runs away to $-\infty$ at the rate $(\log\frac{1}{a})^2$. Jaffe (2000) says that early progress in two dimensions owed much to Nelson's 1965 paper, which showed the stability (being bounded below) of the Hamiltonian of $\phi^4$ with space taken to be a circle. Known

The argument that it is integrable even though the lower bound runs away has three steps. (i) The cut-off $V_a$ is bounded below by formula (5). (ii) The variance of $V - V_a$ becomes small like a power of $a$, as in the convergence of Table 1. (iii) For a polynomial of degree 4 in a Gaussian field with standard deviation $s$, the probability of exceeding $t$ is of order $\exp(-\text{const}\cdot(t/s)^{1/2})$. $V < -M$ needs $V - V_a$ to be large and negative, and choosing $a$ to match $M$ makes that probability decrease faster than $e^{-pM}$. The conclusion is (Summers 2016 § 3.1) Known

$$e^{-\lambda V} \in L^p(\mathrm{d}\mu_C) \qquad (\text{for all } p < \infty) \quad (6)$$

Here $X \in L^p(\mathrm{d}\mu_C)$ means $E[|X|^p] < \infty$, that is, the expectation of the $p$-th power is finite ($E$ is the expectation in the free-field measure $\mu_C$). For $p = 1$ the normalising constant $Z = E[e^{-\lambda V}]$ is finite, and the measure $e^{-\lambda V}\,\mathrm{d}\mu_C / Z$ can be made.

Figure 1 As a is refined, the lower bound $-6c_a^2$ (dashed) runs away to the left, but in two dimensions the distribution of $V_a$ does not move. Switch to three dimensions and it spreads




```
d = 2, a = 1/4, lower bound −1.48, standard deviation 0.406, 0.1% point −0.969, log E[e^(−V)] =
0.0575 (20000 samples)
```

Computed The numbers in this figure were computed on this machine. $V_a = a^d \sum_x :\phi_x^4:$ was evaluated on 20000 samples of the free lattice field (6000 for a = 1/64 in three dimensions) in a periodic box of side 1 with m = 2. The horizontal axis is $V_a$, the vertical axis the density.

Computed Table 2 The distributions of Figure 1 (excerpt). The standard deviation is exact from formula (3); the 0.1% point and $\log E[e^{-V}]$ (λ = 1) are from the samples (the error of the latter is ±0.002 in two dimensions and ±0.005–0.17 in three). In brackets, the Gaussian approximation $\mathrm{Var}(V)/2$.

| d, a | Lower bound | Standard deviation | 0.1% point | $\log E[e^{-V}]$ |
|---|---|---|---|---|
| 2, 1/4 | −1.48 | 0.4060 | −0.969 | 0.0575 |
| 2, 1/16 | −3.15 | 0.3860 | −0.991 | 0.0479 |
| 2, 1/64 | −5.37 | 0.3788 | −0.946 | 0.0505 |
| 2, 1/128 | −6.70 | 0.3779 | −0.923 | 0.0500 |
| 3, 1/4 | −5.91 | 0.6806 | −1.944 | 0.176 (0.232) |
| 3, 1/16 | −97.20 | 1.3599 | −4.152 | 0.865 (0.925) |
| 3, 1/32 | −390.73 | 1.9229 | −5.866 | 1.746 (1.849) |
| 3, 1/64 | −1566.46 | 2.7194 | −8.151 | 3.54 (3.70) |

In two dimensions, even as the lower bound drops from −1.5 to −6.7, the 0.1% point stays at −0.9 to −1.0 and $\log E[e^{-V}]$ settles at about 0.05. The lower bound is far away, but the probability of getting there is extremely small: this is the content of formula (6). In three dimensions, each halving multiplies the standard deviation by $\sqrt{2}$ and

roughly doubles $\log E[e^{-V}]$, approaching the Gaussian approximation. A constant growing like $1/a$ has to be subtracted at order $\lambda^2$, and this corresponds to the $\delta$ of the problem statement (§ 06). Computed

04

## Two roads: field equations and statistical mechanics

The first construction (the Glimm–Jaffe series of four papers, 1968–1972) took the form of solving the field equations as operators in Minkowski space-time. The equation in two dimensions is

$$\partial_t^2\phi - \partial_x^2\phi + m_0^2\phi + 4\lambda\,{:}\phi^3{:} = 0, \qquad {:}\phi^3{:} = \lim_{\kappa\to\infty}\left(\phi_\kappa^3 - 3c_\kappa\phi_\kappa\right) \tag{7}$$

($\phi_\kappa$ is the field with Fourier components cut at $|k| \le \kappa$, $c_\kappa \sim \log\kappa$. Jaffe 2000 § 3). One shows that $H(g) = H_0 + \lambda\int {:}\phi(0,x)^4{:}\,g\,\mathrm{d}x$, cut off in space by a function $g$, is essentially self-adjoint on Fock space (a unique self-adjoint operator is fixed by the cut-off expression), bounded below, and has a simple lowest eigenvalue, and then uses the fact that signals do not exceed the speed of light to obtain a Poincaré-invariant vacuum as $g \to 1$ (finite-volume step → infinite-volume step. Summers 2016 § 2). The representation is not equivalent to the Fock representation (Haag's theorem of 4-09), but it is equivalent when restricted to a bounded region. Known

In the early 1970s, Nelson's Markov fields (only named here) and the Osterwalder–Schrader axioms (4-10) opened the road of making the Euclidean measure

$$\mathrm{d}\mu_\Lambda = \frac{1}{Z}\,e^{-\lambda U_\Lambda}\,\mathrm{d}\mu_C, \qquad U_\Lambda = \int_\Lambda {:}P(\phi){:}\,\mathrm{d}^2x \tag{8}$$

(given meaning by formula (6)). On a two-dimensional lattice, the difference term, for pairs of neighbouring points $\langle xy\rangle$, becomes

$$\tfrac12\sum_{\langle xy\rangle}(\phi_x-\phi_y)^2 = \sum_{\langle xy\rangle}\left(\tfrac12\phi_x^2+\tfrac12\phi_y^2\right) - \sum_{\langle xy\rangle}\phi_x\phi_y \tag{9}$$

and the last term, inside $e^{-S}$, appears as $e^{+\phi_x\phi_y}$ and makes neighbours prefer the same sign. It is a ferromagnet whose spins range over all real numbers (the title of Guerra–Rosen–Simon 1975 says 'as classical statistical mechanics'). There correlation inequalities (of GKS type: inequalities saying that correlations are non-negative in systems where neighbours prefer the same sign) apply, and if $P$ has the form of an even polynomial minus $h\phi$ ($h \ge 0$), the Schwinger functions are non-negative and, with suitably chosen boundary conditions, increase with the box. Together with a uniform upper bound, the infinite-volume limit is obtained by monotone convergence (the

infinite-volume step), with no restriction on the size of the coupling (Summers 2016 § 3.1), though the problem statement says that correlation inequalities tend to be limited to scalar bosons and abelian gauge theories (JW § 6.1). Reflection positivity is inherited by the limit (the axiom step, 7-01). Known

05

## Weak coupling: the cluster expansion, the mass gap, scattering

Glimm–Jaffe–Spencer (1974) carried it through to the uniform-gap step. When $\lambda/m_0^2$ is small enough, they showed the infinite-volume limit and the OS axioms by the cluster expansion, and obtained exponential decay of correlations (formula (9) of 7-01). For the mass operator $M = \sqrt{H^2 - P^2}$ ($P$ is the momentum)

$$\operatorname{spec} M \subset \{0\} \cup \{m\} \cup [m_1, \infty), \qquad m \to m_0, \quad m_1 \to 2m_0 \quad (\lambda \to 0) \tag{10}$$

has been shown (Summers 2016 § 3.1). 0 is the vacuum, the isolated $m$ is the one-particle state, and above $m_1$ are two or more particles. With an isolated one particle, Haag–Ruelle scattering theory (the general theory that assembles scattering states in the far past and far future from states of isolated particles; only named here) can be used, and Osterwalder–Sénéor (1976) showed that the scattering matrix is non-trivial and that the perturbation series is its asymptotic expansion (Summers 2016 § 3.1). On the other hand, the cluster expansion used in the proofs of existence and of the gap is not a power series in $\lambda$ but a convergent expansion whose $n$-th term has the form $\lambda^n G_n(\lambda)$ ($G_n$ bounded as $\lambda \to 0$) (Jaffe 2000 § 4). The perturbation series diverges (4-12) and is only an asymptotic expansion of the exact answer. The mechanism is treated in 7-04. Known

Example: if $\lambda = 0$, the two-point function is $C(x)$, which decays at large distance like $e^{-m_0|x|}$, and the gap is $m_0$. The weak-coupling result says that this gap is not closed by a small interaction. The source of the gap is the $m_0$ written into the action (§ 09).

06

## Three dimensions: one more renormalisation

The counting of diagrams in perturbation theory gives an idea of what increases (4-03, 4-04). A connected diagram with $n$ vertices and $E$ external lines has $I = 2n - E/2$ internal lines and $L = n - E/2 + 1$ loops. A loop integral carries momentum to the power $d$ and a propagator to the power $-2$, so the superficial degree of divergence is

$$\omega = dL - 2I = d + (d-4)\,n - \frac{d-2}{2}\,E \tag{11}$$

and the diagrams with $\omega \geq 0$ are the ones that can diverge in the ultraviolet ($\omega = 0$ is logarithmic). A diagram with one vertex consists only of lines joined to itself, and vanishes with Wick products.

Figure 2 Moving the dimension d changes the types of diagram that can diverge (coloured frames; dark for ω > 0, light for ω = 0). Dashed: types with one vertex, which vanish with Wick products. "tree": diagrams with no loop. "—": types with no connected diagram

| E\n | n = 1 | n = 2 | n = 3 | n = 4 | n = 5 | n = 6 |
|---|---|---|---|---|---|---|
| E = 0 | **ω = 2** | **ω = 1** | **ω = 0** | ω = −1 | ω = −2 | ω = −3 |
| E = 2 | **ω = 1** | **ω = 0** | ω = −1 | ω = −2 | ω = −3 | ω = −4 |
| E = 4 | tree | ω = −1 | ω = −2 | ω = −3 | ω = −4 | ω = −5 |
| E = 6 | — | tree | ω = −3 | ω = −4 | ω = −5 | ω = −6 |
| E = 8 | — | — | tree | ω = −5 | ω = −6 | ω = −7 |

`d = 3.00, types still divergent after Wick (n ≥ 2): 3`

Computed The values in this figure are formula (11) computed on the spot. Table 3 agrees with the computation on this machine.

Computed Table 3 The types (n, E) with ω ≥ 0, and ω.

| d | One vertex (vanishes with Wick products) | Types remaining after Wick |
|---|---|---|
| 2 | (1, 0), (1, 2): 0 | none |
| 3 | (1, 0): 2, (1, 2): 1 | (2, 0): 1, (2, 2): 0, (3, 0): 0 |
| 4 | (1, 0): 4, (1, 2): 2 | E = 0, 2, 4 for every n (ω = 4, 2, 0) |

Two of the types remaining in three dimensions are exactly the columns of Table 1. (2, 0) is the vacuum bubble joining two vertices by four lines, $\int C^4$ $(1/a)$, and (2, 2) is the self-energy joining them by three lines, $\int C^3$ (logarithmic). Moving $d$ in Figure 2 from 2.66 to 2.67 brings (2, 0) into the types that can diverge. The borderline is $d = 8/3$, which agrees with the borderline $n = 4$ of formula (4). Computed

The problem statement says that in two dimensions $\alpha, \gamma$, and in three dimensions all five, are made to depend on the cutoff (JW § 6.2). Known The problem statement does not say which constant corresponds to which diagram, but reading the order in $\lambda$ as the number of vertices, $\alpha, \gamma$ correspond to $n = 1$, $\beta, \delta$ to $n = 2$ and $\varepsilon$ to $n = 3$, which matches the number of types in Table 3. Computed Jaffe (2000 § 6) also says that in three dimensions, besides Wick products, one needs a logarithmically divergent mass

counterterm and two vacuum-energy counterterms, divergent linearly and logarithmically. $\lambda$ itself is not renormalised. Known

Moreover, in three dimensions the Wick-ordered Hamiltonian is not a densely defined operator (an operator defined on a subspace containing vectors arbitrarily close to any vector), the state vectors have to be rescaled by a constant going to 0 with the cutoff (what the problem statement calls wave-function renormalisation), and the representation is not equivalent to the Fock representation (Jaffe 2000 § 6). Jaffe calls this entirely different from two dimensions ( § 04), where it is equivalent to the Fock representation on bounded regions. The tool of control is the localisation of Glimm–Jaffe (1973) into phase-space cells (boxes divided by scale in both position and momentum), and the induction from estimates at one scale to estimates at twice the scale leads to the idea of the renormalisation group (ibid.; 7-05). On that basis Feldman–Osterwalder (1976) and Magnen–Sénéor (1976) independently showed the OS axioms in the limit with the cutoff and volume removed, when $\lambda/m_0^2$ is small (Summers 2016 § 3.2; the former puts the mass gap in its title). The restriction on $\lambda/m_0^2$ was later removed by other methods by Seiler–Simon (1976) and Park (1977), and Burnap (1977) showed the existence of one-particle states so that Haag–Ruelle scattering theory can be used (Summers 2016 § 3.2; original papers from bibliographic data only). Known

07

## Once more by other roads

$\phi^4_2$ and $\phi^4_3$ then became examples for checking whether new methods can reproduce known results. Each method includes different stages in its output. Brydges–Fröhlich–Sokal (1983) used the lattice, the Schwinger–Dyson equations and correlation inequalities to show existence and non-triviality at weak coupling as subsequential limits of lattice theories. Stochastic quantisation (7-09) makes the measure $e^{-S}$ as the invariant measure (a measure whose distribution does not change under time evolution by this equation) of

$$\partial_t \phi = -\frac{\delta S}{\delta \phi} + \sqrt{2}\,\xi \qquad (\xi \text{ is space-time white noise}) \tag{12}$$

Hairer (2014), with regularity structures, constructed solutions of the stochastic quantisation equation (12) corresponding to $\Phi^4_3$ in finite volume (as local solutions up to a time at which they might blow up). The abstract describes this as solving the problem of constructing a natural Markov process symmetric with respect to that measure, but the text does not show that the solutions are global or that the measure is invariant; it says these are expected. Gubinelli–Hofmanová (2021) showed on $\mathbb{R}^3$ that every limit point is non-Gaussian, reflection positive and translation invariant, and the

abstract says they verified the OS axioms except for rotation invariance and clustering. In finite volume (the three-dimensional torus) there is also a method writing the measure variationally by stochastic optimal control (Barashkov–Gubinelli 2020). Note that the problem statement, for the existence part of the Yang–Mills problem, does not accept weak existence by compactness alone as a solution, unless the properties of the limit (mass gap, axioms) are shown separately (JW footnote 2). Known

08

## Detour: strongly coupled φ⁴₂ and the phase transition

A section you may skip.

Two-dimensional $\phi^4$ exists for every positive $\lambda$, but for $\lambda \gg 1$ the solution splits in two (JW § 6.3). Rearranged, it takes a form with two wells

$$\lambda'\Big(\phi^2 - \frac{1}{\lambda'}\Big)^2 + \tfrac{1}{2}\sigma'\phi^2 + c' \qquad (\lambda' \ll 1) \quad (13)$$

and by setting up a cluster expansion in each well and showing that the probability of going between wells is small, one obtains two phases in which the symmetry $\phi \to -\phi$ is broken (Glimm–Jaffe–Spencer 1976). As the critical value $\lambda_c$ is approached from below, the gap decreases monotonically and continuously to 0, and near it 'practically nothing has been proved' (JW § 6.3). The gap is a quantity that can close when the coupling is changed, and in the language of 4-11 the critical point is where the correlation length goes to infinity. Pushing the view of formula (9) further, $\phi^4_2$ cut off by a lattice and finite volume can be approximated by collections of Ising spins (Simon–Griffiths 1973). This work also extends the Lee–Yang theorem (the theorem that derives, from the location of the zeros of the partition function, that there is no phase transition in an external field) to $\phi^4_2$, and from it follows that the vacuum of the model with an external field $h \neq 0$ added is unique (Summers 2016 § 3.1). The gap of the Ising model is in 2-09. Known

09

## What can be carried to four dimensions, and what cannot

Known Table 4 Four theories (the mass dimension of the coupling is formula (7) of 7-01, the types of divergence are Table 3).

| | φ⁴₂ | φ⁴₃ | φ⁴₄ | Yang–Mills (four dimensions) |
|---|---|---|---|---|
| Mass dimension of the coupling | 2 | 1 | 0 | 0 |

| | $\varphi^4_2$ | $\varphi^4_3$ | $\varphi^4_4$ | **Yang–Mills (four dimensions)** |
|---|---|---|---|---|
| Types of diagram that can diverge ($\phi^4$ after Wick) | none | 3 | The three kinds E = 0, 2, 4 at every n (finitely many kinds of counterterm) | Appear at every order (finitely many kinds of counterterm. 4-04) |
| Coupling at short distance | Acts weakly (super-renormalisable) | | Not asymptotically free | Becomes weak (4-06) |
| Source of the gap | The mass $m_0$ in the action | | — | No mass in the action |
| Construction | Axioms, gap, scattering | Axioms, gap, one-particle states (Summers § 3.2) | The lattice limit is Gaussian (7-08) | Not yet before 2026 (7-07). The construction published in 2026-11 goes as far as constructing a theory that satisfies the Wightman axioms (6-01). The mass gap is not shown |

Four things can be carried over. (1) Reading the form of the counterterms from perturbation theory and making them depend on the cutoff. (2) Piling up estimates scale by scale (phase-space cells became the sequence of renormalisation group steps. 7-05). (3) Properties given by inequalities, such as reflection positivity, are inherited by the limit. (4) If exponential decay of correlations is shown by a weak-coupling expansion, it becomes the gap.

At the centre of what cannot be carried over is the source of the gap. For $\phi^4_2$ and $\phi^4_3$, $m_0 > 0$ was written into the action and it was shown that $|m - m_0|$ is small. The Yang–Mills action has no mass term, and the expected scale of the gap depends on the coupling in the form $e^{-\text{const}/g^2}$, which cannot be expanded in powers of $g$ (4-07). The mass that would be the foothold of a weak-coupling expansion is absent from the start. In addition, in four dimensions divergent diagrams appear at every order, and the field must preserve gauge invariance as a connection (5-01). Known Physics

On the other hand, the problem statement cites the fact that four-dimensional $\phi^4$ is not asymptotically free as evidence of triviality, and names Yang–Mills, whose coupling becomes weak at short distance, as the natural candidate for constructive field theory in four dimensions (JW § 6.2). Where it stops is seen in 7-10.

## What this chapter can say

| | Content |
|---|---|
| **Said** | The lattice $\int C^n$ converges for n = 2, 3, 4 in two dimensions; in three dimensions n = 2 converges, n = 3 is logarithmic and n = 4 grows like 1/a (a ≥ 1/256, 1/64) Computed |
| **Said** | In two dimensions, even as the lower bound $-6c_a^2$ drops, the left tail of $V_a$ and $\log E[e^{-V_a}]$ do not move; in three dimensions $\log E[e^{-V_a}]$ grows like 1/a (a ≥ 1/128, 1/64) Computed |
| **Said** | The types of diagram that can still diverge after Wick are none for d = 2 and three for d = 3, and for d = 4 the three kinds E = 0, 2, 4 appear at every n. The three in three dimensions match the descriptions of the counterterms in the problem statement and in Jaffe (2000), reading the order of the constants as the number of vertices Computed Known |
| **Said** | Weakly coupled $\phi^4_2$ and $\phi^4_3$ satisfy the axioms and have a mass gap, and strongly coupled $\phi^4_2$ has two phases (according to the problem statement and the survey) Known |
| **Not said** | The proofs of formula (6), the cluster expansion and phase-space cells (outline only; original papers from bibliographic data only) |
| **Not said** | The existence of limits from the numbers in Tables 1 and 2 (computations with finite a and finite samples) |
| **Not said** | Near the critical point; rotation invariance and clustering on the road of stochastic quantisation (later literature not followed) |
| **Not said** | Global solutions and invariance on the road of stochastic quantisation (literature after Hairer 2014 not followed) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Problem statement § 6.1–6.3, footnote 2 | Primary source (text checked) | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000) |
| History, formula (7), counterterms and representation in three dimensions, phase-space cells | Survey (text checked) | A. Jaffe, Constructive quantum field theory, in *Mathematical Physics 2000*, World Scientific (2000) |
| Formulas (2), (6), (8), (10), the order of steps, correlation inequalities, scattering and the extension of the Lee–Yang theorem in § 3.1, removal of the restriction and one-particle states in § 3.2 | Survey (text checked) | S. J. Summers, A Perspective on Constructive Quantum Field Theory, arXiv:1203.3991 (v2, 2016) |

| Item | Kind | Source / tools |
|---|---|---|
| Original papers in two dimensions | Bibliographic data only (reference lists of the problem statement and the survey) | Nelson, proceedings of the 1965 conference *Mathematical Theory of Elementary Particles* (ed. R. Goodman, I. Segal, MIT Press 1966). Glimm–Jaffe, Phys. Rev. 176 (1968) 1945–1951; Ann. Math. 91 (1970) 362–401; Acta Math. 125 (1970) 203–267; J. Math. Phys. 13 (1972) 1568–1584. Nelson, Proc. Sympos. Pure Math. XXIII, AMS (1973) 413–420. Osterwalder–Schrader, Comm. Math. Phys. 31 (1973) 83–112; 42 (1975) 281–305. Guerra–Rosen–Simon, Ann. Math. 101 (1975) 111–259. Glimm–Jaffe–Spencer, Ann. Math. 100 (1974) 585–632. Osterwalder–Sénéor, Helv. Phys. Acta 49 (1976) 525–535 |
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| Other constructions | Brydges–Fröhlich–Sokal from bibliographic data and a summary of the abstract only. Hairer checked in the abstract and the text of arXiv version v4 (Remark 1.18), the others in the abstracts | Brydges–Fröhlich–Sokal, Comm. Math. Phys. 91 (1983) 141–186. Hairer, Invent. Math. 198 (2014) 269–504, doi:10.1007/s00222-014-0505-4, arXiv:1303.5113. Barashkov–Gubinelli, Duke Math. J. 169 (2020) 3339–3415, arXiv:1805.10814. Gubinelli–Hofmanová, Comm. Math. Phys. 384 (2021) 1–75, arXiv:1810.01700 |
| Phase transition | Bibliographic data only | Glimm–Jaffe–Spencer, Ann. Phys. 101 (1976) 610–630, 631–669. Simon–Griffiths, Comm. Math. Phys. 33 (1973) 145–164. Decrease towards the critical point (references of JW § 6.3): Glimm–Jaffe, Phys. Rev. D 10 (1974) 536–539. Guerra–Rosen–Simon, Comm. Math. Phys. 41 (1975) 19–32. McBryan–Rosen, Comm. Math. Phys. 51 (1976) 97–105 |
| $\phi^4$ in four dimensions | Bibliographic data only (treated in 7-08) | Aizenman–Duminil-Copin, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973 |
| Tables 1–3, Figure 1 | Computed | Python (numpy, scipy) on this machine: `wick_powers.py`, `wick_mc.py` (free fields on periodic lattices generated by FFT, fixed random seed. Run separately for each (d, a), each within 5 minutes; the heaviest, a = 1/64 in three dimensions, takes about 3 minutes 40 seconds), `fig_data.py`, `power_count.py` |

| Item | Kind | Source / tools |
|---|---|---|
| Figures 1 and 2 | Figures | canvas, SVG and plain JavaScript. Figure 1 is the frequencies of the samples of Table 2 (densities in steps of 0.01), Figure 2 computes formula (11) |

Next chapter to read: 7-04 The cluster expansion (the mechanism of the tool used in § 05).

# The cluster expansion — a tool that splits into weakly linked small lumps and rewrites the logarithm as a convergent sum

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — The formula that writes the logarithm of the partition function as a sum over "connected lumps" (clusters), and its convergence condition. That convergence gives the infinite-volume limit, analyticity and exponential decay of correlations all at once. That continuum fields, too, can be decoupled cube by cube.

**Prerequisite chapters** — 7-03. We also use the infinite-volume step and the uniform-gap step of 7-01, 4-11 and 1-08 (Gaussian measures). The language of graphs is defined on the spot.

**Said first** — This is a tool for when there is a small parameter (weak coupling, high temperature, strong coupling on the lattice), and it does not reach the infinite-volume limit or the gap of four-dimensional Yang–Mills (the infinite-volume limit of the construction published on 2026-11-10 is taken along subsequences of volumes by compactness, not by an expansion; 12-05). The computations concern a one-dimensional model and small graphs.

The order of this chapter

1. What the tool is for — if independent, the logarithm is a sum. Weak links are added in as corrections
2. The smallest example: the one-dimensional hard-core lattice gas — coefficients, radius of convergence 1/4, the singularity on the side of negative activity (a table of computations)
3. Polymer models — the common form of a gas of incompatible lumps
4. Taking the logarithm: Ursell functions and connected graphs — only connected clusters remain
5. Why it can converge: cancellation and trees — the signs cancel, and only the number of trees remains (a moving figure)

6. Criteria for convergence — inequalities such as Kotecký–Preiss, and the range they guarantee (a moving figure)
7. Detour: the same inequality as the local lemma of probability — the Lovász local lemma and the zeros of the independent set polynomial
8. What the expansion gives: infinite volume, analyticity, exponential decay — the decay rate of correlations is the "mass gap" of this model
9. To continuum fields: cut into cubes and decouple by interpolation — rewritten as a gas of lumps of cubes
10. Uses and limits for gauge fields — works on the strong-coupling side, and does not reach the infrared in four dimensions

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01

## What the tool is for

What the infinite-volume step and the uniform-gap step need are estimates independent of the size of the box, and quantities far apart becoming nearly independent (formula (9) of 7-01). The problem statement says that because perturbation series usually diverge, expansion methods collectively called "cluster expansions" were developed, and writes 'Refined estimates yield the rate of exponential decay of Green's functions, magnitude of masses, …' (JW § 6.1). Known

The core is a single point. If the blocks are independent of each other, the partition function is a product and the logarithm is a sum.

$$Z_\Lambda = \prod_j Z_j \quad \Longrightarrow \quad \log Z_\Lambda = \sum_j \log Z_j \tag{1}$$

If it is a sum, quantities per unit volume do not depend on the box, and distant blocks have no effect. Real blocks are weakly linked, and the cluster expansion adds those links in as corrections for each "set of connected blocks" and shows that the sum converges. As a metaphor, it is a thin gas that hardly ever collides. More precisely, two things are needed: (i) the correction linking $n$ of them is of the order of a small number to the power $n$, and (ii) the number of ways of linking, divided by $n!$, is of exponential order ( § 05).

**Example:** if each block is independently "empty" or "one particle (weight $z$)", then $Z = (1+z)^N$. $\log Z/N = z - \frac{z^2}{2} + \cdots$ converges for $|z| < 1$ and stops at $z = -1$, where $Z = 0$. The reach of the series is decided by where the partition function vanishes for complex $z$.

02

## The smallest example: the one-dimensional hard-core lattice gas

This example is used throughout the chapter. Place particles at the points $x = 1, \dots, N$ ($n_x \in \{0,1\}$), forbid neighbours from being occupied together (**hard-core repulsion**), and give each particle the weight $z$ (the **activity**).

$$\Xi_N(z) = \sum_{\substack{n \in \{0,1\}^N \\ n_x n_{x+1} = 0}} z^{n_1 + \cdots + n_N} \tag{2}$$

If points $N$ and 1 are also regarded as neighbours to make a ring, one can write $\Xi_N = \operatorname{tr} T^N$ with the transfer matrix $T = \begin{pmatrix} 1 & \sqrt{z} \\ \sqrt{z} & 0 \end{pmatrix}$ (transfer matrices are in 2-08 § 03; the example of the Ising model is in 2-09, and lattice gauge theory in 5-03). The entries are $T_{nn'} = z^{(n+n')/2}$ ($n, n' \in \{0,1\}$; only $n = n' = 1$ gives 0). So in $\operatorname{tr} T^N = \sum_{n_1, \dots, n_N} \prod_{x=1}^{N} T_{n_x n_{x+1}}$ ($n_{N+1} = n_1$), the weight of each $n_x$ is picked up twice, half each time, giving $z^{n_x}$, and simultaneous occupation of neighbours vanishes with 0. This is exactly formula (2) on the ring. Of the eigenvalues $\lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1+4z})$, the larger survives. We call the following $p(z)$ the **pressure** (the physical pressure times the inverse temperature, the log partition function per unit volume).

$$p(z) = \lim_{N \to \infty} \frac{1}{N} \log \Xi_N = \log \frac{1 + \sqrt{1+4z}}{2} = \sum_{n \geq 1} b_n z^n, \qquad b_n = \frac{(-1)^{n-1}}{n} \binom{2n-1}{n} \tag{3}$$

Computed Table 1 The coefficients $b_n$ (agree with the closed form up to $n \leq 24$ by series operations with rational numbers).

| $n$ | 2 | 3 | 4 | 6 | 20 | 60 |
|---|---|---|---|---|---|---|
| $b_n$ | -3/2 | 10/3 | -35/4 | -77 | -3.4×10⁹ | -8.1×10³² |
| $\lvert b_n / b_{n-1} \rvert$ | 1.500 | 2.222 | 2.625 | 3.056 | 3.705 | 3.901 |

The ratio approaches 4, and the radius of convergence is $1/4$. The radius of convergence of a power series is the distance to the nearest point to the origin at which the function stops being analytic in the complex plane (a theorem of complex analysis) Known . Here that point is $z = -1/4$, where the inside of the square root vanishes and the eigenvalues coincide. There is no phase transition for $z > 0$, and yet the series cannot go beyond $1/4$. **The reach of the expansion is decided by a singularity on the side of non-physical negative activity.** The zero of $\Xi$ for a chain of length $n$ nearest to the origin is $-1/(4\cos^2 \frac{\pi}{n+2})$ (Scott–Sokal 2005, Ex. 3.2), which for $n = 5, 10, 20$ is $-0.3080, -0.2679, -0.2552$. Computed

03

## Polymer models

We extract the structure. Prepare a collection $\mathcal{P}$ of finite lumps (**polymers**) $\gamma$, a relation of being **incompatible**, $\gamma \not\sim \gamma'$ (overlapping, touching and so on; each is incompatible with itself), and complex activities $z_\gamma$, and count only the sets in which every two are compatible.

$$\Xi_\Lambda = 1 + \sum_{n\ge 1} \frac{1}{n!} \sum_{(\gamma_1,\dots,\gamma_n)\in\Lambda^n} z_{\gamma_1}\cdots z_{\gamma_n} \prod_{j<k} \mathbf{1}\{\gamma_j \sim \gamma_k\} \tag{4}$$

$\Lambda$ is a finite collection, and $\mathbf{1}\{\cdot\}$ is 1 if the condition is true and 0 if false. This formalism is said to have been introduced by Gruber–Kunz (1971) (Fernández–Procacci 2007 § 1). Known

**Example:** in § 02, a polymer = one occupied point, incompatible = the same point or a neighbour. In the high-temperature Ising model without external field (2-08), the polymers are connected sets of edges with even degree at every point, they are incompatible if they share a vertex, and the activity is $(\tanh\beta)^{\text{number of edges}}$. Strong coupling in lattice gauge theory (5-04) has connected sets of plaquettes, and continuum fields at weak coupling ( § 09) have connected sets of cubes. The reason the lumps are small becomes the small parameter of the expansion.

04

## Taking the logarithm: Ursell functions and connected graphs

In the device that goes by the name of Mayer (said to go back to Mayer 1937), one writes $\mathbf{1}\{\gamma_j \sim \gamma_k\} = 1 + \zeta_{jk}$ ($-1$ if incompatible, 0 if compatible) and expands the product. Each term corresponds to a **graph** (points, and edges joining pairs of points) on the points $1,\dots,n$, and in the logarithm only connected graphs remain. The reason is that the weight of a graph splits into the product of the weights of its connected components. Organising by the ways of dividing the points into connected components, between the sums over all $n$ with $1/n!$ attached, $\text{sum over all graphs} = \exp\big(\text{sum over connected graphs}\big)$ (the **exponential formula**) holds, and taking the logarithm leaves only the connected ones. Known

$$\log\Xi_\Lambda = \sum_{n\ge1}\frac{1}{n!}\sum_{(\gamma_1,\dots,\gamma_n)} \varphi^{\mathrm{T}}(\gamma_1,\dots,\gamma_n)\, z_{\gamma_1}\cdots z_{\gamma_n}, \qquad \varphi^{\mathrm{T}} = \sum_{\substack{G\subset G_\gamma \\ \text{connected, spanning}}} (-1)^{|E(G)|}$$

$G_\gamma$ is the graph joining $j,k$ when $\gamma_j \not\sim \gamma_k$, the sum is over connected subgraphs using all the points, and $|E(G)|$ is the number of edges (1 for $n=1$). $\varphi^{\mathrm{T}}$ is the **Ursell**

**function** (unrelated to the field $\phi$), and it is 0 if $G_\gamma$ is not connected. Only sets connected by incompatibility, that is **clusters**, contribute (Fernández–Procacci (2.3) (2.4). Still a formal series).

**Example:** in § 02, averaging over translations, $b_n = \frac{1}{n!}\sum_{x_2,\dots,x_n} \varphi^{\mathrm{T}}(0, x_2, \dots, x_n)$. For $n = 2$, each of $x_2 = -1, 0, 1$ gives $-1$ with one edge, so $b_2 = -3/2$. The $(0, 0)$ forbidden in configurations appears in the logarithm. Summing by brute force up to $n \leq 6$ agreed with Table 1 in rational numbers. Computed

05

## Why it can converge: cancellation and trees

The sum in formula (5) is huge. When every two are incompatible, $G_\gamma$ is the **complete graph** $K_n$ (joining all pairs), there are nearly as many connected spanning subgraphs as graphs in all, and if they remained as $\pm 1$ the sum would not converge even after dividing by $n!$. But the sum cancels, and only $(-1)^{n-1}(n-1)!$ remains.

Computed Table 2 $K_n$ (brute force for $n \leq 6$, a recurrence for $n = 10$).

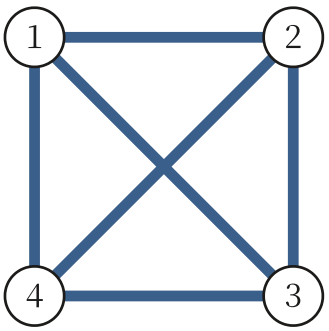
| $n$ | Connected spanning subgraphs | Spanning trees $n^{n-2}$ | Alternating sum |
|---|---|---|---|
| 4 | 38 | 16 | −6 |
| 10 | 34 496 488 594 816 | 100 000 000 | −362 880 |

The mechanism is due to Penrose (1967). Dividing the connected spanning subgraphs into intervals $[T, R(T)]$, one for each **spanning tree** (a subgraph without cycles connecting all the points) $T$, the sum of signs within an interval vanishes by the factor $\prod_{e \in R(T)\setminus T}(1-1)$, and only the trees with $R(T) = T$ (Penrose trees) remain (Fernández–Procacci, Prop. 5). Known

$$\varphi^{\mathrm{T}}(\gamma_1, \dots, \gamma_n) = (-1)^{n-1}\,\#\{\text{Penrose trees}\} \quad\Longrightarrow\quad |\varphi^{\mathrm{T}}| \leq \#\{G_\gamma\text{ 's spanning trees}\} \tag{6}$$

The number of Penrose trees agreed with the absolute value of the alternating sum by brute force for $n \leq 6$ Computed . Even for $K_n$ there are $n^{n-2}$ spanning trees, and $n^{n-2}/n!$ is of order $e^n$. **Since only an exponential remains after dividing by** $n!$, (5) can converge if the activities are small. Identities that write the sum in terms of trees or forests from the start were refined from Brydges–Federbush (1978) to the forest formula of Brydges–Kennedy (1987) (Abdesselam–Rivasseau 1994).

Figure 1 Choose a graph and press "Play": the connected spanning subgraphs are counted in order of increasing number of edges, and the running total of the signs $(-1)^{|E|}$ changes (bar: green +1, red −1)




K4: 38 subgraphs, no. 38

6 edges → sign +1

running total -6



K4: 38 subgraphs, no. 38, 6 edges → sign +1, running total -6

Computed The numbers in this figure were computed on this machine. The running total for $K_4$ is −16 at 3 edges (16 spanning trees), −1 at 4 (15 of them), −7 at 5 (6 of them) and −6 at 6. $C_4$ gives −3, $P_4$ −1 and $K_3$ +2.

06

## Criteria for convergence

Summing up the tree estimates gives a condition. The widely used condition of Kotecký–Preiss (1986) assigns to each lump a "budget" $a(\gamma) \geq 0$ and requires

$$\sum_{\gamma' \not\sim \gamma} |z_{\gamma'}|\, e^{a(\gamma')} \;\leq\; a(\gamma) \qquad \text{(for all } \gamma\text{)} \tag{7}$$

(the left-hand side includes $\gamma$ itself). If it holds, the series obtained by differentiating $\log \Xi_\Lambda$ with respect to $z_{\gamma_0}$ (the sum of clusters containing the lump $\gamma_0$; the sum per polymer) is at most $e^{a(\gamma_0)}$ independently of $\Lambda$, even with each term replaced by its absolute value (Fernández–Procacci (2.6)(2.7)(2.14)(2.15)). $\log \Xi_\Lambda$ itself grows with the volume, but because this per-polymer upper bound does not depend on $\Lambda$, (5) converges absolutely with estimates uniform in $\Lambda$ (ibid., the sentence right after (2.7)). The change in the logarithm when the lump $\gamma_0$ is removed is also at most $|z_{\gamma_0}|e^{a(\gamma_0)}$ (ibid. (2.11)). Read it as "the total weighted activity of the incompatible partners fits within one's own budget". The conditions of the same type of Dobrushin (1996) and Fernández–Procacci (2007) widen the range of application in that order. Known

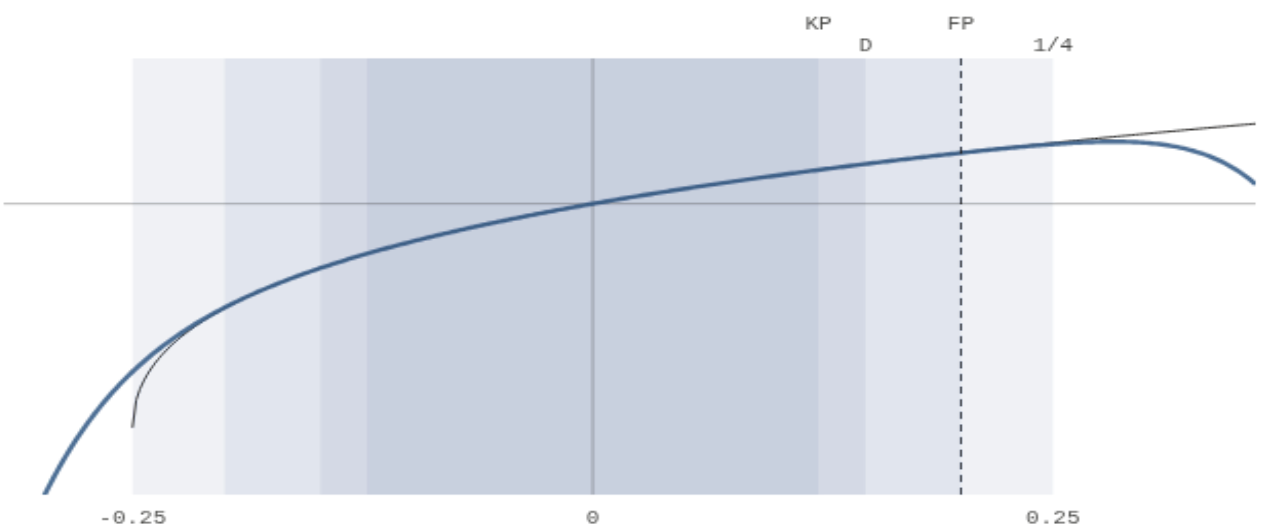
Computed Table 3 The range guaranteed for the gas of §02 (maximum degree $\Delta = 2$). Numerical maximisation agrees with the formulas of Fernández–Procacci Table 1.

| Condition | Form for this gas | Guaranteed $\lvert z\rvert$ |
|---|---|---|
| Kotecký–Preiss | $3\lvert z\rvert e^{a} \leq a$ | $1/(3e) = 0.1226$ |

| Condition | Form for this gas | Guaranteed $\lvert z\rvert$ |
|---|---|---|
| Dobrushin | $\lvert z\rvert \le \mu/(1+\mu)^3$ | $4/27 = 0.1481$ |
| Fernández–Procacci | $\lvert z\rvert \le \mu/(1+3\mu+\mu^2)$ | $1/5 = 0.2000$ |
| Exact radius | Formula (3) | $1/4 = 0.2500$ |

$1+3\mu+\mu^2$ counts the mutually compatible subsets of three incompatible points (the empty set, the three single points, $\{x-1, x+1\}$), and $(1+\mu)^3$ is coarser by also counting the sets that are not. The $1/4$ in the last row is the exact value $(\Delta-1)^{\Delta-1}/\Delta^{\Delta}$ on the infinite tree with every degree $\Delta$ (Shearer / Scott–Sokal; the starred entry in Fernández–Procacci Table 1) with $\Delta = 2$ put in. This tree with $\Delta = 2$ is the line of integers itself, so it agrees with the $1/4$ of formula (3). All three conditions above are **sufficient conditions**.

Figure 2 As the number of terms $N$ increases, the partial sum of series (3) (thick line) lies on the exact pressure (thin line) for $\lvert z\rvert < 1/4$, and runs wild outside (bands are the range of each condition)




```
N = 10, z = 0.200: partial sum 0.15732, difference 3.86e-4
```

Computed The numbers in this figure were computed on this machine. $\lvert S_N(z) - p(z)\rvert$ for $N = 5, 20, 40$ is 3e−3, 1.5e−5, 6.4e−8 at $z = 0.2$ and 0.029, 0.063, 0.88 at $z = 0.3$.

07

## Detour: the same inequality as the local lemma of probability

A section you may skip.

The **Lovász local lemma** of combinatorics (Erdős–Lovász 1975) is a condition for the probability that none of the "bad events" $A_x$ occurs to be positive. The way the events depend on each other is represented by a graph. Write $\Gamma(x)$ for the set of points adjacent to the point $x$.

**Dependency graph:** when each $A_x$ is independent of the events other than $x$ and its neighbours $\Gamma(x)$ (taken all together), this graph is called a dependency graph. The theorem has

the following form (Scott–Sokal 2005, Thm 1.1; $A_x^c$ is the event that $A_x$ does not occur).

$$0 \le r_x < 1, \quad \mathbb{P}(A_x) \le r_x \prod_{y \in \Gamma(x)} (1 - r_y) \text{ (for all } x) \implies \mathbb{P}\Big(\bigcap_x A_x^c\Big) \ge \prod_x (1 - r_x) > 0 \tag{8}$$

Scott–Sokal showed the following equivalence. That the conclusion of the local lemma (the probability that none occurs is positive) holds for **every** family of events with dependency graph $G$ and $\mathbb{P}(A_x) \le p_x$ is equivalent to the partition function of the hard-core gas on $G$ not vanishing on the polydisc of radii $p_x$ (the tuples of complex numbers $(z_x)$ with $|z_x| \le p_x$ for every $x$). This partition function is the sum $\sum_S \prod_{x \in S} z_x$ over sets $S$ of mutually non-adjacent points (the **independent set polynomial**; the multivariate version with an activity for each point), and if $G$ is a path it becomes the gas of § 02. They also show that the usual proof of the local lemma corresponds to the induction on "having no zeros" found implicitly by Shearer (1985) and explicitly by Dobrushin (abstract).

Setting $r = \mu/(1+\mu)$, the hypothesis of (8) is Dobrushin's form in Table 3, and for a sequence of events depending only on their left and right neighbours $p \le 4/27$ is enough. The best is $1/4$, bounded from above by Example 8.1 of Scott–Sokal § 8.4 (the zeros of finite paths approach $-1/4$) and from below by the exact radius of Table 3 (Shearer's value $(\Delta-1)^{\Delta-1}/\Delta^{\Delta}$ with $\Delta = 2$). The symmetric version often seen in textbooks also follows from here. When each event depends on at most $d$ other events and every probability is at most $p$, setting $r_x = 1/(d+1)$ in (8) gives $r(1-r)^d = \frac{1}{d+1}\big(\frac{d}{d+1}\big)^d \ge \frac{1}{e(d+1)}$, so $ep(d+1) \le 1$ is enough. For $d = 2$ this is $1/(3e)$, the Kotecký–Preiss value of Table 3. In this way all four values of Table 3 also appear on the side of the local lemma. Known

08

## What the expansion gives: infinite volume, analyticity, exponential decay

If condition (7) holds independently of the volume, three things follow. (i) The limit of the pressure exists and is analytic in the disc. (ii) Only clusters that touch the edge or go round feel the size of the box. (iii) In truncated correlations only clusters touching both $x$ and $y$, that is, connected from $x$ to $y$ by a chain of incompatibility, remain, and each link contributes a factor of order $|z|$ (Ueltschi 2004, Thm 3). Known

$$\big|\langle A_x B_y\rangle - \langle A_x\rangle\langle B_y\rangle\big| \;\le\; C\, e^{-m|x-y|} \tag{9}$$

**Example:** counting the gas of § 02 on a ring of 14 points, the coefficients of $\log \Xi_{14}/14$ agree with $b_n$ for $n \le 13$, and first differ at $n = 14$ (going round needs all 14 points). The correction is $\frac{1}{N}\log\big(1 + (\lambda_-/\lambda_+)^N\big)$, and for small $z$, $\lambda_-/\lambda_+ \approx -z$, so it is of order $(-z)^N/N$ (the difference in the coefficient at $n = 14$ is exactly $1/14$). The truncated correlation of the density is proportional to $(\lambda_-/\lambda_+)^{|x-y|}$, and the decay rate is $m(z) = \log(\lambda_+/|\lambda_-|)$. Computed

Computed Table 4 The decay rate $m(z)$ and the estimate from chains $\log(1/|z|)$.

| $z$ | −0.249 | −0.1 | 0.01 | 0.25 | 1 |
|---|---|---|---|---|---|
| $m(z)$ | 0.1267 | 2.0634 | 4.6249 | 1.7627 | 0.9624 |
| $\log(1/\lvert z\rvert)$ | 1.3903 | 2.3026 | 4.6052 | 1.3863 | 0 |

For small $z$, $m \approx \log(1/z)$ and the picture of chains is right; as $z \to -1/4$, $m \to 0$ (decay stops at the singularity that fixed the radius). $z = 1$ is outside the radius and still has $m > 0$: convergence is a sufficient condition.

$m(z)$ is the ratio of eigenvalues of the transfer matrix, the "mass gap" of this model (the uniform-gap step of 7-01, 4-11). In field theory too, if truncated correlations in the time direction decay like $e^{-mt}$ uniformly in the volume, and the vectors $\psi$ (with the vacuum component subtracted) made from the observables for which this decay can be shown are dense in the subspace orthogonal to the vacuum, then the Hamiltonian $H$ of the OS reconstruction (4-10) has no spectrum in $(0, m)$. Known

The tool is the **spectral measure**. For each $\psi$ there is a measure $\mu_\psi$ on $[0,\infty)$ with $\langle \psi, e^{-tH}\psi\rangle = \int e^{-t\lambda}\,\mathrm{d}\mu_\psi(\lambda)$ (the spectral theorem of 1-10; if there is a basis of eigenvectors $e_k$, $\mu_\psi$ puts the weight $|\langle e_k,\psi\rangle|^2$ on each eigenvalue $\lambda_k$). If $\int e^{-t\lambda}\mathrm{d}\mu_\psi \le Ce^{-mt}$ holds for all $t$, then looking at $t\to\infty$, $\mu_\psi$ cannot put weight on $\lambda < m$. Since this holds for a dense set of $\psi$, no spectrum remains in $(0,m)$. The converse is formula (2) of § 5 of the problem statement.

09

## To continuum fields: cut into cubes and decouple by interpolation

For continuum fields, the covariance $C(x,y)$ of the free field links even distant points. Cut the box into unit cubes $b_i$ (indicator functions $\chi_i$), and multiply the parts straddling cubes by $s_{ij} \in [0,1]$.

$$C(\mathbf{s})(x,y) = \sum_i \chi_i(x) C(x,y) \chi_i(y) + \sum_{i \neq j} s_{ij}\, \chi_i(x) C(x,y) \chi_j(y), \qquad s_{ij} = s_{ji} \tag{10}$$

If all the $s$ are 1 this is the original covariance, and if all are 0 the cubes are independent (Abdesselam–Rivasseau 1994 (V.A.5)). The two are joined by Taylor's formula (ibid. (III.1)). The index $\ell$ in formula (11) runs over pairs of cubes $\{i,j\}$, and $s_\ell = s_{ij}$. $f(\mathbf{s})$ is the partition function with covariance $C(\mathbf{s})$, $I$ is a set of pairs, and $\psi_I(h)$ is the point with $s_\ell = h_\ell$ for $\ell \in I$ and $s_\ell = 0$ for $\ell \notin I$.

$$f(1,\dots,1)=\sum_{I\subset\{\text{pairs}\}}\prod_{\ell\in I}\int_0^1 \mathrm{d}h_\ell\,\Big(\prod_{\ell\in I}\frac{\partial}{\partial s_\ell}\Big)f\big(\psi_I(h)\big)\qquad (11)$$

For a single pair, $f(1)=f(0)+\int_0^1 f'(h)\,\mathrm{d}h$, and (11) repeats this pair by pair. $\partial/\partial s_\ell$ ($\ell=\{i,j\}$) brings down one line joining the cubes $b_i, b_j$, of size about $|C|$, and with mass $m_0$ about $e^{-m_0|x-y|}$. The differentiated pairs form lumps, the integral splits lump by lump, and it becomes a gas whose polymers are connected sets of cubes $Y$ (ibid. (V.A.7)(V.A.8)).

However, the interpolated covariance must stay positive definite (the condition for it to make sense as a Gaussian measure), so in practice one uses the **forest formula** (the Taylor forest formula of Brydges–Kennedy, ibid. Thm III.1). This is the version in which the parameter joining two cubes is set to the minimum of $h_\ell$ along the path joining them in the tree of the forest, and this keeps positive definiteness (ibid. (V.A.9), Thm IV.5). In this form, for any constant $K$, with an ultraviolet cutoff in place and the coupling $g$ taken small enough (depending on $K$), $\sum_{Y\ni 0}|A(Y)|K^{|Y|}\le 1$ (ibid. (V.A.10); $|Y|$ is the number of cubes in $Y$). It was Glimm–Jaffe–Spencer who brought this type of expansion into constructive field theory (ibid. § I), and the results are in 7-03 § 05 and § 08. Known

**Example:** the tool is **Gaussian integration by parts** (Gaussian distributions are in 1-08 § 06). For a standard normal $\phi$, the derivative of the density $e^{-\phi^2/2}$ is $-\phi\,e^{-\phi^2/2}$, so integration by parts gives $\mathbb{E}[\phi F(\phi)]=\mathbb{E}[F'(\phi)]$. Extending this to Gaussian variables with mean 0 and covariance matrix $C(s)$ gives $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_s[H]=\frac12\sum_{a,b}\frac{\mathrm{d}C_{ab}}{\mathrm{d}s}\,\mathbb{E}_s[\partial_a\partial_b H]$ Known . For Gaussian variables $(\phi_1,\phi_2)$ with variance 1 and covariance $sc$, $\mathrm{d}C_{12}/\mathrm{d}s=\mathrm{d}C_{21}/\mathrm{d}s=c$, so $\frac{\mathrm{d}}{\mathrm{d}s}\mathbb{E}_s[F(\phi_1)G(\phi_2)]=c\,\mathbb{E}_s[F'(\phi_1)G'(\phi_2)]$. At $s=0$ the two are independent, so integrating in $s$ from 0 to 1 gives

$$\operatorname{Cov}\big(F(\phi_1),G(\phi_2)\big)=c\int_0^1\mathbb{E}_s\big[F'(\phi_1)\,G'(\phi_2)\big]\,\mathrm{d}s,\qquad \big|\operatorname{Cov}\big|\le|c|\,\sup|F'|\,\sup|G'|\qquad (12)$$

and the correlation of lumps is bounded by the covariance $|c|$ that links them. For $F=G=\cos$ the value is $e^{-1}(\cosh c-1)$ ($0.15932358$ at $c=0.9$), and numerical integration of the left- and right-hand sides agreed to 8 digits. Computed

10

## Uses and limits for gauge fields

**Strong coupling on the lattice.** For small $\beta$ the expansion in lumps of plaquettes works (5-02, 5-04), and the existence and analyticity of the infinite-volume limit and estimates of confinement have been shown (Osterwalder–Seiler 1978). But Chatterjee

(2018) cautions that this area law holds for every group including $U(1)$, and is not evidence of non-abelian confinement. Known

**Phase-space expansion.** When there is no small parameter at the unit scale, the field is split by scale, expanded in the "phase-space cells" of each scale, and the results piled up (Battle–Federbush 1982; the 'phase-space localization' of JW § 6.1). In lattice gauge theory, Bałaban's series of papers is this (7-05). Known

**The infrared limit.** In four dimensions the coupling becomes strong at long distances. Magnen–Rivasseau–Sénéor 1993 § I writes 'there is no easy solvable model of these phenomenons around which to expand', and the problem statement also says that a gap uniform in the volume needs new ideas (JW § 6.5). The small $\beta$ where the expansion converges is the shore opposite to the continuum limit $\beta \to \infty$. More precisely, the expansion converges on a finite interval on the side of small $\beta$, and the continuum limit is taken at $\beta \to \infty$ (5-09).

Seam The reach, and where we want to go

In § 02 the singularity that stopped the expansion was on the side of negative activity. On the four-dimensional $U(1)$ lattice there really is a phase at large $\beta$ in which the area law fails (Guth 1980; a complete proof and the massless phase are Fröhlich–Spencer 1982). For the non-abelian case (the Wilson action of $SU(2)$ and $SU(3)$), it is believed that there is no phase transition that loses confinement Physics , but it is not a theorem. Where the convergence of the expansion stops says nothing about whether there is a phase transition beyond it. In the organisation of this book, what is missing is one step between the weak-coupling ultraviolet (7-05 to 7-07) and the strong-coupling expansion, where the coupling becomes $O(1)$ (7-10).

## What this chapter can say

| | Content |
|---|---|
| **Said** | The formula of the expansion, Penrose's identity, the three convergence conditions and their consequences (analyticity, exponential decay) Known |
| **Said** | The coefficients, radius $1/4$, thresholds and decay rate of the one-dimensional hard-core lattice gas. Agreement of the alternating sum of $K_n$ with the Penrose trees ($n \le 6$). Formula (12) Computed |

| | Content |
|---|---|
| **Not said** | The content of the convergence proofs for continuum fields and for strong coupling on the lattice (original papers not followed; 5-04). An expansion usable in the infrared in four dimensions (none is known) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Quotations from the problem statement (§ 5, § 6.1, § 6.5) | Text checked | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) |
| Formulas (4)–(7), the conditions of Dobrushin and FP, the formulas of Table 3 | Text checked | R. Fernández, A. Procacci, CMP 274 (2007) 123–140, arXiv:math-ph/0605041 |
| Formulas (10)(11), the forest formula / decay of correlations | Text checked | A. Abdesselam, V. Rivasseau, arXiv:hep-th/9409094 (1994). D. Ueltschi, Moscow Math. J. 4 (2004) 511–522 |
| The local lemma, the equivalence, the zeros of chains, $\lambda_c \leq 1/4$, Shearer's value | Abstract, § 1, § 4.1, § 5.2, § 8.4 checked | A. D. Scott, A. D. Sokal, J. Stat. Phys. 118 (2005) 1151–1261, arXiv:cond-mat/0309352 |
| Original sources, lineage | Bibliographic data only | Kotecký–Preiss, CMP 103 (1986) 491. Dobrushin, AMS Transl. (2) 177 (1996) 59. Gruber–Kunz, CMP 22 (1971) 133. Penrose, in *Statistical Mechanics: Foundations and Applications* (T. A. Bak ed.), Benjamin (1967) 101–109. Mayer, J. Chem. Phys. 5 (1937) 67. Brydges–Federbush, J. Math. Phys. 19 (1978) 2064. Brydges–Kennedy, J. Stat. Phys. 48 (1987) 19. Battle–Federbush, Ann. Phys. 142 (1982) 95. Glimm–Jaffe–Spencer, in *Constructive Quantum Field Theory* (G. Velo, A. Wightman eds.), Lecture Notes in Physics 25, Springer (1973) (Part II: The cluster expansion), and Ann. Math. 100 (1974) 585. Erdős–Lovász, in *Infinite and Finite Sets* II, Colloq. Math. Soc. J. Bolyai 10, North-Holland (1975) 609–627. Shearer, Combinatorica 5 (1985) 241 |
| Strong coupling on the lattice, the phases of $U(1)$, the quotation on the infrared | From the abstracts and the checks of this book's materials | Osterwalder–Seiler, Ann. Phys. 110 (1978) 440 (text not obtained). Chatterjee, arXiv:1803.01950 § 4. Guth, PRD 21 (1980) 2291 (bibliographic data from Chatterjee's reference list). Fröhlich–Spencer, CMP 83 (1982) 411. Magnen–Rivasseau–Sénéor, CMP 155 (1993) 325 |

| Item | Kind | Source / tools |
|---|---|---|
| The exponential formula, spectral measures, Gaussian integration by parts, radius of convergence and singularities | Known | Standard facts (1-08, 1-10) |
| Tables, zeros, rings, formula (12), figures | Computed | Python on this machine: `hardcore1d.py`, `graphs.py`, `decouple.py`. The figures are plain JavaScript |

Next chapter to read: 7-05 The sequence of renormalisation group steps (piling up the expansion into a sequence of scales).

# The sequence of renormalisation group steps — integrating out one scale at a time, splitting into small and large fields as we go

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — The idea of controlling the ultraviolet limit by repeating a transformation that coarsens the lattice one step at a time (blocking). Splitting the field at each step into "small fields" and "large fields". That gauge fields need averages of group elements and local gauges.

**Prerequisite chapters** — 7-01 (the ultraviolet-limit step) and 4-04 (the running of the coupling). We also use the polymers of 7-04, the $b_0$ of 4-06 § 02, and two-dimensional $U(1)$ from 7-02 (independence of the plaquette angles, the heat kernel). Plaquettes and the Wilson action are the language of 5-01, but what is needed is defined on the spot.

**Said first** — This explains the skeleton of the method and does not follow Bałaban's proofs. In four dimensions even the main theorem of this method carries a hypothesis, and it does not reach uniqueness of the continuum limit, infinite volume or the mass gap ( § 07, § 09). How the construction published on 2026-11-10 handles one step of renormalisation and large fields is read in Parts 9 and 11. The details for gauge fields are within the range of the introductions and the statements of theorems in the primary sources, and an exposition for scalar fields (Dimock). The computations concern two-dimensional $U(1)$, free fields and the one-loop running, not numbers for non-abelian theories in four dimensions.

The order of this chapter

1. Why a sequence of steps — there is no small number common to all scales, so the scales are integrated one step at a time (formula (1))
2. One step of the transformation: blocking — fixing the averages loosely and integrating the fine variables changes the form of the action in one step (Table 1)

3. For free fields one step can be written exactly — minimal configurations, and the locality of the Green's function produced by the constraint on averages alone (Table 2)
4. Small fields and large fields (a moving figure) — the region is split at each step, and large fields carry a small factor (Figure 1, Table 3)
5. What gauge fields add — cannot be rescaled, cannot be added and divided, the gauge is chosen locally (formula (8))
6. The running of the coupling and the number of steps (a moving figure) — in four dimensions the sum of corrections keeps growing with the number of steps (Figure 2, Table 4)
7. The form of the induction — keep estimates of the same form at each step, with constants independent of the step and the lattice spacing (formula (10))
8. Detour: the name "phase-space cells" — counting the degrees of freedom integrated at each step (formula (11))
9. Another road, and where it stops — construction with a continuum cutoff, and the scale where the coupling becomes $O(1)$

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01

## Why a sequence of steps

What we wanted to show in the ultraviolet-limit step was that expectation values are bounded independently of the lattice spacing $a$ (formula (8) of 7-01). In four-dimensional Yang–Mills, by asymptotic freedom (4-06), the coupling $g$ is small at fine scales and becomes $O(1)$ at coarse scales. Every scale has a small number, but there is no small number common to all scales. So the scales are integrated one step at a time.

$$a \;\to\; La \;\to\; L^2 a \;\to\; \cdots \;\to\; L^K a = a_0, \qquad K = \log_L \frac{a_0}{a} \quad (1)$$

The transformation with factor $L$ is repeated $K$ times, and at each step only the coupling of that scale is used as the small number ($L$ is unrelated to the size of the box in 7-01, and $a_0$ is fixed). Writing $\rho_k$ for the weight (effective density) after step $k$, $Z = \int \rho_0 = \int \rho_1 = \cdots = \int \rho_K$. The problem statement explains this transformation as "integrating out the large-momentum degrees of freedom and restoring the scale" (JW § 6.5), and the introduction of Bałaban 1985 says it follows the form proposed by K. Wilson. Known

**Example:** with $L = 2$, $a_0 = 1$ and $a = 2^{-20}$, $K = 20$. Each halving of $a$ adds one step, so what must be shown are estimates **independent of** $K$.

02

## One step of the transformation: blocking

In one step, one fixes the variables of the coarse lattice and integrates the fine variables so as to keep those values. Written for a scalar field, the average over a block $B(y)$ of side $L$ points is (2), and Dimock, who expounds Bałaban's method, fixes the average loosely with a Gaussian weight ($\alpha > 0$ is arbitrary, $N$ a constant that preserves the integral).

$$(Q\Phi)(y) = \frac{1}{L^d} \sum_{x \in B(y)} \Phi(x) \quad (2)$$

$$\tilde{\rho}_{k+1}(\Phi') = \frac{1}{N} \int \exp\Big( -\frac{\alpha}{2L^2} \big\| \Phi' - Q\Phi \big\|^2 \Big)\, \rho_k(\Phi)\, \mathrm{d}\Phi \quad (3)$$

Then the lattice is shrunk to unit spacing and the field rescaled, giving $\rho_{k+1}$ (Dimock I (26)(27)). Known

**Example:** look at two-dimensional $U(1)$. An element of $U(1)$ can be written as $e^{i\theta}$ with the angle $\theta$. A unit square $p$ of the lattice is called a **plaquette**; the product of the variables on its four sides taken in order along the orientation is written $U(\partial p) = e^{i\theta_p}$, and $\theta_p$ is called the angle of the plaquette (5-01 § 03). Putting the weight $e^{\beta \cos \theta_p}$ on each plaquette is the **Wilson-type** action ($\beta$ is the inverse lattice coupling; the larger it is, the weaker the coupling. A different quantity from the β function of § 06). In two dimensions with free boundary the plaquette angles are independent (7-02 § 02), and the angle of a $2 \times 2$ block is the sum of the four angles. For a sum of independent angles, the Fourier coefficients of the distribution are the products of the individual coefficients. Using the coefficient of one angle

$$c_n(\beta) = \frac{\int_{-\pi}^{\pi} e^{in\theta}\, e^{\beta \cos\theta}\, \mathrm{d}\theta}{\int_{-\pi}^{\pi} e^{\beta \cos\theta}\, \mathrm{d}\theta}$$

the coefficient of the angle of the block becomes $c_n(\beta)^4$ in one step. Setting $I_n(\beta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{in\theta} e^{\beta\cos\theta}\, \mathrm{d}\theta$, $c_n = I_n(\beta)/I_0(\beta)$, and $I_n$ is a special function called the modified Bessel function. If the action were still of Wilson type after one step, the $\beta'$ fixed by $c_1$ should also match $c_2$.

Computed Table 1 One step ($L = 2$) of two-dimensional $U(1)$ with Wilson type. $\beta'$ is the value fixed by $c_1(\beta') = c_1(\beta)^4$, and the mismatch is $c_2(\beta')/c_2(\beta)^4 - 1$.

| $\beta$ | 2 | 10 | 50 | 100 | 400 |
|---|---|---|---|---|---|
| $\beta'$ | 0.488 | 2.996 | 12.892 | 25.383 | 100.377 |

| $\beta$ | 2 | 10 | 50 | 100 | 400 |
|---|---|---|---|---|---|
| Mismatch | 2.43 | 6.6e−2 | 2.8e−4 | 3.2e−5 | 4.8e−7 |

It does not match. The form of the action changes in one step, and new terms are born. Computed If the weight is chosen from the start so that its coefficients are $c_n = e^{-n^2/2\beta}$, then $c_n^4 = e^{-n^2/(2\cdot\beta/4)}$, and the form is kept exactly. This weight is the **heat kernel** on the circle (also called Villain type): the distribution on the circle obtained by adding up copies of the density of the normal distribution with variance $1/\beta$ shifted by $2\pi$, that is, the normal distribution **wrapped** round the circle (7-02 § 04). In general one follows the main local terms and gathers the rest into corrections $E_k(X)$ for each lump $X$ (Dimock I (105)(108); the form of the polymers of 7-04). Known

03

## For free fields one step can be written exactly

If the weight is Gaussian (a free field), (3) can be written in closed form for any number of steps. The density after $k$ steps is determined by the configuration $\phi_k$ that minimises the action of the fine field given the coarse values $\Phi$ (the **minimal configuration**, or background field) (Dimock I (45)–(52)).

$$\rho_k(\Phi) = Z_k\, e^{-S_k(\Phi,\,\phi_k)}, \qquad \phi_k = \alpha_k G_k Q_k^{\mathsf T}\Phi, \qquad G_k = \big(-\Delta + \bar\mu_k + \alpha_k Q_k^{\mathsf T} Q_k\big)^{-1} \tag{4}$$

$\Delta$ is the lattice Laplacian (not the $\Delta$ of the mass gap), $Q_k$ the average over $k$ steps, $Q_k^{\mathsf T}Q_k$ the projection onto functions constant on blocks, $\bar\mu_k$ the mass squared, and $\alpha_k = \alpha(1-L^{-2})/(1-L^{-2k})$ (Dimock I (32)) a constant gathering the weights of $k$ steps. Fluctuations about the minimal configuration are Gaussian variables, and integrating them leaves a constant $Z_k$. The point is the locality of $G_k$. Even if $\bar\mu_k$ is close to 0, the term acting on the averages plays the role of a mass, and when $x$ is in a cube $\Delta_y$ and the support of $f$ in a cube $\Delta_{y'}$ ($\Delta_y$ is a large cube of side $M = L^m$, and $d(y,y')$ the distance measured in units of these cubes)

$$\big|(G_k f)(x)\big| \le C\, e^{-\gamma\, d(y,y')}\, \|f\|_\infty \tag{5}$$

holds (Dimock I (85), where $\gamma = \frac{1}{2}\gamma_0$. The literature is Bałaban, CMP 89 (1983) and others). Known The massless $(-\Delta)^{-1}$ changes in two dimensions only with the logarithm of the distance.

**Example:** on a two-dimensional periodic lattice (32 blocks on a side), we lined up $(-\Delta + \ell^{-2}P)^{-1}$ ($P$ is the projection onto the averages over blocks of side $\ell$) by the distance $r$ in block units.

Computed Table 2 $G(r)/G(0)$. The bottom row is $\gamma$ from fitting $e^{-\gamma r}$ to the values for $r = 4$ to 12 (the decay rate per block). The rightmost column puts the mass $\ell^{-2}$ everywhere.

| $r$ | $\ell = 2$ | $\ell = 4$ | $\ell = 8$ | **Mass ($\ell = 8$)** |
|---|---|---|---|---|
| 1 | 0.181 | 0.135 | 0.111 | 0.111 |
| 4 | 4.29e−3 | 3.12e−3 | 2.53e−3 | 2.94e−3 |
| 8 | 5.10e−5 | 3.47e−5 | 2.76e−5 | 3.87e−5 |
| Decay rate | 1.089 | 1.106 | 1.111 | 1.064 |

The decay rate is about 1.1 per block and hardly depends on $\ell$. Although the constraint acts only on the block averages, it is about the same as putting a mass everywhere (1.055–1.064 for $\ell = 2, 4, 8$ by the same fit), and if anything slightly faster. This locality, independent of the step, is why the cluster expansion (7-04) can be used at each step. Computed

04

## Small fields and large fields

Analysis in perturbative form can only be used when the field is small. The distinctive feature of Bałaban's method is that at each step the lattice is split into the **small-field region** $\Omega$ and its outside, and the sum is taken over all ways of splitting (Dimock II § 1).

$$1 = \sum_{\Omega} \zeta(\Omega^c, \Phi)\, \chi(\Omega, \Phi) \quad (6)$$

$\chi$ is the indicator function that the small-field condition holds on $\Omega$, and $\zeta$ that the condition fails somewhere in each cube of $\Omega^c$. In the words of the introduction of Dimock I ( § 1.1), 'The large field region is treated crudely but makes a small contribution due to the fundamental stability of the interaction'. The next step splits within the small-field region of the previous step, so the regions shrink, $\Omega_1 \supset \Omega_2 \supset \cdots$. Known

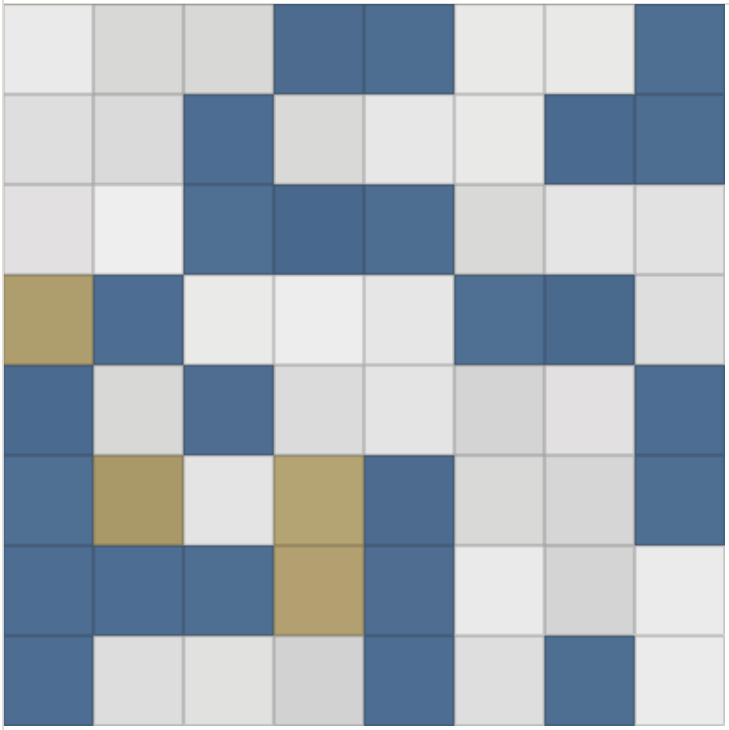
For gauge fields the condition is put on plaquettes. At the first step (bare coupling $g_0$), Bałaban 1985 imposes on each plaquette $p$

$$|U(\partial p) - 1| < \varepsilon_1, \qquad \varepsilon_1 = g_0\, p(g_0), \qquad p(g) = c\,(1 + \log g^{-1})^{p_0}, \quad p_0 > 2 \quad (7)$$

($c$ is a sufficiently large constant), and takes as the small-field region the union of blocks sufficiently far from the plaquettes where it fails (ibid. (4)(7) on pp. 256–257; notation adapted to this book). Known

We estimate the sizes as a rough guide. The weight is roughly $e^{-|U(\partial p)-1|^{2}/g^{2}}$, so the typical size is about $g$, and the threshold is $p(g)$ times that. Large fields then carry, as a rough guide, a factor of order $e^{-O(1)\,p(g)^{2}}$. This is smaller than any power of $g$. The exposition for scalar fields says that a factor $e^{-O(1)\,p_0^{2}}$ appears for each large-field cube (Dimock II § 1; the $p_0 = (-\log \lambda_0)^{p}$ there is the size of the threshold, a different symbol from the exponent $p_0$ in (7)). Known **Example:** with $c = 1$, $p_0 = 3$ and $g = 0.1$, $p \approx 36.0$. Taking $O(1)$ to be 1, $e^{-p^{2}} \approx e^{-1298}$, which is in a different league from $g^{10} = e^{-23}$. Computed

Figure 1 As the step k advances, how the size of the block angles (shade of the background), large fields (■ dark fill) and the parts that left the small-field region at earlier steps (■ light fill) grow



Step 2 (4×4), β_k = 4.000, ε = 0.50, large fields 25/64, outside the region 29/64

Computed The numbers in this figure were computed on this machine (a 32×32 sample of standard normals embedded at about 0.1 resolution; the angles $\theta_p = z_p/\sqrt{\beta}$ and the block sums are computed on the spot).

Computed Table 3 The sample of Figure 1 ($\beta = 64$, $\varepsilon = 0.5$; step 0 is 0/1024). "Exact" is the probability under the wrapped normal distribution with variance $4^{k}/\beta$.

| Step $k$ | $\beta_k$ | Large fields (sample) | Same (exact) | Outside the region |
|---|---|---|---|---|
| 1 | 16 | 13/256 | 0.043 | 13/256 |
| 2 | 4 | 25/64 | 0.312 | 29/64 |
| 3 | 1 | 13/16 | 0.613 | 15/16 |

The model of the figure is the heat-kernel type of § 02, and the effective coupling at step $k$ is exactly $\beta_k = \beta/4^{k}$. In two dimensions the coupling strengthens at each step, and with the threshold fixed, large fields fill everything within a few steps. The place to

stop before the coupling becomes $O(1)$ is the $a_0$ of (1). In four dimensions the strengthening is much slower ( § 06).

05

## What gauge fields add

Gauge fields add three things. Known **(i) Group elements cannot be rescaled.** Bałaban 1985 writes 'we use improvement of regularity properties of typical gauge fields instead of non-existing scalings'. **(ii) Group elements cannot be added and divided.** The variables on coarse edges are made from those on fine edges, but the average of group elements leaves the group. How to make the average is itself the subject of a whole paper (Bałaban, CMP 98 (1985) 17–51, bibliographic data only). For charged fields, one averages gauge-covariantly by multiplying by phases along paths (formula (20) of Dimock's paper on QED). **(iii) The gauge is chosen locally, scale by scale.** To write $U = e^{igA}$ and expand, the gauge must be fixed, but no globally continuous choice exists (Singer 1978, 3-04, 4-08). On the lattice the group is compact, so the definition of $Z$ needs no gauge fixing, and only local choices are needed. According to the problem statement, the gauge is chosen for each place in space-time and each scale ('phase cells'), and is rechosen with the steps from an ultraviolet gauge to an infrared gauge (JW § 6.5; describing the work in three dimensions).

The reason it is local is elementary. Define the **operator norm** of a matrix $A$ as the factor by which it stretches vectors the most, $\|A\| = \max_{|v|=1} |Av|$. A unitary matrix does not change lengths, so $\|U\| = 1$. For unitary matrices $U, V, W$

$$\|UV - 1\| \le \|U - 1\| + \|V - 1\|, \qquad \|WUW^{-1} - 1\| = \|U - 1\| \quad (8)$$

(the former from $UV - 1 = U(V - 1) + (U - 1)$ and $\|U(V-1)\| \le \|U\|\,\|V-1\| = \|V - 1\|$, the latter because neither $W$ nor $W^{-1}$ changes lengths). Paper Under a gauge transformation the plaquette variable changes to the form $WU(\partial p)W^{-1}$, so by the latter, condition (7) measured in the operator norm does not change under gauge transformations. On a two-dimensional block of side $\ell$, setting the vertical edges and the bottom horizontal edges to 1 (complete axial gauge), a horizontal edge at height $y$ is the product of the inverses of the variables of the $y$ plaquettes directly below it, so repeated use of (8) gives $\|U_b - 1\| < \ell\,\varepsilon_1$. Paper **Example:** for $\varepsilon_1 = 0.05$, $\ell = 8$ gives within 0.4. $\ell = 64$ gives 3.2, which says nothing since always $\|U - 1\| \le 2$. The gauge is effective over a range of about a block.

06

## The running of the coupling and the number of steps

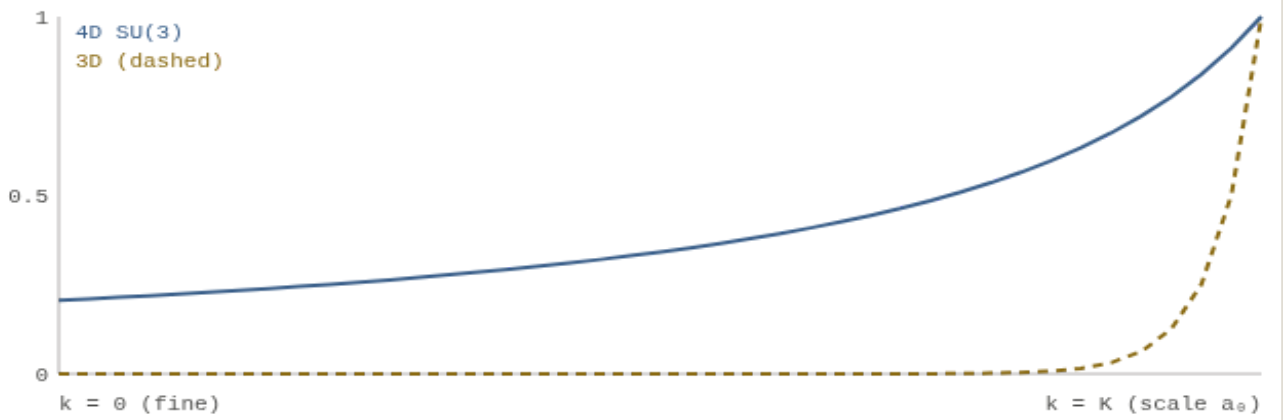
The main thing that changes from step to step is the effective coupling $g_k$. In four dimensions, cutting into steps the one-loop running ($1/g^2(\mu) = 1/g^2(\mu_0) + 2b_0 \ln(\mu/\mu_0$ ), obtained by integrating the β function $\mu\, \mathrm{d}g/\mathrm{d}\mu = -b_0 g^3 + O(g^5)$, $b_0 = 11N/48\pi^2$ of 4-06 § 01; the same formula from the physics side is formula (2) of 17-01) gives

$$\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - 2b_0 \ln L \quad (9)$$

and it strengthens by a fixed amount at each coarsening. In three dimensions $g^2$ has the dimension of mass, and the dimensionless coupling at step $k$, $g_k^2 = g^2 L^k a$, is geometrically small on the fine side. Known

From here on this is a rough-guide argument using only formula (9); it is not Bałaban's estimate itself, and not a claim at the level of a theorem. The difference shows up in the sum. If each step produces a correction of order $g_k^2$, then in three dimensions $\sum_k g_k^2$ is finite, but in four dimensions $g_k^2 \approx 1/(2b_0 \ln L \cdot (K-k))$ and the sum keeps growing like the logarithm of the number of steps, so it cannot be piled up as an error. What this rough guide shows is that corrections of that size must be renormalised into the running of the coupling itself and followed. If the remainder is of order $g_k^3$, the sum stays bounded. The title of the paper extending to four dimensions (Bałaban, CMP 109 (1987); from the reference list of JW) contains 'a coupling constant renormalization in 4D'. Known

Figure 2 Moving the number of steps K (a = 2^−K): how the sequence of effective couplings g_k² (g² = 1 at the right end) and its sums change. In three dimensions g_k² = 2^(k−K)




K = 40, four dimensions: $g_0^2$ = 0.206, Σg² = 16.99, Σg³ = 11.88, three dimensions: Σg² = 2.000

Computed The numbers in this figure were computed on this machine (partial sums of formula (9); they agree with Table 4).

Computed Table 4 $SU(3)$, $L = 2$, $g_K^2 = 1$ at the right end. $g_0^2$ is the value at the finest step.

| $K$ | $g_0^2$ | $\sum g_k^2$ (4D) | $\sum g_k^3$ (4D) | $\sum g_k^2$ (3D) |
|---|---|---|---|---|
| 10 | 0.509 | 7.76 | 6.63 | 1.999 |
| 40 | 0.206 | 16.99 | 11.88 | 2.000 |
| 160 | 0.061 | 29.54 | 16.12 | 2.000 |
| 320 | 0.031 | 36.38 | 17.56 | 2.000 |

**Example:** to go from $g^2 = 1$ at the right end to $g^2 = 0.1$ at the finest step takes about 93 steps for $SU(3)$ ($a \approx 9 \times 10^{-29}$) and about 140 steps for $SU(2)$. $\sum g_k^2$ grows by about 7.2 each time $K$ doubles, while $\sum g_k^3$ settles at about 21.2 and the three-dimensional $\sum g_k^2$ at 2. Computed

07

## The form of the induction: what is assumed and what is shown at each step

The proof is an induction on the step $k$. The hypothesis is that "$\rho_k$ satisfies a fixed form and estimates". The form consists of the main action on the small-field region, exponentially small corrections $E_k(X)$ localised on lumps $X$, and the sum over large-field regions (Dimock II § 1). One shows that the same form and estimates hold after one step (3), and the key is that the constants do not depend on $k$ or $a$. The estimate of the main theorem of the three-dimensional paper has the following form (Bałaban 1985 (5); notation adapted to this book).

$$\chi(U)\,\exp\Big(-\frac{1}{g_k^2}A\big(U_k(U)\big)-C\,|T_k|\Big)\;\le\;\rho_k(U)\;\le\;\exp\big(C\,|T_k|\big) \tag{10}$$

$A$ is the Wilson action, $U_k(U)$ the minimal configuration, $\chi$ the indicator function of the small-field condition, $|T_k|$ the number of points of the lattice at step $k$, and $C$ depends neither on $a$ nor on $k$. Integrating gives upper and lower bounds on the partition function uniform in $a$ (ibid. (6)). In four dimensions, the 1989 paper states ultraviolet stability in four dimensions as its main theorem (the Conditional Ultraviolet Stability Theorem; Theorem 1 of Bałaban 1989) under the hypothesis 'If the sequence of the effective coupling constants is contained in an interval ]0, γ] with a sufficiently small positive γ'. Known

**Example:** for three-dimensional $\phi^4$, Dimock's ultraviolet stability theorem (Dimock's Theorem 1) is of the same type: with the counterterms chosen, $e^{-c\,\mathrm{Vol}} \le Z/Z_{\mathrm{free}} \le e^{c\,\mathrm{Vol}}$ holds independently of the ultraviolet cutoff and the volume (7-03 § 06).

Seam The type of the estimate, and what it lets us say

What follows from estimates of type (10) goes only as far as uniform boundedness of the partition function and the effective action. The 1989 paper says in its

introduction that the proof of the basic inequality that removes the hypothesis (Theorem 2 of Bałaban 1989, by a second-order perturbative calculation) is unpublished, and leaves the analysis of expectation values such as Wilson loops as future work. MRS 1993 comments that from this the ultraviolet limit of smoothed Wilson loops should follow at the level of subsequences, but not necessarily uniquely. The problem statement does not accept weak existence by compactness alone as a solution, unless the properties of the limit are shown separately (JW footnote 2). Three dimensions in 7-06, four dimensions in 7-07.

08

## Detour: the name "phase-space cells"

A section you may skip.

This method is also called "phase cell renormalization" (Chatterjee 2018 § 6). Known We explain what the name means in the words of this chapter. The fluctuations at step $k$ are localised in space to blocks of side $L^k a$, and in momentum to about $1/(L^k a)$. So one counts roughly one degree of freedom for each box in which the product of the widths in position and momentum is $O(1)$: a phase-space cell. The content is the exponential localisation of (5). For three-dimensional scalar fields, the same idea appeared as the localisation into phase-space cells of Glimm–Jaffe (1973) (7-03 § 06).

$$n_k = L^{d(K-k)}\bigl(1 - L^{-d}\bigr), \qquad \sum_{k=0}^{K-1} n_k = L^{dK} - 1 \tag{11}$$

$n_k$ is the number of variables integrated at step $k$ per unit volume. **Example:** in four dimensions with $L = 2$ and $K = 20$ there are $2^{80} \approx 1.2 \times 10^{24}$ points, and the first step integrates 15/16 of them. Computed The same words also appear in 7-04 § 10.

09

## Another road, and where this method stops

Magnen–Rivasseau–Sénéor (MRS 1993) treated four-dimensional $SU(2)$ with a continuum momentum cutoff (with an infrared cutoff, in the trivial topological sector). They use axial gauge for large fields and a background-field-dependent gauge for small fields, and restore with counterterms the gauge invariance broken by the cutoff. The introduction states explicitly that the paper does not give a detailed proof ('we do not provide a detailed proof'). We saw the introduction's comment on Bałaban's results in the Seam of § 07. Known The advantage of starting from the lattice is that gauge invariance and reflection positivity are kept (JW § 6.5, 5-03). In exchange, the

task remains of restoring the broken rotation symmetry in the limit (the axiom step of 7-01).

Where it stops is the right end of (1): at scales longer than $a_0$ the expansion has no foothold. MRS give as their reason for not removing the infrared cutoff 'this would lead to large values of the coupling constant, and presumably to so-called non-perturbative effects corresponding to confinement', and the problem statement says that a gap uniform in the volume needs new ideas (JW § 6.5). **Example:** increasing $K$ in Figure 2 does not move $g_K^2 = 1$ at the right end. A way of adding steps beyond the right end is not among the tools of this chapter. Continued in 7-10 and 7-11.

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## What this chapter can say

| | Content |
|---|---|
| **Said** | The forms of formulas (3)–(7) and (10) (from Dimock's exposition and the statements of the theorems of Bałaban 1985). The hypothesis of the 1989 main theorem (from the introduction) Known |
| **Said** | The values in Tables 1–4 and the figures (two-dimensional models and the arithmetic of formula (9)) Computed |
| **Said** | Inequality (8) and the estimate of edges in the two-dimensional complete axial gauge (proved on the spot) Paper |
| **Not said** | The content of the averages, the choice of gauge and the handling of large fields for gauge fields (the texts of many papers in the series not obtained) |
| **Not said** | The estimate of the large-field factor $e^{-O(1)p(g)^2}$ in § 04 and the argument on the sum of corrections in § 06 (both rough guides, not at the level of theorems) |
| **Not said** | Completion of the proof in four dimensions by Bałaban's method (the main theorem carries a hypothesis; 7-07). That Table 4 is the actual sequence of effective couplings. Infinite volume and the gap |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The transformation, gauges, reflection positivity, footnote 2 | Text checked | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) § 6.5 |
| The main theorem in three dimensions, (7)(10), Wilson's renormalisation group | Text pp. 255–257 checked | T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275 |

| Item | Kind | Source / tools |
|---|---|---|
| The hypothesis of the main theorem in four dimensions, the unpublished basic inequality (Theorem 2) | Abstract and introduction checked | T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 |
| Other papers of Bałaban's series (averages, gauge fixing, four dimensions I, II and others) | Bibliographic data only | CMP 89 (1983); 95, 96 (1984); 98, 99 (two papers), 102 (277–309) (1985); 109 (1987) 249–301; 116, 119 (1988); 122 (1989) 175–202 (cross-checked with the reference lists of Dimock I and Chatterjee) |
| (2)–(6), the form of the induction, the theorem for $\phi^4_3$, averages for charged fields | I § 1–3, II § 1, QED § 2.3 checked; III from the abstract | J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705; Quantum electrodynamics on the 3-torus I, arXiv:math-ph/0210020 |
| The continuum cutoff, the comment on Bałaban, the reservation about the infrared, the explicit statement that there is no detailed proof | pp. 325–327 checked | J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383 |
| The name phase cell renormalization | § 6 checked | S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 |
| The impossibility of global gauge fixing | Bibliographic data only | I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12 |
| Localisation into phase-space cells for three-dimensional scalar fields | Bibliographic data only (attribution from 7-03 § 06) | J. Glimm, A. Jaffe, Fortschr. Phys. 21 (1973) 327–376 |
| Tables 1–4, the sample of the figure, the numbers in § 04 and § 08 | Computed | Python (numpy, scipy) on this machine: `wilson_step.py`, `greens.py`, `villain2d.py`, `running.py`, `numbers.py` (the numbers in § 04 and § 08). The figures are canvas and plain JavaScript |

Next chapter to read: 7-06 Ultraviolet stability of three-dimensional Yang–Mills (the place where this method was first used: what was shown, and what is not yet constructed).

# Ultraviolet stability of three-dimensional Yang–Mills — the lattice estimates became uniform, but there is no measure yet

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — That in three dimensions the coupling constant has the dimension of an inverse length, and the diagrams that can diverge are finitely many. What the theorem of Bałaban (1985) estimated uniformly on the lattice. Why there is still no continuum measure and no observables.

**Prerequisite chapters** — 7-05 The sequence of renormalisation group steps. We also refer to 7-01, 5-01 and 4-04.

**Said first** — This introduces known results and follows the proofs only in outline. The numbers in the tables and figures are computations for free (Gaussian) fields and for the counting of steps, and show nothing about the measure of three-dimensional Yang–Mills. The non-abelian three-dimensional continuum measure has not been constructed, even in finite volume.

The order of this chapter

1. Three dimensions in between — two dimensions is constructed, for four a construction in the form of correlation functions was published in 2026-11, three has reached uniform ultraviolet estimates
2. The coupling constant has a length — the coupling seen on the lattice is $g^2 a$, weaker the finer it is
3. Counting divergences — the types that can diverge are finitely many, and one constant is the only counterterm needed. A table of computations
4. The theorem of Bałaban 1985 — upper and lower bounds on the effective densities independent of the lattice spacing. A moving figure
5. The outline of the proof — small fields and large fields, local axial gauges, the one place semisimplicity is used

6. What the theorem does not say — the distance from uniform estimates to limits, observables and axioms
7. The wall of observables: Wilson loops in three dimensions — line integrals collapse, but with smoothed fields a value remains. A table of computations, a moving figure
8. Another road: stochastic quantisation and the state space — the road approaching the measure from the equation, and the state space made from smoothed fields
9. Detour: lattice numerics in 2+1 dimensions — numerically there are glueball masses and a string tension (at the level of physics)
10. Where three dimensions stands — what has been reached at each stage, and how it differs from four dimensions

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01

## Three dimensions in between

Pure Yang–Mills in two dimensions is solved exactly on the lattice, and the continuum measure has been constructed (7-02). In four dimensions, in the line of work of Bałaban and MRS, the ultraviolet side had come as far as "the level of control" in finite volume, and was unfinished (7-07). The committee's assessment is that the construction published on 2026-11-10 built a theory satisfying the Wightman axioms in the form of correlation functions of local observables (6-01). A continuum probability measure has not been built (6-04). Three dimensions lies in between, and corresponds to 2+1-dimensional space-time.

The survey says that two and three dimensions are thought to be 'substantially simpler' because they are super-renormalisable (only finitely many diagrams can diverge; § 03), and then writes 'a construction of the 3D YM measure and a description of its gauge-invariant observables, even on T3, remains open' (Chevyrev 2022 § 1.1). Known

To say it first in one sentence: what has been shown in three dimensions is **uniform estimates** on the lattice, and what remains is turning the estimates into a **limit** and deciding what to measure in the limit (**observables**). The place is the ultraviolet-limit step of the six steps. Saying "it has been done in two and three dimensions" is accurate for scalar fields (7-03), and for abelian groups too the continuum limit has been shown (Table 3). For the non-abelian case, "three dimensions has reached uniform ultraviolet estimates" is accurate.

02

## The coupling constant has a length

In this chapter the coupling is put in front of the action, and $A$ does not contain $g$. $A$ is a 1-form with values in anti-Hermitian matrices (matrices satisfying $X^\dagger = -X$; the Lie algebra of $SU(N)$ has this form, 1-03), $F_A = dA + A \wedge A$ and $|F_A|^2 = -\sum_{\mu,\nu} \operatorname{tr} F_{\mu\nu}^2 \geq 0$ (3-03). The relation to the Hermitian $A_\mu$ of the physics chapters (3-02, written with the coupling absorbed) is $A = -iA_\mu dx^\mu$, $F_A = -iF$.

The action $S(A) = \frac{1}{2g^2} \int |F_A|^2 \, d^d x$ is a dimensionless number. $A$ has the dimension of length to the power $-1$, the curvature to the power $-2$, and the volume element to the power $d$, so the dimension of the coupling constant, and the dimensionless coupling seen on a lattice of spacing $a$, are fixed.

$$[g^2] = (\text{length})^{d-4}, \qquad g_0^2 = g^2 a^{4-d} = g^2 a \quad (d = 3) \ (1)$$

The formula on the right is the starting point of Bałaban (1985). $g_0^2 = g^2 a$ is a relation fixed by dimensions, and the constant of proportionality depends on the normalisation of the action. In this book's conventions (5-01, appendix A-01), writing the Wilson action of $SU(N)$ as $\beta \sum_p \left(1 - \frac{1}{N} \operatorname{Re} \operatorname{tr} U_p\right)$ gives $\beta = 2N/g_0^2$. The normalisation of tr is not necessarily the same as in the paper's action, so it may differ by a constant factor from the paper's $g_0$. In either convention, the three-dimensional $\beta = 2N/(g^2 a)$ is proportional to $1/a$. Known

Example: in units with $g^2 = 1$, $a = 2^{-20}$ gives $g_0^2 \approx 10^{-6}$. The finer the lattice, the weaker the coupling becomes by itself, without waiting for the logarithmic slowness of asymptotic freedom (4-06). At the length $1/g^2$ the coupling is of order 1, and since this is the only length scale, if the theory exists and has a mass, it is a constant multiple of $g^2$ (for four dimensions see 4-07). Physics

For how this book uses this, see 15-01 § 01 (in four dimensions the sum of losses diverges logarithmically).

03

## Counting divergences

We count the diagrams that can diverge in perturbation theory (4-04; for scalar fields 7-03). In perturbation theory one substitutes $A \to gA$. The action takes the form $\frac{1}{2} \int |dA|^2$ with a three-point vertex carrying $g$ and a four-point vertex carrying $g^2$. A one-particle-irreducible diagram (one that does not fall into two pieces when any single line is cut) with $E$ external gluon lines and $L$ loops carries $g^{E-2+2L}$, so subtracting the dimension of the power of $g$ from the dimension of momentum,

$$\omega(E, L) = d - E + (4 - d)(1 - L) \quad (2)$$

is the superficial degree of divergence. In four dimensions $\omega = 4 - E$, independent of $L$, and divergences appear at every order. In three dimensions $\omega = 4 - E - L$, which drops by 1 with each additional loop.

Computed Table 1 $\omega(E, L)$ in three dimensions. Bold is $\omega \geq 0$. $E = 1$ vanishes by colour symmetry. External lines are gluons only (types with external ghost lines are omitted).

| E\L | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 0 (vacuum) | **3** | **2** | **1** | **0** | −1 |
| 2 | **1** | **0** | −1 | −2 | −3 |
| 3 | **0** | −1 | −2 | −3 | −4 |
| 4 | −1 | −2 | −3 | −4 | −5 |

With external lines restricted to gluons, there are seven types with $\omega \geq 0$ in three dimensions (three in two dimensions, infinitely many in four).

### One constant is the only counterterm

Of the seven, the divergences of the three other than the vacuum (one and two loops with $E = 2$ and one loop with $E = 3$) need no counterterm under a regularisation that keeps gauge invariance. The reason lies in the form of the counterterms. A counterterm is a gauge-invariant local term (the integral of a polynomial in the field and its derivatives at one point) whose coefficient can be written as a power of $g^2$. In three dimensions $g^2$ has the dimension of length to the power $-1$, so with the conventions of § 02 the terms are restricted to dimension 3 or less, and the only such term is a constant. The candidate term linear in $F$, of dimension 2, would have to be a linear form on the Lie algebra invariant under the adjoint action ($X \mapsto hXh^{-1}$), and for a semisimple group there is only 0 (in matrices, $\operatorname{tr} F = 0$). The candidate of dimension 3, the Chern–Simons term, changes sign under reversal of orientation (parity), so it does not arise from an action invariant under reversal. Known

This argument belongs to continuum perturbation theory. On the lattice rotation symmetry is broken (only the rotations and reflections of the cubic lattice remain), and it cannot be applied as it is. What corresponds to it on the lattice is Bałaban's "one place semisimplicity is used", seen in § 05. Bałaban's starting density also contains, besides the Wilson action, only constants including 'vacuum energy renormalization counterterms'. A vacuum diagram with $L$ loops is of order $g_0^{2(L-1)}$, and the last one, $L = 4$, corresponds to $g_0^6$, which matches the paper's expansion up to order 6 in $g_0$ (a cross-check made in this chapter).

04

## The theorem of Bałaban 1985

### The statement of the theorem

A torus $\mathbb{T}$ of fixed side is approximated by lattices of spacing $a$ (Bałaban's paper writes the lattice spacing as $\varepsilon$). The starting density is

$$\rho_0(U) = \exp\big[ - A(U)/g_0^2 - E_0\big]$$

(formula (1) of the paper), where $A$ is the Wilson action and $E_0$ is a constant including the normalisation and the vacuum-energy counterterms ($E$ in the paper; we write $E_0$ to distinguish it from the number of external lines in § 03). This corresponds to the single constant being enough as counterterm in § 03. The renormalisation transformation $T$ (blocking that coarsens by a factor $L$; 7-05) produces $\rho_{k+1} = T\rho_k$, and one stops at $L^K a = a_0$ ($a_0$ a positive constant fixed by $g$ alone). Ultraviolet stability means that the upper and lower bounds on the last density do not depend on $a$, and formula (3) of the paper is

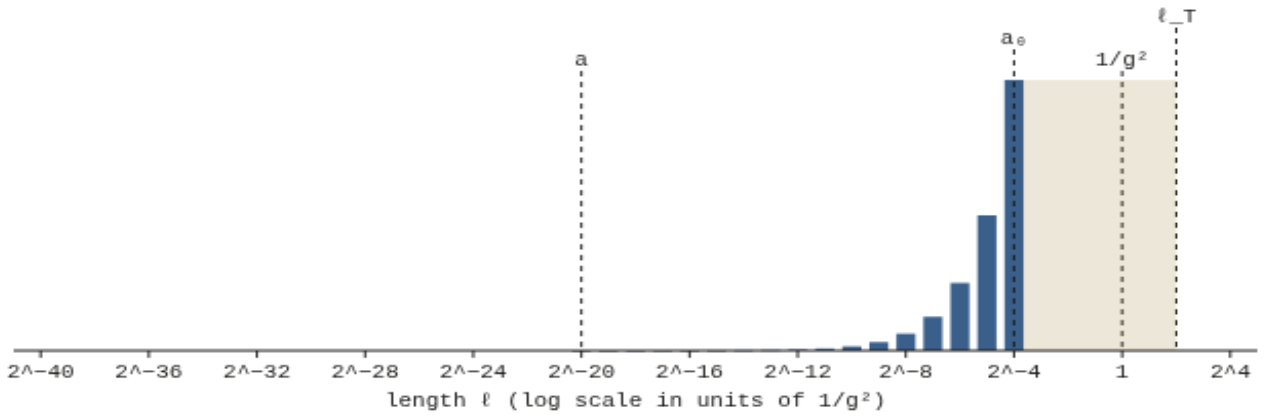
$$\chi(U)\, e^{-O(1)|\mathbb{T}|} \;\le\; \rho_K(U) \;\le\; e^{O(1)|\mathbb{T}|} \quad \text{(3)}$$

Here $U(\partial p)$ is the product of the link variables once round the boundary of the plaquette $p$, $\chi$ is the function (indicator function) that is 1 when $|U(\partial p) - 1|$ is small on every plaquette of the last coarse lattice (spacing $a_0$) and 0 otherwise, $|\mathbb{T}|$ is the volume, and $O(1)$ is a constant fixed by $g$ and $a_0$ alone. Known

The paper notes that this $O(1)$ goes to $\infty$ as $g \to 0$, rewrites it in a form with the terms of the action divergent in $g$ written out, and extends it to all steps $k$. That inequality (5) is the main theorem (its form is formula (10) of 7-05): 'The lattice approximations of the three-dimensional pure Yang-Mills theory with a semi-simple compact group Lie G are ultraviolet stable in the sense that the sequence of densities ρk ... satisfies the bounds (5).' The corollary is 'uniform in ε bounds for the partition function'. The author cautions that 'the proof is very sketchy'. Known

### The sum over steps

Figure 1 Refining the lattice spacing $a$ adds steps (blue bars = the coupling $g_k^2$ seen at that step) at the left end. The yellow band is the scales left outside the steps (from $a_0$ to the side of the torus $\ell_T = 4/g^2$)




a = 2^−20/g², number of steps K = 16  $g_0^2$ = 9.5e-7  Σ $g_k^2$ = 0.124999 (below 0.125), points of the fine lattice ≈ 10^19.9

Computed The numbers in this figure were computed on this machine (in units of $g^2 = 1$; $L = 2$ and $g^2 a_0 = 1/16$ are values chosen for illustration).

The coupling seen on the lattice at step $k$ is $g_k^2 = g^2 L^k a$ (the paper's $g_k = g(L^k a)^{1/2}$), shrinking by a factor $1/L$ at each step towards the fine side. A new step is only added at the left end with a smaller coupling than before, and the sum does not exceed $\frac{L}{L-1} g^2 a_0$ (for $g^2 a = 2^{-8}, 2^{-20}, 2^{-40}$ the steps are 4, 16, 36 and the sums 0.121094, 0.124999, 0.125000). Computed

In four dimensions the same sum grows like the logarithm of the number of steps (§ 06 of 7-05; about 7.2 more each time the number of steps doubles). Above $a_0$ there is a finite-dimensional integral on a lattice whose number of points does not depend on $a$ (262144 points in the figure), and since it is over a compact group it is bounded. But if the torus is larger than $1/g^2$, scales where the coupling exceeds 1 enter there. Computed

05

## The outline of the proof

The proof stands on a group of earlier papers (propagators, averaging, spaces of regular configurations, variational problems) and follows the author's own method for the three-dimensional Higgs model. Known

**Small fields and large fields.** For each plaquette, 1 is partitioned according to whether $|U(\partial p) - 1|$ is small or large (1 is written as the sum of two functions, "small" and "large", and each case is estimated). A large plaquette makes $A(U)/g_0^2$ large, so in the end a small factor $\exp(-\frac{1}{4} p^2(g_j))$ appears for each large-field plaquette. $g_j = g(L^j a)^{1/2}$ is the coupling at step $j$, and $p(g)$ is the multiplier for the large-field threshold, which grows like a power of $\log g^{-1}$ (formula (7) of 7-05). This factor becomes small faster than any power of $g_j$.

**Inside the small fields.** In each block one takes an axial gauge (a gauge setting the link variables in a chosen direction to 1 within the block) by the Faddeev–Popov procedure (a method of fixing the gauge to one and multiplying the integral by a compensating factor; 4-08) (locally, since there is no global gauge fixing; 3-04), and splits into the configuration minimising the action (the background field) and fluctuations. The fluctuations become a Gaussian measure with exponentially decaying covariance plus local interactions, and since $g_0 = g a^{1/2}$ is small, an expansion to order 6 in $g_0$ and vacuum diagrams with up to 6 vertices are enough. The paper writes that this integral could properly be expressed by the cluster expansion (7-04), but it uses the simplifications given by super-renormalisability.

**Two features.** Lattice fields with values in a group cannot be rescaled, so instead of scale transformations one uses 'improvement of regularity properties of typical gauge fields'. Semisimplicity is used in just one place: from the fact that the only element of the Lie algebra invariant under the adjoint action is 0, one derives 'there are no relevant variables in the effective action' (there are no quantities in the effective action that grow as the steps advance and so need to be tracked). This is the lattice-side way of saying the "the only counterterm is a constant" of § 03 (this chapter's reading). The problem statement introduces this work as choosing the gauge again for each phase-space cell and each scale, from ultraviolet-oriented to infrared-oriented (JW § 6.5). Known

06

## What the theorem does not say

Formula (3) says only that the effective densities and the partition function are uniformly bounded, and says nothing about the limit.

$$\sup_a |\log Z_a| < \infty \quad /\!\!\Longrightarrow \quad \lim_{a\to 0}\langle \mathcal{O}\rangle_a \text{ exists} \quad /\!\!\Longrightarrow \quad \text{axioms and gap} \ (4)$$

**From estimates to a limit.** From uniform estimates, compactness (the property that a convergent subsequence can be taken from a bounded sequence) gives the prospect of subsequential limits, but not uniqueness (there is also a usage that calls the existence of subsequences "ultraviolet stability"; Chatterjee 2018 § 6). Footnote 2 of the problem statement says that for the four-dimensional problem 'weak-existence (compactness)' alone is not accepted as a solution (unless the properties of the limit are shown separately). Known

**What to measure.** The problem statement writes 'These results need to be extended to the study of expectations of gauge-invariant functions of the fields' (JW § 6.5). In three dimensions this is the problem of defining observables (next section).

**Beyond.** The constant $O(1)|\mathbb{T}|$ is proportional to the volume and says nothing about infinite volume (the infinite-volume step). Reflection positivity holds on the lattice (Osterwalder–Seiler 1978, 5-03), but recovering rotation symmetry and the gap (the axiom step and the uniform-gap step) are untouched. The paper of Chevyrev–Shen, which treats the dynamics of stochastic quantisation in two dimensions, also writes in § 1.2 of its 2026 version 'the construction of the 3D YM measure in the continuum is open, even in finite volume'. Known

07

## The wall of observables: Wilson loops in three dimensions

### Line integrals are not defined

In two dimensions the Wilson loop $W(\gamma) = \operatorname{tr} \mathcal{P} \exp \oint_\gamma A$ is the natural observable, and the area law holds exactly (7-02). Three-dimensional Yang–Mills is thought to behave at short distances like the Gaussian free field (GFF; 1-09), whose covariance is of order $1/|x-y|$, so the variance of the line integral over a segment is $\int_0^1 \int_0^1 ds\, dt/|t-s| = \infty$ (Chevyrev 2022 § 3.1; the abstract of Cao–Chatterjee also says 'line integrals of the GFF do not exist in such dimensions'). Known

We check this in the simplest case. Making the U(1) free field (a Gaussian measure) on a lattice, with a square loop of side 1 and coupling $e^2 = 1$, independently of the gauge $\langle W \rangle = \exp(-\frac{1}{2} \operatorname{Var} \oint A)$, and

$$\operatorname{Var} \oint_\gamma A = \oint_\gamma \oint_\gamma G(x-y)\, dx \cdot dy, \qquad G_t(r) = \frac{\operatorname{erf}\left(r/\sqrt{8t}\right)}{4\pi r} \quad (d=3) \tag{5}$$

$G$ is the inverse of the lattice Laplacian, and $G_t$ is the continuum version of the covariance of the field smoothed by the heat flow $A \mapsto e^{t\Delta} A$ (the linearisation of the Yang–Mills flow). $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, ds$ is the error function, which approaches 1 when $x$ is large. The smoothing width is about $\sqrt{8t}$, the length in the argument of erf; for two points farther apart than that, $G_t$ is almost the same as $1/(4\pi r)$, and for two points closer together it is held to a finite value.

Computed Table 2 $\langle W \rangle$ for the square of side 1. t is the time of the heat flow. The bottom row is formula (5) integrated in the continuum.

| a | Two dimensions | Three dimensions | Three dimensions t = 0.01 | Four dimensions |
|---|---|---|---|---|
| 1/4 | 0.6065 | 0.4837 | 0.6919 | 1.96e−1 |
| 1/16 | 0.6065 | 0.3150 | 0.7396 | 5.22e−4 |
| 1/128 | 0.6065 | 0.1631 | 0.7421 | 1.67e−28 |

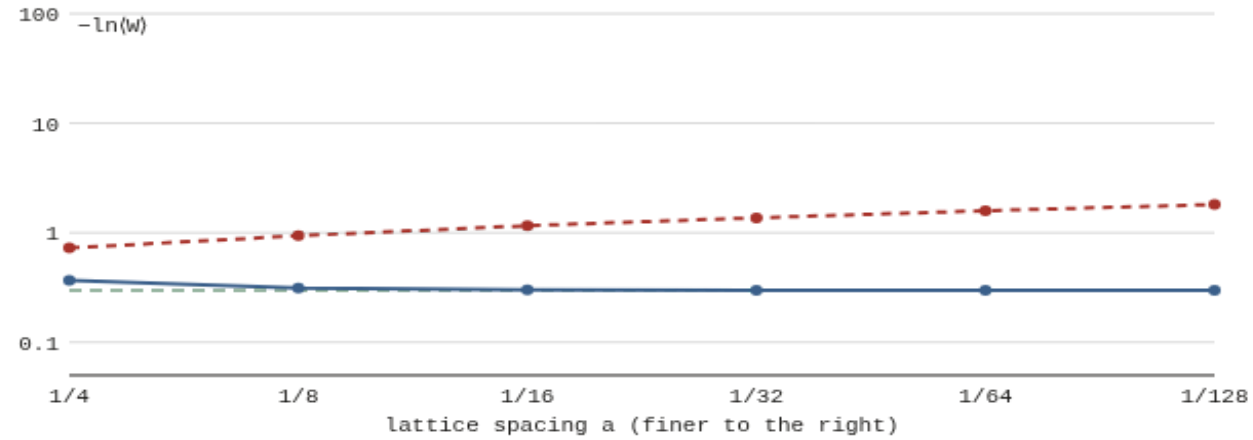
| a | Two dimensions | Three dimensions | Three dimensions t = 0.01 | Four dimensions |
|---|---|---|---|---|
| Continuum | $e^{-1/2}$ | 0 | 0.7421 | 0 |

### Reading Table 2

In two dimensions, on every lattice $\mathrm{Var} = 1$ (the area) and $\langle W \rangle = e^{-1/2}$ comes out exactly. In three dimensions $\mathrm{Var}$ grows in proportion to $\ln(1/a)$, and $\langle W \rangle \to 0$. The slope of $\mathrm{Var}$ against $\ln(1/a)$ goes from $0.613$ to $0.635$ (approaching $2/\pi \approx 0.637$). The increment each time $a$ is halved goes from $0.425$ to $0.440$, approaching $(2/\pi)\ln 2 \approx 0.441$. In four dimensions $\mathrm{Var} \approx 1/a$. Computed

With smoothing, the lattice values converge to the continuum values in both three and four dimensions. This is the smallest example of the starting point of Cao–Chatterjee and CCHS, which define observables as "Wilson loops of the smoothed field". Computed

Figure 2 Moving the time of the heat flow $t$: the smoothed Wilson loop (blue) settles to the continuum value (green dashed) as the lattice is refined. The red dashed line is unsmoothed




d = 3  t = 0.01, a = 1/128: unsmoothed −ln⟨W⟩ = 1.81, smoothed value 0.298 (continuum 0.298)

Computed The numbers in this figure were computed on this machine (the same computation as Table 2).

08

## Another road: stochastic quantisation and the state space

### The road of solving an equation

Instead of making the measure directly, there is the road of solving a stochastic partial differential equation that should have it as its invariant measure (a measure whose distribution does not change as time passes) (7-09). Chandra–Chevyrev–Hairer–Shen (CCHS 2024), for three-dimensional Yang–Mills–Higgs, extended gauge equivalence to a non-linear metric space of distributions, constructed local solutions (solutions existing only for a short time) of the equation renormalised with regularity structures (the theory of solutions of singular stochastic partial differential equations, treated in

7-09), and showed 'there is a unique choice of renormalisation counterterms such that these solutions are gauge covariant in law'. Known

The Higgs field may be absent, and a statement restricted to pure Yang–Mills is in Chevyrev 2022 Remark 1.2. As a result, a Markov process on the orbit space (a stochastic process whose future behaviour is determined only by the present state) is defined, with the possibility of finite-time blow-up (the solution becoming infinite in finite time).

### The state space

How the state space is made is the answer to the previous section. Line integrals cannot be used, so the deterministic Yang–Mills flow (with the DeTurck term; a term in the gauge direction added to put it in the form of a heat equation) is taken as "smoothing compatible with gauge", and gauge equivalence is defined for the fields after smoothing (Chevyrev 2022 § 3.1). The space of orbits is a Polish space (a space with a complete metric and a countable dense subset) in two dimensions, but in three dimensions it is only known to be 'completely Hausdorff (and separable)' (any two distinct points can be separated by a continuous function, and there is a countable dense subset) (ibid. Remark 1.5). In the non-abelian case the space of orbits is non-linear, and the difficulty begins with the state space (ibid. § 1.1, 3-04).

What remains is the invariant measure, and the survey writes that showing it exists uniquely would give a construction of the three-dimensional Yang–Mills measure in finite volume (ibid. § 4). Cao–Chatterjee (2024) showed that, assuming the theory behaves like the GFF at short distances, the theory can be realised on the unit torus as 'random distributional gauge orbits', and smoothed Wilson loops can be defined. This too is a result about the state space. Known

09

## Detour: lattice numerics in 2+1 dimensions

A section you may skip.

On the physics side, three-dimensional Yang–Mills is thought to have confinement and a mass gap. Teper (arXiv 1998, Phys. Rev. D 1999) computed lattice theories with SU(2) to SU(5) in 2+1 dimensions by the Monte Carlo method (5-07), and extrapolated the ratios of glueball masses to the string tension to the continuum limit. According to the abstract, the ratios hardly depend on N, and the string tension does not go to 0 even for large N. Since $\beta = 2N/(g^2 a)$, the correlation length in lattice units $\xi/a = \beta/\left(2N\,(m/g^2)\right)$ only grows in proportion to $\beta$ (exponentially in four dimensions; 5-05). Physics

The problem statement lists Feynman's paper in 2+1 dimensions (1981) among the literature on the view that the gap comes from the quartic term $(A \wedge A)^2$ (JW § 6.6). Three-dimensional U(1)

lattice gauge theory has been shown to confine static charges at every coupling with the lattice spacing kept fixed (Göpfert–Mack 1982; for four-dimensional U(1) see 5-06). Known Its continuum limit is the free electromagnetic field (Gross 1983), so this confinement is thought not to survive in the continuum limit. Physics

10

## Where three dimensions stands

Known Table 3 What has been reached in three dimensions (stages as in 7-01; categories as in Table 2 of 7-01).

| Stage | Non-abelian Yang–Mills | Abelian U(1) (for reference) |
|---|---|---|
| 1 Regularisation | Closed | Closed |
| 2 Finite-volume measure | Closed (lattice). Making it directly in the continuum is counted under "Partial" in the ultraviolet-limit step | Closed (lattice) |
| 3 Ultraviolet | Partial: uniform lattice estimates (Bałaban 1985). The continuum measure is not constructed. Stochastic quantisation as far as the dynamics (CCHS 2024) | Continuum limit (Gross 1983, in a different kind of convergence; the limit is a free field). With Higgs, King 1986 |
| 4 Infinite volume | Not yet | Limit with Higgs (King 1986) |
| 5 Axioms | Not yet | Not checked here |
| 6 Gap | Not yet (present numerically. Physics ) | Confinement at every coupling on the lattice (a fixed) (Göpfert–Mack 1982). This property is thought not to survive in the continuum limit (the ultraviolet-limit step) Physics |

The U(1) column is for reference. What is written in the row for the uniform-gap step is confinement (the force between static charges not weakening with distance), and confinement and the mass gap are different properties (17-02 § 02).

The difference from four dimensions (7-07) is that what weakens the coupling in the ultraviolet is the dimension itself. What remains in common is turning estimates into limits, deciding the quantities to measure in the limit, and estimates uniform in the volume. In infinite volume, three dimensions too has the coupling of order 1 at the length $1/g^2$, and enters the same kind of difficulty as the infrared in four dimensions (this chapter's reading; 7-10).

## What this chapter can say

| | Content |
|---|---|
| **Said** | The coupling seen on the lattice is $g^2 a$. With external lines restricted to gluons, there are seven types that can diverge Computed . From the dimensions of gauge-invariant local terms, the only counterterm needed is one vacuum-energy constant Known |
| **Said** | The theorem of Bałaban (1985): upper and lower bounds on the effective densities uniform in the lattice spacing, and uniform estimates of the partition function (text checked) Known |
| **Said** | The Wilson loop of the U(1) free field collapses at the rate $\ln(1/a)$ in three dimensions and $1/a$ in four, and when smoothed by the heat flow converges to the continuum value (in the range $a \geq 1/128$) Computed |
| **Said** | The non-abelian three-dimensional continuum measure is not constructed, even in finite volume (Chevyrev 2022, Chevyrev–Shen) Known |
| **Not said** | The details of Bałaban's proof (the author himself calls it a sketch). Whether smoothed Wilson loops have a limit in an interacting theory (Table 2 is a free field) |
| **Not said** | The mass gap and confinement in three dimensions (the numerics are at the level of physics). Complete coverage of the literature from 2024 on |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| § 02– § 05 | Primary source (text checked) | T. Bałaban, Ultraviolet stability of three-dimensional lattice pure gauge field theories, Comm. Math. Phys. 102 (1985) 255–275 |
| Quotations in § 05, § 06, § 09 | Primary source (text checked) | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, the official problem statement of the Clay Mathematics Institute (2000) |
| Quotations in § 01, § 07, § 08 | Survey (text checked) | I. Chevyrev, Stochastic quantisation of Yang–Mills, arXiv:2202.13359 (2022) § 1.1, Remark 1.2, Remark 1.5, § 3.1, § 4 |
| § 06, the abelian group in Table 3 | Survey (text checked) | S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 (2018) § 4, § 6 |
| The status of the continuum measure in § 06 | Primary source ( § 1.2 text checked) | I. Chevyrev, H. Shen, Invariant measure and universality of the 2D Yang–Mills Langevin dynamic, Comm. Pure Appl. Math. (2026), arXiv:2302.12160 (v3, 2026) |
| § 07, § 08, § 09 | Checked in the abstracts | A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Stochastic quantisation of Yang–Mills–Higgs in 3D, Invent. Math. 237 (2024) 541–696, arXiv:2201.03487. S. Cao, S. Chatterjee, A state space for 3D Euclidean Yang–Mills theories, Comm. Math. Phys. 405 (2024), arXiv:2111.12813. M. Teper, Phys. Rev. D 59 (1999) 014512, arXiv:hep-lat/9804008 |

| Item | Kind | Source / tools |
|---|---|---|
| Abelian groups, Feynman, reflection positivity | Bibliographic data only | L. Gross, CMP 92 (1983) 137–162. C. King, CMP 102 (1986) 649–677; 103 (1986) 323–349. M. Göpfert, G. Mack, CMP 82 (1982) 545–606 (this one checked as far as the abstract: Villain action, positive string tension at every coupling. INSPIRE. Text not read. The year is the Crossref publication year; INSPIRE gives 1981). R. P. Feynman, Nucl. Phys. B 188 (1981) 479–512. K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 |
| Tables, figures | Computed | Python on this machine: `powercount.py` (Table 1), `ladder.py` (the sum over steps in § 04, Figure 1), `wilson_free.py` (Table 2, Figure 2. Integrates the lattice Green's function with scipy from the Bessel representation of the heat kernel, reproducing Var = 1 in two dimensions to 1e−10. About 10 seconds). The figures are canvas and plain JavaScript |

Next chapter to read: 7-07 Four dimensions (the four-dimensional version of the same method).

# Four dimensions: Bałaban and Magnen–Rivasseau–Sénéor — the ultraviolet side has reached "the level of control" in finite volume, but in these two lines of work the construction is not complete

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — That in four dimensions the coupling weakens only logarithmically, so the coupling itself must be renormalised. What Bałaban's 1989 main theorem assumes and what it concludes. What MRS 1993 did with a continuum cutoff. How far "the level of control" reaches, and what lies beyond it.

**Prerequisite chapters** — 7-05 The sequence of renormalisation group steps. We also refer to 7-06, the ultraviolet-limit step of 7-01 and 4-06.

**Said first** — This introduces known results. Bałaban's four-dimensional series is treated from bibliographic data only, apart from the abstract and introduction of the 1989 paper. MRS is read as far as the introduction and the section on the set-up. The numbers in the tables and figures are arithmetic of the running of the coupling. In the two lines of work in this chapter, the construction of four-dimensional Yang–Mills is not complete, even on a torus of finite volume. On 2026-11-10, a Lean formalisation of a construction with a different design was published, together with the committee's assessment report. It is read in Part 6 and Parts 8–14; this chapter records what had been reached before it (only the right-hand column of Table 4 is the published construction).

The order of this chapter

1. What changes in four dimensions — the coupling weakens only logarithmically
2. Bałaban's four-dimensional series — from the foundations to four dimensions, 13 papers over six years
3. Reading the 1989 main theorem — a theorem with a hypothesis
4. The hypothesis in numbers — a moving figure, a table

5. Three readings of "ultraviolet stability" — estimates, subsequences, uniqueness
6. MRS 1993: with a continuum cutoff — axial gauge and the Slavnov identities
7. Setting the two roads side by side — the lattice and the continuum cutoff, the first upper bound on the field
8. What "the level of control" means — a moving figure, a table
9. Detour: how to check a proof of several hundred pages

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01

## What changes in four dimensions

In three dimensions, refining the lattice weakened the coupling by itself (7-06 § 02). In four dimensions the coupling $g$ has no dimension of length, and only asymptotic freedom weakens the coupling at fine scales (4-06). The coupling $g(\ell)$ at length scale $\ell$ obeys, for pure gauge $SU(N)$,

$$\ell \frac{dg}{d\ell} = b_0 g^3 + b_1 g^5 + \cdots, \qquad b_0 = \frac{11N}{48\pi^2}, \qquad b_1 = \frac{17N^2}{384\pi^4} \quad (1)$$

(the PDG formula (9.3) converted to the conventions of this book). Known Keeping only one loop,

$$\frac{1}{g^2(\ell)} = \frac{1}{g^2(\ell')} + 2b_0 \ln \frac{\ell'}{\ell} \quad (2)$$

and each halving of the scale increases $1/g^2$ by $2b_0 \ln 2$. **Example:** halving the lattice spacing from $g^2 = 0.2$, for $SU(3)$ $1/g^2$ goes from 5 to 5.0966 and $g^2$ decreases by 1.9%; for $SU(2)$ it decreases by 1.3%. In three dimensions $g_0^2 = g^2 a$ would halve. Computed

Because of this slowness, the sum of corrections of order $g_k^2$ at each step keeps growing like the logarithm of the number of steps, and cannot be piled up as an error (7-05 § 06). The corrections must be renormalised into the coupling itself, and the title of the first paper of the four-dimensional series contains 'a coupling constant renormalization in four dimensions'. Known

02

## Bałaban's four-dimensional series

Bałaban's work is a series that piles four-dimensional papers (1987–89) on foundations shared with three dimensions (1983–85), and the survey calls it 'a long series of papers

spanning six years' (Chatterjee 2018 § 6). Known

Known Table 1 The order of the series (bibliographic data cross-checked with Chatterjee 2018, Dimock 2011 and Crossref).

| Year | Journal CMP | Content (from the titles) | Checked |
|---|---|---|---|
| 1983–85 | 89, 95, 96, 98, 99 (two papers), 102 | Foundations: propagators, averaging, gauge fixing, background fields and so on | Bibliographic data only |
| 1985 | 102, 255–275 | Ultraviolet stability in three dimensions (7-06 § 04) | Text pp. 255–257 |
| 1987 | 109, 249–301 | The effective action in the small-field approximation, and coupling constant renormalisation in four dimensions | Bibliographic data only |
| 1988 | 116, 1–22; 119, 243–285 | Same, II: cluster expansions; convergent renormalisation expansions | Bibliographic data only |
| 1989 | 122, 175–202 | Renormalisation of large fields I: the basic step of the R operation | Bibliographic data only |
| 1989 | 122, 355–392 | Renormalisation of large fields II: localisation, exponentiation and estimates | Abstract and introduction |

The problem statement calls this extension 'a qualitative breakthrough', and summarises it as 'an analysis of asymptotic freedom to control the renormalization group flow as well as obtaining quantitative estimates on effects arising from large values of the gauge field' (JW § 6.5). Going by the titles, the 1987 paper handles small fields and the running of the coupling, and the two 1989 papers handle large fields (the $\Omega^c$ of 7-05 § 04). Known

For how this book uses this, see 15-02 § 01 (a map of the five stages of the series).

## 03 Reading the 1989 main theorem

This is the theorem in the introduction of the last paper (the Conditional Ultraviolet Stability Theorem; Theorem 1 of Bałaban 1989).

'Theorem 1. If the sequence of the effective coupling constants is contained in an interval ]0, γ] with a sufficiently small positive γ, then the effective densities $\rho_k$ have the form, and satisfy all the conditions and bounds, described in Sect. 2 [III].'

The sequence of effective couplings is the couplings $g_k$ for each step $k = 0, \dots, K$ (from the fine steps to the unit lattice). "Form, conditions and estimates" refers to the form of the induction in 7-05 § 07. Extracting only the logical form,

$$\underbrace{0 < g_k \le \gamma \ \ (k = 0, \dots, K)}_{\text{hypothesis}} \implies \underbrace{\rho_k \text{ satisfies the inductive representation and estimates}}_{\text{Theorem 1}} \tag{3}$$

The author continues that the hypothesis can be removed because it follows from basic inequalities that come from Theorem 2 of an earlier paper, but 'The proof of

Theorem 2, which is based on second order perturbative calculations, is very awkward and long ... and has not been published yet, so we have Theorem 1 with the assumption' (Theorem 2 is the basic inequality for removing the hypothesis). As of 1989, ultraviolet stability in four dimensions is a **theorem with a hypothesis**. Within what we searched (the bibliographic data of the author's papers from 1989 on), no paper containing the proof of the basic inequality (Theorem 2) was found. Known

From the theorem follow 'the ultraviolet stability bounds of the same type as for superrenormalizable models in [16]', with constants $E_{\pm}$ 'independent of k, $T_{\eta}$, $U_{k}$'. The analysis of expectation values of loop variables and the like is left as future work. Known

This means estimates of the same type as for super-renormalisable models (reference [16] of the original paper). Three-dimensional Yang–Mills is super-renormalisable (formula (7) of 7-01), so the form can be read as being of the same type as formula (10) of 7-05. $T_{\eta}$ is the paper's lattice and $|T_{\eta}|$ its number of points, corresponding to the $|T_{k}|$ of formula (10) of 7-05. $U_{k}$ is the field configuration at step $k$. The point is that the constants depend neither on the step, nor on the size of the lattice, nor on the field (this paragraph is this chapter's reading).

**Example:** the effective couplings $g_{k}$ of the paper are defined step by step within the paper, and are not the running (2) itself. Reading them with the one-loop running (2), $g_{k}$ is larger on the coarser side, and the maximum of the sequence is $g_{K}$ at the unit lattice, so hypothesis (3) shrinks to the single condition $g_{K} \leq \gamma$ (this chapter's reading).

04

## The hypothesis in numbers

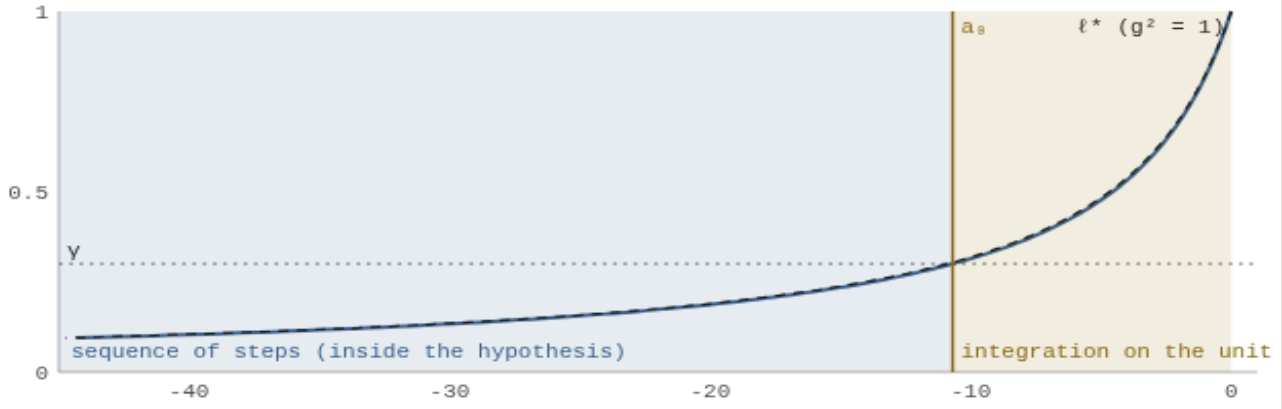
If the scale of the unit lattice where the sequence of steps stops is $a_{0}$ (formula (1) of 7-05), the hypothesis is that the coupling is small at the scale of the unit lattice, written here as $g^{2}(a_{0}) \leq \gamma$. Extending formula (1) to the coarse side, write $\ell_{*}$ for the length at which $g^{2} = 1$. $\ell_{*}$ is a **rough-guide length**, extended to where perturbation theory cannot be trusted, and an indication of the scale at which physics thinks confinement and masses appear. Physics Integrating formula (1) with $u = g^{2}$,

$$\ln \frac{\ell_{*}}{a_{0}} = \int_{\gamma}^{1} \frac{du}{2u^{2}\,(b_{0} + b_{1}u)} \;\approx\; \frac{1}{2b_{0}}\left(\frac{1}{\gamma} - 1\right) \quad (4)$$

The right-hand one is the one-loop approximation. The statement of the theorem only says that $\gamma$ is 'sufficiently small', so the $\gamma$ below are values for illustration. Here $\gamma$ is

used as an upper bound on $g^2$ (it differs from the $g_k \leq \gamma$ of formula (3) only by a relabelling of $\gamma$).

Figure 1 Moving γ changes the distance between the scale $a_0$ where the sequence of steps stops (vertical line) and the rough guide $\ell^*$ for $g^2 = 1$ (right end)




SU(2), γ = 0.30, $\log_{10}(a_0/\ell^*)$ = −10.70 (two loops), −10.91 (one loop), points of the unit lattice on a torus of side $\ell^* \approx$ 10^42.8

Computed The numbers in this figure were computed on this machine (formula (4); solid line two loops, dashed line one loop; agrees with Table 2).

Computed Table 2 $\log_{10}(\ell_*/a_0)$ (one loop / two loops), and the number of points of the unit lattice on a four-dimensional torus of side $\ell_*$ (two loops).

| $\gamma$ | SU(2) | SU(3) | Points SU(2) | Points SU(3) |
|---|---|---|---|---|
| 0.5 | 4.68 / 4.55 | 3.12 / 3.00 | 10^18.2 | 10^12.0 |
| 0.3 | 10.91 / 10.70 | 7.27 / 7.06 | 10^42.8 | 10^28.2 |
| 0.2 | 18.70 / 18.42 | 12.47 / 12.18 | 10^73.7 | 10^48.7 |
| 0.1 | 42.08 / 41.67 | 28.06 / 27.64 | 10^166.7 | 10^110.6 |

In the range of Table 2 ($\gamma = 0.1$ to 0.5), $a_0$ lies 3 to 42 orders of magnitude finer than $\ell_*$, and the four-dimensional torus of side $\ell_*$ has $10^{12}$ to $10^{167}$ points of the unit lattice.

Computed

How to read it (this chapter's reading). The sequence of steps controls only scales finer than $a_0$. Beyond that is a finite-dimensional integral on the unit lattice, which is bounded since the group is compact, but the constant has the form $e^{E|T|}$. $|T|$ is the number of points of the unit lattice, corresponding to the last step of the $|T_k|$ of formula (10) of 7-05 (the $|T_\eta|$ of § 03). For a torus of side $\ell_*$, $|T|$ is of the size of the number of points above. Ultraviolet stability is not information about the scale $\ell_*$.

> 05

### Three readings of "ultraviolet stability"

The same words are used with different strengths. Known **As estimates:** 'The ultraviolet stability means that the actions $\rho_K$ have bounds independent of the lattice spacing $\varepsilon$' (Bałaban 1985). **As subsequences:** 'The existence of a convergent subsequence is known as ultraviolet stability' (Chatterjee 2018 § 6). **As "a form":** for MRS and Bałaban, 'a form of ultraviolet stability' was shown (Chevyrev 2022 § 1.1).

The tool that goes from estimates to subsequences is compactness, and its simplest form is the Bolzano–Weierstrass theorem.

$$\sup_K |x_K| < \infty \implies \text{for some } K_1 < K_2 < \cdots,\ x_{K_j} \text{ converges} \quad (5)$$

The limit need not be unique. **Example:** $x_K = \sin K$ has $|x_K| \le 1$ but does not converge: at $K = 33, 77$ it is 0.99991, 0.99952, and at $K = 11, 55$ −0.99999, −0.99976, so one can take a subsequence approaching 1 and one approaching −1. Computed

The introduction of MRS comments on Bałaban's result as follows. 'From this result the existence of an ultraviolet limit for gauge invariant observables such as "smoothed Wilson loops" should follow, at least through a compactness argument using a subsequence of approximations; but the limit is not necessarily unique.' Footnote 2 of the problem statement does not accept weak existence by compactness as a solution, unless the properties of the limit (gap and axioms) are shown separately (JW § 6.5; see also 7-06 § 06). Known

06

## MRS 1993: with a continuum cutoff

Magnen–Rivasseau–Sénéor did not use a lattice, and started from a continuum momentum cutoff. The object is the regularised axial-gauge Schwinger functions of pure four-dimensional $SU(2)$, in the trivial topological sector, with a fixed infrared cutoff, and with the ultraviolet cutoff removed. Known We define the words in turn.

- **Axial gauge**: a choice of gauge that sets the component in one coordinate direction to 0 ($A_4 = 0$ and so on; gauge fixing is in 4-08).
- **Trivial topological sector**: treating only configurations with instanton number (3-05) 0.
- **Schwinger functions**: the expectation values of products of fields with one gauge fixed, $\langle A(x_1) \cdots A(x_n) \rangle$.

The centre is the estimate of the size of the field. The counterterms restoring the gauge invariance broken by the cutoff hold the field to about $\lambda^{-1}$ ($\lambda$ is the coupling). In axial gauge, the measure after gauge fixing remains a positive weight. This is the **positivity** of axial gauge, and MRS give 'lack of positivity' as the reason Feynman gauge and the like cannot be used as starting points. Together with this positivity, the field is about $\lambda^{-1/2}$ in the sense of probability. So one treats it by splitting into small-field and large-field regions. Written with the curvature of MRS,

$$F = dA - \lambda[A,A], \qquad \frac{\lambda|A|^3}{|A|^2} = \lambda|A|, \qquad \frac{\lambda^2|A|^4}{|A|^2} = \lambda^2|A|^2 \quad (6)$$

Expanding the action $|F|^2$ gives quadratic, cubic and quartic terms. The two on the right are the ratios of the cubic and quartic terms to the quadratic term, rough guides to size without counting derivatives (this chapter's reading). At $|A| \sim \lambda^{-1}$ both ratios are 1, and at $|A| \sim \lambda^{-1/2}$ they are $\lambda^{1/2}$ and $\lambda$. At $\lambda^{-1}$ the non-linear terms are as large as the quadratic term, and MRS too write that information at $\lambda^{-1}$ alone is 'not enough to start perturbation theory'. **Example:** for $\lambda = 0.01$, at $|A| \sim 100$ the ratios are 1 and 1, and at $|A| \sim 10$ 0.1 and 0.01. Computed

In the small-field region the gauge is replaced by a background-field-dependent one suited to perturbative calculation. The conclusion is 'The ultraviolet limit as $\rho \to \infty$ of the Schwinger functions … exists; furthermore these functions in the ultraviolet limit satisfy the Slavnov identities'. The Slavnov identities are relations that invariance under small gauge transformations leaves among gauge-fixed correlation functions (4-08). Known

The authors themselves also write the reservations. 'we do not provide a detailed proof of this statement', and a detailed proof 'would presumably be several hundred pages long'. It is restricted to axial gauge, does not treat instantons, and does not remove the infrared cutoff. Removing it would make the coupling large, leading 'presumably to so-called non-perturbative effects corresponding to confinement'. Known

07

## Setting the two roads side by side

MRS say the role of their cutoff is 'quite similar to the role played by compactness of integration over the gauge group in Balaban'. As formulas,

$$\text{lattice: } U_\ell \in G \ \text{(compact)} \Rightarrow \|U_\ell - 1\| \le 2, \qquad \text{continuum: counterterms} \Rightarrow |A| \lesssim \lambda^{-1} \quad (7)$$

and both are first upper bounds on the field. Known The lattice side is written with group elements and the continuum side with Lie algebra elements, so we match the

sizes. Writing $U_\ell = e^{igB_\ell}$ ($B_\ell$ a Lie algebra element) for $SU(2)$, the rotation angle can be taken from 0 to $\pi$, so $|B_\ell| \le \pi/g$. That is, on the lattice side too the Lie-algebra field is held to about $1/g$, and both sides take the form "field $\lesssim$ 1/coupling" (this chapter's reading). **Example:** for $SU(2)$, $\|U - 1\| = 2$ happens at $U = -1$, rotation angle $\pi$. Computed Also in common is that upper bounds alone cannot start perturbation theory.

Known Table 3 The two roads.

| | **Bałaban (1987–89)** | **MRS (1993)** |
|---|---|---|
| Regularisation | Lattice (Wilson action) | Momentum cutoff (breaks gauge invariance, restored by counterterms) |
| Group, infrared | Compact Lie group $G$ (the set-up of the three-dimensional paper; details in four dimensions not checked), on a torus | $SU(2)$, fixed infrared cutoff |
| What was controlled | Upper and lower bounds on the effective densities and the partition function, independent of the lattice spacing | The ultraviolet limit of axial-gauge Schwinger functions and the Slavnov identities |
| Gauge | Chosen locally for each place and scale (the idea is in 7-05 § 05; the concrete choice in the four-dimensional papers not checked in the text) | Axial for large fields, background-field-dependent for small fields |
| Reflection positivity | Holds for the Wilson measure at each lattice spacing (JW § 6.5, 5-03). Holding in the limit is not treated | Not yet. The authors say it could probably be shown by taking a lattice as the first cutoff |
| Proof | The main theorem carries a hypothesis ( § 03) | No detailed proof (the authors) |

MRS point out that Bałaban's work does not treat gauge-fixed Schwinger functions, and write 'one can consider in fact that the ultraviolet problem for the Yang-Mills field theory is not yet understood until this point is clarified'. Conversely, MRS leave reflection positivity and the infrared for later. Known

08

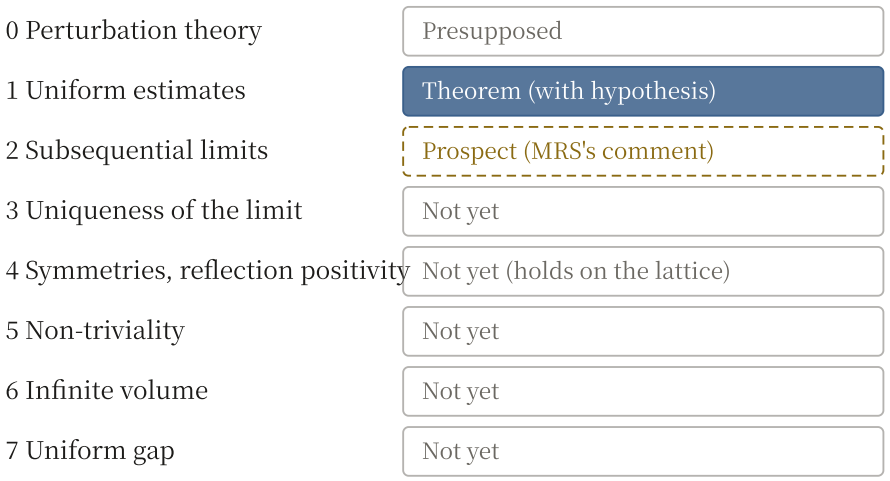
## What "the level of control" means

We divide what lies inside and beyond the ultraviolet-limit step (7-01) into eight levels, from the level of perturbation theory to the level of a uniform gap (the rows of Table 4. The level of non-triviality is that the limit is not a free (Gaussian) field, and the problem statement further requires the short-distance behaviour to agree with the predictions of asymptotic freedom. The level of a uniform gap is a mass gap uniform in the volume). The numbers in formula (8) are the row numbers of Table 4. Putting the words of the literature on arrows,

$$1 \xrightarrow{\text{should follow (MRS)}} 2 \xrightarrow{\text{not necessarily unique}} 3 \longrightarrow 4,5 \longrightarrow 6,7 \quad (8)$$

and only the level of uniform estimates has a theorem (in four dimensions, with a hypothesis). Known

Figure 2 Choosing a work changes the status of each level (theorem, with hypothesis, sketch, prospect, not yet)

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Bałaban four dimensions: the level of uniform estimates is a theorem with a hypothesis, the level of subsequential limits "should follow"
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Known Table 4 The levels of control ("prospect" is what the literature says "should follow" or "could probably be shown". What the problem statement asks for is the existence and properties of the theory, and the level of uniform estimates is a step of method). The right-hand column is the construction published on 2026-11-10, where "Theorem" means "the committee's assessment is that it is proved in Lean" (6-01, 6-04).

| Level | Bałaban 4D | MRS | Problem statement | Published construction (2026-11-10) |
|---|---|---|---|---|
| 0 Perturbation theory | Presupposed | Presupposed | Presupposed | Presupposed |
| 1 Uniform estimates | Theorem (with hypothesis) | Sketch | (usually needed as a method) | Theorem (multistep estimates; Part 11) |
| 2 Subsequential limits | Prospect | Sketch | Not enough alone (footnote 2) | Theorem |
| 3 Uniqueness | Not yet | Sketch | Required | Converges along the whole sequence of spacings chosen by the family. Universality with respect to other sequences of spacings not yet (6-04) |
| 4 Symmetries, reflection positivity | Not yet (holds for the Wilson measure at each lattice spacing) | Slavnov a sketch, OS positivity a prospect | Required | Theorem (each component of the phases satisfies a variant of the OS axioms; Part 12) |

| Level | Bałaban 4D | MRS | Problem statement | Published construction (2026-11-10) |
|---|---|---|---|---|
| 5 Non-triviality | Not yet | Not yet | Required | Theorem |
| 6 Infinite volume | Not yet | Not yet (the infrared cutoff is not removed) | Required ($\mathbb{R}^4$) | Theorem along subsequences of volumes. Convergence and uniqueness not yet (6-04) |
| 7 Uniform gap | Not yet | Not yet | Required ($\Delta > 0$) | Not yet (only polynomial estimates; 6-04) |

"Has reached the level of control" means the state in which the level of uniform estimates goes through (in four dimensions, with a hypothesis) and parts of the levels of subsequential limits, uniqueness, and symmetry and reflection positivity are written as prospects or sketches. The problem statement writes of $T^4$ 'while the construction is not complete, there is ample indication that known methods could be extended to construct Yang–Mills theory on T4', and says that a detailed construction on a compact space alone would be 'a major breakthrough' (JW § 6). A 2004 report also writes 'The state of the art remains the works of Balaban and of Magnen, Rivasseau and Sénéor' (Douglas 2004). Known The committee's assessment is that the construction published on 2026-11-10, as in the right-hand column of Table 4, reaches the levels from uniform estimates to non-triviality, and infinite volume along subsequences of volumes. It does not reach the level of a uniform gap (6-01).

**Example:** as a yardstick, for weakly coupled three-dimensional $\phi^4$, everything corresponding to the levels from uniform estimates to a uniform gap is a theorem (JW § 6.2, 7-03 § 06). What corresponds to the level of non-triviality is that the limiting measure is non-Gaussian ('a remarkable non-Gaussian measure' of JW § 6.2). That the level of non-triviality is a property that must be shown separately is treated in 7-08, and that the setters ask for new ideas at the level of infinite volume and the level of a uniform gap in 7-10.

09

## Detour: how to check a proof of several hundred pages

A section you may skip.

The two works share one difficulty: length. Adding up the pages of the papers in Table 1 gives

$$\underbrace{221}_{\text{7 foundation papers}} + \underbrace{21}_{\text{three dimensions}} + \underbrace{184}_{\text{4D, 5 papers}} = 426 \text{ pages} \quad (9)$$

(sums of page ranges). Computed MRS judge Bałaban's work 'not easily accessible ... the results are scattered over many publications; hence to check the consistency of all the arguments is

very difficult', and conclude that writing their own detailed proof would be 'an extremely valuable task'. Known

One response is rewriting in an easier model: Dimock, saying 'others have been slow to adopt the approach', expounded the method for three-dimensional $\phi^4$ (7-05). Another is tools that check long proofs by machine; the idea is in the guide to Lean.

## What this chapter can say

| | Content |
|---|---|
| **Said** | The statement and hypothesis of the Conditional Ultraviolet Stability Theorem (Theorem 1) and the unpublished basic inequality (Theorem 2), the set-up and reservations of MRS 1993, the assessments of the setters and the surveys (checked in primary sources) Known |
| **Said** | The arithmetic of formulas (1)(2)(4)(6), Table 2 and Figure 1, the examples and numbers in § 01 and § 04– § 07, formula (9) Computed |
| **Not said** | The content of Bałaban's four-dimensional proof and the details of the set-up. The value of $\gamma$. That the $a_0$ of Table 2 coincides with the scale of the paper's unit lattice (this chapter's reading) |
| **Not said** | Publication of the proof of the basic inequality (Theorem 2) (not found within what we searched = the bibliographic data of the author's papers from 1989 on). Whether there has been progress on the four-dimensional ultraviolet side in the literature since 2004 (the 2026 survey was read from bibliographic data and abstract only, and not used for assessment; the construction published on 2026-11-10 is read in Volume 2) |
| **Not said** | Uniqueness of the limit, axioms, non-triviality, infinite volume and the mass gap in four dimensions. On none of these does this chapter, or this book, show anything of its own. For the construction published on 2026-11-10, the committee's assessment is that the axioms, non-triviality and infinite volume along subsequences of volumes are proved in Lean. Uniqueness of the volume limit and the mass gap are not shown (6-04) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Theorem 1, Theorem 2 | Abstract and introduction pp. 355–356 | T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 |
| The definition of ultraviolet stability (three dimensions) | Text pp. 255–257 | T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275 |
| Other papers of the series (Table 1) | Bibliographic data only | CMP 89 (1983); 95, 96 (1984); 98, 99, 102 (1985); 109 (1987); 116, 119 (1988); 122 (1989) 175–202. Titles cross-checked with Chatterjee's reference list and Crossref |

| Item | Kind | Source / tools |
|---|---|---|
| Whether the basic inequality (Theorem 2) was published | Bibliographic data only | Crossref author search (T. Bałaban, 1989 and later). The only paper involving gauge fields is T. Bałaban, M. O'Carroll, R. Schor, J. Math. Phys. 32 (1991) 3199–3208 (block renormalisation of fermions in an external gauge field), and nothing corresponding to the proof of the basic inequality was found |
| The content of MRS and its comment on Bałaban | p.325–328 | J. Magnen, V. Rivasseau, R. Sénéor, Construction of $YM_4$ with an infrared cutoff, Comm. Math. Phys. 155 (1993) 325–383 |
| The setters' assessment, footnote 2, the non-Gaussianity of $\phi^4_3$ | Text | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) § 6, § 6.2, § 6.5 |
| 'state of the art' | Text | M. R. Douglas, Report on the Status of the Yang–Mills Millenium Prize Problem, Clay Mathematics Institute (2004) |
| The usage of ultraviolet stability, reference list | § 6 | S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950 |
| 'a form of ultraviolet stability' | § 1.1 | I. Chevyrev, Stochastic quantisation of Yang–Mills, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359 |
| Motivation of the exposition, reference list | § 1.1 | J. Dimock, The renormalization group according to Balaban, I. Small fields, arXiv:1108.1335 |
| $b_0, b_1$ | § 9.1.1 formula (9.3) | S. Navas et al. (Particle Data Group), Phys. Rev. D 110 (2024) 030001 |
| The 2026 survey | Bibliographic data and abstract only (not used for assessment) | M. R. Douglas, The Yang–Mills Millennium problem, Nature Reviews Physics 8 (2026) 86–97 |
| Table 2, Figure 1, examples, formula (9) | Computed | Python on this machine: `running_gamma.py` (formula (4), checked against numerical integration), `ratios_quartic.py` (the ratios of formula (6) and $\lvert B\rvert \le \pi/g$ for $SU(2)$) |

Next chapter to read: 7-08 Triviality of $\phi^4$ in four dimensions (why the level of non-triviality is a property that must be shown separately).

# Triviality of φ⁴ in four dimensions — the limit made on the lattice collapses to a Gaussian

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — That "trivial" means Gaussian. The counting that makes four dimensions the borderline, and a way of seeing it through the meeting of two random walks. The hypotheses and conclusion of the theorem of Aizenman–Duminil-Copin (2021). Why "non-trivial" becomes a separate requirement for Yang–Mills.

**Prerequisite chapters** — 7-03 Constructing scalar fields (see also 2-08 and 7-07). Only the one-loop formula of § 08 uses 4-04 Divergences and renormalisation (and Parseval's identity from 1-11).

**Said first** — The proofs are followed only in outline. The numbers in the tables are computations for free fields and simple random walks, not for the Ising model. § 08 and § 09 are a one-loop picture, not theorems. This chapter shows nothing about whether Yang–Mills is trivial or non-trivial.

The order of this chapter

1. What trivial means: Gaussians and Wick's rule
2. From the lattice to the continuum: approaching the critical point
3. Counting dimensions: the tree-diagram bound and $L^{4-d}$
4. Random currents: the deviation from Wick is the meeting of clusters
5. The meeting of two random walks: rare, but they meet many times at once
6. The bubble diagram: the logarithm of four dimensions
7. Reading the theorem: its range and a chronology
8. The physics picture: the one-loop running coupling
9. Reading the table and figure: the collapsing coupling and the Landau pole (Detour: the side that becomes free in the infrared)
10. The requirement of non-triviality: what Yang–Mills needs

01

## What trivial means: Gaussians and Wick's rule

A Euclidean field theory is a probability measure, and its content appears in the correlation functions $S_n(x_1,\dots,x_n)=\langle\Phi(x_1)\cdots\Phi(x_n)\rangle$ (7-01). For a Gaussian measure, the correlations at an even number of points are determined just by dividing the two-point functions into pairs (1-08).

$$S_{2n}(x_1,\dots,x_{2n})=\sum_{\pi}\prod_{j=1}^{n}S_2\big(x_{\pi(2j-1)},x_{\pi(2j)}\big) \tag{1}$$

The sum is over all the ways $\pi$ of grouping the $2n$ points in twos. This Wick rule means there is no interaction, and for that reason Gaussian fields have been called **trivial** (ADC § 1.1). The first quantity measuring the deviation is the four-point Ursell function. Known

$$U_4(x,y,z,t)=S_4(x,y,z,t)-\big[S_2(x,y)S_2(z,t)+S_2(x,z)S_2(y,t)+S_2(x,t)S_2(y,z)\big] \tag{2}$$

In models with the Lee–Yang property (the zeros of the partition function in the magnetic field lie on the imaginary axis; the Lee–Yang theorem of 2-08), Gaussianity follows if $U_4$ vanishes in the limit (Newman 1975, ADC § 6.3). Known **Example:** for a single spin taking $\pm1$ with equal probability, $\langle\sigma^4\rangle-3\langle\sigma^2\rangle^2=-2$, and for a sum of $N$ independent ones the ratio $U_4/\mathrm{Var}^2$ collapses like $-2/N$ (1-08). Spins near the critical point are strongly correlated, so this reason cannot be used, and three-dimensional $\phi^4$ gives a non-Gaussian measure (JW § 6.2, 7-03).

02

## From the lattice to the continuum: approaching the critical point

The object is the model with a real number $\phi_x$ at each point of the lattice $\mathbb{Z}^d$, with neighbours linked by a ferromagnetic coupling $J\ge0$ (ADC formula (1.12); the lattice version of formula (1) of 7-03).

$$\langle F\rangle=\frac{1}{Z}\int F(\phi)\,\exp\Big(\sum_{\{x,y\}}J_{xy}\phi_x\phi_y\Big)\prod_x e^{-\lambda\phi_x^4+b\phi_x^2}\,\mathrm{d}\phi_x \tag{3}$$

As $b=2\lambda\to\infty$, the distribution at a point concentrates on $\pm1$ and it becomes the Ising model (ADC formula (1.20)). Taking the scale at which we look to be $L$ in lattice units (the lattice spacing in continuum units is $a=1/L$), we follow the smeared sum divided

by the standard deviation $\sqrt{\Sigma_L}$ of the sum over the box $\Lambda_L$ ($\Lambda_L = [-L, L]^d$, ADC formulas (1.14)(1.15)).

$$T_{f,L}(\phi) = \frac{1}{\sqrt{\Sigma_L}} \sum_{x \in \mathbb{Z}^d} f(x/L)\, \phi_x, \qquad \Sigma_L = \Big\langle \Big( \sum_{x \in \Lambda_L} \phi_x \Big)^2 \Big\rangle \quad (4)$$

The volume is first taken to infinity and then $L \to \infty$, and $(J, \lambda, b)$ may be moved along the way. To get smooth correlations at scale $L$, one approaches the critical point to make the correlation length at least of order $L$. If the correlation length stays finite in lattice units, the limit is white noise (ADC footnote 1, 2-08). For the Ising model ($b = 2\lambda \to \infty$), $J$ is written $\beta J$, $\beta$ is called the inverse temperature, and $\beta_c$ the critical value at which the correlation length diverges (2-08). $S_2 \to 0$ at large distance is the side $\beta \le \beta_c$ (ADC formula (1.10)), and this chapter is restricted to that side. Known

Marginal Triviality Theorem (THEOREM 1.2 of Aizenman–Duminil-Copin 2021)

For $d = 4$, every random field reachable by the construction above with $S_2 \to 0$ at large distance is a generalised Gaussian process. Known

A **generalised Gaussian process** is a distribution-valued random field for which every finite collection of $T f(\Phi)$, one for each test function $f$ (the limit as $L \to \infty$ of the $T_{f,L}$ of formula (4)), follows a multivariate Gaussian distribution (Gaussian measures in 1-09).

## 03 Counting dimensions: the tree-diagram bound and $L^{4-d}$

Triviality in five or more dimensions was shown by two inequalities (Aizenman 1981, 1982, Fröhlich 1982): the **tree-diagram bound**, which bounds the four-point deviation by "the diagram in which four legs meet at one point $u$", and the infrared bound, by which the two-point function does not exceed the free-field form even at the critical point (Ising model, ADC formulas (1.22)(1.23)).

$$|U_4(x,y,z,t)| \le 2 \sum_{u \in \mathbb{Z}^d} S_2(u,x) S_2(u,y) S_2(u,z) S_2(u,t), \qquad S_2(x,y) \le \frac{C}{|x-y|^{d-2}} \ (\beta = \beta_c$$

If the four points are at mutual distances of order $L$, then $S_4 \sim S_2^2 \sim L^{-2(d-2)}$; on the right-hand side the sum over $u$ gives $L^d$ and the four factors $L^{-4(d-2)}$.

$$\frac{|U_4|}{S_4} \lesssim \frac{L^d\, L^{-4(d-2)}}{L^{-2(d-2)}} = L^{4-d} \quad (6)$$

For $d > 4$ this goes to 0, and apart from details this is the backbone of the theorem in five or more dimensions (ADC § 1.3). For $d = 4$ it is $L^0$, and the primary source writes 'clearly inconclusive for d = 4'. The problem statement also writes that in five or more dimensions scalar fields with a quartic self-interaction do not exist, and that the proof applies to one-component fields (JW § 6.2). Known The same exponent also appears in the dimension of the coupling $[\lambda] = 4 - d$ in formula (7) of 7-01; for $d > 4$ it points to the coupling ceasing to act at long distances. Physics

04

## Random currents: the deviation from Wick is the meeting of clusters

The origin of the tree-diagram bound is the **random current representation**. Taylor-expanding the weight $e^{\beta J \sigma_x \sigma_y}$ edge by edge, viewing the degree $n(x,y)$ as an integer on the edge (a current), and summing over spins, only the configurations survive in which the set $\partial n$ of points with odd total degree (sources) coincides with the set $A$ of points whose correlation is taken (ADC formula (3.5)).

$$\Big\langle \prod_{x \in A} \sigma_x \Big\rangle = \frac{\sum_{\partial n = A} w(n)}{\sum_{\partial n = \varnothing} w(n)}, \qquad w(n) = \prod_{\{x,y\}} \frac{(\beta J_{xy})^{n(x,y)}}{n(x,y)!} \tag{7}$$

Configurations without sources are collections of loops, and the two-point correlation is "the effect of inserting one path joining $x$ and $y$". An identity that exchanges sources (the switching lemma) gives the following (ADC formula (3.11)).

$$U_4(x,y,z,t) = -2\, S_2(x,y)\, S_2(z,t)\, \mathbf{P}^{xy,zt}\big[\, \mathcal{C}(x) \cap \mathcal{C}(z) \;/= \varnothing \,\big] \tag{8}$$

$\mathbf{P}^{xy,zt}$ is the law of two independent currents $n_1, n_2$ with sources $\{x,y\}$ and $\{z,t\}$ respectively (the weights of formula (7) turned into probabilities). $\mathcal{C}(x)$ is the set of points connected to $x$ by following edges where the sum $n_1 + n_2$ is positive, and likewise $\mathcal{C}(z)$. $\mathcal{C}(x) \cap \mathcal{C}(z) \;/= \varnothing$ means that $x$ and $z$ are connected in $n_1 + n_2$. The deviation from Wick's rule is exactly the probability of this meeting. Known

The two clusters are not independent. So one adds currents without sources and replaces them by two independent clusters. Doing so can only increase the probability of intersecting (ADC formulas (3.13)(3.14)). Bounding that probability by the expected number of intersection points and using the switching lemma once more gives the first inequality of formula (5) (the tree-diagram bound) (ADC § 3.2). Known

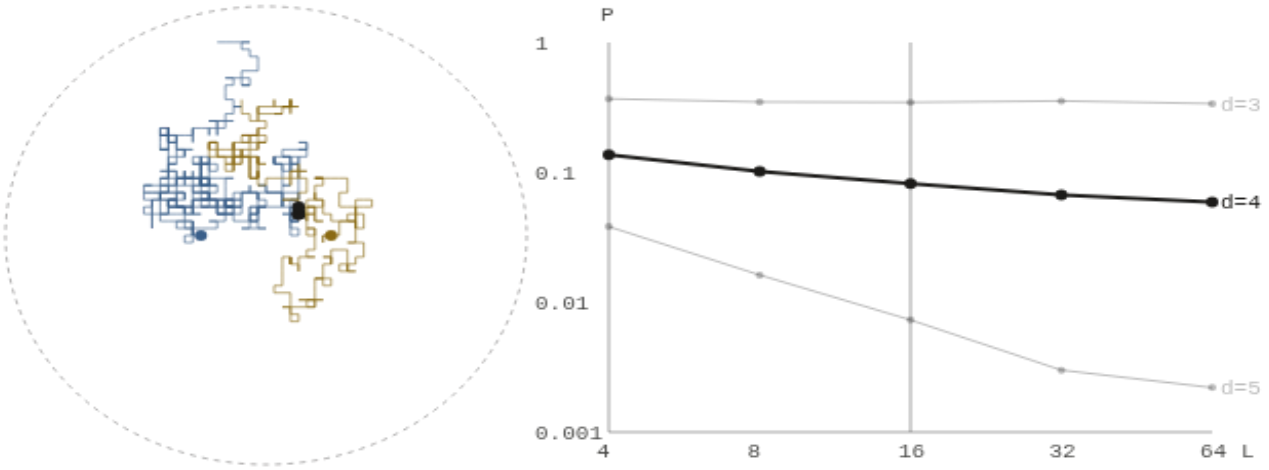
05

## The meeting of two random walks: rare, but they meet many times at once

Intuition comes from picturing the two clusters as random walks joining $x$ and $y$, and $z$ and $t$ (the primary source also uses this analogy as a guide). In four dimensions, consider two trajectories starting from two points a distance $L$ apart and walking until they leave a region of radius about $L$ (about $L^2$ steps). The probability of intersecting goes to 0 like $1/\log L$, yet the expected number of intersection points does not become smaller than a positive constant independent of $L$. Intersections are rare, but when they intersect the number of intersection points grows like $\log L$ (ADC § 3.2). Known

The rough guide is that the trajectory of a random walk is two-dimensional (about $L^2$ points within radius $L$), and the borderline for two two-dimensional objects to meet is $2 + 2 = 4$. More precisely, the content of this guide is that the expected number of intersection points behaves like the $L^{4-d}$ of formula (6) (the $E$ of Table 2).

Figure 1 Choosing the dimension and L changes the two random walks (projected onto the first two coordinates) and the points that coincide in d dimensions (black dots). On the right (below on narrow screens) is the intersection probability P (the computation of Table 2; logarithmic vertical axis). The number of intersection points is the value for one sample, not an average




```
d = 4, L = 16, intersections in this sample: 5 (sample value), average: P = 0.082, E = 0.3
```

Computed The points P and Table 2 were computed on this machine. The pictures of random walks are samples from random numbers with fixed seeds (the first picture was chosen to be an intersecting example). Most of the places that look overlapping in the projection do not coincide in $d$ dimensions.

Computed Table 2 Two simple random walks started from $\pm(L/2)e_1$ and run until they leave the ball of radius $2L$: the mean number of common lattice points $E$, the intersection probability $P$, and the mean given intersection $E/P$ (6000 pairs each; the standard error of $P$ is 0.003–0.0044; $L = 8, 32$ are in Figure 1).

| L | E (d=3) | E (d=4) | E (d=5) | P (d=4) | E/P (d=4) |
|---|---|---|---|---|---|
| 4 | 1.75 | 0.32 | 0.064 | 0.137 | 2.35 |
| 16 | 5.86 | 0.30 | 0.013 | 0.082 | 3.67 |
| 64 | 23.03 | 0.27 | 0.004 | 0.059 | 4.61 |

$E$ is proportional to $L$ in three dimensions, decreases like $1/L$ in five, and is nearly constant in four, just as the $L^{4-d}$ of formula (6). In four dimensions $P$ decreases slowly and $E/P$ increases: the shape "rare, but when they meet they meet many times at once". With finite $L$ this does not confirm the rate $1/\log L$. Computed

06

## The bubble diagram: the logarithm of four dimensions

The key to improving the tree-diagram bound in four dimensions is the bubble diagram $B_L$ cut off at distance $L$ (ADC formula (1.25)): the diagram that joins two two-point functions and goes once round. For the free-field two-point function $G$ (the inverse of the lattice Laplacian) it is proportional to the expected number of times two random walks from the same point visit the same point, and since $G(x) \approx 1/(4\pi^2|x|^2)$ far away in four dimensions, it grows logarithmically. Known

$$B_L = \sum_{x \in \Lambda_L} S_2(0,x)^2, \qquad \sum_{x \in \Lambda_L} G(x)^2 \approx \int_{1<|x|<L} \frac{\mathrm{d}^4 x}{16\pi^4 |x|^4} = \frac{\ln L}{8\pi^2} + \text{const} \tag{9}$$

Computed Table 1 The lattice bubble of the free field $\sum_{x \in [-L,L]^d} G(x)^2$ (infinite lattice, mass 0, $G(x) = \int_0^\infty \prod_i e^{-2t} I_{x_i}(2t)\,\mathrm{d}t$ integrated numerically. $I_n$ is the modified Bessel function, and $e^{-2t} I_n(2t)$ is the probability that a one-dimensional continuous-time random walk jumping left and right each at rate 1 is at $n$ at time $t$). The increment is the difference when $L$ doubles; formula (9) gives $\ln 2/(8\pi^2) = 0.00878$.

| L | d=3 | d=4 | Increment for d=4 | d=5 |
|---|---|---|---|---|
| 4 | 0.4504 | 0.05123 | 0.00738 | 0.01866 |
| 8 | 0.8388 | 0.05926 | 0.00803 | 0.01898 |
| 16 | 1.6161 | 0.06766 | 0.00839 | 0.01916 |
| 32 | 3.1711 | 0.07624 | 0.00858 | 0.01925 |

The four-dimensional increment approaches 0.00878; in three dimensions the increment doubles each time ($\propto L$), and in five it halves and converges. Computed The improved tree-diagram bound (Theorem 1.3 of Aizenman–Duminil-Copin) gives, for the four-dimensional nearest-neighbour ferromagnetic Ising model with $\beta \le \beta_c$, $L \le \xi(\beta)$, and four points mutually farther apart than $L$, with some $c, C > 0$, the following (ADC formula (1.24)).

$$|U_4(x,y,z,t)| \le \frac{C}{B_L(\beta)^c} \sum_{u \in \mathbb{Z}^4} S_2(u,x) S_2(u,y) S_2(u,z) S_2(u,t) \tag{10}$$

If the two-point function nearly satisfies the infrared bound (the case expected in four dimensions), $B_L$ is of order $\log L$ and $|U_4|/S_4 = O((\log L)^{-c})$; if it is much smaller, formula (5) alone suffices. Beyond four points, it closes by the characteristic function

of the quantity in formula (4) being close to Gaussian with error $(\log L)^{-c}$ (the proposition that the characteristic function is close to Gaussian; ADC Proposition 1.4). The centre of the proof is a multi-scale analysis showing that, given intersection, the number of intersection points grows at least like $(\log L)^{c}$ (ADC § 4, § 6). Known

07

## Reading the theorem: its range and a chronology

The Marginal Triviality Theorem (Theorem 1.2 of ADC) does not depend on the choice of counterterms. The result rests not on computing counterterms but on dimension-dependent relations among the Schwinger functions (the $S_n$ of § 01) that can appear in any limit (ADC § 1.1). It includes "hard" $\phi^4$ and the Ising model (the Griffiths–Simon construction, which makes $\phi^4$ from averages of blocks of spins). On the other hand, the object is restricted to limits of nearest-neighbour, ferromagnetic, one-component models with a lattice cutoff, and it is not a theorem about every method of construction. Nor does it give the exact size of the logarithmic correction or the covariance of the limit (ADC § 1.2). Known

Known Table 3 The main results on triviality (classification from ADC §1.1–1.2. §1.1 also counts some of the small-$\lambda$ results below among the partial results for $d = 4$. Journals at the end).

| Year | Authors | Content |
|---|---|---|
| 1981–82 | Aizenman; Fröhlich | Gaussian for $d > 4$ |
| 1983 | Aizenman–Graham; Aragão de Carvalho–Caracciolo–Fröhlich | Partial results for $d = 4$ |
| 1985–87 | Gawędzki–Kupiainen; Feldman–Magnen–Rivasseau–Sénéor; Hara–Tasaki | Small $\lambda$ (renormalisation group) |
| 2014 | Bauerschmidt–Brydges–Slade | Small $\lambda$, $n$ components |
| 2021 | Aizenman–Duminil-Copin | $d = 4$, all $\lambda$ and Ising |

Since then, a one-page correction (2024) and an extension to effective dimension four and above (Panis 2026) have appeared (bibliographic data only).

08

## The physics picture: the one-loop running coupling

The reason on the physics side is that the quartic coupling in four dimensions is not asymptotically free (JW § 6.2). What follows is a one-loop picture, not a theorem. Physics If you have not read 4-04, you may accept just the right-hand end of formula (11) and read on. Write the interaction as $\frac{u}{4!}\phi^4$ (for formula (3), $u = 24\lambda$). At one loop the four-point coupling shifts by one inserted bubble; there are three ways of inserting

it (the three pairs of formula (2)), each with symmetry factor $\frac{1}{2}$, and the bubble is the same $\frac{1}{8\pi^2}\ln$ as in formula (9) (the sum over positions $\sum G^2$ and the integral over momenta are equal by Parseval's identity; 1-11). With $\Lambda$ the cutoff and $\mu$ the scale we look at,

$$u(\mu) \approx u(\Lambda) - \frac{3}{2}u(\Lambda)^2\frac{\ln(\Lambda/\mu)}{8\pi^2}, \qquad \frac{1}{u(\mu)} = \frac{1}{u(\Lambda)} + \frac{3\ln(\Lambda/\mu)}{16\pi^2} \implies u(\mu) < \frac{16\pi^2}{3\ln(\Lambda/\mu)} \quad (11)$$

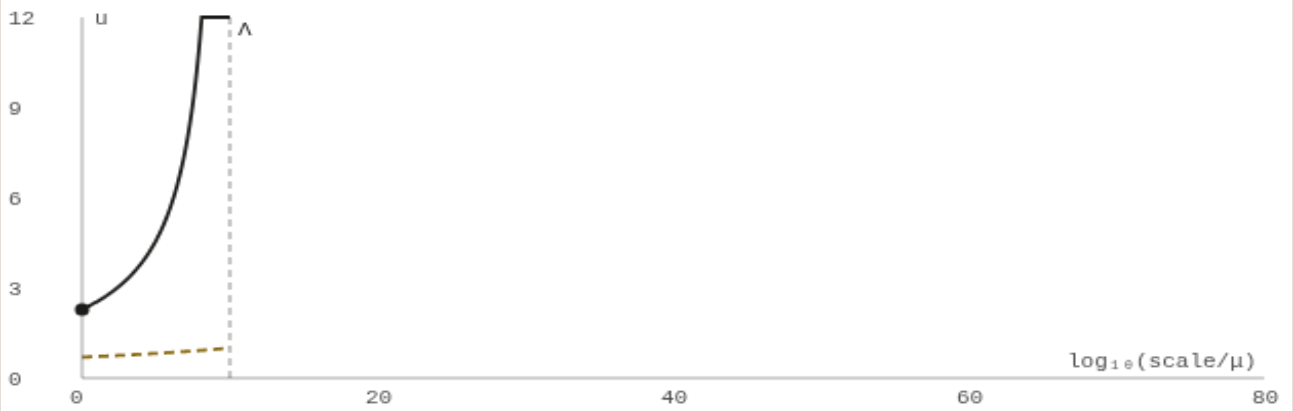
(the middle is the integral of the flow equation $\mu\, du/d\mu = 3u^2/16\pi^2$). However large the bare coupling $u(\Lambda)$ is made, raising the cutoff sends $u(\mu)$ to 0. Conversely, keeping $u(\mu) = 1$ and extending upwards, the denominator becomes 0 at $\ln(\Lambda/\mu) = 16\pi^2/3 \approx 52.6$ ($\Lambda/\mu \approx 10^{22.9}$) (the Landau pole). For Yang–Mills the sign is reversed (formula (2) of 7-07), and fixing $g^2(\mu)$ and raising $\Lambda$ only sends the bare coupling $g^2(\Lambda)$ to 0. Physics Computed

09

## Reading the table and figure: the collapsing coupling and the Landau pole

We look at formula (11) in numbers. The two left columns of Table 4 are $\phi^4$ and the right end is Yang–Mills; Figure 2 draws the same quantities as functions of the scale.

Figure 2 Raising the cutoff Λ (vertical line): for $\phi^4$ the coupling at the scale μ (left end) collapses, and for Yang–Mills the bare coupling (on the vertical line) goes to 0 (one loop). For $\phi^4$ the solid line is bare coupling $u(\Lambda) = \infty$, the dashed line $u(\Lambda) = 1$




φ⁴, $\log_{10}(\Lambda/\mu) = 10$, $u(\mu) < 2.286$ (even with bare coupling ∞), if $u(\Lambda) = 1$: 0.696

Computed The numbers in this figure were computed on this machine (formula (11) and formula (2) of 7-07; they agree with Table 4).

Computed Physics Table 4 $u(\mu)$ of formula (11), and for $SU(3)$ the bare coupling $g^2(\Lambda)$ keeping $g^2(\mu) = 1$ ($b_0 = 11 \cdot 3/48\pi^2$).

| $\log_{10}(\Lambda/\mu)$ | $u(\Lambda)=\infty$ | $u(\Lambda)=1$ | $g^2(\Lambda)$ |
|---:|---:|---:|---:|
| 2 | 11.430 | 0.920 | 0.609 |
| 10 | 2.286 | 0.696 | 0.238 |
| 40 | 0.572 | 0.364 | 0.072 |
| 80 | 0.286 | 0.222 | 0.038 |

The rate of collapse is $1/\ln\Lambda$, the same type of slowness as the $(\log L)^{-c}$ of formula (10). The one-loop formula cannot be trusted where the coupling is large, and is not a proof of triviality. In 2000 the problem statement wrote 'all insights from random walks, perturbation theory, and renormalization analysis point toward triviality' (JW § 6.2). The answer as a theorem is § 02– § 07. Known

### Detour: the side that becomes free in the infrared

A subsection you may skip.

The primary source writes that what may be regarded as a 'disappointment' from the point of view of constructive field theory is, from the point of view of statistical mechanics, 'a positive and constructive result' (ADC § 1.2). This is because it answers that the large scales of the four-dimensional critical Ising model are described by Gaussians. Known

Reading formula (11) towards long distances, $u$ goes to 0 logarithmically. **Example:** the $u(\Lambda)=1$ column of Table 4 reads 0.696 at distances 10 orders of magnitude longer than the scale where $u=1$, and 0.222 at 80 orders. The coupling stays small, so this is the side where the renormalisation group (7-05) is easier to control. Physics The 'asymptotically free' in the title of Gawędzki–Kupiainen in Table 3 ('Massless Lattice $\Phi^4_4$ Theory: Rigorous Control of a Renormalizable Asymptotically Free Model') can be read as meaning that in the massless (the 'Massless' of the title) model the coupling goes to 0 on the long-distance side (formula (11) read towards the infrared). Physics Yang–Mills is the exact opposite: the coupling becomes strong at long distances, and the gap is thought to come from that strong side (7-07 § 01). Physics

10

## The requirement of non-triviality: what Yang–Mills needs

The problem statement asks that 'a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$', and requires that the correlation functions agree at short distances with the predictions of asymptotic freedom and perturbative renormalisation (JW § 4, 0-03). Known In the prediction of perturbation theory, at short distances the coupling goes to 0 logarithmically, like $g^2(\ell)\approx 1/(2b_0\ln(\ell_0/\ell))$, but is not identically 0 (4-06; an example is the right end of Table 4). Physics The meaning of the requirement can be organised into three points (this chapter's reading).

1. **Non-triviality is a property to be shown separately.** It does not follow from ultraviolet estimates or from the existence of a limit (the level of non-triviality in 7-07). Four-dimensional $\phi^4$ is an actual example that, within the family of nearest-neighbour, ferromagnetic, one-component lattice models, becomes Gaussian however the parameters are moved (if $S_2 \to 0$ at large distance).
2. **The theorem for $\phi^4$ does not show that Yang–Mills is trivial.** The problem statement contrasts the two by whether they are asymptotically free, and calls Yang–Mills the 'natural candidate' (JW § 6.2). The committee's assessment is that the four-dimensional Yang–Mills construction published on 2026-11-10 also shows non-triviality (6-01; the evidence that it is Yang–Mills is in Part 13). The Gaussian limit of the $SU(2)$ Yang–Mills–Higgs model comes from a way of taking the limit that sends the gauge coupling to 0 in step with the lattice spacing, and non-Gaussian limits are said to be open (Chatterjee 2026).
3. **The tools of the proof cannot be carried over as they are.** The random current representation is a structure of Ising-type models.

The state before 2026 was as in the problem statement's 'at present we do not know any non-trivial relativistic field theory that satisfies the Wightman (or any other reasonable) axioms in four dimensions' (JW § 6), and the survey's 'there is yet no construction of a continuum limit of a lattice gauge theory in any dimension higher than two where Wilson loop variables have been shown to have nontrivial behavior' (Chatterjee 2018 § 6). Known For the construction published on 2026-11-10, the committee's assessment is that non-triviality is also proved (the second item above). The whole of where things stop is seen in 7-10.

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## What this chapter can say

| | Content |
|---|---|
| **Said** | That "trivial" means Gaussian, with the four-point Ursell function as the marker. That the limits approaching the critical point of one-component $\phi^4$ with a lattice ultraviolet cutoff and of the Ising model were shown Gaussian for $d > 4$ in 1981–82 and for $d = 4$ in 2021 ($d > 4$ from the description in ADC § 1.1 and JW § 6.2; $d = 4$ checked in the text of ADC) Known |
| **Said** | That the counting of formula (6) appears in the free-field bubble (Table 1) and the random-walk samples (Table 2). The arithmetic of Table 4 and Figure 2 Computed . The contrast of $\phi^4$ and Yang–Mills in the one-loop picture Physics |
| **Not said** | Triviality of $\phi^4_4$ outside lattice, nearest-neighbour, one-component models. The exact size of the logarithmic correction, the covariance of the limit, the content of the correction. The values for the Ising model in Tables 1 and 2. Non-triviality or triviality of four-dimensional Yang–Mills (neither this chapter nor this book shows either) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Formulas (1)–(10), Theorems 1.2 and 1.3, Proposition 1.4, the quotations in § 03, § 07, § 09 | Text (arXiv version 4) | M. Aizenman, H. Duminil-Copin, Marginal triviality of the scaling limits of critical 4D Ising and $\phi^4_4$ models, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973 |
| The other literature in Table 3 | Bibliographic data only | Aizenman, PRL 47 (1981) 1–4, CMP 86 (1982) 1–48; Fröhlich, NPB 200 (1982) 281–296; Aizenman–Graham, NPB 225 (1983) 261–288; Aragão de Carvalho–Caracciolo–Fröhlich, NPB 215 (1983) 209–248; Gawędzki–Kupiainen, CMP 99 (1985) 197–252; FMRS, CMP 109 (1987) 437–480; Hara–Tasaki, JSP 47 (1987) 99–121; BBS, JSP 157 (2014) 692–742 |
| Newman, the infrared bound, the correction, the extension | Bibliographic data only | Newman, CMP 41 (1975) 1–9; Fröhlich–Simon–Spencer, CMP 50 (1976) 79–95; Glimm–Jaffe, Positivity of the $\phi^4_3$ hamiltonian, Fortschr. Phys. 21 (1973) 327–376 (ADC cites these two as [19, 21] as the source of the infrared bound (1.23)); Ann. of Math. 199 (2024) 479; R. Panis, Ann. Probab. 54 (2026) no. 2 |
| The requirement of the problem statement, the contrast with $\phi^4$, the present state | Text § 4, § 6, § 6.2 | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) |
| Non-trivial continuum limits of gauge theories, the Gaussian limit of YMH | § 6, abstract | S. Chatterjee, Yang–Mills for probabilists, arXiv:1803.01950; A scaling limit of SU(2) lattice Yang–Mills–Higgs theory, Prob. Math. Phys. 7 (2026) 339–381, arXiv:2401.10507 |
| Tables 1, 2, 4, Figures 1 and 2, the Landau pole | Computed | Python on this machine: `bubble.py`, `rw_intersect.py`, `running.py` |

Next chapter to read: 7-09 Stochastic quantisation and regularity structures (subcritical and critical: the face, on the side of equations, of four dimensions being the borderline).

# Stochastic quantisation and regularity structures — the road of making the measure "by moving it", and why it stops short of four dimensions

> This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — The idea of making the measure $e^{-S}$ from a stochastic time evolution that leaves it invariant. Why products stop being definable for fields, and the prescriptions in two and three dimensions. Where subcriticality $d < 4$ comes from, and what has been obtained for Yang–Mills.

**Prerequisite chapters** — 7-01 What constructive field theory is, 1-09 Probability in infinite dimensions. We also refer to 7-03 (Wick products), 1-03 (the Lie bracket) and 1-07 (connections and curvature). The notation for gauge fields and the terms for stochastic processes used in § 07 and § 08 are defined on the spot, one line each.

**Said first** — This introduces known results and follows the proofs only in outline. The numbers are a one-dimensional model, counting of degrees and counting of symbols, and show nothing about the Yang–Mills measure. Three-dimensional Yang–Mills has reached the dynamics (local solutions and a Markov process), and the measure has not been constructed, even in finite volume. Four dimensions is critical and lies outside the general theory of regularity structures.

The order of this chapter

1. Why make the measure "by moving it" — the balance between the flow down the slope and the noise leaves $e^{-S}/Z$ unmoved. A moving figure of a ball in one dimension
2. The stochastic heat equation and the free field — if linear, the stationary distribution is the Gaussian free field. Accounting for the roughness of the noise
3. Products cannot be defined — products of distributions only when the sum of degrees is positive. A table of the "margin" in each dimension

4. Two dimensions: splitting the solution in two — the rough part is made first with Wick products, and the rest solved by a fixed-point theorem
5. Three dimensions: regularity structures — the Taylor polynomials are replaced by symbols built from the noise
6. Subcritical and critical — the non-linear term vanishes under magnification for $d < 4$. A moving figure of the number of symbols growing with dimension
7. Gauge fields: the DeTurck term and covariance — a term smoothing in the direction of the orbits is added, and the renormalisation constants are made compatible with covariance
8. What has been reached for Yang–Mills — the state space, the Markov process, the invariant measure in two dimensions, and a table of the remaining problems
9. Detour: coming down from infinity — estimates that forget large initial values are the key to making invariant measures
10. What can be carried to four dimensions — the viewpoint and the state space can be carried over; the premise of local subcriticality cannot

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01

## Why make the measure "by moving it"

To draw points from a finite-dimensional density $e^{-S(x)}/Z$ ($Z$ is the normalising constant), one keeps shaking the point with noise while letting it go down the slope $S$.

$$dX_t = -\nabla S(X_t)\,dt + \sqrt{2}\,dW_t \quad (1)$$

$W_t$ is Brownian motion (1-09). Formula (1) means: advance in small steps $\Delta t$ by the rule

$$X_{t+\Delta t} = X_t - \nabla S(X_t)\,\Delta t + \sqrt{2\Delta t}\;N$$

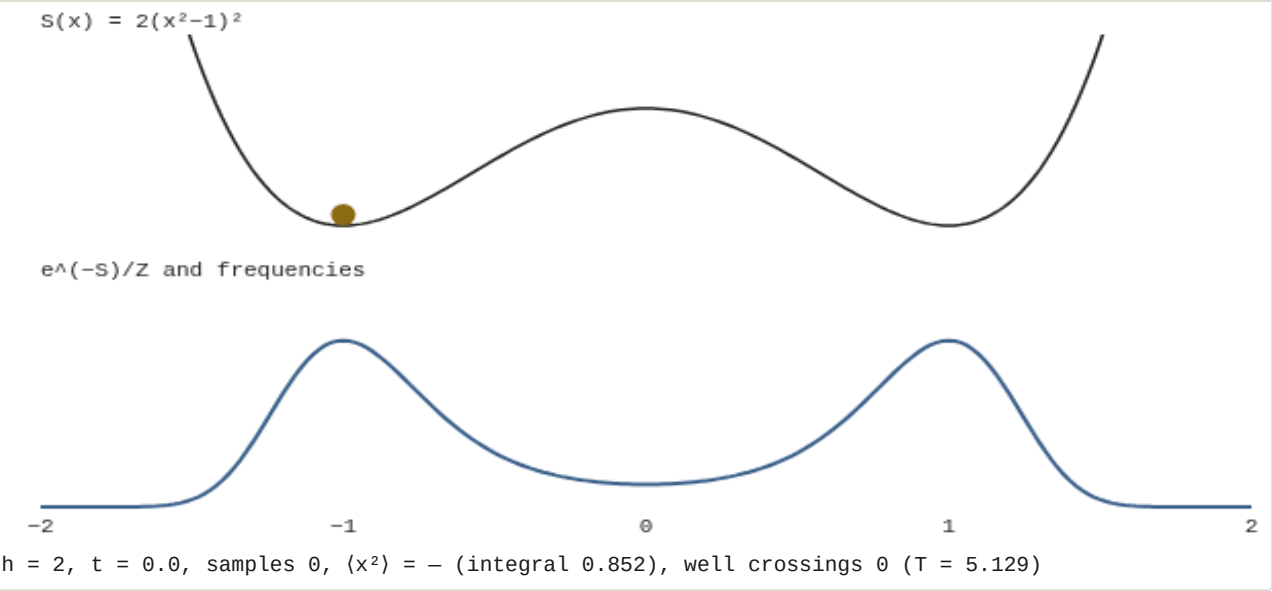
($N$ is a standard normal random number drawn afresh each time), and take the limit $\Delta t \to 0$. The ball in Figure 1 also moves by this rule (the Euler–Maruyama method). Formula (1) is called the **Langevin equation**, and the time evolution it defines the Langevin dynamics.

If many points run at once, the density $\rho(t,x)$ of the swarm obeys the Fokker–Planck equation $\partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla S)$. The right-hand side has the form of the divergence of a flow, so the total number of points is conserved (a conservation law obeyed by the density). For $\rho = e^{-S}/Z$ the inside of the bracket is 0 and nothing moves. This is the

**invariant measure**. As a metaphor, it is the balance between the flow down the slope and the flow spread by the noise. Precisely, when $S$ grows fast enough at infinity, the law of $X_t$ approaches $e^{-S}/Z$ wherever it starts. Known

Making the measure of a field as the invariant measure of an equation of this form is **stochastic quantisation**, proposed in the context of gauge theory by Parisi–Wu (1981) (Chevyrev 2022 § 1.2). The formula in 7-02 is the same thing with time rescaled, and the Monte Carlo method (5-07) rests on the same idea. The example is $S_h(x) = h(x^2 - 1)^2$, with wells at $x = \pm 1$ and a hill of height $h$.

Figure 1 The knob changes the height of the hill $h$ (five settings 0.5, 1, 2, 3, 4), and "Play" shakes the ball by formula (1). The bars below are the frequencies of where the ball has been, and the line is $e^{-S_h}/Z$




Computed $Z$, $\langle x^2 \rangle$ and $T$ in this figure are values computed on this machine. The ball is moved on the spot by the Euler–Maruyama method. The simulation in Table 1 is 2000 chains started from $x = -1$ (step 0.002, times 20–120), and $T$ is the mean hitting time from $-1$ to $+1$, $\int_{-1}^{1} e^{S(y)} \int_{-\infty}^{y} e^{-S(z)} dz\, dy$.

| h | ⟨x²⟩ integral | ⟨x²⟩ simulation | T |
|---|---|---|---|
| 0.5 | 0.8935 | 0.8932 | 3.35 |
| 1 | 0.8327 | 0.8316 | 3.57 |
| 2 | 0.8521 | 0.8497 | 5.13 |
| 3 | 0.8893 | 0.8872 | 8.88 |
| 4 | 0.9177 | 0.9152 | 17.3 |

$\langle x^2 \rangle$ agrees within 0.3%. On the other hand $T$ grows exponentially when $h$ is large (80.9 and 437.5 for $h = 6, 8$), and for $h = 4$ the chain average $\langle x \rangle$ is still −0.33 at time 10, biased towards the left well. Computed

How it grows for large $h$ is given by Kramers' formula: from the curvatures at the bottom of the wells and the top of the hill, $S_h''(\pm 1) = 8h$ and $S_h''(0) = -4h$,

$$T \approx \frac{2\pi}{\sqrt{S_h''(-1)\,|S_h''(0)|}}\, e^{h} = \frac{\pi}{2\sqrt{2}\,h}\, e^{h}$$

Known Forming $T \cdot h/e^{h}$ from the computed $T$ gives 1.27, 1.20, 1.17 for $h = 4, 6, 8$, slowly approaching the coefficient $\pi/(2\sqrt{2}) \approx 1.11$. Computed That an invariant measure exists and that it is approached quickly are separate problems, and in field theory too, estimates uniform in time are the hard part (§ 08).

02

## The stochastic heat equation and the free field

For a field on the torus $T^d$ with $S(\phi) = \int \left(\frac{1}{2}|\nabla\phi|^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4}\phi^4\right)dx$, formula (1) becomes

$$\partial_t \phi = \Delta\phi - m^2\phi - \lambda\phi^3 + \sqrt{2}\,\xi \quad (2)$$

($\Delta$ is the Laplacian, not the $\Delta$ of the mass gap). $\xi$ is space-time white noise, a random distribution formally satisfying $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\,\delta(x-y)$ (1-09). The linear **stochastic heat equation** with $\lambda = 0$ splits into independent $d\hat\phi_k = -(|k|^2 + m^2)\hat\phi_k\,dt + \sqrt{2}\,dW_k$ for each Fourier component, and the stationary distribution is the Gaussian free field with covariance $(-\Delta + m^2)^{-1}$ (the starting measure of 7-03). Known

In the parabolic scaling, which counts time as two lengths, white noise has regularity just below degree $-(d+2)/2$, and solving with the heat equation raises it by 2 (Hairer 2014 § 8). The linear solution $\Psi$ lies in

$$\Psi \in \mathcal{C}^{\,1-d/2-\kappa} \qquad (\kappa > 0 \text{ arbitrarily small}) \quad (3)$$

For $0 < \alpha < 1$, $\mathcal{C}^\alpha$ is the smoothness $|f(x) - f(y)| \lesssim |x-y|^\alpha$; for $\alpha < 0$ it is the space of distributions whose values smeared with a bump function of width $r$ stay of order $r^\alpha$. For $d = 1$ it is $\frac{1}{2} - \kappa$ (the roughness of Brownian motion), for $d = 2$ $-\kappa$ (no longer a function), and for $d = 3$ $-\frac{1}{2} - \kappa$.

03

## Products cannot be defined

The trouble is $\phi^3$. Products of distributions have the following limit (Hairer 2014 § 1, Chevyrev 2022 § 1.3; $\alpha, \beta$ are not integers). Known

$$(f, g) \mapsto fg \text{ extends continuously to } \mathcal{C}^\alpha \times \mathcal{C}^\beta \iff \alpha + \beta > 0 \quad (4)$$

For $\Psi \in \mathcal{C}^{-\kappa}$ with $d = 2$, $\Psi^2$ cannot be defined; the mean of the cut-off $\Psi_\varepsilon^2$ diverges like $\log(1/\varepsilon)$, and subtracting a constant (Wick products; 7-03) lets the limit be taken. The **margin** in Table 2 is "the degree of the non-linear term − the degree of the noise"; if positive, the non-linear term stays a correction smoother than the noise at fine scales.

Computed Table 2 Accounting of degrees (excluding $\kappa$; output in rational numbers by `trees.py`). $d = 4$ is critical, $d = 5$ supercritical.

| d | Noise ξ | Ψ | φ³ | A∂A | Margin φ⁴ | Margin YM |
|---|---|---|---|---|---|---|
| 2 | −2 | 0 | 0 | −1 | 2 | 1 |
| 3 | −5/2 | −1/2 | −3/2 | −2 | 1 | 1/2 |
| 4 | −3 | −1 | −3 | −3 | 0 | 0 |
| 5 | −7/2 | −3/2 | −9/2 | −4 | −1 | −1/2 |

The margin is $4 - d$ for $\phi^4$ and $(4 - d)/2$ for $A\partial A$ ($A^3$ is the same as $\phi^3$), positive for $d < 4$. Hairer takes this as the condition of **local subcriticality**, and checks that for $\phi^4$ it is equivalent to $d < 4$ (Hairer 2014 Assumption 8.3, § 8). Known

04

## Two dimensions: splitting the solution in two

In two dimensions the solution is split into "the roughest part of the noise" and the rest, setting $\phi = \Psi + v$.

$$\partial_t v = \Delta v - m^2 v - \lambda\left(v^3 + 3v^2\Psi + 3v :\Psi^2: + :\Psi^3:\right) \quad (5)$$

$:\Psi^n:$ is the Wick product in $\mathcal{C}^{-\kappa}$, which can be made first by probabilistic computation. The right-hand side is in $\mathcal{C}^{-\kappa}$, so $v \in \mathcal{C}^{2-\kappa}$, and $v \cdot :\Psi^2:$ has a positive sum of degrees and falls under (4), so it can be solved by an ordinary fixed-point theorem (for short times, the right-hand side of the equation written in integral form is a contraction, and there is exactly one fixed point). Returning to $\phi$, it is the limit of the equation with $\phi^3$ replaced by $\phi^3 - 3C_\varepsilon\phi$ ($C_\varepsilon$ of order $\log(1/\varepsilon)$). Da Prato–Debussche (2003) obtained **strong solutions** in two dimensions in this way (solutions satisfying the equation for each given realisation of the noise). Known

The construction of the process goes back further. Jona-Lasinio–Mitter (1985), on a square with Dirichlet boundary, made a **Markov process** (a stochastic process whose future distribution is determined only by the present state) converging to the $\phi_2^4$ measure as time goes to infinity, as a weak solution (one where the noise may be replaced too, satisfying the equation in law) of the equation smoothed by a power of an operator (Summers 2016 § 5). Hairer 2014 § 1 lists Albeverio–Röckner (1991) and Da Prato–Debussche (2003) as earlier work in two dimensions. Known

With $a = 1 - d/2$, $v \in \mathcal{C}^{2+3a}$, so $v \cdot {:}\Psi^2{:}$ needs $5a + 2 > 0$, that is $d < 14/5$. In three dimensions a single split is not enough (Hairer 2014 § 8). Computed

05

## Three dimensions: regularity structures

A smooth function can be written near each point by Taylor polynomials. Hairer (2014) replaced these polynomials by a set of distributions built from the noise for each problem (in the words of the abstract, "a model built for the problem at hand").

$$\phi(y) - \phi(x) \approx \sum_{\tau} \phi_\tau(x)\,(\Pi_x \tau)(y) \qquad (\text{error of order } |y-x|^{\gamma}) \quad (6)$$

$\tau$ are **symbols** such as $\Psi$, $\mathcal{I}(\Psi^3)\Psi$ ($\mathcal{I}$ is the operation of solving with the heat equation) and polynomials $X$, and $\Pi_x\tau$ is the distribution obtained by recentring it at the point $x$. Products that cannot be defined move into the algebra of symbols, and what the analysis needs is only the finitely many $\Pi_x\tau$ that can be made by probabilistic computation. The operation subtracting their divergences appears as the action of a "renormalisation group" (same abstract; the algebra is Bruned–Hairer–Zambotti 2019). For three-dimensional $\phi^4$

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + (3C_1^\varepsilon - 9C_2^\varepsilon)\,u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon \quad (7)$$

with $C_1^\varepsilon \sim \varepsilon^{-1}$ and $C_2^\varepsilon \sim \log \varepsilon$ (ibid. (9.21), Remark 10.23). The symbols of negative degree are the 8 symbols $\Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_i, \mathcal{I}(\Psi^3)\Psi, \mathcal{I}(\Psi^2)\Psi^2, \mathcal{I}(\Psi^3)\Psi^2$ (ibid. § 10.5, the proof of Theorem 10.22). There is also another road, paracontrolled distributions (Gubinelli–Imkeller–Perkowski 2015), and later results on invariant measures and the axioms are in Table 4 and 7-03. Known

06

## Subcritical and critical

Magnifying as $\phi_\delta(t,x) = \delta^{(d-2)/2}\phi(\delta^2 t, \delta x)$ (the gauge field $A$ is magnified the same way), the law of the noise and $\Delta$ do not change; only the coefficients of the non-linear terms change. The right of formula (8) is the Yang–Mills coupling $g$. With the action $\frac{1}{g^2}\int |F_A|^2$ and $A$ replaced by $gA$, the $A\partial A$ term of formula (9) in § 07 gets $g$ and the $A^3$ term gets $g^2$.

$$\lambda \longrightarrow \lambda\,\delta^{4-d}, \qquad g^2 \longrightarrow g^2\,\delta^{4-d} \quad (8)$$

For $d < 4$ the non-linear terms vanish at fine scales $\delta \to 0$, and the equation becomes a perturbation of the stochastic heat equation. This is subcriticality, which corresponds to super-renormalisability (Chevyrev 2022 § 1.3: subcritical is equivalent to $d < 4$). The general theory of regularity structures assumes local subcriticality (Hairer 2014 Assumption 8.3). Known A figure setting the margin $4 - d$ beside other quantities that switch at four is the dimension knob in 1-12 § 10.

$d = 4$ is critical, and the general theory covers neither model. For $\phi^4$, Hairer 2014 § 1 writes that it is not expected to have a non-trivial meaning in four dimensions, and Aizenman–Duminil-Copin later showed that the limit from the lattice is trivial (7-08). Known Yang–Mills is asymptotically free, and the coupling is thought to become logarithmically small at fine scales. This is a result about the perturbative β function (4-06), not a non-perturbative theorem. Physics

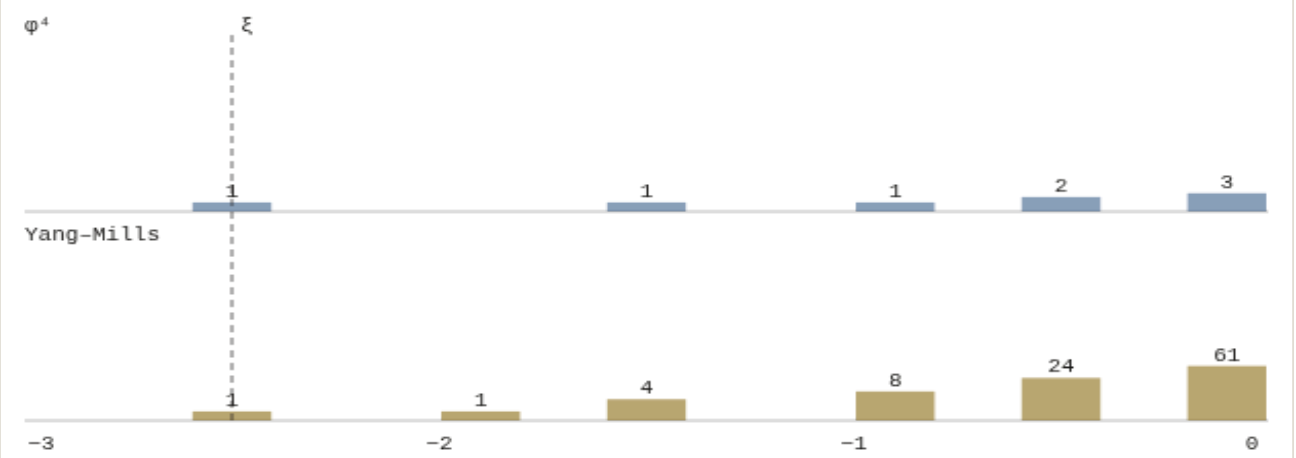
### Seen through the number of symbols

The approach to criticality can also be seen in the number of symbols of § 05. Build symbols by the following three rules, and count those with negative degree. $\kappa$ is treated as a positive infinitesimal, and the directions of $X_i$ and the component indices of the field are not distinguished.

1. **Ingredients.** The noise $\Xi$ (degree $-(d+2)/2 - \kappa$), and for each symbol $\tau$ already made, $\mathcal{I}(\tau)$ with $\mathcal{I}$ (the operation of § 05 of solving with the heat equation) applied (degree +2) and its derivative $\partial\mathcal{I}(\tau)$ (degree +1). $\Psi = \mathcal{I}(\Xi)$.
2. **Products.** Ingredients are placed in seats matching the form of the non-linear term and multiplied (degrees add). For $\phi^4$, one to three $\mathcal{I}(\cdot)$ in the three seats of $\phi^3$. For Yang–Mills, up to one each of $\mathcal{I}(\cdot)$ and $\partial\mathcal{I}(\cdot)$ in the two seats of $A\partial A$, or one to three $\mathcal{I}(\cdot)$ in the three seats of $A^3$.
3. **Empty seats.** If even one seat is empty, a polynomial $X$ (degree +1, any power) may be placed there.

**Example** ($\phi^4$ for $d = 3$): $\Xi$ is $-\frac{5}{2} - \kappa$ and $\Psi$ is $-\frac{1}{2} - \kappa$. $\Psi^3$, filling the three seats with $\Psi$, is $-\frac{3}{2} - 3\kappa$, and $\mathcal{I}(\Psi^3)$, with $\mathcal{I}$ applied to it, is $\frac{1}{2} - 3\kappa$. $\mathcal{I}(\Psi^3)\Psi$, putting $\mathcal{I}(\Psi^3)$ and $\Psi$ in two seats, is $-4\kappa$, just below 0, so it is counted. Putting $X$ in the empty third seat gives $1 - 4\kappa$, positive, so it is not counted. This is repeated until no new negative symbols appear.

Figure 2 Moving the dimension $d$ with the knob changes the number of symbols of negative degree at each degree (top $\phi^4$, bottom Yang–Mills. Bar heights are logarithms of the counts; the dotted line is the degree of the noise)




d = 3.00, noise degree −2.500, symbols of negative degree: φ⁴ 8, Yang–Mills 99

Computed The numbers in this figure and Table 3 were computed on this machine (`trees.py`; the number of "shapes" by the rules above; — means not computed).

| d | 2 | 2.8 | 3 | 22/7 | 10/3 | 3.5 | 3.7 | 3.8 |
|---|---|---|---|---|---|---|---|---|
| φ⁴ | 4 | 5 | 8 | 10 | 22 | 44 | 525 | 38561 |
| YM | 11 | 27 | 99 | 163 | 1918 | 65197 | — | — |

$\phi^4$ agrees at $d = 3$ with the 8 of § 05. $\mathcal{I}(\Psi^3)\Psi^2$, added at $d = 14/5$, is the borderline where the split of § 04 breaks down, and the two added at $d = 22/7$ are the borderline that Hairer 2014 § 9.2 flags as $\alpha = -18/7$. Yang–Mills, with half the margin, grows much faster, and at $d = 4$ stacking $\mathcal{I}(\tau)\Psi^2$ does not change the degree, so there are infinitely many. Computed For $\phi^4$, Chandra–Moinat–Weber take the noise in $\mathcal{C}^{-3+\delta}$ to imitate non-integer dimensions, and give a priori estimates over the whole subcritical range. Known

07

## Gauge fields: the DeTurck term and covariance

The components of the gauge field $A = (A_1, \dots, A_d)$ are Lie algebra elements (matrices), and $[X, Y] = XY - YX$ is the Lie bracket (1-03). $d_A = d + [A, \cdot]$ is the covariant derivative, $F_A$ its curvature (the field strength), and $d_A^*$ the adjoint of $d_A$ (the operator corresponding to the transpose with respect to the inner product) (1-07). Applying formula (1) to the action $S(A) = \int |F_A|^2$ and recounting time by a factor 2 gives $\partial_t A = -d_A^* F_A + \xi$. Even without noise, this equation is not **parabolic** (the type that, like the heat equation $\partial_t u = \Delta u$, smooths out fine fluctuations in every direction over time). This is because the action does not change under gauge transformations, so no smoothing force acts in the direction of the orbits (Chevyrev 2022 § 1.2). Adding the term tangent to the orbits $-d_A d^* A$ ($d^* A = -\sum_i \partial_i A_i$) (Zwanziger 1981, Donaldson 1985; for Ricci flow, DeTurck 1983)

$$\partial_t A_i = \Delta A_i + [A_j,\, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] + \xi_i \quad (9)$$

makes it parabolic (ibid. (1.3); in shorthand $\partial_t A = \Delta A + A\partial A + A^3 + \xi$). Formally, $B = A^g$, the solution $A$ moved by a time-dependent gauge transformation $g$, also satisfies the same equation with the noise $\mathrm{Ad}_g \xi = g\xi g^{-1}$ (rotated by sandwiching with $g$), and $\mathrm{Ad}_g \xi$ has the same law as $\xi$. This **gauge covariance in law** is the ground for defining a process on the orbits. Known

In renormalisation, a mass term $(C^\varepsilon_{\mathrm{bphz}} + \mathring{C})A$ is added to the equation with the noise smoothed at width $\varepsilon$. $C^\varepsilon_{\mathrm{bphz}}$ (the BPHZ constant) is the part depending on $\varepsilon$; in two dimensions it converges to a finite value by cancellation, and in three dimensions it diverges like $\varepsilon^{-1}$ (Chevyrev 2022 Remark 1.10). $\mathring{C}$ (the bare mass) is the finite part independent of $\varepsilon$, and can be chosen freely. Both this mass term and the smoothing of the noise break covariance. Chandra–Chevyrev–Hairer–Shen (CCHS) showed, in two dimensions (2022) and three dimensions (2024), that choosing $\mathring{C}$ to be a certain value $\check{C}$ makes the two breakings cancel in the limit. Known

Whether $\check{C}$ is fixed uniquely has been answered in two stages. CCHS showed that there is a unique choice in the sense of a certain coupling (a construction that places the solutions from two gauge-equivalent initial values on the same probability space with rotated noise and compares them) (Chevyrev 2022 Theorem 1.14). That there is only one mass renormalisation giving solutions gauge covariant in law was shown in two dimensions by Chevyrev–Shen (Comm. Pure Appl. Math. 2026, Theorem 2.15 of arXiv:2302.12160) and in three dimensions by Chevyrev–Shen (Arch. Ration. Mech. Anal. 2026, arXiv:2503.03060), strengthening the result of CCHS. Known

**Example:** for $G = U(1)$ the equation is $\partial_t A = \Delta A + CA + \xi$, and on $T^d$ only $C = 0$ is covariant (with $C \neq 0$, equivalent initial values with $A_1$ shifted by the constant $2\pi$ give different expectations of $e^{i\int A_1}$; ibid. § 1.4). Known

08

## What has been reached for Yang–Mills

A **state space** for the solutions to live in is also needed. In two dimensions line integrals along segments make sense, and the space of orbits is a Polish space (a complete metric space with a countable dense subset; convenient for handling probability measures) (CCHS 2022). In three dimensions line integrals of the Gaussian free field already diverge ($\mathbb{E}|A(\ell)|^2 \asymp \int_0^1\int_0^1 |\ell|/|t-s|\,ds\,dt = \infty$), so one uses a non-linear space in which gauge equivalence is defined for fields smoothed by the Yang–Mills flow (the deterministic flow obtained by removing the noise from formula (9)) (Chevyrev 2022 § 3.1, 7-06). What was obtained is a Markov process on the space of

orbits ( § 04), in a form that leaves open the possibility of blow-up in finite time. In two dimensions Chevyrev–Shen (Comm. Pure Appl. Math. 2026) showed that the Yang–Mills measure on $T^2$ is its unique invariant measure, by moment estimates using solvability, gauge fixing on the lattice, and Bourgain's argument (7-02). Gauge fixing, avoided in the dynamics, is used on the lattice in the proof of the invariant measure. Known

Known Table 4 What stochastic quantisation has reached (finite volume is, as a rule, the torus).

| Model | Local solutions, process | Invariant measure | Infinite volume, axioms |
|---|---|---|---|
| $\phi^4_2$ | Da Prato–Debussche 2003 (strong solutions. Processes came earlier: Jona-Lasinio–Mitter 1985, Albeverio–Röckner 1991) | A process converging to the $\phi^4_2$ measure (known from constructive field theory): Jona-Lasinio–Mitter 1985 (square, Dirichlet boundary, equation smoothed by a power of an operator) | Known by other methods (7-03) |
| $\phi^4_3$ | Hairer 2014 | Mourrat–Weber 2017 | Gubinelli–Hofmanová 2021 (except rotation invariance and clustering) |
| YM two dimensions | CCHS 2022 (with possible blow-up) | Chevyrev–Shen (Comm. Pure Appl. Math. 2026): the YM measure is the unique invariant measure | $T^2 \to \mathbb{R}^2$ not yet by this method |
| YM three dimensions | CCHS 2024 (likewise; also with Higgs. Uniqueness of the mass renormalisation: Chevyrev–Shen, Arch. Ration. Mech. Anal. 2026) | Open | Not yet |
| YM four dimensions | Critical, outside the general theory of regularity structures | | |

The open problems of the survey are the existence of the invariant measure (in three dimensions, the construction of a Yang–Mills measure, which does not yet exist even in finite volume), $T^d \to \mathbb{R}^d$ (not trivial even in two dimensions), non-trivial principal bundles, and $N \to \infty$ (Chevyrev 2022 § 4). Chevyrev–Shen (Comm. Pure Appl. Math. 2026) attribute the restriction to two dimensions to the moment estimates required by Bourgain's argument, and leave extension to higher dimensions open. Reflection positivity is another task (7-01). Known

09

## Detour: coming down from infinity

A section you may skip.

The standard way to make an invariant measure is to take a limit from uniform estimates on time averages (Krylov–Bogoliubov), and what works is an estimate that "from any initial value however large, it comes back to a fixed size in a fixed time". The simplest example is $\dot{x} = -x^3$.

$$x(t) = \frac{x_0}{\sqrt{1 + 2x_0^2 t}} \longrightarrow \frac{1}{\sqrt{2t}} \qquad (x_0 \to \infty) \quad (10)$$

The size does not exceed $1/\sqrt{2t}$, and the initial value is forgotten. At $t = 0.01$ the solutions from $x_0 = 10, 10^3, 10^6$ are 5.7735, 7.0709, 7.0711, lining up at the limit 7.0711. The solution $x_0 e^{-t}$ of the linear $\dot{x} = -x$ keeps remembering its initial value (for $x_0 = 10^6$, $9.90 \times 10^5$). Computed

Mourrat–Weber (2017) showed an estimate of this type (the "coming down from infinity" of the title) for three-dimensional $\phi^4$ and made the invariant measure (abstract). For Yang–Mills, even for abelian $U(1)$ the spatial average $\int A_i$ of the solution of formula (9) moves like Brownian motion and there is no invariant probability measure; an invariant measure appears only after passing to orbits (Chevyrev 2022 Remark 1.17). The estimate needed in the non-abelian case in three dimensions is the content of the open problems of § 08. Known

10

## What can be carried to four dimensions

**What can be carried over:** the viewpoint of seeing the measure as the invariant measure of a dynamics compatible with gauge. The state space that compares after smoothing by the Yang–Mills flow (on the lattice side too, Lüscher 2010 writes that theory and numerics suggest the flowed field becomes a smooth renormalised field Physics ). That the DeTurck term was originally a device of Ricci flow is a clue for thinking, in 16-06, about "what corresponds to Ricci flow".

**What cannot be carried over:** the premise of local subcriticality. In four dimensions the margin is 0, and the symbols do not stay finite in number (Table 3). Even in three dimensions, the existence of the invariant measure, infinite volume and reflection positivity remain. The mass gap is a property of the spectrum of the infinite-volume Hamiltonian (4-11), and does not come directly from an invariant measure in finite volume. How four dimensions stops is summarised in 7-10.

## What this chapter can say

| | Content |
|---|---|
| **Said** | The simulation of formula (1) agrees with the integral within 0.3%, and the time to cross between wells grows exponentially with $h$ ($T \cdot h/e^{h}$ is 1.17 at $h = 8$, approaching the Kramers coefficient $\pi/(2\sqrt{2}) \approx 1.11$). The margin is $4 - d$ for $\phi^4$ and $(4-d)/2$ for $A\partial A$. The symbols of negative degree number 8 for $\phi^4$ at $d = 3$ (agreeing with Hairer 2014), 38561 at $d = 3.8$, and 65197 for Yang–Mills at $d = 3.5$ Computed |
| **Said** | The Yang–Mills measure on $T^2$ is the unique invariant measure of the Langevin dynamics. Three-dimensional Yang–Mills(–Higgs) has reached local solutions and a Markov process on the orbit space. The mass renormalisation that makes it gauge covariant in law is unique in both two and three dimensions Known |
| **Not said** | The number of symbols is a number of "shapes", defined differently from the "symbols that can contribute to renormalisation" counted in the literature. The three-dimensional Yang–Mills measure and the extension to four dimensions (open). The content of the proofs of regularity structures. Complete coverage of the literature from 2025 on |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Degrees, products, formula (7), symbols | Text checked (§ 1, § 8, § 9, § 10) | M. Hairer, Invent. Math. 198 (2014) 269–504, arXiv:1303.5113 |
| Quotations, formula (9), Remark 1.10, Theorem 1.14, Remark 1.17 | Survey, text checked | I. Chevyrev, J. Math. Phys. 63 (2022) 091101, arXiv:2202.13359 |
| The invariant measure in two dimensions, uniqueness of the mass renormalisation | Text checked (§ 1, Theorems 2.15 and 2.16, the title of § 9.1) | I. Chevyrev, H. Shen, Comm. Pure Appl. Math. (2026), arXiv:2302.12160 |
| Results in § 04– § 09 | Checked in the abstracts | CCHS: arXiv:2006.04987 (Publ. IHÉS 2022), 2201.03487 (Invent. Math. 2024). Chevyrev–Shen: 2503.03060 (Arch. Ration. Mech. Anal. 250 (2026) Paper No. 11). Da Prato–Debussche, Ann. Probab. 31 (2003). Mourrat–Weber: 1601.01234. Gubinelli–Hofmanová: 1810.01700. Chandra–Moinat–Weber: 1910.13854 |
| Jona-Lasinio–Mitter | From the survey (original papers from bibliographic data only) | S. J. Summers, arXiv:1203.3991v2 (2016) § 5. Jona-Lasinio–Mitter, CMP 101 (1985) 409–436 |
| Albeverio–Röckner | Bibliographic data only (reference list of Hairer 2014) | S. Albeverio, M. Röckner, Probab. Theory Related Fields 89 (1991) 347–386 |

| Item | Kind | Source / tools |
|---|---|---|
| § 01, § 05, § 07, § 10 | Bibliographic data only (reference list of Chevyrev 2022) | Parisi–Wu, Sci. Sinica 24 (1981). Zwanziger, Nucl. Phys. B 192 (1981). DeTurck, J. Diff. Geom. 18 (1983). Donaldson, PLMS 50 (1985). Gubinelli–Imkeller–Perkowski, Forum Math. Pi 3 (2015). Bruned–Hairer–Zambotti, Invent. Math. 215 (2019) |
| Lüscher's "suggest" in § 10 | Checked in the abstracts | M. Lüscher, JHEP 2010(8) 071 |
| Kramers' formula in § 01 | Standard result (original paper not checked) | An approximation to the mean time to cross a hill, fixed by the second derivatives at the bottom of the well and the top of the hill. The ratio to the $T$ of Table 1 is computed by `kramers.py` |
| Tables 1–3, the figures, the numbers in § 09 | Computed | Python on this machine: `langevin.py` (about 1 minute), `kramers.py` ($T$ and $T \cdot h/e^h$ for $h = 0.5$ to 8), `trees.py` (counting in rational numbers; checked against the 8 for $\phi^4$ at $d = 3$ and the borderline at $d = 22/7$; about 30 seconds), `fig2_pack.py`, `comedown.py`. The figures are canvas and plain JavaScript |

Next chapter to read: 7-10 Where four dimensions stops. Stochastic quantisation as a candidate for a new concept is looked at in 16-06.

# Where four dimensions stops — the ultraviolet is a prospect in finite volume; what was stuck was infinite volume and a uniform gap (the map before the 2026 publication)

This chapter is a map of open questions. § 01, § 03, § 06, § 08 and § 09 are established knowledge.

**What this chapter shows** — That the setters (Jaffe–Witten 2000) distinguished in their wording between the construction on $T^4$, a "prospect", and $T^4 \to \mathbb{R}^4$ and a uniform gap, for which "no present ideas point the direction". What has been reached in four dimensions at each of the six stages, and the words of the literature. Reading the clustering formula of the problem statement with the numbers of a free field.

**Prerequisite chapters** — 7-07 Four dimensions: Bałaban and MRS. The six stages are in 7-01, non-triviality in 7-08, the definition of the gap in 4-11.

**Said first** — This chapter rearranges the words of known literature step by step, and proves no theorem. In the literature before the 2026 publication, the construction of four-dimensional Yang–Mills is not complete, even on $T^4$. Literature after 2000 is limited to what we checked in the text or the abstract. For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet-limit step, the infinite-volume step along subsequences of volumes, and the axiom step are proved in Lean. Where things are stuck has moved to uniqueness of the infinite-volume limit and the uniform-gap step (6-04; in Volume 3, 16-02 and 16-03).

The order of this chapter

1. The setters' two sentences — the construction on $T^4$ has "ample indication"; for a uniform gap and $T^4 \to \mathbb{R}^4$ "no present ideas point the direction"
2. A map by stage — overlaying the six stages with what has been reached in four dimensions and the words of the literature (a moving figure)
3. Regularisation and finite volume: closed on the lattice — on the lattice it becomes a finite-dimensional integral, but rotation symmetry is broken

4. The ultraviolet-limit step: a "prospect" in finite volume — the closed form is the limit of renormalised Wilson loops; what remains is uniqueness and non-triviality
5. The infinite-volume step: it was stuck here — there is no step that crosses, independently of the volume, the scales where the coupling grows to $O(1)$
6. The axiom step: the axioms and the additional conditions of the problem statement — reflection positivity is inherited; the problem statement adds local fields, asymptotic freedom, non-triviality and $m < \infty$
7. The uniform-gap step: a gap uniform in the volume — the setters write that a uniform gap could open the road to infinite volume
8. Reading the clustering formula JW (2) — seeing $C < \Delta$ and "sufficiently far" with the numbers of a free field (computation, a moving figure)
9. Detour: a gap not visible classically — in Debye screening, a change of variables makes the mass visible

01

## The setters' two sentences

§ 6 of the problem statement (Jaffe–Witten 2000, JW below) divides the state of four dimensions into two sentences. Known

'Some results are known for Yang–Mills theory on a 4-torus $T^4$ approximating $R^4$, and, while the construction is not complete, there is ample indication that known methods could be extended to construct Yang–Mills theory on $T^4$.' / 'Yet, even if this were accomplished, no present ideas point the direction to establish the existence of a mass gap that is uniform in the volume. Nor do present methods suggest how to obtain the existence of the infinite volume limit $T^4 \to R^4$.'

The construction on the torus $T^4$ (a box with opposite faces glued together) is not complete, but there are ample signs that it can be made by extending known methods. For a gap uniform in the volume and for $T^4 \to \mathbb{R}^4$, no present idea points the direction. Writing $\mu_L$ for the measure on a box of side $L$ and $\Delta_L$ for the gap of the Hamiltonian, the objects are

$$\underbrace{\mu_L = \lim_{a \to 0} \mu_{a,L} \quad (L \text{ fixed})}_{\text{ample indication}} \qquad \underbrace{\inf_L \Delta_L > 0, \qquad \lim_{L \to \infty} \mu_L}_{\text{no ideas point the direction}} \tag{1}$$

($\mu_{a,L}$ is the measure at lattice spacing $a$; formula (6) of 7-01). **Example:** the setters distinguish the strength of their assessments in words (Table 1). This chapter maps that difference onto the stages.

Known Table 1 Words of strength in the problem statement (checked in the text of the Clay PDF).

| Words | Gloss | About what | Section |
|---|---|---|---|
| 'ample indication' | enough signs | The construction on $T^4$ | § 6 |
| 'no present ideas point the direction' | no idea points the way | A gap uniform in the volume | § 6 |
| 'New ideas are needed' | new ideas required | A gap uniform in the volume | § 6.5 |
| 'may play a fundamental role' | can be fundamental | Uniform gap → infinite-volume limit | § 5 |

02

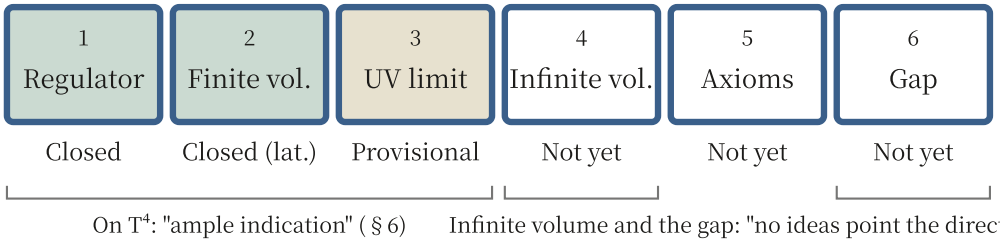
## A map by stage

We overlay the six steps of 7-01 with what has been reached in four dimensions and the words of the literature. What has been reached is one of three: **Closed** (there is a theorem), **Provisional** (a theorem under a hypothesis, or a sketch without a detailed proof) and **Not yet** (no theorem). The steps from the regularisation step to the ultraviolet-limit step are inside the double limit

$$\mu = \lim_{L\to\infty} \lim_{a\to 0} \mu_{a,L} \quad (2)$$

the infinite-volume step is outside it, and the axiom step and the uniform-gap step are properties of $\mu$. The first sentence before formula (1) points to the inside, the second to the outside and the uniform-gap step.

Figure 1 Choosing a work outlines the stages it mentions and shows the points (pressing a stage box or a row of Table 2 shows the words for that stage). "Play" cycles through the works




**Problem statement 2000**
Regulator step: Wilson defined a gauge-invariant action on the lattice (JW §6.5).
Finite vol. step: For a compact group and space-time, the lattice reduces the functional integral to a finite-dimensional integral (JW §6.5).
UV limit step: The construction on $T^4$ is not complete but there is ample indication (JW §6).
Infinite vol. step: Present methods do not suggest $T^4 \to R^4$ (JW §6).
Axioms step: Reflection positivity holds in the Wilson approximation, is hard to recover once lost, and setting up the Hilbert space is part of the problem (JW §6.5).
Gap step: For a gap uniform in the volume no present ideas point the direction, and new ideas are needed (JW §6, §6.5).

Known The sentences in this figure are the same as in Table 2 (from the texts of the problem statement, the report, the papers and the surveys). The brackets below show that the first sentence of §6 points to the steps from regularisation to the

ultraviolet limit, and the second to the infinite-volume step and the uniform-gap step.

Known Table 2 What has been reached at each step for four-dimensional Yang–Mills (as a continuum theory; from the literature before the 2026-11-10 publication; the published construction is in the paragraph after the table).

| Stage | Reached | Words of the literature (key points) | Chapter |
|---|---|---|---|
| 1 Regulator | Closed | Wilson defined a gauge-invariant action on the lattice (JW § 6.5) | 5-01 |
| 2 Finite vol. | Closed (lat.) | For a compact group and space-time, the lattice reduces the functional integral to a finite-dimensional integral (JW § 6.5). The continuum measure is not constructed on $T^3$ even in three dimensions (Chevyrev 2022) | 7-06 |
| 3 UV limit | Provisional | The construction on $T^4$ is not complete but there is ample indication (JW § 6). The ultraviolet limit is a sketch, with no detailed proof (MRS 1993). As far as Douglas knows there has been no breakthrough in recent years, and the state of the art is still Bałaban and MRS (Douglas 2004). There is no construction of a non-trivial continuum limit above two dimensions (Chatterjee 2018). In four dimensions, "a form of ultraviolet stability" (Chevyrev 2022) | 7-07 |
| 4 Infinite vol. | Not yet | Present methods do not suggest $T^4 \to R^4$ (JW § 6). The infrared cutoff is not removed; removing it makes the coupling large, and there is no solvable model to serve as a foothold (MRS 1993). Uniqueness at large β is generally unknown for d > 2 (Chatterjee 2018) | 5-04 |
| 5 Axioms | Not yet | Reflection positivity holds in the Wilson approximation, is hard to recover once lost, and setting up the Hilbert space is part of the problem (JW § 6.5). With an infrared cutoff the OS axioms cannot be treated as a whole (MRS 1993) | 4-10 |
| 6 Gap | Not yet | For a gap uniform in the volume no present ideas point the direction, and new ideas are needed (JW § 6, § 6.5). The lattice correlation-length problem (Chatterjee's Problem 5.1): finite correlation length at every β, infinite as $\beta \to \infty$ (Chatterjee 2018). | 5-09 |

Within what this chapter has read, the four-dimensional results after 2000 are of two kinds. One is on the side of a fixed lattice spacing ( § 05). The other is a special scaling limit of an $SU(2)$ model with a Higgs field added (one of the matter fields that couple to the gauge field; 3-07), taken by moving the lattice spacing, the coupling and the length of the Higgs field together (in any dimension $d \geq 2$). That limit is Gaussian (a free field), and the author writes 'The question of constructing a non-Gaussian scaling limit remains open' (Chatterjee 2026). Known Within what this chapter has read, there is no literature before the 2026 publication that claims to close the steps from the continuum ultraviolet-limit step to the uniform-gap step for pure Yang–Mills.

For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet-limit step (convergence along the whole sequence of spacings chosen by the family), the infinite-volume step along subsequences of volumes, and the axiom step (each component of the phases satisfies a variant of the OS axioms, from which a

Wightman theory is reconstructed) are proved in Lean. The uniform-gap step and uniqueness of the infinite-volume limit are not shown (6-01, 6-04; for reconstruction, 12-08).

03

## Regularisation and finite volume: closed on the lattice

The regularisation step and the finite-volume step are closed even in four dimensions, on the lattice. JW § 6.5 writes 'the lattice approximation reduces the functional integration to a finite-dimensional integral', and says that reflection positivity also holds (Osterwalder–Seiler 1978). Known In the notation of Chatterjee (2018), with $P$ the set of plaquettes (the smallest squares) and $U_p$ the product of group elements around $p$,

$$S(U) = \sum_{p \in P} \operatorname{Re}\operatorname{tr}\left(I - U_p\right), \qquad 0 \le \operatorname{Re}\operatorname{tr}(I - U_p) \le 2N, \qquad e^{-2N\beta|P|} \le Z = \int e^{-\beta S(U)}$$

This is because the eigenvalues of $U_p \in SU(N)$ have absolute value 1, and the Haar measure $\mathrm{d}U_\ell$ has total weight 1. **Example:** a four-dimensional periodic lattice with 4 points on a side has $6 \times 4^4 = 1536$ plaquettes, so for $SU(2)$, $e^{-6144\beta} \le Z \le 1$. Computed

In exchange, the lattice breaks rotation symmetry, and a continuum cutoff breaks gauge invariance (7-07 § 06). On reflection positivity, JW § 6.5 writes that if it is lost in regularisation there are few ways of recovering it (quoted in 0-03 § 03). Known A four-dimensional regularisation keeping all four of gauge invariance, reflection positivity, Euclidean invariance and ultraviolet control at once was not found in the literature this chapter read (this chapter's reading).

04

## The ultraviolet-limit step: a "prospect" in finite volume

We read the content in 7-07 (Bałaban 1989 is estimates with a hypothesis, MRS 1993 a sketch). The setters write in § 6.5: Known

'One presumably needs to revisit known results at a deep level, simplify the methods, and extend them.'

Review, simplify, extend. What is asked for at the ultraviolet-limit step is not new ideas but this work. The closed form can be written with Wilson loops $W_\gamma$ (the $\frac{1}{N}\operatorname{tr}$ of the product of group elements along a closed path $\gamma$; 5-01). For lattice paths $\gamma_a \to \gamma$

$$\langle W_{\gamma_a} \rangle^{\mathrm{ren}}_{a,L} \xrightarrow[a \to 0]{} \langle W_\gamma \rangle_L \qquad \text{(for the whole sequence, not a subsequence, to a limit that}$$

(ren is the renormalisation of the next paragraph; the form follows Chatterjee 2018 § 5, § 6).

Why renormalisation is needed can be seen in the continuum string-tension problem of Chatterjee 2018 (Problem 5.2; the continuum limit). On an infinite four-dimensional lattice, writing $\gamma_{R/a,T/a}$ for the rectangular path of physical size $R \times T$ ($R/a$ and $T/a$ steps on the lattice), the problem is to find sequences, as $\beta \to \infty$, such that

$$\log \left\langle W_{\gamma_{R/a,\,T/a}} \right\rangle = -c\,(R+T) - \sigma\, RT + o(1), \qquad a = a(\beta) \to 0, \quad c = c(\beta) \to \infty, \quad \sigma \neq$$

($\sigma$ is the string tension of the continuum theory; the original's symbol is $d$). $R+T$ is proportional to the perimeter, and $RT$ is the area. Since $c \to \infty$, as it stands $\log\langle W\rangle \to -\infty$, that is, the expectation collapses to 0. Adding back the term $c(R+T)$ proportional to the perimeter before taking the limit is the renormalisation here. Known But Chatterjee himself writes 'it is not clear to me whether the renormalization term c(R + T) is indeed necessary', so whether this term is really needed is also an open question.

**Example:** for the abelian $U(1)$ four-dimensional lattice there is a construction of the continuum limit (Driver 1987). Chatterjee 2018 § 6 describes Driver as using in four dimensions the method Gross showed for three-dimensional $U(1)$ as convergence in a sense different from the usual one. Known For an abelian group the terms of the lattice action higher than quadratic (self-interactions of the field) do not survive in the continuum limit, and the limit is expected to be the free electromagnetic field: that is this chapter's explanation (Driver's text not read).

For the non-abelian case there are two steps before reaching formula (4). (i) Ultraviolet stability only gives convergence of subsequences (Chatterjee 2018 § 6). Convergence of the whole sequence, that is, uniqueness of the limit, is separate work (7-07 § 05). (ii) Even if the limit exists, it may be Gaussian (a free field). Not being a free field, that is, non-triviality, is also separate work (7-08 § 10). Footnote 2 does not accept weak existence by compactness as a solution, unless the properties of the limit are shown separately (end of § 6.5). The committee's assessment is that the construction published on 2026-11-10, for correlations of local observables, gets past (i) with convergence along the whole sequence of spacings chosen by the family, and also shows the non-triviality of (ii). It does not, however, build the Wilson-loop observables of formula (4) themselves (6-04).

05

## The infinite-volume step: it was stuck here

The setters' words are a single sentence: 'Nor do present methods suggest how to obtain the existence of the infinite volume limit' ( § 6). From the inside, MRS 1993 § I says removing the infrared cutoff makes the coupling large, and writes 'there is no easy solvable model of these phenomenons around which to expand'. Known Laying out the sequence of renormalisation group steps (7-05) along the axis of scales,

$$\underbrace{a \to 2a \to \cdots \to a0}_{g^2 \le \gamma\text{: stage 3 (provisional)}} \quad \longrightarrow \quad \underbrace{a0 \to \cdots \to \ell_*}_{g^2 \text{ grows to } O(1)} \quad \longrightarrow \quad \underbrace{\ell_* \to \cdots \to L \to \infty}_{\text{stage 4}} \tag{5}$$

$a_0$ is the coarsest scale the hypothesis reaches ($g^2(a_0) \le \gamma$; as in 7-07, $\gamma$ is used as an upper bound on $g^2$), and $\ell_*$ is the rough-guide length where extending the running gives $g^2 = 1$; for $SU(2)$ with $\gamma = 0.3$, $\ell_*/a_0 \approx 10^{10.7}$ (two loops; Table 2 of 7-07 § 04). Computed Reading $\ell_*$ as the scale of confinement is a physics estimate. Physics In the organisation of this book, what is missing is the middle: one step that crosses the scales where the coupling grows to $O(1)$ with estimates independent of the volume (7-04 § 10). The construction published on 2026-11-10 takes the infinite-volume limit along subsequences of volumes from estimates independent of the volume (Part 12, 12-03, 12-05). Uniqueness of the limit and exponential decay uniform in the volume are not shown (6-04).

**Example:** with the lattice spacing fixed, the infinite-volume step is closed on the strong-coupling side (Osterwalder–Seiler 1978). Shen–Zhu–Zhu 2023 showed, for $SU(N)$ with $|\beta| < 1/(16(d-1))$ ($1/48 \approx 0.0208$ when $d = 4$), uniqueness of the infinite-volume measure and exponential decay of correlations. This $\beta$ is the value in 't Hooft's convention. In this convention the inverse coupling in front of the action is written $\beta N$, so converting to the $\beta$ of formula (3) multiplies it by $N$, and the condition is $|\beta| < N/48$ ($N^2/48$ in this book's Wilson $\beta$; the conversions among the five conventions are in Table 4 of 5-01). On the weak-coupling side even uniqueness is generally unknown for non-abelian groups (Chatterjee 2018 § 3), and exponential decay has been shown only for finite groups (Adhikari–Cao 2025, 5-10). $U(1)$ has a non-confining phase at large $\beta$ (Guth 1980, Fröhlich–Spencer 1982), and that there is no phase transition in the non-abelian case is only believed Physics (5-06).

06

## The axiom step: the axioms and the additional conditions of the problem statement

The axiom step is the Osterwalder–Schrader axioms (4-10). Reflection positivity holds at each lattice spacing, and being an inequality it is inherited by the limit (7-01 § 07).

Recovering rotation symmetry is separate work, and setting up the Hilbert space is itself part of the problem (JW § 6.5). Known How the published construction satisfies these is read in Part 12 (12-01 Lattice reflection positivity, 12-06 Euclidean invariance, 12-08 Reconstruction and the Wightman axioms).

§ 4 adds conditions. (i) Local fields corresponding to gauge-invariant local polynomials in the curvature such as $\mathrm{Tr}\, F_{ij}F_{kl}(x)$ (footnote 1: there is no natural one-to-one correspondence, because of renormalisation). (ii) Agreement at short distances with the predictions of asymptotic freedom and perturbative renormalisation (including the stress tensor and the operator product expansion). (iii) Non-triviality (7-08). And

$$\mathrm{spec}\, H \cap (0, \Delta) = \varnothing \quad (\Delta > 0), \qquad m = \sup\{\Delta\}, \qquad m < \infty \ (6)$$

($H \geq 0$ is the Hamiltonian, $m$ the mass). **Example:** since $H \geq 0$, $m < \infty$ is the same as the spectrum of $H$ containing a positive point, that is, $H \neq 0$. If $m = \infty$, there is no spectrum in $(0, \infty)$, and $H = 0$. Further, using the form that states uniqueness of the vacuum with respect to translations (exactly one translation-invariant unit vector; the wording of Axiom 3 in nLab's "Wightman axioms"), the spectral condition (energy and momentum lie in the forward cone) makes the momentum 0 as well, every vector becomes translation invariant, and the Hilbert space consists of the vacuum alone. Under axioms of this form, $m < \infty$ can be read as a condition excluding the empty theory (this chapter's reading). The summary in § 3 of the problem statement takes the form "a Poincaré-invariant vacuum vector unique up to a phase", from which the same conclusion does not follow immediately. The example that fails (iii) is four-dimensional $\phi^4$, whose limit from the lattice is Gaussian (Aizenman–Duminil-Copin 2021).

07

## The uniform-gap step: a gap uniform in the volume

On a lattice in a finite box the gap exists automatically (7-01 § 08), and the question is its independence of the side $L$. JW § 6.5 writes 'New ideas are needed to prove the existence of a mass gap that is uniform in the volume of space-time. Such a result presumably would enable the study of the limit as $\mathrm{T}^4 \to \mathrm{R}^{4}$' ( § 5 points the same way; Table 1). Known

The setters write that a uniform gap in finite volume (the uniform-gap step) could open the road to infinite volume (the infinite-volume step) (the 'may play a fundamental role' of § 5, the 'presumably would enable' of § 6.5). In 7-01 the infinite-volume step was placed first because the gap is defined for the theory on $\mathbb{R}^4$, not because of the order of proof. The mechanism is a one-line estimate. For the box

Hamiltonian $H_L$, vacuum $\Omega_L$ and gap $\Delta_L$, separating the vacuum component from the rest by the spectral decomposition,

$$\left|\langle \Omega_L,\, A\, e^{-tH_L} B\, \Omega_L\rangle - \langle \Omega_L, A\,\Omega_L\rangle\langle \Omega_L, B\,\Omega_L\rangle\right| \;\le\; \|A^*\Omega_L\|\,\|B\,\Omega_L\|\, e^{-\Delta_L t} \tag{7}$$

(correlations separated by imaginary time $t$), and if $\Delta_L \ge \Delta_* > 0$ independently of $L$, the rate of decay does not depend on $L$ either. Known **Example:** on the strip of the two-dimensional Ising model, $E_1 - E_0$ closes like $1/L$ at the critical point and closes exponentially at low temperature (2-09 § 07, § 08). For weakly coupled $\phi^4_2$ and $\phi^4_3$ the expansion in a small parameter gave the infinite-volume step and the uniform-gap step at once (7-04 § 08), but the long distances of four-dimensional Yang–Mills have no such parameter.

08

## Reading the clustering formula JW (2)

JW § 5 writes, as a consequence of the gap, 'For any positive constant C < Δ and for any local quantum field operator O … such that ⟨Ω, OΩ⟩ = 0, one has (2) |⟨Ω, O(x)O(y)Ω⟩| ≤ exp(−C|x − y|), as long as |x − y| is sufficiently large' ($x, y$ are points of space. Below this inequality is written JW (2), to distinguish it from formula (2) of this chapter). Known The point is $C < \Delta$ and a "sufficiently large" distance. The right-hand side needs no constant because $C$ is taken strictly smaller than $\Delta$, so constant factors and power factors alike lose, far enough away, to $e^{-(\Delta - C)r}$. The equal-time correlation of a free scalar field of mass $\Delta$ is

$$\langle \Omega, \phi(\underset{\sim}{x})\phi(\underset{\sim}{y})\Omega\rangle = \frac{\Delta\, K_1(\Delta r)}{4\pi^2 r} \;\sim\; \frac{\sqrt{\pi/2}}{4\pi^2}\,\frac{\Delta^{1/2}}{r^{3/2}}\, e^{-\Delta r} \qquad (r = |\underset{\sim}{x} - \underset{\sim}{y}| \to \infty) \tag{8}$$

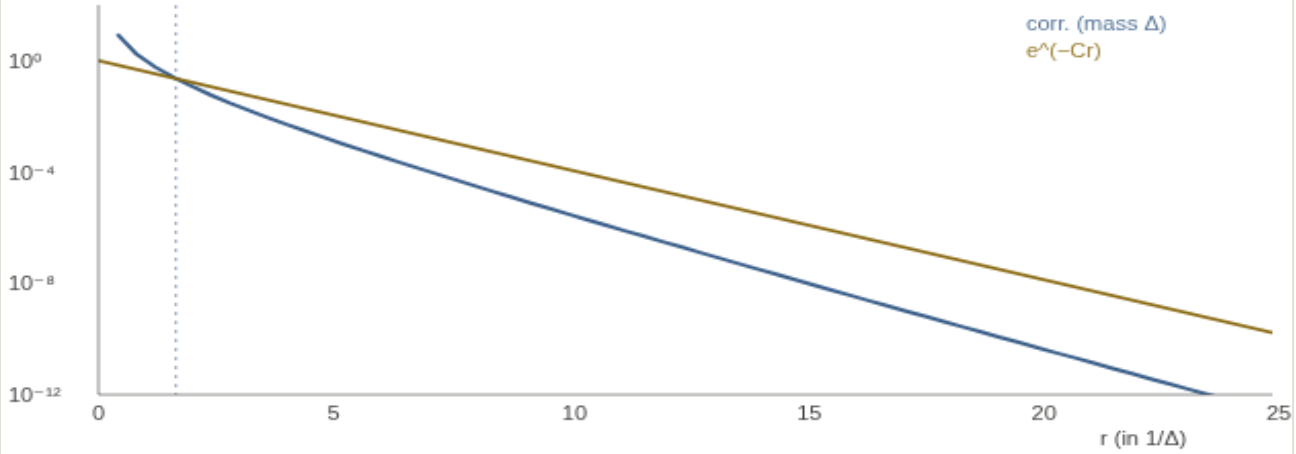
which has the same form as the Euclidean two-point function ($K_1$ is the modified Bessel function of the second kind. The point field is a stand-in for a local operator; smearing changes the form far away only by a constant factor). Normalising to 1 at $r = 1/\Delta$, we compare with $e^{-Cr}$, and also set mass 0 (correlation proportional to $1/r^2$) beside it.

Computed Table 3 The range of $r$ in which the normalised correlation is at most $e^{-Cr}$ ($r$ in units of $1/\Delta$, examined for $r \le 5000$).

| $C/\Delta$ | Mass $\Delta$ | Mass 0 |
|---|---|---|
| 0.10 | $r \ge 1.04$ | $1.05 \le r \le 90$ |
| 0.50 | $r \ge 1.25$ | $1.43 \le r \le 8.6$ |
| 0.90 | $r \ge 1.65$ | never holds |

| $C/\Delta$ | Mass $\Delta$ | Mass 0 |
|---|---|---|
| 1.00 | $r \geq 1.82$ | never holds |
| 1.05 | $1.93 \leq r \leq 132$ | never holds |
| 1.20 | $2.47 \leq r \leq 17.8$ | never holds |

Figure 2 Moving C/Δ changes the distance at which the line $e^{-Cr}$ and the correlation (logarithmic vertical axis) swap places. "Compare with Δ = 0" overlays the massless correlation




C/Δ = 0.90: with mass Δ, r ≥ 1.65 is where it holds

Computed The numbers in this figure were computed on this machine (formula (8). The vertical dotted lines are the distances where JW (2) starts to hold and starts to fail, for mass Δ).

How to read it: (a) for $C < \Delta$ it holds beyond $r$ of about 2. (b) For $C = \Delta$ it still holds thanks to the power $r^{-3/2}$, but the problem statement only speaks of $C < \Delta$. (c) For $C > \Delta$ it always fails far away. (d) For mass 0, every $C > 0$ fails far away. The slope far away, $-\frac{\mathrm{d}}{\mathrm{d}r}\log$, is $1.153\Delta$ at $r = 10$ and $1.015\Delta$ at $r = 100$, approaching $\Delta$ from above.

Computed Lattice measurements sum over space to extract the zero-momentum component, remove the same kind of power factor from correlations in the time direction, and look at sums of exponentials $e^{-mt}$ (5-08). For a free field, the correlation in the time direction obtained by integrating the Euclidean two-point function over space is exactly $e^{-\Delta|t|}/(2\Delta)$. Known To cross from the uniform-gap step to the infinite-volume step, JW (2) and formula (7) must hold independently of the size of the box.

09

## Detour: a gap not visible classically

A section you may skip.

JW § 6.6 writes '... a theory in which the mass gap is not classically visible, to demonstrate it may require a non-classical change of variables or "duality transformation."' Linearising the

classical Yang–Mills equations for small fields gives the wave equation, with frequency $\omega(k) = |k|$: massless (3-03). The example given is Debye screening in the Coulomb gas (a gas of positive and negative point charges exerting on each other a potential $1/r$ inversely proportional to the distance $r$). That the force between test charges decreases exponentially is shown by rewriting the partition function as that of sine-Gordon field theory (the theory of a scalar field $\phi$ with a $\cos(\lambda\phi)$ term in the action) and approximating the bottom of the well by a parabola (Brydges–Federbush 1980). Known

$$-z\cos(\lambda\phi) = -z + \tfrac{1}{2}\, z\lambda^2\, \phi^2 + O(\phi^4) \quad (9)$$

The coefficient $z\lambda^2$ of $\phi^2$ plays the role of the mass squared ($z$ is the activity = how readily particles appear; with chemical potential $\mu$ and temperature $T$, $z \propto e^{\mu/k\mathrm{B}T}$), and a mass invisible in the language of $1/r$ becomes visible as the curvature of the well. Precisely, (9) is an approximate form, and the proof estimates the conditions under which it works. The same section also lists the quartic term $(A \wedge A)^2$ (Feynman 1981), bound states in quantum mechanics with flat directions (Simon 1983; 2-05), and large-$N$ duality (17-04).

## What this chapter can say

| | Content |
|---|---|
| **Said** | The sentences of the problem statement § 3– § 6.6 and its footnotes, the 2004 report (text of the Clay PDF), MRS 1993, Chatterjee 2018 (Problems 5.1 and 5.2, § 6) and Chevyrev 2022. Shen–Zhu–Zhu and Adhikari–Cao from the text of § 1 (formulas (1.2)(1.3), introduction), Chatterjee 2026 from the sentences of the abstract Known |
| **Said** | In the literature before the 2026 publication, in the four-dimensional continuum, the ultraviolet-limit step is provisional (with a hypothesis, or a sketch) in finite volume, and the infinite-volume, axiom and uniform-gap steps are not yet. With the lattice spacing fixed, the infinite-volume step and the uniform-gap step are closed on the strong-coupling side. The one-line estimates of formulas (3) and (7) Known |
| **Said** | JW (2) read with a free field: Table 3, Figure 2 and the slope far away. $\ell_*/a_0 \approx 10^{10.7}$ (the computation of 7-07) Computed |
| **Not said** | That the road from a uniform gap to the infinite-volume limit actually goes through for four-dimensional Yang–Mills (the setters too write 'may', 'presumably') |
| **Not said** | "The missing step", reading $m < \infty$ as a condition excluding the empty theory (depends on the form of uniqueness of the vacuum), the view that no regularisation keeps all four (this book's readings). That the limit of Driver 1987 is the free electromagnetic field (text not read). Complete coverage of the literature after 2000 (Douglas's 2026 survey from bibliographic data only). For none of the steps from the ultraviolet-limit step to the uniform-gap step in four dimensions does this book show anything of its own (the reading of the published construction is in Volume 2) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Problem statement (§ 4–§ 6.6, footnotes 1 and 2, reference list) | Text | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) |
| The 2004 report | Text | M. R. Douglas, *Report on the Status of the Yang–Mills Millenium Prize Problem*, Clay Mathematics Institute (2004) |
| Reservations on the infrared cutoff and the axioms | Text pp. 325–327 | J. Magnen, V. Rivasseau, R. Sénéor, Comm. Math. Phys. 155 (1993) 325–383 |
| The action, uniqueness, Problems 5.1 and 5.2, ultraviolet stability and the description of Driver (§ 6) | Text | S. Chatterjee, *Yang–Mills for probabilists*, arXiv:1803.01950 (2018) |
| The form stating uniqueness of the vacuum with respect to translations (§ 06) | Text | nLab, *Wightman axioms* (Axiom 2, Axiom 3) |
| Ultraviolet stability, three dimensions | Text § 1.1, § 4 | I. Chevyrev, arXiv:2202.13359, J. Math. Phys. 63 (2022) 091101 |
| The theorem of Bałaban 1989 | Abstract and introduction (7-07) | T. Bałaban, Comm. Math. Phys. 122 (1989) 355–392 |
| Results at strong coupling and for finite groups | Text § 1 (formulas (1.2)(1.3), introduction; cross-checked in 16-01) | H. Shen, R. Zhu, X. Zhu, arXiv:2204.12737, CMP 400 (2023) 805–851 / A. Adhikari, S. Cao, arXiv:2202.10375, Ann. Probab. 53 (2025) 140–174 |
| The result for Yang–Mills–Higgs | Abstract only | S. Chatterjee, arXiv:2401.10507, Probab. Math. Phys. 7 (2026) 339–381 |
| Other literature in § 03–§ 06 and § 09 | Bibliographic data only | Osterwalder–Seiler, Ann. Phys. 110 (1978) 440–471 / Guth, Phys. Rev. D 21 (1980) 2291–2307 / Fröhlich–Spencer, CMP 83 (1982) 411–454 / Driver, CMP 110 (1987) 479–501 / Aizenman–Duminil-Copin, Ann. of Math. 194 (2021) 163–235 / Brydges–Federbush, CMP 73 (1980) 197–246 / Feynman, Nucl. Phys. B188 (1981) 479–512 / Simon, Ann. Phys. 146 (1983) 209–220 |
| Douglas's 2026 survey (the table of scope) | Bibliographic data only | M. R. Douglas, Nature Reviews Physics 8 (2026) 86–97 (bibliographic data from search results) |

| Item | Kind | Source / tools |
|---|---|---|
| Table 3, Figure 2, the slope far away, the example of formula (3) | Computed | Python on this machine (SciPy's $K_1$; agrees with the integral representation via the heat kernel to a relative difference within $2 \times 10^{-8}$. Step $10^{-4}$ for $r \le 50$, step $10^{-2}$ for $50 < r \le 5000$). The example of formula (3) is arithmetic |
| $\ell_*/a_0 \approx 10^{10.7}$ (§ 05) | Computed | The computation of 7-07 (Table 2; $SU(2)$, $\gamma = 0.3$, two loops) |

Next chapter to read: 7-11 Obstacles other than divergence, which lists the reasons the infinite-volume, axiom and uniform-gap steps were stuck as obstacles distinct from divergence. A chronology and a table by dimension are in 16-01.

# Obstacles other than divergence — problems remain even on a lattice that contains not a single infinity

> This chapter organises established knowledge. § 02 and § 08 are a map of open questions.

**What this chapter shows** — Six obstacles that remain even on a lattice with the ultraviolet divergences removed, and how much of each is a problem of divergence. Seeing three of them by computing the smallest examples (orders of perturbation theory and $e^{-c/\alpha}$, a sum of signs equal to 0 in gauge fixing, a vacuum overlap that goes to 0 through volume alone).

**Prerequisite chapters** — 7-10 Where four dimensions stops. The tools are 2-06 (Fock space), 3-04 (the gauge orbit space), 4-08 (gauge fixing), 4-10 (reflection positivity) and 5-01 (the lattice).

**Said first** — This chapter rearranges obstacles found in the literature, and removes none of them. Counting them as six, and judging whether each is a divergence, is this book's organisation. The computations are smallest examples, not computations for four-dimensional Yang–Mills.

The order of this chapter

1. The test — fix the lattice spacing
2. No small parameter in the infrared — change the group
3. Invisible to perturbation theory — Table 1, Figure 1
4. Not collapsing to a free field — triviality and 0/0
5. The state space is curved — the part reducible to divergence, and the 0/0 that remains (Table 2)
6. Reflection positivity — an example where it fails
7. The structure of infinitely many degrees of freedom — vacuum overlaps (Figure 2)
8. The six side by side — Table 4
9. Detour: topological sectors and θ

01

## The test: does it remain with divergence removed?

The explanation that "the difficulty is handling infinities" is one side of it. This chapter puts each obstacle to the following test. **Does the obstacle remain even with the lattice spacing $a$ kept fixed (with the ultraviolet cutoff left in place)?** If it remains, no handling of divergences will dispose of it.

Divergences were quantities that go to infinity when the cutoff is removed, like the variance of the free field at a point, $G_a(0) \approx 0.1549/a^2$ (in four dimensions) (7-01 § 02). Lattice gauge theory in a finite box with fixed lattice spacing is a finite-dimensional integral.

$$\langle F \rangle_{\beta,\Lambda} = \frac{1}{Z_{\beta,\Lambda}} \int F(U)\, e^{-\beta S(U)} \prod_{\ell \in \Lambda} \mathrm{d}U_\ell, \qquad 0 < Z_{\beta,\Lambda} \leq 1 \quad (1)$$

$\Lambda$ is the set of edges of the box, $\mathrm{d}U_\ell$ the Haar measure of total weight 1, $S$ the Wilson action, and $\beta$ the inverse lattice coupling (not an inverse temperature; large $\beta$ is weak coupling) (formula (3) of 7-10). There is not a single infinity.

**Example:** even so, for the theory with only the box extended to $\mathbb{Z}^4$, Chatterjee's talk slides (no date given; the PDF was created in September 2020) write 'Even showing any kind of correlation decay at large β for non-Abelian theories in 3D and 4D is open.' Known Since then, Adhikari–Cao have shown decay of correlations at weak coupling for finite groups. Known Within the literature this chapter has read, what is open now is the case of continuous non-abelian Lie groups ($SU(N)$ and so on). The results are summarised in Table 4.

02

## No small parameter in the infrared

Many mathematical estimates are expansions in a small number, but the coupling of four-dimensional Yang–Mills is weak at short distances (asymptotic freedom) and strong at long distances, and no foothold for an expansion is found on the infrared side (7-10 § 05). The small $\beta$ (strong coupling) where the expansion converges is the shore opposite to the continuum limit $\beta \to \infty$. More precisely, the range of $\beta$ where the strong-coupling expansion converges is limited to near 0, and does not overlap with $\beta \to \infty$, which the continuum limit needs.

The target on the lattice is the lattice correlation-length problem of Chatterjee (2018) (Problem 5.1): for the length $\xi(\beta)$ (in lattice units) over which correlations of gauge-

invariant local quantities decrease like $e^{-|x-y|/\xi}$,

$$0 < \xi(\beta) < \infty \quad \text{(for all } \beta > 0), \qquad \lim_{\beta \to \infty} \xi(\beta) = \infty \quad (2)$$

is to be shown (four dimensions, compact non-abelian Lie group). Known The first half is that there is a gap at every lattice spacing; the second half is the prospect of a continuum limit (5-09).

**Example:** changing the group breaks (2). $U(1)$ enters a massless phase at large $\beta$, and the area law fails (Guth 1980; a complete proof is Fröhlich–Spencer 1982). Known For finite groups correlations decrease exponentially at weak coupling (Adhikari–Cao) Known , but they are thought not to confine and to have no continuum limit. Physics $SU(N)$ is believed to be in the confining phase at every $\beta$. Physics

An argument that works equally for every group goes wrong for $U(1)$, so what ought to work is "for a continuous non-abelian group, the gap survives even with the transition point at $\beta = \infty$" (5-06, 5-10). What is missing is an estimate that crosses, independently of the volume, the scales where the coupling grows to $O(1)$.

03

## Invisible to perturbation theory

In the physics picture, the mass gap is a constant multiple of the single scale $\Lambda$ of the theory (4-07), and at one loop in the renormalisation group it is related to the coupling $g$ at scale $\mu$ as follows. Physics

$$\frac{\Lambda}{\mu} = \exp\Big(-\frac{1}{2b_0 g^2}\Big) = \exp\Big(-\frac{cN}{\alpha}\Big), \qquad b_0 = \frac{11N}{48\pi^2}, \quad \alpha = \frac{g^2}{4\pi}, \quad cN = \frac{6\pi}{11N} \quad (3)$$

$b_0$ is the coefficient of the running of the coupling $\mu\,\mathrm{d}g/\mathrm{d}\mu = -b_0 g^3 + \cdots$ (4-06). The right-hand side approaches 0 faster than any power.

$$\lim_{\alpha \to 0+} \alpha^{-n}\, e^{-c/\alpha} = 0 \qquad (n = 0, 1, 2, \dots) \quad (4)$$

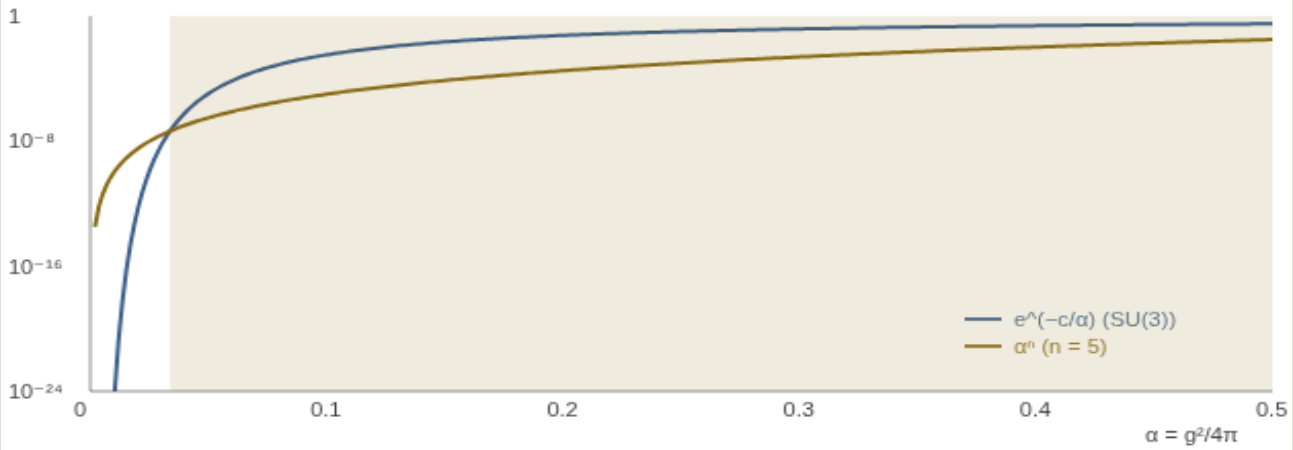
So all Taylor coefficients at $\alpha = 0$ are 0, and perturbation theory answers $\Lambda = 0$ at every order (for quantum mechanics, 2-06). Known

**Example:** the order $n_* = cN/\big(\alpha \ln(1/\alpha)\big)$ at which $\alpha^n$ equals $e^{-cN/\alpha}$ grows as the coupling gets smaller.

Computed Table 1 $e^{-cN/\alpha}$ and the order $n_*$ ($c_2 = 0.8568$, $c_3 = 0.5712$).

| $\alpha$ | $SU(2)$ | $n_*$ | $SU(3)$ | $n_*$ |
|---|---|---|---|---|
| 0.30 | 5.75e−2 | 2.37 | 1.49e−1 | 1.58 |
| 0.10 | 1.90e−4 | 3.72 | 3.31e−3 | 2.48 |
| 0.05 | 3.61e−8 | 5.72 | 1.09e−5 | 3.81 |
| 0.02 | 2.48e−19 | 10.95 | 3.95e−13 | 7.30 |
| 0.01 | 6.16e−38 | 18.61 | 1.56e−25 | 12.40 |

Figure 1 Moving $n$ moves the band where $e^{-c/\alpha}$ exceeds $\alpha^n$ (vertical axis in common logarithms). To the left of the band there always remains a range where $\alpha^n$ is larger




SU(3), n = 5: 0.0337 < α < 0.878 is the only range where e^(−c/α) exceeds αⁿ. For α < 0.0337, αⁿ is larger

Computed The numbers in this figure were computed on this machine (formula (3) and bisection for the crossing points).

What is missing is not the value of an infinity but information that does not appear in the series. Divergence of the series is a separate phenomenon. The problem statement (Jaffe–Witten 2000, JW below) writes in § 6.1 that perturbation series are in many examples 'divergent expansions'. Beneke (1999) writes 'The sum of a convergent series may still differ from R by exponentially small terms exp(−1/α)'. Known Even a convergent series can miss $e^{-c/\alpha}$ (4-12, 7-12).

04

## Not collapsing to a free field

Even if divergences are handled and a limit is taken, a free field (a Gaussian measure) is not an answer (JW § 4 asks for a non-trivial theory). From the two- and four-point correlations $S_2$, $S_4$ of a field with mean 0, form

$$u_4(x_1,x_2,x_3,x_4) = S_4 - \big[S_2(x_1,x_2)S_2(x_3,x_4) + S_2(x_1,x_3)S_2(x_2,x_4) + S_2(x_1,x_4)S_2(x_2,$$

and for a Gaussian $u_4 \equiv 0$ (Wick's rule; 7-08 § 01).

**Example:** for four-dimensional $\phi^4$, the infinite-volume scaling limit with the ultraviolet cut on a lattice and approaching the critical point is Gaussian (Aizenman–Duminil-Copin 2021). Known The divergences have been handled, and still the limit is a free field. A special scaling limit of $SU(2)$ Yang–Mills–Higgs is also Gaussian, and a non-Gaussian limit is said to 'remain open' (Chatterjee 2026). Known

What to hold fixed is a problem of the same kind. The mass in physical units, $m_{\text{lat}}(\beta)/a(\beta)$, has the form 0/0, and what has meaning is the limit of dimensionless ratios such as $m/\sqrt{\sigma}$, that is, ratios of rates of approach to 0 (5-09). The requirement is in 7-08 § 10.

05

## The state space is curved

The place to put the measure is the orbit space, the space of connections divided by gauge transformations, which is not a linear space (3-04). An orbit is a set of connections carried into one another by gauge transformations. Singer (1978) showed that on $S^4$ ($\mathbb{R}^4$ with one point at infinity added), for non-abelian compact groups, one cannot continuously choose exactly one connection from each orbit ('No gauge fixing is possible.'). Known "Curved" in the title is a metaphor; precisely, it refers to these two things (not being a linear space, and representatives not being choosable globally and continuously).

### The part reducible to divergence

With the formal $\mathrm{d}A$ (meant as Lebesgue measure in infinitely many dimensions), $Z = \infty$ (Chatterjee 2018), and the volume of the orbits is also expected to be infinite (Singer 1978). The lattice disposes of this. Formula (1) is finite without fixing the gauge.

### The 0/0 that remains

What remains is when gauge fixing is needed in weak-coupling analysis. Bałaban uses local gauges that change from cell to cell (7-05 § 05). Fixing a single covariant (Landau-type) gauge on the whole lattice BRST-symmetrically produces the following 0/0 (Neuberger 1987; the formula is (2) of Mehta–Daleo–Hauenstein–Seaton 2014). Known We define the words used, one line each (details in 4-08).

- **Landau-type gauge**: a fixing that builds a function $F$ (a function of the gauge transformation) measuring how far the transformed edge variables are from 1, and chooses its critical points as representatives of the orbit. The $F$ of the example below has this form.

- **Gribov copies**: when there are several critical points of $F$ on one orbit, each one of them.
- **Faddeev–Popov matrix** $M_{\mathrm{FP}}$: the matrix of second derivatives of $F$ at a critical point. The gauge-fixed integral has $\det M_{\mathrm{FP}}$ as its weight.
- **BRST-symmetric fixing**: a way of fixing so that a formal symmetry (BRST) remains after gauge fixing. Then the contribution of one orbit is the sum of $\operatorname{sign}\det M_{\mathrm{FP}}$ over all the copies.
- **Euler characteristic** $\chi$: the number obtained by adding up the critical points of a smooth function (with non-degenerate critical points) on a compact manifold without boundary (1-05), each with the sign of the determinant of the matrix of second derivatives ($+1$ if the number of negative eigenvalues is even, $-1$ if odd). It does not depend on the choice of function (the Poincaré–Hopf theorem). For a surface it is "the number of minima and maxima − the number of saddles", and for the sphere it is 2. On a circle a minimum counts $+1$ and a maximum $-1$, so it is 0.

$$Z_{\mathrm{GF}} = \sum_{\text{Gribov copies}} \operatorname{sign}\det M_{\mathrm{FP}} = \chi(G)^N = 0 \tag{6}$$

$N$ is the number of lattice points, and a gauge transformation places one element of the group $G$ at each point, so the space moved over is the product of $N$ copies of $G$. The sum of signs is its Euler characteristic $\chi(G)^N$, and for compact connected Lie groups of positive dimension $\chi(G) = 0$. Gauge-invariant quantities are constant on orbits, so for gauge-fixed expectation values both numerator and denominator become 0: 0/0.

**Example:** for a smooth function on the circle $S^1 = U(1)$, maxima and minima alternate and the sum of signs is $\chi(S^1) = 0$. On a ring of $N$ points we counted all the critical points of $F = \sum_j \left(1 - \cos(\phi_j + \theta_{j+1} - \theta_j)\right)$ in formula (3) of Mehta et al. ($\theta_j$ is the angle of the gauge transformation at point $j$, with $\theta_0 = 0$ to remove the freedom of rotating everything uniformly; $\phi_j$ are the edge angles, uniform random numbers, 300 orbits).

Computed Table 2 Gauge fixing on a $U(1)$ ring.

| $N$ | Number of copies | Sum of signs | Number of minima |
|---|---|---|---|
| 2 | 2 | 0 | 1 |
| 3 | 6 | 0 | 1–2 |
| 4 | 12 | 0 | 2 |
| 8 | 280 | 0 | 4 |
| 12 | 5544 | 0 | 6 |

The number of copies was the same on every orbit examined, growing exponentially as $\frac{N}{2}\binom{N}{N/2}$ for even $N$, and the sum of signs was always $0$. Computed Restricting to minima avoids $0/0$, but that the number of minima can vary from orbit to orbit ($N = 3$) is another difficulty (Mehta et al. § I). The $0$ comes from topology. Singer himself named as a way round it localisation, gluing local covers together with a partition of unity (4-08, 7-12).

06

## Few regularisations keep reflection positivity

The road back from Euclidean to space-time depends on reflection positivity $\langle \Theta F\, F\rangle \geq 0$ ($\Theta$ is time reflection, $F$ a quantity on the positive-time side) (4-10). The Wilson action keeps it (Osterwalder–Seiler 1978) and breaks rotation symmetry. Continuum regularisations are the reverse: the problem statement writes that there are few ways to recover reflection positivity lost in regularisation, giving dimensional regularisation and Pauli–Villars as examples (JW § 6.5; quoted in 0-03 § 03). Known

**Example:** fixing the spatial momentum $p$ of a free field, the time dependence of the two-point function is $G(t) = e^{-\omega|t|}/(2\omega)$ ($\omega = \sqrt{p^2 + m^2}$). For a Gaussian field, reflection positivity is, for positive times $t_i$ and real numbers $f_i$,

$$Q(f) = \sum_{i,j} f_i f_j\, G(t_i + t_j) \;\geq\; 0 \tag{7}$$

If $G$ is a sum of exponentials with positive weights $\rho(E)$, then $Q = \sum_E \rho(E)\big(\sum_i f_i e^{-Et_i}\big)^2 \geq 0$. Pauli–Villars subtracts a term with a heavy mass $M$, giving $G_{\mathrm{PV}}(t) = e^{-\omega t}/(2\omega) - e^{-\Omega t}/(2\Omega)$ ($\Omega = \sqrt{p^2 + M^2}$), so for $f$ with $\sum_i f_i e^{-\omega t i} = 0$, $Q = -\frac{1}{2\Omega}\big(\sum_i f_i e^{-\Omega t_i}\big)^2 < 0$. With $p = 0$, $m = 1$, $M = 5$, $t = (0.1, 0.2)$ and $f = (1, -e^{0.1})$, $Q = -4.0 \times 10^{-3}$. Computed As $M \to \infty$ the violation shrinks, but it cannot be used at the stage where the cutoff is kept.

That Wilson loops must be smoothed by the heat flow above two dimensions (Cao–Chatterjee) is an ultraviolet problem, but regularisations that keep positivity, recovery of rotation symmetry, and putting observables valued on curves rather than points (Chatterjee 2018) into the axioms are problems of structure. Known

07

## The structure of infinitely many degrees of freedom

With finitely many oscillators the irreducible representation of the canonical commutation relations is essentially unique (Stone–von Neumann; 2-06), but with

infinitely many, countless inequivalent representations appear. Known One consequence is Haag's theorem, by which an interacting theory cannot be put on the Fock space of the free field (in the interaction picture) (Haag 1955, Hall–Wightman 1957; 4-09).

The mechanism can be seen without infinities. The overlap of the ground states of oscillators with frequencies $\omega \neq \omega'$ is $\left(2\sqrt{\omega\omega'}/(\omega+\omega')\right)^{1/2} < 1$, and the overlap of the vacua of free fields of masses $m$ and $m'$ on a three-dimensional periodic lattice with $L$ points on a side and spacing $a$ is the product over modes.

$$\langle \Omega_m, \Omega_{m'} \rangle = \prod_{k} \left( \frac{2\sqrt{\omega_k \omega_k'}}{\omega_k + \omega_k'} \right)^{1/2}, \qquad (a\omega_k)^2 = (am)^2 + 4\sum_{j=1}^{3} \sin^2 \frac{\pi n_j}{L} \tag{8}$$

The product runs over the $L^3$ modes $k = (n_1, n_2, n_3)$, $n_j = 0, \dots, L-1$ ($\omega_k'$ is the same with $m'$). The logarithm for each mode is negative.

**Example:** we compare two limits. A keeps the lattice spacing fixed ($am = 0.5$) and enlarges the box; B keeps the box fixed (side $2/m$) and refines the lattice. Both increase the number of points on a side, $L$.

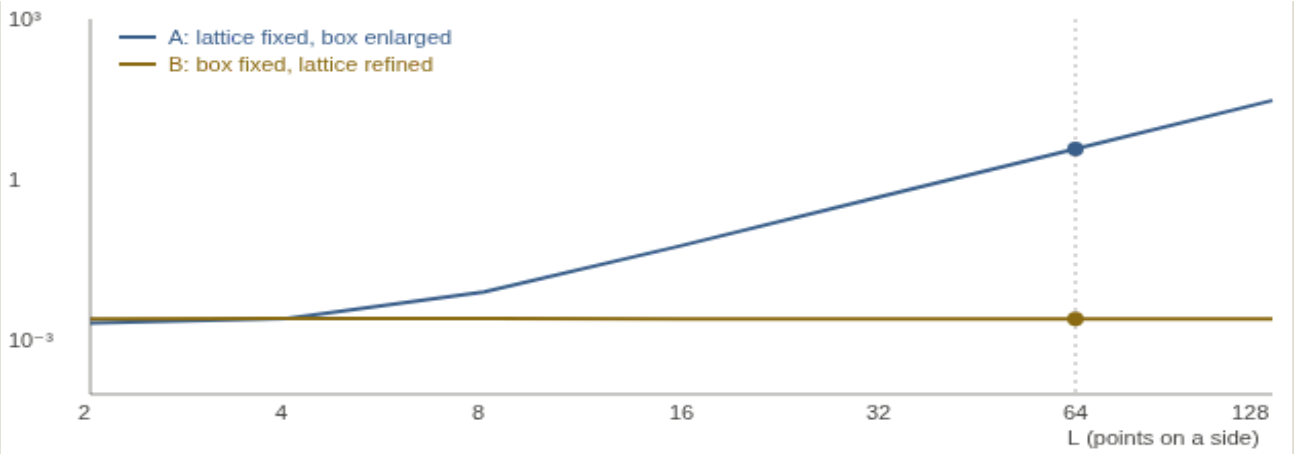
Computed Table 3 The vacuum overlap $\langle \Omega_m, \Omega_{m'} \rangle$ for $m' = 1.2m$.

| $L$ | Overlap in A (box side $0.5L/m$) | Overlap in B (lattice spacing $2/(Lm)$) |
|---|---|---|
| 4 | 0.9974 | 0.99741 |
| 32 | 0.6237 | 0.99745 |
| 64 | 0.02289 | 0.99746 |
| 128 | 7.53e−14 | 0.99747 |

In A, $-\ln\langle \Omega_m, \Omega_{m'} \rangle$ is proportional to $L^3$ ($1.44 \times 10^{-5}$ per point), and the overlap goes to 0. Computed

The reason the overlap settles to a positive value in B lies in the high-momentum modes. When $\omega_k$ is large, the contribution of one mode to the logarithm is about $-(m'^2 - m^2)^2/(64\omega_k^4)$, and $\omega_k$ grows nearly in proportion to $|k|$, so the whole is a sum of the form $\sum_k |k|^{-4}$. In three dimensions the shell $|k| \approx R$ has about $R^2$ modes, and the contribution of each shell decreases like $R^{-2}$, so this sum converges. Computed

Figure 2 Choosing $m'/m$ changes $-\ln\langle \Omega_m, \Omega_{m'} \rangle$ (log–log). "Play" increases $L$. A has slope 3 (proportional to the volume), B is flat




```
m'/m = 1.2, L = 64: overlap in A 0.0229 (box side 32/m) / overlap in B 0.99746 (lattice spacing 0.0313/m)
```

Computed The numbers in this figure were computed on this machine (formula (8); m′/m = 1.1, 1.2, 1.5, 2).

The overlap goes to 0 because of the volume, not the ultraviolet (that free fields of different masses give inequivalent representations is said to be a generalisation of Haag's theorem; Reed–Simon). Constructive field theory makes the measure as a density in a finite box on a lattice, and reconstructs the state space from its limit (4-10). The state space is an output, not an input (Chatterjee 2018's 'we do not even know for certain the right spaces').

We contrast this with a related phenomenon. The vacua of this section (at a fixed time, on a lattice in three space dimensions) have positive overlap in finite volume (B of Table 3). Comparing two free fields of different masses as measures on Euclidean four-dimensional space-time, they are mutually singular even in a finite box (torus), because of the ultraviolet (the detour in 1-09, formula (11)). How it acts divides between space dimension 3 and space-time dimension 4.

08

## The six side by side: how much is divergence

Table 4 The six obstacles and their divergent part (the part that only arises when the ultraviolet cutoff is removed; the judgements are this book's organisation).

| Obstacle | With $a$ fixed | Divergent part | What is missing |
|---|---|---|---|
| 1 Infrared parameter | Remains | None | A uniform estimate crossing the scales where the coupling is $O(1)$ |
| 2 Invisible to perturbation theory | Remains | None | Handling the $e^{-c/\alpha}$ that does not appear in the series |
| 3 Non-triviality | A question about the continuum limit | None | A proof that it is not a free field |

| Obstacle | With $a$ fixed | Divergent part | What is missing |
|---|---|---|---|
| 4 State space | Local gauges and 0/0 | The formal $Z = \infty$, the volume of the orbits | Something to replace global gauge fixing |
| 5 Reflection positivity | Recovering rotations | Observables of rough fields | Regularisations that keep positivity, axioms for observables on curves |
| 6 Infinitely many degrees of freedom | Remains (orthogonal through volume) | None in this example (free fields, different masses). For measures on four-dimensional space-time the ultraviolet acts too (1-09) | Making the state space from the theory |

In the literature before the 2026 publication, the ultraviolet side was not complete either (Bałaban's four-dimensional result went only as far as estimates with a hypothesis, and subsequential limits were a prospect; 7-07 § 05). For the construction published on 2026-11-10, the committee's assessment is that the ultraviolet side and the infinite-volume limit along subsequences of volumes are proved in Lean (6-01). Even so, the side of a gap uniform in the volume, for which the setters asked for new ideas, is not shown (7-10 § 01, 6-04). Among the obstacles of Table 4, the obstacle of the infrared parameter lies on the side that the published construction has not crossed either (this book's reading). Outside the six are the correspondence with local fields (JW footnote 1) and topological sectors (detour). The seams are in 16-05 (a glossary of names in the bilingual table of 16-05), the tasks in 16-07.

09

## Detour: topological sectors and θ

A section you may skip.

Smooth connections divide into sectors by an integer topological charge (the second Chern number) (3-05). On the lattice things change. For two-dimensional $U(1)$, from the angle $\theta_p \in (-\pi, \pi]$ of the plaquette $p$ (the principal value of the argument of $U_p$), form

$$Q = \frac{1}{2\pi} \sum_p \theta_p \ \in \ \mathbb{Z} \quad (9)$$

which is an integer on a periodic lattice. The sum of the edge angles as they are is 0, since each edge appears twice with opposite orientations, and the shifts to principal values are integer multiples of $2\pi$. Paper But when the configuration is moved continuously, $Q$ jumps when $\theta_p$ crosses $\pm\pi$, and it does not jump if one restricts to configurations in which every plaquette is close to 1 (the action density is small). Lüscher (1982) showed this for non-abelian lattice fields in four dimensions, and also showed that $Q$ returns to the Chern number in the classical continuum limit. Known Typical configurations are rough, so sectors need a condition of this kind. The construction of Magnen–Rivasseau–Sénéor (1993, MRS) is restricted to the trivial sector (7-07 § 06).

A θ term multiplying the weight by $e^{i\theta Q}$ is invisible to perturbation theory. At $\theta = \pi$, it has been argued that because of an anomaly the vacuum cannot be a trivial, non-degenerate gapped state (Gaiotto–Kapustin–Komargodski–Seiberg 2017). Physics The problem statement does not mention θ, and implicitly refers to $\theta = 0$.

## What this chapter can say

| | Content |
|---|---|
| **Said** | The words of the primary sources for the six obstacles, and that they remain even with the lattice spacing fixed Known |
| **Said** | Tables 1–3, Figures 1–2, the Pauli–Villars example Computed , that the $Q$ of formula (9) is an integer Paper |
| **Not said** | Counting six and the judgements of Table 4 (this book's organisation). Formula (3), $\Delta \propto \Lambda$, $\theta = \pi$ (at the level of physics). The computations are smallest examples, and formula (8) is not a proof of Haag's theorem. This book has removed none of the obstacles |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Quotations from the problem statement, the lattice correlation-length problem, the talk slides | Text | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Math. Inst. (2000) / S. Chatterjee, *Yang–Mills for probabilists*, arXiv:1803.01950 (2018) / same author, *Yang–Mills on the lattice: New results and open problems* (talk slides; PDF created 2020-09; no date given) |
| Exponentially small terms, impossibility of gauge fixing | Text | M. Beneke, *Renormalons*, Phys. Rept. 317 (1999) 1–142 / I. M. Singer, CMP 60 (1978) 7–12 |
| Formula (6), the function on the ring, the caution about restricting to minima | Text § I–II | D. Mehta, N. S. Daleo, J. D. Hauenstein, C. Seaton, Phys. Rev. D 90 (2014) 054504, arXiv:1406.6678 |
| Finite groups, $U(1)$, triviality, Gaussian limits, Wilson loops, topological charge, $\theta = \pi$ | Abstracts | A. Adhikari, S. Cao, Ann. Probab. 53 (2025) 140–174, arXiv:2202.10375 / J. Fröhlich, T. Spencer, CMP 83 (1982) 411–454 / M. Aizenman, H. Duminil-Copin, Ann. of Math. 194 (2021) 163–235, arXiv:1912.07973 / S. Chatterjee, arXiv:2401.10507, Probab. Math. Phys. 7 (2026) 339–381 / S. Cao, S. Chatterjee, arXiv:2111.12813 / M. Lüscher, CMP 85 (1982) 39–48 / J. Magnen, V. Rivasseau, R. Sénéor, CMP 155 (1993) 325–383 / D. Gaiotto, A. Kapustin, Z. Komargodski, N. Seiberg, JHEP 05 (2017) 091 |

| Item | Kind | Source / tools |
|---|---|---|
| Original sources for 0/0, Gribov, the phases of $U(1)$, reflection positivity | Bibliographic data only | H. Neuberger, Phys. Lett. B 183 (1987) 337 / M. Schaden, Phys. Rev. D 59 (1999) 014508 / V. N. Gribov, Nucl. Phys. B 139 (1978) 1–19 / A. H. Guth, Phys. Rev. D 21 (1980) 2291–2307 / K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440–471 |
| Infinitely many degrees of freedom and uniqueness of representations | Text § 4.3 | Stanford Encyclopedia of Philosophy, “Quantum Field Theory” |
| Haag's theorem and the generalisation to different masses | Bibliographic data only (checked via a secondary source) | R. Haag, Mat.-Fys. Medd. 29 (1955) no. 12 / D. Hall, A. S. Wightman, Mat.-Fys. Medd. 31 (1957) no. 5 / M. Reed, B. Simon, *Methods of Modern Mathematical Physics* II (1975), Theorem X.46 (from a secondary source; text not checked) |
| Table 1, Figure 1 | Computed | Python on this machine. Formula (3) and $n_*$, bisection for the crossing points |
| Table 2 | Computed | Critical points enumerated combinatorially, with signs and minima decided by eigenvalues ($N = 3$ agrees with Newton's method) |
| The Pauli–Villars example | Computed | Formula (7) evaluated at two points (negative for $M/m = 2, 5, 10, 100$) |
| Table 3, Figure 2 | Computed | Formula (8) computed over all modes ($L \leq 180$). The oscillator formula agrees with numerical integration |

Next chapter to read: 7-12 Detour: "if 1/0 is given a symbol", which looks again at the $e^{-c/\alpha}$ of § 03 and the 0/0 of § 05 from the side of symbols. From the physics side, 17-01.

# Detour: "if 1/0 is given a symbol" — assigning a value erases the manner of approach, and the scale expected for the gap has the form $e^{-1/g^2}$

This chapter organises established knowledge (sources carry the “Known” label in the text). This book's own investigation is mainly in Part 15 and Volume 3 (Parts 16–28 and the appendices).

**What this chapter shows** — What systems that assign a value to $1/0$ give up. The manner of approach carried by $0/0$, and the blow-up that keeps it. $e^{-1/g^2}$, which looks like 0 at every order, and its tie to the poles of the Borel plane.

**Prerequisite chapters** — 4-05 The algebra of divergence, 4-12 Detour: perturbation series diverge. What is used is defined on the spot.

**Said first** — This is an organisation of known mathematics and literature, not a chapter that shows anything about Yang–Mills. "Making $1/0$ a symbol very probably does not change the core" is an assessment, not a theorem. $\Delta \propto \Lambda$ and renormalons are expectations at the level of physics.

The order of this chapter

1. Three readings of the question
2. Systems that assign a value to 1/0, and their price
3. Fixing the path fixes the value
4. 0/0 has a direction: the blow-up
5. Where replacements that keep the direction have worked
6. The algebra of divergence carries more than 1/0
7. A quantity that looks like 0 at every order
8. 1/0 appears in the Borel plane
9. Recording the discrepancy: transseries and renormalons
10. What a symbol should carry — Detour: from the side of logic

01

## Three readings of the question

"If $1/0$ were built into mathematics as a symbol, wouldn't the situation of field theory, troubled by divergences, change?" The question can be read in three ways. (i) **Assign a value**: give $1/0$ or $0/0$ a single value ($\infty$, $0$, a new symbol $\perp$ and so on). (ii) **Replace**: exchange the singular point for the set of directions of approach, or a boundary. (iii) **Change the axioms**.

(iii) becomes a different problem, because the problem statement names the properties: 'Existence includes establishing axiomatic properties at least as strong as those cited in [45, 35]' (the Wightman and Osterwalder–Schrader axioms) (0-03). Known This chapter examines (i) and (ii), and arrives at the following.

$$\text{assigning a value} \Rightarrow \text{the manner of approach disappears}$$
$$\text{replacement keeping direction} \Rightarrow \text{worked in geometry and perturbation theory} \quad (1)$$

**Example:** $\sin x/x$ has the form $0/0$ at $x = 0$, yet the limit is fixed at $1$. On a line there are two ways of approach, from the left and from the right, and both give the same value. In the plane it is not so (§ 04).

02

## Systems that assign a value to 1/0, and their price

There are several systems that assign a value to $1/0$, all built without contradiction, but each gives something up (Table 1).

Known Table 1 Systems that assign a value to $1/0$ (IEEE 754 computed; transreal from bibliographic data only).

| System | $1/0$, $0/0$ | What it gives up |
|---|---|---|
| The Riemann sphere | $1/0 = \infty$ | $0 \cdot \infty$ and $\infty - \infty$ undefined. Order breaks at $\infty$ |
| Wheel (Carlström) | $0/0$ is also an element | $0x = 0$ does not hold in general. $x - x = 0x^2$, $x/x = 1 + 0x/x$ |
| Meadow (Bergstra et al.) | $0^{-1} = 0$ | Axiomatised just by adding formula (2) to the axioms of a commutative ring. $x\,x^{-1} \not= 1$ is possible even with $x \not= 0$ |
| Transreal (Anderson et al.) | $\pm\infty$, $\Phi = 0/0$ | Not a field. The distributive law is said to fail in part |
| IEEE 754 floating point | $1/\pm 0 = \pm\infty$, $0/0 = \mathrm{NaN}$ | Reflexivity of equality and "$x = y \Rightarrow 1/x = 1/y$" |
| Division-by-zero calculus (Saitoh et al.) | $b/0 = 0$ | Fixes the path of regularisation to one (§ 03) |

A meadow adds to a commutative ring (a set in which addition, subtraction and multiplication obey the usual rules; division is not assumed) the operation of inverse $x \mapsto x^{-1}$, and imposes only

$$(x^{-1})^{-1} = x, \qquad x \cdot (x \cdot x^{-1}) = x \tag{2}$$

From this follows $0^{-1} = 0$. A field (a commutative ring in which one can divide by elements other than 0; $\mathbb{Q}$ and so on) with the convention $0^{-1} = 0$ is a meadow, and so is its direct product (componentwise operations). For example, for $x = (1, 0)$ in $\mathbb{Q} \times \mathbb{Q}$, $x^{-1} = (1, 0)$ and $x \cdot x^{-1} = (1, 0) \;/=\; (1, 1)$. Computed

Checking IEEE 754 double precision on this machine: in NumPy 1.26, `1/0`, `1/(-0)`, `0/0` give `inf`, `-inf`, `nan`; `0.0 == -0.0` is `True` and `nan == nan` is `False`; and Python's `1.0/0.0` raises the exception `ZeroDivisionError`. Computed The same numbers give different answers. "The value of $1/0$" is a convention of the system. The signed zero is a symbol that keeps one bit of "which side it came from", and in exchange the reciprocals differ although $0 = -0$. The tools of Yang–Mills (Hilbert spaces, spectra; 1-10) rely on $0 \cdot x = 0$, subtraction and equality. Nearby are non-standard analysis, which has $1/\varepsilon$ but does not define $1/0$, and Colombeau's algebras, which allow products of distributions at the cost of weakening equality (1-11).

03

## Fixing the path fixes the value

§ 2.1 of Saitoh's draft (2023) regards $b/a$ as the solution of the equation $ax = b$, and fixes it by Tikhonov regularisation. Known

$$x\lambda(a, b) = \arg\min_x \left[\lambda x^2 + (ax - b)^2\right] = \frac{ab}{\lambda + a^2}, \qquad \frac{b}{a} := \lim_{\lambda \to +0} x\lambda(a, b) \tag{3}$$

If $a \;/=\; 0$ it is the ordinary $b/a$; if $a = 0$, $x\lambda = 0$, so $b/0 = 0$. Replacing the penalty by $\lambda(x - c)^2$ gives

$$x\lambda = \frac{\lambda c + ab}{\lambda + a^2} \xrightarrow[\lambda \to +0]{} \begin{cases} b/a & (a \;/=\; 0) \\ c & (a = 0) \end{cases} \tag{4}$$

and the answer at $a = 0$ is $c$ itself. **Example:** for $(a, b) = (0, 1)$, always $x\lambda = c$ from $\lambda = 1$ down to $10^{-8}$. For $(a, b, c) = (2, 1, 5)$, $1.4,\ 0.511,\ 0.5001,\ 0.5$, approaching $b/a$. Computed

Formula (3) is also the trivial case of the solution of the equation $ax = b$ by the Moore–Penrose generalised inverse (draft § 2.1), and is a consistent definition. What can be said is that the value of $b/0$ comes not from the equation but from the added penalty:

the path of regularisation. What renormalisation in field theory tracks is this dependence on the path (§ 06).

04

## 0/0 has a direction: the blow-up

In the plane the limit changes with direction.

$$f(x,y) = \frac{2xy}{x^2+y^2} \qquad \big((x,y) \;/=\; (0,0)\big) \quad (5)$$

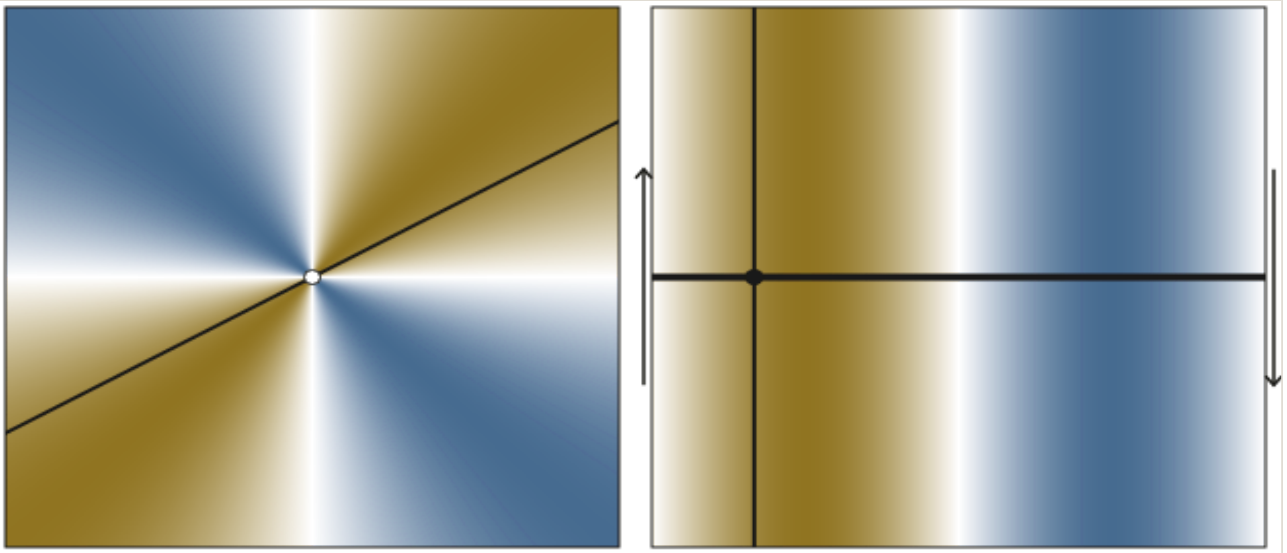
On the half-line $(r\cos\theta,\ r\sin\theta)$ at angle $\theta$, $f = \sin 2\theta$, independent of $r$. Fixing one value at the origin does not make $f$ continuous. Known

The **blow-up** is the operation of replacing the origin by the set of directions (the projective line $\mathbb{RP}^1$: the circle with $\theta$ and $\theta+\pi$ regarded as the same). A direction is written $[u:v]$. $[u:v]$ is $(u,v) \;/=\; (0,0)$ viewed up to non-zero multiples, that is, the name of a line through the origin ($[1:2] = [-2:-4]$). Precisely, one makes the set of pairs of a point and a direction

$$\widetilde{\mathbb{R}^2} = \big\{\, ((x,y),[u:v]) \;:\; xv = yu \,\big\} \quad (6)$$

Away from the origin the direction is fixed uniquely, so it is the same as the original plane; only over the origin does a circle sit, and the whole is an open Möbius band (1-05). On it, $f = 2uv/(u^2+v^2)$ is continuous, including on the circle over the origin.

Figure 1 Turning the direction $\theta$ with the knob shows the value $\sin 2\theta$ on approaching the origin from that direction, at once in the plane on the left and in the blow-up on the right. On the right the horizontal axis is $\theta \in [0, \pi)$ and the vertical the signed distance $r$; the central horizontal line $r = 0$ is the circle that replaces the origin. The left and right edges are glued upside down, as the arrows indicate. "Play" turns the direction



```
θ = 30°: approaching along the line in this direction, f → sin 2θ = 0.866 (the black dot on the central horizontal line of the blow-up)
```

Computed The values in this figure compute formula (5) within the figure (shade is $|f|$, the two colours the sign).

The picture on the right represents the point $(r\cos\theta,\ r\sin\theta)$ by $\theta \in [0, \pi)$ and signed $r$. Points with $r < 0$ lie on the side of direction $\theta + \pi$, but $\sin 2\theta$ does not change under $\theta \to \theta + \pi$, so the values form vertical stripes and do not break when crossing the central horizontal line $r = 0$. $\theta = \pi$ is the same direction as $\theta = 0$, and there the sign of $r$ is exchanged, so $(\pi, r)$ on the right edge is glued to $(0, -r)$ on the left edge. Since it is glued upside down, the band becomes a Möbius band. A symbol that assigns one value to $0/0$ has no place to carry these stripes.

05

## Where replacements that keep the direction have worked

The examples where replacing the singular point has worked all keep the manner of approach. Known

- **Adding one point at infinity.** By the removable singularity theorem of Uhlenbeck (1982), solutions of the Yang–Mills equations with finite action on $\mathbb{R}^4$ come, up to gauge transformations, from smooth solutions on $S^4 = \mathbb{R}^4 \cup \{\infty\}$ (3-05). Intuitively, the finite action constrains the manner of approach to infinity.
- **Replacing collisions by directions.** Fulton–MacPherson (1994) replace the diagonals (where two or more points coincide) of configuration space (all tuples of positions of $n$ points) by a boundary that records the direction and scale of collision. Part II of Axelrod–Singer (1994) used a compactification of this type to show topological invariance of the perturbation theory of Chern–Simons theory, one of the gauge theories in three dimensions. Bergbauer–Brunetti–Kreimer (2009) showed that extension to wonderful models (De Concini–Procesi; a generalisation of the Fulton–MacPherson type of compactification) agrees with Epstein–Glaser renormalisation.

The same holds within Yang–Mills. Singer (1978) noted that, because there is no continuous global gauge fixing, 'an infinity in the determinant of the Jacobian' arises in the Faddeev–Popov method, and named the road of gluing local gauge fixings together with a partition of unity (3-04, 4-08). The lattice $\Delta = m_{\text{lat}}/a$ has the form $0/0$ as $a \to 0$, and what has meaning is the limit of the dimensionless ratio $m_{\text{lat}}/\sqrt{\sigma_{\text{lat}}}$ (5-09). What has worked is on the side of geometry and perturbation theory, and it has not yet reached non-Gaussian measures and the mass gap.

06

## The algebra of divergence carries more than 1/0

Ultraviolet divergences come as families of regularised quantities. A small model:

$$I(p;\eta) = \int_{\eta}^{\infty} \frac{e^{-px}}{x}\,dx,$$
$$I(p;\delta) = \int_{0}^{\infty} x^{\delta-1} e^{-px}\,dx = \Gamma(\delta)\,p^{-\delta} \quad (7)$$

The left diverges like $\ln(1/\eta)$ by cutting the lower end, the right like $1/\delta$ by shifting the power. Written as "1/0" they are the same, but their rates and forms differ, and still their difference goes to the same value.

$$I(1;\cdot) - I(2;\cdot) \;\longrightarrow\; \ln 2 = 0.693147\ldots \quad (8)$$

Computed Table 2 Two regularisations (SciPy on this machine).

| $\eta$ | $I(1;\eta)$ | Difference | $\delta$ | $I(1;\delta)$ | Difference |
|---|---|---|---|---|---|
| $10^{-2}$ | 4.037930 | 0.683222 | $10^{-2}$ | 99.432585 | 0.686831 |
| $10^{-4}$ | 8.633225 | 0.693047 | $10^{-4}$ | 9999.422883 | 0.693083 |
| $10^{-8}$ | 17.843465 | 0.693147 | $10^{-6}$ | 999999.422785 | 0.693147 |

Choosing a scale $\mu$ and subtracting as $I_R(p;\mu) = \lim_{\eta\to 0}[I(p;\eta) + \ln(\eta\mu)] = -\gamma - \ln(p/\mu)$ ($\gamma$ is Euler's constant), the finite part depends on $\mu$ but the difference does not. Computed This freedom, and how the finite part moves when $\mu$ is moved, are the seed of the renormalisation group (4-04).

In field theory, symbolisations that carry "how it diverges" have already been built mathematically (4-05): the BPHZ subtraction, the Epstein–Glaser method in which no infinity ever appears, and the Hopf algebra of Connes–Kreimer, which Birkhoff-decomposes the pole $1/\varepsilon$ of dimensional regularisation as a formal symbol. Perturbative renormalisability of Yang–Mills was established in 1971–72 by 't Hooft, 't Hooft–Veltman, Lee–Zinn-Justin and others (followed by an organisation through BRS symmetry), and Costello gave another proof in the BV formalism. Known Symbolisation has been realised in a form that carries more than $1/0$, and the problem still remains. What remains is not a problem of assigning values.

07

## A quantity that looks like 0 at every order

$$h(x) = \begin{cases} e^{-1/x} & (x > 0) \\ 0 & (x \le 0) \end{cases} \quad (9)$$

is a calculus staple. For $x > 0$, $h^{(n)}(x) = P_n(1/x)\, e^{-1/x}$ with $P_{n+1}(u) = u^2(P_n(u) - P_n'(u))$ (for example $P_2 = u^4 - 2u^3$), and since $u^k e^{-u} \to 0$ as $u \to \infty$, $h^{(n)}(0) = 0$ for every $n$. The Taylor series is identically 0, yet $h$ is not 0 for $x > 0$. Known

$e^{-1/x}$ goes to 0 faster than any power, but it starts winning late: the maximum of $e^{-1/x}/x^k$ is $k^k e^{-k}$ at $x = 1/k$. **Example:** $e^{-1/x}/x^{10}$ is $4.5 \times 10^5$ at $x = 0.1$, $2.1 \times 10^4$ at $x = 0.05$, and only at $x = 0.02$ does it reach $1.9 \times 10^{-5}$. Computed

In physics this form appears in the scale $\Lambda$. The one-loop running coupling is $\alpha_s(\mu^2) = (b_0 \ln(\mu^2/\Lambda^2))^{-1}$, and the review of the Particle Data Group (PDG) explains $\Lambda$ as 'the scale where the perturbatively-defined coupling would diverge' (the "1/0" of the one-loop running coupling). Solving the other way,

$$\begin{aligned} \Lambda &= \mu\, \exp\!\Big(-\frac{1}{2b_0\, \alpha_s(\mu^2)}\Big) \\ &= \mu\, \exp\!\Big(-\frac{8\pi^2}{\beta_0\, g^2}\Big), \quad \beta_0 = \frac{11N}{3} \end{aligned} \tag{10}$$

(pure $SU(N)$, $\alpha_s = g^2/4\pi$; PDG's $b_0 = (33 - 2n_f)/12\pi$ with $N = 3$, $n_f = 0$; 4-07). (10) has the form of (9) as a function of $g^2$, so quantities proportional to $\Lambda$ look like 0 at every order of a power series in $g$. The mass gap is expected to be proportional to $\Lambda$. Physics **Example:** for $SU(3)$, $g^2 = 1$ gives $\Lambda/\mu = 7.6 \times 10^{-4}$ and $g^2 = 2$ gives $2.8 \times 10^{-2}$ ($g^2$ is hypothetical). Computed § 6.6 of the problem statement also writes 'the mass gap is not classically visible' (7-10).

08

## 1/0 appears in the Borel plane

The series with coefficients $n!$

$$F(x) \sim \sum_{n=0}^{\infty} n!\, x^n \tag{11}$$

does not converge for any $x > 0$, even though each term is finite. The terms decrease until $n$ reaches $1/x$, and increase beyond. By Stirling's formula the smallest term is of order $\sqrt{2\pi/x}\, e^{-1/x}$, and the accuracy of truncation levels off there (Figure 2). The **Borel sum** forms $B(t) = \sum t^n = 1/(1-t)$ by dividing the coefficients by $n!$, and returns via $\int_0^\infty e^{-t} t^n\, dt = n!$.

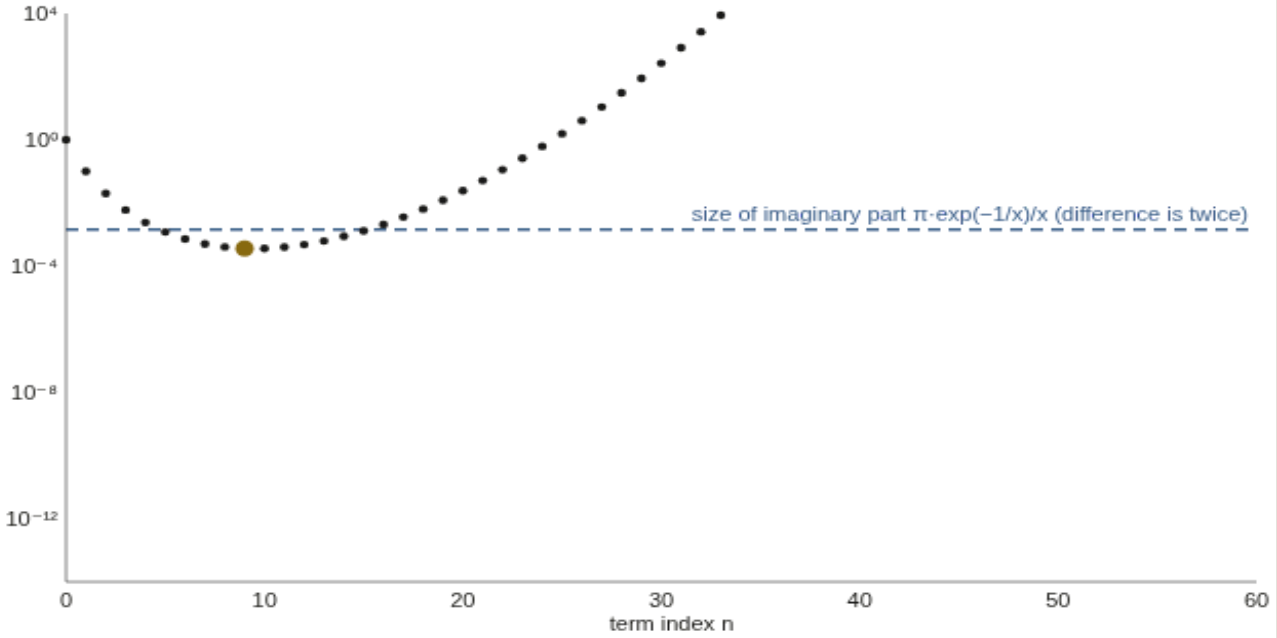
$$S(x) = \int_0^\infty e^{-t}\, B(xt)\, dt = \int_0^\infty \frac{e^{-t}}{1 - xt}\, dt \tag{12}$$

On the path of integration, at $t = 1/x$, there is a pole. The "1/0" of this chapter appears here: in the Borel plane. Avoiding the pole by a small semicircle $t = 1/x + \rho e^{i\phi}$, going over the top moves $\phi$ from $\pi$ to 0, and going underneath from $-\pi$ to 0, so $\int dt/(t - 1/x) = \int i\, d\phi$ is $-i\pi$ above and $+i\pi$ below. The integrand is $\frac{1}{1-xt} = -\frac{1}{x} \cdot \frac{1}{t-1/x}$, so the sign is reversed, and putting $e^{-t} \approx e^{-1/x}$ near the pole, the contribution of the semicircle is $+i\pi e^{-1/x}/x$ above and $-i\pi e^{-1/x}/x$ below. Writing $S_+$ for the sum going over the top and $S_-$ for the sum going underneath,

$$S_\pm(x) = \mathrm{PV}\int_0^\infty \frac{e^{-t}\,dt}{1-xt} \pm i\pi\,\frac{e^{-1/x}}{x},$$
$$\mathrm{PV}\int_0^\infty \frac{e^{-t}\,dt}{1-xt} = \frac{e^{-1/x}\,\mathrm{Ei}(1/x)}{x} \tag{13}$$

and they disagree. PV is the principal value, approaching symmetrically from both sides of the pole, and $\mathrm{Ei}(y) = \mathrm{PV}\int_{-\infty}^{y} e^s\, ds/s$ is the exponential integral. The two differ only in the sign of the imaginary part, whose size is $\pi e^{-1/x}/x$. The difference between above and below is $S_+ - S_- = 2\pi i\, e^{-1/x}/x$, twice that in size. Both the error of optimal truncation and the discrepancy are of the size of $e^{-1/x}$ times a power of $x$ (Table 3). Computed

Figure 2 Changing $x$ with the knob moves the valley of the terms $n!\,x^n$ (log scale), set beside the size of the imaginary part of the Borel sum $\pi e^{-1/x}/x$ (dashed; the difference between above and below is twice that). "Play" moves $x$ back and forth




```
x = 0.100: the terms are smallest at n = 9 (3.63e-4). Size of imaginary part π·exp(−1/x)/x =
1.43e-3 (the difference between the sums above and below is twice that).
```

Computed The values in this figure compute the terms $n!\,x^n$ and $\pi e^{-1/x}/x$ within the figure.

Computed Table 3 The numbers in the table were computed on this machine (SciPy). $N^*$ is the index of the smallest term. The rightmost column is the size of the imaginary part of the Borel sum; the difference between above and below is twice that.

| $x$ | Principal value PV | $N^*$ | Smallest term | $\min_N \lvert S_N - \mathrm{PV}\rvert$ | Size of imaginary part $\pi e^$ |
|---|---|---|---|---|---|
| 0.05 | 1.055956 | 19, 20 | 2.32e−8 | 7.70e−9 | 1 |
| 0.10 | 1.131470 | 9, 10 | 3.63e−4 | 1.20e−4 | 1 |
| 0.15 | 1.239251 | 6 | 8.20e−3 | 2.74e−3 | 2 |
| 0.20 | 1.353831 | 4, 5 | 3.84e−2 | 1.26e−2 | 1 |

09

## Recording the discrepancy: transseries and renormalons

To keep the discrepancy, add $e^{-1/x}$ as an independent unit.

$$F(x) = \sum_{n \geq 0} n!\, x^n \;+\; \sigma\, \frac{e^{-1/x}}{x} \;+\; \cdots \tag{14}$$

The choice of how to avoid the pole moves to the choice of the coefficient $\sigma$, and the information about the manner of approach is recorded not in the value of $1/0$ but in the coefficient of $e^{-1/x}$. Such formal sums are called **transseries**, and research in this direction resurgence (4-12). Known

In Yang–Mills the perturbation series is thought to diverge like $\alpha_s^n\, n!$ (renormalons). The $n!$ of (infrared) renormalons is thought to be a contribution from the region of the loop-momentum integral where the momentum is close to $\Lambda$, near the point where the one-loop running coupling diverges. There are said to be separate ultraviolet renormalons, coming from the side of large momenta. The PDG writes 'closely connected with non-perturbative contributions', while saying they are 'not usually visible' at the orders available. Physics Results of resurgence at the level of physics exist for the two-dimensional $CP(N-1)$ model on $\mathbb{R} \times S^1$ (twisted boundary conditions; Dunne–Ünsal 2012) and for four-dimensional gauge theories with adjoint fermions compactified on $\mathbb{R}^3 \times S^1$ (Argyres–Ünsal 2012), but not for pure Yang–Mills on $\mathbb{R}^4$.

10

## What a symbol should carry

Systems that assign values hold together by weakening the rules of operations or fixing the path to one, and throw away the information about the manner of approach. What has worked are replacements that keep the manner of approach, and symbols that carry the rate of divergence (the Birkhoff decomposition of $1/\varepsilon$, the renormalisation group), both on the side of geometry and perturbation theory. Known The scale of the gap and the expected $\Lambda$ have the form $e^{-1/g^2}$, invisible at every order. Physics The obstacles that remained before 2026 — estimates uniform in the volume, the infinite-volume limit, non-triviality, reflection positivity, the gauge orbit space — are not problems of assigning values (7-11). The present state, after the published construction (2026), is in 7-11 and 6-04.

### Detour: from the side of logic

A subsection you may skip.

An extension that produces no new theorems about sentences of the original language is a **conservative extension**. Theorem 8.8 of Nelson's (1977) internal set theory IST is 'Every internal theorem of IST is a theorem of ZFC' (a result of Powell). ZFC is the axiom system of ordinary mathematics, and an internal theorem is a theorem written in sentences that do not use the predicate "standard" newly added by IST. IST handles $1/\varepsilon$ and does not define $1/0$. Known But "conservative symbols add no theorems, so they are powerless" is not decisive. The Hopf algebra of graphs, regularity structures and Perelman's entropy are all extensions by definition within ordinary mathematics. Even so, regularity structures were used to construct solutions of singular stochastic partial differential equations, Perelman's entropy in the proof of the geometrisation conjecture, and the Hopf algebra gave a clear view of the combinatorics of renormalisation. Symbols can only be assessed at the level of "what they carry".

This book's reading (unverified)

"The mass gap is not a problem of $1/0$ but a problem of $e^{-1/g^2}$." Confidence is moderate. Transseries too are formal objects, and by themselves give neither a measure nor estimates uniform in the volume. For pure Yang–Mills on $\mathbb{R}^4$, resurgence is not established even at the level of physics.

## What this chapter can say

| | Content |
|---|---|
| **Said** | The systems of Table 1 weaken the rules of operations or fix the path to one (wheel, meadow and division-by-zero calculus from the texts, IEEE 754 within what was checked on this machine; transreal from bibliographic data only) Known Computed |
| **Said** | The blow-up of formula (5), that the Taylor coefficients of formula (9) are 0, the known examples of § 05 and § 06 Known |
| **Said** | The numbers in Tables 2 and 3 and in § 02, § 03 and § 07. That the discrepancy of the Borel sum and the error of optimal truncation are both of the size of $e^{-1/x}$ times a power Computed |
| **Not said** | The divergence of the perturbation series of four-dimensional Yang–Mills, renormalons, $\Delta \propto \Lambda$ (expectations of physics). That symbolising $1/0$ is powerless in principle (an assessment). That a symbol carrying $e^{-1/g^2}$ reaches the gap (this book's reading). This book shows nothing about Yang–Mills |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Problem statement (§ 6.6, the specification of the axioms) | Text | A. Jaffe, E. Witten, *Quantum Yang–Mills Theory*, Clay Mathematics Institute (2000) |
| Running coupling, $b_0$, renormalons | Text § 9.1.1, footnote 1 | Particle Data Group, *Review of Particle Physics* (2024), § 9 |
| Wheel, meadow, division-by-zero calculus | Text | J. Carlström, Research Reports in Math. 2001:11, Stockholm Univ. (journal version Math. Struct. Comput. Sci. 14 (2004) 143–184) / Bergstra–Hirshfeld–Tucker, arXiv:0901.0823 § 1 / S. Saitoh, *Division by Zero Calculus; an extended version* (draft 2023) § 2.1 |
| IST, Singer's remark | Text | E. Nelson, Bull. Amer. Math. Soc. 83 (1977) 1165–1198 (p. 1196) / I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12 |
| Others | Abstracts | Axelrod–Singer, J. Differential Geom. 39 (1994) 173–213 / Bergbauer–Brunetti–Kreimer, arXiv:0908.0633 / Connes–Kreimer, Comm. Math. Phys. 210 (2000) 249–273 / Epstein–Glaser, Ann. IHP A 19 (1973) 211–295 / Dunne–Ünsal, arXiv:1210.2423, JHEP 11 (2012) 170 ($\mathbb{R} \times S^1$ and twisted boundary conditions checked in the section headings of the text) / Argyres–Ünsal, arXiv:1206.1890, JHEP 08 (2012) 063 |
| Original sources of division-by-zero calculus | Bibliographic data only | M. Kuroda, H. Michiwaki, S. Saitoh, M. Yamane, Int. J. Appl. Math. 27 (2014) 191–198 / S. Saitoh, Springer Proc. Math. Stat. 177 (2016) 151–182 (proceedings of ISAAC 2015). Both from references [58] and [115] of the draft |

| Item | Kind | Source / tools |
|---|---|---|
| Renormalisability | Bibliographic data only | G. 't Hooft, Nucl. Phys. B 33 (1971) 173–199 and B 35 (1971) 167–188 / 't Hooft–Veltman, Nucl. Phys. B 50 (1972) 318–353 / B. W. Lee, J. Zinn-Justin, Phys. Rev. D 5 (1972) 3121–3137 (sequels 3137, 3155) / Becchi–Rouet–Stora, Ann. Phys. 98 (1976) 287–321 / Costello, AMS (2011) |
| Others | Bibliographic data only | Anderson–Völker–Adams, Proc. SPIE 6499 (2007) / IEEE Std 754-2019 / Colombeau, North-Holland (1984) / Uhlenbeck, Comm. Math. Phys. 83 (1982) 11–29 / Fulton–MacPherson, Ann. of Math. 139 (1994) 183–225 / 't Hooft–Veltman, Nucl. Phys. B 44 (1972) 189–213 (dimensional regularisation) / M. Beneke, Phys. Rep. 317 (1999) 1–142, arXiv:hep-ph/9807443 (renormalons) / M. Hairer, Invent. Math. 198 (2014) 269–504 (regularity structures) / G. Perelman, arXiv:math/0211159 (entropy) |
| Tables 2 and 3, the examples in § 02, § 03 and § 07, Figures 1 and 2 | Computed | Python 3, NumPy 1.26 and SciPy 1.11 on this machine ($E_1$, $\Gamma$, Ei; the principal value agrees to 12 digits with numerical integration with a Cauchy weight; the meadow checked with rational numbers). The figures compute formula (5) and $n!\,x^n$ within the figure |

Next chapter to read: 8-01 The lattice, the Wilson measure, conventions. From Part 8, Volume 2 turns to the definitions of the objects of the published construction. The view from the side of physics is in Volume 3, 17-01 How physics sees it; Part 17 begins with how physics handles the scale $\Lambda$.

# The lattice, the Wilson measure, conventions

This chapter is existing knowledge (a reading of the published construction). § 05 is this book's study.

**What this chapter shows** — the definitions of the objects the published construction takes as its starting point — a group $G \subset U(m)$, a periodic lattice, plaquette holonomies, the Wilson measure — and its conventions ($\beta = 1/g_0^2$, the inner product $\langle X, Y\rangle_{\mathfrak{g}} = -m^{-1}\,\mathrm{Re}\,\mathrm{Tr}(XY)$, the Casimir constant $C_{\mathfrak{g}}$). At the end, the conversion to this book's convention (the standard $\beta = 4/g^2$ for $SU(2)$) is put into one table.

**Prerequisite chapters** — 5-01 Wilson's lattice (introduction), 19-01 The lattice, link variables and the Wilson measure (this book's conventions), 4-06 Asymptotic freedom (the one-loop coefficient). The citation conventions of this volume are in 6-00.

**Said first** — this is a chapter of definitions and conventions. The sequence of lattice spacings and the sequence of volumes are treated in the next chapter, 8-02, and the continuum limit in 8-05. Results of the published construction are cited at the level of "the committee's assessment is that it is proved in Lean"; what this book has checked for itself goes no further than reading the definitions and the form of the statements.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — group, lattice, Wilson measure: the objects set up by the published statement
2. Continuum conventions — β = 1/g² and the inner product: how the plaquette term approaches the square of the curvature
3. The Casimir constant C_g: definition, value for SU(N), dependence on the size of the representation

4. Conversion table — the published construction, this book, the standard convention: what the same number means in each convention
5. What this book's study had seen: comparison with the conventions of this book's Lean
6. What we have not read: with locations

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01

## The statement — group, lattice, Wilson measure

The published construction was made public as the Lean repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the committee's assessment, `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10, hereafter "the assessment"), evaluates it. In this part we read the objects that construction places on the lattice and the tools that carry it from the lattice to the continuum. Below, relative file paths are paths inside this repository.

**The group.** The input is a closed subgroup $G$ of $U(m)$ for some $m$, whose real Lie algebra $\mathfrak{g}$ is non-abelian and simple (it has no proper non-zero ideal) Source (`GaugeGroup` and `GaugeGroup.IsSimple` in `PublicTheory.lean`, lines 25–45). $m$ is "the size of the matrices", a number distinct from the rank of the group and from $\dim \mathfrak{g}$. $G = SU(2)$ is given by its two-dimensional defining representation, with $m = 2$ Source (`su2` in `ConcreteSU2.lean`).

**The lattice.** There are two indices: the spacing index $N$ and the volume index $p$. A side has $L_p D_N$ points, and the lattice is

$$\Lambda_{N,p} = \big(\mathbb{Z}/(L_p D_N)\mathbb{Z}\big)^4, \qquad a_N = D_N^{-1} \quad (1)$$

A lattice point $y$ is placed at the continuum point $a_N \cdot y$ (each coordinate read through its representative between $-\frac{1}{2}L_pD_N$ and $\frac{1}{2}L_pD_N$) Source (`Site`, `mesh` and `place` in `PublicWilson.lean`, lines 11 and 50–52). The side length in continuum units is $a_N \cdot L_p D_N = L_p$. How $D_N$ and $L_p$ are chosen is read in 8-02.

**Plaquettes and the Wilson energy.** A link variable $U_{y,\mu} \in G$ is placed on each positively oriented edge, and the plaquette holonomy is

$$U_{\square} = U_{y,\mu}\, U_{y+e_\mu,\nu}\, U_{y+e_\nu,\mu}^{-1}\, U_{y,\nu}^{-1} \qquad (\mu < \nu) \quad (2)$$

Source (`plaquette`, lines 20–23). The Wilson energy is the sum over all lattice points and all pairs of directions $\mu < \nu$

$$E(U) = \sum_{y \in \Lambda_{N,p}} \sum_{\mu<\nu} \left(1 - \frac{1}{m} \operatorname{Re} \operatorname{Tr} U_{\square}\right) \quad (3)$$

Source (`wilsonEnergy`, lines 25–27; the denominator is `g.n+1`, which is $m$). Each plaquette is counted exactly once, $6|\Lambda_{N,p}|$ plaquettes in all.

**The Wilson measure.** The product of Haar probability measures $h$, multiplied by the density $e^{-\beta E}$ and normalised,

$$d\mu^{\mathrm{W}}_{N,p}(U) = Z^{-1}_{N,p} \exp\bigl(-\beta_{N,p} E(U)\bigr) \prod_{e} dh(U_e) \quad (4)$$

is the lattice theory Source (`wilsonLaw`, lines 29–33) Assessment (the assessment § 4, equation (3)). The public structure `WilsonFamily` (lines 35–46) has as its fields the sequence of volumes `period`, the sequence of spacing denominators `denominator` (both strictly increasing), the coupling `beta N p`, the Haar measure (a probability measure, left-invariant), the type of labels `Token` and the insertions `insertion`. Labels and insertions are the subject of 8-03 and 8-04.

This structure and the conditions it satisfies, `WilsonRegularity` (`ChallengeSpec.lean`, lines 12–22), are part of the statement of the public theorem `JointExistence` (same file, lines 186–188) Source. In other words, "which measure on which lattice to start from" is written into the public statement itself.

The lattice-side definitions have been checked against an independently built Lean formulation of lattice gauge theory (`mrdouglasny/lgt`). According to the committee's record, that check relabels lattice points, links and plaquettes and shows in Lean that the action, the partition function and the probability measure agree, with the coupling conversion $\beta_{\text{submission}} = m\,\beta_{\text{lgt}}$ Assessment (the assessment § 11). The text is in `same_plaquette` and `coupling_conversion` (lines 35 and 55) of the committee's annotated example `LatticeAndScale-20261028.lean` Source.

02

## Continuum conventions — β = 1/g² and the inner product

The assessment § 3 sets the conventions of continuum Yang–Mills as follows Assessment. For a smooth $\mathfrak{g}$-valued connection $A$ the curvature is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the invariant inner product on $\mathfrak{g}$ is

$$\langle X, Y\rangle_{\mathfrak{g}} = -\frac{1}{m}\operatorname{Re}\operatorname{Tr}(XY) \quad (5)$$

The formal measure is

$$Z^{-1}\exp\Big(-\frac{1}{2g^2}\int_{\mathbb{R}^4}\sum_{\mu<\nu}\|F_{\mu\nu}(x)\|_{\mathfrak{g}}^2\,d^4x\Big)\,\mathcal{D}A \quad (6)$$

In this convention, the $\beta$ of equation (4) is read as $\beta = 1/g_0^2$ ($g_0$ is the coupling at the cutoff) Assessment (the assessment § 4). The assessment states explicitly that constructing a probability measure corresponding to (6) is not claimed. What is constructed are correlation functions of local observables (8-05).

Why $\beta = 1/g^2$ can be seen from the Taylor expansion of the plaquette term Proved here . When the links come from parallel transport by a smooth connection, the plaquette holonomy at spacing $a$ can be written $U_{\mu\nu} = \exp X$, $X = a^2F_{\mu\nu}(x) + O(a^3)$. Since $X$ is anti-Hermitian, $\operatorname{Re}\operatorname{Tr} X = 0$, and expanding $\exp X$ to second order gives

$$1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr}U_{\mu\nu} = -\frac{1}{2m}\operatorname{Re}\operatorname{Tr}(X^2) + O(a^8) = \frac{a^4}{2}\|F_{\mu\nu}\|_{\mathfrak{g}}^2 + O(a^5) \quad (7)$$

The sum over lattice points $\sum_y$ corresponds to $a^{-4}\int d^4x$, so

$$\beta\,E(U) \;\approx\; \frac{\beta}{2}\int\sum_{\mu<\nu}\|F_{\mu\nu}\|_{\mathfrak{g}}^2\,d^4x \quad (8)$$

and comparing with (6) gives $\beta = 1/g^2$. Equation (4) of the assessment § 4.1, $a^{-4}\big(1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr}U_{\mu\nu}\big) \to -\frac{1}{2m}\operatorname{Tr}F_{\mu\nu}(0)^2$, is the same fact as (7) Assessment .

Measuring lengths with this inner product is not a mere convention. The coefficients that appear later — the slope of the running coupling, the Casimir constant, the coefficient of the trace anomaly — are all written relative to this inner product. Changing the convention changes the numerical values of the coefficients, not their meaning. The table in § 04 is the correspondence between those values.

03

## The Casimir constant C_g

What the public statement `WilsonRegularity.running` asks for is only that $\beta_{N,p}/\log(1/a_N)$ approach some positive slope $s$ — that a positive slope **exists** Source (`ChallengeSpec.lean`, lines 13–14). The construction meets this with $s = 11C_{\mathfrak{g}}/(24\pi^2)$ Source (`regular` in `HierarchyBridge.lean`, line 18) Assessment (equation (7) of

Theorem 4.1 in the assessment). Here $C_{\mathfrak{g}}$ is the adjoint Casimir constant relative to the inner product (5), the number determined by the identity

$$-\operatorname{tr}\big(\operatorname{ad} x \circ \operatorname{ad} y\big) = C_{\mathfrak{g}}\,\langle x, y\rangle_{\mathfrak{g}} \qquad (x, y \in \mathfrak{g}) \quad (9)$$

In Lean it is defined, for a basis $b_i$ orthonormal for the inner product (5), as $C_{\mathfrak{g}} = -\sum_i \operatorname{tr}(\operatorname{ad} b_i \circ \operatorname{ad} b_i)/\dim \mathfrak{g}$ (`casimir` in `YM/Gauge/Colors.lean`, line 180), and the fact that (9) holds for simple $\mathfrak{g}$ is `trace_adjoint` (`YM/Gauge/Casimir.lean`, line 140) Source . That the inner product is (5) can be read from `trace_color` (`Colors.lean`, line 71; $-\operatorname{Re}\operatorname{Tr}(cd) = m\langle c, d\rangle$) Source . The slope constant is `slopeConstant` $= 11C_{\mathfrak{g}}/(24\pi^2)$ (`Colors.lean`, line 184), and $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$ (`YM/Pert/Asymptotics/RawAxis.lean`, line 13); $2b_0$ is the slope Source .

**The value for the defining representation of $SU(N)$.** The Killing form of $\mathfrak{su}(N)$ is $\operatorname{tr}(\operatorname{ad} x\ \operatorname{ad} y) = 2N\operatorname{Tr}(xy)$. With $m = N$ in (5), $-2N\operatorname{Tr}(xy) = C_{\mathfrak{g}} \cdot (-\operatorname{Tr}(xy)/N)$, so

$$C_{\mathfrak{su}(N)} = 2N^2, \qquad C_{\mathfrak{su}(2)} = 8 \quad (10)$$

Proved here (we also checked this numerically for $N = 2, 3, 4$: $8, 18, 32$). The slope for $SU(2)$ is $11 \cdot 8/(24\pi^2) = 11/(3\pi^2) \approx 0.3715$.

**Dependence on the size of the representation.** Since (5) divides the matrix trace by $m$, the inner product, and with it the value of $C_{\mathfrak{g}}$, changes with the choice of $U(m)$ into which the same Lie algebra is embedded. For example, if the same $\mathfrak{su}(2)$ is carried by the three-dimensional adjoint representation ($SO(3) \subset U(3)$, $m = 3$), (9) gives $C_{\mathfrak{g}} = 3$ Proved here (checked numerically). The meanings of $\beta$ and $g$ then also change to match (5). **$C_{\mathfrak{g}} = 8$ is the value in the convention "$SU(2)$ carried by its two-dimensional representation".** Do not carry the number alone into formulas written in another convention.

**Comparison with the standard convention.** In the standard convention for $SU(N)$, $\beta_{\mathrm{std}} = 2N/g_{\mathrm{std}}^2$ and the one-loop coefficient is $b_0^{\mathrm{std}} = 11N/(48\pi^2)$ (Gross–Wilczek, Politzer; 4-06). Rewriting the one-loop running of asymptotic freedom, $1/g_{\mathrm{std}}^2 \approx 2b_0^{\mathrm{std}} \log(1/a)$, in terms of $\beta$ gives $\beta_{\mathrm{std}}/\log(1/a) \to 2N \cdot 2b_0^{\mathrm{std}} = 11N^2/(12\pi^2)$, which agrees with $11C_{\mathfrak{g}}/(24\pi^2) = 11 \cdot 2N^2/(24\pi^2)$ after inserting (10) Proved here .

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04
```

### Conversion table — the published construction, this book, the standard convention

The object of Volume 3 of this book is $SU(2)$, and its weight is written $\exp\left(\beta \sum_p \operatorname{Re} U_p\right)$ ($\operatorname{Re} U_p = \frac{1}{2}\operatorname{tr} U_p$, the real part of a quaternion) (19-01 § 05). Taking $G = SU(2)$, $m = 2$ in the published construction, $\operatorname{Re}\operatorname{Tr} U/m = \frac{1}{2}\operatorname{tr} U = \operatorname{Re} U$, so

$$e^{-\beta E(U)} = e^{-6|\Lambda|\beta} \exp\left(\beta \sum_p \operatorname{Re} U_p\right) \quad (11)$$

and the constant disappears under normalisation Proved here . **For $SU(2)$, the $\beta$ of the published construction and the $\beta$ of this book are the same number.** Only the convention for writing that number in terms of $g$ differs.

| Quantity | Published construction ($G \subset U(m)$) | This book ($SU(2)$) | Standard convention ($SU(N)$, definir |
|---|---|---|---|
| Plaquette weight | $\exp\left(-\beta \sum (1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_\square)\right)$ | $\exp\left(\beta \sum \operatorname{Re} U_p\right)$ (same as the left, up to a constant factor) | $\exp\left(-\beta_{\text{std}} \sum (1 - \frac{1}{N}\operatorname{Re}\operatorname{Tr} U_p)\right)$ |
| $\beta$ and $g$ | $\beta = 1/g^2$ (inner product (5)) | $\beta = 4/g^2_{\text{std}}$ | $\beta_{\text{std}} = 2N/g^2_{\text{std}}$ |
| Between the $g$'s | $g^2_{\text{std}} = 2m\,g^2$. For $SU(2)$, $g^2_{\text{std}} = 4g^2$, and the number $\beta$ is the same in all three Proved here | | |
| Casimir | $C_{\mathfrak{g}}$ ((9), relative to the inner product (5)). $2N^2$ for the defining representation of $SU(N)$, 8 for $SU(2)$ | (not used) | $C_A = N$ (in the convention $\operatorname{tr} T^a T^b$ |
| One-loop coefficient | $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$. $11/(6\pi^2)$ for $SU(2)$ | (written in units of $\beta$; next row) | $b_0^{\text{std}} = 11N/(48\pi^2)$. $11/(24\pi^2)$ for $SU$ |
| Slope of the running of $\beta$ | $\beta/\log(1/a) \to 2b_0 = 11C_{\mathfrak{g}}/(24\pi^2)$. $11/(3\pi^2) \approx 0.3715$ for $SU(2)$ | One step with side 2: $\frac{11}{3\pi^2}\ln 2 \approx 0.2575$ (15-01) | $2N \cdot 2b_0^{\text{std}} = 11N^2/(12\pi^2)$ |
| Independent lattice formulation | $\beta = m\,\beta_{\text{lgt}}$ Assessment | — | — |
| Coefficient of the trace anomaly (13-02) | $3\dim_{\mathbb{R}} \mathfrak{g}/\pi^4$. $9/\pi^4 \approx 0.0924$ for $SU(2)$ | (no counterpart) | — |

The ratio of the two $b_0$'s is 4, which comes from $g^2_{\text{std}} = 4g^2$. The slope written in terms of $\beta$ agrees in all three conventions (because $\beta$ is the same number). This book's one-step value with side 2 is slope $\times \ln 2$; since the sides in the published construction are

odd numbers of at least 3 (8-02), the one step with side 2 itself is not treated there. Only the coefficient of $\log b$ can be compared; the $O(1)$ constant of one step cannot Inferred .

The unit of volume also needs care. The unit of length in the published construction is "the spacing at step 0", with $a_0 = 1$ (8-02). The lattice of Volume 3 of this book is a single torus with spacing 1. The "1" is the same, but in the published construction the physical unit is fixed by matching $\beta$ to a value $\bar{\beta}$ determined at step 0 (10-01). No such matching enters this book's lattice spacing 1.

05

## What this book's study had seen

This book's Lean treated only $SU(2)$, as unit quaternions (19-01). The form of the weight, the meaning of $\beta$ and the product of Haar measures are, as in § 04, the same as in the case $m = 2$ of the published construction. The differences are the following three.

- This book's lattice is a single torus with the spacing fixed at 1; there is neither a sequence of spacings nor a sequence of volumes. The published construction has a family with two indices $(N, p)$ from the start.
- This book's Lean defines neither the inner product on the Lie algebra nor the Casimir constant. The one-loop value $\frac{11}{3\pi^2} \ln 2$ was used only on the side of paper and numerics (15-01 and elsewhere).
- The published construction states the same statement for any simple closed matrix group, and $SU(2)$ is a special case ( `su2Theory` in `ConcreteSU2.lean` ).

We found no disagreement at the level of conventions. Numbers written in this book's $\beta$ (the strong-coupling window $|\beta| < 0.0556$, Part 20, and others) can be read as the same numbers in the $\beta$ of the published construction. The published construction, however, keeps $\beta$ large at every step (10-01) and does not pass through the strong-coupling window.

06

## What we have not read

- We have not read the content of the transfer from `GaugeGroup` to the construction's internal type `Gauge` ( `g.core` , `GaugeGroup.coreSimple` ). Nor have we read the proof of $C_{\mathfrak{g}} > 0$ ( `casimir_pos` ).
- The expansion (7) is this book's hand computation. We read the Taylor conditions used when the construction chooses representatives of labels (8-03), but we have not

looked for a declaration stating (7) for the plaquette term itself.

- On the committee's side, `submissions/openai-20261028/checks/ImportedAFGauge.lean` (lines 81–86) holds the one-loop coefficient in another convention, $\mu\, dg/d\mu = -b_0 g^3/(16\pi^2)$, $b_0 = \frac{11}{3} C_2(\mathrm{adj})$ Source . For $SU(N)$ this is $11N/3$, and dividing by $16\pi^2$ gives the same as the standard $b_0^{\mathrm{std}} = 11N/(48\pi^2)$ Inferred . We have not read where in the committee's checks the coefficient in this convention is used.
- For the check against the independent lattice formulation (end of § 01), we read only the statements in the annotated example, not the proof of the check.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The public statement starts from $G \subset U(m)$ (with simple Lie algebra), the lattice (1), the plaquettes (2), the energy (3) and the family of measures (4) (`PublicWilson.lean`, lines 20–46) | Which $\beta_{N,p}$ to choose (that belongs to the construction; 10-04) |
| Assessment $\beta = 1/g_0^2$, and the inner product is (5) (the assessment § 3, § 4) | The existence of a continuum probability measure corresponding to (6) (the assessment states explicitly that it is not claimed) |
| Source The definition of $C_{\mathfrak{g}}$ and (9) (`casimir`, `trace_adjoint`) | That the value of $C_{\mathfrak{g}}$ does not depend on the representation (it does; § 03) |
| Proved here $C_{\mathfrak{g}} = 2N^2$ for the defining representation of $SU(N)$, 8 for $SU(2)$. The slope is $11/(3\pi^2)$ | A list of values for other embeddings and other groups |
| Proved here For $SU(2)$, the $\beta$ of the published construction and the $\beta$ of this book are the same number ((11)) | A comparison of the $O(1)$ constants of one step (the sides differ) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Lattice, plaquettes, energy, measure, `WilsonFamily` | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 10–46, `ChallengeSpec.lean` lines 12–22 and 186–188, `PublicTheory.lean` lines 25–45, `ConcreteSU2.lean` |
| Conventions (5)(6), $\beta = 1/g_0^2$, equations (3)(4) | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 3, § 4, § 4.1 |

| Item | Kind | Source / tools |
|---|---|---|
| $C_{\mathfrak{g}}$, the slope, $b_0$ | Source | `YM/Gauge/Colors.lean` lines 71, 180 and 184, `YM/Gauge/Casimir.lean` line 140, `YM/Pert/Asymptotics/RawAxis.lean` line 13 |
| Check against the independent lattice formulation | Assessment | The assessment § 11 |
| Check against the independent lattice formulation (annotated example) | Source | On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 35 and 55 |
| The Killing form of $\mathfrak{su}(N)$, the one-loop coefficient | Cited | D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346 (4-06) |
| Values of $C_{\mathfrak{g}}$ ($N = 2, 3,$ 4, adjoint embedding), the bookkeeping of 24 versus 48/16 | Proved here | Numerical record of this book's study (`numcheck.py`; builds matrix traces and the adjoint action numerically and computes the ratio in (9)) |

**Terms used in this chapter** — Wilson measure, plaquette holonomy, Casimir constant, one-loop coefficient, convention. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-02 The set-up — the sequence of spacings, the sequence of volumes, growing sides — how the two indices of the family of lattices are chosen.

# The set-up — the sequence of spacings, the sequence of volumes, growing sides

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study.

**What this chapter shows** — how the two indices of the family of lattices $\Lambda_{N,p}$ — the spacing index $N$ and the volume index $p$ — are determined, in the public statement and on the construction's side. The spacing is $a_N = 1/D_N$, $D_N = \prod_{k<N} b_k$, where the ratios $b_k$ are odd numbers of at least 3 that do not decrease with the index and grow without bound. The side of the volume, $L_p$, is measured in units in which the spacing at step 0 is 1.

**Prerequisite chapters** — 8-01 The lattice, the Wilson measure, conventions. The idea of renormalisation steps is in 7-05, and this book's block map in 19-04.

**Said first** — the sequence of spacings is chosen by the construction, and the published construction does not say that "every sequence of spacings gives the same limit". The detailed accounting of why the sides grow is in 10-05, and the order in which constants are chosen is in 11-08.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — what the public statement asks for: two strictly increasing sequences, and the running condition
2. The direction of the steps and the size of the lattice: step N is the finest, step 0 has spacing 1
3. The sequence of ratios — odd, non-decreasing, growing: what can be read from the conditions of the calendar

4. Why the sides grow (outline): dividing the one-step error by log b
5. The sequence of volumes and the unit: L_p = 4(M+p+1), where test functions are placed
6. What this book's study had seen: fixed blocks with side 2 and side 3
7. What we have not read

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01

## The statement — what the public statement asks for

The public structure `WilsonFamily` (8-01) has a sequence of volumes $p \mapsto L_p$ (`period`) and a sequence of spacing denominators $N \mapsto D_N$ (`denominator`), and asks only that both be positive integers that are **strictly increasing** Source (`PublicWilson.lean`, lines 36–39). The spacing is $a_N = 1/D_N$ (`mesh`, line 50).

The first of the conditions `WilsonRegularity` imposed on this family is the running of the coupling Source (`ChallengeSpec.lean`, lines 13–14).

$$\exists\, s > 0 \;\; \forall \varepsilon > 0 \;\; \exists N_0 \;\; \forall N \geq N_0 \;\; \forall p : \qquad \left| \frac{\beta_{N,p}}{\log(1/a_N)} - s \right| < \varepsilon \quad (1)$$

Note the order of the quantifiers. $N_0$ is determined by $\varepsilon$ alone and does not depend on the volume index $p$. That is, the running is **uniform in the volume**. The construction meets it with the slope $s = 11C_{\mathfrak{g}}/(24\pi^2)$ (the value the assessment writes as twice the one-loop coefficient; 8-01 § 03) Source (`regular` in `HierarchyBridge.lean`, lines 16–25) Assessment (equation (7) of Theorem 4.1 in the assessment). The proof of the running is read in 10-06.

The assessment states the following plainly about this family Assessment (the assessment § 4, § 4.3).

- The construction **chooses** the two sequences. The ultraviolet convergence theorem does not assert convergence for every possible sequence of spacings.
- Convergence holds along the **whole** of the chosen sequence of spacings (not along a subsequence). There is, however, no universality result about the type of the sequence of spacings.

The appendix of the assessment (a glossary of the Lean statements) describes the construction's choice as $L_p = 4(M + p + 1)$ ($M$ fixed large enough to satisfy the size conditions), $D_N = \prod_{k<N} b_k$, $b_N \geq 3$ odd with $b_{N+1} \geq b_N$, and $\log b_N / \log D_N \to 0$

Assessment (`WilsonFamily.period` and `denominator` in Appendix A of the assessment). In § 02– § 05 we compare this with the definitions on the construction's side.

02

## The direction of the steps and the size of the lattice

The set-up on the construction's side is collected in the structure `Schedule` Source (`YM/Lattice/Scale.lean`, lines 28–34). Its fields are the sequence of ratios `block` (each $b_m$ odd with $b_m \geq 3$) and the sequence of volumes `period` (divisible by 4, strictly increasing). From these,

$$D_0 = 1, \quad D_{m+1} = D_m\, b_m, \qquad a_m = \frac{1}{D_m}, \qquad n_{m,p} = L_p\, D_m \quad (2)$$

are defined Source (`denominator` lines 39–41, `a` line 99, `side` line 67). $n_{m,p}$ is the number of points on a side of the lattice at step $m$, and the lattice at step $m$ is $(\mathbb{Z}/n_{m,p})^4$. The physical side length is

$$a_m \cdot n_{m,p} = L_p \qquad (\text{for all } m) \quad (3)$$

and does not depend on the step Proved here (also checked in the committee's example, `side_length_in_physical_units`, `LatticeAndScale-20261028.lean` line 92 Source).

**The direction of the steps.** The theory with cutoff $N$ starts from the Wilson measure at step $N$ (the finest, spacing $a_N$) and is coarsened down to step 0 (spacing $a_0 = 1$) by applying one renormalisation step (9-01) $N$ times. In one step from step $m$ to step $m-1$, the number of points on a side is divided by $b_{m-1}$. Since $D_m$ is a product of odd numbers, a coarse point can be placed at the **centre** of a fine block (9-02).

| Symbol | Meaning | Lean side |
|---|---|---|
| $N$ | Cutoff index (the finest step) | The index `N` of the public `WilsonFamily` |
| $m \in \{0, \dots, N\}$ | Renormalisation step ($m = N$ is bare, $m = 0$ has spacing 1) | The step index of `History` |
| $b_m$ | Ratio from step $m+1$ to step $m$ (side of the block) | `Schedule.block m` |
| $D_m,\ a_m$ | Denominator and spacing | `Schedule.denominator`, `Schedule.a` |
| $L_p$ | Physical side length (in units of the spacing at step 0) | `Schedule.period p` (the public `period`) |
| $\log(1/a_m) = \sum_{k<m} \log b_k$ | Scale of the steps | `Schedule.scale` (`scale_succ`, line 124) |

The denominator $D_N$ of the public statement is exactly `Schedule.denominator` here Source (`wilsonFamily` in `WilsonBridge.lean`, lines 20–30).

03

## The sequence of ratios — odd, non-decreasing, growing

`Schedule` itself asks only for "odd and at least 3". The other properties come from the calendar `Calendar` and its condition `Calendar.Fits` Source (from `YM/RG/History.lean`, lines 64–77 and 97). A calendar has a sequence of tuples of step constants $j \mapsto \mathrm{stage}(j)$, a sequence of candidate sides $j \mapsto \mathrm{factor}(j)$, and step depths $J(m) \geq 4$, and `Fits` asks for the following.

- `sameFactor`: $b_m = \mathrm{factor}(J(m+1))$. `monotoneL`: factor is monotone. `mono`: $J$ is monotone. `stepSlow`: $J(m+1) \leq J(m) + 1$. `deep`: $J(m) \to \infty$.
- `mesh` (`Gentle`, `Scale.lean` lines 129–130): $\log b_m / \log(1/a_m) \to 0$.
- `driftSmall`: for the drift of the tuple of constants at step $m+1$, $\mathrm{drift} / \log b_m \leq J(m+1)^{-2}$.
- `logCost`: $\log\bigl(1 + \mathrm{paths} + \mathrm{width}^{-1} + \mathrm{columns} + \mathrm{relaxation} + \mathrm{supportExponent}\bigr) / \log b_m \leq J(m+1)^{-2}$.
- Size conditions (`sizeT`, `fitsPatch`): the side of the lattice exceeds 100 times the size of the cover and the radius of the patches, and so on.

We list what can be read from these.

**Non-decreasing.** $b_m = \mathrm{factor}(J(m+1))$, and since both factor and $J$ are monotone, $b_{m+1} \geq b_m$ Proved here.

**Growing without bound.** The field columns of a tuple of constants is at least 1 (`columns_ge`), so the numerator of `logCost` is at least $\log 2$. Then $\log b_m \geq J(m+1)^2 \log 2$, and $J \to \infty$ gives $b_m \to \infty$ Inferred (a derivation from the definitions, using that the other terms in the numerator are non-negative). The appendix of the assessment says "$b_N$ is chosen to satisfy complicated but explicit inequalities written in terms of the constants appearing in the RG transformation" Assessment.

**Growing slowly.** `Gentle` says that $b_m$ is small compared with $D_m$ ($\log b_m = o(\log D_m)$). `mesh_inequality` in `Scale.lean` (line 133) derives from it that there is $c > 0$ with $c\, a_m^2 \leq a_{m+1}$ Source. This means the spacing does not shrink too abruptly from one step to the next.

The short-distance analysis of the trace (13-03) imposes a further condition `PostT` on the calendar. One part of it, `bloom`, is $b_i \le \log \hat\beta_i$ ($\hat\beta_i$ is the one-loop running at step $i$), and asks that the sequence of ratios grow more slowly than a logarithm Source (`YM/TraceExtension/Terrace/Base.lean`, lines 29–34).

In summary, the sequence of ratios is "odd and at least 3, non-decreasing, growing without bound, but slowly compared with $\log D_m$". No explicit formula for $b_m$ appears in the statement (§ 07).

04

## Why the sides grow (outline)

The details are read in 10-05. Here we give only the skeleton.

The change of the coupling in one step carries a deviation (drift) from the one-loop value. The one-step formula has the form $|\beta_{m+1} - \beta_m - s \log b_m| \le \mathrm{drift}$ Source (`History.Transfer`, `History.lean` line 223), where drift is a real-valued field of the tuple of step constants. This book reads the construction as bounding this deviation only by a step constant ($O(1)$) Inferred. If the side were fixed at a constant $b$, the sum over $N$ steps would be $N \cdot O(1) = \frac{\log D_N}{\log b} \cdot O(1)$, of the same size as $\log D_N$, and the slope in (1) would not be determined Inferred. Since `driftSmall` asks that $\mathrm{drift}_m \le \log b_m / J(m+1)^2$, the sum of deviations is

$$\sum_{m<N} \mathrm{drift}_m \le \sum_{m<N} \frac{\log b_m}{J(m+1)^2} = o\Big(\sum_{m<N} \log b_m\Big) = o(\log D_N) \quad (4)$$

Inferred (from $J \to \infty$ and $\sum \log b_m = \log D_N \to \infty$; the precise form is in 10-05). The design keeps the one-step error at $O(1)$ and removes it by dividing by $\log b$. The two-loop coefficient is not used.

A second reason is that the step constants themselves increase with the step. To handle local fields of degree $d$ (8-03), the Taylor order of the steps beyond a reference step must be at least $d+1$ Source (`Makes.stages` in `YM/Source/Construction.lean`, lines 47–49: $d+3 \le J(r(d))$, order $\ge d+1$ from step $r(d)$ on). Since fields of every degree are obtained, the step depth $J$ grows without bound, and the sides grow along with it Inferred.

The price is that **the sequence of spacings is a special one, adapted to the construction**. This is the content of "no universality is claimed" in § 01.

05

## The sequence of volumes and the unit

The side of the volume is $L_p = \mathrm{page}(M + p) = 4(M + p + 1)$ Source (`YM/Price/Waits.lean`, lines 87 and 102). $M$ is a natural number taken large enough to satisfy the calendar, and the assessment says it is "fixed so as to satisfy the minimum size and the initial volume estimates" Assessment .

**The unit.** The unit of length is the spacing at step 0, with $a_0 = 1$. The committee has checked that every cutoff $N$ and every volume $p$ use the same yardstick, as the following four facts Assessment (the assessment § 11, part of the committee's combined target).

1. The reference spacing is $a(0) = 1$.
2. For all $N, p$, the RG trajectory arrives at step 0 at the same reference coupling $\beta_{N,p}(0) = \bar{\beta} > 0$ (10-04).
3. The quadratic term of the effective action is normalised to the identity matrix (10-01).
4. The coefficient of a label of degree $d$ equals the label itself at the reference step $r(d)$, independently of the cutoff and the volume (8-04).

So the length "1" is not just a lattice graduation; it is fixed as a physical unit by matching the coupling to $\bar{\beta}$ at step 0.

**Where test functions are placed.** At volume $p$, only test functions whose support lies **strictly** inside $\big((-L_p/2, L_p/2)^4\big)^n$ can be used Source (`suitable` in `PublicWilson.lean`, lines 75–77). A fixed test function always satisfies this condition for $p$ large enough.

**Distances longer than spacing 1.** Renormalisation stops at step 0 and does not proceed to scales coarser than spacing 1. The volume limit is handled by reflection positivity, the Chessboard Estimate, the transfer operator and compactness (Part 12).

06

## What this book's study had seen

This book's block map has two designs, with side 2 and side 3, both with a fixed side (19-04). The box at step $m$ has the form $(\mathbb{Z}/2^m L)^4$, and the direction of the steps is the same as in the published construction (the fine side has the larger index).

- The side-3 design (odd side, centred) is in some respects close to how the published construction is built. The difference is that in the published construction the side

grows.

- In this book's chain, the sequence of bare couplings $\beta_N$ was given from outside, and an estimate across steps was placed as a premise (28-04). The circumstance that one-step errors accumulate when the side is fixed is the same as in § 04; this book had placed it on the side of the premises.
- The idea of fixing the unit of volume by a matching at step 0 points in the same direction as this book's "matching" (matching a quantity determined by the unit length to a fixed value). The quantity used for the matching differs (14-03).

07

## What we have not read

- The explicit formula for $b_m$. We read only the statements of `Wait.ready` (`YM/Price/Waits.lean` line 137), which builds the calendar, and of `Program.exhaust` and `Rolls`.
- The size of $M$, and the content of the "initial volume estimates" that determine it.
- The values of the fields of the tuple of step constants `Stage` (paths, width, columns, relaxation and so on): which of them increase with $J$, and how fast.
- (4) is this book's inference. The conditions `slow` (drift/log L $\le$ 1/j$^2$) and `kick` (drift/log L $\le$ s/4) of the same accounting inside the construction, `YM.RG.Supply.Large` (`YM/Price/Intent.lean` lines 39–47), have the same form as `driftSmall` and do not contradict (4). We have not matched the whole accounting.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The public statement asks for strictly increasing $L_p$, $D_N$ and a running (1) uniform in the volume | How $\beta_{N,p}$ depends on $p$, beyond the slope of the running |
| Source $b_m$ is odd and at least 3, $D_m = \prod b_k$, $a_m = 1/D_m$, and $L_p$ is a multiple of 4, $4(M+p+1)$ | The explicit values of $b_m$ and $M$ |
| Proved here From the conditions of the calendar, $b_m$ is non-decreasing. Inferred $b_m \to \infty$ | An explicit form of the growth rate |
| Assessment Convergence holds along the whole of the chosen sequence of spacings | Universality with respect to the choice of the sequence of spacings (the assessment states explicitly that it is open) |

| Can say | Cannot say |
|---|---|
| Assessment The same yardstick for all $N, p$ (reference spacing, reference coupling, kinetic term, calibration of labels) | How things depend on $\bar{\beta}$ when it is changed |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Public sequences and the running condition | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 36–39, 50, 75–77, `ChallengeSpec.lean` lines 13–14, `HierarchyBridge.lean` lines 16–25, `WilsonBridge.lean` lines 20–30 |
| The set-up and the calendar | Source | `YM/Lattice/Scale.lean` lines 28–34, 39–41, 67, 99, 124, 129–133, `YM/RG/History.lean` lines 64–77, 97–118, 223, `YM/Price/Waits.lean` lines 87 and 102, `YM/Source/Construction.lean` lines 47–49 |
| Absence of universality, choice of the sequences, common yardstick | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4, § 4.3, § 11, Appendix A (entry for `WilsonFamily`) |
| Step-independent side, common yardstick (annotated example) | Source | On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 90–105 |

**Terms used in this chapter** — renormalisation step, side of a block, calendar, universality, physical unit. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-03 Labels of local fields and their lattice representatives — what is called a "field" on this lattice.

# Labels of local fields and their lattice representatives

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study.

**What this chapter shows** — that in the published construction local fields are given in three stages: labels (algebraic objects), lattice representatives (polynomials in finitely many links), and fields (classes of labels distinguishable by correlations). The space of fields $F_d = V_d/N_d$ is defined intrinsically as a quotient by the null space of the correlations, and in each degree it is linearly isomorphic to the space of polynomials of the free Maxwell field.

**Prerequisite chapters** — 8-01 (the lattice and conventions), 8-02 (the sequence of spacings). Classical curvature and covariant derivatives are in 3-02, and fields in the axioms in 4-09.

**Said first** — this chapter goes only as far as the **definitions** of labels and representatives. The renormalisation coefficients multiplying the representatives are in 8-04, and how correlations are built is in 8-05. The linear isomorphism with the Maxwell field means "the number of fields is the same"; it does not mean that the correlations are the same as those of the free field.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — fields in the public statement: an abstract type of labels, and finite-dimensional spaces of fields
2. Labels — curvature polynomials up to degree d: how weights are assigned, and the basis
3. Lattice representatives — polynomials that match in the Taylor expansion: a finite stencil and seven conditions

4. The space of fields — dividing by the null space of the correlations: an intrinsic definition, unchanged across phases
5. The linear isomorphism with the free Maxwell field: what is the same and what is not
6. What this book's study had seen: the plaquette average is a naive representative of degree 4
7. What we have not read

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01

## The statement — fields in the public statement

In the public theorem `JointExistence`, fields appear in the following form Source (`PublicWilson.lean` lines 44–46 and `ChallengeSpec.lean` lines 28–33 in `yang-mills-group/OpenAI-YM-20261028`).

- The family of lattices `WilsonFamily` has a type `Token` (labels) and insertions `insertion` that return a complex number for a label, a lattice point and a configuration.
- The continuum-side hierarchy `Hierarchy` has, for each degree $d$, a finite-dimensional complex normed space `Field d` ($= F_d$), a linear map `toLabel` into the space of labels, and `project`, which sends labels to fields.
- The reconstruction field (`Reconstructed`, same file, lines 120–123) asks that `toLabel` be injective, that every field be the image of some label (`represents`), and that the space of labels be exhausted by the images of fields (`covers`).

Reading the public statement alone, `Token` is a type whose content is unknown. **Which curvature polynomial a label stands for is not written in the public structure** Source (`PublicWilson.lean` line 44 has only `Token : Type`). It is written on the construction's side (`YM.Register.Token`, `YM.Labels.Frame`) and in the committee's checks Assessment (the assessment § 3, and the entry `WilsonFamily.Token` in Appendix A). This chapter reads the content on the construction's side.

The summary in the assessment § 3 is as follows Assessment . Local observables are gauge-invariant polynomials in the curvature and its covariant derivatives at a point, defined not as functions or distributions but as **abstract labels**, elements of a vector space. Labels have a weight (degree), and the labels of weight at most $d$ form a finite-dimensional complex vector space $V_d$. There are linear relations among labels coming from the Bianchi identity and elsewhere, and these relations are formulated through

correlation functions. The fields of the main theorem are elements of the sum over all degrees of the quotients $F_d = V_d/N_d$.

02

## Labels — curvature polynomials up to degree d

**Weight.** Give the curvature $F_{\mu\nu}$ weight 2 and each covariant derivative weight 1, and let the weight of a product be the sum. The curvature with $k$ covariant derivatives applied, $D_{\lambda_1}\cdots D_{\lambda_k}F_{\mu\nu}$, has weight $k+2$. $V_d$ is the set of all gauge-invariant polynomials in which every monomial has weight at most $d$. Examples Assessment (the assessment § 3):

$$-\operatorname{Tr}(F_{\mu\nu}F_{\rho\sigma}) \in V_4, \qquad -\operatorname{Tr}\big((D_\lambda F_{\mu\nu})(D_\kappa F_{\rho\sigma})\big) \in V_6 \ (1)$$

The weight corresponds to the "engineering dimension" of a field (it scales like length to the power $-d$). Since $V_d \subset V_{d+1}$, the degree is an upper bound.

On the construction's side, polynomials are classified by a pair of numbers (a bidegree) Source (`weight_slot` in `YM/Labels/Grading.lean`, lines 76–79): the first is the weight above, the second the number of curvature factors (one curvature variable with $k$ derivatives has $(k+2, 1)$).

**Basis and labels.** The structure `Frame G d` chooses a basis of $V_d$ Source (`YM/Labels/Basis.lean`, lines 50–57). There are three conditions.

- Basis element 0 is the identity field $\mathbf{1}$ (the constant polynomial 1).
- Each basis element is real (unchanged by complex conjugation).
- Each basis element belongs to a single bidegree, with weight at most $d$.

A label is a pair $(d, c)$, where $c$ is the vector of coefficients in this basis Assessment (`WilsonFamily.Token` in Appendix A of the assessment). Having the identity field at the head of the basis matters when renormalisation deals with mixing with the identity field (8-04).

The committee's annotated example shows that every gauge-invariant polynomial appears as a label of some degree $d$ (no upper bound on the degree is imposed overall) Source (`every_registered_polynomial_appears`, line 41 of `submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` in `yang-mills-group/YM`, revision 55ba9fc, on the assessment side). The same example writes that this is exhaustiveness "with respect to the submitted algebraic dictionary", that identifying the dictionary with the intended local fields is a matter of human review, and that

completeness for non-local observables such as line defects is not claimed (lines 35–38).

03

## Lattice representatives — polynomials that match in the Taylor expansion

Labels are algebraic objects and have no correlations. To measure on the lattice, a **lattice representative** is chosen for each label. The idea can be seen in the example in the assessment § 4.1 Assessment . For smooth links, $U_{\mu\nu} = 1 + a^2 F_{\mu\nu}(0) + O(a^3)$, so

$$a^{-4}\Big(1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}\Big) \;\longrightarrow\; -\frac{1}{2m}\operatorname{Tr} F_{\mu\nu}(0)^2 \quad (2)$$

(the same computation as (7) in 8-01). Hence $1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$ is a **candidate representative** of the label $-\frac{1}{2m}\operatorname{Tr} F_{\mu\nu}^2$. The assessment notes, however, that the construction does not choose this naive plaquette term itself as the representative, because of additional symmetries.

The definition on the construction's side has the following two stages.

**Stencil and polynomial.** For each degree $d$, choose a finite stencil (a finite set of lattice offsets) $S_d \subset \mathbb{Z}^4$ and a linear map from coefficient vectors to polynomials. The variables of the polynomial are the matrix entries of the links on the edges in the stencil and their complex conjugates Source ( `SectionData` in `YM/Source/Local.lean`, lines 103–105; the variable index `Index` is at lines 92–93). The stencil and the polynomial do not depend on $N, p$ Assessment ( `WilsonFamily.insertion` in Appendix A). The representative at a point $x$ is the stencil translated to $x$ ( `places` ).

**Seven soundness conditions.** `SectionData.Sound` asks for the following Source ( `Local.lean`, lines 140–147).

1. The stencil contains the origin.
2. The stencil is invariant under the hypercubic group (signed permutations of coordinates).
3. The polynomial is gauge invariant.
4. The coefficient vector of the identity field is mapped to the constant 1.
5. It is compatible with the action of the hypercubic group (rotating the configuration is the same as rotating the coefficient vector).
6. Complex conjugation of the coefficients is mapped to complex conjugation of the values (in particular, a real coefficient vector has real values; `ordinary_real` ).

7. **The Taylor condition** (`Projects`, lines 85–90): for every "jet" $J$ (the coefficients of a polynomial gauge potential), the value $f(h)$ of the polynomial on the links built from parallel transport at scale $h$, and the value $g(h)$ of the continuum label at $J$ shrunk by scale $h$ (the coefficient of order $i$ multiplied by $h^{i+2}$), have the same derivatives at $h = 0$ up to order $d$.

Condition 7 is the precise form of "the Taylor expansion matches up to order $d$". In the words of the assessment, "on links obtained by parallel transport of a polynomial gauge potential, the Taylor expansion in the spacing agrees up to order $d$ with the rescaled continuum label" Assessment . This book reads conditions 2 and 5 as keeping the lattice symmetry (the hypercubic group) at the level of representatives, corresponding to the "additional symmetries" in the assessment Inferred .

On the side of the public statement, only three properties of the insertions are asked for Source (`ChallengeSpec.lean`, lines 15–22): continuity in the configuration, invariance under gauge transformations, and, for each label, a finite stencil independent of $N, p, x$ such that the insertion depends only on the links in it (`localSupport`).

04

## The space of fields — dividing by the null space of the correlations

Different labels can have the same correlations (the Bianchi identity, the equations of motion and so on). The construction defines this identification **through the correlation functions themselves** Source (`YM/Euclid/Null.lean`).

For a table of continuum correlations $S$ (the $S^{\mathrm{mix}}$ of 8-05), a label $v$ of degree $d$ is **null** if, for any number of spectator labels $B_1, \dots, B_n$ and any mutually distinct points $x_0, \dots, x_n$,

$$S\big(v(x_0)\, B_1(x_1) \cdots B_n(x_n)\big) = 0 \quad (3)$$

(`criterion`, lines 36–38; the null space `null` at line 34 is the kernel of the linear map listing all correlations at separated points). The space of fields is the quotient

$$F_d = V_d / N_d \quad (4)$$

(`Field`, line 74). Its norm is the quotient norm coming from the maximum norm of the basis coefficients Assessment (`Hierarchy.Field` in Appendix A).

Two injections appear, so we separate them. One is the map from the quotient $F_d$ to the totality of correlations at separated points, induced by the map whose kernel is the null space (`spectra`, `Null.lean` line 93); it is injective by the definition of the quotient

Source . The other is `toLabel` in the public statement, a linear map from the space of fields into the space of labels of the reconstructed theory (`ChallengeSpec.lean` line 32). Its injectivity is asked for as the public field `Reconstructed.injective` (line 120), and the construction sets `toLabel := S.pin` Source (`HierarchyBridge.lean` line 32). The committee's example notes about the former that "injectivity alone does not show that too many fields have not been killed", and says that the substantive result is the comparison of dimensions in § 05 below Source (`OperatorContent-20261028.lean` lines 71–86).

**Choosing a phase loses no fields.** In 8-05 the mixed theory $S^{\mathrm{mix}}$ is split into an average of components (phases) $S^{\xi}$. The committee's example shows that in every phase $\xi$ the null space is the same as that of $S^{\mathrm{mix}}$ Source (`each_phase_retains_the_dictionary`, line 103). The quantifier is "every phase", not "almost every phase", which is a stronger statement than the mixed theory being an average of phases.

Defining fields by a quotient is the natural choice of treating as the same whatever the correlations cannot distinguish. The space of fields does not depend on the choice of lattice representatives; it is determined by the continuum correlations alone.

05

## The linear isomorphism with the free Maxwell field

The assessment § 3 says that the construction has shown that, in each degree $d$, $Fd$ is **linearly isomorphic** to the space of gauge-invariant polynomials coming from classical Maxwell theory Assessment . The space compared with is the polynomials in the free curvature and its derivatives of weight at most $d$, with two identified when they agree on formal curvature data satisfying the source-free Maxwell equations and the Bianchi identity (`Maxwell.shellSector G d`).

$$Fd \;\cong_{\mathbb{C}\text{-linear}}\; (\text{polynomials on the Maxwell shell, weight} \le d) \quad (5)$$

The committee's example shows (5) as a Lean statement Source (`interacting_field_dictionary`, `OperatorContent-20261028.lean` lines 89–91; from `local_field_content` at line 45 of `submissions/openai-20261028/checks/CompletenessAudit.lean` on the assessment side). On the free side, it is also shown that the space compared with has no extra null elements (`free_residual_vanishes`, line 62).

**What is the same and what is not.** The same example states the following explicitly Source (lines 84–86).

- (5) is an **isomorphism of vector spaces**, not an equality of interacting and free correlations.
- It is neither an isomorphism of algebras nor a unitary equivalence of quantum field theories.
- Nor is it a comparison with an independently constructed Hilbert space of standard Yang–Mills.

What (5) says is that "in each degree, the number of independent fields is the same as for the free Maxwell field". Put differently, there are exactly as many relations among labels that vanish in the correlations as there are relations of the free field (the equations of motion and the Bianchi identity) Inferred . That the number of fields is neither too large nor too small confirms, at the level of counting, the requirement of the problem statement that "the space of fields corresponds to curvature polynomials" (item 2 of the assessment § 1).

This comparison is part of the committee's combined target (`combined_acceptance`) Assessment (the field-content check in the assessment § 9.3). On the construction's side, the source of the comparison is the witness field `F.phase.transport.comparator` (`Maxwell.Ranks`) Source (the field `comparator` at line 40 of `YM/Spec/Closed.lean`).

06

## What this book's study had seen

This book's Lean has **no** labels of local fields, no space of fields, and no comparison with the Maxwell field. This book's observables were functions of the links of a coarse lattice — the plaquette average, Wilson loops, the ratio of twisted partition functions $\theta_M$ (Parts 19–21).

- The plaquette average $\frac{1}{6}\sum_{\mu<\nu}\operatorname{Re}U_{\mu\nu}$ is, in view of (2), a **naive representative** (plus a constant) of a label of degree 4 (of the form $\operatorname{Tr}F^2$) Inferred . In the published construction, however, correlations converge only for the renormalised insertions of the next chapter, 8-04, whose coefficients should include mixing with the identity field (a term of order $a^{-4}$). To speak of the continuum limit of this book's plaquette average, this mixing would have to be reread in this book's ledger (14-06).
- Wilson loops and twisted partition functions do not belong to the space of local labels. The committee's example also explicitly excludes line defects.
- The pattern of defining an object intrinsically by dividing by the null space of the correlations is one that could also be used to define continuum objects for this book's non-local observables Inferred .

07

## What we have not read

- The explicit form of the representative polynomials (the size of the stencil, which polynomial). We have not read where concrete instances of `SectionData` are built.
- The details of the parallel-transport flow in the Taylor condition (`Hol.flow`, `links`) and of the definition of reading the continuum label (`read`).
- The proof of (5) (`Maxwell.Ranks` and its source), including where the basis of the space compared with is counted.
- Where the second component of the bidegree (the number of curvature factors) is used, other than in the renormalisation scaling (`scaled` in 8-04).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source In the public statement, labels are an abstract type and fields are finite-dimensional spaces `Field d` | The meaning of labels from the public statement alone (it lies on the construction's side and in the committee's checks) |
| Source Representatives are polynomials in the links on a finite stencil, satisfying seven conditions (including the Taylor condition) | The explicit form of the representatives |
| Source $F_d = V_d/N_d$, where $N_d$ consists of the labels all of whose correlations at separated points vanish | Relations where points coincide (correlations are only at separated points; 8-05) |
| Source In each degree, $F_d$ is linearly isomorphic to the space of polynomials on the Maxwell shell, the same in every phase | Agreement of correlations, isomorphism of algebras, unitary equivalence; completeness for line defects |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Public labels, fields, conditions on insertions | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 44–46, `ChallengeSpec.lean` lines 15–33 and 117–123 |
| Weight, basis, representatives, the Taylor condition | Source | `YM/Labels/Grading.lean` lines 76–79, `YM/Labels/Basis.lean` lines 50–57, `YM/Source/Local.lean` lines 85–105 and 140–147 |
| The null space and the quotient | Source | `YM/Euclid/Null.lean` lines 34–38 and 74 |

| Item | Kind | Source / tools |
|---|---|---|
| Description of labels, comparison with Maxwell, examples (1)(2) | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 3, § 4.1, § 9.3, Appendix A |
| The committee's examples (exhaustiveness, linear isomorphism, retention across phases) | Source | On the assessment side, `submissions/openai-20261028/docs/examples/OperatorContent-20261028.lean` lines 35–38, 41, 62, 71–91, 103, and `submissions/openai-20261028/checks/CompletenessAudit.lean` line 45 |

**Terms used in this chapter** — label, lattice representative, stencil, Taylor condition, null space of the correlations, space of fields, Maxwell shell. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-04 Renormalised insertions — what coefficients the representatives must be multiplied by for the correlations to converge.

# Renormalised insertions

This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study.

**What this chapter shows** — that in the published construction it is not the naive representatives of labels but the **renormalised insertions** whose correlations converge, and how their coefficients are determined. The insertion is carried exactly to the coarser step one step at a time, and the coefficients at the fine step are chosen through the inverse of the transport so that at the reference step the coefficient is the label itself.

**Prerequisite chapters** — 8-03 Labels of local fields and their lattice representatives. The idea of divergences and renormalisation is in 4-04, and one renormalisation step in 9-01 and 9-06.

**Said first** — there is no explicit formula for the coefficients (the assessment says so too). How mixing with the identity field enters can be read only as far as the types in the source allow; we have not been able to read the form of the mixing matrix. The line of proof that the coefficients converge in $N$ is treated in 11-10.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. The statement — the form of the insertion: a sum of representatives multiplied by $a^{d}_{N}$ and coefficients
2. Why coefficients are needed — mixing: why the correlations of naive representatives diverge
3. Transport and its inverse — calibration at the reference step: carry the insertion exactly, and read it at the reference step

4. Where mixing with the identity field and lower-degree fields sits: within the coordinates of the space of coefficients
5. Stability in the cutoff (outline): finite-dimensional linear maps and shrinking remainders
6. What is not said
7. What this book's study had seen
8. What we have not read

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01

## The statement — the form of the insertion

The public statement asks only three things of an insertion $I_{N,p}(v,x;U)\in\mathbb{C}$: continuity, gauge invariance, and locality on a finite stencil (8-03 § 03) Source . What an insertion concretely is lies on the construction's side.

The assessment § 4.2 writes as follows Assessment . For a label $v\in V_d$ of weight $d$ and a basis $v1,\dots,vM$ of $Vd$, the ultraviolet convergence statement associates to $v$

$$I_{N,p}(v,x;U)=\sum_{j=1}^{M} c_j(N,p;v)\,L_{vj}(x;U) \tag{1}$$

a **renormalised lattice observable**. The coefficients are linear in $v$, contain powers of the spacing, and may diverge in the limit. The convergence theorem says that renormalisation coefficients $c_j(N,p;v)$ making the correlations converge **exist**. The submission gives formulas expressing the coefficients through lattice renormalisation transformations, but does not express them explicitly as sums or integrals.

The entry `WilsonFamily.insertion` in Appendix A goes one step further Assessment . At sufficiently fine spacing, the renormalisation transport gives an **invertible linear map** from coefficients at spacing $a_N$ to coefficients at the reference spacing $a_{r(d)}$ ($r(d)\le N$, with $r(d)$ depending only on $d$). The construction applies its inverse to $c$ (so that the transported coefficients equal $c$) and multiplies by a scale factor coordinate by coordinate. Writing the resulting coefficients as $\tilde c_{N,p,j}$,

$$I_{N,p}\big((d,c),x;U\big)=a_N^{-d}\sum_{j=1}^{k}\tilde c_{N,p,j}\,P_{d,j}(x;U) \tag{2}$$

where $P_{d,j}$ are the lattice representatives of 8-03. The code defines this map and shows that it is invertible, but gives no explicit formula for the matrix entries of the inverse.

02

## Why coefficients are needed — mixing

Consider the plaquette example of 8-03. The expectation of $P = 1 - \frac{1}{m}\operatorname{Re}\operatorname{Tr} U_{\mu\nu}$ is not determined by the Taylor expansion of a smooth configuration, $a^4 \cdot \frac{1}{2}\|F\|^2$, alone. Quantum fluctuations occur at the scale of the spacing $a$, and their contribution enters the expectation of $P$ as a quantity far larger than $a^4$. At weak coupling $\langle P \rangle$ is of order $g_0^2$, and since $g_0^2 \sim 1/(2b_0 \log(1/a))$, it becomes small only logarithmically in the spacing Inferred. The expectation of $a^{-4}P$ then diverges like $a^{-4}/\log(1/a)$. This is a manifestation of the degree-4 field $\operatorname{Tr} F^2$ **mixing with the identity field 1 (degree 0)**. For the same reason, in general a field of degree $d$ mixes with lower-degree fields of the same symmetry with coefficients $a^{-(d-d')}$ (renormalisation of composite operators; 4-04).

So the coefficients of a linear combination of naive representatives must be adjusted along with $a$ to cancel the mixing. To write the cancellation explicitly, one would need to know the mixing coefficients in every degree. Instead of writing them explicitly, the published construction takes the approach of **letting the renormalisation transformation itself carry the coefficients**.

03

## Transport and its inverse — calibration at the reference step

**Density with an insertion.** The construction treats the insertion as a first-order perturbation of the density. At step $m$, the information about the insertion is held in a tuple (`Coefficient`) Source (`YM/Source/Insertion.lean`, lines 26–30):

- local coefficients $x \in \mathbb{C}^k$ (coordinates in the basis of 8-03; coordinate 0 is the identity field),
- a regular remainder (a sequence of functions with supports),
- a large-field remainder (the part attached to forced activities).

The density with an insertion is the density $\rho_m$ at step $m$ (9-06) multiplied by "$x_0$ + the non-identity coefficients placed on local representatives + the regular remainder", with the large-field part added Source (`density`, lines 47–49; the docstring says "the formula written with $\lambda^2 = 0$ for the source $\lambda$"). The non-identity coefficients are placed with scaling by powers of the small-field threshold $t_m$ at step $m$ (9-07)

(`exponentialEntry`, lines 37–39; the scaling map `scaled` has the number of curvature factors as its exponent; `YM/Labels/Reference.lean` line 58).

**Transport.** For each label of degree $d$, cutoff $N$ and volume $p$, the construction has a family of transports `TransferData` Source (`Insertion.lean`, lines 73–75): `entries m v`, which at each step $m$ returns the tuple at step $m$ for an input $v$, and a linear isomorphism of the space of coefficients `terminal`. The main fields of the condition `Based` (lines 89–114) are as follows.

| Field | Content |
|---|---|
| `bare_coord`, `bare_germ` | At step $N$ the local coefficients are the input $v$ itself, and the remainders are $0$ |
| `encode` | The density with an insertion at step $N$ is the bare Wilson weight multiplied by the initial insertion (with a factor $a_N^d$) |
| `unnormalized` | The density with an insertion at step $m = b_m^d \cdot$ (the density with an insertion at step $m+1$ pushed forward by the one-step kernel). **An equality, not an approximation** |
| `reads` | The local coefficients at the reference step $r$ are $\mathbf{terminal}(v)$ |
| `unit_coord` | The identity field goes to the identity field: $a_N^d \mathbf{1}$ at step $N$ is mapped to $a_m^d \mathbf{1}$ at step $m$ |
| `back`, `front` | Estimates in both directions on the size of the coefficients at adjacent steps |
| `linear` | Coefficients and remainders are linear in $v$ |

That is, `terminal` is the linear map "place coefficients $v$ at step $N$, carry them by exact push-forward down to step $r$, and the local coefficients at step $r$ become $\mathbf{terminal}(v)$".

**Calibrating with the inverse.** The insertion for a label $v$ uses $\mathbf{terminal}^{-1}(v)$ as its input Source (`cylinder`, line 87; `Family.O`, `YM/Source/Construction.lean` line 42):

$$I_{N,p}(v) = \mathbf{initial}\big(\mathbf{terminal}^{-1}_{N,p}\, v\big), \qquad \mathbf{initial}(w) = a_N^{-d} \cdot P_d\big(\mathbf{scaled}_{t_N^{-1}}\, w\big) \quad (3)$$

(`initial`, lines 77–79). The reading at the reference step is then $\mathbf{terminal}(\mathbf{terminal}^{-1} v) = v$, **the label itself** (`normalized`, `Construction.lean` lines 65–68; the hypotheses are `Makes` and $r(d) \le N$) Source. The construction's witness condition `Makes.calibrated` (line 62) asks for this `Based` for all $d, N, p$ (with $r(d) \le N$).

The committee checks this as one of the normalisations common to all volumes Assessment (the fourth item of the assessment § 11: "the normalised coefficient of a label $v$ of degree $d$ equals $v$ at the reference step $r(d)$, independently of the cutoff and the volume"). The text is in `insertion_has_the_prescribed_reference_value` (line 105) of the committee's annotated example `LatticeAndScale-20261028.lean` Source.

Comparing (2) and (3), the $\tilde{c}_{N,p}$ of the assessment corresponds to $\mathrm{scaled}_{t_N^{-1}}\left(\mathrm{terminal}_{N,p}^{-1} c\right)$ Inferred . The "scale factor coordinate by coordinate" is the threshold $t_N$ at step $N$ raised to the power of the number of curvature factors.

04

## Where mixing with the identity field and lower-degree fields sits

The space of coefficients consists of coordinates in the basis of labels of degree at most $d$ (the `Frame` of 8-03). Basis element 0 is the identity field, and the other basis elements have weight at most $d$, so **lower-degree fields are in the same space** Source (`YM/Labels/Basis.lean`, lines 50–57). The mixing of § 02 can therefore be read as appearing as **off-diagonal entries** of the matrices of the step-by-step linear maps (`leanings` in § 05) Inferred .

- `unit_coord` fixes the column of the identity field (the identity field goes to the identity field, multiplied only by the ratio of the step scales $a^d$) Source .
- Components flowing from other fields into the identity field (of the type $a^{-4}$) are allowed as coordinate 0 Inferred . The mixing of $\mathrm{Tr}\, F^2$ with $\mathbf{1}$ in § 02 should enter here.
- The estimates on the remainders are written in terms of the size of the part with the identity-field component removed, `core` (`Reference.lean` line 123) Source .

Within what we read, we found no statement writing the mixing as a triangular matrix, and no statement subtracting the identity-field component. The inverse map $\mathrm{terminal}^{-1}$ works by cancelling the mixing all at once.

05

## Stability in the cutoff (outline)

The coefficients in (3) differ for each $N$. For the correlations to converge as $N \to \infty$, the transport maps must be stable in $N$. The construction's line of argument is as follows Source (details in 11-10).

1. The map at each step (`Club`, `YM/FullVolume/Mark/Nursery.lean` lines 11–16): a linear isomorphism `leanings` from the coefficients at step $m+1$ to those at step $m$, depending on the Gram of the remainder at step $m+1$. Its inverse is uniformly bounded (`hail`). The remainder map `flow` halves differences (`trail`).
2. Fix the value at the reference step $r$ and carry it to the fine side by inverse maps (`Gear.spends`, `Mark/Sages.lean` line 66): if it is Cauchy at the reference step, the

map at each step converges in $N$, and the inverses are bounded, then it is Cauchy at every step above.

3. As a result, the coefficients and the values of the insertions at steps $m \geq r$ are Cauchy in $N$ (`Branch.winds`, `stays`, `Gables/Master.lean` lines 44 and 50).

The maps are finite-dimensional linear maps, and the remainders shrink from above, so no compactness is needed in this part. There is no statement of a rate; the convergence is qualitative Source .

06

## What is not said

- An explicit formula for the coefficients $\tilde{c}_{N,p,j}$ (stated explicitly by the assessment) Assessment .
- That the mixing matrix is triangular, or that the identity-field component is written as a subtraction (not within what we read).
- That the coefficients and the limit do not depend on the choice of the reference step $r(d)$.
- That the coefficients agree with the renormalisation of composite operators in perturbation theory.

07

## What this book's study had seen

This book's Lean has **no** renormalisation of composite operators.

- This book's observables (the plaquette average, the ratio of twisted partition functions) were used without renormalisation. In the chain that matches the plaquette average at a coarse step to a fixed value at step $m_0$ (Part 28), the design was such that no coefficient needed to multiply the observable itself.
- This book treated the linearisation of the block map and the covariance of the push-forward (15-13). There the linearisation is how field fluctuations are carried; it was not used as a map on the space of coefficients of local observables.
- The view "give the renormalisation of composite fields as the inverse of the linearisation of the renormalisation transport" is an idea this book did not have (14-05). It is also an entry point for speaking of the continuum limit of this book's plaquette average.

08

## What we have not read

- The content of the step-by-step linear isomorphisms (`Gram.reads`, `Mark/Hinge.lean`) and the form of their matrices.
- How `terminal` is actually built (concrete instances of `Family.factory`).
- The value of the reference step $r(d)$. We have read nothing beyond `Makes.stages` asking for $d + 3 \leq J(r(d))$.
- How the large-field remainder of the density with an insertion (`marked`) is reorganised under one step of push-forward.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Assessment The insertion is a linear combination of representatives; the coefficients are determined by the inverse of the renormalisation transport and a scale factor, with no explicit formula | An explicit formula for the coefficients |
| Source The transport is the exact push-forward by the one-step kernel; the reading at the reference step is $\mathrm{terminal}(v)$, and the insertion takes $\mathrm{terminal}^{-1}(v)$ as its input | The explicit form of terminal |
| Source At the reference step the coefficient is the label itself (independent of cutoff and volume) | Independence of the choice of reference step |
| Inferred Mixing with the identity field enters as components into coordinate 0 of the linear maps | The form of the mixing matrix (whether triangular) |
| Source The coefficients at steps $m \geq r$ are Cauchy in $N$ | The rate of convergence |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The insertion tuple, density, transport, calibration | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `YM/Source/Insertion.lean` lines 26–49 and 73–114, `YM/Source/Construction.lean` lines 24–30, 42, 62, 65–68, `YM/Labels/Reference.lean` lines 58 and 123 |
| Stability of the coefficients | Source | `YM/FullVolume/Mark/Nursery.lean` lines 11–16, `YM/FullVolume/Mark/Sages.lean` line 66, `YM/FullVolume/Gables/Master.lean` lines 44 and 50 |

| Item | Kind | Source / tools |
|---|---|---|
| Equations (1)(2), absence of an explicit formula, common normalisation | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4.2, § 11, Appendix A (`WilsonFamily.insertion`) |
| Common normalisation (annotated example) | Source | On the assessment side, `yang-mills-group/YM`, revision 55ba9fc: `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` line 105 |
| Mixing of composite operators | Cited | Introduction in 4-04 |

**Terms used in this chapter** — renormalised insertion, renormalisation transport, reference step, calibration, identity field, mixing. Their meanings are in the A-02 Glossary.

Next chapter to read: 8-05 Correlation functions and the order of limits — in what order of limits the correlations of renormalised insertions are carried to the continuum.

# Correlation functions and the order of limits

This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study.

**What this chapter shows** — the form of the convergence statement `Converges` inside the public statement `JointExistence`. Limits of the smeared lattice correlations $C_{N,p}(v;f)$ are taken in the order (i) $N \to \infty$ at fixed volume $p$ (along the whole chosen sequence of spacings), then (ii) along a subsequence of volumes, giving a mixed theory $S^{\mathrm{mix}}$. $S^{\mathrm{mix}}$ is a probability average of components $S^{\xi}$.

**Prerequisite chapters** — 8-02 (spacings and volumes), 8-03 (labels and fields), 8-04 (renormalised insertions). Distributions and test functions are in 1-11, and the OS axioms in 4-10.

**Said first** — all correlations are defined only where the points are mutually distinct. The volume limit is along a subsequence, and it has not been shown that the limit is independent of the choice of subsequence. Nor is it said that the order of the two limits may be exchanged. The content of the phase decomposition and of reconstruction is treated in Part 12.

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Smeared lattice correlations: the formula and the admissible test functions
2. The convergence statement — two limits and their order: the three fields of `Converges`
3. The quantifiers of the public theorem — a subsequence and one hierarchy: $\forall p$ $\exists q$ $\exists T$
4. Why test functions at separated points: singularities and contact terms

5. The mixed theory and its components: $S^{\mathrm{mix}} = \int S^{\xi}\, d\mu$
6. What is not said
7. What this book's study had seen
8. What we have not read

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01

## Smeared lattice correlations

For a sequence of labels $v = (v_1, \dots, v_n)$ and a test function $f$, the lattice correlation is defined as

$$C_{N,p}(v; f) = a_N^{4n} \sum_{x_1,\dots,x_n \in \Lambda_{N,p}} f(a_N x_1, \dots, a_N x_n)\, \mathbb{E}_{\mu^{\mathrm{W}}_{N,p}}\Big[\prod_{j=1}^{n} I_{N,p}(v_j, x_j; U)\Big] \quad (1)$$

Source (`PublicWilson.lean` in `yang-mills-group/OpenAI-YM-20261028`: `finiteMoment` lines 54–57, `smear` lines 79–81) Assessment (equation (6) of the assessment § 4.3). The sum runs over all ordered $n$-tuples of lattice points, and $a_N^{4n}$ is the factor that matches the lattice sum to the integral $\int d^{4n}x$. $I_{N,p}$ is the renormalised insertion of 8-04, and $\mu^{\mathrm{W}}_{N,p}$ the Wilson measure of 8-01.

**Admissible test functions** Source (lines 58–77):

- Smooth, compactly supported complex functions (`SeparatedTest n`) on the open set $\{(x_1, \dots, x_n) : x_i \neq x_j \ (i \neq j)\}$ (`separated n`). The support stays away from where points coincide.
- They are not restricted to product form $f_1(x_1) \cdots f_n(x_n)$.
- When used at volume $p$, the support must lie inside $\big((-L_p/2, L_p/2)^4\big)^n$ (`suitable p`). A fixed $f$ always satisfies this for $p$ large enough.

02

## The convergence statement — two limits and their order

For a strictly increasing sequence of volume indices $k \mapsto p_k$ (`volumeOrder`) and continuum correlations $S$, `Converges` asks for the following three things Source (`PublicWilson.lean`, lines 85–90). For every $n$, every sequence of labels $v$ and every test function $f$:

1. For $k$ large enough, $f$ fits in volume $p_k$.
2. There is a sequence $\mathrm{row}(p)$ such that, for **every** volume $p$ in which $f$ fits, $C_{N,p}(v;f) \to \mathrm{row}(p)$ $(N \to \infty$, over **all** $N)$.
3. $\mathrm{row}(p_k) \to \int f(x)\, S(v;x)\, dx$ $(k \to \infty)$.

In the assessment's notation Assessment (`Properties.wilson` in Appendix A)

$$\lim_{k\to\infty} \lim_{N\to\infty} C_{N,p_k}(v;f) = \int_{(\mathbb{R}^4)^n} f(x)\, S^{\mathrm{mix}}(v;x)\, dx \quad (2)$$

**The cutoff is inside and the volume outside**, and this order is fixed.

**The inner limit.** Theorem 4.1 of the assessment states that the ultraviolet limit at fixed volume, $S_p(v;f) = \lim_N C_{N,p}(v;f)$, exists along the whole chosen sequence of spacings, and moreover that the kernel $S_p(v;x)$ is $C^\infty$ away from coincident points (modulo the period $L_p$) Assessment . The assessment summarises the proof as using the construction's renormalisation normalisation to compare sufficiently fine cutoffs with each other and show that the sequence is Cauchy in $N$. That line of argument is read in 11-10.

**The outer limit.** Theorem 5.1 of the assessment states that the volume limit can be taken along a subsequence, and that **a single subsequence works for all correlations at once** Assessment . The tool is Arzelà–Ascoli compactness: for each compact set $K$ of configurations at separated points, boundedness and equicontinuity uniform in the volume,

$$\sup_{p \ge p_0(K)} \sup_{x\in K} |S_p(v;x)| \le M_K, \qquad |S_p(v;x) - S_p(v;y)| \le \omega_K(|x-y|) \quad (3)$$

are shown (equation (11) in the assessment § 5). The assessment writes that these two estimates come from general tools — reflection positivity, the Chessboard Estimate, the transfer-operator representation and Cauchy estimates — and that the input specific to the construction is control of the renormalised observables uniform in the spacing Assessment . The content is in 12-04 and 12-05.

The committee's annotated example states explicitly that this statement "makes no pointwise claim at coincident points" and "does not say that the ultraviolet and infrared limits can be exchanged" Source (`submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` lines 113–116 in `yang-mills-group/YM`, revision 55ba9fc, on the assessment side).

> 03

## The quantifiers of the public theorem — a subsequence and one hierarchy

The public theorem has the following form Source (`ChallengeSpec.lean`, lines 186–188).

```
∀ g (simple), ∃ C : WilsonFamily g, WilsonRegularity C ∧
  ∀ p (strictly increasing sequence of volumes), ∃ q (strictly increasing),
    ∃ T : Hierarchy C, T.Properties (p ∘ q)
```

We split the reading into three points.

- **The family comes first.** The family of lattices $C$ (spacings, volumes, couplings, insertions) is fixed once, before the sequence of volumes is asked for.
- **A subsequence for every sequence of volumes.** For each requested sequence of volumes $p$, taking a further subsequence $q$, (2) holds along $p \circ q$. This is the form of a compactness statement.
- **One hierarchy.** On that subsequence, one hierarchy $T$ (fields, correlations, the reconstructed theory, the OPE, the trace field) satisfies four properties at once — convergence `wilson`, reconstruction `reconstruction`, the OPE `ope`, and non-freeness `nonfree` (`Properties`, lines 177–181).

$T$ is chosen for each $p$, and it is not said that different $p$ give the same $T$. The assessment writes that whether the limiting correlations depend on the chosen subsequence is unresolved, and notes that the Jaffe–Witten problem statement regards subsequential limits obtained by compactness alone as insufficient (unless arguments showing properties of the limit are added) Assessment (the assessment § 5).

The conclusion of the construction's theorem `constructed` contains more than the public statement, and among it is the statement `FullVolume.PositionsUV C Sequence.full` Source (`Construction.lean` line 24). The definition (`YM/FullVolume/Profiles.lean`, lines 25–36) has the form: for each volume $p$, there is a kernel smooth at separated points, the smeared correlations converge to that kernel along the whole sequence of cutoffs, and the differenced correlations converge to the derivatives of the kernel Source (we read only the form of the definition). In the public statement too, the cutoff runs along the whole sequence and only the volume is a subsequence. The stronger form with kernels and differences exists only in the construction's theorem.

04

### Why test functions at separated points

Correlations of local fields become singular as points approach each other. For example, the two-point function of a degree-4 field is naively of size $|x-y|^{-8}$, with logarithmic corrections on top (13-02). This is not locally integrable on $\mathbb{R}^4 \times \mathbb{R}^4$, so to smear it with test functions that include coincident points, terms supported on the coincidence set (contact terms) must be determined. Contact terms are not uniquely determined by the theory (the ambiguity of renormalisation; 4-04).

By restricting the convergence statement from the start to test functions at separated points, the published construction removes this problem from the convergence statement. That a distribution extending to coincident points **exists** is included in the construction's theorem as a separate statement (extension of contact terms, existence only) (13-04).

The committee's independent check of the OS axioms uses a different space of test functions — Schwartz functions all of whose derivatives of every order vanish on all diagonals Assessment (equation (21) in the assessment § 10). This contains the compactly supported functions at separated points, but contact terms on the diagonals are not visible.

05

## The mixed theory and its components

The limit $S^{\mathrm{mix}}$ in (2) is called a "mixed" theory, because the property that correlations of two groups moved far apart factorise into a product (clustering) has not been shown Assessment (the assessment § 6). Without clustering, the theory obtained by OS reconstruction need not have a unique vacuum.

The construction gets around this by a **phase decomposition** Assessment (the assessment § 6.1, and `Hierarchy.Character` in Appendix A). In the OS Hilbert space built from $S^{\mathrm{mix}}$, it shows that the C*-algebra generated by operators compressed to the zero-energy subspace $H_0$ is commutative, and takes its Gelfand spectrum $X$ (a compact metric space) as the space of phases. For a probability measure $\mu$ on $X$ representing the vacuum state (`phaseMeasure` in the source; the assessment writes $\nu$),

$$S^{\mathrm{mix}}(v;x) = \int_X S^{\xi}(v;x)\, d\mu(\xi) \qquad (x \text{ a configuration of separated points}) \quad (4)$$

In the public statement, correlations come in two kinds, `correlation none` ($S^{\mathrm{mix}}$) and `correlation (some ξ)` ($S^{\xi}$), and (4) is the reconstruction field `mixture` Source (`ChallengeSpec.lean` lines 41 and 133–134). The same collection of fields includes that

$\mu$ is a probability measure (`normalized`) with full support (`fullSupport`), and that $\xi \mapsto S^{\xi}(v;x)$ is continuous (`characterContinuity`) (lines 124–125 and 131–132).

Each component $S^{\xi}$ satisfies a variant of the OS axioms that includes clustering, and one phase $\xi^*$ (`selected`) is chosen to reconstruct a Wightman theory. The content is in 12-07 and 12-08.

**The limit is non-zero.** The committee has checked that, for a fixed test function $f$ at separated points, the two-point lattice correlation of the renormalised trace field has a non-zero limit in the same order of limits Assessment (equation (23) in the assessment § 11):

$$\lim_{k\to\infty}\ \lim_{N\to\infty} \left|C_{N,p(q(k))}(\Theta,\Theta;f)\right|^{1/8} = \left|S_2^{\mathrm{mix}}(\Theta,\Theta;f)\right|^{1/8} > 0 \quad (5)$$

The committee's example writes that this is neither a spectral mass, nor a mass gap, nor a rate of convergence, and, since the test function and its normalisation may depend on the witness, not an intrinsic phase-independent scale either (`LatticeAndScale-20261028.lean` lines 136–157).

06

## What is not said

- Universality with respect to the choice of the sequence of spacings Assessment .
- Uniqueness of the volume limit (independence of the choice of subsequence). The first of the open questions in the assessment § 8 Assessment (16-02).
- Clustering of $S^{\mathrm{mix}}$. Equality of the components with one another. That each component is a subsequential limit of states in the original periodic volumes Source (`LeadingTrace-20261028.lean` lines 151–153).
- Exchanging the order of the two limits. Values at coincident points.
- A continuum probability measure (there are correlations, but no measure on field configurations is constructed; the second item of the assessment § 8) Assessment .

07

## What this book's study had seen

The chain of Volume 3 of this book took as its object the **distribution of block variables** at a coarse step (the push-forward measure) (28-01), not correlations of local fields at separated points.

- What is shared: the form of first considering the limit that removes the cutoff in a box of fixed physical size, and taking the limit along a subsequence by compactness.

- What differs: this book's implications obtained the existence of subsequential limits from compactness alone and carried the properties in the fields of assumptions. The published construction obtains the ultraviolet limit at fixed volume **along the whole sequence, not a subsequence** (its source is the uniqueness from "anchor and contraction" in 11-10).
- This book has nothing corresponding to the phase decomposition; its non-degeneracy condition (matching the expectation of the plaquette average to a fixed value) was designed to play the same role as the non-zero limit (5), with a different observable Inferred .

08

## What we have not read

- How the differences inside `FullVolume.PositionsUV` are built (`squall`, `Cluster`, `Meets`), and their proofs.
- How $p_0(K)$ in (3) is determined, and the details of why the lower bound on the volume depends on $K$.
- The proof of `fromWilson`, beyond `wilsonLimit` (`HierarchyBridge.lean` lines 44–56), which derives the public `Converges` from the construction.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Correlations have the form (1) and are smeared with smooth compactly supported test functions at separated points (not restricted to product form) | Convergence with test functions that include coincident points |
| Source The order of limits is "fix the volume and remove the cutoff along the whole sequence → subsequence of volumes" | Exchanging the order |
| Source The family is fixed first, and for every sequence of volumes there are a subsequence and one hierarchy | That the hierarchy does not depend on the subsequence |
| Source $S^{\mathrm{mix}} = \int S^{\xi} d\mu$, where $\mu$ is a probability measure with full support | Clustering of $S^{\mathrm{mix}}$, agreement of the components |
| Assessment The two-point lattice correlation of the trace field has a non-zero limit (5) | That (5) is an intrinsic value of a mass or a scale |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Correlations, test functions, the convergence statement, the public theorem | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e): `PublicWilson.lean` lines 54–90, `ChallengeSpec.lean` lines 41, 117–134, 177–188, `Construction.lean` lines 15–31, `HierarchyBridge.lean` lines 44–56 |
| Theorems 4.1 and 5.1, equations (6)(11)(23), the phase decomposition, open questions | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 4.3, § 5, § 6, § 6.1, § 8, § 10, § 11, Appendix A |
| The committee's annotated examples | Source | On the assessment side, `submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` lines 113–116 and 151–153, `submissions/openai-20261028/docs/examples/LatticeAndScale-20261028.lean` lines 136–157 |
| The Jaffe–Witten problem statement | Cited | A. Jaffe, E. Witten (official problem statement of the Clay Mathematics Institute, 2000). 0-03 |

**Terms used in this chapter** — smeared correlation, test function at separated points, order of limits, subsequence, mixed theory, phase (component), clustering. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-01 The block kernel — in Part 9 we read the content of what was called "one renormalisation step" here.

# The block kernel — one step is a probability kernel, not a map

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — what mathematical object "one renormalisation step" is defined to be in the published construction: the formula of the probability kernel $T_\beta(U, V)$ from fine configurations $U$ to coarse configurations $V$, the role of the penalty coefficient $c_{\mathrm{pen}}$, the fact that the $\beta$ entering the kernel is the value at the fine step, and which targets these declarations are used for.

**Prerequisite chapters** — 8-01 The lattice, the Wilson measure, conventions, 8-02 The set-up, 7-05 The sequence of renormalisation group steps. For comparison, 19-04 The block map and one step of renormalisation.

**Said first** — results of the published construction are written at the level of "the committee's assessment is that it is proved in Lean". What this book has checked goes no further than reading the definitions and the form of the statements in the source. The content of the predictor, the gauge fixing and the weights that enter the kernel is read in 9-02 to 9-04. Whether the kernel is of positive type with respect to reflections is not found in the statements of the source, and this book has not examined it either.

The order of this chapter

1. **The statement — one step is a probability kernel**: `StepData.density` and `push`, and the theorem that it is a kernel
2. **Reading the kernel's formula as three factors**: the probability of the choices, the predictor, Gaussian-type noise
3. **The penalty coefficient**: any positive number, and the main theorem quantifies over each
4. **The β of the kernel is the β of the fine step**: the one-step identity and preservation of the partition function
5. **Which targets it belongs to**: a table of declarations and records
6. **What this book's study had seen**: the difference from the push-forward by a deterministic block map

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — one step is a probability kernel

The published construction starts from the Wilson measure at lattice spacing $a_N = 1/D_N$ and coarsens it down to the step with spacing 1 by applying the block transformation $N$ times (step numbering as in 8-02: $m = N$ is the finest and $m = 0$ the coarsest). This "one step" is not a map that determines a single coarse configuration $V$ from a fine configuration $U$. It is defined as **a kernel that draws the coarse configuration at random** Source .

We set notation. For an element $u$ of the gauge group $G \subset U(d_G)$, write

$$\mathrm{price}(u) = 1 - \frac{\mathrm{Re}\,\mathrm{tr}\,u}{d_G}, \qquad \mathrm{chord}(u) = \frac{\|u-1\|_F}{\sqrt{d_G}}, \qquad \mathrm{price}(u) = \tfrac{1}{2}\,\mathrm{chord}(u)^2 \tag{1}$$

($\|\cdot\|_F$ is the Frobenius norm; the last identity is `chord_price` in `YM/Gauge/MatrixGroup.lean`, line 117) Source . price is a function of the same form as a single plaquette term of the Wilson action, $1 - \mathrm{Re}\,\mathrm{tr}\,U_p/d_G$; it is 0 at $u = 1$ and non-negative on the group. For SU(2), $d_G = 2$, and it agrees with this book's $1 - \mathrm{Re}\,U_p$ (19-01) Proved here .

The one-step kernel is defined in `StepData.density` (line 71) of `YM/RG/Block.lean` in the source, in the following form Source .

$$T_\beta(U,V)=\sum_{\tau} P(\tau\mid U)\prod_{b}\frac{1}{Z_\beta}\exp\!\Big(-c_{\mathrm{pen}}\,\beta\,\operatorname{price}\big(V_b\,Q_b(U,\tau)^{-1}\big)\Big),\qquad Z_\beta=\int_G e^{-c_{\mathrm{pen}}}$$

The sum over $\tau$ runs over finitely many discrete choices (pairs of "frame labels" and "path labels", read in 9-03 and 9-04), $P(\tau\mid U)$ is the probability of that choice, the product over $b$ runs over all positively oriented links of the coarse lattice, $Q_b(U,\tau)\in G$ is the **predictor** of a coarse link built from the fine configuration (9-04), and $h$ is the Haar measure normalised to a probability. The transport of functions by the kernel (push-forward) is

$$(\mathcal{T}_\beta F)(V)=\int T_\beta(U,V)\,F(U)\prod_{\ell} dh(U_\ell)\qquad (3)$$

whose name in the source is `StepData.push` (same file, line 77) Source . The output is, by convention, read as "a density with respect to the product of Haar measures on the coarse lattice" (docstring, lines 74–76).

That it is a kernel is stated as a theorem. `StepData.isKernel` (same file, line 203) assumes the three obligations `Patch.Branches` of 9-03 and 9-04 (uniqueness of the minimiser of the frame, uniqueness of the predictor, non-emptiness of the labels drawn at large field) and concludes, for every $\beta$, the following five things Source : $T_\beta\ge 0$; measurable in $(U,V)$; bounded above; integrable in $V$ for each $U$; $\int T_\beta(U,V)\,dV=1$. It is shown separately that, for each fixed penalty coefficient, the three obligations can be met for all patches (`Axial.viable` in § 03) Source .

## 02 Reading the kernel's formula as three factors

Formula (2) writes, in the language of probability, a procedure that draws the coarse configuration in the following order Inferred .

1. **Draw the choices.** Looking at the fine configuration $U$, draw the discrete choice $\tau$ with probability $P(\tau\mid U)$. `likelihood` in the source (line 49) is the product of the probabilities of the frame labels at each coarse point, `frameMass`, and of the path labels on each coarse link, `predictorMass`. At small field the labels are not used, and a uniform probability is put in formally (9-03, 9-04).
2. **Build the predictor.** From $U$ and $\tau$, build one group element $Q_b(U,\tau)$ for each coarse link (`prediction`, line 44).
3. **Add noise.** Draw each coarse link $V_b$ independently, link by link, with the weight $e^{-c_{\mathrm{pen}}\beta\,\operatorname{price}(V_bQ_b^{-1})}$ centred at $Q_b$ (`noise`, line 68).

The third factor is a probability measure in $V$ for each $Q$, because by right invariance of the Haar measure $\int Z_\beta^{-1} e^{-c_{\mathrm{pen}}\beta\,\mathrm{price}(vQ^{-1})}\,dh(v) = 1$ (`noise_mass`, line 146) Source. Together with the sum of the first factor being 1 (`chosen`, line 123), this gives $\int T\,dV = 1$ Proved here.

Let us look at the width of the noise. Writing $V_b Q_b^{-1} = e^X$ ($X$ a small element of the Lie algebra), with the assessment's inner product $\langle X, Y\rangle = -\,\mathrm{Re}\,\mathrm{tr}(XY)/dG$ (the assessment § 3), $\mathrm{price}(e^X) = \frac{1}{2}\langle X, X\rangle + O(|X|^3)$ Proved here. So the weight is roughly $\exp(-\frac{1}{2}c_{\mathrm{pen}}\beta\langle X, X\rangle)$, and $V_b$ is scattered around $Q_b$ with width about $(c_{\mathrm{pen}}\beta)^{-1/2}$ Inferred. At weak-coupling steps where $\beta$ is large, the kernel concentrates strongly near the predictor, but does not collapse to a point.

**A deterministic map (for comparison)**

The coarse configuration is fixed as a single $V = B(U)$. The push-forward is $\int F(B(U))\,d\mu(U)$; written as a kernel, it is the point measure $\delta_{B(U)}$ in $V$.

**A probability kernel (the published construction)**

$V$ fluctuates around the predictor with width $(c_{\mathrm{pen}}\beta)^{-1/2}$. No $\delta$-constraint is imposed. The docstring of `Block.lean` also states that no division by a density is done outside the kernel.

03

## The penalty coefficient — any positive number, and the main theorem quantifies over each

The $c_{\mathrm{pen}}$ in formula (2) is `KernelChoices.penalty` in the source (`YM/RG/Patch.lean`, lines 37–45). `KernelChoices` is a structure bundling the penalty coefficient and three thresholds (the frame threshold `frame_cut`, the predictor threshold `predictor_cut`, the axial-gauge factor `axial_factor`) as positive numbers, and its docstring says it is "chosen once" Source. This book writes the penalty coefficient as $c_{\mathrm{pen}}$.

The penalty coefficient can be taken to be any positive number. `Axial.viable` (`YM/Branch/Pick.lean`, line 137) says that for any $c_{\mathrm{pen}} > 0$ (the argument is named $\kappa$ in the source) there is a tuple whose `penalty` is that value, whose frame and predictor thresholds are determined by the gauge group alone, and whose axial-gauge factor is 16, such that the three obligations `Branches` hold for **all patches** (independently of the side $L$ and the smoothing exponent $J$) Source.

The construction's theorem `YangMills.constructed` (`Construction.lean`, lines 15–16) takes $b > 0$ as an argument and builds a family whose penalty coefficient is exactly $b$. That is, the main conclusion is quantified over each penalty coefficient Source . The public theorem `YangMills.jointExistence` (`Solution.lean`, line 4) uses it with 1 put in for this $b$ Source . The docstring says "the penalty coefficient is fixed before the family is chosen" (`Construction.lean`, line 13), and the step constants are chosen after the penalty coefficient is fixed Source .

The penalty coefficient also matters outside the kernel. The precision matrix of the one-step free field is defined as $K + c_{\mathrm{pen}} Q^{\mathsf{T}} Q$ (`baseMatrix`, `YM/Push/FREE_RESOLVENT/Flat.lean` lines 80–81), and the lower bound on the precision on the tree-gauge slice has the form $\min(a, c_{\mathrm{pen}})/(1 + \beta_L + \gamma_L)$ (`treeFare`, `tree_gap`, same file, lines 189–198) Source . If the penalty coefficient is small, this lower bound and the decay constants of the propagator worsen in proportion Inferred . The detailed form of the small-field step is read in 11-09. We find no estimate in the source's statements that is uniform as $c_{\mathrm{pen}} \to 0$ or $c_{\mathrm{pen}} \to \infty$ Source .

04

## The β of the kernel is the β of the fine step

Which value is put into the $\beta$ of formula (2) is determined on the side of the one-step identity. `History.Transfer` in `YM/RG/History.lean` (lines 223–227) imposes, for the density representation $\rho_m$ at step $m$ (9-06),

$$\rho_m = \mathcal{T}_{\beta_{m+1}}\, \rho_{m+1}, \qquad \left|\beta_{m+1} - \beta_m - s \log b_m\right| \le \mathrm{drift}_{m+1} \quad (4)$$

Source . What enters the kernel is the coupling of the **fine step**, $\beta_{m+1} = \beta(N, p, m+1)$ (line 225). Here $b_m$ is the side at step $m$, $s = 11 C_g/(24\pi^2)$ is the slope of 10-02, and drift is a step constant.

Two things can be read from this choice.

- The kernel differs from step to step. Since $\beta(N, p, m+1)$ depends on the cutoff $N$ and the volume index $p$ (Part 10), the kernel changes with $N$ even for one step at the same step $m$ Inferred . The "push-forward measure" is an object whose map itself changes with $N$ (11-10).
- The partition function is preserved. Since the total mass of the kernel is 1, $\int \mathcal{T}_\beta F = \int F$ (`push_integral`, `YM/RG/Push.lean` line 37) Source . Repeating this from step $N$, `History.partition_eq` (`History.lean`, line 242) says that integrating the representation at any step gives the Wilson partition function $Z_N(\beta_N)$ at step $N$ Source .

The first formula of the one-step identity (4) is an **equality**, not an approximation Source . The measure does not change in one step; it is the representation (9-06) that is rewritten.

05

## Which targets it belongs to

The source's `verification/declaration-usage.tsv` records as `used` the declarations reached from any of the 17 headline theorems listed in the verification record, and as `outside` those not reached. `used` does not mean "used by the public theorem" (6-02) Source .

| Declaration (location) | Content | Record | Relation to the targets |
|---|---|---|---|
| `StepData.density` (`RG/Block.lean`:71) | The kernel (2) | used | Enters the one-step identity of `History.Selected`, a field of the construction theorem's conclusion `C.Makes` Source |
| `StepData.isKernel` (:203) | That it is a probability kernel | used | Reached from one of the headline theorems. We have not traced the chain of the existence route |
| `Axial.viable` (`Branch/Pick.lean`:137) | The obligations hold for any penalty coefficient | used | As above |
| `History.Transfer`, `Selected` (`RG/History.lean`:223, 234) | The one-step identity and the sequence of steps | used | The field `wilson` of `Family.Makes` (`Source/Construction.lean`:45); the conclusion of the construction's theorem Source |
| `History.partition_eq` (:242) | Preservation of the partition function | used | Reached from one of the headline theorems |

Neither the kernel nor the sequence of steps appears in the statement of the public theorem `JointExistence` (the assessment's glossary, page 23) Assessment . The kernel is an object inside the construction that builds the **witness** of the public theorem.

06

## What this book's study had seen

One step in Volume 3 and Part 15 of this book is a **deterministic** block map $B$ (19-04). It fixes one coarse link by pulling a weighted average of path products back to the sphere, and carries the measure by the push-forward $B_{*}\mu$ (19-04, push-forward). In the language of kernels, this book's step is a point measure in $V$, whereas a step of the

published construction is a fluctuation of width $(c_{\mathrm{pen}}\beta)^{-1/2}$ around the predictor Inferred .

This book's study had two block designs. One is the current design with side 2 and overlapping weights $(1/8, 3/4, 1/8)$ (28-05); the other has side 3, blocks centred on coarse points, and uniform weights. For the latter, it has been machine-checked in Lean that the measure obtained by pushing the SU(2) Wilson measure forward $m$ steps is reflection positive with respect to reflections in planes of lattice points (for bounded measurable functions depending only on coarse links lying strictly between two planes of coarse lattice points) Lean : `Block3.rp_push`, `Block3.rp_push_prob` (periodic lattice with even side, any real $\beta$). This is a formalisation of a result of a known type (a local map commuting with reflections preserves reflection positivity under push-forward).

This argument uses that the deterministic map commutes with reflections, and that the map is local. Since a step of the published construction is a kernel, the same argument does not transfer as it is Inferred . A separate question arises: whether the kernel is of positive type with respect to reflections. On the published construction's side too, we found no statement of reflection positivity for the one-step kernel within what we read Source . Reflection positivity is stated for the original Wilson measure (12-01). The comparison of designs is summarised in 14-02.

07

## What we have not read

- Reflection positivity of the kernel: we have not searched the whole repository for a statement on whether the one-step kernel is of positive type with respect to reflections (we read within `YM/RG/` and `YM/Branch/`).
- Whether `isKernel` and `Axial.viable` lie on the existence route of the public theorem: we have not traced the chain. We know only that, according to the record, they are reached from one of the headline theorems (with `SharpGram.habitat` as the witness).
- How the step constants change when the penalty coefficient is varied (we go only as far as reading the docstring saying that the constants are chosen after the penalty coefficient is fixed).
- We found no explanation in the source of why the kernel's $\beta$ is taken on the fine side; the reading in § 04 is an inference from the definitions.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source One step is defined as the probability kernel (2), and the push-forward is (3) (`StepData.density`, `push`) | A statement treating the kernel as a limit of deterministic maps (not in the source) |
| Source Under the three obligations the kernel is a probability kernel (`isKernel`), and the obligations hold for all patches for any penalty coefficient (`Axial.viable`) | Estimates uniform in the penalty coefficient |
| Source The construction's theorem is quantified over each penalty coefficient $b > 0$, and the public theorem uses $b = 1$ | A conclusion in a form independent of the penalty coefficient |
| Source The $\beta$ entering the kernel is the value at the fine step (`History.lean`:225) | The reason for that choice (not in the source's docstrings) |
| Source The partition function is preserved by one step (`push_integral`, `partition_eq`) | Reflection positivity of the one-step kernel (no statement found) |
| Lean In this book's side-3 centred design, the push-forward measure is reflection positive for planes of lattice points (`Block3.rp_push`) | A claim of the same form for the probability kernel (this book has not examined it) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Kernel, push-forward, being a kernel | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Block.lean`: `StepData.density` line 71, `push` line 77, `noise_mass` line 146, `isKernel` line 203 |
| price and chord | Source | Same repository, `YM/Gauge/MatrixGroup.lean` lines 112, 115 and 117 |
| Penalty coefficient and three thresholds, obligations | Source | Same repository, `YM/RG/Patch.lean` lines 37–45 and 182–185, `YM/Branch/Pick.lean` line 137 `Axial.viable` |
| Quantification over the penalty coefficient | Source | Same repository, `Construction.lean` lines 13–16 `YangMills.constructed`, `Solution.lean` lines 4–6 |
| Precision matrix and the lower bound on the tree slice | Source | Same repository, `YM/Push/FREE_RESOLVENT/Flat.lean` lines 80–81 and 189–198 |
| One-step identity, partition function | Source | Same repository, `YM/RG/History.lean` lines 223–227 and 242, `YM/RG/Push.lean` line 37 |
| Dependency record | Source | Same repository, `verification/declaration-usage.tsv`, `verification/Usage.lean` |
| Conventions and glossary | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 3, Appendix A |

| Item | Kind | Source / tools |
|---|---|---|
| This book's deterministic block map and push-forward | Lean | The source table of 19-04 |
| Reflection positivity of the side-3 centred design | Lean | This book's Lean bundle (`Block3.rp_push`, `Block3.rp_push_prob`; components `Block3.down_refl`, `down_local`), a bundle that passed the check by the Lean kernel |

**Terms used in this chapter** — block kernel, probability kernel, push-forward, predictor, penalty coefficient, price, chord, one-step identity, preservation of the partition function, headline theorem, used and outside. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-02 Patches and weights — we read the patches centred at coarse points used to build the predictor, and the weights given by convolutions of the uniform distribution.

# Patches and weights — centred blocks and convolutions of the uniform distribution

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the definitions of the box of fine points attached to each coarse point (the patch) and of the weights $w_J$ on it, used inside the one-step kernel (9-01): that the coarse point sits at the centre of its block, that the weights are the $J$-fold convolution of a uniform distribution, that the support overlaps heavily with neighbouring blocks, and that $J$ grows to infinity with the step.

**Prerequisite chapters** — 9-01 The block kernel, 8-02 The set-up. Convolution of probabilities is in 1-08.

**Said first** — the weights and patches are definitions, and there are almost no theorems here. The reasons for the choice of weights (why convolutions, why $J \geq 4$) are not written in the source's docstrings, and the reading in § 04 includes inference.

The order of this chapter

1. **The statement — definitions of patches and weights**: `Patch`, `radius`, `weight`
2. **The coarse point sits at the centre of its block**: odd side and the embedding $x \mapsto Lx$
3. **Convolutions of the uniform distribution**: the form along one axis, and the example $L = 3,\ J = 4$
4. **The radius of the support and the overlap**: $J(L-1)/2$ versus the distance $L$ to the neighbouring coarse point
5. **J grows with the step**: `Calendar.op` and `Fits`
6. **What this book's study had seen**: side 2 with transverse-shift weights; side 3, centred, uniform
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment.

01

## The statement — definitions of patches and weights

The shape of one step is determined by the structure `Patch` (lines 29–34) in `YM/RG/Patch.lean` of the source. It has four ingredients Source.

- The side $L$: odd, with $L \geq 3$ (`odd_side`, `coarse`). The spacing of the coarse lattice is $L$ times that of the fine lattice, and the side of the fine torus is $L$ times the side of the coarse torus (`StepData.sizes`).
- The smoothing exponent $J$: $J \geq 4$ (`order_cut`).
- The patch radius $R = (J+3)L$ (`radius`, line 51). The patch is the box of the fine lattice centred at the origin with side $2R+1$, $\{z \in \mathbb{Z}^4 : |z|_\infty \leq R\}$ (`sites`, line 53).
- The weights $w_J$ (`weight`, line 87): a probability on the points of the patch.

We write the definition of the weights as a formula. Take the small box $C = \{z : |z|_\infty \leq (L-1)/2\}$ (`Small` in the source is `box (L/2)`, and since $L$ is odd the radius is $(L-1)/2$), and let $w_J$ be the distribution of the sum $z_1 + \cdots + z_J$ when a $J$-tuple $(z_1, \ldots, z_J)$ of points of $C$ is drawn uniformly Source.

$$w_J(x) = \frac{\#\{(z_1,\ldots,z_J) \in C^J : z_1 + \cdots + z_J = x\}}{|C|^J} = \big(u_C^{*J}\big)(x), \qquad u_C = \frac{1}{|C|}\mathbf{1}_C. \tag{1}$$

$u_C$ is the uniform distribution on $C$, and $*$ is convolution. The source shows $\sum_x w_J(x) = 1$ in `weight_sum` (line 90) and non-negativity in `weight_nonneg` (line 96) Source. The sum $z_1 + \cdots + z_J$ lies in the box of radius $J(L-1)/2$, which is inside the patch (radius $(J+3)L$), so the weights are defined on the points of the patch (`roll_sites`, line 63) Source.

The docstring says of the shape of the patch that it is "a single shape, also used for analytic conventions before the ratio of sides is fixed" (line 28) Source.

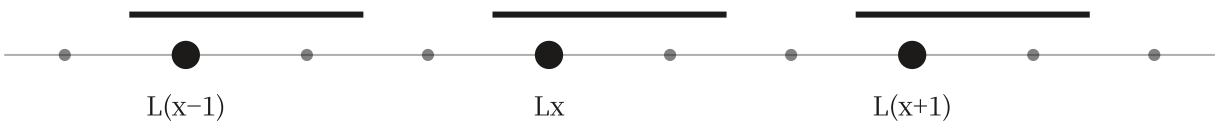
02

## The coarse point sits at the centre of its block

A coarse point $x$ is mapped to the fine point $Lx$. The source's `Ratio.embed` (`YM/Lattice/Subdivision.lean`, line 29) takes the representative `lift x` of the coarse point, centred, multiplies it by $L$, and wraps it back onto the fine torus Source . The patch is then placed, centred at the origin, on the configuration seen from the point $Lx$ (`expandAt`, `YM/Lattice/Path.lean` line 224) (`StepData.frames`, `YM/RG/Block.lean` lines 36–37) Source .

Since $L$ is odd, the fine lattice splits, with neither overlaps nor gaps, into cubes $Lx + C$ of side $L$ centred at the coarse points Proved here . Each coarse point sits right in the middle of its own cube. With an even side this kind of splitting is impossible, and the coarse point falls at a corner of the block or at a position with no fine point.

Figure 1 Blocks of side $L = 3$ (thick intervals) and coarse points (●) seen along one axis. The coarse point $Lx$ sits at the centre of its own block.




The figure is a cross-section along one axis. In four dimensions, $L^4$ fine points form one block.

03

## Convolutions of the uniform distribution

Since $C$ is the product of the intervals $\{-(L-1)/2, \dots, (L-1)/2\}$ along the four axes, $u_C$ is a product of one-axis uniform distributions, and the convolution also splits axis by axis Proved here .

$$w_J(x) = \prod_{i=1}^{4} v_J(x_i), \qquad v_J = \left(\tfrac{1}{L}\mathbf{1}_{\{-(L-1)/2,\dots,(L-1)/2\}}\right)^{*J}. \tag{2}$$

Take $L = 3,\ J = 4$ as an example. The one-axis distribution is the 4-fold convolution of the uniform distribution on $\{-1, 0, 1\}$:

$$v_4(-4), \dots, v_4(4) = \frac{1,\ 4,\ 10,\ 16,\ 19,\ 16,\ 10,\ 4,\ 1}{81} \tag{3}$$

Proved here . It is a hill, largest at the centre and decreasing towards the ends. The variance along one axis is $J(L^2-1)/12$, and for large $J$, $v_J$ approaches a discrete

Gaussian by the central limit theorem Proved here .

The weights are invariant under the hypercubic group (permutations of axes and sign flips). The source's `weight_invariant` ( `YM/Branch/Transport.lean`, line 135) shows $w_J(s \cdot x) = w_J(x)$ for signed permutations $s$ Source . The proof has the form: the uniform distribution $u_C$ is invariant, so its convolutions are too. This invariance is one of the origins of the symmetries of the one-step kernel (9-05).

The weights are used on the predictor side (9-04) Source .

- Weighted average: the predictor of a coarse link is built from the $w_J$-weighted average of the transports $Y(z)$ at the points $z$ of the patch ( `bondCost`, `Patch.lean` line 159).
- Probability of drawing a single one: when the average cannot be used, a point $z$ is drawn with probability $w_J(z)$ and $Y(z)$ is used as it is ( `predictorMass`, line 177).

The average in the gauge fixing (9-03) does not use the weights; it is a uniform average over all pairs of points of the patch Source .

04

## The radius of the support and the overlap

The support of $w_J$ is the box of radius $J(L-1)/2$ Proved here . The distance to the neighbouring coarse point is $L$, so if $J(L-1)/2 \geq L$ the support reaches the centre of the neighbouring block. For $J \geq 4,\ L \geq 3$, $J(L-1)/2 \geq 2(L-1) \geq L$, so it always reaches Proved here .

| $L$ | $J$ | Support radius $J(L-1)/2$ | Patch radius $(J+3)L$ | Layers of coarse points reached by the support |
|---|---|---|---|---|
| 3 | 4 | 4 | 21 | 1 |
| 3 | 10 | 10 | 39 | 3 |
| 5 | 4 | 8 | 35 | 1 |
| 9 | 6 | 24 | 81 | 2 |

The last column of the table is $\lfloor J(L-1)/(2L) \rfloor$: how many layers of coarse points, counting from the neighbour, the support reaches Proved here . As $J$ grows, the fine region seen by the predictor of one coarse link spreads to about $J/2$ layers in units of the coarse lattice. The patch radius $(J+3)L$ is $J+3$ layers in units of the coarse lattice, taken about $J/2+3$ layers wider than the support (about $J/2$ layers) ( `Patch.radius`, `YM/RG/Patch.lean` line 51) Source . Since the frame (9-03) is defined on all points of the patch, there is a margin outside the points used for the predictor's average Inferred .

The overlap is large. A single fine link enters the averages of the predictors of many coarse links. Even though the kernel's formula ((2) in 9-01) is a product of independent noise for each coarse link, the predictors $Q_b(U,\tau)$ are related to one another through shared fine links Inferred . How distant coarse links become independent (or do not) in the push-forward measure cannot be read from the definition of the kernel alone.

That the lattice is large enough is imposed by the conditions of the calendar. The field `fitsPatch` of `Calendar.Fits` (`YM/RG/History.lean`, line 110) requires $100 \cdot (\text{patch radius} + 1) \le$ the side of the fine lattice Source . A patch never wraps around the torus and overlaps itself.

05

## J grows with the step

One step from step $m+1$ to step $m$, `Calendar.op m p` (`History.lean`, lines 88–91), uses the patch with side $L = b_m$ (the side at step $m$) and smoothing exponent $J = J(m+1)$ Source . $J$ is the calendar field `J : ℕ → ℕ`, and `Calendar.Fits` (lines 97–118) imposes the following Source .

- `mono`: $J$ is non-decreasing. `stepSlow`: it increases by at most 1 per step. `deep`: $J \to \infty$.
- `sameFactor`: the side $b_m$ is the value `factor (J (m+1))`, determined by $J(m+1)$ alone. `monotoneL`: that value is non-decreasing in $J$.
- `deepOrder` and others: the step constants `Stage`, such as the Taylor order, are indexed by $J$ and monotone in $J$.

So $J$ is also "the index of the tuple of step constants", and the side $b_m$, the number of convolutions in the weights, and the order of the expansion of the action (9-06) all grow together with the same index Source . From $J \to \infty$ and the `logCost` condition, the side $b_m \to \infty$ follows (10-05) Inferred . Why the sides grow is read in 10-05.

As the number of convolutions increases, the weights become smoother, and the step in the weights at the edge of a block becomes smaller Inferred . The source's docstrings do not give the reason for choosing convolutions.

06

## What this book's study had seen

We set it beside the two designs of this book's study (this book's designs are in 19-04, weights and 28-05).

| Item | Published construction | This book's current design | This book's side-3 design |
|---|---|---|---|
| Side | Odd $\geq 3$, growing with the step | 2 (fixed) | 3 (fixed) |
| Position of the coarse point | Centre of the block | On a fine point (corner of the block) | Centre of the block |
| What the weights are on | Points of the patch (all translations in four dimensions) | Transverse shifts of the path $-1, 0, +1$ | Uniform average within the block |
| Shape of the weights | $J$-fold convolution of a uniform distribution ($J \geq 4$, growing) | $(1/8,\ 3/4,\ 1/8)$ | Uniform |
| Overlap with neighbours | Large (support radius $J(L-1)/2 \geq L$) | Present (shift of 1 sideways) | — |

Centring the point goes in the same direction as this book's side-3 design. In the side-3 design, using that centring the coarse points fits with reflections in planes of lattice points, reflection positivity of the push-forward measure is closed in Lean (9-01 § 06, `Block3.rp_push`) Lean . The weights of the published construction have a growing support that overlaps heavily with neighbours, so they do not fit the premises of this book's reflection-positivity argument (locality of the map, and not crossing the reflection plane) Inferred . The idea of smoothing the weights by convolution and letting $J$ grow is one this book's study did not have.

07

## What we have not read

- Why $J \geq 4$, and why the patch radius is $J + 3$ layers. The source's docstrings give no reason, and we have not traced where they are used in downstream estimates (the Taylor order, estimates on supports).
- In which estimates the smoothness of the weights (the number of convolutions) matters. We infer that it is used in the small-field step (11-09), but have not checked.
- How the dependence between heavily overlapping predictors is reflected in the locality of the activities (the supports of the regular activities in 9-06).
- For the explicit sequence of sides $b_m$ (the construction in `YM/Price/Waits.lean`), we read only the statements.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The patch is a box with odd side $L \geq 3$, $J \geq 4$ and radius $(J+3)L$, and the coarse point $x$ sits at the centre, at the fine point $Lx$ | The reasons for $J \geq 4$ and $J+3$ |
| Source The weights are the $J$-fold convolution of a uniform distribution: total 1, non-negative, invariant under the hypercubic group (`weight_sum`, `weight_invariant`) | The reason for choosing convolutions (not in the docstrings) |
| Proved here The support radius $J(L-1)/2$ is always at least $L$ for $J \geq 4$, so it overlaps the neighbouring block | How the overlap affects independence in the push-forward measure |
| Source One step at step $m$ uses $L = b_m$, $J = J(m+1)$; $J$ is non-decreasing, increases by at most 1 per step, and grows to infinity | The explicit growth of $J$ (we read only the statements of the calendar's construction) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Patch, radius, weights | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Patch.lean`: `Patch` line 29, `radius` line 51, `roll_sites` line 63, `weight` line 87, `weight_sum` line 90, `bondCost` line 159, `predictorMass` line 177 |
| Embedding and view of configurations | Source | Same repository, `YM/Lattice/Subdivision.lean` line 29 `Ratio.embed`, `YM/Lattice/Path.lean` line 224 `expandAt`, `YM/Lattice/Box.lean` line 13 `box` |
| Invariance of the weights | Source | Same repository, `YM/Branch/Transport.lean` line 135 `weight_invariant` |
| Patches per step and the calendar | Source | Same repository, `YM/RG/History.lean` lines 88–91 `Calendar.op`, lines 97–118 `Calendar.Fits` |
| The numbers in (3) and in the table | Proved here | Hand computation in the text (coefficients of $(1+x+x^2)^4$) |
| This book's two designs | Lean | The source table of 19-04, and this book's Lean bundle (`Block3.paths`, `Block3.wts`, `Block3.rp_push`) |

**Terms used in this chapter** — patch, smoothing exponent, convolution of the uniform distribution, support radius, centre of the block, hypercubic group, calendar. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-03 Two kinds of gauge fixing — we read in which frame the fine configuration is viewed on the patch (symmetric minimisation at small field, and a randomly drawn axial gauge at large field).

# Two kinds of gauge fixing — symmetric minimisation, and a randomly drawn axial gauge

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the definition of the "frame" that enters the one-step kernel (9-01). A frame is a group element at each point of the patch; the fine configuration is re-viewed through it before the predictor of a coarse link is built. We read the small-field frame (symmetric minimisation, Landau type) and the large-field frame (an axial gauge whose root and axis order are drawn at random), the condition separating the two, and how, at large field, drawing the symmetry-breaking choices at random and summing over them keeps the discrete symmetries of the kernel.

**Prerequisite chapters** — 9-01 The block kernel, 9-02 Patches and weights. Gauge fixing in general is in 4-08.

**Said first** — we have not read the content of the proofs of uniqueness of the minimiser (`Branch.align`) and of the existence of labels that may be drawn (`Axial.existence`). Nor have we read the concrete values of the thresholds. What we can say goes as far as the definitions and the form of the statement that uniqueness holds if the threshold is taken small enough.

The order of this chapter

1. **The statement — frames and their two kinds**: `Patch.frame`, `usesMin`, `frameMass`
2. **The small-field frame — symmetric minimisation**: an average over all pairs of points and the 24 axis orders
3. **The large-field frame — a randomly drawn axial gauge**: labels for root and axis order, the condition for labels that may be drawn
4. **The branching condition**: the chord of transports in the central axial gauge, and a threshold
5. **Why two kinds**: uniqueness, continuity, discrete symmetries
6. **What this book's study had seen**: symmetric averages over families of paths

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

## The statement — frames and their two kinds

A **frame** on a patch $B$ (9-02) is a collection of group elements, one per point, $h : B \to G$ (`Patch.Frame` in the source, `YM/RG/Patch.lean` line 99; the type itself contains no normalisation). The normalisation at the centre, $h(0) = 1$, is part of the minimisation condition `minimizing` (line 121) at small field, and the axial-gauge frame at large field satisfies it automatically by formula (3). A frame is a gauge transformation on the patch; the fine configuration is re-viewed through it before the predictor is built (9-04) Source .

We set notation. For an axis order $l$ (an ordering of the four axes, 24 in all; `Order` is a permutation of $\{0,1,2,3\}$, `YM/Lattice/Box.lean` line 39) and two points $z, z'$, let $\Gamma_l(z \to z')$ be "the coordinate path from $z$ to $z'$ going straight along one axis at a time in the order $l$" (`route`, same file, line 44). The product of a configuration $U$ along a path (transport) is written $U(\Gamma)$ Source .

The frame is defined in two kinds (`Patch.frame`, lines 153–155) Source .

$$h = \begin{cases} h^{\mathrm{sym}}(U) & \text{(small field: } \texttt{usesMin} \text{ holds)}, \\ h^{\mathrm{ax}}_{(r,l)}(U) \ \text{ with the label } (r,l) \text{ drawn at random} & \text{(large field)}. \end{cases} \tag{1}$$

The probability of the label $(r, l)$ is `frameMass` (lines 148–151). At small field a uniform probability $1/|B \times \mathrm{Order}|$ is put on all labels (the labels are not used); at large field a uniform probability $1/|E(U)|$ is put on the set $E(U)$ of "labels that may be drawn"

Source . This probability is the frame part of $P(\tau \mid U)$ in the kernel's formula ((2) in 9-01).

That the two frames are well defined is two of the three obligations `Patch.Branches` (lines 182–185): that the small-field minimiser is unique (`frame_unique`), and that the set of labels that may be drawn is non-empty (`axial_exists`) Source . For the former, `RG.frame_unique` (`YM/Branch/Uniform.lean` line 18) says "if the frame threshold is at most some $\delta > 0$ determined by the group alone, it is unique for all patches"; the latter is met by `Axial.viable` (`YM/Branch/Pick.lean` line 137) with factor 16 Source .

02

## The small-field frame — symmetric minimisation

The small-field frame is the frame minimising the following function (`score`, lines 114–115; `minimizing`, lines 121–122) Source .

$$\mathcal{S}U(h) = \frac{1}{|B|^2 \cdot 24} \sum_{z \in B} \sum_{z' \in B} \sum_{l} \operatorname{price}\Big(h(z)\, U\big(\Gamma_l(z \to z')\big)\, h(z')^{-1}\Big), \qquad h(0) = 1. \tag{2}$$

Here $\operatorname{price}(u) = 1 - \operatorname{Re}\operatorname{tr} u/dG$ ((1) in 9-01). The sum runs over **all pairs of points** of the patch and **all axis orders**. A minimiser exists by compactness of the set of frames fixed to 1 at the centre (the group is compact) (`min_exists`, line 124), and the chosen one is called `selected` (line 129) Source .

This minimisation is one discrete form of the Landau gauge. The Landau gauge looks only at links between neighbouring points and minimises $\sum \operatorname{price}(h(z) U_{zz'} h(z')^{-1})$, whereas (2) joins every pair of points by coordinate paths, so it is a Landau type on the complete graph over the patch Inferred . By looking at all pairs of points uniformly and averaging over the 24 orders, (2) becomes a function invariant under the hypercubic group (permutations and reflections of axes) Inferred .

The minimiser satisfies a self-consistency equation. The source's `Branch.self_consistent` (`YM/Branch/Synchronize.lean` line 100) says that at each point $z$, $h(z)$ is the group element closest to "the average of the values of $h$ carried from all other points" Source . The docstring of the lemma that pins the minimiser down to one, `Branch.align` (same file, line 232), says it uses that the difference of averages lies in the normal slice, together with an estimate on the derivative of the nearest-point map, to show that the difference of two minimisers contracts in the sup norm Source . We have not read the content of the proof.

03

## The large-field frame — a randomly drawn axial gauge

At large field, one label $(r, l)$ (a root $r \in B$ and an axis order $l$) is drawn, and the frame is built from values transported from the root along coordinate paths (`axial`, lines 102–103, and the second branch of `frame`) Source .

$$A_{r,l}(z) = U\big(\Gamma_l(r \to z)\big), \qquad h^{\mathrm{ax}}_{(r,l)}(z) = A_{r,l}(0)^{-1} A_{r,l}(z). \quad (3)$$

This is the axial gauge from the root $r$ (one kind of tree gauge), normalised from the left so that $h = 1$ at the centre 0 Inferred . In the axial gauge, every link on the coordinate paths leaving the root becomes 1, and the values of the remaining links can be written as products of curvatures (plaquette values).

The labels that may be drawn are those for which the sum of the chords of the links in the axial gauge is at most a fixed multiple of the sum of the chords of the plaquettes (`eligible`, lines 143–146) Source .

$$E(U) = \Big\{ (r,l) : \sum_{\text{links } e \text{ in the patch}} \mathrm{chord}\big(U_e^{h^{\mathrm{ax}}_{(r,l)}}\big) \;\le\; c_{\mathrm{ax}}\,(2R+1) \sum_{\text{plaquettes } p \text{ in the pat}}$$

$c_{\mathrm{ax}}$ is `axial_factor`, and $R$ is the patch radius. `Axial.existence` (`Pick.lean` line 80) shows that for every configuration there is a root whose sum is bounded by (number of axes) × (number of layers) × 4 times the plaquette sum, and with four axes, taking the factor 16 gives that $E(U)$ is non-empty (`Axial.eligible`, same file, line 119; if the factor is at least 16, for every axis order there is a root that may be drawn) Source . The meaning is "choose an axial gauge in which the size of the links is bounded by the size of the curvature". At large field some plaquette values are large; including the places where they are not, a frame is taken in which the total size of the link values can be measured by the total size of the curvature Inferred .

The choice of label breaks the symmetry. Once a root position and an axis order are chosen, the action of the hypercubic group moves them to another label. So instead of fixing one label, **the labels that may be drawn are drawn uniformly, and the kernel sums over them**. A group element $s$ maps the set of labels that may be drawn one-to-one onto the set of labels (`orient` in 9-05), so the summed kernel is symmetric Source .

04

## The branching condition

Which frame is used is decided by `usesMin` (lines 109–110) Source .

$$\mathtt{usesMin}(U) \iff \forall\, l,\ k,\ z,\ z' : \quad \mathrm{chord}\Big(A_{0,l}(z)\, U\big(\Gamma_k(z \to z')\big)\, A_{0,l}(z')^{-1}\Big) < \delta_F. \quad (5)$$

That is, the small-field frame is used when, seen in **the axial gauge rooted at the centre** 0 (for every axis order $l$), the transport along every coordinate path joining every pair of points of the patch lies at distance less than $\delta_F$ from the identity. $\delta_F$ is `frame_cut` Source .

The condition is written not in terms of the size of plaquettes but in terms of the **chord of transports**. Even if every plaquette value is small, the transport can become large if curvature accumulates over the size of the patch (radius $(J+3)L$) Inferred . (5) is a condition rewritten in gauge-invariant form via the axial gauge, and does not change under gauge transformations of the configuration (gauge invariance is in 9-05).

| | **Small field (`usesMin`)** | **Large field (otherwise)** |
|---|---|---|
| Frame | The minimiser of (2) (unique) | The axial gauge (3) for each label $(r, l)$ |
| Probability of labels | Uniform (not used) | Uniform on the labels that may be drawn, $E(U)$ |
| How symmetry is kept | (2) is invariant, and the minimiser is unique | Sum over labels |
| Threshold used | $\delta_F$ (small enough for uniqueness of the minimiser) | Factor $c_{\mathrm{ax}} = 16$ |

05

## Why two kinds

From the source's docstrings and definitions, we read the roles of the two kinds as follows Inferred .

- **Why minimisation is used at small field.** The minimiser is unique ( § 01), and as the unique minimiser of a symmetric function it commutes with the action of the symmetries. Moreover, it is expected to move continuously (analytically) with the configuration, which suits the expansion of the small-field step (11-09).
- **Why minimisation is not used at large field.** When transports are large, the minimiser need not be unique, and the choice can be discontinuous in the configuration. The axial gauge is the explicit formula (3) and has no uniqueness problem.
- **Why draw at random.** The axial gauge breaks symmetry through the choice of root and order. Choosing by a single rule would make the kernel non-invariant under the hypercubic group. Drawing labels uniformly and summing restores the symmetry at the level of the kernel.

The docstring calls the small-field branch "the open branch" (`Patch.lean` line 108), and says that the branching threshold and the existence of uniqueness are "proof

obligations, not assumed for any gauge group" (lines 10–12) Source . At the branching boundary the frame switches discontinuously, but the kernel is defined measurably as a probability sum over the two branches (`density_meas` in 9-01), and the design is that this is enough Inferred .

06

## What this book's study had seen

This book's deterministic block map (19-04, families of paths) does no gauge fixing. It builds a coarse link as a weighted average of products along fine paths, and makes the map commute with the discrete symmetries by taking the family of paths itself symmetric under permutations and reflections of the transverse directions. A path product is gauge transformed only at its two ends, so gauge covariance follows directly from the path products.

The published construction fixes the gauge with a frame and then averages (9-04). The aim is the same (commuting with discrete symmetries, and gauge covariance); the route is different.

- This book "averages paths symmetrically"; the published construction "minimises the frame symmetrically (small field) / draws the symmetry-breaking choices at random and sums (large field)".
- This book's block map uses the same formula at large field, only pulling back inside the sphere when the average becomes small (19-04). A design that treats large field as a separate branch inside the definition of the map is something this book's study did not have.
- The step-by-step gauges in this book's study mixed the Landau slice, the axial gauge and the tree gauge depending on the place (28-05 § 05). The published construction puts them in at once inside the definition of the one-step kernel, in the form of two frames switched by a condition.

This book's study did not have the idea of drawing the large-field frame at random. The comparison is summarised in 14-02 and 14-05.

07

## What we have not read

- The content of the proof of `Branch.align` (`YM/Branch/Synchronize.lean` line 232), and the value of the frame threshold $\delta_F$.
- The details of the proof of `Axial.existence` (`YM/Branch/Pick.lean` line 80) (we read as far as its form: choosing, layer by layer, the slice with the smallest plaquette sum,

and inducting).

- Where the statement that the frame moves analytically with the configuration within small field is located (we saw only the name `YM/PREC/FrameResponse` on the side of derivatives of the predictor).
- How the discontinuity of the frame at the branching boundary is kept from affecting the estimates of the small-field step.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The frame comes in two kinds: if the chord of the transport along every coordinate path in the central axial gauge is below the threshold, the symmetric minimisation (2); otherwise the randomly drawn axial gauge (3) | The value of the threshold $\delta_F$ |
| Source If the frame threshold is at most a value determined by the group alone, the small-field minimiser is unique for all patches (`RG.frame_unique`) | The content of that proof (not read) |
| Source With factor 16, the set (4) of labels that may be drawn is non-empty (`Axial.viable`, `Axial.eligible`) | Whether the factor 16 is necessary |
| Inferred At large field, drawing the symmetry-breaking choices at random and summing keeps the symmetry of the kernel (the statement is in 9-05) | The mechanism by which the discontinuity of the frame does not affect the estimates |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Frames, labels, score, branching, labels that may be drawn | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Patch.lean`: lines 99–155 (`Frame`, `axial`, `via`, `usesMin`, `score`, `minimizing`, `selected`, `facesum`, `axialCost`, `eligible`, `frameMass`, `frame`), obligations lines 182–185 |
| Coordinate paths and axis orders | Source | Same repository, `YM/Lattice/Box.lean` lines 30–44 (`axisRoute`, `Order`, `route`) |
| Uniqueness of the minimiser | Source | Same repository, `YM/Branch/Uniform.lean` line 18 `RG.frame_unique`, `YM/Branch/Synchronize.lean` line 100 `self_consistent`, line 232 `align` |
| Existence of labels that may be drawn | Source | Same repository, `YM/Branch/Pick.lean` line 80 `Axial.existence`, line 119 `Axial.eligible`, line 137 `Axial.viable` |
| This book's symmetric averages over families of paths | Lean | The source table of 19-04 |

**Terms used in this chapter** — frame, symmetric minimisation, Landau gauge, axial gauge, label, labels that may be drawn, branching condition, chord, hypercubic group. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-04 The predictor and projection to the group — we read the procedure that averages the transports re-viewed through the frame with weights and projects onto the group to build the predictor of a coarse link.

# The predictor and projection to the group — a weighted average of transports, onto the group

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — how the centre $Q_b(U,\tau)$ of the one-step kernel (9-01) is built: the transports along parallel straight lines joining corresponding points of two neighbouring patches, their weighted average, the projection onto the group (nearest point), and the condition for switching to drawing a single one at random (both the spread and the distance from the average to the group). And that the predictor is a different thing from the background field of a Bałaban-type constrained minimisation.

**Prerequisite chapters** — 9-01 The block kernel, 9-02 Patches and weights, 9-03 Two kinds of gauge fixing.

**Said first** — we have not read the value of the predictor threshold $\delta_P$, nor the estimate that the predictor moves smoothly with the configuration. "Not a background field" in § 05 is a comparison of definitions; it says nothing about estimates between the two objects.

The order of this chapter

1. **The statement — definition of the predictor**: `connector`, `bondCost`, `best`, `predict`
2. **Transports along parallel straight lines**: joining corresponding points of neighbouring patches
3. **Minimising the weighted average is projection onto the group**: `Branch.min_iff`, and the identity spread + distance
4. **The condition for switching to drawing a single one**: both the spread and the distance from the average to the group
5. **The predictor is not a background field**: an explicitly built central value, versus a stationary point inside the analysis
6. **What this book's study had seen**: pull-back and projection coincide for SU(2)

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

## The statement — definition of the predictor

The **predictor** $Q_b(U,\tau) \in G$ of a coarse link $b=(x,i)$ (from the coarse point $x$ in direction $i$) is built in `StepData.prediction` in the source (`YM/RG/Block.lean` line 44) in the following three stages Source .

1. **The collection of transports** (`connector`, lines 39–42): for each point $z$ of the patch,

$$Y_b(z) = h_x(z)\, U\big(Lx+z \;\to\; Lx+z+Le_i \text{ straight line}\big)\, h_{x+e_i}(z)^{-1}. \quad (1)$$

   $h_x$ is the frame of the patch at the coarse point $x$ (9-03), and $L$ is the side.

2. **The weighted cost** (`bondCost`, `YM/RG/Patch.lean` line 159): for a group element $u$,

$$\mathcal{C}_b(u) = \sum_z w_J(z)\ \mathrm{price}\big(Y_b(z)\,u^{-1}\big). \quad (2)$$

   One of its minimisers is $u_b^*$ (`best`, line 169). A minimiser exists by compactness (`bond_min`, line 164).

3. **Branching** (`predict`, lines 173–175):

$$Q_b = \begin{cases} u_b^* & \mathcal{C}_b(u_b^*) < \delta_P, \\ Y_b(r)\ \ (r \text{ drawn with probability } w_J(r)\ ) & \text{otherwise.} \end{cases} \quad (3)$$

   $\delta_P$ is the predictor threshold `predictor_cut`. The probability of drawing a single one is `predictorMass` (lines 177–179); on the branch that uses the average, the label $r$ is not used and a uniform probability is put.

That the minimiser is unique on the branch using the average is the second of the three obligations `Branches`, `predictor_unique`. `RG.predictor_unique` (`YM/Branch/Uniform.lean` line 58) says that if the predictor threshold is at most some value determined by the group alone, it is unique for all patches Source .

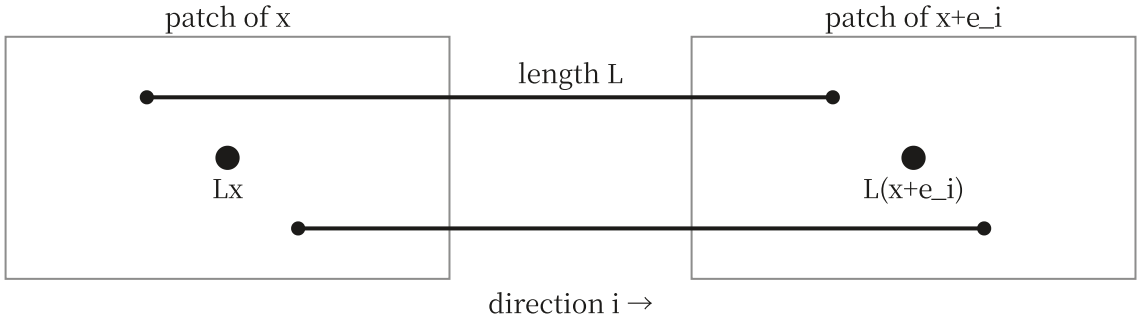
02

## Transports along parallel straight lines

The straight line in formula (1) goes from the point $Lx + z$ of the patch at the start of the coarse link, exactly $L$ steps in direction $i$, to the **corresponding point** $L(x + e_i) + z$ of the patch at the end Source . Since $z$ ranges over all points of the patch (a four-dimensional box of radius $(J + 3)L$), the lines include all translations in four dimensions. Both shifts along direction $i$ (longitudinal shifts) and transverse shifts are included Source .

The frames of the respective patches are multiplied at the two ends of the transport. Since a frame is a gauge transformation on the patch, $Y_b(z)$ is "the transport from the point $z$ seen in the frame at the start to the point $z$ seen in the frame at the end" Inferred . Applying a gauge transformation $g$ to the fine configuration moves the frame as $h_x(z) \mapsto g(Lx)\, h_x(z)\, g(Lx + z)^{-1}$ (since it is normalised to $h = 1$ at the centre, the value $g(Lx)$ at the coarse point attaches on the left), and $Y_b(z)$ changes to $g(Lx)\, Y_b(z)\, g(L(x + e_i))^{-1}$, only by end elements **independent of** $z$ (`bond_gauge` in 9-05) Inferred . So averaging over $z$ does not break gauge covariance.

Figure 1 The collection of transports for a coarse link $b = (x, i)$. A point $z$ of the starting patch and the same point $z$ of the ending patch are joined by a straight line of length $L$ in direction $i$ (only two are drawn here).




Since all points of the patch are used, the predictor of one coarse link depends on the fine links in the union of the two patches Inferred . The weights $w_J$ act on the central part of radius $J(L - 1)/2$ (9-02), but the frame is determined by the whole patch.

03

## Minimising the weighted average is projection onto the group

Viewing a group element as a $dG \times dG$ matrix $\mathrm{mat}(u)$, set the average matrix

$$M_b = \sum_z w_J(z)\ \mathrm{mat}\big(Y_b(z)\big) \tag{4}$$

In general $M_b$ is not a group element. The source's `Branch.min_iff` (`YM/Branch/Center.lean` line 72) says that when the weights sum to 1,

$$u \text{ is a minimiser of } (2) \iff \mathrm{mat}(u) \text{ is the group element nearest to } M_b \text{ in the Frobenius nor}$$

Source . "Nearest" is `Gauge.Nearest` (`YM/Gauge/Nearest.lean` line 26), meaning $\|M - \mathrm{mat}\,u\|_F \le \|M - \mathrm{mat}\,v\|_F$ for all $v$ Source . So the predictor on the branch using the average is the **projection onto the group** of the average matrix, $\Pi_G(M_b)$.

The core of the proof is a single identity (`hk` inside `min_iff`, lines 76–77) Source . Expanding with $\|\mathrm{mat}(u) - \mathrm{mat}(v)\|_F^2 = 2dG\,\mathrm{price}(uv^{-1})$ (`price_diff`, line 63) gives

$$2dG\,\mathcal{C}_b(v) = \underbrace{\sum_z w_J(z)\,\big\|\mathrm{mat}\,Y_b(z) - M_b\big\|_F^2}_{\text{weighted spread}} \;+\; \big\|M_b - \mathrm{mat}(v)\big\|_F^2 \tag{6}$$

Proved here . The first term does not depend on $v$, so minimising $\mathcal{C}_b$ is the same as minimising the second term, that is, the distance from $M_b$ to the group. Uniqueness of the minimiser is derived by `Branch.weighted_unique` (`Center.lean` line 95) from the fact that if $\mathcal{C}_b$ is small enough at some group element, $M_b$ lies in a tube near the group, where the nearest point is unique (`nearest_tube`, `Nearest.lean` line 141) Source .

04

## The condition for switching to drawing a single one

The switching condition in (3) is $\mathcal{C}_b(u_b^*) \ge \delta P$ (the negation of `usesBest`, `Patch.lean` line 171) Source . Putting the minimiser into (6),

$$2dG\,\mathcal{C}_b(u_b^*) = \sum_z w_J(z)\big\|\mathrm{mat}\,Y_b(z) - M_b\big\|_F^2 + \mathrm{dist}_F\big(M_b, G\big)^2 \tag{7}$$

Proved here , so the switch to drawing a single one happens when

$$\text{weighted spread} + \text{distance from the average to the group}^2 \;\ge\; 2dG\,\delta P \tag{8}$$

Proved here . **Both terms matter.** Writing only "when the average is far from the group" drops the contribution of the spread. For example, if the transport values split into two

distant places in the group, the spread is large even if the average happens to be close to the group, and the branch drawing a single one is entered Inferred .

On this branch, a point $r$ is drawn with weight $w_J(r)$, and that single transport $Y_b(r)$ is used as the predictor as it is. It is a group element, so no projection is needed. The drawing probability enters the sum in the kernel ($P(\tau \mid U)$ in 9-01), and since the weights are invariant under the hypercubic group, the summed kernel is symmetric (9-05) Source .

| Branch | Condition | Predictor | Probability of labels |
|---|---|---|---|
| Average | Left side of (8) $< 2d_G\delta_P$ | $\Pi_G(M_b)$ (unique) | Uniform (not used) |
| Single | Left side of (8) $\geq 2d_G\delta_P$ | $Y_b(r)$ | $w_J(r)$ |

05

## The predictor is not a background field

In Bałaban's programme (7-07), one builds the configuration minimising the fine action with the coarse variables held fixed (the constrained minimiser, the background field), and integrates the fluctuations around it. The predictor of the published construction is a different object Source .

- The predictor is a **central value** built from the fine configuration by explicit rules (frame, parallel lines, weighted average, projection, or drawing a single one). No minimisation of the action enters (the minimisation of the frame is a minimisation of an average of price, not of the action).
- The coarse configuration is not constrained to the predictor; it fluctuates around it in Gaussian fashion (9-01).

What corresponds to a stationary point appears separately, inside the **analysis** of the small-field step. The field `stationary` of `Kin` (lines 74–78) in `YM/PCalc/Loop.lean` is a stationary point of the "coupled action", the fine action plus the penalty term, and its properties (real analyticity, a lower bound on the Hessian, agreement of jets up to order 4 with the coarse action) are carried as fields of `Kin.Planned` (lines 93–116) Source . This is an object for setting up the integration variables, and does not enter the definition of the kernel. It is read in detail in 11-09.

06

## What this book's study had seen

This book's block map (19-04, the non-linear block map) builds a weighted average of path products $q_b(U)$ (a quaternion), and returns it to the sphere by the pull-back with

threshold $r_0 \in (0, 1]$, $\rho_{r_0}(q) = q/\max(|q|, r_0)$. For SU(2), there is a relation between these two constructions Proved here .

A real linear combination of elements of SU(2) is a real linear combination of unit quaternions, hence a quaternion $q$ (the $2 \times 2$ matrix $q_0\mathbf{1} + i\underset{\sim}{q} \cdot \underset{\sim}{\sigma}$). The Frobenius norm is $\|q\|_F^2 = 2|q|^2$, and SU(2) is the sphere $|q| = 1$. So the nearest point to $q \neq 0$ is $q/|q|$ Proved here . That is,

$$\Pi_{SU(2)}(q) = \frac{q}{|q|} = \rho_{r_0}(q) \qquad (|q| \geq r_0). \quad (9)$$

Where the average is at least the threshold, this book's pull-back and the published construction's projection are the same map Proved here . They differ in three points.

- **What is averaged**: this book uses products along a fixed family of paths (paths shifted by $\pm 1$ sideways); the published construction uses transports along parallel straight lines re-viewed through the frame (all points of the patch, including longitudinal shifts).
- **Where the average cannot be used**: for $|q| < r_0$ this book returns the value inside the sphere, $q/r_0$, widening the configuration space to the ball. The published construction switches branch by spread + distance$^2$ and stays with a group value by drawing a single one at random.
- **After the central value**: this book makes the central value itself the coarse link (deterministic); the published construction adds Gaussian-type fluctuations around the central value (9-01).

On the analysis side of this book's study (Part 15, 28-05 § 05), a constrained minimiser of the action was used as the background. Defining the kernel with an explicitly built central value and using stationary points only as an analytic tool is a division this book's study did not have (14-05).

07

## What we have not read

- The value of the predictor threshold $\delta P$ (we read as far as `weighted_unique` building it from the radius of the nearest-point tube of the group).
- The estimate that the predictor moves smoothly with the configuration within small field (we saw only the name `YM/PREC/FrameResponse`).
- How the branch drawing a single one is paid for in the large-field estimate (11-04).
- What kernel the inclusion of longitudinal shifts in the parallel lines produces in the first-order part (linearisation) of the predictor. We have not read the formula for the

counterpart of this book's path kernel (19-04) (a candidate is `probe` in `YM/Push/FREE_RESOLVENT/Flat.lean`).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The predictor is built from the weighted average of the transports (1) along parallel straight lines re-viewed through the frame; if the minimum of the cost (2) is at least the threshold, a single one is drawn with probability $wJ$ | The value of the threshold $\delta$ $P$ |
| Source The minimiser of (2) is the nearest point on the group to the average matrix (4) (`Branch.min_iff`), unique if the threshold is small enough (`RG.predictor_unique`) | An estimate that the minimiser moves smoothly with the configuration |
| Proved here The switch happens when "weighted spread + distance from the average to the group$^2$" is at least $2dG\delta P$ (both terms matter) | — |
| Source The predictor is not a constrained minimiser of the action; a stationary point enters separately, inside the analysis, as `Kin.stationary` | An estimate of the difference between the predictor and the stationary point |
| Proved here For SU(2), the projection onto the group is $q/\|q\|$, which agrees with this book's pull-back for $\|q\| \geq r0$ | How the difference in the families of averaged transports matters |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The collection of transports, the predictor | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/Block.lean` lines 39–45 (`connector`, `prediction`) |
| Cost, minimiser, branching | Source | Same repository, `YM/RG/Patch.lean` lines 159–179 (`bondCost`, `bond_min`, `best`, `usesBest`, `predict`, `predictorMass`) |
| Being a projection, the identity | Source | Same repository, `YM/Branch/Center.lean` line 63 `price_diff`, line 72 `min_iff` (identity at lines 76–77), `weighted_unique`; `YM/Gauge/Nearest.lean` line 26 `Nearest`, line 141 `nearest_tube` |
| Uniqueness of the predictor | Source | Same repository, `YM/Branch/Uniform.lean` line 58 `RG.predictor_unique` |
| Stationary point (inside the analysis) | Source | Same repository, `YM/PCalc/Loop.lean` lines 74–116 (`Kin`, `Kin.Planned`) |
| (6)–(9) | Proved here | Computations in the text |

| Item | Kind | Source / tools |
|---|---|---|
| This book's block map and pull-back | Lean | The source table of 19-04 |

**Terms used in this chapter** — predictor, transport, parallel straight lines, weighted cost, projection onto the group, nearest point, spread, background field, stationary point, pull-back. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-05 Symmetries — from how the frame, weights and predictor are built, we read that the one-step kernel is invariant under the hypercubic group and coarse translations, and covariant under gauge transformations.

# Symmetries — invariance under the hypercubic group and coarse translations, gauge covariance

This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the exact form of the two equalities satisfied by the one-step kernel $T_\beta(U, V)$ (invariance under discrete spacetime symmetries, and covariance under gauge transformations), and which components they come from. The mechanism of drawing the symmetry-breaking choices at large field at random and summing over them does its work here.

**Prerequisite chapters** — 9-01 to 9-04. The hypercubic group is in 1-02, and gauge transformations in 19-01.

**Said first** — the symmetry group is the set of all maps of coarse lattice points $x \mapsto s \cdot x + b$ ($s$ a signed permutation, $b$ a coarse point). Both reflections in planes through coarse lattice points and reflections in planes through the midpoints of coarse links are included ( § 02). This is an equality of **covariance as a symmetry** of the kernel, not a statement of reflection positivity. Continuous rotations are not stated at the lattice level; they are raised separately in the limit (12-06).

The order of this chapter

1. **The statement — two equalities**: `StepData.channel` and `StepData.covary`
2. **The form of the group action**: the 384 signed permutations, shifts by coarse points only, and reflections in two kinds of planes
3. **Origin (1) — invariance of the weights and patches**
4. **Origin (2) — uniqueness of the minimiser, and the sum over labels**
5. **Origin (3) — orientation reversal and inverses, gauge covariance**
6. **What this book's study had seen**: symmetric families of paths, and commuting with reflections in the side-3 design
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

## The statement — two equalities

The source's `YM/Branch/Covariant.lean` assumes the three obligations `Branches` (9-03, 9-04) and shows two equalities for the one-step kernel Source .

$$T\beta\big(\mathrm{pull}_{s,\,Lb}\,U,\ \mathrm{pull}_{s,\,b}\,V\big) = T\beta(U,V) \qquad (\texttt{StepData.channel},\ \text{line } 50\ )\ (1)$$

$$T\beta\big(g\cdot U,\ (g\circ\iota)\cdot V\big) = T\beta(U,V) \qquad (\texttt{StepData.covary},\ \text{line } 56\ )\ (2)$$

(1) holds for every signed permutation $s$, every coarse point $b$ and every $\beta$; (2) holds for every gauge transformation $g$ of the fine lattice (a group element at each point) and every $\beta$. $\iota$ is the embedding sending a coarse point to the fine point $Lx$ (9-02), and $g\cdot U$ is the action $U_{z,z+e}\mapsto g(z)U_{z,z+e}g(z+e)^{-1}$ on each link Source .

(2) can be read as "if the fine configuration is gauge transformed and the coarse configuration is gauge transformed by the values $g(Lx)$ at the coarse points, the kernel does not change". A gauge transformation of the coarse lattice is the restriction of a fine gauge transformation to the coarse points Inferred .

These two are required at each step as the fields `symmetric` and `gauge` of the calendar structure `Calendar.BlockControl` ( `YM/RG/History.lean` , lines 120–128), and `Calendar.blocks_go` ( `Covariant.lean` line 70) supplies them from (1)(2) Source . Since `BlockControl` is a field of the structure `Levels` of the sequence of steps, the sequence of steps of `History` under the construction theorem's conclusion `Family.Makes` uses kernels with these symmetries at every step Source .

02

## The form of the group action

A signed permutation $s$ is a pair of a permutation of the four axes and a sign flip on each axis (`Signed` in the source, `YM/Space/Index.lean` line 23). There are $4! \times 2^4 = 384$ elements, and this is the symmetry group of the four-dimensional hypercube (the hyperoctahedral group) Proved here .

The action on configurations, $\mathrm{pull}_{s,b}$ (`YM/Lattice/Path.lean` line 147), pulls a configuration back by the affine map of points $x \mapsto s \cdot x + b$ (`affine`, line 145). On links whose orientation is reversed, the value traversed backwards, that is the inverse, is read (`Step.pull_read`) Source .

$s$ acts about the origin, and then the shift by $b$ follows. It is not a rotation **about** the coarse point $b$ Source . Since $b$ in (1) ranges over all coarse points, the group contains all of the following maps Proved here .

- **Coarse translations**: with $s = 1$, $x \mapsto x + b$. On the fine side this is a translation by $Lb$.
- **Reflections in planes through lattice points**: taking $s$ to be the sign flip of axis $i$ (`Signed.mirror` in the source, `YM/Space/Index.lean` line 55) and $b = 0$ gives $x_i \mapsto -x_i$, the reflection in the plane $x_i = 0$. With $b = 2c\,e_i$ it is the reflection in the plane $x_i = c$ (a plane through coarse lattice points).
- **Reflections in planes through the midpoints of coarse links**: with the same $s$ and $b = e_i$, $x_i \mapsto 1 - x_i$. This is the reflection in the plane $x_i = \frac{1}{2}$, which passes through the midpoint of the coarse link $(0, e_i)$. On the fine lattice it is the plane $x_i = L/2$; since $L$ is odd, this is a block boundary, a plane through the midpoints of fine links. With $b = (2c + 1)e_i$ it is the plane $x_i = c + \frac{1}{2}$.
- **Permutations of axes, and their combinations with translations**: the case where $s$ contains a permutation. For example, with $b = 0$ it is a hypercubic rotation about the origin.

So (1) contains reflections in **both** kinds of planes of the coarse lattice (planes through lattice points and planes through the midpoints of links) Proved here . These two kinds have the same form as the two kinds of planes for which reflection positivity of the original Wilson measure is stated (12-01). However, (1) is an equality of **covariance**, saying that the kernel commutes with symmetries; it does not state reflection positivity of the push-forward measure Source . We found no statement of reflection positivity of the one-step kernel within what we read (9-01 § 07).

03

### Origin (1) — invariance of the weights and patches

The patch is a box centred at the origin, and the weights are convolutions of a uniform distribution (9-02). Both are invariant under signed permutations. The source builds the permutation of the points of the box `offset` (`YM/Branch/Transport.lean` line 18), and shows $w_J(s \cdot z) = w_J(z)$ in `weight_invariant` (line 135) Source .

This invariance acts on the predictor side. The cost $C_b(u) = \sum_z w_J(z)\,\mathrm{price}(Y_b(z)u^{-1})$ ((2) in 9-04) does not change value when the collection of transports is reordered by $z \mapsto s \cdot z$, so the minimiser and the probability of drawing a single one commute with the reordering (`bond_pull`, `YM/Branch/Predict.lean` line 37) Source .

04

### Origin (2) — uniqueness of the minimiser, and the sum over labels

The symmetry arises differently on the two branches of the frame (9-03) Source .

- **Small field**: the function minimised, $\mathcal{S}_U(h)$ ((2) in 9-03), is an average over all pairs of points and all axis orders, so it is invariant under signed permutations (`via_channel`, `Transport.lean` line 26). The branching condition is also invariant (`min_channel`, line 34). **Since the minimiser is unique**, the minimiser of the rotated configuration equals the rotated minimiser (`selected_channel`, line 56). Without uniqueness, there would be no guarantee that the chosen minimiser commutes with the symmetries Inferred .
- **Large field**: a label $(r, l)$ of the axial gauge moves under a group element $s$ to $(s \cdot r,\ s \circ l)$ (`orient`, line 20). The set of labels that may be drawn also moves one-to-one (inside the proof of `frames_channel`, lines 77–90). Since the probability of labels is the uniform distribution on the set, the probability of a label for the rotated configuration equals the probability of the moved label. Changing the variable of the kernel's sum $\sum_\tau$ as $\tau \mapsto s \cdot \tau$ (`withinChoice`, `Predict.lean` line 88; `Equiv.sum_comp`) then recovers the value of the kernel (`mass_channel`, `Covariant.lean` line 22).

An axial gauge fixed by a single label breaks the symmetry, but by **putting the probability of drawing the labels into the kernel and summing**, the kernel becomes invariant as an equality Inferred . The branch of the predictor that draws a single one (9-04) is recovered in the same way, since the weights $w_J$ are invariant.

05

## Origin (3) — orientation reversal and inverses, gauge covariance

A sign flip can move a coarse link $b$ to a negatively oriented link. In that case one uses that the predictor of the oppositely oriented coarse link is the inverse. `bond_inv` (`Predict.lean` line 44) says that inverting the collection of transports inverts the predictor and leaves the probability of drawing a single one unchanged Source . The proof is $\mathbf{price}(k^{-1}u) = \mathbf{price}(ku^{-1})$ (invariance of $\mathbf{price}$ under inverses and conjugation, `price_inv`, `price_conj`) Source . The noise factor is recovered by the same invariance (`noise_channel`, `Covariant.lean` line 37).

Gauge covariance (2) comes from three components Source .

1. The frame: under a fine gauge transformation, the frame moves by having the value at the coarse point attached on the left, and the probability of labels does not change (`frame_gauged`, `Transport.lean` line 111). The branching condition and the set of labels that may be drawn are gauge invariant.
2. The predictor: sandwiching the collection of transports between constants $u, v$ on both sides sandwiches the predictor between the same $u, v$, and leaves the probability of drawing a single one unchanged (`bond_gauge`, `Predict.lean` line 54).
3. The noise: $\mathbf{price}\big(uxv^{-1}(uyv^{-1})^{-1}\big) = \mathbf{price}(xy^{-1})$ (invariance under conjugation, at the end of the proof of `covary`).

As seen in 9-04 § 02, the collection of transports $Y_b(z)$ is sandwiched between end elements $g(Lx)$ and $g(L(x+e_i))^{-1}$ that do not depend on $z$, so the form of the predictor component (bond_gauge) can be used as it is Proved here .

| Component of the kernel | Origin of the spacetime symmetry (1) | Origin of the gauge covariance (2) |
|---|---|---|
| Weights $w_J$, patch | `weight_invariant`, `offset` | — |
| Small-field frame | Invariance of score + uniqueness of the minimiser (`selected_channel`) | `frame_gauged` |
| Large-field frame | Permutation of labels `orient` + sum over labels | `frame_gauged` |
| Predictor (average, single) | `bond_pull`, `bond_inv` | `bond_gauge` |
| Noise | `noise_channel` (invariance of price under inverses and conjugation) | Invariance of price under conjugation |

This symmetry is carried over to the representation side as well. The main action is written in a form averaged over signed permutations (`inChannels` in 9-06), and that the families of activities are moved into one another by permutations (`Activities.Moves`) is part of the conditions on the representation Source . It is also

from this symmetry that, in the small-field step, the sum of the log-Hessians in the curvature directions takes the form $\delta_{ab}\Delta$ (becomes isotropic) (11-09) Inferred .

06

## What this book's study had seen

This book's block map was made to commute with the discrete symmetries by taking the family of paths symmetric under permutations and reflections of the transverse directions (19-04, families of paths). Gauge covariance follows from path products being gauge transformed only at their two ends.

The symmetry statements closed in this book's Lean are the following Lean .

- The current design (side 2): when the fine step is large enough, the push-forward Wilson measure is invariant under coarse translations (for expectations of bounded continuous functions; Lean: `hsym_translation`, no assumption on the coupling).
- The side-3 centred design: commuting with reflections in planes of lattice points (all steps, no gauge transformation; Lean: `Block3.blockMap_refl_cfg`, `Block3.down_refl`), coarse translations (`Block3.blockMap_trans`, `down_trans`), sign flips (`Block3.blockMap_flip`), and boundary reflections (`Block3.down_reflB`).

Since this book's map is deterministic, its symmetries can be stated as equalities of maps (of the form $B(\Theta U) = \Theta B(U)$). Since the published construction uses a kernel, its symmetries are stated as the kernel equality (1), and the configuration-by-configuration symmetry-breaking choices (axial-gauge labels, the point drawn singly) are put into the kernel together with their probabilities and recovered by the sum. The aim is the same; the route is different. This book's study did not have the idea of recovering symmetry by averaging the symmetry-breaking choices over a probability (14-05).

07

## What we have not read

- Whether reflection positivity (positive type with respect to reflections) is stated for the one-step kernel. (1) is covariance as a symmetry, and we found no positivity statement within what we read. Reflection positivity is stated for the original Wilson measure, for two kinds of planes (12-01).
- The details of the proofs of `selected_channel`, `min_channel` and `good_channel` (we read only the statements and how they are used).
- Where (1)(2) are used in the proof that carries the symmetry of the representation (`Activities.Moves`) from step to step (we go as far as the record that `push_columns`

in `Push/SectorMeasure` uses them).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Under the obligations `Branches`, the kernel is invariant under maps of coarse lattice points $x \mapsto s \cdot x + b$ (the 384 signed permutations and coarse points $b$), which include coarse translations and reflections in both planes through lattice points and planes through the midpoints of coarse links (`StepData.channel`) | Reflection positivity of the one-step kernel (this equality is covariance as a symmetry, not positivity) |
| Source Under the same assumptions, the kernel is covariant under fine gauge transformations and their restriction to coarse points (`StepData.covary`) | — |
| Source `BlockControl` at each step requires the two equalities, and `blocks_go` supplies them | The details of the proof carrying the symmetry of the representation from step to step |
| Inferred The origins are the invariance of the weights and patches, the uniqueness of the small-field minimiser, the sum over large-field labels, and the invariance of price under inverses and conjugation | — |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The two equalities and their supply | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/Branch/Covariant.lean`: `prediction_channel` line 13, `mass_channel` line 22, `noise_channel` line 37, `channel` line 50, `covary` line 56, `Calendar.blocks_go` line 70 |
| Per-step requirements | Source | Same repository, `YM/RG/History.lean` lines 120–134 (`BlockControl`, `Levels`) |
| Signed permutations and affine pull-back | Source | Same repository, `YM/Space/Index.lean` line 23 `Signed`, `YM/Lattice/Path.lean` lines 145–147 `affine`, `pull` |
| Permutations of patches and labels, symmetry and gauge of the frame | Source | Same repository, `YM/Branch/Transport.lean`: `orient` line 20, `offset` line 18, `via_channel` line 26, `min_channel` line 34, `selected_channel` line 56, `frames_channel` line 77, `frame_gauged` line 111, `weight_invariant` line 135 |
| Symmetry and gauge of the predictor | Source | Same repository, `YM/Branch/Predict.lean`: `bond_pull` line 37, `bond_inv` line 44, `bond_gauge` line 54, `withinChoice` line 88 |

| Item | Kind | Source / tools |
|---|---|---|
| This book's symmetry statements | Lean | This book's Lean bundle (`hsym_translation`, `Block3.blockMap_refl_cfg`, `down_refl`, `blockMap_trans`, `down_trans`, `blockMap_flip`, `down_reflB`), a bundle that passed the check by the Lean kernel |

**Terms used in this chapter** — signed permutation, hyperoctahedral group, coarse translation, gauge covariance, uniqueness of the minimiser, sum over labels, orientation reversal. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-06 The density representation — we read the "representation" that writes the density carried by one step as a vacuum term, a main action, regular activities and a gas of forced activities.

# The density representation — the vacuum term, the main action, regular activities, the gas of forced activities

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the four parts of the form `Rep` in which the density $\rho_m$ at step $m$ is written, and the conditions imposed on it (normalisation, budgets, symmetry); that $\beta$ is defined as the coefficient of the kinetic term in this representation; that at the finest step the representation agrees with the bare Wilson weight; and that the representation is not said to be unique.

**Prerequisite chapters** — 9-01 The block kernel, 9-05 Symmetries. Gases of polymers (clusters) are in 7-04.

**Said first** — this chapter is about the **definition** of the representation and the form of its conditions. The proof that this form is preserved from step to step is read in Part 11 (11-01 to 11-09). The details of the small-field threshold and the truncated action are in 9-07.

The order of this chapter

1. **The statement — the form of the representation**: `Rep` and `Rep.eval`
2. **The main action**: truncated plaquette terms, squares of transported curvature, cubic and quartic atoms, one-loop atoms
3. **Two normalisations**: the coefficient of the kinetic term `metric_zero` and the one-loop part `loop_jets`
4. **Regular activities and the gas of forced activities**: `Activities` and `gas`
5. **Conditions imposed on the representation, and what is not said**: `Conv`, `Lawful`, no uniqueness
6. **What this book's study had seen**: the push-forward measure itself, versus coefficients inside a representation

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — the form of the representation

When a density is carried by the one-step kernel (9-01), it can become an arbitrary function of the configuration. The published construction requires, as a field of the structure of the sequence of steps, that the density at each step can be written in a **fixed form**. That form is `Rep` (lines 165–171) in `YM/RG/History.lean`, a tuple of six fields (path data, truncation parameters, activities, budget point, $\beta$, $v$) Source . `Rep.eval` (line 179) is

$$\rho(U) = \exp\Big(v\,|\Lambda| \;-\; S^{\mathrm{ch}}_{\beta}(U) \;-\; \sum_{i} \mathbf{1}\big[\text{support } X_i \text{ is healthy}\big]\, E_i(U)\Big) \cdot \mathrm{gas}(U) \tag{1}$$

Source . We name the four parts.

| Part | Name | Source | Chapter in this book |
|---|---|---|---|
| $v\|\Lambda\|$ | Vacuum term ($\|\Lambda\|$ is the number of points) | `Rep.v` | This chapter |
| $S^{\mathrm{ch}}_{\beta}$ | Main action (averaged over signed permutations) | `PathData.inChannels` | § 02, § 03, 9-07 |
| $\sum_i \mathbf{1}[\cdot]E_i$ | Sum of regular activities (healthy places only) | `Activities.additive` | § 04, 11-09 |
| gas | Gas of forced activities | `Activities.gas` | § 04, 11-06 |

"Healthy" means that no flag is raised at the points of the support. A flag is a point touching a plaquette whose chord exceeds the step threshold $t_m$ (9-07) Source .

The sequence of steps `History` has one representation for each $(N, p, m)$ (lines 204–206). In the structure `History.Selected` (lines 234–240), the field `represents` requires that the representation satisfy the condition `Conv` at every step $m \le N$; `starts` requires that at the finest step $m = N$, $v = 0$ and the representation equal the bare Wilson weight $e^{-\beta N S_W(U)}$; and `steps` requires the one-step identity ((4) in 9-01) Source . `Selected` is the field `wilson` of the construction theorem's conclusion `Family.Makes` (`YM/Source/Construction.lean` line 45) Source .

02

## The main action

The main action is `PathData.principal` (`YM/RG/PathActions.lean`, lines 236–238) averaged over the signed permutations $R$ (384 elements, 9-05) (`inChannels`, lines 241–242) Source .

$$S_\beta^{\mathrm{ch}}(U) = \frac{1}{384} \sum_R S_\beta^{\mathrm{pr}}(\mathrm{pull}_{R,0}\, U), \tag{2}$$

$$S_\beta^{\mathrm{pr}}(U) = \beta \Big[ \sum_p \Big( \mathrm{clip}_t\big(\mathrm{chord}\, U_p\big) + \mathbf{1}[\mathrm{kept}]\, \tfrac{1}{2} \|\ell_p(U)\|^2 \Big) + \sum_x \big( \text{cubic atoms} + \text{quartic atoms} \big)$$

We read the four ingredients in turn Source .

- **The truncated plaquette term** $\mathrm{clip}_t(\sigma)$ (`clipped`, line 218): if the plaquette chord $\sigma = \mathrm{chord}\, U_p$ is at most the threshold $t$, it is $c_{\mathrm{dir}} \sigma^2/2$; beyond it, the linear $c_{\mathrm{lin}}\, t\, \sigma$. The form and the condition at the kink, $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$, are read in 9-07.
- **The square of the transported curvature** $\frac{1}{2}\|\ell_p\|^2$: a Lie-algebra value for each plaquette $p$, $\ell_p(U) = \sum_k r_{p,k}\ \mathrm{Ad}\big(U(\gamma_k)\big) F(U_{p_k})$ (`ell`, line 190). $F(u)$ is the projection of $(u - u^{-1})/2$ onto the Lie algebra (`Gauge.curvature`, `YM/Gauge/Colors.lean` line 173), $\gamma_k$ are paths, $p_k$ are other plaquettes, and the list of coefficients $r_{p,k}$ is a "row" of the path data (`PathData.row`, line 151; `Slot`, lines 38–56). It is a term that carries the curvature of distant plaquettes along paths and adds them up. This term enters only when the configuration satisfies the condition kept (the chord of every plaquette is at most $t\, e^{\mathrm{dist}/\text{relaxation length}}$, lines 210–211).
- **Cubic and quartic atoms**: sums, with coefficients, of products of three or four Lie-algebra values (`Atom`, lines 84–105). They are set to 0 where flags are raised (`placed` uses `masked`, lines 107–108).
- **One-loop atoms**: atoms with two legs, **not multiplied by** $\beta$. This is the part carrying the one-loop contribution, which does not depend on $\beta$.

The coefficients of the main action (rows, atoms) are held in the path data `PathData` for each step $m$ and cutoff $N$, and do not depend on the volume index $p$ (`Selected.path_shared`, line 235) Source .

03

## Two normalisations

The $\beta$ in representation (1) is **a single real number** multiplying the main action. As it stands, $F^2$ terms could be shifted between $\beta$ and the one-loop atoms or the regular activities, so two normalisations stop such shifting (the fields `metric` and `loop_normal` of the path-data condition `PathData.Bounds`, lines 173–185) Source .

**The coefficient of the kinetic term** (`metric_zero`, lines 161–162). Writing the zero-momentum sum of the row coefficients as $S^{(0)}_{k,i} = \sum_{\text{rows of plaquette type } i} r$ (`symbol0`, line 158), for all plaquette types $i, j$,

$$c_{\mathrm{dir}}\,\delta_{ij} + \sum_{k} S^{(0)}_{k,i}\,S^{(0)}_{k,j} = \delta_{ij}. \quad (4)$$

For smooth small configurations, $\operatorname{chord} U_p \approx |F_p|$ and $\ell_p \approx \sum_j S^{(0)}_{p,j} F_j$ (zero momentum), so the quadratic part of the main action at zero momentum becomes $\frac{\beta}{2}$ $\sum_{ij}\big(c_{\mathrm{dir}}\delta_{ij} + (S^{(0)\mathsf{T}} S^{(0)})_{ij}\big)F_iF_j = \frac{\beta}{2}\sum_i F_i^2$ Inferred . (4) is the normalisation for "reading $\beta$ exactly as the coefficient of the kinetic term".

**The one-loop part** (`loop_jets`, lines 167–169). For the coefficients of the one-loop atoms, it requires, for each pair of plaquette type and colour, that the sum of the coefficients be 0 and that the first moment in each direction (the sum weighted by the difference of the positions of the two legs) also be 0 Source . This means that no $F^2$ terms at zero and first order in momentum remain in the one-loop part, that is, all marginal $F^2$ terms are collected into $\beta$ Inferred .

The source's docstring (`History.lean`, lines 25–33) describes $\beta$ as "the actual scalar in the base part of the representation, under the flat coefficient arrays and the normalisation of the kinetic metric, the one-loop kernel with its leading Fourier coefficients removed, and the condition that affine remainders vanish; not a number tied independently to the measure" Source . The discussion of "the coupling at step $m$" that starts from this definition is read in 10-01.

04

## Regular activities and the gas of forced activities

The family of activities `Activities` (`YM/RG/Activities.lean`, lines 62–66) is two kinds of sequences of functions, addressed by natural numbers Source .

**Regular activities** $E_i$: the address is an anchor point and a support (`RegularAddress`, lines 37–40). In the representation they enter only when the support is healthy (`regularOccurrence` is `masked`, line 131). The condition `RegularFamily` (lines 70–75) requires locality on the support, gauge invariance, $C^{\text{order}}$ smoothness at healthy places, **vanishing at the flat configuration** ($E_i(1) = 0$), and **vanishing log-Hessian in affine directions**. The last two express that this is the form **after** the constant has been moved into the vacuum term $v$ and the marginal quadratic term into $\beta$ (the rereading in the small-field step, 11-09) Inferred .

**Forced activities** $K_i$: the address is a support and a "stock" (a non-empty set of points responsible for flags; `ForcedAddress`, lines 42–47). The field `permit` of the condition `ForcedFamily` (lines 77–84) requires that if $K_i \neq 0$, the flags in the support coincide exactly with the stock Source . The gas is

$$\mathrm{gas}(U) = \sum_{\substack{b\text{: disjoint supports,} \\ \bigcup_{i\in b} \mathrm{stock}_i = \mathrm{flags}(U)}} \prod_{i\in b} K_i(U) \tag{5}$$

(`admissible`, `gasOccurrence`, `gas`, lines 114–129) Source . It is a sum only over collections of forced activities with non-overlapping supports that **exactly cover** all the flags. If there are no flags, only the empty collection remains, and $\mathrm{gas} = 1$ (`gas_one`, line 135) Source . All large-field contributions go into this gas, and at small field with no flags the representation reduces to the exponential part of (1).

The size budget is measured by point-by-point weighted sums (`EBound`, `KBound`, lines 98–104) Source . For regular activities per anchor point, and for forced activities per point contained in the support, $e^{c_{\mathrm{sup}}\cdot(\text{cover number})} \times (\text{size})$ is summed and kept within the budget. $c_{\mathrm{sup}}$ is the step constant `supportExponent`; the cover number is the smallest number of balls of radius $B_m = \lceil (\log \hat{\beta}_m)^2 \rceil$ in the lattice maximum distance (centred cubes of side $2B_m + 1$), arranged so that their union is connected, needed to cover the support (`coverNumber`, `YM/Lattice/Support.lean` line 71; `BudgetPoint.mass`, `YM/RG/Masks.lean` line 144); and the "size" of a regular activity is the supremum, over healthy configurations, of finite-order derivatives in directions of size $t_m$ (`supBounds`, lines 164–166). Since these are point-by-point sums, the budget has a volume-independent form Inferred .

```
05
```

## Conditions imposed on the representation, and what is not said

The condition `History.Conv` (`History.lean`, lines 213–218) satisfied by the representation at step $m$ has four parts Source .

| Field | Content |
|---|---|
| `vac` | The budget point (threshold $t_m$, cover scale $B_m$, order) is the fixed value for step $m$, `budget m` |
| `clip` | The truncation parameters are step constants |
| `good` | The budget of the path data (`Bounds`) and the budget of the activities (`Valid`). The budget for regular activities is $\hat{g}_m^{11/5}$ (`allowance`), that for forced activities $e^{-P_m^{1/5}}$ (`forcedAllowance`) |
| `band` | $\beta(N,p,m) \in [\hat{\beta}_m/2,\ 2\hat{\beta}_m]$ |

Here $\hat{\beta}_m = \bar{\beta} + s\log D_m$, $\hat{g}_m = \hat{\beta}_m^{-1/2}$, $P_m = \exp((\log\log\hat{\beta}_m)^2)$ (definition of `Calendar`, lines 79–86) Source . The budget of activities also includes a symmetry condition (`Activities.Moves`, lines 87–96: activities are moved into one another, addresses included, by signed permutations).

The representation at the finest step is the bare Wilson weight itself (`starts`): the jet up to order 4 of the main action of the path data agrees with the Wilson energy, and the coefficients of the one-loop atoms are 0 (the fields `peak` and `initial` of `Kin.Planned`, `YM/PCalc/Loop.lean` lines 100 and 102) Source . The main action at a coarse step is tied to the fine step by the condition that "its jet up to order 4 at the flat configuration agrees with the stationary value of the fine main action plus the penalty term" (`ClassicalMatch`, the field at line 108 of the same file), but this is a condition; it does not define the main action as an expansion of the stationary value Source .

**What is not said.** The source's docstrings say in two places that the representation is not claimed to be unique Source .

- `PathActions.lean`, lines 25–26: uniqueness of the fitted representation as a function is neither claimed nor used.
- `History.lean`, lines 29–31: the formulation is of the form that such a representation and continuation **exist**; neither uniqueness nor reproduction of the implementation's choice of paths is required.

Since the same measure (the same density $\rho_m$) can have different representations, the $\beta$ in the representation is not determined by the measure alone. This point is the subject of 10-01.

```
06
```

## What this book's study had seen

One renormalisation step in this book's study took the push-forward measure $B_*\mu$ itself as its object (19-04, push-forward). It did not write the theory at a coarse step in a representation of fixed form. The statements to be shown (19-07) are also written only in terms of the push-forward measure and expectations of observables.

On the side of the list of premises (28-04), there were two proposals on paper for the definition of the effective coupling — "read it as a coefficient of the blocked action" and "read it as a renormalised coupling" — and the choice was open (28-05 § 06). The published construction fixes the former form, with the two normalisations (4) and `loop_jets`, and requires the existence of a representation at each step. In exchange, it does not say that the representation is unique, and $\beta$ becomes a number that depends on the representation.

Holding forced activities in the representation as a gas that "exactly covers the flags" gives an axis of comparison with this book's study's treatment of large fields (the rows for the large-field estimate and the gas in the list of premises). The comparison is in 14-04.

07

## What we have not read

- How the rows $r_{p,k}$ and paths $\gamma_k$ of the path data are concretely built (the body of the "explicit renormalisation map" for the main terms). Candidate locations are `YM/Push/BETA_TRANSPORT`, `LOOP_COEFFICIENT`, `FREE_RESOLVENT` and `STATIONARY_UNIFORM`; we have not read them.
- The value of the relaxation length in the condition kept (`Clip.relaxation`), and where among the step constants it comes from.
- The construction that brings the forced activities back within the budget $e^{-P_m^{1/5}}$ after one step (the subject of 11-06; this book's study has not checked the balance either).
- The proof that path data satisfying the normalisation (4) exist at each step.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source That the density at each step is written in a representation of the form (1) is required as a field of the structure of the sequence of steps (and is part of the construction theorem's conclusion) | Uniqueness of the representation (the source states that it does not claim it) |
| Source The main action is the average over signed permutations of the truncated plaquette terms, squares of transported curvature (only when kept), cubic and quartic atoms, and one-loop atoms not multiplied by $\beta$ | How the path data are concretely built |
| Inferred The normalisation (4) makes $\beta$ read as the coefficient of the kinetic term at zero momentum, and `loop_jets` leaves no marginal $F^2$ in the one-loop part | The proof of existence of path data satisfying the normalisations (not read) |
| Source With no flags the gas is 1 (`gas_one`); the gas is a sum over collections with non-overlapping supports that exactly cover the flags | Bringing the forced activities back within budget after one step |
| Source At the finest step the representation is the bare Wilson weight; the budgets are $\hat{g}^{11/5}$ and $e^{-P^{1/5}}$; $\beta$ lies in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$ | The proof that the budgets are preserved from step to step (Part 11) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Representation, sequence of steps, conditions | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean`: docstring lines 25–33, `Calendar` lines 79–86, `Rep` lines 165–179, `History` line 204, `Conv` lines 213–218, `Selected` lines 234–240 |
| Main action and normalisations | Source | Same repository, `YM/RG/PathActions.lean`: docstring lines 25–26, `Slot` line 38, `Atom` line 84, `Clip` line 139, `PathData` line 150, `symbol0` line 158, `metric_zero` line 161, `loop_jets` line 167, `Bounds` line 173, `ell` line 190, `kept` line 210, `clipped` line 218, `principal` line 236, `inChannels` line 241 |
| Curvature | Source | Same repository, `YM/Gauge/Colors.lean` line 173 `curvature` |
| Activities, gas, budgets | Source | Same repository, `YM/RG/Activities.lean` lines 37–135; `YM/RG/Masks.lean` lines 125–166 (`BudgetPoint`, `mass`, `Regular`, `supBounds`) |
| Conditions at the finest step, conditions on the coarse main action | Source | Same repository, `YM/PCalc/Loop.lean` lines 100, 102 and 108 (fields of `Kin.Planned`) |
| Relation to the construction theorem's conclusion | Source | Same repository, `YM/Source/Construction.lean` lines 44–45 `Family.Makes.wilson` |

| Item | Kind | Source / tools |
|---|---|---|
| This book's push-forward measure and statements to be shown | Lean | The source tables of 19-04 and 19-07 |

**Terms used in this chapter** — representation, vacuum term, main action, truncated plaquette term, transported curvature, atom, one-loop atom, normalisation of the kinetic term, regular activity, forced activity, stock, flag, healthy, gas, budget, cover number. Their meanings are in the A-02 Glossary.

Next chapter to read: 9-07 The small-field threshold and the truncated action — we read the step-dependent threshold that separates large and small fields in the representation, and the form that truncates the plaquette term to linear.

# The small-field threshold and the truncated action — a step-dependent threshold

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the formula and properties of the threshold $t_m$ that decides, inside the representation at step $m$ (9-06), whether a plaquette is "large"; the flags raised at points touching plaquettes beyond the threshold, and "healthy" supports with no flags; the form that truncates the plaquette term of the main action to linear above the threshold, with the kink condition $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$; and that the threshold goes to 0 with the step while staying much larger than the fluctuation size $\hat{\beta}_m^{-1/2}$.

**Prerequisite chapters** — 9-06 The density representation, 8-02 The set-up.

**Said first** — the truncated action is the **form** of the main part of the representation; it does not change the measure. The one-step identity remains an equality, and the truncated difference goes into the activities. It cannot be read as saying that truncation makes the difficulties of large fields disappear. The large-field estimate itself is read in Part 11 (11-03, 11-04).

The order of this chapter

1. **The statement — the step threshold**: `Calendar.threshold` and the conditions of the calendar
2. **The size of the threshold**: it goes to 0, but stays larger than the fluctuation size by the factor $P_m$
3. **Flags and healthy configurations**: `flags`, `Healthy`, `masked`
4. **The truncated action**: the form of `clipped` and the kink condition
5. **The threshold depends on the step**: two thresholds within one step
6. **What this book's study had seen**: a fixed threshold versus a step threshold
7. **What we have not read**

Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — the step threshold

The calendar `Calendar` in `YM/RG/History.lean` of the source (lines 79–86) defines the following quantities for each step $m$ Source.

$$\hat\beta_m = \bar\beta + s\log D_m, \qquad \hat g_m = \hat\beta_m^{-1/2}, \qquad P_m = \exp\big((\log\log\hat\beta_m)^2\big), \qquad t_m = \hat g_m\, P_m. \quad (1$$

$\bar\beta$ is the value matched at step 0 (10-01), $D_m$ is the reciprocal of the lattice spacing at step $m$, and $s = 11C_g/(24\pi^2)$ is the one-loop slope (`hatBeta`, `hatCoupling`, `monitor`, `threshold`). $\hat\beta_m$ is the coupling at step $m$ expected from the one-loop running, and $\hat g_m$ is the size of that coupling. $t_m$ is the **small-field threshold** at step $m$, used as the threshold of the budget point of the representation (`budget m`) (`Levels.budget`, lines 141–156) Source.

The calendar condition `Calendar.Fits` requires $1 < \hat\beta_m$ (`delayed`) and $t_m \le 1$ (`smallField`) at every step (lines 107–108) Source. The latter is supplied by the condition on $\bar\beta$, `Supply.Bar.threshold` (`YM/Price/Intent.lean` line 127): if $x \ge \bar\beta$ then $x^{-1/2}e^{(\log\log x)^2} \le 1$ Source. Since $\hat\beta_m \ge \bar\beta$, the threshold is at most 1 at every step Proved here.

The same calendar also defines the cover scale $B_m = \lceil(\log\hat\beta_m)^2\rceil$ (`coverSize`), the budget for regular activities $\hat g_m^{11/5}$ (`allowance`), and the budget for forced activities $e^{-P_m^{1/5}}$ (`forcedAllowance`), all as functions of the same $\hat\beta_m$ (lines 82–85) Source.

02

## The size of the threshold

$P(x) = e^{(\log\log x)^2}$ grows faster than any power of $\log x$ and slower than any positive power of $x$ Proved here , because $\log P = (\log\log x)^2$ grows faster than $\log(\log x)^k = k \log\log x$ and slower than $\log x^\epsilon = \epsilon \log x$. `cover_pay` and `inactive_price` in the source's `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean` state forms of this property ($b^n / P^a \to 0$ and so on) Source .

Hence, as $\hat\beta \to \infty$,

$$t = \hat g\, P \longrightarrow 0, \qquad \frac{t}{\hat g} = P \longrightarrow \infty, \qquad \hat\beta\, t^2 = P^2 \gg (\log\hat\beta)^k \ \text{(for all } k\text{)} \quad (2)$$

Proved here . At weak-coupling steps, the typical fluctuation size of a plaquette is of order $\hat g$ Inferred . The threshold is larger than that by the factor $P$, so a plaquette beyond the threshold is "a plaquette far from typical". On the other hand, since $t \to 0$, the small-field expansion of the action can be used below the threshold Inferred . The action weight of one plaquette beyond the threshold is of order $\beta t^2 \approx P^2$, and the form is chosen so that this outweighs the loss from counting covers (of the order of a power of $\log\hat\beta$) Inferred .

| $\hat\beta$ | $P$ | $t = \hat\beta^{-1/2} P$ | $\hat\beta t^2 = P^2$ | $B = \lceil(\log\hat\beta)^2\rceil$ | $e^{-P^{1/5}}$ |
|---|---|---|---|---|---|
| $10^5$ | 392 | 1.24 | 1.5e5 | 133 | 3.7e-2 |
| $10^6$ | 987 | 0.987 | 9.7e5 | 191 | 1.9e-2 |
| $10^8$ | 4.86e3 | 0.486 | 2.4e7 | 340 | 4.2e-3 |
| $10^{12}$ | 6.08e4 | 0.0608 | 3.7e9 | 764 | 1.2e-4 |
| $10^{24}$ | 9.79e6 | 9.8e-6 | 9.6e13 | 3054 | 1.4e-11 |

The table gives values of (1) evaluated with a calculator (2–3 significant figures) Proved here . $t \le 1$ requires $\hat\beta$ of roughly $10^6$ or more. This fits the calendar's design of taking $\bar\beta$ large (10-05). No concrete value of $\bar\beta$ is in the source; it is chosen in existence form Source .

03

## Flags and healthy configurations

A **flag** is raised at each point touching a plaquette beyond the threshold $t$. The source's `flags` (`YM/RG/Masks.lean`, lines 34–36) is

$$\mathrm{flags}_t(U) = \{x : \exists p\ (\text{plaquette}),\ x \in \partial p \text{ and } \mathrm{chord}(U_p) > t\} \quad (3)$$

where $\partial p$ is the four vertices of the plaquette Source . Plaquette values only undergo conjugation under gauge transformations of the configuration, so flags are gauge invariant (`gauge_flags`, line 43) Source . A set of points $X$ is **healthy** if there are no flags in $X$ (`Healthy`, line 41) Source .

Flags are used in three places in the representation Source .

- Regular activities $E_i$ and cubic, quartic and one-loop atoms enter only when their support (for atoms, their footprint) is healthy; otherwise they are set to 0 (`masked`, lines 70–72; `Atom.placed`, `Activities.regularOccurrence`).
- The gas of forced activities is a sum over collections that exactly cover all the flags ((5) in 9-06). With no flags the gas is 1.
- The transported-curvature term $\frac{1}{2}\|\ell_p\|^2$ enters only when $\operatorname{chord} U_q \le t\, e^{\operatorname{dist}(p,q)/\text{relaxation length}}$ for every plaquette $q$ (kept) (`PathActions.lean`, lines 210–211). The threshold is looser for more distant plaquettes; it is a condition that looks at the whole configuration.

The size of a regular activity is measured by the supremum, over healthy configurations, of finite-order derivatives in directions of size $t$ (`dirBound`: the size in the direction of link $e$ is at most $t(1+\operatorname{dist}(e,o))^{\text{power}}$) (`Masks.lean`, lines 141–166) Source . The threshold $t$ enters both the decision about flags and the yardstick for the size of activities.

04

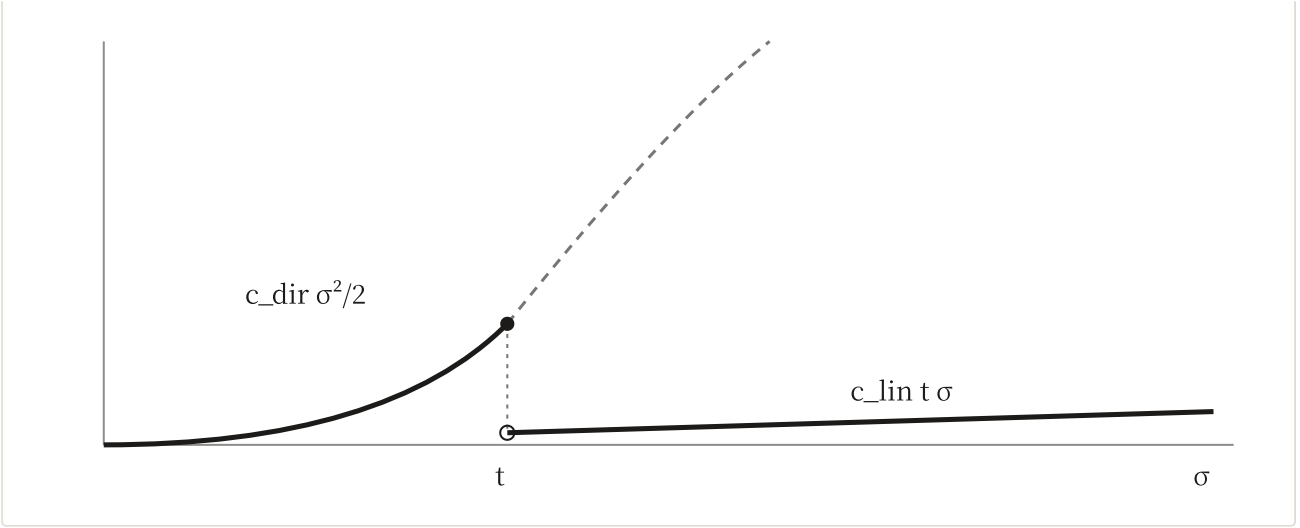
## The truncated action

The plaquette term of the main action, as a function of the chord $\sigma = \operatorname{chord} U_p$, is truncated to linear above the threshold $t$ (`clipped`, `PathActions.lean` lines 218–219) Source .

$$\operatorname{clip}_t(\sigma) = \begin{cases} \dfrac{c_{\text{dir}}}{2}\,\sigma^2 & \sigma \le t, \\ c_{\text{lin}}\, t\, \sigma & \sigma > t, \end{cases} \qquad 0 < c_{\text{lin}} < \frac{c_{\text{dir}}}{16}. \tag{4}$$

$c_{\text{dir}}, c_{\text{lin}}$ and the relaxation length are the step constants `Clip` (lines 139–146), and the kink condition `kink` is $c_{\text{lin}} < c_{\text{dir}}/16$ Source . $c_{\text{dir}}$ also enters the normalisation of the kinetic term ((4) in 9-06).

Figure 1 The form of the truncated plaquette term (4) (example with $c_{\text{lin}} = c_{\text{dir}}/20$). At $\sigma = t$ the value jumps down, and beyond it is a straight line of slope $c_{\text{lin}}t$. The dashed curve is the untruncated quadratic term.




(4) is not continuous at $\sigma = t$. It jumps down from the left value $c_{\mathrm{dir}} t^2/2$ to the right value $c_{\mathrm{lin}} t^2 < c_{\mathrm{dir}} t^2/16$ Proved here . The action is neither monotone nor convex in $\sigma$ Proved here . The term of a plaquette beyond the threshold is bounded below by $c_{\mathrm{lin}} t\,\sigma \geq c_{\mathrm{lin}} t^2$, and multiplied by $\beta$ it is of order $\beta c_{\mathrm{lin}} t^2 \approx c_{\mathrm{lin}} P^2$ (under the band $\beta \in [\hat{\beta}/2, 2\hat{\beta}]$) Inferred .

We write what the truncated action means, following the source's definitions Source .

- Only **the main part of the representation** is truncated. The one-step identity $\rho_m = \mathcal{T}_{\beta_{m+1}} \rho_{m+1}$ remains an equality, and the measure does not change. The truncated difference goes to the side of the regular and forced activities.
- The large-field estimate is written using the truncated action, in the form of an inequality uniform over configurations satisfying the gauge slice and region conditions (`Teller.Grants` in 11-04, with a funding assumption).

The linear tail is a form that keeps the plaquette term from growing quadratically above the threshold. If a quadratic action were used in the large-field region, negative terms in the decomposition of the action within one step (the balance of fine plaquettes, penalty and coarse plaquettes) could grow quadratically, and this can be read as holding them down to linear Inferred . We have not read where the number 16 in the kink comes from.

05

## The threshold depends on the step

Since $\hat{\beta}_m$ increases with $m$ (larger at finer steps), $t_m$ becomes smaller at finer steps Proved here . The threshold is a threshold on chords measured in lattice units, not a quantity in physical units Inferred . At each step, the threshold, the cover scale, the budgets and the truncation parameters (`Conv.clip` is the step constant `(params m).clip`) are determined afresh (`Conv` in 9-06) Source .

Within one step (from step $m+1$ to step $m$), **two thresholds** appear at once Source .

| Threshold | Value | Where it is used |
|---|---|---|
| Fine side | $t_{m+1}$ (smaller) | Flags of the incoming representation. In the large-field estimate, deciding seeds (marks of large field) and the left side of the inequality on the fine-plaquette side (`fiery` in the source) |
| Coarse side | $t_m$ (larger) | Flags of the outgoing representation. The right side of the inequality on the coarse-plaquette side, and the region margin $5\lceil(\log \hat{\beta}_m)^2\rceil$ |

Which threshold is used in which inequality was checked statement by statement against the source in this book's study (cited in 11-03 and 11-04; the threshold symbols chapter by chapter in Part 11 are in the table correspondence of threshold symbols in 11-01). The two thresholds are linked in the form that wherever a large-field activity coming from the previous step sits always becomes a seed of this step (`flagged` in `Allocation`) Source .

Because the threshold depends on the step, the budgets are redefined at each step as functions of that step's coupling. The one-step certificate has the form of showing only that "the output fits into the fixed budget of the next step" (11-07) Inferred . Conditions uniform in the number of steps are paid for on the calendar's side (the order in which sides and constants are chosen) (10-05) Source .

06

## What this book's study had seen

In this book's study, the threshold separating "large plaquettes" was a fixed value independent of the step Proved here .

- The theorem on the rarity of large plaquettes (22-01) counts plaquettes with $\operatorname{Re} U_p < 0$. It can be written in terms of the Wilson measure alone, and the threshold is fixed.
- The pull-back threshold $r_0 \in (0,1]$ of the block map (19-04) is also a fixed parameter.
- On the side of the list of premises, there was a row placing a step-dependent threshold, but its form had not been settled (Table 3 of 28-05 § 08).

The threshold (1) of the published construction is determined explicitly as a function of the step coupling, in a form that leaves a margin $P$, absorbing polylogarithmic losses, on both sides between the fluctuation size $\hat{g}$ and 1 Inferred . The budget exponent $11/5$ (between the quadratic 2 and the first non-marginal degree 3) and $e^{-P^{1/5}}$ are also functions of the same $\hat{\beta}_m$. The design of resetting thresholds and budgets at each step as functions of the coupling is something this book's study did not have (14-05). Its relation to where this book's study got stuck in the large-field estimate is treated in 14-04.

07

## What we have not read

- The relation between the 16 of the kink condition and the axial-gauge factor 16 (9-03). It is the same number, but we have not checked whether they are connected.
- The value of the relaxation length (`Clip.relaxation`), and the role of the condition `slow_selector` ($12 \le$ relaxation length $\times$ width, `History.lean` line 59).
- In which estimate the exponent 2 in $(\log\log)^2$ and the $1/5$ of the budget are first needed. We read as far as the statement of `dictionary_fit` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean` line 78), which states an upper bound of the form $e^{-AP^{\delta}}$ for sufficiently large coupling, for any exponent $0 \le \delta < 2$.
- An explicit form of the lower bound on $\bar{\beta}$ (it is chosen in existence form, and the source has no number).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The threshold at step $m$ is $t_m = \hat{\beta}_m^{-1/2} e^{(\log\log\hat{\beta}_m)^2}$, and by the calendar conditions $t_m \le 1$ at every step | An explicit lower bound on $\bar{\beta}$ |
| Proved here $t_m \to 0$, $t_m/\hat{g}_m = P_m \to \infty$, and $\hat{\beta} t^2 = P^2$ exceeds any power of $\log\hat{\beta}$ | In which estimates the exponents 2 and $1/5$ are needed |
| Source Flags are points touching plaquettes beyond the threshold (gauge invariant). Regular activities and atoms enter only on healthy supports | — |
| Source The plaquette term is truncated to linear above the threshold, with $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$. It jumps down at $\sigma = t$ and is neither monotone nor convex | Where the 16 comes from |
| Source Only the main part of the representation is truncated; the one-step identity remains an equality, and the difference goes into the activities | The reading that truncation makes the difficulties of large fields disappear (this cannot be written) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Step quantities and threshold, calendar conditions | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean` lines 79–86 (`hatBeta`, `hatCoupling`, `monitor`, `coverSize`, `threshold`, `allowance`, `forcedAllowance`), lines 107–108 (`delayed`, `smallField`), lines 141–156 `Levels.budget` |
| Threshold condition on $\bar{\beta}$ | Source | Same repository, `YM/Price/Intent.lean` lines 123–127 `Supply.Bar` |
| Growth of $P$ | Source | Same repository, `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean` (`cover_pay`, `inactive_price`; statements only) |
| Flags, healthy, masking | Source | Same repository, `YM/RG/Masks.lean` lines 34–72 (`flags`, `Healthy`, `gauge_flags`, `masked`), lines 141–166 |
| Truncated action and kept | Source | Same repository, `YM/RG/PathActions.lean` lines 139–146 `Clip`, lines 210–219 `kept`, `clipped` |
| Numbers in Table 1 | Proved here | Values of (1) evaluated in floating point |
| This book's fixed thresholds | Lean | The source tables of 22-01 and 19-04 |

**Terms used in this chapter** — small-field threshold, flag, healthy, truncated action, kink condition, cover scale, budget, polylogarithmic margin, fine-side and coarse-side thresholds. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-01 What "the coupling at step m" is — we read that $\beta$, defined as a coefficient in the representation together with its two normalisations, is a number not determined by the measure alone.

# What "the coupling at step m" is — a coefficient in the representation

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the definition of the number written $\beta(N, p, m)$ in the published construction: that it is the field `beta` of the representation `Rep`, a coefficient taken after two normalisations (kinetic term, one-loop part) have stopped $F^2$ from being shifted around; the bare coupling $\beta_{N,p} = \beta(N, p, N)$, the target value $\bar{\beta}$ at step 0, and the decrease per step; and that it is not an invariant readable from the measure alone.

**Prerequisite chapters** — 9-06 The density representation, 8-01 The lattice, the Wilson measure, conventions. The physics of asymptotic freedom is in 4-06.

**Said first** — this is a chapter of definitions. That "the coupling at step $m$" depends on the representation is stated explicitly by the source itself; it is not this book's inference. That the representation is not unique is a separate matter from whether there is another definition of the coupling determined by the measure, and on the latter the source says nothing.

The order of this chapter

1. **The statement — definition of the coupling at step m**: `History.β` and `Rep.beta`
2. **Two normalisations**: the coefficient of the kinetic term, and the leading Fourier coefficients of the one-loop part
3. **The direction of the steps, and the target value at step 0**: the bare coupling $\beta_{N,p}$, $\bar{\beta}$, the decrease per step
4. **Not a number determined by the measure alone**: three places in the source's docstrings
5. **Converting conventions**: $\beta = 1/g_0^2$, $C_g$, and the relation to this book's $\beta$

6. **What this book's study had seen**: matching quantities definable from the measure alone
7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — definition of the coupling at step m

For each cutoff $N$ (lattice spacing $a_N = 1/D_N$) and volume index $p$ (physical side $L_p$), the published construction has a representation of the density at steps $m = N, N-1, \ldots, 0$, `rep N p m` (`YM/RG/History.lean`, lines 204–206). "The coupling at step $m$" is the field `beta` of that representation Source.

$$\beta(N,p,m) \;:=\; (\texttt{rep}\ N\ p\ m).\texttt{beta} \qquad (\texttt{History.}\beta,\ \text{line 211}).\ (1)$$

In the form of the representation ((1) in 9-06), this number multiplies the main action $S_\beta^{\mathrm{pr}}$, and is the overall coefficient of the truncated plaquette terms, the squares of transported curvature, and the cubic and quartic atoms. It sits in a different field from the one-loop atoms, which are not multiplied by $\beta$, the regular activities, and the gas of forced activities Source.

The value at the finest step is the **bare coupling**. `Family.beta N p` is defined as $\beta(N,p,N)$ (`YM/Source/Construction.lean` line 38), and becomes the field `WilsonFamily.beta` of the family in the public theorem Source. The assessment's glossary describes this field as "the coupling multiplying the Wilson energy; for each $N, p$, the construction chooses the bare coupling so that the coupling after $N$ renormalisations equals a fixed reference value $\bar{\beta}$" (Appendix A, page 23) Assessment.

02

## Two normalisations

To call one coefficient in the representation "the coupling", one must make it impossible to shift $F^2$ terms between it and the other fields. For example, by adding a term proportional to $\sum_p F_p^2$ to the one-loop atoms and decreasing $\beta$, the same density could be written with a different $\beta$ Inferred . The source stops this with two fields of the path-data condition `PathData.Bounds` ( `YM/RG/PathActions.lean`, lines 161–169 and 181–185) Source .

- **Normalisation of the kinetic term** ( `metric_zero` ): the matrix of quadratic coefficients at zero momentum is the identity matrix ((4) in 9-06). This corresponds to what the assessment calls "the normalisation of the kinetic term" Inferred . With it, the quadratic part of the main action for smooth small configurations is $\frac{\beta}{2}\sum F^2$ Inferred .
- **Normalisation of the one-loop part** ( `loop_jets` ): the zeroth- and first-order Fourier coefficients (the sum and the first moment) of the coefficients of the $\beta$-independent one-loop atoms are 0. No marginal $F^2$ term remains in the one-loop part Inferred .

On the side of the regular activities too there are the conditions of vanishing at the flat configuration and vanishing log-Hessian in affine directions ( `RegularFamily`, 9-06 § 04) Source . Together the three have the effect that all marginal $F^2$ coefficients are collected into $\beta$ Inferred . The assessment writes in § 7.4: "integrating out fluctuations shorter than length $r$ changes the coefficient of the square-of-curvature term of the effective action; the inverse of its square root in a fixed normalisation is the running coupling" Assessment .

03

## The direction of the steps, and the target value at step 0

The structure of the sequence of steps `History.Selected` ( `History.lean`, lines 234–240) imposes the following on the sequence of couplings Source .

| Field | Content |
|---|---|
| `starts` | The representation at step $N$ (the finest) is the bare Wilson weight $e^{-\beta N,p S_{\mathrm{W}}}$, with $v = 0$ |
| `steps` | The one-step identity and $\lvert\beta(N,p,m+1) - \beta(N,p,m) - s\log b_m\rvert \le \mathrm{drift}_{m+1}$ ( `Transfer`, lines 223–227) |
| `represents` | $\beta(N,p,m) \in [\hat\beta_m/2,\ 2\hat\beta_m]$ ( `Conv.band` ) |
| `reaches` | $\beta(N,p,0) = \bar\beta$ for all $N,p$ |

| Field | Content |
| --- | --- |
| `running` | $\lvert\beta(N,p,m)-\hat\beta_m\rvert \le \varepsilon\log(1/a_m)$ (for $m$ large enough, uniformly in $N,p$; `Running`, lines 229–230) |

Here $s = 11C_g/(24\pi^2)$ (`slopeConstant` in the source, `YM/Gauge/Colors.lean` line 184) and $\hat\beta_m = \bar\beta + s\log D_m$ Source . Each time one step coarser is taken, the coupling **decreases** by about $s\log b_m$ (the coupling constant $g^2 = 1/\beta$ increases). The finer the step, the larger $\beta$; this is the direction of asymptotic freedom Inferred .

At step 0 (lattice spacing 1) the coupling is matched to the same value $\bar\beta$ for all $N,p$. The bare coupling $\beta_{N,p}$ is a number chosen to satisfy this condition (how it is chosen is in 10-04) Source . $\bar\beta$ is the calendar field `Calendar.bar` (line 67) and is taken large (table in 9-07: about $10^6$ or more for the threshold to be at most 1). At every step the coupling stays in the band $[\hat\beta_m/2, 2\hat\beta_m]$, and renormalisation does not pass through the strong-coupling region Inferred .

`History.Selected` is the field `wilson` of the construction theorem's conclusion `Family.Makes` (`Construction.lean` line 45) Source . The public theorem's statement `WilsonRegularity.running` follows from `running` (10-06).

04

## Not a number determined by the measure alone

The source's docstrings state in three places that this coupling is not an invariant of the measure Source .

1. `YM/RG/History.lean`, lines 25–33: the definition of the inverse coupling here is the actual scalar in the base part of the representation, under conditions such as the normalisation of the flat coefficients and the kinetic metric, the one-loop kernel with its leading Fourier coefficients removed, and the vanishing of affine remainders. **It is not a number tied independently to the measure.** The formulation is of the form that such a representation and continuation **exist**, and uniqueness is not required. One should not infer claims about the usual beta function beyond `Running`.
2. The docstring of `YM/UVNext/COUPLING_RESET/Reset.lean`: nothing is said about the existence of an intrinsic readout of $\beta$ on a measure for which paths and activities have not been fixed.
3. The docstring of `YM/Pert/Asymptotics/RawAxis.lean`: it extracts the leading logarithm of the same running inverse-action coefficient, and does not say, beyond the extracted comparison, that this equals a coupling in a physical position-space scheme.

Since the representation is not unique (9-06 § 05), the same density $\rho_m$ can have a different $\beta$ in a different representation Inferred . So "the coupling at step $m$" is a number inside the sequence of steps `History` chosen by the construction. The matching condition $\beta(N, p, 0) = \bar{\beta}$ is also a condition matching a coefficient in that representation, not a condition fixing a property of the measure at step 0 Inferred .

Couplings visible from the side of the measure are defined separately, in two ways (10-06) Source .

- $g_W(r)$: starting from the sequence of bare couplings, renormalise down to the step just below length $r$, take the limit, and take the $-1/2$ power of the coefficient of the representation. This too is a coefficient of a representation.
- A second coupling built from continuum correlations: normalise the derivative of the scalar two-point function and the integrated tensor–scalar–scalar correlation to their free-field values, and measure with the derivative of the tensor–scalar correlation (the assessment § 7.4) Assessment . This is written only in terms of limiting correlations. The assessment writes that the construction proves that the ratio of this coupling to $g_W$ tends to 1 Assessment . In Lean this is the field `calibrationX` of the structure `FieldComparison` (`YM/TraceExtension/Endpoint.lean` line 49); which targets' conclusions it remains in is summarised in 10-06 § 05.

05

## Converting conventions

In the assessment's convention, the Wilson measure is $\exp\bigl(-\beta \sum_p (1 - \operatorname{Re}\operatorname{tr} U_p / d_G)\bigr)$, the inner product on the Lie algebra is $\langle X, Y\rangle = -\operatorname{Re}\operatorname{tr}(XY)/d_G$, and the coupling is read as $\beta = 1/g_0^2$ ( § 3, § 4) Assessment . $C_g$ is the Casimir constant of the adjoint representation, which the source fixes by $-\operatorname{tr}(\operatorname{ad} x\ \operatorname{ad} y) = C_g \langle x, y\rangle$ (`YM/Gauge/Casimir.lean` line 140, `Colors.lean` lines 180–182) Source .

For SU(2) ($d_G = 2$), taking the basis $i\sigma_a$, orthonormal for this inner product, gives $C_g = 8$, and $s = 11 \cdot 8/(24\pi^2) = 11/(3\pi^2) \approx 0.3715$ Proved here (numerical check done in floating point). Since the weight $1 - \operatorname{Re}\operatorname{tr} U_p/2$ is the same as this book's $1 - \operatorname{Re} U_p$ (19-01), the $\beta$ of the published construction and the $\beta$ of this book are the same number, as coefficients of the same weight Proved here . The value $11/(3\pi^2)$ that Volume 3 of this book uses as the one-step slope agrees with this $s$ (23-01) Proved here . This book's $\beta$ is $4/g_{\text{std}}^2$ and the published construction's $\beta$ is $1/g^2$; since $g_{\text{std}}^2 = 4g^2$, they are the same number (8-01 § 04) Proved here .

The value of $C_g$ depends on which $U(dG)$ the group is placed in (the representation). Because the convention divides the inner product by $dG$, $C_g = 2N^2$ for the defining representation of $SU(N)$ (8 for $SU(2)$, 18 for $SU(3)$), and placing it by another representation gives another value Proved here . For the full conversion ($g^2_{\mathrm{std}} = 2dG\, g^2$; for $SU(2)$, $\beta$ is the same number in all three conventions), the conversion table in 8-01 § 04 is the reference. `submissions/openai-20261028/checks/ImportedAFGauge.lean` in the repository on the assessment side (`yang-mills-group/YM`) writes $b_0$ in another convention, but we have read only its docstring.

06

## What this book's study had seen

The "matching" in this book's study took the form of matching **a quantity definable from the measure alone** to a fixed value (one of the statements to be shown, 19-07, matching).

- The expectation of the coarse plaquette average of the push-forward measure: the expectation of the average of $\mathrm{Re}$ over coarse plaquettes under the measure pushed forward to step $m_0$ by the deterministic block map. In the side-3 centred design, it is closed in Lean that for each $k$ there is a non-negative coupling making this value equal to a target $\kappa_0 \in [0, 1)$ Lean : `Block3.tuned3_exists` (the assumptions are only the design values and arithmetic; monotonicity, divergence and uniqueness of the coupling are not claimed). For the current design too, continuity of the expectation and existence of a solution between its values are closed (`continuous_Yc`, `exists_tuning`).
- The vortex ratio $\theta_M$: a ratio of twisted partition functions; for every $\kappa_0 \in [0, 1)$ there is a solution of $\theta(\beta) = \kappa_0$ (`tuning_all`, 21-07) Lean .

Both are formalisations of arguments of a known type (intermediate values of continuous functions). We tabulate the differences.

| | Published construction | This book's study |
|---|---|---|
| Quantity matched | The $F^2$ coefficient of the representation at step 0 | Expectation of the coarse plaquette average; the vortex ratio |
| Writable from the measure alone | No (a representation is needed first) | Yes |
| Relation to the $F^2$ coefficient | It is the definition itself | Needs to be shown separately (not shown) |
| Range of the coupling | Within the weak-coupling band at all steps | All of $\beta \geq 0$ (including strong coupling) |

What they share is the idea of fixing the scale by matching a quantity determined by the unit length to a fixed value Inferred . The published construction's rereading tool (log-ratios at two configurations and strong monotonicity, 10-03) could serve as a model for linking "the value of an observable" and "the $F^2$ coefficient" step by step Inferred . The comparison is summarised in 14-03.

07

## What we have not read

- The source gives no concrete example of non-uniqueness of the representation (two representations of the same density), and this book has not built one either. The shifting example in § 02 is an inference.
- How the limiting correlations change when $\bar{\beta}$ is changed (whether they map into each other by a change of length). The shooting toward nearby targets in the source's `YM/Extension/UV/` lies outside the dependencies (outside), and we have not read what it means.
- The content of `submissions/openai-20261028/checks/CommonScaleAudit.lean` (`calibrated_reset`, line 67) on the assessment side (only the form of the statement).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The coupling at step $m$ is the field `beta` of the representation (the coefficient of the main action), and the bare coupling is its value at step $N$ | A reading as a coupling determined by the measure alone (denied by the source) |
| Source The two normalisations, of the kinetic term and of the one-loop part, stop $F^2$ from being shifted | That the representation is unique (not claimed) |
| Source At step 0, $\beta = \bar{\beta}$ for all $N, p$; at all steps it lies in the band $[\hat{\beta}_m/2, 2\hat{\beta}_m]$; it decreases by about $s \log b_m$ per step (fields of the construction theorem's conclusion) | How things depend on $\bar{\beta}$ when it is changed |
| Proved here For SU(2), $C_g = 8$ and $s = 11/(3\pi^2)$; the coefficient of the same weight as this book's $\beta$ | Comparison of values in a different inner-product convention (on the assessment side we read only the docstring) |
| Lean This book's matching quantities (expectation of the coarse plaquette average, the vortex ratio) can be written from the measure alone, and existence of solutions is closed in Lean | Their relation to the $F^2$ coefficient |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Step coupling, sequence of steps, docstrings | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/RG/History.lean` lines 25–33, 67, 79, 204–240 |
| Normalisations | Source | Same repository, `YM/RG/PathActions.lean` lines 161–185 |
| Bare coupling, fields of the construction theorem | Source | Same repository, `YM/Source/Construction.lean` lines 38 and 44–45 |
| Slope and Casimir | Source, Proved here | Same repository, `YM/Gauge/Colors.lean` lines 180–184, `YM/Gauge/Casimir.lean` line 140. SU(2) values by hand computation and floating-point check |
| Docstrings (rereading, leading logarithm) | Source | Same repository, head of `YM/UVNext/COUPLING_RESET/Reset.lean`, head of `YM/Pert/Asymptotics/RawAxis.lean` |
| The assessment's conventions and description of the coupling | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 3, § 4, § 7.4, Appendix A |
| This book's matching quantities | Lean | This book's Lean bundle (`Block3.tuned3_exists`, `continuous_Yc`, `exists_tuning`, `tuning_all`), a bundle that passed the check by the Lean kernel. 19-07, 21-07 |

**Terms used in this chapter** — step coupling, bare coupling, representation, normalisation of the kinetic term, normalisation of the one-loop part, matching, reference value $\bar{\beta}$, band, running coupling, Casimir constant. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-02 The one-step map and the one-loop computation — we read where the main part $s \log b$ of how much the coupling decreases in one step is computed as a shell integral.

# The one-step map and the one-loop computation — the shell integral and the coefficient 11C_g/(24π²)

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — the condition $|\alpha + s \log b_m| + 1 \le \mathrm{drift}_{m+1}$ on the one-step change of the coupling $\alpha(N, m)$; the computation (with formulas) of the shell integral $(2\pi)^{-4} \int_{\mu < |k| \le 1}$ that produces its main part; the estimate bounding the difference between the continuum shell integral and the lattice Brillouin-zone integral by a constant independent of $L$; and an entry point to why the two-loop coefficient is not needed.

**Prerequisite chapters** — 10-01 The coupling at step m, 4-06 Asymptotic freedom. Four-dimensional polar coordinates and averages over the sphere are in 1-11.

**Said first** — we read the statement of the shell integral and the statement of the difference from the lattice, but have not traced the proof linking them to the one-step change $\alpha$ (the link between `alphaPay` and `Kin.alpha`). The whole of the "one-loop" part of the one-step map (the Gaussian integral around the stationary point) is read in 11-09.

The order of this chapter

1. **The statement — the condition on the one-step change**: `Kin.Planned.kinetic` and the one-step map of the coupling
2. **The one-loop effective action and second derivatives**: `FiniteBound`
3. **The shell integral**: the main part of the integrand and the angular average, `poleSpin`
4. **The difference from the lattice**: `absoluteGeode`, splitting into inner ball, shell and outside; the side-independent constant `HorizCalc`
5. **Two loops are not used**: the one-step error stays a constant
6. **What this book's study had seen**: the one-step coefficient for side 2, $11 \ln 2/(3\pi^2)$

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — the condition on the one-step change

In one step (from step $m+1$ to step $m$), the coupling changes in the following form Source .

$$\beta(N,p,m) = \beta(N,p,m+1) + \alpha(N,m) + d, \qquad |d| \le 1. \tag{1}$$

$\alpha(N,m)$ is the **one-loop change**, `Kin.alpha` in the source (`YM/PCalc/Loop.lean` line 75). It does not take the volume index $p$ as an argument Source . $d$ is the correction made when the representation is reread, read in 10-03. $|d| \le 1$ is the field `δ` of the one-step structure `Directed.Move` (`YM/PCalc/Band.lean` line 55: $|u.\beta - v.\beta - \alpha| \le 1$), supplied by `CouplingReset.advance` from $|d| \le$ width and width $\le 1$ (`ResetPrices.unit`, `YM/UVNext/COUPLING_RESET/Prices.lean` line 43) Source .

The one-loop change is subject to the condition `Kin.Planned.kinetic` (same file, line 110) Source .

$$\left|\alpha(N,m) + s\log b_m\right| + 1 \;\le\; \mathrm{drift}_{m+1}. \tag{2}$$

$s = 11C_g/(24\pi^2)$, $b_m$ is the side at step $m$, and $\mathrm{drift}_{m+1}$ is a step constant. Combining (1) and (2) gives $|\beta(N,p,m+1) - \beta(N,p,m) - s\log b_m| \le \mathrm{drift}_{m+1}$, which is the second formula of the one-step condition of the sequence of steps, `History.Transfer` (`YM/RG/History.lean`, lines 226–227) Proved here . The docstring describes drift as "the

per-step size of the remainder from the universal one-loop main part, not an identification of a particular one-loop computation" (lines 220–222) Source .

That $\alpha$ does not depend on the volume is one of the reasons the running estimate is uniform in $p$ (10-06) Inferred .

02

## The one-loop effective action and second derivatives

The meaning of the one-loop change can be read from the condition `FiniteBound` (`Loop.lean`, lines 62–71) Source . Fix the coarse field $u$ and do the Gaussian integral over the fine field around the stationary point (the stationary point of the fine action plus the penalty term, 9-04 § 05); this gives in the effective action of the coarse field

$$\Gamma^{1\mathrm{loop}}(u) = \text{one-loop atoms}(u) + \tfrac{1}{2}\log\det \mathrm{Hess}(u) - \log(\text{Jacobian of the coordinates}) \quad (3)$$

(`thermal`, lines 46–49). The field `wave_price` of `FiniteBound` requires that the second derivative of $\Gamma^{1\mathrm{loop}}$ taken in two plane-wave directions agree with the second derivative of $\alpha \times (\text{quadratic part of the kinetic term}) + (\text{one-loop atoms})$ (`draft`, line 51) to within (number of points) $\times$ constant $\times\, e^{-\gamma\cdot(\mathrm{side})}$ Source . That is, $\alpha$ is the change in the coefficient of the kinetic term of the one-loop effective action Inferred .

Since the second derivative is taken with plane waves, $\alpha$ can be written as an integral over momentum space. In units of the fine lattice, taking the second-order coefficient in the external momentum, it becomes an integral over the Brillouin zone with an infrared cutoff $\mu = 1/L$ that cuts off wavelengths longer than the coarse scale Inferred . The source calls this coefficient `alphaPay` (`YM/Next/STATIONARY/Films/LeadingRay.lean` line 699) Source .

03

## The shell integral

The main part of the integrand, homogeneous of degree $-4$ in the momentum $k \in \mathbb{R}^4$, is (`poleVec`, `YM/Next/STATIONARY/OneLoop/Bubbles.lean` line 125) Source

$$\Pi(k) = 4C_g\Big(\frac{k_1^2}{|k|^6} - \frac{4k_1^2k_0^2}{|k|^8} - \frac{1}{|k|^4}\Big). \quad (4)$$

$k_0$ is the direction of the external momentum (the direction in which the second derivative is taken), and $k_1$ is the component corresponding to the colour/polarisation direction of the field Inferred . `poleSpin` (`YM/Next/STATIONARY/OneLoop/MomentumIntegral.lean` line 326) says that for $0 < \mu \le 1$

$$\frac{1}{(2\pi)^4}\int_{\mu<|k|\le 1}\Pi(k)\,d^4k=-\,s\,\log\frac{1}{\mu},\qquad s=\frac{11C_g}{24\pi^2}\quad(5)$$

Source (the factor $(2\pi)^{-4}$ is `cartWeight`, line 28).

We check this by hand Proved here . Writing $\langle\cdot\rangle$ for the average over the unit sphere in four dimensions, symmetry gives

$$\langle\hat k_1^2\rangle=\frac14,\qquad\langle\hat k_1^2\hat k_0^2\rangle=\frac{1}{d(d+2)}=\frac{1}{24}\quad(d=4).\quad(6)$$

So the average of $\Pi$ over the sphere $|k|=r$ is $\frac{4C_g}{r^4}\left(\frac14-\frac{4}{24}-1\right)=-\frac{11C_g}{3r^4}$. The volume element of a four-dimensional spherical shell is $2\pi^2r^3\,dr$, so

$$\int_{\mu<|k|\le1}\Pi\,d^4k=-\frac{11C_g}{3}\cdot2\pi^2\int_\mu^1\frac{dr}{r}=-\frac{22\pi^2C_g}{3}\log\frac{1}{\mu},\qquad\frac{1}{16\pi^4}\cdot\left(-\frac{22\pi^2C_g}{3}\right)=-\frac{11C_g}{24\pi^2}=-s.$$

This book also checked (6) and (5) by averaging over 400,000 uniform points on the unit sphere ($\langle\hat k_1^2\rangle\approx0.2503$, $\langle\hat k_1^2\hat k_0^2\rangle\approx0.0417$; at $\mu=0.05$ the difference in the integral is of order $10^{-4}$; $C_g=8$ for SU(2)) Proved here .

Setting $\mu=1/L$, the shell integral is $-s\log L$. In one step with side $L$, the main part decreases the coupling by $s\log L$. This is the $-s\log b_m$ in (2) Inferred . Of the three terms, $-1/|k|^4$ is the largest (coefficient $-1$), and the first two ($+1/4$ and $-1/6$) are corrections Proved here . That the integral of $1/|k|^4$ in four dimensions is logarithmic is the computational form of the coupling being dimensionless and running logarithmically Inferred .

04

## The difference from the lattice

The actual $\alpha$ is an integral over the lattice Brillouin zone, and its integrand is not the main part $\Pi$ itself. `absoluteGeode` (`MomentumIntegral.lean` line 346) splits the zone into three and bounds the difference Source .

- Inner ball $|k|\le\mu$: by the size assumption on the integrand $g$, `Waves1`: $|g(k)|\le A/(|k|+\mu)^4$ (`Bubbles.lean` line 276), the product with the volume $\sim\mu^4$ is $O(A)$.
- Shell $\mu<|k|\le1$: by the assumption on the difference from the main part, `Waves2`: $|g(k)-\Pi(k)|\le D/|k|^3+D\mu^2/|k|^6$ (line 277), $\int_\mu^1(r^{-3}+\mu^2r^{-6})r^3dr=O(1)$.
- Outside (the rest of the Brillouin zone): $O(A)$ by `Waves1`.

The conclusion is $|\alpha_{\mathrm{pay}} + s \log \mu^{-1}| \le (2\pi)^{-4}\, c(A, D)$, whose right side does not depend on $\mu$ Source . `HorizCalc` (`YM/Next/STATIONARY/Level/Transition.lean` line 20) is a statement restating this in terms of the side: there are $L_{\min}$ and $D \ge 0$ such that $|\alpha_{\mathrm{pay}} + s \log L| \le D$ for every odd $L$ at least $L_{\min}$ Source (the statement further contains "for all" over sequences of sides, path data, floor conditions and so on, and "there exists" for the quantities estimated; we have omitted these here). And `pouring` (`YM/Next/STATIONARY/FullPlanned.lean` line 17) says that if the gauge group is simple, the bundle of estimates of this form, `Fluent G`, holds Source .

What matters is that the difference constant $D$ **does not depend on the side** $L$ Source . The one-step error is $O(1)$, small compared with $\log L$.

05

## Two loops are not used

In standard perturbation theory the running of the coupling is $\mu\, dg/d\mu = -b_0 g^3 - b_1 g^5 - \cdots$ (4-06), and the $O(1)$ part of the one-step change contains two-loop and higher contributions and finite lattice effects Cited . The published construction does not compute this $O(1)$ Source . (2) only places $\alpha$ within the constant drift of the main part $-s \log b$; it does not estimate the content of drift.

If the side were fixed, the error over $n$ steps would be $n \cdot O(1)$, of the same size as the main part $s\, n \log b$, and the slope $s$ would not be determined. The published construction lets the side grow with the step, making $\mathrm{drift}/\log b_m \to 0$, so that the sum of errors is $o(\cdot)$ of the main part (10-05) Source . It is a design that determines the one-loop slope without using the two-loop coefficient. In exchange, the sequence of spacings becomes a special one (the assessment § 4.3: there is no universality with respect to the type of the sequence of spacings) Assessment .

The running theorem too states only the main part. The two-loop coefficient, $\log \log$ corrections, and boundedness of $\beta - (\bar\beta + s \log D)$ are not stated (10-06) Source .

06

## What this book's study had seen

Part 15 and Volume 3 of this book took the main part of the decrease of the coupling in one step with side 2 to be $\frac{11}{3\pi^2} \ln 2$ (SU(2)) (15-01, 23-01). Numerically, $11 \ln 2/(3\pi^2) \approx 0.2575$.

With conventions aligned, the slope $s$ of the published construction is $11/(3\pi^2)$ for SU(2) (10-01 § 05; the conversion is in 8-01 § 04), and the amount for one step with side 2, $s \ln 2$, agrees with this book's value Proved here . The premise is that this book's $\beta$ is the coefficient of $\sum_p (1 - \operatorname{Re} U_p)$ (19-01). The agreement is a hand computation of this book's study, not a Lean statement.

- The published construction uses only odd sides of at least 3, so side 2 is not treated Source .
- Only the coefficient of $\log b$ agrees; the $O(1)$ constant of one step cannot be compared Inferred .
- The ultraviolet attempts of this book's study (Part 15) fixed the side at 2 and looked for a form paying the net bracket of one step (main part $\pm$ what one step misses) uniformly over steps. They did not take the route of growing the side and removing the $O(1)$ by dividing it by $\log b$ (14-05).

07

## What we have not read

- The proof linking `alphaPay` and `Kin.alpha` (beyond `YM/Next/STATIONARY/FullPlanned.lean`, from `under_contract` on).
- The exact definition of `alphaPay` (the content of `leapTip` and `profitable`), and the meaning of $k_0, k_1$ in (4) (the reading in § 03 is an inference).
- The supply of the constants in the assumptions `Waves1` and `Waves2` (`astronomy`, `saunter`, `vintageWilson`).
- Whether these declarations lie on the existence route of the public theorem. According to the record, `poleSpin`, `absoluteGeode`, `HorizCalc`, `pouring` and `Kin.Planned.kinetic` are used (witness `SharpGram.habitat`); we have not traced the chain of the existence route.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The one-step change is $\beta_m = \beta_{m+1} + \alpha + d$, $\lvert d\rvert \le 1$, $\lvert\alpha + s \log b_m\rvert + 1 \le \mathrm{drift}_{m+1}$, and $\alpha$ does not depend on the volume | The content of drift (by design, it is not estimated) |
| Source $(2\pi)^{-4} \int_{\mu<\lvert k\rvert\le 1} \Pi = -s \log(1/\mu)$ (`poleSpin`) | The two-loop coefficient |

| Can say | Cannot say |
|---|---|
| Proved here The angular averages $1/4$, $1/24$ and the shell volume $2\pi^2 r^3\,dr$ give the coefficient $11C_g/(24\pi^2)$ (also checked numerically) | — |
| Source The difference between the lattice integral and the main part is bounded by a constant independent of the side (`absoluteGeode`, `pouring`) | The link between `alphaPay` and `Kin.alpha` (not traced) |
| Proved here For SU(2), the amount for one step with side 2, $s \ln 2 = 11 \ln 2/(3\pi^2)$, agrees with this book's value (hand computation with aligned conventions) | A comparison of the $O(1)$ constants of one step |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Condition on the one-step change, effective action | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/PCalc/Loop.lean` lines 46–51 (`thermal`, `draft`), lines 62–71 `FiniteBound`, line 75 `Kin.alpha`, line 110 `kinetic` |
| Main part and shell integral | Source | Same repository, `YM/Next/STATIONARY/OneLoop/Bubbles.lean` lines 125, 276, 277, `MomentumIntegral.lean` lines 28, 174, 326, 346 |
| Side-independent constant | Source | Same repository, `YM/Next/STATIONARY/Level/Transition.lean` line 20 `HorizCalc`, `FullPlanned.lean` line 17 `pouring`, `Films/LeadingRay.lean` line 699 `alphaPay` |
| One-step condition of the sequence of steps | Source | Same repository, `YM/RG/History.lean` lines 220–227 |
| (6)(7) and the numerical check | Proved here | Hand computation in the text; averages over 400,000 uniform points on the sphere (record of floating-point computation) |
| Absence of universality | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4.3 |
| This book's one-step value for side 2 | Proved here | 15-01, 23-01 |

**Terms used in this chapter** — one-loop change, drift, one-loop effective action, shell integral, angular average, Brillouin zone, infrared cutoff, two loops, slope $2b_0$. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-03 Rereading — we read the procedure that, after one step, rereads the $\beta$ of the representation from the log-ratio of the density at two configurations.

# Rereading — reading the coupling from the log-ratio of the density at two configurations

> This chapter is existing knowledge (a reading of the published construction). § 06 is this book's study, and § 07 is a map of what is unresolved.

**What this chapter shows** — how the correction $d$ in the one-step change $\beta_m = \beta_{m+1} + \alpha + d$ ((1) in 10-02) is determined: a single equation matching the log-ratio of the values of the push-forward density $H$ at the flat configuration $\mathbf{1}$ and at a test configuration $V$ to the formula of the representation; and, from the fact that its left side increases in $d$ with slope at least $K(V)/2$ (strong monotonicity), that the root is unique, depends continuously on the input, and satisfies $|d| \le 1$.

**Prerequisite chapters** — 10-01 The coupling at step m, 10-02 The one-step map and the one-loop computation, 9-06 The density representation.

**Said first** — rereading determines only two scalars ($\beta$ and the vacuum term $v$). That the representation equals the density **at every configuration** comes from a separate remainder estimate (`Remainder`). The content of that estimate (the expansion of the small-field step) is within the scope of 11-09; in this chapter we read only the form of the statement. We have not read the concrete form of the test configuration $V$.

The order of this chapter

1. **The statement — rereading two scalars**: `Template`, `residual`, `correct`, `advance`
2. **Two configurations**: $v$ at the flat configuration, $\beta$ at the test configuration
3. **Strong monotonicity and a unique root**: `Response` and `solve`
4. **Bound on the shift and continuity**: `posterior`, `balance`, $|d| \le 1$
5. **From equality at two points to equality everywhere**: `ShapeControl` and `realizes`
6. **What this book's study had seen**: a tool linking the values of observables and the $F^2$ coefficient

7. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment .

01

## The statement — rereading two scalars

After one step, the density at step $m$ is the function pushed forward by the kernel, $H = \mathcal{T}_{\beta_{m+1}\rho_{m+1}}$ (9-01). When rewriting it in the representation at step $m$ (9-06), the source first fixes the rest of the representation (path data, truncation parameters, budget point, family of activities), and rereads the two scalars $\beta$ and $v$ last (`YM/UVNext/COUPLING_RESET/Reset.lean`, `Advance.lean`) Source .

The exponential part of the representation is written split into a form linear in $\beta$ (`Template`, `Reset.lean` lines 43–47 and 60–62) Source .

$$-\log\frac{\rho(U)}{e^{v|\Lambda|}} = \beta\,K(U) + B(U) + A_d(U) \qquad (\text{flag-free } U). \tag{1}$$

$K$ is the part of the main action multiplied by $\beta$ (`leg`; `kinetic` in the source), $B$ is the main action at $\beta = 0$, that is, the one-loop atoms (`background`), and $A_d$ is the sum of regular activities (`add`). The activities are allowed to depend on the correction $d$ (`terms : ℝ → ℝ → Activities`). With no flags the gas is 1 (9-06).

The rereading equation is `residual` in the source (lines 64–65) Source :

$$\Phi_x(d) = \big(\beta_x^{\mathrm{pred}} + d\big)K(V) + B(V) + A_{x,d}(V) - \big[\log H_x(\mathbf{1}) - \log H_x(V)\big] = 0. \tag{2}$$

$x$ is the parameter of the curve (the shooting bare coupling, 10-04), $\beta_x^{\mathrm{pred}} = \beta_{m+1}(x) + \alpha(N,m)$ is the one-loop prediction (`predictor`, `Advance.lean` line 51), and the bracket

is the log-ratio of the density at the two configurations (`contrast`, `Reset.lean` line 50).

The theorem `CouplingReset.correct` (`Advance.lean` line 157) says that, under its assumptions (the condition on the test configuration `Usable`, the funding of the input `Income`, and that the input is a curve at step $m+1$), the root $d(x)$ of (2) exists, is continuous in $x$, and the reread representation equals $H_x$ at the two points $\mathbf{1}$ and $V$ Source . `advance` (line 175) says further that, under the remainder assumption `Remainder`, the reread representation satisfies the one-step identity (equality at every configuration), and $|d(x)| \le \text{width} \le 1$ Source .

02

## Two configurations

Rereading uses only two configurations Source .

- **The flat configuration $\mathbf{1}$** (every link the identity): every plaquette value is 1, the main action and the one-loop atoms are 0, the regular activities are 0 by the condition $E_i(\mathbf{1}) = 0$, there are no flags, and the gas is 1. The value of the representation is just $e^{v|\Lambda|}$, so $v = \log H(\mathbf{1})/|\Lambda|$ can be read off (`scalar`, `scalar_eq`, lines 51 and 85).
- **The test configuration $V$**: a configuration with no flags in which the $\beta$ part of the main action is large compared with the number of points. The condition `Usable` (`Advance.lean`, lines 55–58) requires $\text{flags}_{t_m}(V) = \varnothing$ and $K(V) \ge |\Lambda|\,\eta\,t_m^2$ ($\eta$ a step constant, $t_m$ the threshold of 9-07). That such a $V$ exists is supplied by `usable` (line 60).

The two equalities matching the values of the representation at the two configurations to $H$ have, for the two unknowns $v, \beta$, a separated form: $v$ is determined at the flat configuration, and $\beta$ by the log-ratio (2) (`contrast_eq`, line 88) Source . Since a log-ratio is used, multiplying the density by a positive constant only moves $v$ and leaves $\beta$ unchanged (docstring, `rescale_*`) Source .

Since only flag-free small-field configurations are read, $\beta$ is read as "the coefficient of the kinetic term of the effective action at small field" Inferred . The docstring says that agreement at other configurations requires, besides the separation, the price of a remainder estimate (head of `Reset.lean`) Source .

03

## Strong monotonicity and a unique root

The source's `Response` (`Root.lean`, lines 27–31) is a structure bundling the following four properties of a two-variable function $\Phi(x, d)$ Source .

$$\mu > 0, \qquad a \le b \;\Rightarrow\; \Phi(x,b) - \Phi(x,a) \ge \mu\,(b-a), \qquad d \mapsto \Phi(x,d) \text{ continuous}, \qquad x \mapsto \Phi(x$$

The second is **strong monotonicity** (slope at least $\mu$). `correct` builds a `Response` for (2) with $\mu = K(V)/2$ Source . The slope of (2) in $d$ is the sum of $K(V)$ and the dependence of the activities on $d$, $\partial_d A_{x,d}(V)$. The source's `response` (`Reset.lean` line 131) is a sufficient condition of the form: if the change in the activities is at least $-(K(V) - \mu)(b - a)$, the slope is at least $\mu$; its price is paid as a field of `Income` Source .

By strong monotonicity there is at most one root. Existence comes from the intermediate value theorem: `solve` (`Root.lean` line 60) says that if at each $x$ there is a bracket with $\Phi(x, \mathrm{lo}) \le 0 \le \Phi(x, \mathrm{hi})$, then a root $d(x) \in [\mathrm{lo}, \mathrm{hi}]$ exists and is continuous in $x$ Source .

Note that strong monotonicity is a property in the **correction** $d$. The docstring (above `response`, lines 127–130) says that since strong monotonicity includes the contributions of paths already fixed, **monotonicity in the incoming bare coupling is not required** Source . On the shooting side (10-04) too, monotonicity of the map from the bare coupling to the coupling at step 0 is not used.

04

## Bound on the shift and continuity

Strong monotonicity bounds the shift of the root by the value of the equation Source (`Response.inverse`, `posterior`, `Root.lean` lines 42 and 49).

$$|b - a| \le \frac{|\Phi(x,b) - \Phi(x,a)|}{\mu}, \qquad |d(x) - a| \le \frac{|\Phi(x,a)|}{\mu}. \quad (4)$$

Setting $a = 0$, the size of the correction is bounded by $|\Phi(x, 0)|$, how far the equation is off at the one-loop prediction itself Proved here . `balance` (line 72) says that if $|\Phi(x,0)| \le \mu r$, then $\pm r$ is a bracket and the root lies in $[-r, r]$ Source . `correct` uses this with the width $r$ (`width`; `ResetPrices.radius` in the source's price table), giving the conclusion of `advance`, $|d| \le$ width, and further width $\le 1$ (`q.unit`) Source .

From the first part of (4), the root depends continuously on the input (`Response.continuous`, `Root.lean` line 54) Source . If $x \mapsto \Phi(x, a)$ is continuous, so is $x \mapsto d(x)$. This is one of the components that make the couplings at each step a

continuous curve (`Curve`) when the bare coupling is varied in the shooting method (10-04) Inferred .

| Tool | Property used | What comes out |
|---|---|---|
| Intermediate value theorem (`solve`) | Bracket and continuity | Existence of the root |
| Strong monotonicity | Slope $\geq \mu$ | Uniqueness of the root |
| `posterior` | Strong monotonicity | Bound on the correction $\lvert\Phi(x, 0)\rvert/\mu$ |
| `Response.continuous` | Strong monotonicity + continuity in $x$ | Continuity of the root |

05

## From equality at two points to equality everywhere

Matching at two configurations alone does not show that the representation equals the density at every configuration. The source fills this gap with a separate assumption Source .

`ShapeControl` (`Reset.lean`, lines 115–117) is the assumption that, for each $x$ and each configuration $U$, the difference between the shape part of the representation (without $e^{v|\Lambda|}$) and $H_x(U)/H_x(\mathbf{1})$ is bounded by a constant times the value $|\Phi_x(a)|$ of (2). `realizes` (line 119) concludes that, since the right side is 0 at the root $d(x)$, the representation **equals $H_x$ at every configuration** Source .

That is, the content of the one-step identity (an equality with no approximation, 9-01 § 04) is divided as: (i) showing separately, by the expansion of the small-field step, that "the shape of the representation agrees except for the scalars" (the field `shape` of `Remainder`), and (ii) matching the two remaining scalars by rereading Inferred . The expansion of the small-field step consists of three identities that carry both the error of the Gaussian approximation and the non-Gaussian remainder inside the integrand, and yields `Move.ident` (11-09) Source .

Which targets it belongs to: according to the record, `correct`, `advance`, `residual` and `solve` are used (witness `SharpGram.habitat`), and the direct user of `advance` is `AnalyticBridge.VerifiedReset.Return.reset` Source . This book's study traced the source and confirmed through the chain that `reset` lies on the existence route of the public theorem (from `rodB`) Source .

06

## What this book's study had seen

The matching in this book's study (10-01 § 06) viewed the expectation of the coarse plaquette average of the push-forward measure, or the vortex ratio, as a continuous function of the bare coupling $\beta$, and matched it to a target by the intermediate value theorem. What is closed in Lean goes only as far as existence of a solution; monotonicity and uniqueness in $\beta$ are not claimed (`Block3.tuned3_exists`, `tuning_all`) Lean .

The published construction's rereading gets uniqueness in a different place.

- What is varied is not the bare coupling but **the correction $d$ inside the representation**. With the other parts of the representation fixed, the log-ratio equation increases in $d$ with slope at least $K(V)/2$, and the root is unique.
- No monotonicity in the bare coupling is required, and the shooting (10-04) uses only existence.

As a tool for carrying, step by step, the relation "value of an observable $\leftrightarrow$ $F^2$ coefficient" that was missing in this book's study (the table in 10-01 § 06), rereading by a log-ratio at two configurations and strong monotonicity could serve as a model Inferred . However, the published construction's log-ratio is a quantity inside the representation, a different thing from this book's matching quantities (expectations writable from the measure alone). There is no statement linking the two, either on this book's side or on the published construction's side Inferred . The comparison is summarised in 14-03.

07

## What we have not read

- The concrete form of the test configuration $V$ (`YM/UVNext/COUPLING_RESET/Probe.lean`; we read only the docstring), and the content of the proof of `usable`.
- How the funding of the input `Income` and the remainder assumption `Remainder` are supplied on the existence route (on the `AnalyticBridge/VERIFIED_RESET_STEP` side).
- The location and form of the statement of the uniqueness `scalars_unique` mentioned in the docstring (head of `Advance.lean`).
- The value of the width `ResetPrices.radius`, and where among the step constants it comes from.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source $v$ is reread at the flat configuration, and $\beta$ from the log-ratio of the density at the flat configuration and a flag-free test configuration (equation (2)) | The concrete form of the test configuration |
| Source Under the assumptions, (2) increases in $d$ with slope at least $K(V)/2$, and the root is unique and continuous in the input (`correct`, `solve`) | Monotonicity in the bare coupling (not required) |
| Source Under the remainder assumption, the reread representation equals the density at every configuration, and $\lvert d\rvert \le 1$ (`advance`, `realizes`) | Supply of the remainder assumption (within the scope of 11-09) |
| Source That rereading lies on the existence route was confirmed in this book's study by tracing the chain in the source | — |
| Inferred Rereading could serve as a model for a tool linking the values of observables and the $F^2$ coefficient step by step | A statement linking this book's matching quantities and the coefficients of the representation (on neither side) |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Shape, equation, equality at two points, equality everywhere | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/UVNext/COUPLING_RESET/Reset.lean`: `Template` line 43, `contrast` line 50, `scalar` line 51, `leg`, `background`, `add` lines 60–62, `residual` line 64, `scalar_eq` line 85, `contrast_eq` line 88, `passes` line 99, `ShapeControl` line 115, `realizes` line 119, `response` line 131 |
| Strong monotonicity and roots | Source | Same repository, `Root.lean`: `Response` line 27, `inverse` line 42, `posterior` line 49, `continuous` line 54, `solve` line 60, `balance` line 72, `balance_solve` line 80 |
| One-step rereading | Source | Same repository, `Advance.lean`: `predictor` line 51, `Usable` line 55, `usable` line 60, `Income` line 78, `Remainder` line 97, `correct` line 157, `advance` line 175 |
| Dependency record | Source | Same repository, `verification/declaration-usage.tsv` |
| Existence of this book's matching | Lean | This book's Lean bundle (`Block3.tuned3_exists`, `tuning_all`). 21-07 |

**Terms used in this chapter** — rereading, log-ratio, flat configuration, test configuration, strong monotonicity, root, bracket, bound on the shift, remainder estimate, one-step identity. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-04 The shooting method — we read how the bare coupling that makes the coupling at step 0 equal to $\bar{\beta}$ is chosen, by nested intervals of the per-step

bands and the intermediate value theorem.

# The shooting method — choosing the bare coupling by nested intervals and the intermediate value theorem

> This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study, and § 08 is a map of what is unresolved.

**What this chapter shows** — the procedure that, for each cutoff $N$ and volume $p$, views the couplings at each step as a continuous curve $x \mapsto \beta_m(x)$ parametrised by the bare coupling $x$ at step $N$, builds nested intervals from the per-step bands, and at step 0 uses the intermediate value theorem to take $x$ with $\beta_0(x) = \bar{\beta}$; that it uses only the continuity of the curves and the one-step error inequality $\mathrm{drift} \le (s/4)\log b$; and where uniqueness is stated and where it is not.

**Prerequisite chapters** — 10-01 The coupling at step m, 10-03 Rereading. The intermediate value theorem is within the prerequisites of Part 1.

**Said first** — the source has two kinds of shooting band. The one on the existence route of the public theorem is the band in `YM/FullVolume/Arc.lean`; the band in `YM/Shoot/` (which collapses to a point at step 0) belongs to another headline theorem. This chapter mainly describes the former, and describes the latter in § 05 to distinguish them. Shooting gives existence only; uniqueness of the bare coupling is not stated on the shooting route.

The order of this chapter

1. **The statement — a bare coupling making the coupling at step 0 equal to $\bar{\beta}$ exists**: `Arc.bought`, `covers`, `rodB`
2. **Bands, and the inequality for crossing a band in one step**: `low`, `high`, `crosses`
3. **Nested intervals**: `narrow` (the intermediate value theorem twice), and induction from step $N$ to step 0
4. **Taking a point at step 0**: `nearby` and `covers`, choice

5. **Not confusing the two shooting bands**: how it differs from the band in `Shoot/`
6. **On uniqueness**: not stated on the shooting route; a separate field of uniqueness within a fixed recursion
7. **What this book's study had seen**: matching by global endpoints
8. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source , and the level "it is proved" is marked Assessment . Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — a bare coupling making the coupling at step 0 equal to $\bar{\beta}$ exists

The field `reaches` of the structure of the sequence of steps `History.Selected` requires that the coupling at step 0 be $\bar{\beta}$ for all $N, p$ (10-01). The shooting method is what chooses the bare coupling $\beta_{N,p}$ for this. On the existence route of the public theorem, it is done by the following three Source .

1. `FullVolume.Arc.bought` ( `YM/FullVolume/Arc.lean` line 95): under its assumptions (the recursion materials `Issues` , a simple group, the one-step error condition `Drifting` ), there is an "arc" `Arc` for each $m \le N$ Source .
2. `Arc.covers` (line 71): for the arc at step 0, the interval $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ is contained in the image of "bare coupling $x \mapsto$ coupling at step 0" Source .
3. `FullVolume.rodB` ( `YM/FullVolume/Stage/Rods.lean` line 26): using `bought` and `covers` , and the fact that $\bar{\beta}$ lies in the interior of the interval ( `centers` , `Arc.lean`

line 25), it **chooses** (`choose`) for each $N, p$ one bare coupling that makes the coupling at step 0 equal to $\bar{\beta}$, and assembles the sequence of steps `Selected` Source .

`rodB` is one of the declarations that this book's study confirmed, by tracing the chain, to lie on the existence route of the public theorem, and according to the record it is the direct user of `Arc.bought` and `covers` Source . `Issues` does not remain an assumption: on the existence route it is supplied by `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean` line 76) Source .

02

## Bands, and the inequality for crossing a band in one step

The band at step $m$ is set as follows (`low`, `high`, `Arc.lean` lines 21–22) Source .

$$\mathrm{low}_m = \tfrac{3}{4}\bar{\beta} + \tfrac{5}{8}\cdot s\log D_m, \qquad \mathrm{high}_m = \tfrac{5}{4}\bar{\beta} + \tfrac{3}{2}\cdot s\log D_m. \quad (1)$$

At step 0, $D_0 = 1$, so the band is $[\tfrac{3}{4}\bar{\beta}, \tfrac{5}{4}\bar{\beta}]$ and does not collapse to a point. At every step the band lies within the band of the representation $[\hat{\beta}_m/2, 2\hat{\beta}_m]$ ($\hat{\beta}_m = \bar{\beta} + s\log D_m$) (`fits`, line 31) Source . Comparing coefficients: $\tfrac{3}{4} \ge \tfrac{1}{2}$, $\tfrac{5}{8} \ge \tfrac{1}{2}$, $\tfrac{5}{4} \le 2$, $\tfrac{3}{2} \le 2$ Proved here .

The one-step error condition `Drifting` (`YM/Shoot/Reach.lean`, lines 70–71) is $\mathrm{drift}_{i+1}/\log b_i \le s/4$ Source . Combined with the one-step relation $|\beta_{i+1} - \beta_i - s\log b_i| \le \mathrm{drift}_{i+1}$ (10-02), it gives the following (`crosses`, `Arc.lean` line 75) Source .

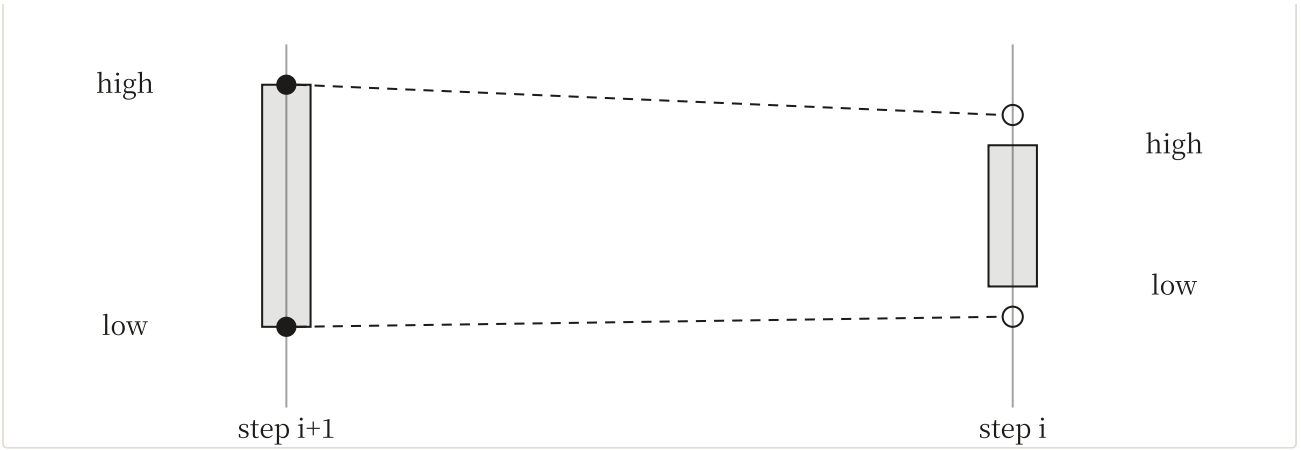
$$\beta_{i+1} = \mathrm{low}_{i+1} \;\Rightarrow\; \beta_i < \mathrm{low}_i, \qquad \beta_{i+1} = \mathrm{high}_{i+1} \;\Rightarrow\; \beta_i > \mathrm{high}_i. \quad (2)$$

We check it Proved here . Writing $\ell = \log b_i$, $\mathrm{low}_{i+1} = \mathrm{low}_i + \tfrac{5}{8}s\ell$ and $\mathrm{high}_{i+1} = \mathrm{high}_i + \tfrac{3}{2}s\ell$. In one step $\beta$ decreases by $s\ell \pm \tfrac{1}{4}s\ell$, so

$$\beta_i \le \mathrm{low}_i + \tfrac{5}{8}s\ell - s\ell + \tfrac{1}{4}s\ell = \mathrm{low}_i - \tfrac{1}{8}s\ell, \qquad \beta_i \ge \mathrm{high}_i + \tfrac{3}{2}s\ell - s\ell - \tfrac{1}{4}s\ell = \mathrm{high}_i + \tfrac{1}{4}s\ell. \quad (3)$$

The floor of the band drops by only $\tfrac{5}{8}s\ell$ in one step, while the coupling drops by at least $\tfrac{3}{4}s\ell$. The ceiling of the band drops by $\tfrac{3}{2}s\ell$, while the coupling drops by at most $\tfrac{5}{4}s\ell$. So a trajectory at the edge of the band at a fine step leaves the band at the step one coarser.

Figure 1 The band and trajectories in one step (step $i+1 \to$ step $i$). Trajectories at the floor and ceiling of the left step (●) leave the band at the right step (○). By the intermediate value theorem, the values of the trajectories in between cover the band.




The figure redraws the vertical positions step by step so that the shape of the band is visible; the scale is not exact.

03

## Nested intervals

`Arc` (`Arc.lean`, lines 58–67) is, for step $m$, the following tuple Source : an interval $[l, u]$, and for each $x \in [l, u]$ a trajectory from step $N$ to step 0, $x \mapsto$ (representation at each step). The conditions are: the coupling at step $N$ is $x$ ($x$ is the bare coupling); at each step from $m$ to $N$, $x \mapsto \beta_i(x)$ is a continuous curve (`Curve`); at each step it is in the band and the one-step relation holds; and at step $m$, $\beta_m(l) = \mathrm{low}_m$ and $\beta_m(u) = \mathrm{high}_m$.

`bought` builds arcs by induction downward from step $N$ Source .

1. **Step $N$**: using the continuous curve of `Issues.initial` (`YM/Shoot/Criteria.lean`, lines 100–101) (the representation parametrised by the bare coupling), take the interval $[\mathrm{low}_N, \mathrm{high}_N]$. The coupling at step $N$ is $x$ itself.
2. **From step $k+1$ to step $k$**: `Issues.advances` (lines 102–106) builds a continuous curve at step $k$ from the continuous curve at step $k+1$ (one renormalisation step and rereading, 10-03). By (2), the trajectories at the ends of the interval leave the band at step $k$. `narrow` (`Reach.lean` line 18) then uses the intermediate value theorem twice to choose a subinterval $[l', u'] \subset [l, u]$ with $\beta_k(l') = \mathrm{low}_k$, $\beta_k(u') = \mathrm{high}_k$, and $\beta_k$ taking values in the band $[\mathrm{low}_k, \mathrm{high}_k]$ on $[l', u']$.

`narrow` chooses by taking the infimum $w$ of the set of points where $\beta_k = \mathrm{high}_k$, and then the supremum $z$ of the set of points in $[l, w]$ where $\beta_k = \mathrm{low}_k$ (`sInf`, `sSup`) Source . Then in the interior of $[z, w]$, $\beta_k$ stays between the two end values Proved here .

The intervals narrow in a nested way as one goes down the steps, and only trajectories inside the band at every step remain. What is used is **only the continuity of the**

**curves and the one-step error inequality** Source . No estimate on derivatives of the one-step map is used. The docstring (head of `Arc.lean`) describes it as "shooting before the one-step derivative estimates" Source .

04

## Taking a point at step 0

For the arc at step 0, $\beta_0(l) = \mathrm{low}_0 = \frac{3}{4}\bar{\beta}$ and $\beta_0(u) = \mathrm{high}_0 = \frac{5}{4}\bar{\beta}$. Applying the intermediate value theorem to the continuous curve $x \mapsto \beta_0(x)$, every value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ is in the image (`covers`) Source . Since $\bar{\beta} > 0$, $\bar{\beta}$ is in the interior of this interval (`centers`) Source . So there is a bare coupling $x$ with $\beta_0(x) = \bar{\beta}$, and `rodB` takes one by the axiom of choice Source .

From the trajectory chosen in this way, all the fields of the sequence of steps `Selected` (`starts`, `steps`, `represents`, `reaches`, `running`) are assembled (`Rods.lean`, lines 39–46). `running` comes from the sum of the one-step errors (`Shoot.trajectory`, 10-05) Source .

Since the band is not collapsed at step 0 and a width around $\bar{\beta}$ is kept, shooting can be done with the same calendar even if the target is replaced by another value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ Inferred . The docstring says "whether one can shoot at a slightly moved target is not a question of prescribing another calendar" (head of `Arc.lean`) Source . However, the arc states only a **forward curve**; it does not say that the inverse map (value at step 0 → bare coupling) is a function (same place) Source .

05

## Not confusing the two shooting bands

The source contains another shooting, `YM/Shoot/Reach.lean` Source . Its band has floor $\bar{\beta} + \frac{1}{2} \cdot s \log D_m$ and ceiling $\bar{\beta} + 2 \cdot s \log D_m$ (`floorBeta`, `ceilBeta`), and collapses to the single point $\bar{\beta}$ at step 0. `gates` (line 81) plays the same role as (2), and `steps` (line 112) and `successful` (line 156) take the left end of the interval. `Recipe.races` (`YM/Shoot/Race.lean` line 67) assembles `Selected` from this. The docstring says "it is not claimed that `Issues` exists for every family" (`Race.lean` line 5).

| | `FullVolume/Arc.lean` | `Shoot/Reach.lean` |
|---|---|---|
| Floor | $\frac{3}{4}\bar{\beta} + \frac{5}{8} \cdot s \log D_m$ | $\bar{\beta} + \frac{1}{2} \cdot s \log D_m$ |
| Ceiling | $\frac{5}{4}\bar{\beta} + \frac{3}{2} \cdot s \log D_m$ | $\bar{\beta} + 2 \cdot s \log D_m$ |

| | `FullVolume/Arc.lean` | `Shoot/Reach.lean` |
|---|---|---|
| Step 0 | $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ (not collapsed); $\bar{\beta}$ taken via `covers` | The single point $\bar{\beta}$; left end of the interval |
| One-step condition | The same `Drifting` ($\text{drift} \leq \frac{1}{4} \cdot s \log b$) | |
| Record | used; direct user `rodB` (witness: a headline on the existence side) | used; witness `parityEndpoint` (on the `Graces.winnow` side) |

The existence route does not use `Shoot.successful`; it uses `Arc.bought` (according to the record, the direct user of `Arc.bought` is `rodB`) Source . The two share the interval-narrowing tool `narrow`. What we cite as the shooting of the existence route is the band of `Arc.lean`.

06

## On uniqueness

The shooting route (`Arc.bought`, `covers`, `rodB`) gives only the **existence** of the bare coupling Source . `rodB` takes one by choice and says nothing about whether there are other solutions. The docstring of `Rods.lean` also says "this first purchase is only a surjection; uniqueness is something later analyticity estimates should impose" Source .

On the other hand, the source also has fields that state uniqueness. The field `unique` (line 142) of `FullVolume.Final.Jewel` (`YM/FullVolume/Finale/Program.lean` line 133) says that for each $p, N$ and $\zeta \in [-1, 1]$ there is **exactly one** "successful" bare coupling making the coupling at step 0 equal to $\bar{\beta} + \zeta \cdot w$ ($w$ is the width `beamWidth`). The field `tuning` (line 143) says that this bare coupling is given by an analytic function of $\zeta$ (on the unit disc) Source . They are supplied by `wins` (line 60) and `bareTrial` (line 70), with the structure `Winter` as assumption Source . The reading we confirmed in the source, free of contradiction, is as follows.

- "Successful" (`successful`, lines 57–58) means that within a **fixed recursion** (`T.L.tunes`), there is a competing trajectory from step $N$ to step 0. The docstring (lines 17–20) states explicitly that "this is uniqueness only within this recursion, not uniqueness among all conceivable representations" Source .
- The core of uniqueness is `Winter.unique` (`YM/FullVolume/Glide/Vac.lean` line 40): two competing trajectories with equal couplings at step 0 agree at every step. This is a claim of the same type as the uniqueness in 11-10 (anchoring at step 0) Inferred .
- According to the record, `wins` is used (it is used to build `Jewel`, with a headline on the existence side as witness), while the fields `Jewel.unique` and `Jewel.tuning` themselves are outside (read from no headline theorem) Source .

In summary, the place on the existence route of the public theorem where the bare coupling is chosen uses only existence and choice, not uniqueness. Uniqueness and analyticity within a fixed recursion exist as fields of a separate structure `Jewel`, and there are lemmas `wins` and `bareTrial` (`Program.lean` lines 60 and 70, used) that fill those fields. The fields themselves (`unique`, `tuning`) are read from no headline theorem (outside) `Source` . Neither says "unique among all representations".

07

## What this book's study had seen

The "existence of a matched sequence" closed in Lean in this book's study (10-01 § 06) derives existence from the **global endpoints** of a single continuous function `Lean` : the expectation of the coarse plaquette average is 0 at $\beta = 0$ and approaches 1 as $\beta \to \infty$ (`Block3.Y3_zero`, `Block3.tendsto_Y3_one`), and a value $\kappa_0$ in between can be reached by the intermediate value theorem (`Block3.tuned3_exists`). The vortex ratio has the same form (`tuning_all`).

The published construction's shooting builds nested intervals from the **local edges of the band** at each step and the one-step error inequality. There are three differences `Inferred` .

- Since this book knows the endpoint values globally, no shooting induction is needed. Instead, it passes through all $\beta \geq 0$ (including strong coupling).
- The published construction carries every step within the weak-coupling band, and obtains per-step continuity (`Issues`) from the analysis of renormalisation.
- Both give existence only, and use neither uniqueness nor monotonicity (uniqueness within the published construction's fixed recursion is in § 06).

That the crux of the shooting is a single inequality, "one-step error $\leq \frac{1}{4} \cdot s \log b$", gives an axis of comparison with where the sum of errors across steps got stuck in this book's study (Part 15) (14-03).

08

## What we have not read

- The content of how `Roof.Lyrics.supplier` closes `Issues` (we read as far as the names of the materials `Graces.hops`, `Chord.spores`, `moves`).
- The assumptions of `offers` (`Glide/Vac.lean` line 65), which supplies `Winter`, and whether the recursion of `Jewel` is the same as that of `rodB`.

- The content of the proof of uniqueness at step 0, `one` (`YM/FullVolume/Gain/Resolve.lean` line 40).
- How the band coefficients $\frac{3}{4}, \frac{5}{8}, \frac{5}{4}, \frac{3}{2}$ of `Arc` were re-chosen from those of `Shoot/` (the docstring gives no reason).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Under the assumptions, for each $N, p$ there are bare couplings making the coupling at step 0 equal to every value in $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$ (`Arc.bought`, `covers`) | That the inverse map is a function (not claimed) |
| Proved here If the one-step error is $\leq \frac{1}{4} \cdot s \log b$, trajectories at the edge of the band leave the band in one step ((3)) | — |
| Source The existence route (`rodB`) uses the band of `Arc.lean` and takes one by choice; the band of `Shoot/` belongs to another headline | — |
| Source There are fields of uniqueness and analyticity within a fixed recursion (`Jewel.unique`, `tuning`); their supply `wins` is used, the fields themselves are outside | Uniqueness among all representations (denied by the source) |
| Lean Existence of this book's matching comes from global endpoints and the intermediate value theorem (`Block3.tuned3_exists`, `tuning_all`) | Uniqueness and monotonicity on this book's side |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Shooting on the existence route | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/FullVolume/Arc.lean`: `low`, `high` lines 21–22, `nearby` line 24, `centers` line 25, `fits` line 31, `Arc` line 58, `covers` line 71, `crosses` line 75, `bought` line 95 |
| Choice and assembly of the sequence of steps | Source | Same repository, `YM/FullVolume/Stage/Rods.lean` line 26 `rodB` (docstring lines 5–14), `YM/FullVolume/C/Songbook.lean` line 76 `Roof.Lyrics.supplier` |
| Recursion materials and one-step condition | Source | Same repository, `YM/Shoot/Criteria.lean` lines 63, 88, 99 (`band`, `Relays`, `Issues`), `YM/Shoot/Reach.lean` line 18 `narrow`, line 70 `Drifting` |
| The other shooting | Source | Same repository, `YM/Shoot/Reach.lean` lines 81, 112, 156, `YM/Shoot/Race.lean` lines 5 and 67 |

| Item | Kind | Source / tools |
|---|---|---|
| Uniqueness within a fixed recursion | Source | Same repository, `YM/FullVolume/Finale/Program.lean` docstring lines 17–20, `successful` line 57, `wins` line 60, `bareTrial` line 70, `Jewel` line 133 (`unique` line 142, `tuning` line 143); `YM/FullVolume/Glide/Vac.lean` lines 31–40 `Winter`; `YM/FullVolume/Ray/Gain.lean` line 22 `sun`; `YM/FullVolume/Gain/Resolve.lean` line 72 `ask` |
| Dependency record | Source | Same repository, `verification/declaration-usage.tsv` (`Arc.bought`, `covers`, `rodB`, `wins` are used; `Jewel.unique`, `tuning`, `july` are outside; `Shoot.successful` has witness `parityEndpoint`) |
| (3) and the figure | Proved here | Computation in the text |
| Existence of this book's matching | Lean | This book's Lean bundle (`Block3.Y3_zero`, `tendsto_Y3_one`, `tuned3_exists`, `tuning_all`), a bundle that passed the check by the Lean kernel |

**Terms used in this chapter** — shooting method, band, arc, nested intervals, intermediate value theorem, choice, one-step error, uniqueness within a fixed recursion, forward curve. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-05 Why the sides grow, and the calendar — we read the design of growing the sides so that the one-step error, kept a constant, is removed by dividing by $\log b$, and the order in which constants are chosen.

# Why the sides grow, and the calendar — dividing the one-step error by log b to remove it

> This chapter is existing knowledge (a reading of the published construction). § 07 is this book's study, and § 08 is a map of what is unresolved.

**What this chapter shows** — that the sides $b_m$ are odd numbers of at least 3, non-decreasing, and grow to infinity; the reason (keeping the one-step error drift at $O(1)$ while making $\text{drift}/\log b_m \to 0$, so that the sum of errors is $o(\log D_m)$) and the computation by which the running estimate `Running` follows; the calendar conditions and the order in which constants are chosen; and, as the price, that the sequence of spacings becomes a special one.

**Prerequisite chapters** — 10-02 The one-step map and the one-loop computation, 10-04 The shooting method, 8-02 The set-up.

**Said first** — growing the sides does not remove the need for conditions uniform in the number of steps. Conditions uniform over steps are paid for on the calendar's side (the entry coupling of each step, lower bounds on the sides, margins and so on). The whole calendar (including large-field constants) is read in 11-08. This chapter is limited to what concerns the running of the coupling.

The order of this chapter

1. **The statement — the sequence of sides and the calendar conditions**: `Schedule`, `Calendar.Fits`, `Supply.Large`
2. **Dividing the one-step error by log b to remove it**: the computation in `Shoot.trajectory`
3. **The sides grow to infinity**: `logCost` and the smoothing exponent
4. **Gentle — one step is small compared with the whole**: $\log b_m / \log D_m \to 0$
5. **The order in which constants are chosen**: `Sheet` → `Rolls` → `Protection` → `Delays` → `Bar` → `Wait` → `Paid`
6. **The price — the sequence of spacings becomes special**
7. **What this book's study had seen**: the net one-step bracket with a fixed side

8. **What we have not read**

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Labels: Source = we read the source of the published construction / Inferred = this book's inference / Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For theorems of the published construction, the wording of the statement is marked Source, and the level "it is proved" is marked Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## The statement — the sequence of sides and the calendar conditions

How the steps are taken is `Schedule` (`YM/Lattice/Scale.lean`, lines 28–34) Source: the sides $b_m$ are odd with $b_m \geq 3$, $D_m = \prod_{k<m} b_k$, the lattice spacing is $a_m = 1/D_m$, $\mathrm{scale}(m) = \log D_m$ (`scale`, line 115), and the physical sides $L_p$ are multiples of 4, strictly increasing. The condition `Gentle` (lines 129–130) is $\log b_m / \log D_m \to 0$. The assessment's glossary explains that the construction takes $L_p = 4(M + p + 1)$ ($M$ fixed large enough) (Appendix A, page 23) Assessment.

Of the calendar conditions `Calendar.Fits` (`YM/RG/History.lean`, lines 97–118), those concerning the running of the coupling are the following Source.

| Field | Content |
|---|---|
| `mesh` | `Gentle` |
| `sameFactor`, `monotoneL` | $b_m$ is a value determined by the smoothing exponent $J(m+1)$ alone, non-decreasing in $J$ |
| `mono`, `stepSlow`, `deep` | $J$ is non-decreasing, increases by at most 1 per step, and $J \to \infty$ |
| `driftSmall` | $\mathrm{drift}_{m+1} / \log b_m \leq 1/J(m+1)^2$ |
| `logCost` | $\log(1 + \text{sum of step constants}) / \log b_m \leq 1/J(m+1)^2$ |
| `delayed`, `smallField` | $1 < \hat{\beta}_m$, threshold $t_m \leq 1$ (9-07) |

The condition on the side of candidate sides, `Supply.Large` (`YM/Price/Intent.lean`, lines 39–48), imposes, for a smoothing exponent $j$ and a side $L$, $\mathrm{drift}/\log L \le s/4$ (`kick`: the shooting condition `Drifting`, 10-04), $\mathrm{drift}/\log L \le 1/j^2$ (`slow`), and $\log(1+\text{step constants})/\log L \le 1/j^2$ (`fits`) Source. `Rolls.eventually_large` (line 55) says that for each $j$ every sufficiently large odd $L$ satisfies these, and `Rolls.exists_rolls` (line 79) says that a sequence of sides exists Source. The assessment describes this as "the block ratios are chosen to satisfy complicated but explicit inequalities using constants that appear in the renormalisation transformation" (Appendix A) Assessment.

02

## Dividing the one-step error by log b to remove it

The one-step relation (10-02) is $|\beta_{m+1}-\beta_m - s\log b_m| \le \mathrm{drift}_{m+1}$, and the matching condition is $\beta_0 = \bar\beta$. Since $\hat\beta_m = \bar\beta + s\sum_{k<m}\log b_k$, the triangle inequality gives

$$\left|\beta(N,p,m) - \hat\beta_m\right| \;\le\; \sum_{k=1}^{m} \mathrm{drift}_k \qquad (1)$$

Proved here. The source's `Shoot.trajectory` (`YM/Shoot/Race.lean` line 23) shows this as the lemma `hd`, and calls the sum `βCost` Source.

If the side were fixed ($b_k = b$, with drift also a constant $\delta$), the right side would be $m\delta$ and the main part $s\log D_m$ would be $s\,m\log b$; their ratio stays at $\delta/(s\log b)$ and does not go to 0 Proved here. The slope $s$ would not be determined.

The published construction sets $\mathrm{drift}_{k+1} \le \log b_k / J(k+1)^2$ by `driftSmall`, and uses $J \to \infty$. The proof of `trajectory` (`Race.lean`, lines 32–52) takes, for $\varepsilon > 0$, a step $j$ beyond which $1/J(k+1)^2 \le \varepsilon/2$, writes

$$\sum_{k\le i}\mathrm{drift}_k \;\le\; \sum_{k\le j}\mathrm{drift}_k + \frac{\varepsilon}{2}\sum_{j\le k<i}\log b_k = \mathrm{const}_j + \frac{\varepsilon}{2}\left(\log D_i - \log D_j\right) \qquad (2)$$

and, since $\log D_i \to \infty$, concludes that from the step where $\mathrm{const}_j \le \frac{\varepsilon}{2}\log D_i$ on, the sum is at most $\varepsilon \log D_i$ Source. This is `History.Running`: $|\beta(N,p,m) - \hat\beta_m| \le \varepsilon\log(1/a_m)$ (for $m$ large enough, uniformly in $N,p$) Source. The sum of errors becomes $o(\cdot)$ of the main part, and the slope $s$ is determined without ever computing the two-loop coefficient Inferred.

The crux of (2) is that the one-step error drift **may be large** ($O(1)$, and may even increase with the step). What is made small is the ratio $\mathrm{drift}/\log b$, and that is paid for by growing the sides Inferred.

03

## The sides grow to infinity

The source's `Schedule` does not directly require the sides to grow (only $b_m \geq 3$ and `Gentle`). But $b_m \to \infty$ follows from the calendar conditions Inferred : by `logCost`, $\log(1 + \cdots + \text{columns} + \cdots)/\log b_m \leq 1/J^2$, and the step constant `columns` is at least 1 (`Stage.columns_ge`, `History.lean` line 60), so the numerator is at least $\log 2$, hence $\log b_m \geq J(m+1)^2 \log 2 \to \infty$ Proved here . This derivation presupposes that a calendar satisfying the calendar conditions exists (`Wait.ready`) Inferred .

The step constants (the Taylor order `order`, the power `power`, the support exponent and so on) are indexed by the smoothing exponent $J$ and increase monotonically in $J$ (`deepOrder` and others) Source . The field `stages` of the construction theorem's conclusion `Family.Makes` requires that, from the reference step for local fields of degree $d$ on, the Taylor order be at least $d+1$ (`YM/Source/Construction.lean`, lines 47–49) Source . The form is: grow the step constants to handle local fields of every degree, and absorb their logarithmic loss $\log(1 + \text{step constants})$ with the logarithm of the side (`logCost`) Inferred .

04

## Gentle — one step is small compared with the whole

`Gentle`, $\log b_m / \log D_m \to 0$, is the condition that even as the sides grow, the logarithm of one step is negligible compared with the logarithm of the whole Source . With it, even if a length $r$ is read at "the step just below $r$", the ratio of $\log(1/r)$ to $\log D_m$ tends to 1 Inferred . The lemma `pace_log` of the theorem on the short-distance effective coupling $g_W(r)$ (10-06) uses not `Gentle` but a stronger condition in the same direction, the field `bloom` of the additional calendar condition `PostT` on the trace side: $b_i \leq \log \hat{\beta}_i$ (`YM/TraceExtension/Terrace/Base.lean` line 31) Source . With it, the logarithm of one step, $\log b_{i-1}$, is negligible compared with $\hat{\beta}_i$.

From `Gentle` also follows a lower bound on the ratio of lattice spacings, $c\, a_m^2 \leq a_{m+1}$ (`mesh_inequality`, `Scale.lean` line 133; the docstring says "it need not be assumed separately") Source . The sides grow slowly, in a way bounded by a ratio of logarithms.

| Condition | Role | Chapter |
|---|---|---|
| $\text{drift}/\log b \leq s/4$ | Nesting of the shooting bands (leaving the band in one step) | 10-04 |
| $\text{drift}/\log b \leq 1/J^2 \to 0$ | Making the sum of errors $o(\log D)$ (running) | This chapter § 02 |

| Condition | Role | Chapter |
|---|---|---|
| $\log(1+\text{constants})/\log b \le 1/J$ 2 | Absorbing the loss from growing step constants; $b \to \infty$ | This chapter § 03 |
| $\log b_m/\log D_m \to 0$ | The logarithm of one step is small compared with the whole | This chapter § 04 |
| $b_i \le \log \hat{\beta}_i$ (`PostT.bloom`, trace side) | The ratio does not break down when the coupling is read at the step just below $r$ | 10-06 § 04 |

05

## The order in which constants are chosen

`YM/Price/Intent.lean` defines the materials for building the calendar as a sequence of structures arranged in the order in which they are chosen Source .

1. **`Sheet`** (lines 26–36): the kernel choices (penalty coefficient and three thresholds), the table of step constants `profile`, the initial exponent, and the lower bound on the sides `floorL`. The docstring says "orders and constants are chosen **before** the block ratios" (head of file).
2. **`Rolls`** (lines 50–53): a side $L_j$ for each smoothing exponent $j$ (satisfying `Large`, non-decreasing in $j$).
3. **`Protection`** (lines 88–92): margins $\theta_j \le 1/j$ (decreasing), fixed after $L$ is chosen. The docstring says "θ that accommodates the powers after L".
4. **`Delays`** (lines 115–120): deadlines after the margins, such as the lower bound on the entry coupling of each step, `betaEntry`.
5. **`Bar`** (lines 123–127): $\bar{\beta}$. $\bar{\beta} > 1$, at least the entry of the first step, and threshold $\le 1$ for $x \ge \bar{\beta}$.
6. **`Wait`** (from line 134): the working window (the width of the $o(\cdot)$).
7. **`Paid`** (lines 145–159): that the calendar fits these: `betaEntry` $\le \hat{\beta}_m$ at each step (`pays`), lower bounds on the volume, and so on.

`Wait.ready` (`YM/Price/Waits.lean` line 137) says that a sequence of steps `Levels` satisfying all the requirements exists Source . The docstring says that this existence result "works for any constants already chosen, so after all the assignments and ratios before $L$, the delays after $L$ can be chosen arbitrarily large", and at the same time says "it does not check whether the renormalisation argument actually fits this order of dependence" (head of `Intent.lean`) Source . That the order fits is paid for as an assumption on the side of each step's argument (that the step constants do not depend on $L$) Inferred .

Conditions uniform over steps are paid for within this order Inferred . The constants of the one-step certificate (Part 11) may grow depending on the step constant $J$; in exchange, the calendar guarantees that the entry coupling $\hat{\beta}_m$ at step $m$ is at least `betaEntry` (`pays`). One step is performed only where the coupling is large enough.

06

## The price — the sequence of spacings becomes special

The sequence of sides is a special sequence chosen to fit the renormalisation constants. The assessment writes in § 4.3 that "convergence holds along the whole sequence of spacings used in the proof, but there is no universality result about the type of the sequence of spacings", and at the start of § 4 that "since the construction chooses the two sequences (spacings and volumes), the ultraviolet convergence theorem does not assert convergence for every sequence of spacings" Assessment .

The design of growing the sides is a trade: in exchange for determining the one-loop slope without a two-loop estimate, it does not say whether the limiting theory is the same for another sequence of spacings (for example, halving with side 2) Inferred . This universality relates to the questions of 16-02.

07

## What this book's study had seen

The ultraviolet attempts of this book's study (Part 15) fixed the side at 2 and looked for a form paying, uniformly over steps, the decrease of the coupling in one step by the main part $s \ln 2$ (with $s = 11/(3\pi^2)$ for SU(2)) $\pm$ what one step misses (the net one-step bracket of the coupling, 15-03). With a fixed side, as in § 02, the sum of the $O(1)$ one-step errors is of the same size as the main part, and to determine the slope the one-step error itself had to be estimated in a form summable over steps (for example, a form decreasing with the step) Inferred .

The published construction takes a different route: it leaves the one-step error at $O(1)$, grows the sides, and divides by $\log b$. The price is the universality of § 06, and the conditions uniform over steps move to the calendar's side. This route is an idea this book's study did not have (14-05). Both this book's current design (side 2) and the side-3 design fix the side, and this book has no Lean for a design with growing sides.

08

## What we have not read

- How the sequence of sides is concretely built (`Program.exhaust` in `YM/Price/Waits.lean`, line 41; we read only the statement), and how slowly the sides grow.
- Where in the table of step constants the value of drift comes from (`Stage.drift` is a field of the table; we have not read how it is computed).
- The roles of `scaleEntry`, `shells` and `growth` in `Delays` (on the large-field side; 11-08).
- Which declaration guarantees that the order of dependence fits the actual argument (the docstring says it is not checked).

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The sides are odd numbers of at least 3, non-decreasing in $J$; `Gentle`; drift$/\log b \le 1/J^2$; $\log(1+\text{constants})/\log b \le 1/J^2$ | The explicit form of the sequence of sides |
| Proved here The difference between the coupling and the one-loop running is bounded by the sum of one-step errors; with a fixed side, the sum is of the same size as the main part | — |
| Source Since the error ratio goes to 0, the running `Running` holds uniformly in $N, p$ (`Shoot.trajectory`) | The two-loop coefficient; log log corrections |
| Proved here From the calendar conditions, $b_m \to \infty$ ($\log b_m \ge J(m+1)^2 \log 2$) | Premises other than the existence of the calendar itself (supplied by `Wait.ready`) |
| Source Constants are chosen in the order step constants → sides → margins → delays → $\bar\beta$ | A check that this order fits the actual argument (the docstring says it is not checked) |
| Assessment There is no universality with respect to the type of the sequence of spacings | Agreement of the limit for another sequence of spacings |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| How the steps are taken | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `YM/Lattice/Scale.lean` lines 28–34 `Schedule`, line 115 `scale`, line 129 `Gentle`, line 133 `mesh_inequality` |

| Item | Kind | Source / tools |
|---|---|---|
| Calendar conditions and step constants | Source | Same repository, `YM/RG/History.lean` lines 45–61 `Stage`, lines 97–118 `Calendar.Fits` |
| Derivation of the running | Source | Same repository, `YM/Shoot/Race.lean` lines 19–52 (`βCost`, `trajectory`) |
| Order of choice | Source | Same repository, `YM/Price/Intent.lean`: docstring at the head, `Sheet` line 26, `Large` line 39, `Rolls` line 50, `eventually_large` line 55, `exists_rolls` line 79, `Protection` line 88, `Delays` line 115, `Bar` line 123, `Wait` line 134, `Paid` line 145; `YM/Price/Waits.lean` line 137 `Wait.ready` |
| Degree of local fields and depth of steps | Source | Same repository, `YM/Source/Construction.lean` lines 47–49 `Makes.stages` |
| Universality, description of block ratios | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4, § 4.3, Appendix A |
| This book's one step with side 2 | Proved here | The source table of 15-03 |

**Terms used in this chapter** — side, calendar, drift, smoothing exponent, Gentle, running, sum of errors, step constants, order of dependence, universality. Their meanings are in the A-02 Glossary.

Next chapter to read: 10-06 The running theorem and the short-distance effective coupling — we read where the public statement `WilsonRegularity.running` follows from the running of the sequence of steps, and the theorem on the short-distance effective coupling $g_W(r)$, separating what is inside and outside the dependencies.

2026-11-13 · chapter Yang–Mills and the mass gap Part 10 Matching the coupling, and asymptotic freedom The running theorem and the short-distance effective coupling · Kiichi, Shiori and Rin

# The running theorem and the short-distance effective coupling — inside and outside the dependencies

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

**What this chapter shows** — The route from the running of the sequence of steps (`History.Running`) to `WilsonRegularity.running`, which enters the statement of the public theorem: $\beta_{N,p}/\log(1/a_N) \to s$ (uniformly in $p$). The headline theorem of the verification record, `Pert.Asymptotics.comparison`. The definition of the short-distance effective coupling $g_W(r)$ and its theorem, and its ratio to a second coupling built from the continuum correlations. And which target each of these enters.

**Chapters assumed** — 10-01, 10-05. For the physics of asymptotic freedom see 4-06; for how the limits are taken, 8-05.

**Said first** — The running theorem states only the one-loop leading part; it does not state the two-loop coefficient, the $\log\log$ correction, or the boundedness of $\beta - (\bar{\beta} + s \log D)$. It has not been shown that the short-distance effective coupling $g_W$ is independent of how the limit is taken (the infinite-volume subsequence). The theorem on $g_W$ does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target ( § 05 sets out the three places separately).

The order of this chapter

1. **Statement — the running theorem**: the public statement `WilsonRegularity.running`
2. **The route of the running**: `Running` → `zeroUp_bound` → `regular`
3. **The headline theorem comparison**: the $-1/2$ power of the coupling at step $i$, and $\sqrt{\log(1/a_i)}$
4. **The short-distance effective coupling g_W(r)**: the coefficient at the limiting step, and the step just below $r$

5. **Inside and outside the dependencies**: two Lean forms, and the statement of the public theorem
6. **What this book's study had seen**
7. **What has not been read**

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Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The source is the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0); the assessment is `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10). For the theorems of the published construction, the wording of the statements is given under Source, and the level "it is proved" under Assessment. Notation: the one-loop slope is written $s = 11C_g/(24\pi^2)$ (the quantity the assessment writes as twice the one-loop coefficient; `slopeConstant` in the source).

01

## Statement — the running theorem

The public theorem `JointExistence` says that for each simple gauge group there is a Wilson family $C$ satisfying the condition `WilsonRegularity C` (glossary of the assessment, p. 23) Assessment. The field `running` of that condition (`ChallengeSpec.lean`, lines 13–15) is Source

$$\exists\, s_* > 0 \;\; \forall \varepsilon > 0 \;\; \exists N_0 \;\; \forall N \geq N_0 \;\; \forall p : \qquad \left| \frac{\beta_{N,p}}{\log(1/aN)} - s_* \right| < \varepsilon \quad (1)$$

The ratio of the bare coupling to the logarithm of the inverse lattice spacing approaches a positive slope $s_*$ **uniformly in the volume index** $p$. The construction uses $s_* = s = 11C_g/(24\pi^2)$ (the assessment writes this slope as twice the one-loop coefficient) (glossary, p. 25) Assessment. Equation (7) of Theorem 4.1 in the assessment writes the same content as $\lim_N \sup_p |\beta_{N,p}/\log a_N^{-1} - 11C_g/(24\pi^2)| = 0$ Assessment.

This is the leading form of asymptotic freedom. Reading $\beta = 1/g_0^2$, we get $g_0^2 \log(1/aN) \to 1/s$; that is, the bare coupling constant approaches 0 at the rate $1/\sqrt{s \log(1/a)}$ Proved here. Converting the standard-convention formula of 4-06, $1/g_{\mathrm{std}}^2(\mu) = 1/g_{\mathrm{std}}^2(\mu_0) + c_{\mathrm{std}} \ln(\mu/\mu_0)$ (with $c_{\mathrm{std}} = 11N/(24\pi^2)$), into units of $\beta$ gives the same slope: for

$SU(2)$, $\beta = 4/g^2_{\mathrm{std}}$ and $4c_{\mathrm{std}} = 4 \cdot 22/(24\pi^2) = 11/(3\pi^2) = s$ Proved here (for the conversion of conventions see 8-01 § 04).

The running theorem **enters the statement of the public theorem**: `Solution.lean` takes the family and `WilsonRegularity` from the construction theorem and proves `JointExistence`, and the construction theorem obtains it from `YangMills.regular` (`Construction.lean`, line 33) Source .

02

## The route of the running

(1) comes out in three stages Source .

1. **Running of the sequence of steps** `History.Running` (`YM/RG/History.lean`, lines 229–230): $\forall \varepsilon > 0$, for sufficiently large $m$, and for all $N \geq m$ and $p$, $|\beta(N, p, m) - \hat{\beta}_m| \leq \varepsilon \log(1/a_m)$. For the sequence chosen by shooting, `Shoot.trajectory` derives this from the sum of the one-step errors (10-05 § 02).
2. **Ratio form** `Pert.Asymptotics.zeroUp_bound` (`YM/Pert/Asymptotics/RawAxis.lean`, line 15): for sufficiently large $m$ and all $N \geq m$ and $p$, $|\beta(N, p, m)/\log(1/a_m) - s| < \varepsilon$. It uses the fact that the $\bar{\beta}$ in $\hat{\beta}_m = \bar{\beta} + s \log(1/a_m)$ becomes negligible as $\log(1/a_m) \to \infty$.
3. **Public statement** `YangMills.regular` (`HierarchyBridge.lean`, line 16): sets $m = N$ in `zeroUp_bound` to obtain (1).

Only properties of the sequence of steps are used; neither the limiting correlations nor infinite volume is used Source . The uniformity in the volume comes from three facts: the one-step transition $\alpha$ does not depend on the volume (10-02), the shooting aims at the same $\bar{\beta}$ for every $p$, and the upper bound drift on the one-step error is a constant of the step, independent of $p$ Inferred .

**What it does not say** Source : the two-loop coefficient, the $\log\log(1/a)$ correction, or that $\beta_{N,p} - (\bar{\beta} + s \log D_N)$ is bounded. (1) is a limit of ratios and says the difference only up to $o(\log D_N)$. The docstring of `History.lean` also says that it does "not infer the usual beta-function claims beyond `Running`" (lines 32–33).

03

## The headline theorem comparison

Among the 17 headline theorems listed by the verification record (`verification/joint-headlines.txt`) is `YM.Pert.Asymptotics.comparison` (`RawAxis.lean`, line 36) Source . For

the quantity built from the coupling at step $i$, $g_{\mathrm{reg}}(N,p,i)=\beta(N,p,i)^{-1/2}$ (`regulatorRun`, line 35), and for sequences of cutoffs and volumes $N(i)\ge i$ (for sufficiently large $i$) and arbitrary $k(i)$,

$$g_{\mathrm{reg}}\big(N(i),k(i),i\big)\,\sqrt{\log(1/a_i)}\;\longrightarrow\;\frac{1}{\sqrt{s}}\qquad(i\to\infty)\tag{2}$$

holds Source . It says that the coupling at step $i$, read from any cutoff and any volume, lies on the leading part of the one-loop running. The docstring says that, "beyond the extracted comparison, it does not say this equals a physical position-space scheme coupling" (at the top) Source . (2) is a headline theorem, but it is not in the statement of the public theorem `JointExistence` Source .

04

## The short-distance effective coupling g_W(r)

§ 7.4 of the assessment describes the short-distance effective coupling as follows Assessment . Starting from the family of bare couplings of Theorem 4.1, renormalise down to the **step of lattice spacing just below** the length $r$, send the cutoff to 0, take the limit along an infinite-volume subsequence, and define $g_W(r)$ as the $-1/2$ power of the coefficient of the normalised action. Theorem 7.2: for small $r$, and for any chosen infinite-volume subsequential limit, $g_W(r)>0$ and

$$\log(1/r)\,g_W(r)^2\;\longrightarrow\;\frac{1}{s}\qquad(r\downarrow 0).\tag{3}$$

The assessment states that "it has not been shown that $g_W(r)$ is independent of the choice of infinite-volume subsequential limit" Assessment .

We read the form in the source Source .

- **Coefficient at the limiting step**: `Terrace.Arch` and `steep` (`YM/TraceExtension/Terrace/Dusk.lean`, lines 65–71). Along an ultrafilter $\mathcal{L}$ on $(N,p)$, the coupling $\beta(N,p,i)$ at each step $i$ has a limit $Z.\beta_i$. `holds` (line 79) gives $Z.\beta_i\in[\hat\beta_i/2,2\hat\beta_i]$, and `gamut` (line 89) gives $Z.\beta_i/\hat\beta_i\to 1$.
- **The step just below** $r$: `glen` (`YM/TraceExtension/Atlas/Sink.lean`, line 14) is the first step $i$ at which the lattice spacing $a_i$ falls below $r$.
- $g_W$: `Pilot.wilson` (line 44), $g_W(r)=Z.\beta_{\mathrm{glen}(r)}^{-1/2}$. `Coupling` (lines 45–47) has the form "positive for sufficiently small $r$ as $r\downarrow 0$, and $\log(1/r)g^2\to 1/s$". `wilson_link` (line 50) proves `Coupling` for $g_W$ Source .

The proof is short Inferred : `gamut` gives $Z.\beta/\hat{\beta} \to 1$, and `pace_log` (line 21) gives $\log(1/r)/\hat{\beta}_{\mathrm{glen}(r)} \to 1/s$. The latter comes from $\hat{\beta}_i = \bar{\beta} + s \log D_i$ and from the field `bloom` of the additional calendar condition on the trace side, `PostT` : $b_i \le \log \hat{\beta}_i$ (`YM/TraceExtension/Terrace/Base.lean`, line 31). This makes $\log D_i - \log D_{i-1} = \log b_{i-1}$ negligible compared with $\hat{\beta}_i$, so the ratio does not break when read at the step just below $r$ Source (`H.bloom` inside the proof of `pace_log`). `PostT` is also among the hypotheses of `wilson_link` (line 50). Multiplying the two gives (3).

For a second coupling built from the continuum correlations, `Sundial.glo` (`YM/TraceExtension/Sundial/Noon.lean`, line 28), the field `calibrationX` (`YM/TraceExtension/Endpoint.lean`, line 49) states $\mathrm{glo}/gW \to 1$ Source . The assessment writes that "the running law is visible not only in the effective action but also in continuum observables" Assessment .

05

## Inside and outside the dependencies

The assessment does not give the Lean name for Theorem 7.2 Assessment . Within what we read, statements close to (3) appear in two places; together with the headline theorem (2), we set out the three places separately Source (the dependency record is `verification/declaration-usage.tsv`).

| Place | Content | Record | Relation to the t |
|---|---|---|---|
| (a) `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean`:145) | Ratio form: $g(\mu)(s\log\mu)^{1/2} \to 1$ as $\mu \to \infty$. Here $g$ is the inverse square root of the value of the step limit at the first step with $a_m \le 1/\mu$ (`gLedger`, line 130) | outside | **Outside the dependencies** reached from a headline theor |

| Place | Content | Record | Relation to the t |
|---|---|---|---|
| (b) `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean`:50) | $\log(1/r)\, g_W(r)^2 \to 1/s$ ($r \downarrow 0$, at the first spacing step below $r$). Hypothesis includes `PostT` ($b_i \le \log \hat{\beta}_i$) | used (first headline: `continuum_with_position_tests`; direct user: `crest_ready`) | **An intermedia inside the dependencies.** conclusion ent `Crest.charts` `FieldComparisc` and is dropped passing to the trace tests |
| (c) `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean`:36) | (2): the running of the cutoff coupling at step $i$, $\beta(N_i, k_i, i) - 1/2$. A different quantity from $g_W$ | headline | Itself a headlin of the verificat record. Outsid statement of th theorem |

We checked in the source where the conclusion of (b) appears Source : in the field `charts` (lines 20–22) of the intermediate structure `Crest` of the short-distance analysis of the trace (`YM/TraceExtension/Belfry.lean`, line 16), and in the field `wilson` of the comparison structure `FieldComparison` (`YM/TraceExtension/Endpoint.lean`, line 50). This reads as the proof of the trace law (13-02) using the running of the coupling as an intermediate stage Inferred . In `Integration.ManuscriptTraceTests.of_result` (`YM/Integration/ManuscriptTraceTests.lean`, line 77), which passes from these structures to the public trace tests `Tests`, the fields of `Tests` (lines 62–75) contain no comparison of couplings, and the comparison fields are dropped Source .

**Summary for the targets.**

- The public theorem `JointExistence`: the only running that enters its statement is the running of the bare coupling in (1), `WilsonRegularity.running` Source . The public statement asks only that "some positive slope exists"; $s = 11C_g/(24\pi^2)$ is the value the construction satisfies Source .
- The construction theorem `constructed`: its conclusion fields (`Construction.lean`, lines 15–31) contain no statement about the effective coupling Source .
- The committee's combined target: it does not include a check of the coupling (13-03).
- The verification record: (a) is outside the dependencies, (b) is an intermediate stage inside them, and (c) is itself a headline theorem. **The running of $g_W$ does not**

**survive in the conclusion of any headline theorem**, but the running of the bare step coupling in (c) is a headline conclusion.

The one whose wording is close to the statement of Theorem 7.2 of the assessment ($\log(1/r)g_W^2 \to 1/s$; the assessment writes the right-hand side as the reciprocal of twice the one-loop coefficient; $r \downarrow 0$, at the first spacing below $r$) is (b) Inferred. This book treats the short-distance effective coupling as "a result that is in Lean and that the committee states as Theorem 7.2 of the assessment, but that does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target". 13-03 has the same shape.

06

## What this book's study had seen

This book's study took the form of supplying the sequence of bare couplings from outside. The matching in the statements to be shown (19-07) says, in terms of quantities of the measure, that a sequence of couplings exists; that the sequence grows with slope $s$ against $\log(1/a)$ (the running) is neither in the statements nor in Lean. The one-step slope of this book's Volume 3 (value $11/(3\pi^2)$) is a number set as the leading part of the one-step decrease of the coupling (23-01), not a running theorem.

The published construction obtains the running as a theorem about "the sequence of coefficients in the representation", from the matching ($\bar{\beta}$ at step 0), the one-loop one-step estimate, and the calendar of growing sides. The matching quantity of this book's study can be written in terms of the measure alone, but there is no tool linking it to the coefficients of the representation, so a running theorem of the same form cannot be stated on this book's side Inferred. This difference is treated in 14-03 and in the remaining questions of Volume 3 (16-03).

One more point: the published construction carries every step inside the weak-coupling band $[\hat{\beta}/2, 2\hat{\beta}]$ and stops at step 0 (spacing 1). It does not pass through the strong-coupling region by renormalisation. The firm theorems of this book's Volume 3 are on the side of the strong-coupling window (Part 20), and do not overlap with the infrared edge of the published construction (still weak coupling at spacing 1) Inferred (16-04).

07

## What has not been read

- The chain from `wilson_link` to the headline theorem (`crest_ready` → ... → `continuum_with_position_tests`), and whether `Crest` appears in the statement of

the headline theorem or only inside its proof.

- The name of the declaration the assessment takes as the location of Theorem 7.2 (no name is given in the text).
- The content of the definition of the second coupling `glo` (`lampX`), and the proof of `calibrationX`.
- Whether $g_W$ is independent of the choice of subsequential limit (the assessment says this has not been shown).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The running of the bare coupling, $\beta_{N,p}/\log(1/a_N) \to s$ (uniformly in $p$), enters the statement of the public theorem | The two-loop coefficient, the log log correction, boundedness of the difference |
| Source Running of the sequence of steps → ratio form → public statement (`trajectory`, `zeroUp_bound`, `regular`) | — |
| Source The headline theorem `comparison`: the $-1/2$ power of the coupling at step $i \times \sqrt{\log(1/a_i)} \to s^{-1/2}$ | Identification with a physical-scheme coupling (denied by the source) |
| Assessment The short-distance effective coupling: $\log(1/r) g_W(r)^2 \to 1/s$, and its ratio to the second coupling $\to 1$ | Independence from the choice of subsequential limit |
| Source There are three places: `Running.ratio` is outside the dependencies, `wilson_link` is an intermediate stage inside them (its conclusion is dropped on passing to the public trace tests), and `comparison` is a headline theorem (a different quantity). The running of $g_W$ does not survive in any of the conclusions of the public theorem, the construction theorem or the committee's combined target | The declaration where Theorem 7.2 of the assessment lives (the assessment gives no name; the closest wording is `wilson_link`, Inferred ) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Public statement and its supply | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0) `ChallengeSpec.lean` lines 13–15, `HierarchyBridge.lean` line 16 `regular`, `Construction.lean` line 33, `Solution.lean` lines 4–10 |
| Running of the sequence of steps and the ratio form | Source | Same, `YM/RG/History.lean` line 229 `Running`, `YM/Shoot/Race.lean` line 23 `trajectory`, `YM/Pert/Asymptotics/RawAxis.lean` lines 13–36 (`b0`, `zeroUp_bound`, `regulatorRun`, `comparison`) |

| Item | Kind | Source / tool |
|---|---|---|
| Short-distance effective coupling | Source | Same, `YM/TraceExtension/Terrace/Dusk.lean` lines 65–95, `YM/TraceExtension/Atlas/Sink.lean` lines 11–50, `YM/TraceExtension/Belfry.lean` lines 16, 20–22, 31, `YM/TraceExtension/Terrace/Base.lean` lines 29–34 (`PostT`), `YM/Integration/ManuscriptTraceTests.lean` lines 62–77 (`Tests`, `of_result`), `Construction.lean` lines 15–31, `YM/TraceExtension/Sundial/Noon.lean` line 28, `YM/TraceExtension/Endpoint.lean` lines 49–50, `YM/PertExtension/Running.lean` line 145 |
| Dependency record and headlines | Source | Same, `verification/declaration-usage.tsv`, `verification/joint-headlines.txt` |
| Theorem 4.1, Theorem 7.2, glossary | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 4.3, § 7.4, Appendix A |

**Words used in this chapter** — running theorem, asymptotic freedom, slope $2b_0$, uniform in the volume, headline theorem, short-distance effective coupling, second coupling, ultrafilter, subsequential limit, outside the dependencies. Their meanings are in the A-02 glossary.

Next chapter to read: 11-00 Entrance — the induction hypothesis and the one-step certificate — on to Part 11, which shows with two ledgers, the induction hypothesis and the one-step certificate, that the one-step representation is preserved from step to step.

# Entrance — the induction hypothesis and the one-step certificate

This chapter is a guide. § 02 is the convention for levels of citation in Volume 2, and § 05 is this book's own investigation.

**What this chapter shows** — The shape of the "induction hypothesis" that the published construction keeps at each step, and of the "one-step certificate" it produces at each step, as it stacks renormalisation steps from step $N$ (the bare Wilson measure) down to step 0 (spacing 1). The two carry ledgers of different strength. A table shows which part of this mechanism each of the eleven chapters of this part deals with.

**Chapters assumed** — 9-01 The block kernel (the one-step probability kernel), 9-06 The density representation, 9-07 The small-field threshold and the truncated action, 10-04 The shooting method, 10-05 Why the sides grow, and the calendar.

**Said first** — What this part writes is the definitions and statements of the published construction, as far as this book has read them in the source. The certainty of the results of the published construction is at the level "the committee's assessment is that it is proved in Lean"; what this book has checked goes only as far as the form of the definitions and statements. Much remains unread (11-11). This part says nothing about the mass gap or about uniqueness in infinite volume.

The order of this chapter

1. What this part reads — the three components of the induction that descends the steps.
2. Levels of citation and labels — Source Inferred Assessment , and which target a statement enters.
3. The two ledgers — the induction hypothesis is weak, the one-step certificate strong.
4. A map of this part — the eleven chapters and their main declarations.
5. What this book's study had seen — how it differs from stacking up regions.
6. What has not been read — the list is in 11-11.

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## What this part reads

The published construction starts from the Wilson measure on the lattice of spacing $a_N = 1/D_N$, applies renormalisation with block sides $b_{N-1}, \dots, b_0$ in turn, and descends to step $0$, of spacing $a_0 = 1$. One step is not a fixed map but a probability kernel (9-01). On the configurations $V$ of the lattice at step $m$ sits an unnormalised density $\rho_{N,p,m}(V)$, and the steps are linked by push-forward identities Source .

$$\rho_{N,p,m}(V) = \int T_{\beta_{m+1}}(U,V)\,\rho_{N,p,m+1}(U)\,dU, \qquad \rho_{N,p,N} = e^{-\beta_N S_W}. \tag{1}$$

Here $p$ is the volume index, $T_\beta$ is the one-step kernel, and $dU$ is the product of the probability Haar measures. Equation (1) is an identity, not an approximation. What this part reads are the three components for carrying this sequence of identities "with estimates".

1. **Induction hypothesis**: the density at every step has a representation of the same form, and its ingredients carry a ledger for each step (11-01).
2. **One-step certificate**: building the representation at step $m$ from the representation at step $m+1$, and producing equation (1) and the ledgers. It splits into large fields (11-02–11-08) and small fields (11-09).
3. **Convergence at a fixed step**: with step $m$ held fixed and $N \to \infty$, the ingredients of the representation converge along the whole sequence, and from this the lattice correlations converge (11-10).

02

## Levels of citation and labels

Volume 2 reads, by citation, the Lean source of the existence construction for pure Yang–Mills published in November 2026 (repository `yang-mills-group/OpenAI-YM-`

`20261028`, Apache-2.0) and the committee's assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10).

**Level of certainty.** The committee records that the declarations in the dependencies of the 17 headline theorems (about 100,000) were rechecked in a freshly initialised Lean kernel environment (one that loads fixed external versions), and that they use only the three permitted axioms[Assessment] (assessment § 9.1). The public theorem `YangMills.jointExistence` and the committee's own combined target `YMReview.Acceptance.combined_acceptance` have also passed a separate comparison tool (Comparator)[Assessment] (assessment § 9.2). The assessment also states that no separate kernel implementation was used[Assessment] (§ 9.2). Therefore what this part calls "statements of the published construction" are results at the level **the committee's assessment is that they are proved in Lean**. What this book has checked itself goes only as far as reading the form of the definitions and statements in the source. Of the proofs, only the key points have been read.

The labels follow the legend of 6-00: there are three, [Source] (a definition or statement read in the source of the published construction), [Inferred] (this book's inference) and [Assessment] (according to the committee's assessment). [Source] does not guarantee that a proof is correct; that is the level of the committee's assessment.

**Which target a statement enters.** The verification record (`verification/declaration-usage.tsv`) notes for each declaration whether it enters the dependencies of one of the 17 headline theorems (used) or not (outside). "Used" does not mean "used by the public theorem". 6-00 divides the targets into five (the public theorem, the construction theorem, the committee's combined target, reached from one of the headline theorems, outside the dependencies). Many of the declarations in this part are not the conclusions of the first three themselves but components of their proofs. So in this part we indicate how far we have checked with the following three descriptions. The first is for those we have checked, by tracing the declarations literally, to lie on the proof chain of the construction theorem (and of the public theorem, which contains it as a projection); the second and third are exactly the last two distinctions of 6-00.

| Description | Meaning | Examples |
|---|---|---|
| On the chain of the existence theorem | Checked by tracing the declarations literally from `Nonperturbative.existence` (called by the construction theorem `constructed`) | `rodB` → `Roof.Lyrics.supplier` → … → `Return.reset`, `Window.allB`, `sailsW`, `realUV` |
| Reached from one of the headline theorems | "used" in the record. Not checked whether it lies on the chain of the existence theorem | Most of the large-field lemmas (the witness in the record is `SharpGram.habitat`) |
| Outside the dependencies | "outside" in the record. Not used in the proofs of the main results | `Remainder.hole_price`, `Directed.holding` |

Sources are given as "file path (relative to the root of the repository; versions as in the legend of 6-00), declaration name, line". The lines are ones this book checked by opening the source.

03

## The two ledgers

Write the reference coupling at step $m$ as $\hat{\beta}_m = \bar{\beta} + s\log(1/a_m)$ ($s = 11C_g/(24\pi^2)$ is the one-loop slope), $\hat{g}_m = \hat{\beta}_m^{-1/2}$, and the super-polylogarithmic quantity as $P_m = \exp\left((\log\log\hat{\beta}_m)^2\right)$ (the calendar `Calendar`, `YM/RG/History.lean:64-90`) Source . The small-field threshold is $t_m = \hat{g}_m P_m$.

The **induction hypothesis** `History.Conv` (`YM/RG/History.lean:213`) requires the following of the step-$m$ representation Source .

- The ledger of regular activities (small-field corrections) $\le \hat{g}_m^{11/5}$ (`allowance`, :84).
- The ledger of forced activities (large-field corrections) $\le \exp(-P_m^{1/5})$ (`forcedAllowance`, :85).
- The coupling is in the band: $\beta_m \in [\hat{\beta}_m/2,\, 2\hat{\beta}_m]$.

The **one-step certificate** `Directed.Move` (`YM/PCalc/Band.lean:49`) states, for a representation $v$ at step $m+1$ and a representation $u$ at step $m$, the push-forward identity $u.\mathrm{eval} = \mathrm{push}_{\beta_v}(v.\mathrm{eval})$ and the form of the induction hypothesis (the weak ledger), and in addition a **strong** ledger for the forced activities Source .

$$\text{induction hypothesis: } \|K\| \le e^{-P_m^{1/5}}, \qquad \text{one-step certificate: } \|K\| \le e^{-g\,P_{m+1}^{2}}. \tag{2}$$

Here $\|K\|$ is the pointwise sum weighted by coverings (`KBound`, `YM/RG/Activities.lean:102`), and $g > 0$ is a constant the calendar chooses at each step. The strong ledger is rebuilt from the large-field price of that step (the exponent of the factor $e^{-cEP^2}$ that the one-step integral yields per seed; seeds are defined in 11-03, marks in 11-05) (11-05). It passes to the weak ledger by a one-line inequality that lowers the exponent of $P$ (11-07) Source .

The induction hypothesis carries only the weak ledger; within one step a strong estimate is rebuilt and then relaxed to the weak ledger. There is no need to carry the strong ledger as a hypothesis for the next step Inferred . The weak ledger is also a function of step $m$ and is rebuilt at every step. The conditions for making things hold

uniformly in the step are paid for on the side of the calendar's constants (the entry for β, the lower bound on the sides, and so on) (11-08) `Source` .

04

## A map of this part

| Chapter | Subject | Main declarations ( `Source` ) | How it sits in the targets |
|---|---|---|---|
| 11-01 | The form of the density at each step, and the induction hypothesis | `Rep` , `Rep.eval` , `Conv` , `Transfer` , `Selected` | Objects of the chain of the existence theorem ( `Makes` contains `Selected` ) |
| 11-02 | The finest step | `WilsonStartup.due_eq` | Reached from one of the headline theorems |
| 11-03 | Decomposing the action, seed types, regions, partitions of unity | `Tube.locates` , `Stores` , `Seeded` , `Plot` , `Allocation` | Reached from one of the headline theorems |
| 11-04 | Energy per seed | `Bid.Crowds` , `Seals` , `Lapse` , `Teller.Grants` | Reached from one of the headline theorems |
| 11-05 | Allocating prices, and polymer sums | `Buy.quote` , `Cash.armK` | Reached from one of the headline theorems |
| 11-06 | Reorganising activities at the coarse step | `ThinP` , `ThickP` , `Collection.vac` | Reached from one of the headline theorems |
| 11-07 | Converting between ledgers, and persistence across steps | `Time.late_force` , `Postage.saved` , `Time.now` | Reached from one of the headline theorems |
| 11-08 | The calendar — the order in which constants are chosen | `Supply.Large` , `Bid.pricing_order` , `Calendar.Fits` | Reached from one of the headline theorems |
| 11-09 | The small-field step | `Return.earnings` , `atlas` , `Local.actual` , `Return.reset` | On the chain of the existence theorem up to `reset` |
| 11-10 | Convergence at a fixed step, and the step that removes the step index | `Window.allB` , `accord` , `Flux.fanned` , `Sever.echoes` , `Rising.arrives` | On the chain of the existence theorem |
| 11-11 | List of what has not been read | — | — |

We recommend reading in the order of the table. Skipping the large-field chapters (11-02 to 11-08) and going from 11-01 to 11-09 and 11-10 still lets you follow the thread.

### Correspondence of notation

The chapters of Parts 9–11 were each written in notation close to the letters of the source, so in places the same object has a different letter in different chapters, and in places the same letter denotes different objects in different chapters. We collect the correspondences for reading into one table.

| Object | Parts 9 and 10 | Part 11 | Note |
|---|---|---|---|
| Wilson action | $S_{\mathrm{W}}$ (9-06, 10-01) | $S_{\mathrm{W}}$ (equation (1)). Equation (1) of 11-02 writes it as the sum of prices over plaquettes, $\sum_{\square} \mathrm{price}(U_{\square})$ | The same thing. Both are sums over plaquettes before multiplying by β; the weight multiplies by β and puts it in the exponent (S_W in A-01 § 05 is written in the form that includes β) |
| Small-field threshold | $t_m$ (step $m$), $t_{m+1}$ (step $m+1$) (9-07) | Depending on the chapter, $t_m$, $\tau$, $t$, $t_y$ | The table correspondence of threshold letters in 11-01 |
| Covering scale | $B_m = \lceil (\log \hat{\beta}_m)^2 \rceil$ (9-06, 9-07) | $B_m$ (11-01, 11-08, 11-10), $R = \mathrm{radiusB}(x)$ (11-03, 11-04; equal to $B_m$ at $x = \hat{\beta}_m$) | The $R = (J+3)L$ of 9-02–9-04 is the radius of a patch, a different quantity |
| Discrete choice of the kernel (a pair of frame label and path label) | $\tau$ (9-01–9-05: $P(\tau \mid U)$, $Q_b(U, \tau)$) | $k$ (11-03, 11-04: $Q = Q(U, k)$) | The $k$ of 11-06 is a fusion index, a different quantity |
| Linearisation of the predictor inside the precision matrix | $Q$ (9-01: $K + c_{\mathrm{pen}} Q^{\mathsf{T}} Q$) | $\Pi$ (11-09: $K + c_{\mathrm{pen}} \Pi^{\mathsf{T}} \Pi$) | The same matrix (`FreeResolvent.baseMatrix`). The $\Pi_G$ of 9-04 is the projection to the group, a different thing |
| Dimension of the representation | $d_G$ (9-01, 10-01) | $d$ (11-01, 11-02) | The $d$ of 11-05 and 11-10 is a degree, a different quantity |
| Correction to the coupling (deviation from the one-loop prediction) | $d$ ((1) of 10-02, the root of the rereading equation in 10-03) | $u = \beta_F - \mathrm{predictor}$ and $\Delta$ (11-09) | The same quantity (11-09 § 06) |
| Coefficient of the strong ledger of the one-step certificate | — | $g$ (equation (2), 11-08), $d$ (11-07; the letter in the source's `late_force`) | The same coefficient. The $g_m$ of 11-10 is the one-step deviation of the coupling, a different quantity |

| Object | Parts 9 and 10 | Part 11 | Note |
|---|---|---|---|
| Names of the fields of the representation | main action, truncation parameter, ledger point, support (9-06) | path action, truncation constant, budget point, support (11-01 onward) | The same things |

05

## What this book's study had seen

This book's study treated the ultraviolet side as "a chain of block maps" and thought in terms of stacking up large-field regions step by step and fixing them thereafter (the Bałaban type). What was open there was how to handle defects that persist over many steps near configurations whose shape the block map does not change (the "large-field estimate" and "large-field gas" in the premise list, 28-04, 28-06). What this book had in Lean went only as far as the rarity of large plaquettes at a single scale (22-01) and one step of the free field (15-12).

The published construction **answers this obstacle in a different form**. It keeps no history of regions; it recounts seeds at every step and, with an inequality uniform over configurations satisfying the gauge section and the region conditions, produces a factor per seed at every step. It does not change the measure (equation (1) stays an identity) Source (form read) / Inferred (the reading of how this corresponds to this book's obstacle). The full comparison is in 14-04.

06

## What has not been read

At the end of each chapter of this part, we list, with locations, what has not been read on that chapter's subject. The list for the whole part is 11-11. There are three large items.

- The whole of the upper bounds on the integrals that remain after the large-field price has been extracted (from `YM/SectorNext/LARGE_FIELD_REMAINDERS` onward).
- The content of how the one-step theorem is assembled (`Use.Machine.run`, `YM/AnalyticBridge/VERIFIED_RESET_STEP/Downstream.lean:65`).
- The proofs of the lemmas that reorganise the activities at the coarse step (inside `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP`).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The steps are linked by push-forward identities, and the induction hypothesis carries the weak ledgers $e^{-P_m^{1/5}}$ and $\hat{g}_m^{11/5}$ and the coupling band (`Conv`, `Transfer`) | That the representation is unique (the source does not claim uniqueness) |
| Source The one-step certificate states the strong ledger $e^{-gP_{m+1}^2}$ for forced activities (`Move.forced`) | How the strong ledger is used at later steps (not read) |
| Assessment The dependencies of the 17 headline theorems were rechecked in a freshly initialised kernel environment (assessment §9.1; no separate kernel implementation was used, §9.2) | That this book has checked the proofs of the lemmas in this part |
| Inferred The strong ledger is not carried as a hypothesis for the next step; it is rebuilt within a step and then relaxed to the weak ledger | That no condition uniform in the number of steps is needed (it is paid for on the calendar's side) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Lean source of the published construction | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0). This book only read a copy; it neither built nor ran it |
| The committee's assessment | Assessment | `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), §4.3, §9.1, §9.2. Versions as in the legend of 6-00 (code 3456e5e, assessment side 55ba9fc) |
| Calendar and ledgers | Source | `YM/RG/History.lean:64` (`Calendar`), :84 (`allowance`), :85 (`forcedAllowance`) |
| Induction hypothesis, one-step relation, all steps | Source | `YM/RG/History.lean:213` (`Conv`), :223 (`Transfer`), :234 (`Selected`) |
| One-step certificate | Source | `YM/PCalc/Band.lean:49` (`Directed.Move`), :65 (`Program`) |
| Dependency record | Source | `verification/declaration-usage.tsv` (the status and headline-witness columns) |

**Words used in this chapter** — induction hypothesis, one-step certificate, ledger, regular activity, forced activity, push-forward identity, headline theorem. Their meanings are in the A-02 glossary.

Next chapter to read: 11-01 The form of the density at each step — we read, in formulas, the representation that the induction hypothesis carries.

# The form of the density at each step, and the induction hypothesis

> This chapter is established knowledge (a reading of the published construction). § 04 is this book's own investigation and § 05 is a map of open questions.

**What this chapter shows** — As what function of what ingredients the unnormalised density at step $m$ is written, and what ledgers those ingredients carry. The content of the induction hypothesis `History.Conv`, the one-step relation `History.Transfer`, and the statement for all steps `History.Selected`.

**Chapters assumed** — 11-00 (labels and levels of citation), 9-06 The density representation, 9-07 The small-field threshold and the truncated action.

**Said first** — The representation is stated by an existential quantifier and is not unique (the source's docstring says so explicitly). The ledgers are per-step quantities in the form of pointwise sums independent of the volume, but the size of the density itself includes an exponential in the volume. This chapter only reads the form; the proof that the ledgers are preserved is the subject of 11-02 to 11-09.

The order of this chapter

1. Statement — the representation, the induction hypothesis, the one-step relation, all steps
2. The truncated action and the path action — linear above the threshold.
3. The two kinds of activity, and the gas — regular activities only where healthy, forced activities only on the flags.
4. What this book's study had seen
5. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — the representation, the induction hypothesis, the one-step relation, all steps

Notation follows the calendar (11-00 § 03): $\hat{\beta}_m$, $\hat{g}_m = \hat{\beta}_m^{-1/2}$, $P_m = \exp((\log\log\hat{\beta}_m)^2)$, the threshold $t_m = \hat{g}_m P_m$, and the covering scale $B_m = \lceil(\log\hat{\beta}_m)^2\rceil$. Let $\Lambda_m$ be the lattice at step $m$ and $V$ a configuration on it.

**Correspondence of threshold letters.** Within one step (from step $m+1$ to step $m$) two thresholds appear, one on the fine side and one on the coarse side (9-07). The letters differ from chapter to chapter, but each is the value of the calendar threshold $t_m = \hat{g}_m P_m$ at step $m$ or at step $m+1$. In the source, `temp` (`YM/Pressure/Seeds.lean:25`) is $\mathrm{temp}(\hat{\beta}_m) = t_m$, and `fiery` (:26) is $\mathrm{fiery}(\hat{\beta}_m, L) = \mathrm{temp}(\hat{\beta}_m + s\log L) = t_{m+1}$ Source .

| Chapter | Threshold on the coarse side (step $m$) | Threshold on the fine side (step $m+1$) | Note |
|---|---|---|---|
| 9-07, 11-00, 11-01, 11-08 | $t_m$ | $t_{m+1}$ | The calendar's notation as is |
| 11-02 | — | $t$ $(= t_N)$ | The threshold of the finest step $N$ |
| 11-03, 11-04 | $\tau = \mathrm{temp}(x)$ | $t = \mathrm{fiery}(x, L)$ | $x = \hat{\beta}_m$ |
| 11-05 | — | $t = \mathrm{fiery}(x, L)$ | $x = \hat{\beta}_m$ |
| 11-06, 11-07 | $t_m$ | $t_y$ | $y$ is the reference coupling of the fine step |
| 11-09 | $t = \hat{g}P$ | — | $x = \hat{\beta}_m$, $\hat{g} = x^{-1/2}$. The opposite side from the $t$ of 11-03 |

The $t$ in § 02 and § 03 of this chapter is the threshold of the budget point of the representation in question ($t_m$ for the representation at step $m$).

The **representation** `Rep` (`YM/RG/History.lean:165`) is a tuple of six ingredients Source : the path data (the paths entering the action and their coefficients), the truncation constants `Clip` (the "truncation parameters" of 9-06), the family of activities `Activities`, the budget point `BudgetPoint` (the "ledger point" of 9-06: threshold, covering scale, order of derivatives and so on), the coupling $\beta$, and the vacuum constant $v$. Its value is

$$\mathrm{eval}(V) = \exp\Big(v\,|\Lambda_m| - S^{\mathrm{ch}}_\beta(V) - \sum_i \mathbf{1}[\text{healthy}]\,E_i(V)\Big)\cdot \mathrm{gas}(V) \tag{1}$$

(`Rep.exponent`, `Rep.eval`, :176, :179) Source . $S^{\mathrm{ch}}_\beta$ is the path action averaged over the signed orientations of the hypercubic group (384 of them) (`PathData.inChannels`, `YM/RG/PathActions.lean:241`), $E_i$ are the regular activities, and gas is the gas of forced activities.

The **induction hypothesis** `History.Conv N p m` (`YM/RG/History.lean:213`) has four fields Source .

1. The budget point is that of step $m$ (threshold $t_m$, covering scale $B_m$ and so on).
2. The truncation constants are the constants of the step.
3. The representation is `Lawful`: the path data satisfy `Bounds`, the activities are `Valid`, the ledger of regular activities is $\le \hat g_m^{11/5}$, and the ledger of forced activities is $\le e^{-P_m^{1/5}}$.
4. The coupling is in the band: $\beta_{N,p,m} \in [\hat\beta_m/2,\, 2\hat\beta_m]$.

The **one-step relation** `History.Transfer N p m` (:223) consists of two formulas Source .

$$\mathrm{eval}_m = \mathrm{push}_{\beta_{m+1}}\,\mathrm{eval}_{m+1}, \qquad \big|\beta_{m+1} - \beta_m - s\log b_m\big| \le \mathrm{drift}_{m+1}. \tag{2}$$

The push-forward $\mathrm{push}_\beta F(V) = \int T_\beta(U,V)F(U)\,dU$ is the integral against the one-step kernel (`YM/RG/Block.lean:77`), and the $\beta$ of the kernel is the coupling of the step on the **fine side**. The second formula bounds the deviation from the one-loop running ($s = 11C_g/(24\pi^2)$ is the one-loop slope).

The **statement for all steps** `History.Selected` (:234) bundles the following for all $N, p$ and $m \le N$ Source : the path data do not depend on the volume $p$ (`path_shared`); the induction hypothesis at each step (`represents`); at step $N$, with $v = 0$, the

representation equals the bare Wilson weight exactly (`starts`, :237); the one-step relation at each step (`steps`); $\beta = \bar{\beta}$ at step 0 (`reaches`, :239); and the running (`running`).

As a corollary, the partition function is preserved through the steps Source (`History.partition_eq`, :242):

$$\int \mathrm{eval}_{N,p,m}\, dV = Z_{\Lambda_N}(\beta_{N,p,N}) \qquad (m \le N). \tag{3}$$

**Which target it enters.** The certificate of the construction's family, `Family.Makes` (`YM/Source/Construction.lean:44`), contains `path.Selected`, and the family appears in the conclusion of `constructed`. The objects of this chapter lie on the chain of the existence theorem Source .

## 02 The truncated action and the path action

For the holonomy $U_p$ of a plaquette $p$ we use the chord distance $\sigma = \mathrm{chord}(U_p) = \|U_p - 1\|/\sqrt{d}$ ($d$ is the dimension of the representation; $\mathrm{price}(U_p) = 1 - \mathrm{Re}\,\mathrm{tr}\, U_p/d = \sigma^2/2$). The **truncated action** (`PathData.clipped`, `YM/RG/PathActions.lean:218`) is

$$\mathrm{clip}_t(\sigma) = \begin{cases} c_{\mathrm{dir}}\,\sigma^2/2 & (\sigma \le t) \\ c_{\mathrm{lin}}\, t\,\sigma & (\sigma > t) \end{cases}, \qquad 0 < c_{\mathrm{lin}} < c_{\mathrm{dir}}/16 \tag{4}$$

(`Clip`, :139; the last inequality is the field `kink`) Source . Above the threshold $t$, the action drops from quadratic to linear. At $\sigma = t$ the value jumps downward, and the function is neither monotone nor convex Source .

The basic term of a plaquette (`baseTerm`, :221) is $\mathrm{clip}_t(\sigma_p)$, plus the square of the transported curvature $\|\ell_p\|^2/2$ only when the field around the plaquette is small (`kept`, :210: the chords of all plaquettes are bounded by a threshold that loosens with distance from the plaquette). The path action (the "main action" of 9-06) is

$$S\beta(V) = \beta\Big(\sum_p \mathrm{base}_p(V) + \sum_x (\text{cubic and quartic atoms})\Big) + \sum_x (\text{one-loop atoms}) \tag{5}$$

(`principal`, :236), and $S_\beta^{\mathrm{ch}}$ is its average over orientations Source .

Meaning Inferred : by weakening the action to linear above the threshold and keeping it in the representation, the rarity of large-field configurations remains explicitly in the action terms rather than in the gas. Having weakened the action handed to the coarse step produces prices as a "positive remainder" in the one-step integral (11-04).

03

## The two kinds of activity, and the gas

**Flags and health.** At threshold $t$, the set of points touching plaquettes whose chord exceeds $t$ is called the flags $\mathrm{flags}_t(V)$ (`YM/RG/Masks.lean:34`). When a support (the "support" of 9-06) $X$ does not meet the flags, $X$ is healthy (`Healthy`, :41) Source .

A **regular activity** (`RegularAddress`, `E`) is a function $E_i$ with an anchor point and a support, and it enters the density only when healthy (the indicator in `masked`, :70). The condition `RegularFamily` (`YM/RG/Activities.lean:70`) has three parts: smooth up to finite order where healthy, $E_i(1) = 0$ at the flat configuration, and log-Hessian 0 in the affine directions at the anchor Source . The last two are the shape after the marginal terms (constant and quadratic) have been moved into the vacuum term and the coupling (11-09).

A **forced activity** (`ForcedAddress`, `K`) is a function $K_i$ with a support and a non-empty "inventory" inside it. If $K_i(V) \neq 0$, the intersection of the support with the flags equals the inventory exactly (`permit`) Source . The gas is

$$\mathrm{gas}(V) = \sum_{b} \mathbf{1}\Big[\text{supports pairwise disjoint},\ \bigcup_{i\in b} \mathrm{inv}_i = \mathrm{flags}_t(V)\Big] \prod_{i\in b} K_i(V) \tag{6}$$

(`admissible`, `gas`, `YM/RG/Activities.lean:114-129`) Source . If the flags are empty, the gas is 1 (`gas_one`, :135). In other words, forced activities **live only on the flags of the field at that step**.

**Ledgers.** Both kinds of activity carry a ledger in the form of a pointwise sum (`EBound`, `KBound`, :98, :102) Source :

$$\sup_x \sum_{i:\,\mathrm{site}_i = x} e^{c\sup \mathrm{cov} B(X_i)}\, \|E_i\|_{t,k} \le \hat{g}_m^{11/5}, \qquad \sup_x \sum_{i:\,x\in X_i} e^{c\sup \mathrm{cov} B(X_i)}\, \sup|K_i| \le e^{-P_m^{1/5}}. \tag{7}$$

The sum for regular activities is over pieces anchored at $x$; the sum for forced activities is over pieces whose support contains $x$. $c_{\mathrm{sup}}$ is the covering exponent of the budget point (`supportExponent`). $\mathrm{cov}_B(X)$ is the smallest number of balls of radius $B$ in the lattice maximum distance (cubes of side $2B+1$ with a centre), arranged so that their union is connected, needed to cover the support (`coverNumber`, `YM/Lattice/Support.lean:71`), and $\|E\|_{t,k}$ is the supremum over healthy configurations of derivatives up to order $k$ in directions of size $t$ (`supBounds`, `YM/RG/Masks.lean:164`) Source . Because the sums are pointwise, the ledgers themselves do not depend on the volume Source .

04

## What this book's study had seen

This book's study also held the large-field corrections as "activities per hole", and assumed estimates of the form $|z_t(\gamma)| \le \exp(-\gamma_E \beta S^\flat(\gamma))$ at each step and each hole (the "large-field estimate" and "large-field gas" in the premise list, 28-04). There are two differences.

- This book's premises put the rarity $e^{-c\beta}$ on the side of the activities, and stacked up regions step by step. The published construction keeps the rarity on the side of the truncated action, and the ledger of activities is the weak $e^{-P^{1/5}}$ Inferred .
- For this book's deterministic block map, it has not been shown that the coarse density keeps a representation of the form (1) at every step. Carrying the representation itself through all steps is the component this book lacks Inferred (11-10 § 06).

05

## What has not been read

- The meaning of all the fields of `Bounds` for the path data (`YM/RG/PathActions.lean:173`). We have read only as far as the names `metric_zero` and `loop_jets`.
- The role of `extraTest` (an additional condition) for forced activities.
- How the difference between the orientation average $S^{\mathrm{ch}}$ and the path action of a single orientation is handled in the accounting of one-step prices (11-07 § 06).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The density at each step has a representation of the form (1), and the induction hypothesis requires the ledgers (7) and the coupling band | That the representation is unique (the source states only existence) |
| Source Steps are linked by the push-forward identity (2), and the partition function is preserved (3) | Reflection positivity of the push-forward measure (not stated) |
| Source The truncated action is linear above the threshold, with linear coefficient less than $c_{\mathrm{dir}}/16$ | That the truncated action is "an approximation of the effective action" (it is only a component of the representation) |
| Source Forced activities live only on the flags, and the gas is 1 when the flags are empty | That the gas is bounded uniformly in the volume (estimates at the level of the density include an exponential in the volume) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Representation, value, Lawful | Source | `YM/RG/History.lean:165` (`Rep`), :179 (`eval`), :181 (`Lawful`) |
| Induction hypothesis, one-step relation, all steps, partition function | Source | `YM/RG/History.lean:213`, :223, :234, :242 |
| Truncated action | Source | `YM/RG/PathActions.lean:139` (`Clip`), :210 (`kept`), :218 (`clipped`), :221 (`baseTerm`), :241 (`inChannels`) |
| Activities, gas, ledgers | Source | `YM/RG/Activities.lean:62` (`Activities`), :70, :98, :102, :106 (`Valid`), :129 (`gas`), :135 (`gas_one`) |
| Flags, health, ledger weights | Source | `YM/RG/Masks.lean:34`, :41, :70, :144 (`mass`), :164 (`supBounds`) |
| Certificate of the family | Source | `YM/Source/Construction.lean:44` (`Family.Makes`) |

**Words used in this chapter** — representation, truncated action, flags, healthy, regular activity, forced activity, gas, ledger, covering number. Their meanings are in the A-02 glossary.

Next chapter to read: 11-02 The finest step — the first move, splitting the bare Wilson weight into this representation.

# The finest step — splitting the bare Wilson weight into the representation

> This chapter is established knowledge (a reading of the published construction). § 04 is this book's own investigation and § 05 is a map of open questions.

**What this chapter shows** — The induction starts at step $N$ from "the value of the representation equals the bare Wilson weight exactly". We show the plaquette-by-plaquette identity that builds that representation, and that the remaining difference, due, is non-negative and at least $t^2/4$ on large plaquettes. This is why the first forced activities are small.

**Chapters assumed** — 11-01 The form of the density at each step (truncated action, flags, health, the two kinds of activity).

**Said first** — The identity is an equality per plaquette, with no factor of $\beta$. How the first activities are arranged into polymers and entered into the ledgers, this book has read only as far as names and docstrings. The lemmas of this chapter are reached in the verification record from one of the headline theorems, but we have not checked whether they lie on the chain of the existence theorem.

The order of this chapter

1. Statement — the starting condition and the plaquette-by-plaquette identity
2. Outline of the mechanism — splitting into four terms
3. Where the difference goes — the first regular activities and forced activities
4. What this book's study had seen
5. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — the starting condition and the plaquette-by-plaquette identity

**Starting condition.** The field `starts` of the statement for all steps `History.Selected` (`YM/RG/History.lean:237`) requires that the representation at step $N$ has $v = 0$ and that its value equals the bare Wilson weight exactly Source :

$$\mathrm{eval}_{N,p,N}(U) = \exp\Big(-\beta_N \sum_{\square} \mathrm{price}(U_{\square})\Big), \qquad \mathrm{price}(u) = 1 - \tfrac{1}{d}\mathrm{Re}\,\mathrm{tr}\,u = \tfrac{1}{2}\mathrm{chord}(u)^2. \tag{1}$$

Here the subscript $p$ of eval is the volume index, and the sum is over all plaquettes $\square$. The volume index does not appear from equation (2) on, so there we write a plaquette as $p$. The starting field on the side of the one-step theorem, `Recipe.Starter` (`YM/Shoot/Criteria.lean:93`), requires the same two things ($v = 0$, value equal to the bare weight) Source .

**The plaquette-by-plaquette identity** `WilsonStartup.due_eq` (`YM/Extra/WILSON_STARTUP/Ownership.lean:68`) Source : if the truncation constant satisfies $c_{\mathrm{dir}} \le 1$, then for every configuration $U$ and plaquette $p$

$$\mathrm{price}(U_p) = \mathrm{base}_p(U) + \mathbf{1}[\mathrm{healthy}]\, Q_p(U) + (1 - c_{\mathrm{dir}})\,\mathbf{1}[\mathrm{healthy}]\, R_p(U) + \mathrm{due}_p(U). \tag{2}$$

Here $\mathrm{base}_p$ is the basic term of the first step's path data `top` (`Design.lean:34`) (the truncated action, plus the quadratic term when `kept`), $Q_p$ is the quartic term in the curvature (coefficient $1 - c_{\mathrm{dir}}$), $R_p$ is the remainder of the price after removing the quadratic and quartic parts (`surplus`, `Curvature.lean:166`), and "healthy" means that the points the plaquette touches are not in the flags. Three things are stated about the difference $\mathrm{due}_p$ Source .

- `due_range`: $0 \le \mathrm{due}_p \le 2$ (under $c_{\mathrm{dir}} \le 1$).

- `due_large`: on plaquettes whose chord exceeds the threshold ($\mathrm{chord}(U_p) > t$), $\mathrm{due}_p \geq t^2/4$.
- `owed_zero` (:193): the same quantity read in the orientation frame is $0$ outside the $B$-halo of the flags. The hypothesis is the whole of `Decent` ($c_{\mathrm{dir}} \leq 1$, $B \geq 2$, and conditions on the threshold and the loosening length).

**Which target it enters.** `due_eq` and `owed_zero` are "used" in the verification record, and the witness in the record is the headline theorem `SharpGram.habitat`. We have not checked whether they lie on the chain of the existence theorem Source .

02

## Outline of the mechanism — splitting into four terms

Let the chord be $\sigma = \mathrm{chord}(U_p)$ and the threshold $t$. The first path data `top` take the plaquette itself as the path, with transported curvature $\ell_p = \sqrt{1 - c_{\mathrm{dir}}}\,\mathrm{curv}(U_p)$ (`ell_sq`, `Design.lean:86`: $\|\ell_p\|^2 = (1 - c_{\mathrm{dir}})\|\mathrm{curv}\|^2$) Source . The definition of due (`due` in `Ownership.lean`) splits into three cases Source .

| Case | Part entered into the representation | due |
|---|---|---|
| $\sigma > t$ (large plaquette) | Only the truncated action $c_{\mathrm{lin}} t \sigma$ | price $- c_{\mathrm{lin}} t \sigma$ |
| $\sigma \leq t$, healthy and `kept` | $c_{\mathrm{dir}}\sigma^2/2 + \|\ell\|^2/2 +$ quartic + remainder | $0$ |
| $\sigma \leq t$, otherwise | $c_{\mathrm{dir}}\sigma^2/2$ (and, if healthy, quartic and remainder) | The surplus or shortfall of the quadratic term (within $[0, 2]$) |

For small fields (second row), $c_{\mathrm{dir}}$ times the bare price $\sigma^2/2$ enters the representation as the quadratic part of the truncated action, and the remaining $1 - c_{\mathrm{dir}}$ times enters as the squared curvature, the quartic term and the remainder; no difference is left Source (formula).

The **lower bound on large plaquettes** is a short computation (the line of the proof of `due_large`) Source . From the condition on the truncation constants $c_{\mathrm{lin}} < c_{\mathrm{dir}}/16 \leq 1/16$ we get $c_{\mathrm{lin}} \leq 1/4$. Since $t < \sigma$,

$$\mathrm{due}_p = \tfrac{1}{2}\sigma^2 - c_{\mathrm{lin}}\, t\, \sigma \;\geq\; \tfrac{1}{2}\sigma^2 - \tfrac{1}{4}\sigma^2 = \tfrac{1}{4}\sigma^2 > \tfrac{1}{4}t^2. \quad (3)$$

In other words, on a plaquette above the threshold the truncated action puts at most half of the bare price into the representation, and sends at least $t^2/4$ of the rest out as

due.

03

## Where the difference goes — the first regular activities and forced activities

Multiplying identity (2) by $\beta_N$, summing and exponentiating, the bare weight becomes the product "exponential of the truncated action $\times$ exponential of the terms where healthy $\times\ e^{-\beta_N \sum \mathrm{due}}$". The headline docstring in the source says: "all the remaining (non-negative) difference is charged to the forced packets; there is no logarithmic addition in this plan" Source (docstring).

- We read the remainder terms where healthy as becoming the first regular activities (location `YM/Extra/WILSON_STARTUP/Ink.lean`) Inferred .
- We read due as becoming the first forced activities, on polymers formed by joining overlapping $B$-balls of the flags (location `YM/Extra/WILSON_STARTUP/Packets.lean`) Inferred .

Size Inferred : from the definition of the threshold, $\hat{\beta}_N t_N^2 = P_N^2$. The actual coupling $\beta_N$ is in the band $[\hat{\beta}_N/2,\ 2\hat{\beta}_N]$ (`Conv.band`, `YM/RG/History.lean:218`), so $\beta_N t_N^2 \geq P_N^2/2$, and per flagged plaquette $e^{-\beta_N \mathrm{due}} \leq e^{-P_N^2/8}$. The first forced activities start much smaller than the induction-hypothesis ledger $e^{-P_N^{1/5}}$.

04

## What this book's study had seen

The ultraviolet chain of this book's study set step $N$ as the bare Wilson measure and stacked up large-field regions from the first step (the premise "large-field estimate" in 28-04). It did not have this way of building the initial condition: truncating the action at the threshold at the first step and pushing the difference onto activities as a bounded factor. We note it as something worth comparing.

05

## What has not been read

- The construction that arranges the first forced activities into polymers (`YM/Extra/WILSON_STARTUP/Packets.lean`), and their entry into the first ledger (`Treats.start`, location unconfirmed).
- The part that turns the remainder where healthy into the first regular activities (`YM/Extra/WILSON_STARTUP/Ink.lean`), and its ledger $\hat{g}_N^{11/5}$.

- Whether `due_eq` lies on the chain of the existence theorem (the witness in the record is `SharpGram.habitat`).
- The details of the correspondence between `owed` (due read in the orientation frame) and `due`.

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The representation at step $N$ has $v = 0$ and equals the bare Wilson weight (`Selected.starts`) | That this representation is unique |
| Source The plaquette-by-plaquette identity (2): bare price = basic term + quartic and remainder where healthy + due ($c_{\mathrm{dir}} \le 1$) | The assembly from the identity to the first ledgers (not read) |
| Source $0 \le \mathrm{due} \le 2$, and $\mathrm{due} \ge t^2/4$ on large plaquettes | That due is large on every plaquette near the flags (it can be 0 on small plaquettes) |
| Inferred The first forced activities are of order at most $e^{-P_N^2/8}$ per flag | That this size is written as a statement (this book has not seen it) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Starting condition | Source | `YM/RG/History.lean:237` (`Selected.starts`), `YM/Shoot/Criteria.lean:93` (`Starter`) |
| The plaquette-by-plaquette identity and its properties | Source | `YM/Extra/WILSON_STARTUP/Ownership.lean:68` (`due_eq`), `due_large` and `due_range` (same file), :193 (`owed_zero`) |
| The first path data | Source | `YM/Extra/WILSON_STARTUP/Design.lean:23` (`bulk`), :34 (`top`), :86 (`ell_sq`), `Curvature.lean:166` (`surplus`) |
| Dependency record | Source | `verification/declaration-usage.tsv` (`due_eq` and `owed_zero` are used, witness `SharpGram.habitat`) |

**Words used in this chapter** — bare Wilson weight, truncated action, flags, healthy, due (the remaining difference), forced activity, regular activity. Their meanings are in the A-02 glossary.

Next chapter to read: 11-03 Decomposing the one-step action, and regions — inside the one-step integral, the action is split into local rows and configurations are sorted by seed type.

# Decomposing the one-step action, and regions

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

**What this chapter shows** — The algebraic identity that writes the "fine action + penalty − coarse action" appearing in the one-step integral as a sum of local rows, one per plaquette and per edge, and the regions that sort configurations by three sets of seeds (the seed type). The regions overlap and are glued with a smooth partition of unity. The region conditions are imposed globally, and the estimates use a form localised around the seeds.

**Chapters assumed** — 11-01 (truncated action, flags), 9-01 The block kernel (predictor and penalty), 9-04 The predictor and projection to the group.

**Said first** — The row identity is an algebraic equality and contains no estimate. The estimate (energy per seed) is 11-04. The declarations of this chapter are reached in the verification record from one of the headline theorems, but we have not checked whether they lie on the chain of the existence theorem.

The order of this chapter

1. Setting — the one-step integral and three thresholds
2. Statement 1 — the row identity
3. Statement 2 — seed types and regions
4. Statement 3 — the partition of unity, and flags always touch a seed
5. Outline of the mechanism — why overlapping regions, why both global and local
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Setting — the one-step integral and three thresholds

Let the one-step data be $S$ (from the fine lattice $f$ to the coarse lattice $c$, block side $L$). There is a fine configuration $U$, a coarse configuration $V$, and a discrete choice inside the kernel (gauge frame and predictor label) $k$; the predictor $Q = Q(U, k)$ is the value on coarse edges built from the fine field (9-04). We write the "deviation" of a coarse edge $b$ as $\mathrm{wire}_b = \mathrm{chord}(V_b Q_b^{-1})$ (`YM/Pressure/Cost.lean:187`) Source .

Let the reference coupling of the coarse step be $x$ $(= \hat{\beta}_m)$; we use three quantities (`YM/Pressure/Seeds.lean:22-27`) Source .

| Symbol | Definition | Meaning |
|---|---|---|
| $\tau = \mathrm{temp}(x)$ | $x^{-1/2}\exp((\log\log x)^2)$ | Threshold of the coarse step |
| $t = \mathrm{fiery}(x, L)$ | $\mathrm{temp}(x + s\log L)$ ($s$ is the one-loop slope) | Threshold of the fine step |
| $R = \mathrm{radiusB}(x)$ | $\lceil(\log x)^2\rceil$ | Scale of the halo (the neighbourhood around a seed) |

When $x = \hat{\beta}_m$, in the notation of 9-07 and 11-01 we have $\tau = t_m$ (coarse side), $t = t_{m+1}$ (fine side), and $R = B_m$ (covering scale) (correspondence of threshold letters in 11-01).

The "size" of a plaquette is $\mathrm{toll}(U_p) = \mathrm{chord}(U_p)$ (`toll` in `YM/Pressure/Rows.lean:55`).

02

## Statement 1 — the row identity

Write the action of a fine plaquette as $\mathrm{crown}_p^f$ (truncated action + the quadratic term when `kept`), the action of a coarse plaquette as $\mathrm{crown}_q^c$, and the penalty term of a coarse edge as $\mathrm{pen}_b$ (`monetary`, `Rows.lean:112`). The **row identity** `Tube.locates` (`YM/Pressure/Rows.lean:145`) is Source

$$\sum_{f} \mathrm{workF} + \sum_{b} \mathrm{workI} + \sum_{q} \mathrm{workC} + \sum \mathrm{workN} + \sum_{q} (\mathrm{workL} + \mathrm{workD}) = \sum_{p} \mathrm{crown}_{p}^{f} + \sum_{b} \mathrm{pen}_{b} - \sum_{q} \mathrm{c}$$

The proof is only a sum of linear identities (`linarith`). The contents of the rows are as follows[Source]. Let $X_q^c$ be the curvature of the coarse plaquette $q$ multiplied by $\sqrt{c_{\mathrm{dir}}}$, and shrink it by the threshold $\tau$:

$$\mathrm{vision}_q = \min\bigl(1,\ \tau / \mathrm{toll}(V_q)\bigr)\, X_q^c \quad (2)$$

(`thinned`, :71, `vision`, :104); distributing this to the fine plaquettes with the blocking kernel $\Phi$ (`leap`, :46) gives $M_v = \Phi \cdot \mathrm{vision}$.

- $\mathrm{workC}_q = \langle \mathrm{vision}_q, X_q^c \rangle - \frac{1}{2}\|\mathrm{vision}_q\|^2 - \mathrm{crown}_q^c$ (the coarse-plaquette row, :129).
- $\mathrm{workF}_p = \mathrm{crown}_p^f + \frac{1}{2}\|M_v\|^2 - \langle M_v, X_p^f \rangle$ (the fine-plaquette row, :122).
- $\mathrm{workI}_b$: completing the square of the penalty term with the vision distributed to the noise side (:126).
- $\mathrm{workL}_q$: the inner product of vision with the "deviation" between the coarse curvature and the curvature rebuilt from the fine curvature and the noise (:133). $\mathrm{workN}$ and $\mathrm{workD}$ are the remainders needed to close the squares.

Meaning[Inferred]: it compares the coarse curvature with "the blocked fine curvature + noise", completes the square, and distributes the difference into local rows. Each row depends only on the neighbourhood of a single plaquette or edge.

03

## Statement 2 — seed types and regions

A **seed type** `Stores` (`YM/Pressure/Seeds.lean:34`) is a triple of finite sets $A = (\mathrm{costs}, \mathrm{aisles}, \mathrm{flags})$[Source]: the set of large fine plaquettes, the set of coarse edges that deviate greatly from the prediction, and the set of flag points of the coarse step. The **number of seeds** is $\#A = |\mathrm{costs}| + |\mathrm{aisles}| + |\mathrm{flags}|$ (`shares`, :46). Here a "seed" is a working term of the published construction meaning each individual element of these three sets.

The **global region condition** `Seeded A U k V x W` (`YM/Push/SECTOR_MEASURE/Cells.lean:51`)[Source] imposes the following on **every** fine plaquette $p$ and every coarse edge $b$.

$$\begin{aligned} &p \in \text{costs} \Rightarrow \text{toll}(U_p) \geq t/4, \qquad p \in/\ \text{costs} \Rightarrow \text{toll}(U_p) \leq t/2, \\ &b \in \text{aisles} \Rightarrow \text{wire}_b \geq Wt, \qquad b \in/\ \text{aisles} \Rightarrow \text{wire}_b \leq 2Wt, \\ &\text{flags} = \text{flags}_\tau(V) \quad \text{(exactly equal)}. \end{aligned} \tag{3}$$

$W$ is a constant taken large. The plaquette and edge conditions have a margin between $t/4$ and $t/2$ (between $Wt$ and $2Wt$), that is, hysteresis. The flags have no margin and are made to coincide exactly with the flags of the coarse field.

The **local region condition** `Plot A U k V x W` (`YM/Pressure/Seeds.lean:51`) Source imposes the same conditions only on plaquettes and edges touching the radius-$5R$ neighbourhood of where the seeds sit (`dome (5*radiusB x)`), requires the flags to coincide there, and adds that the frame weight of the choice $k$ is positive. The local condition follows from the global one (`Seeded.plot`, `Cells.lean`: if the likelihood of the choice $k$ is positive) Source .

The object of the estimate is the sum of rows over the radius-$2R$ neighbourhood of the seeds,

$$\text{assessed}(A) = \sum_{\text{dome}(2R)} \big(\text{workF} + \text{workI} + \text{workC} + \text{workN} + \text{workL} + \text{workD}\big) \tag{4}$$

(`Stores.assessed`, `Seeds.lean:85`) Source .

04

## Statement 3 — the partition of unity, and flags always touch a seed

The **partition of unity** `Allocation` (`YM/Push/SECTOR_MEASURE/Profile.lean:30`) consists of weights $\text{bill}_A(k, U, V)$, one per seed type, that satisfy Source

$$\text{bill}_A \geq 0, \qquad \sum_A \text{bill}_A = 1, \qquad \text{bill}_A \ /= 0 \ \Rightarrow\ \texttt{Seeded}\ A. \tag{5}$$

and are measurable. Wherever a weight is not 0, the **global condition** holds, and if the choice is live the local condition `Plot` can be extracted (`plots`, :50) Source .

**Flags always touch a seed** (`flagged`, :56) Source : if bill$_A$ $\neq 0$ and a point $i$ is in the flags at the fine-step threshold $t$, then there is a plaquette $p \in$ costs touching $i$. The proof: a flagged plaquette has toll $> t > t/2$, so by the contrapositive of the second part of the global condition (3) it is in costs. Since forced activities coming in from the previous step live only on the flags of the fine step (11-01 § 03), **the inventory points of incoming activities always touch a seed of this step (a plaquette in costs)** Source (statement read) / Inferred (reading of its meaning).

**Which partition is used.** The version that splits by indicator functions, `Allocation.hard` (:68; `booking`, `Cells.lean:87`, cuts exactly at thresholds $t/4$ and $Wt$), is outside the dependencies Source (record). What is "used" in the record is the family of product weights built from a smooth transition function $\chi$,

$$\mathrm{share}_J = \prod_{i \in J} (1 - \chi((r_i/d)^2)) \prod_{i \notin J} \chi((r_i/d)^2), \qquad \sum_J \mathrm{share}_J = 1 \tag{6}$$

(`Chi.share`, `share_sum`, `Profile.lean:126-133`) Source . $\chi$ is a smooth transition equal to 1 for $y \leq 1$ and 0 for $y \geq 4$; if $\chi((r/d)^2) \neq 0$ then $r < 2d$, and if $1 - \chi((r/d)^2) \neq 0$ then $r > d$ (`chosen_lt`, `exterior_gt`), so setting $d = t/4$ matches the margin $[t/4, t/2]$ of (3) Inferred (how $d$ is assigned has not been read).

05

## Outline of the mechanism — why overlapping regions, why both global and local

1. **Overlapping regions.** Cutting exactly at a threshold makes the region change discontinuously for configurations near the threshold. Regions with a margin overlap, and gluing them with a smooth partition of unity avoids boundary problems when regions are differentiated or moved inside the integral Inferred . The source's docstring also says that the indicator-function split is for a measurable partition and is not used as a smooth weight Source (docstring).
2. **Both global and local.** The condition `Seeded` attached to the partition of unity is a condition imposed on the whole configuration. The estimate (energy per seed), on the other hand, uses `Plot`, localised to the $5R$ neighbourhood of the seeds, and the sum of rows (4) over the radius-$2R$ neighbourhood. Because the localised form is extracted from the global condition, the estimate side need only look near the seeds Source (read `plots`) / Inferred (reading of the intent).

3. **Scale of the halo.** $R = \lceil (\log x)^2 \rceil$ is measured by the $\hat{\beta}$ of the coarse step. The local condition uses $5R$ and the estimated sum $2R$ Source .
4. **The integral with one region fixed** is the unit of the large-field analysis of one step. A region is determined by the field at that step alone and is not inherited from the previous step Source (read the fields of `Stores`) / Inferred (reading of its meaning).

06

## What this book's study had seen

This book's study sorted configurations by small-field indicator functions and thought in terms of stacking up the large-field region $\Omega^c$ step by step and fixing it thereafter. The integral with one region $A$ fixed corresponds to this book's "integral with $\Omega^c$ fixed" Inferred . There are two differences.

- The regions of the published construction **are determined by that step alone** and keep no history of regions. The locations of incoming activities are automatically included in this step's seeds by `flagged`.
- The published construction uses overlapping regions with margins and a smooth partition of unity. This book's study cut with indicator functions.

A formula corresponding to the row identity (1), writing the difference of one-step actions by local completion of squares, did not exist on this book's side.

07

## What has not been read

- The assembly of the partition of unity actually used on the main-theorem side (where the `Chi` weights are assembled into an `Allocation` over seed types), and how $d$ is assigned.
- The content of the description of overlaps between regions (`YM/Push/SECTOR_MEASURE/Overlap.lean`).
- How the blocking kernel $\Phi$ (`leap`) and the distribution to the noise $\Phi'$ (`kick`) are built, and the details of the meaning of rows workN and workD.
- The role of the condition that the frame weight of the choice $k$ is positive (field `list` of `Plot`).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Fine action + penalty − coarse action equals a sum of local rows (algebraic identity (1)) | That any of the rows is positive (that is the estimate of 11-04) |
| Source The region condition `Seeded` is imposed globally, and the local `Plot` (radius $5R$) is extracted afterwards | That the region condition says anything about the field outside the halo (`Plot` does not) |
| Source Hysteresis applies only to costs and aisles; flags are made to coincide exactly with the flags of the coarse field | That the regions form a partition of configurations (they overlap) |
| Source If the weight of the partition of unity is not 0, every fine flag point touches a plaquette in costs (`flagged`) | That this lemma lies on the chain of the existence theorem (the witness in the record is `SharpGram.habitat`) |
| Source The indicator-function partition `hard` is outside the dependencies; the smooth `Chi.share_sum` is used | The details of assembling the smooth partition |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Thresholds and halo | Source | `YM/Pressure/Seeds.lean:22` (`radiusB`), :25 (`temp`), :26 (`fiery`) |
| Deviation | Source | `YM/Pressure/Cost.lean:187` (`wire`) |
| Rows and the identity | Source | `YM/Pressure/Rows.lean:46` (`leap`), :71 (`thinned`), :104 (`vision`), :112 (`monetary`), :122, :126, :129, :133 (rows), :145 (`locates`) |
| Seed types, local regions, estimated sum | Source | `YM/Pressure/Seeds.lean:34` (`Stores`), :46 (`shares`), :51 (`Plot`), :85 (`assessed`) |
| Global regions, indicator-function split | Source | `YM/Push/SECTOR_MEASURE/Cells.lean:51` (`Seeded`), :87 (`booking`) |
| Partition of unity | Source | `YM/Push/SECTOR_MEASURE/Profile.lean:30` (`Allocation`), :50 (`plots`), :56 (`flagged`), :68 (`hard`, outside), :85 (`Chi`), :133 (`share_sum`) |

**Words used in this chapter** — row identity, seed, seed type, region, hysteresis, partition of unity, flags, halo. Their meanings are in the A-02 glossary.

Next chapter to read: 11-04 Energy per seed — three inequalities showing that, on a region, the sum of rows in the neighbourhood is positive in proportion to the number of seeds.

# Energy per seed

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

**What this chapter shows** — The exact form (order of quantifiers and hypotheses) of the one-step, configuration-by-configuration inequality $c_E\, t^2 \cdot \#\mathbf{seeds} \le \mathbf{assessed}$ on a region, and the three inequalities that build it. Truncating the action handed to the coarse step to linear produces a positive remainder, and the shrinking of linear quantities by $L^{-2}$ under blocking lets the fine side win.

**Chapters assumed** — 11-03 (row identity, seed types, regions, the three thresholds), 11-01 § 02 (truncated action).

**Said first** — The inequality of this chapter is a one-step, configuration-by-configuration inequality, not a claim across steps. This book has not read the proofs of the three inequalities (location `YM/Work/E05` – `E07`); only the statements were read. The declarations of this chapter are reached in the verification record from one of the headline theorems, but we have not checked whether they lie on the chain of the existence theorem.

The order of this chapter

1. Statement — energy per seed (order of quantifiers and hypotheses)
2. Lower bound for coarse plaquettes — the positive remainder
3. Lower bound for fine plaquettes and the penalty
4. Payment for large coarse plaquettes
5. Assembly, and the factor that comes out of the integral
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — energy per seed (order of quantifiers and hypotheses)

Notation follows 11-03 § 01: the reference coupling of the coarse step $x$, the coarse threshold $\tau = \mathrm{temp}(x)$, the fine threshold $t = \mathrm{fiery}(x, L)$, the halo scale $R = \mathrm{radiusB}(x)$, the seed type $A$, the number of seeds $\#A$, and the sum of rows in the neighbourhood $\mathrm{assessed}(A)$. The coarse threshold $\tau$ and the fine threshold $t$ are $t_m$ and $t_{m+1}$ in the notation of 9-07 (correspondence of threshold letters in 11-01).

The **order bundle** `Teller` (`YM/Pressure/Seeds.lean:89`) is an input bundling the choice of kernel, the step constants of the fine and coarse steps, the number $j$ of convolutions of the block weights, and so on. That the one-step data $S$ and the tube $T$ (the pair of path data for the fine and coarse steps) fit it is called `Funds` (the choice of kernel agrees, $J = j$, the kernel's three obligations `Branches`, `Bounds` and `Roll` for the tube's path data, and so on) Source .

The order of quantifiers in **energy per seed** `Teller.Grants` (`YM/Pressure/Seeds.lean:129`) Source is as follows (`Funds`, `live`, `Serves`, `cash`, :101-127).

$$\exists i\, \forall L \geq i\ \ \forall^{\infty} W\ \ \exists c_E > 0\ \ \forall^{\infty} x\ \ \forall (S, T, F, A, U, k, V) :\ \ c_E\, t^2\, \#A\ \leq\ \mathrm{assessed}(A). \quad (1)$$

$\forall^{\infty}$ means "for all sufficiently large". The range of the last $\forall$ carries all of the following hypotheses Source .

1. The block side of $S$ is exactly $L$ (`S.patch.L = L`).
2. **Lower bound on the lattice side**: side of the coarse lattice $\geq 100(1 + R)$.
3. $S, T$ fit the order (`Funds`).
4. **Gauge section**: the fine field $U$ is 1 on the trunk of the block tree (`SliceSet F U`, :48). It is an inequality on the tree gauge section.
5. The local region condition `Plot A U k V x W` (11-03 § 03).

$c_E$ (in the source's wording $\kappa$ in `Grants`, $\mu$ in the price fields) may depend on $L$ and $W$, and the lower bound on $x$ depends on $L, W, c_E$ Source . The threshold $t$ is the threshold on the **fine side**.

This statement holds for every order without hypotheses: `Next.Fluctuations.grants` (`YM/Next/FLUCTUATIONS/Grants.lean:24`) Source . Its content is `Integrated.pressure_grants` (`YM/Integrated.lean:222`), which assembles it with `Bid.issues` (:155) from ten component statements (E01–E09 in `YM/Bid/Energy.lean`, and the smallness outside, `EBare`). The components are proved under `YM/Work/`, and `EBare` is closed by `bare` in `YM/Next/FLUCTUATIONS/Bare.lean:64` Source .

02

## Lower bound for coarse plaquettes — the positive remainder

**Lower bound for coarse plaquettes**: the structure `Bid.Crowds` (`YM/Bid/Energy.lean:94`) and `def E05` (:101), which requires it for every order Source : $\exists \delta > 0\ \exists B > 0\ \forall L\ \forall W$, for sufficiently large $x$, for configurations satisfying the region (with the hypotheses of § 01) and for every coarse plaquette $q$, setting $\sigma = \mathrm{toll}(V_q)$,

$$\begin{aligned} \tau < \sigma \le \delta \ &\Rightarrow\ \tfrac{c_{\mathrm{dir}}}{4}\,\tau\sigma \ \le\ \mathrm{workC}_q, \\ \sigma \le \tau \ &\Rightarrow\ -B\tau^4 \ \le\ \mathrm{workC}_q, \\ \sigma > \delta \ &\Rightarrow\ -c_{\mathrm{lin}}\,\tau\sigma \ \le\ \mathrm{workC}_q. \end{aligned} \tag{2}$$

All the thresholds are on the **coarse side** $\tau$, and so are the truncation constants.

**Meaning** Inferred : on a medium plaquette ($\tau < \sigma \le \delta$), taking $|\mathrm{curv}| \approx \sigma$, vision is shrunk by the factor $\tau/\sigma$, so the first two terms of the coarse-plaquette row are

$$\langle \mathrm{vision}, X^c \rangle - \tfrac{1}{2}\|\mathrm{vision}\|^2 \approx c_{\mathrm{dir}}\left(\tau\sigma - \tfrac{1}{2}\tau^2\right) \ \ge\ \tfrac{1}{2}c_{\mathrm{dir}}\,\tau\sigma. \tag{3}$$

On the other hand, the coarse-step action is the truncated action $c_{\mathrm{lin}}\tau\sigma$ ($c_{\mathrm{lin}} < c_{\mathrm{dir}}/16$). The difference is a positive remainder of at least $c_{\mathrm{dir}}\tau\sigma/4$. If the coarse-step action had stayed quadratic, $c_{\mathrm{dir}}\sigma^2/2$, the same computation would give $\mathrm{workC} \approx -c_{\mathrm{dir}}(\sigma-\tau)^2/2 \le 0$. **The positive remainder comes from truncating the action handed to the coarse step to linear**.

03

## Lower bound for fine plaquettes and the penalty

**Lower bound for fine plaquettes and the penalty**: `Bid.Seals` (`YM/Bid/Energy.lean:103`; `def E06` is :110) Source : $\exists N\ \forall L \ge N\ \forall^{\infty} W\ \exists d > 0$, for sufficiently large $x$, under the hypotheses of § 01,

$$\begin{aligned} \tfrac{c_{\mathrm{lin}}^{I}}{2}\, t \sum_{p \in \mathrm{costs},\ \mathrm{toll} > t} \mathrm{toll}(U_p) \;+\; d\, t^{2}\, \#\{p \in \mathrm{costs} : \mathrm{toll} \le t\} \;\le\; \sum_{\mathrm{dome}(2R)} \mathrm{workF}, \\ \tfrac{c_{\mathrm{pen}}}{8} \sum_{b \in \mathrm{aisles}} \mathrm{wire}_b^{2} \;\le\; \sum_{\mathrm{dome}(2R)} \mathrm{workI}. \end{aligned} \tag{4}$$

The threshold is on the **fine side** $t$, and $c_{\mathrm{pen}}$ is the kernel's penalty coefficient (in the source's wording `S.kernel.penalty`).

Meaning Inferred : the cross term of the fine-plaquette row $\mathrm{workF} = \mathrm{crown}^{f} + \frac{1}{2}\|M_v\|^{2} - \langle M_v, X^{f}\rangle$ is of order $(C/L^{2})\, t\,\mathrm{toll}$, by the upper bound on size (`Maximum`, :83: $\|M_v\| \le (C/L^{2})\, t$). $L \ge N$ is needed to make this at most half of the linear term of the truncated action, $c_{\mathrm{lin}} t\,\mathrm{toll}$.

04

## Payment for large coarse plaquettes

The third line of the coarse-plaquette lower bound can be negative, particularly on very large coarse plaquettes ($\sigma > \delta$). What pays for its total is the **payment for large coarse plaquettes** `Bid.Lapse` (`YM/Bid/Energy.lean:121`; `def E07` is :130) Source : $\exists C > 0\ \forall L\ \forall W > 0\ \exists Z > 0$, for sufficiently large $x$, under the hypotheses of § 01,

$$\sum_{q \in \mathrm{hit}} (\mathrm{workL} + \mathrm{workD})_q + \frac{C}{L^{2}}\, t \sum_{\substack{p \in \mathrm{costs} \\ \mathrm{toll} > t}} \mathrm{toll}(U_p) + C\, t \sum_{b \in \mathrm{aisles}} \mathrm{wire}_b + Z\, t^{2} R^{4}\, \#A_{\mathrm{sp}} \;\ge\; c_{\mathrm{lin}}^{\mathrm{c}}\, \tau \sum_{\substack{q \\ \mathrm{toll}(V_q) > \delta}} \mathrm{to}$$

The threshold on the left-hand side is the fine-side $t$, and on the right-hand side the coarse-side $\tau$. $\#A_{\mathrm{sp}}$ is the number of seeds made only from especially large plaquettes (of order $\mathrm{toll} > \beta_f^{-0.49}$) and deviations (`spend`), and hit is the set of coarse plaquettes near them Source .

How it is paid Inferred : what fills the right-hand side directly is the crossing row $\mathbf{workL} + \mathbf{workD}$ on hit. The other three terms on the left are the accounting of its error.

- $(C/L^2)\, t \sum \mathrm{toll}$ is paid by the $(c_{\mathrm{lin}}/2)\, t \sum \mathrm{toll}$ of the lower bound for fine plaquettes and the penalty. This is why $L$ is taken large.
- The linear term in the deviations is paid by the quadratic of the lower bound for the penalty row.
- $Zt^2 R^4 \# A_{\mathrm{sp}}$ is paid by the linear action of the especially large plaquettes ($t \cdot \beta^{-0.49} \gg t^2 (\log \beta)^8$).

**The $L^{-2}$ shrinking.** The row sums of the blocking kernel $\Phi$ (sums per fine plaquette) are $C/L^2$, and the column sums (per coarse plaquette) $CL^2$ (the kernel row sums `Allowance`, :61) Source . Since the coarse curvature is roughly $\sum_p \Phi(p,q) X_p^f$, for linear quantities $\sum_q |\mathrm{curv}_q| \lesssim (C/L^2) \sum_p |\mathrm{curv}_p|$ Inferred . Quadratic actions keep their form under blocking, but the total of a linear action shrinks by $L^{-2}$. Collecting $L^4$ fine curvatures with weight $L^{-2}$ in four dimensions is consistent with $\sigma_c \approx L^2 \sigma_f$ for smooth fields Inferred .

05

## Assembly, and the factor that comes out of the integral

The remaining components are as follows Source (names and forms of the statements).

| Group name | Declaration | Content |
|---|---|---|
| E01 | `Payments` (:27) | Large coarse plaquettes lie only near especially large seeds, and their number is $\le K \# A_{\mathrm{sp}}$. The number of coarse flags is $\le 4\times$ (the number of plaquettes above the threshold $\tau$) |
| E03 | `Allowance` (:61) | Row sums of the kernel $\Phi$ are $C/L^2$, column sums $CL^2$ |
| E04 | `Maximum` (:83) | $\|\mathbf{vision}\| \le Ct$, $\|M_v\| \le (C/L^2)t$ |
| E08 | `Stitches` (:134) | Outside, where no seed is hit, both the coarse and fine fields are small in a local gauge |
| Smallness outside | `Belly` (:142) | The negative part of the outside rows is $\ge -\varepsilon t^2 \# A$ (for every $\varepsilon > 0$) |
| E09 | Same file | Assembles `Grants` from the above |

**Extracting a factor from the integral**: `Sector.peel` (`YM/Next/FLUCTUATIONS/Sector.lean:229`) Source says that when (1) holds on a region, for the integral over the region

$$\sum_{k} \int_{\text{region}} g \cdot \text{incoming} \;\le\; \text{pref} \cdot e^{-\beta c E t' \# A} \sum_{k} \int_{\text{region}} g \cdot \text{outgoing} \tag{6}$$

holds. outgoing is the exponential of the rows outside the neighbourhood. If $\beta$ is taken of the order of the fine-step coupling, then $\beta t^2 \approx P^2$ ($P = \exp((\log\log\beta)^2)$), so **a factor of order $e^{-cEP^2}$ per seed** comes out Inferred . This lemma does not say how large the outgoing integral is Source (11-05, 11-11).

$P^2$ grows faster than any power of $\log\beta$, so the halo volume $L^4 R^4$ and polylogarithmic losses can be paid with this factor (`Delay.inactive_price`, `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean:120`) Source (an asymptotic statement).

06

## What this book's study had seen

What was open in this book's study was the risk that, near configurations whose shape the block map does not change, the gains in the action balance out so that rarity does not come out at each step (the "large-field estimate" row in 28-06). The published construction **answers this in a different form**: it truncates the main part of the representation (the action handed to the coarse step) to linear, puts the difference into activities, and sets up an **inequality uniform over configurations satisfying the gauge section and the region conditions**, (1). The one-step map stays an identity, and the measure does not change. Statements (1)–(5) contain no hypothesis that excludes types of exceptional configuration Source .

What this book had in Lean was the rarity of large plaquettes at a single scale (the Wilson measure of the fine lattice itself) (22-01, from the Chessboard Estimate (Fröhlich–Israel–Lieb–Simon), uniform in the volume). It did not have a configuration-by-configuration inequality at the pushed-forward coarse scale.

07

## What has not been read

- The proofs of the three inequalities (`YM/Work/E05` – `YM/Work/E07`). Only the statements and docstrings were read.
- The details of the roles of the shell comparison (`Cops`, group name E02) and of the smallness outside (`Stitches`, E08), and the meaning of the tube fields of `Funds` (`Deep`, `Roll`).

- Whether `Grants` and `peel` lie on the chain of the existence theorem (the witness in the record is `SharpGram.habitat`).
- How the difference between the orientation-averaged action and the crown of a single orientation is counted across steps.

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source In the quantified form (1), $c_E t^2 \# A \leq$ assessed on a region (with the gauge section, the side lower bound and `Funds`). Holds for every order (`grants`) | A claim across steps (it is a one-step, configuration-by-configuration inequality) |
| Source Lower bound for coarse-plaquette rows (2): positive remainder $c_{\mathrm{dir}} \tau \sigma / 4$ on medium plaquettes | An explicit value of $c_E$ (only existence, depending on $L, W$) |
| Source Lower bound for fine plaquettes and the penalty (4), payment for large coarse plaquettes (5) | That this book has read the proofs of (2), (4), (5) |
| Source The integral over a region yields $e^{-\beta c_E t^2 \# A}$ (`peel`) | The size of the remaining integral (`peel` does not say) |
| Inferred The positive remainder comes from truncating the coarse action to linear | That the problem of persistent configurations disappears by this truncation alone (what can be said goes only as far as an inequality uniform over configurations satisfying the section and region conditions) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Order, section, quantifiers, energy per seed | Source | `YM/Pressure/Seeds.lean:48` (`SliceSet`), :89 (`Teller`), :101-127 (`Funds`, `live`, `Serves`, `cash`), :129 (`Grants`) |
| The three inequalities and components | Source | `YM/Bid/Energy.lean:27` (E01), :61 (E03), :83 (E04), :94 (`Crowds`), :101 (`def E05`), :103 (`Seals`), :110 (`def E06`), :121 (`Lapse`), :130 (`def E07`), :134 (`Stitches`), :142 (`Belly`), :155 (`issues`) |
| Holding without hypotheses | Source | `YM/Next/FLUCTUATIONS/Grants.lean:24` (`grants`), `YM/Integrated.lean:222`, `YM/Next/FLUCTUATIONS/Bare.lean:64` |
| Extracting a factor from the integral | Source | `YM/Next/FLUCTUATIONS/Sector.lean:229` (`peel`) |

| Item | Kind | Source / tool |
|---|---|---|
| Asymptotic lemmas | Source | `YM/SectorNext/LARGE_FIELD_REMAINDERS/Delay.lean:62` (`cover_pay`), :120 (`inactive_price`) |
| Dependency record | Source | `verification/declaration-usage.tsv` (`Teller.Grants`, `Crowds`, `Seals`, `Lapse`, `Sector.peel` are used, witness `SharpGram.habitat`) |

**Words used in this chapter** — seed, number of seeds, region, gauge section, positive remainder, blocking kernel, order bundle. Their meanings are in the A-02 glossary.

Next chapter to read: 11-05 Allocating prices, and polymer sums — the factor that came out is split into marks and combinatorial payments.

# Allocating prices, and polymer sums

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — How the per-seed factor $e^{-\beta c_E t^2}$ that came out of the region integral is distributed. Half of the guaranteed lower bound $c_E P^2/2$, namely $c_E P^2/4$, is left in the exit activities as a "mark", and the rest pays for the tree expansion of polymers, the covering combinatorics and the polylogarithmic costs. As a result, the pointwise sum of connected polymers containing a seed is bounded by a quantity proportional to $e^{-c_E P^2/4}$.

**Chapters assumed** — 11-04 Energy per seed (the energy-per-seed constant $c_E$ and `peel`), 7-04 The cluster expansion (the idea of the polymer expansion and tree estimates).

**Said first** — The formulas of the accounting fields were read in the source, but the proof of the polymer-sum estimate (Ursell coefficients and the tree inequality) has been read only as far as names and types. The declarations of this chapter are reached in the verification record from one of the headline theorems, but we have not checked whether they lie on the chain of the existence theorem.

The order of this chapter

1. Statement — the price fields and the mark
2. Statement — the pointwise sum of polymers containing a seed
3. Outline of the mechanism — why the super-polylogarithmic $P$ is enough
4. What follows — conversion into a rate
5. What this book's study had seen
6. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — the price fields and the mark

Notation: the reference coupling of the coarse step $x$, that of the fine step $\beta_f = x + s \log L$ ($s$ is the one-loop slope) (`betaF`), the fine threshold $t = \mathrm{fiery}(x, L)$, and $P = \exp((\log\log\beta_f)^2)$ (`fineMonitor`). Then $\beta_f t^2 = P^2$ Source (by definition). $c_E > 0$ is the energy-per-seed constant (the $c_E$ of 11-04; in the source's wording at this location, μ).

**Income.** The income of one seed is $\mathrm{gain} = \beta\, c_E\, t^2$ (`credit`, `YM/Push/ABSOLUTE_SUMS/Grants.lean:48`), and it depends on the actual $\beta$ Source. Under the band condition $\beta \geq u\,\beta_f$, the income is bounded below by

$$\mathrm{gain} = \beta c_E t^2 \;\geq\; \mathrm{hatWin}(c_E, u) = c_E\, u\, P^2 \quad (1)$$

(`Delay.comparison`, `YM/Push/ABSOLUTE_SUMS/Delay.lean:43`; `hatWin` is :28) Source. With $u = 1/2$, the **guaranteed lower bound** is $c_E P^2/2$.

The **price fields** `Buy.quote` (`YM/Push/ABSOLUTE_SUMS/Buy.lean:31`) fill the fields of the accounting type `Costs` (`Mixed.lean:26`) as follows Source. Where it is used, `Target.costs` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/OptionCost.lean:83`), $u = 1/2$.

| Field | Value ($u = 1/2$) | Role |
|---|---|---|
| gain | $\beta c_E t^2$ ($\geq c_E P^2/2$) | Income of one seed |
| markSeed (mark) | $\mathrm{hatWin}(c_E/2, 1/2) = c_E P^2/4$ | Rarity left in the exit activities |
| packetRatio | $e^{-c_E P^2/8}$ | Polymer ratio (convergence of the tree expansion) |
| rootHit | $e^{-c_E P^2/16} + \gamma$ | Weight of the root of the tree |
| spendSeed | $C \cdot \mathrm{bill}^5$, $\mathrm{bill} = 1 + R + (\log\beta_f)^2$ | Logarithmic cost per seed (polylogarithmic) |
| cap, extraSeed | $1, 0$ | Constants imposed on polymers |
| optionalHit $\gamma$ | $\beta_f$ ($-1/2 < r < 0$) | Additional weight |

The source's docstring says: "use half of the guaranteed income as the mark that is extracted" Source (docstring). Here **the guaranteed income is the lower bound $c_E P^2/2$ of (1)**, and the mark $c_E P^2/4$ is half of it. It is not half of the actual income $\beta c_E t^2$ (since $\beta$ moves within the band).

**Sufficient condition** `Buy.quoted` (`Buy.lean:67`) Source : fixing $c_E > 0$, $z < 2$, a constant $D \geq 0$ and a degree $d$, for sufficiently large $x$, if the logarithmic cost per seed and the logarithm of the polymer constant are at most $\text{fee} = D \cdot \text{bill}^d \cdot P^z$ and $\gamma$ is small, then the combinatorial condition `Buy` holds. `Target.pay_for` (`OptionCost.lean`) applies it with $d = 5$ Source .

02

## Statement — the pointwise sum of polymers containing a seed

The result of the one-step integral is rearranged into a sum over "polymers", finite sets of seeds and letters (such as the addresses of incoming activities) (11-06). The sufficient condition for a geometric series of polymer weights and covering weights is `Packets.income` and `fit` (`YM/Push/ABSOLUTE_SUMS/Charge.lean:22-38`) Source :

$$\text{income}(C, \text{spend}, \text{gain}, m) = C\, e^{(\text{spend}-\text{gain})\, m}, \qquad (\text{covering weight}) = e^{a\, \text{cov} B}. \tag{2}$$

The upper bound on the pointwise sum of connected polymers containing a seed, `Cash.stocks` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean`; `armK` is :41) Source : at each point $x$, the sum of covering weight $\times$ polymer weight over polymers $r$ that contain $x$, carry a mark, and have at least $m$ elements is

$$\sum_{r \ni x,\ \#r \geq m} (\cdots) \ \leq\ \text{armK} = 2^{-m}\, e^{-\text{mark}} \cdot \text{mergeScale} \cdot 2 \tag{3}$$

at most this. $\text{mark} = c_E P^2/4$, and $\text{mergeScale}$ is a quantity that goes to 0 at the rate of a negative power of $\beta$ Source (the type of the statement `tempo`).

**Which target it enters.** `Buy.quote`, `Cash.armK` and `dictionary_fit` are "used" in the verification record, and the witness in the record is `SharpGram.habitat` Source .

03

## Outline of the mechanism — why the super-polylogarithmic $P$ is enough

1. **How the income is split.** The field values are mark $c_E P^2/4$, polymer ratio $e^{-c_E P^2/8}$ and tree root $e^{-c_E P^2/16}$ Source (field values). Reading this as "splitting the guaranteed lower bound $c_E P^2/2$" is this book's reading ($1/4 + 1/8 + 1/16 = 7/16 \le 1/2$); it is not written in the source as a single account Inferred. The mark remains just once in the polymer sum (3) as $e^{-\mathrm{mark}}$, and the polymer ratio and root weight pay for the convergence of the tree expansion of connected polymers Inferred.
2. **Polylogarithmic costs.** The halo volume, the number of coverings, Haar losses and so on are of the order of powers of $\log \beta$ per seed, and are collected in the field spendSeed $= C\,\mathrm{bill}^5 \approx C(\log \beta)^{10}$ Source (read the field) / Inferred (reading of what goes into it).
3. **How $P$ grows.** $P = \exp((\log\log \beta)^2)$ grows faster than any power of $\log \beta$ and slower than any positive power of $\beta$. So $P^2$ always beats the cost $D\,\mathrm{bill}^d P^z$ ($z < 2$), while the threshold $t = \beta^{-1/2} P$ goes to 0 (`Delay.cover_pay`, `inactive_price`) Source (asymptotic statements).
4. **Why it is written as proportions.** The fields are all functions of $c_E$ and $P$, written as lower bounds that do not depend on where $\beta$ sits in the band. The induction hypothesis has no field that carries a decrease over to the next step Source (`Conv`), and $c_E$ and the rate are chosen afresh at each step. Conditions uniform in the step are paid for on the calendar's side (11-08, 11-07) Inferred.

04

## What follows — conversion into a rate

The polymer sum (3) is converted into a "rate" for the exit forced activities Source.

- `eventual_resize` (`YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Rent.lean:34`): if the rate $h \le c_E/16$, then for sufficiently large $x$, $\mathrm{armK} \le c \cdot C e^{-h\,(t_y/\mathrm{gap}\,\beta)^2}$. The source's docstring says: "updating the strong ledger needs a rate proportional to $P^2$, not $P^\delta$ ($\delta < 2$); read it back from the integrated estimate before dropping to a smaller exponent".
- `Pocket`, `pockets` (same file, :70, :76): for a given upper bound $H$ on the rate, choose rate $\le \min(H, c_E/64)$ and $h = c_E/32$.
- `dictionary_fit` (`Bulkhead.lean:78`): for any $0 \le \delta < 2$ and $A \ge 0$, for sufficiently large $x$, $\mathrm{armK} \le e^{-AP(x)^\delta}$. The $P(x)$ here is the value at the coarse-side reference coupling $x$.

The whole chain of conversion is read in 11-07.

05

## What this book's study had seen

This book's study thought of the rarity of large fields in the form $e^{-c/g^2} = e^{-c\beta}$, and assumed that the ledger of activities would carry it (the "large-field estimate" in 28-04). The per-seed price of the published construction is $e^{-cEP^2}$, a much weaker rarity than $e^{-c\beta}$. It suffices because the ledger $e^{-P^{1/5}}$ required by the induction hypothesis is weaker still Inferred . The form of accounting that splits the income in proportions and leaves a mark did not exist on this book's side either.

06

## What has not been read

- The proof of the polymer sum (3): Ursell coefficients and the Penrose-type tree inequality ( `YM/Push/ABSOLUTE_SUMS/Ursell.lean` , `Graph.lean` ), and the dictionary of connected activities ( `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6` ).
- The concrete lemma that puts the Haar loss into the spendSeed field (11-11).
- All the fields of the combinatorial condition `Buy` .
- The definition of $\mathrm{mergeScale}$, and the details of the statement ( `tempo` ) that it goes to 0 as a negative power of $\beta$.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source If $\beta \geq \beta_{\mathfrak{f}}/2$, the income $\beta c E t^2 \geq cEP^2/2$ (the guaranteed lower bound) | The value of the actual income ($\beta$ moves within the band) |
| Source The mark is $cEP^2/4$, half of the guaranteed lower bound $cEP^2/2$; polymer ratio $e^{-cEP^2/8}$; root $e^{-cEP^2/16} + \gamma$ | That the mark is "half of the income" (it is half of the lower bound) |
| Source Pointwise sum of connected polymers containing a seed $\leq 2^{1-m} e^{-cEP^2/4}$ $\mathrm{mergeScale}$ | The content of its proof (this book has not read it) |
| Source Costs up to $D$ $\mathrm{bill}^d P^z$ ($z < 2$) can be paid for sufficiently large $x$ ( `quoted` ) | A statement that the Haar loss goes into this field (unconfirmed) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Income and the guaranteed lower bound | Source | `YM/Push/ABSOLUTE_SUMS/Grants.lean:48` (`credit`), `Delay.lean:28` (`hatWin`), :43 (`comparison`) |
| Price fields | Source | `YM/Push/ABSOLUTE_SUMS/Mixed.lean:26` (`Costs`), `Buy.lean:31` (`quote`), :67 (`quoted`), `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/OptionCost.lean:83` (`Target.costs`) |
| Polymer weights | Source | `YM/Push/ABSOLUTE_SUMS/Charge.lean:22-38` (`income`, `fit`) |
| Polymer sums | Source | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean:41` (`armK`), `stocks`, :78 (`dictionary_fit`) |
| Conversion into a rate | Source | `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Rent.lean:34` (`eventual_resize`), :70 (`Pocket`), :76 (`pockets`) |

**Words used in this chapter** — price, income, guaranteed lower bound, mark, polymer, polymer ratio, covering weight, polylogarithmic cost. Their meanings are in the A-02 glossary.

Next chapter to read: 11-06 Reorganising activities at the coarse step — polymers are split into those that contain a coarse flag and those that do not, and entered into the exit representation.

# Reorganising activities at the coarse step

This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — Where, after the result of the one-step integral has been rearranged as a sum over polymers, each piece goes in the representation at the coarse step. Polymers containing a coarse flag go into the gas of exit forced activities; polymers not containing one go to the side of regular activities. The constants of the latter flow to the vacuum term, and their quadratic parts at the anchor flow to the correction of the coupling.

**Chapters assumed** — 11-01 (representation and gas), 11-05 (polymers and marks).

**Said first** — This chapter reads the **classification and the formulas for where things go** in the reorganisation. The map itself by which the values of polymers not containing a coarse flag enter the addresses of regular activities has not been read by this book. That part is written under Inferred and listed in § 06.

The order of this chapter

1. Statement — two kinds of polymer, and the value of the gas
2. Statement — fusing the hard cluster and the soft cluster
3. Outline of the mechanism — where defects that have returned to small field go
4. What could be read, and the map that has not been read
5. What this book's study had seen
6. What has not been read

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Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — two kinds of polymer, and the value of the gas

The quantities produced by the one-step integral are rearranged into a sum over polymers (`Molecule`: a finite set of seeds and letters). Polymers are split in two by the set of coarse flags of the combined seed types of a polymer $M$ (`roster`) (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Colony.lean:26-27`) Source .

$$\mathrm{ThickP} = \{M : \mathrm{roster}(M).\mathrm{flags} \neq \emptyset\}, \qquad \mathrm{ThinP} = \{M : \mathrm{roster}(M).\mathrm{flags} = \emptyset\} \tag{1}$$

(both with non-empty support). We call polymers containing a coarse flag "flagged polymers" and those not containing one "unflagged polymers".

**The gas of flagged polymers** (`banked`, :43) Source : for a finite set $\mathcal{A}$ of flagged polymers, it takes the value of the gas of those with pairwise disjoint supports only when the union of their flag inventories equals the flags of the coarse field $\mathbf{flags}_{t_m}(V)$ exactly; otherwise it is $0$. This has the same form as the gas of the coarse-step representation (equation (6) of 11-01 § 03).

The values of **unflagged polymers** (`entries`) are arranged as a separate gas (`ignores`, `inlet`), avoiding the supports of the flagged polymers Source (form of the definition).

**The vacuum term** (`Collection.vac`, `YM/AnalyticBridge/VERIFIED_RESET_STEP/Evaluation.lean:29`) Source : the exit vacuum constant is

$$v_{\mathrm{out}} = \tau - \frac{1}{|\Lambda_m|} \sum_i \log\text{-value}_i(1) \tag{2}$$

Here $\tau$ is a constant coming from the normalisation of the integral, and $\log\text{-value}_i(1)$ is the value at the flat configuration of each piece on the regular-activity side.

02

## Statement — fusing the hard cluster and the soft cluster

The addresses of the exit forced activities are made by joining two clusters Source .

- **Hard cluster**: the values of flagged polymers. The pointwise sum is bounded by $\mathrm{strong} = C\, e^{-r(t_y/\mathrm{gap}\,\beta)^2}$ ($r$ is the rate, $y$ the reference coupling of the fine step;

`Pairing.strong`, `YM/Progress/DIRECTED_RECIPE/Economy.lean`).

- **Soft cluster**: small-field logarithmic pieces with support on the unhealthy part of the coarse step. The sum is $\mathrm{mild} = C(1+B)^d P^e \,\mathrm{gap}^p$ $(p > 0)$, which becomes small at the rate of a positive power of $\beta$ (`Pairing.mild`, same file).

The fusion condition `Rates` (`YM/Progress/DIRECTED_RECIPE/Tender.lean:33`) and the theorem `tender` (:124) Source : for an exponent $k \geq 0$ ($\kappa$ in the source), a hard sum $a$ and a soft sum $b \leq 1$, if

$$e^{-k}\left(e^{\,e^{k}a+2b}-1\right) \;\leq\; w \quad (3)$$

(the left-hand side is `Bid.quote`, `YM/Bid/Kitchen.lean:18`) and the covering-number condition $(2B+1)^4(e^k a + 2b) \leq 1$ hold, then the fused activities satisfy `KBound` with ledger $w$. `Pairing.currency` (`Economy.lean:76`) satisfies `Rates` with $w = e^{-k}$, $k = \mathrm{kGoal} = \mathrm{rate} \cdot (t_y/\mathrm{gap}\,\beta)^2$, $a = \mathrm{strong}$, $b = \mathrm{mild}$. Since the rate satisfies $r > \mathrm{rate}$, $e^k\,\mathrm{strong} = Ce^{-(r-\mathrm{rate})(\cdots)^2} \to 0$ Source (statement read) / Inferred (reading of why $e^k a$ becomes small).

Each address contains a coarse flag (the `permit` of forced activities), so we read it as containing at least one hard piece, and $e^{-k}$ is applied once Inferred .

03

## Outline of the mechanism — where defects that have returned to small field go

Consider the case where a plaquette (seed) that was large at the fine step does not exceed the threshold $t_m$ at the coarse step, that is, **returns to small field at the coarse step**. A polymer containing that seed contains no coarse flag, so it is an unflagged polymer and does not enter the gas of forced activities. The gas is 1 when the flags are empty (`gas_one`), so there is no place to keep it as a forced activity in the first place Source .

We read where it goes as follows Inferred .

1. The value of an unflagged polymer goes to the side of regular activities (which live only where healthy).
2. Its value at the flat configuration moves into the vacuum term $v$ by equation (2).

3. Its quadratic part in the curvature direction at the anchor moves into $\beta$ as the correction of the coupling $\Delta$ (`Averaging.delta`, 11-09 § 06).
4. The rest is normalised to "0 at the flat configuration, 0 in the quadratic of the affine directions" (`rebate`) and enters the regular-activity ledger $\hat{g}^{11/5}$. Because of the seed's mark $e^{-cEP^2/4}$, it falls well below the ledger.

If it is still large at the coarse step, it becomes the inventory of a new forced activity as a flagged polymer, and need only fit within the ledger $e^{-P^{1/5}}$ (11-07). This split handles the case distinction of whether a defect has returned with a single local test: whether a coarse flag is contained Inferred .

04

## What could be read, and the map that has not been read

| Component | Level | Location |
|---|---|---|
| Definition of the two kinds of polymer (1) | Source | `Colony.lean:26-27` |
| Value of the gas of flagged polymers (exact agreement with the flags) | Source | `Colony.lean:43` (`banked`) |
| Values of unflagged polymers enter the Ursell terms and, if non-zero, are healthy at the coarse threshold | Source | `Fire.lean:43` (`trained`), `Street.lean:68` (`clean_claim`) (same location) |
| Formula (2) for the vacuum term | Source | `Evaluation.lean:29` |
| Fusion condition (3) and `currency` | Source | `Tender.lean:33`, :124, `Economy.lean:76` |
| The map by which values of unflagged polymers enter the addresses of regular activities (log-values) | Inferred | From `entries` onward. Not read |
| Handling of the addresses of incoming activities (footprint `forcedFoot`, covering cost `forcedExtra`) | Source definitions only | `Inherited.lean:150`, :162 |

**Which target it enters.** `Collection.vac` is "used" in the verification record, with witness `SharpGram.habitat`. $v_F = \mathrm{vac}$ enters the conclusion of the small-field one-step theorem `Return.reset`, and `reset` lies on the chain of the existence theorem (11-09) Source .

05

## What this book's study had seen

Because this book's study fixed large-field regions thereafter, it needed separate treatment for the case where a defect returns to small field at the coarse step. The

published construction makes this case distinction by a local test of whether a coarse flag is contained, and puts what has returned onto the vacuum term and the flow of the coupling Inferred . This pairs with the mechanism by which the correction of the coupling is fixed as a coefficient in the representation (10-03).

06

## What has not been read

- The map that puts the values of unflagged polymers into the addresses of regular activities (from `entries` of `Colony` onward, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6`).
- The handling of incoming activities (`Sales`, `ratesCombine`, `OptionCost.lean`) and what follows.
- The proofs of the reorganisation lemmas (`Marlin`, `Trio`, `Meadow` in `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP`).
- The whole construction that rebuilds new forced activities after one step and fits them into the ledger $e^{-P_m^{1/5}}$ (11-11).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Polymers split in two by whether they contain a coarse flag, and the gas of flagged polymers has a value only when it agrees exactly with the flags | The map by which values of unflagged polymers enter the addresses of regular activities (not read) |
| Source The exit vacuum term is $\tau - \sum \log\text{-value}(1)/\lvert\Lambda\rvert$ | An estimate of the content of $\tau$ (normalising determinants and so on) |
| Source `currency` satisfies the fusion condition (3) for the hard and soft clusters | The details of the proof of fusion |
| Inferred Defects that return to small field at the coarse step flow into the vacuum term and the correction of the coupling | That this flow is written as a statement |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Two kinds of polymer, value of the gas | Source | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Colony.lean:26`, :27, :43 |

| Item | Kind | Source / tool |
|---|---|---|
| Addresses of incoming activities | Source | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Inherited.lean:150` (`forcedFoot`), :162 (`forcedExtra`) |
| Vacuum term | Source | `YM/AnalyticBridge/VERIFIED_RESET_STEP/Evaluation.lean:29` (`vac`) |
| Fusion | Source | `YM/Progress/DIRECTED_RECIPE/Economy.lean:47` (`kGoal`), :76 (`currency`), `Tender.lean:33` (`Rates`), :124 (`tender`), `YM/Bid/Kitchen.lean:18` (`quote`) |

**Words used in this chapter** — polymer, flagged polymer, unflagged polymer, hard cluster, soft cluster, vacuum term, correction of the coupling. Their meanings are in the A-02 glossary.

Next chapter to read: 11-07 Converting between ledgers — the strong estimate at the exit is moved to the weak ledger of the induction hypothesis.

# Converting between ledgers, and persistence across steps

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — The final part of the chain of conversions from price to rate. The forced activities at the exit of one step are first bounded in the strong form $e^{-\text{rate}\cdot P_y^2}$, and this is weakened in two ways: to $e^{-dP_y^2}$, recorded in the one-step certificate, and to $e^{-P_x^{1/5}}$, carried by the induction hypothesis. We then read how defects that persist over many steps are handled under these ledgers.

**Chapters assumed** — 11-00 § 03 (the two ledgers), 11-05 (marks and conversion into a rate), 11-06 (how the exit activities are made).

**Said first** — The weak ledger is also a function of step $m$. The conditions for making things hold uniformly in the step are paid for on the side of the calendar (the entry for β, the lower bound on the sides, the choice of rate). The estimate of incoming activities does not enter the formula for the exit rate, but it is used on the side of the one-step hypotheses to make sums converge. The conversion lemmas of this chapter are reached in the verification record from one of the headline theorems, but we have not checked whether they lie on the chain of the existence theorem.

The order of this chapter

1. Statement — the chain of conversions from price to ledger
2. Statement — the one-line inequality that lowers the exponent
3. Carrying the one-step certificate
4. Outline of the mechanism — defects that persist over many steps
5. What this book's study had seen
6. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — the chain of conversions from price to ledger

Notation: the reference coupling of the fine step $y$, that of the coarse step $x$, $P_y = P(y) = \exp((\log\log y)^2)$, and the threshold $t_y$. Let $c_E$ be the energy-per-seed constant. The conversion within one step is the following chain Source (statements at each stage).

| Order | Quantity | Location |
|---|---|---|
| 1 | $c_E t^2 \#A \le$ assessed on a region | `Teller.Grants` (11-04) |
| 2 | Half of the guaranteed lower bound $c_E P_y^2/2$ becomes the mark $c_E P_y^2/4$ | `Buy.quote` (11-05) |
| 3 | Pointwise sum of polymers containing a seed $\le \mathrm{armK} \propto e^{-c_E P_y^2/4}$ | `Cash.armK` |
| 4 | The estimate with rate $h \le c_E/16$, $\mathrm{armK} \le c\,C e^{-h(t_y/\mathrm{gap}\,\beta)^2}$ | `eventual_resize` |
| 5 | Choose $h = c_E/32$ and rate $\le c_E/64$ | `Pocket`, `pockets` |
| 6 | Fused exit activities $\le e^{-\mathrm{kGoal}}$, $\mathrm{kGoal} = \mathrm{rate}\cdot(t_y/\mathrm{gap}\,\beta)^2$ | `tender`, `Pairing.currency` (11-06) |
| 7 | $e^{-\mathrm{kGoal}} \le e^{-dP_y^2}$ and $\le e^{-P_x^{1/5}}$ | `Time.late_force` (§ 02 below) |

For $\beta$ in the band, $t_y/\mathrm{gap}\,\beta$ is of the order of a constant multiple of $P_y$, so $\mathrm{kGoal}$ is of the order of $\mathrm{rate}\cdot P_y^2$ Inferred (from the comparison $P_y \le 2\,t_y/\mathrm{gap}\,\beta$ inside the proof of `late_force`).

**Summary of the one-step exit**: `Time.now` (`YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:84`) Source bundles the exit ledger of regular activities $\mathrm{forecastE} = \mathrm{credit}\cdot\hat{g}_m^{11/5}$ and the exit estimate of forced activities (`forecastK`, the strong of the hard cluster), and from the same estimate produces two things (relaxing the ledger with `K_rises`): the induction-hypothesis form $\|K\| \le e^{-P_m^{1/5}}$, and the one-step-certificate form $\|K\| \le e^{-d\,P_{m+1}^2}$.

02

## Statement — the one-line inequality that lowers the exponent

`Time.late_force` (`Opening.lean:19`) Source : for $d > 0$ and a constant $c$, for sufficiently large $x$, for every rate satisfying $d \le \mathrm{rate}/4$ and every $\beta$ in the band of $y = x + c$, $r = e^{-\mathrm{kGoal}(y,\beta)}$ satisfies

$$r \le e^{-d\,P_y^{2}}, \qquad r \le e^{-P_x^{1/5}} \tag{1}$$

The $d$ here is the same as the one-step certificate's coefficient in § 01 and § 03 (`Time.now` checks $d \le \mathrm{rate}/4$ for the coefficient chosen by the calendar and uses `late_force`) Source . The second half comes from `Postage.saved` (`YM/Next/SCHEDULE/Salary.lean:178`) Source : when two scales are close (`Alike`: including $P_x \le 2P_y$ and so on), for sufficiently large $y$

$$e^{-d\,P_y^{2}} \le e^{-P_x^{1/5}}. \tag{2}$$

The proof is the single line $P_x^{1/5} \le (2P_y)^{1/5} \le 2P_y \le dP_y^{2}$, using the exponent $1/5 \le 1$ Source . A form in which any exponent below $2$ will do is stated elsewhere: `Cash.dictionary_fit` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean:78`) gives $\mathrm{armK} \le e^{-AP(x)^{\delta}}$ for any $0 \le \delta < 2$ ($P(x)$ is the coarse-side value) Source .

Note: `late_force` does not say that $r$ is an upper bound for the actual activities. That is produced separately by items 1–6 of the chain in § 01 Source .

03

## Carrying the one-step certificate

The field `forced` of the one-step certificate `Directed.Move` (`YM/PCalc/Band.lean:49`) states $\|K\| \le e^{-d\,P_{m+1}^{2}}$ for the forced activities of the representation $u$ at step $m$. $P$ is the value at the **fine step** $m+1$ Source .

The induction hypothesis `Conv` and the input conditions of the one-step theorem (`Program.advances`, :65) carry only the weak ledger; the strong ledger does not enter Source . The lemma that records the strong ledger for each Move, `Directed.holding` (:92), is **outside the dependencies** Source (record). On the chain of the existence theorem, `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean:76-89`) packs the

Move from the one-step result into the step record (`tunes.next`) and carries it along Source . Who uses `Move.forced` at later steps, this book has not checked.

04

## Outline of the mechanism — defects that persist over many steps

**What can be read** Source .

1. The large-field part of the induction hypothesis at each step $m$ is only "the covering-weighted sum of forced activities $\le e^{-P_m^{1/5}}$", and it **has no field for the history of regions** (`Rep`, `Conv`, `YM/RG/History.lean:165-218`).
2. Forced activities are different from 0 only on the flags of that step (`permit`, `admissible`).
3. Wherever the weight of the partition of unity is not 0, the flags of the fine step always touch a seed of this step (`flagged`).
4. The formulas for the exit rate (kGoal, strong, mark) are functions of $c_E$, $P$ and $\beta$, and **the estimate of incoming activities does not enter these formulas** (`Economy.lean:40-47`, `Threshold.lean:66`).
5. The estimate of incoming activities is used on the **hypothesis side** of one step: in the field `intake` of `Target.Invoices` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Issue.lean:73-88`) and in the sum condition `Sales`, which includes the covering cost `forcedExtra` of incoming addresses. It is used to make sums converge.
6. Energy per seed (`Grants`) is stated for all configurations satisfying the gauge section and the region (those satisfying the global region condition). Configurations that keep just exceeding the threshold of each step at every step enter the same inequality.

**Assembling the reading** Inferred : a defect that persists over many steps is recounted as a seed at each step, receives a new price $e^{-cEP^2}$ from the action difference of that step, and if still large at the coarse step becomes the inventory of a new forced activity as a flagged polymer. The rarity paid at the previous step is relaxed to the weak ledger (`K_rises`), and at the next step it is extracted afresh from the action difference of that step. Along the chain of conversions the constants decrease by some fraction at each step, but the induction hypothesis has no field that carries the decrease over to the next step Source (`Conv`), and $cE$ and the rate are chosen afresh at each step.

**Where conditions uniform in the step are paid for** Source : the weak ledger $e^{-P_m^{1/5}}$ and the threshold $t_m$ are both functions of step $m$. The "sufficiently large $x$" conditions needed for this conversion to hold at every step are paid for step by step as constants

of the calendar: the entry for β (`Delays.betaEntry`), the lower bound on the sides (`floorL`), the choice of rate (`Pocket`), and the exit conditions (`Opened`). For details see 11-08.

In summary, for persistent defects the published construction does not build a single estimate uniform in the number of steps; it builds, at every step, an estimate that returns to a weak ledger of the same form, and pushes the part uniform in the step onto the calendar's conditions Inferred .

05

## What this book's study had seen

This book's study stacked up and fixed large-field regions step by step, and thought it necessary to keep the rarity of defects that persist near configurations whose shape the block map does not change uniform in the number of steps (the "large-field estimate" in 28-06). The published construction **answers this in a different form**: it keeps no history of regions, recounts seeds at every step, and, with an inequality uniform over configurations satisfying the gauge section and the region conditions, produces a factor per seed at every step. It does not change the measure.

The components this book's study lacked were the chain of conversions of this chapter and the construction of "two ledgers" Inferred . Replacing this book's ledger $e^{-c/g^2}$ by "a weak ledger and rebuilding at every step" is material for the comparison chapter 14-04.

06

## What has not been read

- The whole construction that rebuilds new forced activities after one step and fits them into the ledger (`YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP`, `V9_JOINT_ENDPOINT_LOAD`, `ANNULUS_RATIO_INSERTIONS`). The balancing of the reading assembled in § 04 lies here.
- How incoming activities are handled inside the one-step integral (`Sales`, `ratesCombine`, from `OptionCost.lean:39` onward).
- Where `Move.forced` is used at later steps.
- That the coarse action subtracted at step $m$ has the same path data as the fine action at step $m-1$, so that action differences are not counted twice across steps. The form including the difference between the orientation-averaged action and the action of a single orientation has not been checked.

- Whether `late_force` and `saved` lie on the chain of the existence theorem (the witness in the record is `SharpGram.habitat`).

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The exit estimate $e^{-\mathrm{kGoal}}$ is at most both $e^{-dP_y^2}$ ($d \le$ rate$/4$) and $e^{-P_x^{1/5}}$ (`late_force`) | That $e^{-\mathrm{kGoal}}$ is itself an upper bound for the activities (another lemma provides that) |
| Source If $P_x \le 2P_y$, then $e^{-dP_y^2} \le e^{-P_x^{1/5}}$ (the proof uses exponent $\le 1$). The form with any $\delta < 2$ is `dictionary_fit` | That "1/5 has some meaning" |
| Source The induction hypothesis carries only the weak ledger, and has no field for the history of regions | Where the strong ledger is used at later steps (not checked) |
| Source The estimate of incoming activities does not enter the formula for the exit rate, and is used in the one-step hypotheses (intake of `Invoices`, `Sales`) | That the estimate of incoming activities is not needed for one step (it is used on the hypothesis side) |
| Source Conditions uniform in the step are paid for on the side of the calendar's constants (entry for β, lower bound on the sides, choice of rate) | That no condition uniform in the step is needed (it is paid for on the calendar's side) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Conversion that lowers the exponent | Source | `YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:19` (`late_force`), :84 (`now`), `YM/Next/SCHEDULE/Salary.lean:178` (`saved`), `YM/Next/SCHEDULE/Dues.lean:57` (`w`), :118 (`Alike`) |
| Any exponent $\delta < 2$ | Source | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean:78` (`dictionary_fit`) |
| Exit rate | Source | `YM/Progress/DIRECTED_RECIPE/Economy.lean:47` (`kGoal`), `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Rent.lean:34`, :70, :76 |
| One-step certificate and carrying it | Source | `YM/PCalc/Band.lean:49` (`Move`), :65, :92 (`holding`, outside the dependencies), `YM/FullVolume/C/Songbook.lean:76` (`supplier`) |

| Item | Kind | Source / tool |
|---|---|---|
| Where the estimate of incoming activities is used | Source | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Issue.lean:73` (`Invoices`) |

**Words used in this chapter** — ledger, strong estimate, weak ledger, rate, one-step certificate, persistent defect, calendar. Their meanings are in the A-02 glossary.

Next chapter to read: 11-08 The calendar — the order in which constants are chosen — how the constants that pay for conditions uniform in the step are chosen.

# The calendar — the order in which constants are chosen

This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — The statements of the one-step estimates stack conditions of the form "if $L$ is sufficiently large", "if $W$ is sufficiently large", "if $x$ is sufficiently large". The calendar is the arrangement of in what order, and to which constants, these "sufficiently" conditions are charged. The order is: step constants → block side → margin of the regions → constant of the energy lower bound → rate → entry for the coupling.

**Chapters assumed** — 10-05 Why the sides grow, and the calendar, 8-02 The set-up, 11-04 and 11-07 (statements carrying "sufficiently large").

**Said first** — The calendar's types fix the order of dependence of the constants by type, but, as the source's docstring says, the types themselves do not check "that the renormalisation argument actually fits that order". That is the job of the proofs on the side of the one-step theorem. The reason the sides grow (dividing the one-step error by $\log b$) is the subject of Part 10; this chapter writes only how it looks from the large-field side.

The order of this chapter

1. Statement — the calendar and its fitting conditions
2. Statement — the types for the order of choosing constants
3. Statement — the order of large-field prices (pricing_order)
4. Outline of the mechanism — why it wins even though $c_E$ depends on $L$
5. What this book's study had seen
6. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — the calendar and its fitting conditions

**Set-up** `Schedule` (`YM/Lattice/Scale.lean:28`) Source : block ratios $b_m$ (odd, $\geq 3$) and volume periods $L_p$ (multiples of 4, strictly increasing). $D_N = \prod_{k<N} b_k$, spacing $a_m = 1/D_m$. The condition `Gentle` (:129) is $\log b_m / \log D_m \to 0$.

**Calendar** `Calendar` (`YM/RG/History.lean:64`) Source : the set-up, the choice of kernel, the reference coupling $\bar\beta$, the table of step constants `stage`, the table of sides `factor`, $\theta$, and the number of convolutions $J$ ($\geq 4$). From these the per-step quantities are fixed ($s = 11C_g/(24\pi^2)$ is the one-loop slope):

$$\hat\beta_m = \bar\beta + s\log\tfrac{1}{a_m},\quad P_m = e^{(\log\log\hat\beta_m)^2},\quad t_m = \hat\beta_m^{-1/2}P_m,\quad B_m = \lceil(\log\hat\beta_m)^2\rceil. \tag{1}$$

Main fields of the **fitting condition** `Calendar.Fits` (:97) Source :

- `mesh` : `Gentle`. `sameFactor` : $b_m$ is the $J(m+1)$-th entry of the table of sides. `monotoneL`, `mono` : the sides and $J$ are non-decreasing. `stepSlow` : $J(m+1) \leq J(m) + 1$. `deep` : $J \to \infty$.
- `delayed` : $\hat\beta_m > 1$. `smallField` : $t_m \leq 1$.
- `sizeT` : side of the lattice at step $m \geq 100(B_m + 1)$. `fitsPatch` : side of the fine lattice $\geq 100$(patch radius $+1$).
- `driftSmall` : drift$/\log b_m \leq J(m+1)^{-2}$. `logCost` : $\log(1 + \text{sum of step constants})/\log b_m \leq J(m+1)^{-2}$.

The last two are the content of the condition that the sides grow: the logarithmic cost of the step constants and the deviation from one loop are divided by $\log b_m$ and made smaller than $J^{-2}$. Together with $J \to \infty$, this gives $b_m \to \infty$ Source (read the fields) / Inferred (reading of the derivation).

02

## Statement — the types for the order of choosing constants

`YM/Price/Intent.lean` fixes the order in which constants are chosen by nesting types Source . The source's headings say "the order and constants chosen before the block ratios", "holds for any constants already chosen", and "the types themselves do not check whether the renormalisation argument fits that order of dependence" Source (docstrings).

| Order | Type | What it fixes |
|---|---|---|
| 1 | `Sheet` (:26) | The choice of kernel, the table of step constants profile, the first index, the table of lower bounds on the sides floorL (before the block ratios) |
| 2 | `Rolls` (:50) | Sides $L_j$. For each $j$, `Large` (:39): odd, $\geq 3$, $\geq \mathrm{floorL}(j)$, the one-step movement (drift) at most $(s/4)\log L$ (`kick`) and $\mathrm{drift}/\log L \leq j^{-2}$ (`slow`), $\log(1+\text{constants})/\log L \leq j^{-2}$. $L_j$ is non-decreasing |
| 3 | `Protection` (:88) | $\theta$ |
| 4 | `Delays` (:115) | The entry for the coupling betaEntry (after the sides) and so on |
| 5 | `Bar` (:123) | The reference value $\bar{\beta}$ ($>1$, at least the first entry `ready` (:126), and $x^{-1/2}e^{(\log\log x)^2} \leq 1$ for all $x$ above it `threshold` (:127)) |
| 6 | `Wait` (:134) | Waits and windows, minimum volume |
| 7 | `Levels` and `Paid` (:145) | The calendar. `pays` (:154): $\mathrm{betaEntry}(J(m+1)) \leq \hat{\beta}_m$. `volume`: lower bound on the volume side |

`Paid.pays` is where the "entry for β" is paid by the reference coupling of the step. The entry value itself (`Delays.betaEntry`, :116) is extracted as an existence from a bundle of properties that "hold for sufficiently large $x$" (`open_shop`); there is neither a formula nor a number for it in the source Source . From the reference-value condition `Bar.threshold` alone, $\bar{\beta}$ is at least about $9\times10^5$ Reading (the larger root of $\frac{1}{2}\log x = (\log\log x)^2$, $x \approx 8.97\times10^5$, found numerically). `logCost` and `driftSmall` of `Calendar.Fits` are the fields of `Large` read with $L = b_m$ Source .

03

## Statement — the order of large-field prices (pricing_order)

`Bid.pricing_order` (`YM/Bid/Energy.lean:173`) Source : if energy per seed `Grants` holds for all $(j,i)$, then there is a table of lower bounds on the sides $k$ such that, for any table of lower bounds $n$ at least $k$, any choice of sides fitting it (`Rolls`), any $\theta$, and any choice of entries $V$, there is a $W$ that raises only the entry ($V.\mathrm{betaEntry} \leq W.\mathrm{betaEntry}$, other fields the same), and the price condition (`Ringing`) holds up to any height.

The $i$ in $\exists i\ \forall L \geq i$ of the `Grants` statement becomes the side lower bound $k$ (including the $N$ of the lower bound for fine plaquettes and the penalty). Since `Grants` holds for every order (`Next.Fluctuations.grants`), this ordering comes out without hypotheses as `pressure_pricing` (`YM/Next/FLUCTUATIONS/Grants.lean`) Source .

We read the order in which the one-step large-field constants are chosen as follows Source (read the dependencies of each type and field) / Inferred (arranging them in one line is this book's doing).

$$\underbrace{L}_{\text{side}} \to \underbrace{W}_{\text{region margin}} \to \underbrace{cE}_{\text{energy lower bound}} \to \underbrace{\text{rate},\ h}_{\text{rate (Pocket)}} \to \underbrace{g \leq \text{rate}/4}_{\text{certificate rate}} \to \underbrace{\text{betaEntry}}_{\text{coupling entry}} \Big($$

- $L$: at least the $i$ of `Grants` (makes the fine-plaquette row beat the cross term $(C/L^2)t$).
- $W$: the $\forall^\infty W$ of `Grants`. The penalty row pays for sufficiently large deviations.
- $cE$: the $\exists cE$ of `Grants` ($\kappa$ in the source). It may depend on $L$ and $W$.
- Rate: `Pocket` gives $h = cE/32$ and rate $\leq cE/64$ (11-05 § 04).
- $g$: the coefficient of the one-step certificate's ledger (`PostCost`, `YM/PCalc/Loop.lean:81`). `Time.now` takes $g \leq \text{rate}/4$ as a hypothesis (`Opening.lean:84`).
- betaEntry: all the "if $x$ is sufficiently large" conditions above are paid here together. The exit condition `Opened` (`Opening.lean:45`) says that the one-step price can be paid for all $x$ above the entry, and `open_shop` raises the entry to produce it Source .

04

## Outline of the mechanism — why it wins even though $cE$ depends on $L$

1. The constant of the energy lower bound $cE$ depends on $L$ and $W$. The losses (halo volume $L^4R^4$, polylogarithmic factors) also depend on $L$ Source . We read the Haar loss as being of the same type, but have not checked where that lemma is Inferred .
2. But the price per seed is $cEP^2$, and $P = \exp((\log\log\beta)^2)$ grows with $\beta$ faster than any power of $\log\beta$. If $L$ is held fixed and the entry for β is raised, $cEP^2$ beats every loss depending on $L$ Inferred .
3. Raising the entry for β becomes, on the calendar, the condition "the reference coupling of step $m$, $\hat{\beta}_m$, is at least the entry" (`pays`). Since $\hat{\beta}_m$ increases with $\bar{\beta}$ and

$\log(1/a_m)$, it is paid by taking $\bar{\beta}$ large (`Bar`) and by $\hat{\beta}$ being larger at finer steps Inferred.

4. The sides $L_j$ grow with the step (`Rolls.grow`). The conditions of `Large` are conditions on ratios relative to the constants of step $j$, so even if the step constants get worse with the step, the sides are grown faster and absorb them Source (read the fields) / Inferred (reading of the meaning).

This whole ordering is the content of "conditions uniform in the step are paid for on the calendar's side" in 11-07. Rebuilding the weak ledger $e^{-P_m^{1/5}}$ at every step goes together with choosing the entry, the sides and the rate afresh at every step.

05

## What this book's study had seen

This book's study also had three moves as candidates: growing the sides, fixing the step constants first, and taking the ledger weak (28-05). Under a design with fixed sides, it was looking for a way to keep the rarity of large fields uniform in the step. The published construction uses all three and fixes the order of dependence of the constants by type. This book had nothing corresponding to these "types for the order of choice".

06

## What has not been read

- The content of the price condition `Ringing` (`Stamp` at `YM/Pressure/Seeds.lean:155`, `Ringing` at :161).
- The explicit form of the condition that $\bar{\beta}$ be taken large (what can be said from the statements is only $\bar{\beta} > 1$ and that it is at least the entry).
- Which estimates require the window of `Wait` (`window`) and the lower bound on the volume `volume`.
- This book has not traced along the chain that a concrete calendar satisfying the whole calendar exists (the construction of a `Calendar` satisfying `Fits`).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The calendar fixes $\hat{\beta}_m$, $P_m$, $t_m$, $B_m$ and the ledgers step by step, and `Fits` imposes conditions on the sides, constants and lattice size | That the types themselves check the order of dependence of the renormalisation argument (denied by the source) |
| Source The order of the constant types is Sheet → Rolls → Protection → Delays → Bar → Wait → Levels, Paid | The roles of all the fields of each type |
| Source `pricing_order`: builds the side lower bound from `Grants` and returns a choice in which only the entry is raised | The content of `Ringing` |
| Inferred Large-field constants are chosen in the order L → W → c_E → rate → g → entry, and "sufficiently large x" is paid at the entry | That this order is written in a single declaration (it was arranged from the dependencies of each type) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Set-up | Source | `YM/Lattice/Scale.lean:28` (`Schedule`), :129 (`Gentle`) |
| Calendar and fitting conditions | Source | `YM/RG/History.lean:64` (`Calendar`), :79-86 (per-step quantities), :97 (`Fits`) |
| Types for the order of choosing constants | Source | `YM/Price/Intent.lean:26` (`Sheet`), :39 (`Large`), :50 (`Rolls`), :115 (`Delays`), :123 (`Bar`), :145 (`Paid`), :154 (`pays`) |
| Order of prices | Source | `YM/Bid/Energy.lean:173` (`pricing_order`), `YM/Next/FLUCTUATIONS/Grants.lean:32` (`pressure_pricing`) |
| Certificate rate and exit condition | Source | `YM/PCalc/Loop.lean:81` (`PostCost`), `YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:45` (`Opened`), :84 (`now`) |

**Words used in this chapter** — calendar, set-up, block ratio, lower bound on the sides, entry for the coupling, rate, region margin. Their meanings are in the A-02 glossary.

Next chapter to read: 11-09 The small-field step — the Gaussian integral away from the seeds, and the ledger of regular activities.

# The small-field step

This chapter is established knowledge (a reading of the published construction). § 08 is this book's own investigation and § 09 is a map of open questions.

**What this chapter shows** — The part of the one-step integral that treats the region away from the seeds as a Gaussian integral. The push-forward value is rewritten into the coarse-step representation, without approximation, through three identities, and fitted into the regular-activity ledger $\hat{g}^{11/5}$. The key is to replace the covariance by a windowed approximate inverse and to keep its error inside the integrand.

**Chapters assumed** — 11-01 (representation and ledgers), 11-03 (regions), 9-03 Two kinds of gauge fixing, 10-02 The one-step map and the one-loop computation.

**Said first** — What is linked by identities is the one-step **representation**. The ledgers (upper bounds on regular and forced activities) are inequalities; this does not mean that the estimates use no approximation. This book has not read the content of the expansion (Ursell coefficients, cumulants) or the lemmas that reorganise the early ledger into the exit ledger. That the stationary point agrees with the coarse action up to fourth order is carried in the statements as a **condition**, and another theorem supplies it.

The order of this chapter

1. Statement — linking the one-step representation by identities
2. Coordinates — tree gauge and relative coordinates
3. The Gaussian part — precision, gap, windowed approximate inverse, decay
4. The stationary point — agreement with the coarse action is carried as a condition
5. Expansion of corrections, and the early representation
6. Extracting the marginal terms
7. The ledger — why $\hat{g}^{11/5}$
8. What this book's study had seen
9. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — linking the one-step representation by identities

Notation: the representation at step $m+1$ is $Y$ (coupling $\beta = \beta_{m+1}$); the reference coupling at step $m$ is $x = \hat{\beta}_m$; $\hat{g} = x^{-1/2}$, $P = \exp((\log\log x)^2)$; the threshold $t = \hat{g}P$ (the coarse-side threshold, corresponding to $t_m$ of 9-07 and $\tau$ of 11-03; correspondence of threshold letters in 11-01); and the covering scale $B = \lceil (\log x)^2 \rceil$. The one-step kernel is $T_\beta(U,V) \propto \exp(-c_{\mathrm{pen}}\beta \sum_e \mathrm{price}(V_e Q_e(U)^{-1}))$ (9-01; $c_{\mathrm{pen}}$ is the penalty coefficient, `kernel.penalty` in the source's wording).

The push-forward value is linked to the coarse-step representation by the following three **identities** Source .

$$\mathrm{push}_\beta Y.\mathrm{eval} \overset{(a)}{=} \mathrm{gaussian}(Y) \overset{(b)}{=} \mathrm{earlyValue}(X, v, \beta, \tau, \alpha) \overset{(c)}{=} F.\mathrm{eval}. \quad (1)$$

a. `Return.earnings` (`YM/AnalyticBridge/VERIFIED_RESET_STEP/Inputs.lean:56`): the push-forward equals sector sum × sum over choices × prefactor × Haar integral over the frozen links × Gaussian integral (`gaussian`, :40).

b. `Return.atlas` (:70): a **proposition** that the Gaussian sector formula equals the "early representation" written with a list of activities $X$ and value $v$. The docstring says it "does not prove which field shapes admit such a list". The instance of this proposition is built by `Rain.sprint` (`YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Rain/Glazing.lean:85`), and `Song.forward` (`Song/Embassy.lean:87`: early representation = push-forward) is its material.

c. `Collection.Local.actual` (`Evaluation.lean:93`): the early representation equals the value of the representation reread after extracting the marginal terms.

The **small-field one-step theorem** `Return.reset` (`Advance.lean:65`) Source takes the proposition (b), the ledger hypotheses and a continuous curve $Y$ at step $m+1$, builds a

continuous curve $F$ at step $m$, and states at each point: the licence condition (`licensed`), the one-step certificate `Move` (including the push-forward identity), for the coupling deviation $u = \beta F - \mathbf{predictor}$ the bounds $|u| \le \mathrm{width}$ and $t^2|u| \le H$ ($u$ equals the $\Delta$ of § 06 and corresponds to the correction $d$ of 10-03), and the vacuum term $v_F = \mathbf{vac}$. The one-step theorem `Use.Machine.run` (`Downstream.lean:65`) calls it, and `Mallow.Spring.moves` (`Mallow/Pediment.lean:58`) passes it the three inputs (ledger `Rain.varnish`, list `Rain.sprint`, continuity `Rain.blinking`) Source .

**Which target it enters.** The road to the existence theorem, `rodB` (`YM/FullVolume/Stage/Rods.lean:26`) → `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean:76`) → `Chord.spores` (:53) → `Spring.moves` → `Machine.run` → `reset`, has been checked by tracing the declarations literally Source . However, the witness in the verification record for `reset`, `Machine.run`, `Spring.moves`, `atlas` and `earnings` alike is `SharpGram.habitat` (the witness is only the first one found depth-first). The proof of `supplier` receives the one-step Move returned by `Spring.moves` and packs it into the step record. The lower-level lemmas of this chapter (`tree_gap`, `fog`, `late_cubic` and so on) have `SharpGram.habitat` as their witness in the record, and we have not checked whether they lie on the chain of the existence theorem.

02

## Coordinates — tree gauge and relative coordinates

1. **Background.** Fix the coarse configuration $V$ and place the background $\mathbf{replica}(V)$ on the fine lattice (`YM/PCalc/Chart.lean:35`). Links inside a block are 1, and links crossing blocks take the value of $V$ Source .
2. **Tree gauge.** Span a tree in each block and fix the links on the tree. The gauge freedom is divided out by gauge transformations with the root fixed Source .
3. **Relative coordinates.** Free links are written $u_i = \exp(a_i)\,\mathbf{replica}(V)_i$ ($a_i$ Lie-algebra valued) (`lifting`, :60) Source .
4. **Frozen links and moving links.** Links touching the neighbourhood of the seeds (radius $10R$) are left with the Haar measure (`Frozen`, `YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Shell.lean:42`), and only the remaining links are set as $a = \bar{a} + \hat{\beta}^{-1/2} z$ and integrated over Gaussian variables Source . The split is determined region by region (by seed type) and is not carried across steps.

03

## The Gaussian part — precision, gap, windowed approximate inverse, decay

**The quadratic part.** For the coordinates $a$ of the moving links, the action is split into a quadratic part $\frac{1}{2}|Da+h|^2$ and a remainder residual (`Remainder.lean:58`). $D$ (`move`, `Affine.lean:208`) has rows for the linearised curvature of fine plaquettes and rows for the linearised penalty of coarse links. The precision is $\mathrm{gram} = D^{\mathsf{T}} D$ Source . On a flat background the precision has the form $K + c_{\mathrm{pen}}\,\Pi^{\mathsf{T}}\Pi$ ($K$ is the kinetic part of the plaquette field, $\Pi$ the linearisation of the predictor) (`FreeResolvent.baseMatrix`, `YM/Push/FREE_RESOLVENT/Flat.lean:81`) Source .

**The gap on the tree section** `tree_gap` (`Flat.lean:198`) Source : if the curvature part has gap $a$, then the gap of the flat precision restricted to the tree section is

$$\mathrm{gap} \;\geq\; \frac{\min(a,\,c_{\mathrm{pen}})}{1+\beta_L+\gamma_L} \tag{2}$$

at least this. $\beta_L$ depends only on the block side $L$, the number of convolutions and the choice of kernel, and $\gamma_L$ only on $L$; neither depends on the volume period. The penalty coefficient $c_{\mathrm{pen}}$ enters directly.

**Decay** `resolve_point` (`YM/Push/FREE_RESOLVENT/Inverse.lean:120`) Source : from a weighted row sum $\sum_j e^{\sigma d(i,j)}|A_{ij}| \leq M$ and a gap $r>0$, for $0 \leq \gamma \leq \min(\sigma/2,\; r\sigma/(4(M+1)))$,

$$|(A+u)^{-1}_{ij}| \;\leq\; e^{-\gamma d(i,j)}\,(r/2+u)^{-1}. \tag{3}$$

This is a Combes–Thomas-type estimate, with constants independent of both dimension and volume.

**The windowed approximate inverse.** The actual integral does not use $\mathrm{gram}^{-1}$; instead the Gram matrix is inverted in a window around each address (pairs whose parent distance is within $R/20$), and the columns are collected and symmetrised into $C$ (`fog`, `Chosen.lean:69`), which is used as the covariance Source . `SectorChange.windows` (`Windows.lean:184`) Source : for a symmetric $A$ with a gap and a weighted row sum, there are $v, C', \lambda > 0$ such that, if the window radius $\ell$ is sufficiently large,

$$\|\mathrm{clouded}(W,A)-A^{-1}\|_{\text{unweighted row sum}}\le C'e^{-v\ell},\qquad \mathrm{clouded}(W,A)\text{ is positive definite with gap }\lambda$$

The difference is measured by the **unweighted** row sum. It is not claimed to be positive definite for an arbitrary window (the docstring says so explicitly).

**Change of variables and where the error goes.** Around the centre $\bar a$ (`meanLog`, `Chosen.lean:72`: the minimiser of the quadratic part approximated with the windowed inverse), set $a=\bar a+\hat\beta^{-1/2}z$ and integrate $z$ against the Gaussian with covariance $C$. The integrand (`dyed`, `Change.lean:63`) contains $e^{-\beta\,\mathrm{residual}}$, the likelihood, the region weight, and the **error of the approximation** $\mathrm{adjustment}=\frac12(\cdots)-\mathrm{guide}(\bar a+gz)/g^2$ (`Remainder.lean:124`: the quadratic form of $C^{-1}-\mathrm{gram}$, the shift of the centre, constants) Source . Because the approximation is not taken outside but kept entirely in the integrand, (a) remains an identity. Because the covariance has finite range, Gaussian expectations over separated sets are exactly independent, which is the foundation of the later expansion Inferred .

04

## The stationary point — agreement with the coarse action is carried as a condition

Stationary points appear in two ways Source .

1. **The centre of the integral** $\bar a$ ( § 03). A local approximation of the minimiser of the quadratic part.
2. **The stationary point of the combined action** $\rho$: a map $\phi\mapsto\rho(\phi)$ from the coarse logarithmic field $\phi$ to the fine moving coordinates, giving the stationary point of the combined action $\mathrm{tested}(\phi,\cdot)$ = (main action at step $m+1$) + $c_{\mathrm{pen}}\cdot$ (penalty). Its properties are written as four conditions (`YM/PCalc/Chart.lean`):
   - `Critical` (:88): $\rho(0)=0$, $\rho$ is real-analytic, and near $\phi$, $\rho(\phi)$ is a stationary point.
   - `PriceGap` (:100): the Hessian at 0 is $\ge\eta$.
   - `Guessed` (:97): the first derivative of $\rho$ equals an explicit linear map.
   - `ClassicalMatch` (:108): for every choice of predictor $i$ and $j\le4$, the $j$-th order jets **at the flat configuration** $\phi=0$ agree: $\phi\mapsto\mathrm{tested}_i(\phi,\rho(\phi))$ and $\phi\mapsto\mathrm{mainTerm\ coarse\ path\ data}(\phi)$.

**This agreement is a condition.** The four conditions are **carried in the form of hypotheses** as fields of `Kin.Planned` (`YM/PCalc/Loop.lean:93`) (`critical`, `gap`,

`classicalPart` and so on), via the field `support` of the one-step theorem bundle `Directed` (`YM/PCalc/Band.lean:34`) and the field `paths` of the supply bundle (`YM/FullVolume/Roll/Fiscal.lean:37`), and `sailsW` (`Fiscal.lean:97`) closes them Source . The proof of `sailsW` uses `Stationary.verdant` and `Stationary.pouring`, and its hypotheses are only the simplicity of the group, the kernel's three obligations, the draft order and the sequence of constants. In the record, `sailsW` is reached from `continuum_with_position_tests` (called by the existence theorem) Source .

The main action of the coarse step is **path data given separately** (field `layer` of the supply bundle). Rather than building the coarse action from the stationary value, the form is to carry the condition that the given coarse action and the jet of the stationary value agree up to fourth order, and a separate theorem satisfies it Source . The action on the fine side is the main action at step $m + 1$, and it agrees with the Wilson action only at step $N$ (field `peak` of `Planned`: agreement up to fourth order at step $N$; field `top` of the supply bundle `Imperial` (`YM/FullVolume/Roll/Fiscal.lean:38`): the path data at step $N$ are those of the first step, 11-02) Source . This is a different object from the predictor (the explicit map that defines the kernel, 9-04).

05

## Expansion of corrections, and the early representation

After the Gaussian integral, what remains is a product of local factors integrated against a finite-range covariance. We read the components of the expansion as hard-core Ursell coefficients and the Penrose-type tree inequality (`YM/Push/ABSOLUTE_SUMS`), Gaussian moments and cumulants of local words (`YM/Primitives/CONNECTED_GAUSSIAN_CUMULANTS`), and jets of connected activities (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6`) Inferred (from names and docstrings).

The result is collected into a single dictionary and becomes the **identity** of `Song.forward`, "early representation = push-forward" Source . The early ledger `Song.loaded` (`Embassy.lean:108`) bounds the pieces of regular activities to the order of $\mathrm{octaveNeed}$ (`Song/Litho.lean:60`: the sum of $|\alpha|t^2$, $\beta$·cubic atoms, $\beta$·quartic atoms and one-loop atoms). Its size is $\leq \beta^{-7/16}$, larger than the exit ledger (`suns`, `Boulevard/Dawn.lean:76`) Source . After the same list is reorganised, it enters the exit ledger $\mathrm{forecastE} = \mathrm{credit} \cdot \hat{g}^{11/5}$ (`Rain.varnish`, `Glazing.lean:56`). Pieces with large support, multiplied by the weight $e^{-(w/4)B}$, are at most $\beta_f^{-23/20}$ (the third conclusion of `suns`, `Dawn.lean:76`; the bounding quantity is `gold`, :30) Source .

06

## Extracting the marginal terms

The early representation is rearranged as follows (`Local.actual`, `reset`) Source .

1. **Vacuum term**: collecting the flat values of each regular-activity piece gives $v = \tau - \sum_i E_i(1)/|\Lambda|$ (`vac`, `Evaluation.lean:29`).
2. **Shift of the coupling**: collecting the log-Hessians of each piece in the curvature direction at the anchor gives, by hypercubic symmetry, the form $\delta_{ab}\Delta$ (`Averaging.isotropic`, `Isotropy.lean:77`). $\Delta$ (`delta`, :75) is moved into the coefficient of the kinetic term, $\beta_{\text{out}} = \beta_{\text{in}} + \alpha + \Delta$. $\alpha$ is the one-loop coefficient, held in advance by the path-data side.
3. **Normalising regular activities**: each piece is replaced by $\text{rebate} = (E - E(1)) - (\text{quadratic polynomial at the anchor})$ (`Residual.lean:28`). Regular activities are then $0$ at the flat configuration with Hessian $0$ in the affine directions, and return to the form of the condition `RegularFamily` of 11-01 § 03.
4. **How the coupling is fixed**: `reset` solves for $\beta$ so that the density ratio matches at two test configurations (the flat configuration and a small-field configuration with no flags), and shows that the solution equals "predictor $+\Delta$" (`yield_shape`, `Advance.lean:25`). `yield_shape` says that the left-hand side of the rereading equation ((2) of 10-03) equals a positive constant multiple of $d - \Delta$, so the root is $d = \Delta$. This $\Delta$ corresponds to the correction $d$ of 10-03, and the $u$ of § 01 also equals $\Delta$ (proof of `reset`, `Advance.lean:54, :84`) Source . The coupling is fixed as the coefficient of the kinetic term in the representation (10-03).

07

## The ledger — why $\hat{g}^{11/5}$

The regular-activity ledger is measured by "the supremum of finite-order derivatives in directions of size $t$", so a term of degree $k$ in the field, $c \cdot (\text{field})^k$, contributes about $c\,t^k$. The sizes of what enters the regular activities can be read as follows Inferred (the supporting statements are Source ).

| What enters | Size | Basis |
|---|---|---|
| Fifth-order and higher remainder of the tree level | $\beta t^5 \approx \hat{g}^3 P^5$ | The condition of jet agreement up to fourth order |
| Third-order and higher remainder of one loop | $t^3 \approx \hat{g}^3 P^3$ | `FiniteBound` (`YM/PCalc/Loop.lean:62`) matches second derivatives |
| Cubic remainder of $\alpha$ | $(\log L)\, t^3$ | `late_cubic`, `CubicGate` (`YM/UVNext/COUPLING_RESET/Precision.lean:71`, :96) |

| What enters | Size | Basis |
| --- | --- | --- |
| Pieces with large support | $e^{-(w/4)B}$. $\mathrm{octaveNeed} \le \beta_f^{-23/20}$ | `suns`, `gold` |

The small pieces (first three rows) are of order $\hat{g}^{3-\varepsilon}$, and the pieces with large support of order $\hat{g}^{2.3}$ ($\beta_f^{-23/20}$); all fit within the $\hat{g}^{11/5}$ ledger. Meanwhile, the quadratic (marginal) coefficients are not left in the regular activities but moved to $\beta$ ($\Delta$ and $\alpha$), and the constants to $v$. $11/5$ can be read as a number between the marginal degree $2$ and the first non-marginal degree $3$, leaving room on both sides to absorb polylogarithms Inferred. `late_cubic` (:71) says that for any $C$, for sufficiently large $x$, $C\theta^3 \le \hat{g}^{11/5}$ ($\theta = \hat{g}P$) Source. The loss from rereading is prepaid by the step constant $\mathrm{credit} = 1/(64(1+\mathrm{base}))$ (`Tables.lean:36`) Source.

08

## What this book's study had seen

This book has one step of the free field in Lean: the push-forward covariance $CTC^\dagger$ of the linearised block map, an upper bound on the covariance of the plaquette field, and a contraction uniform in the step (15-12, 15-13). The non-Gaussian step (vertex estimates, localisation, the loss of margin in hand-over) exists only as a paper study.

- What corresponds to this book's "linearised block map" is the linearisation $\Pi$ of the predictor. The gap (2) and the decay (3) of the flat precision $K + c_{\mathrm{pen}}\Pi^{\mathsf{T}}\Pi$ correspond to this book's upper bound on the covariance of the plaquette field Inferred.
- The published construction has no structure that decomposes the covariance across steps and uses it (this book's contraction uniform in the step). It rebuilds the finite-range covariance of each step at each step Inferred.
- Replacing the covariance by a windowed approximate inverse and keeping the error in the integrand carries this book's "loss of margin in hand-over" as a term of the integrand rather than as a loss in the estimate Inferred.
- By redefining the ledger at every step as a function of that step's $\hat{\beta}$ and measuring it with a norm of finite-order derivatives, the construction makes the induction hypothesis not carry the losses in threshold and radius that accumulated step by step in this book's paper study Inferred.

09

## What has not been read

- The content of the reorganisation from the early ledger to the exit ledger (`Marlin`, `Trio`, `Meadow`, `innings` of `Rain`), especially where the $\beta$·cubic atoms cancel.
- The content of the proof supplying `Kin.Planned` (from `YM/Next/STATIONARY` to `sailsW`).
- The Gram gap on non-flat backgrounds (`YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Gap.lean`) and how it connects to (2).
- The specifics of the type of the expansion (the dictionary of connected activities, the estimate of Ursell coefficients).
- The estimate of the normalising determinant (the content of $\tau$ in the vacuum term).
- The relation between the linearisation of the predictor and the kinetic term (transported curvature) of the coarse path data.

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The one-step representation is linked by three identities: push-forward = Gaussian sector formula = early representation = reread representation (the instance of the list is `Rain.sprint`) | That the estimates use no approximation (the ledgers are inequalities) |
| Source Gap on the tree section (2), decay (3), error of the windowed approximate inverse (4) (unweighted row sum) | Positive definiteness for an arbitrary window |
| Source Agreement of jets up to fourth order at the flat configuration between the stationary value and the coarse action is carried as a condition of `Kin.Planned`, supplied by `sailsW` | Building the coarse action from the stationary value (the coarse action is path data given separately) |
| Source Marginal terms: flat values go to the vacuum term, the quadratic at the anchor $\Delta$ to the coupling, and the rest is normalised by rebate | The content of the reorganisation lemmas |
| Inferred The ledger $\hat{g}^{11/5}$ is a number between the marginal degree 2 and the first non-marginal degree 3 | That the reason for choosing $11/5$ is written in the source |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Three identities and the one-step theorem | Source | `YM/AnalyticBridge/VERIFIED_RESET_STEP/Inputs.lean:40`, :56, :70, `Evaluation.lean:29`, :93, `Advance.lean:25`, :65, `Downstream.lean:65` |
| Instance of the list and its material | Source | `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Rain/Glazing.lean:56` :85, `Song/Embassy.lean:87`, :108, `Song/Litho.lean:60`, `Boulevard/Dawn.lean:30`, :76 `Mallow/Pediment.lean:58` |
| Road to the existence theorem | Source | `YM/FullVolume/Stage/Rods.lean:26`, `YM/FullVolume/C/Songbook.lean:53`, :76 |
| Coordinates | Source | `YM/PCalc/Chart.lean:35`, :60, `YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Shell.lean:42` |
| Gaussian part | Source | `YM/Push/FREE_RESOLVENT/Flat.lean:81`, :198, `Inverse.lean:120`, `YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Affine.lean:208`, `Remainder.lean:58`, :124, `Windows.lean:184`, `Chosen.lean:69`, :72, `Change.lean:63` |
| Conditions on the stationary point and their supply | Source | `YM/PCalc/Chart.lean:88`, :97, :100, :108, `YM/PCalc/Loop.lean:62`, :74, :93, `YM/PCalc/Band.lean:34`, `YM/FullVolume/Roll/Fiscal.lean:37`, :97 |
| Marginal terms and the ledger | Source | `YM/AnalyticBridge/VERIFIED_RESET_STEP/Isotropy.lean:75`, :77, `Residual.lean:28`, `Tables.lean:36`, `YM/UVNext/COUPLING_RESET/Precision.lean:71`, :96 |

**Words used in this chapter** — relative coordinates, tree gauge, precision, gap on the tree section, windowed approximate inverse, stationary point, jet agreement, early representation, vacuum term, shift of the coupling. Their meanings are in the A-02 glossary.

Next chapter to read: 11-10 Convergence at a fixed step, and the step that removes the step index — the argument that holds the step fixed and lets $N \to \infty$.

# Convergence at a fixed step, and the step that removes the step index

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

**What this chapter shows** — Why the ingredients of the representation at step $m$ (coupling, activities, path data) converge along the whole sequence in the cutoff $N$. The source is not "an estimate comparing $N$ and $N'$ quantitatively" but compactness, uniqueness by anchoring at step 0 and contracting from above, and the subsequence principle. From there, via uniform convergence of densities, images of observables, and the step that removes the step index, the ultraviolet limit at fixed volume comes out along the whole sequence.

**Chapters assumed** — 11-01 (representation), 10-04 The shooting method ($\beta = \bar{\beta}$ at step 0), 8-05 Correlation functions and the order of limits, 8-04 Renormalised insertions.

**Said first** — The convergence is qualitative; no rate is in the statements. It is convergence for each volume $p$, with no statement uniform in $p$. The limiting objects (the limiting measure, the limiting representation) are not defined. This book has not read the source of the one-step map's contraction constant $1/16$ (the analytic estimates of the one-step map).

The order of this chapter

1. Statement — convergence of ingredients, uniform convergence of densities, the estimate that removes the step index, the ultraviolet limit
2. Outline of the mechanism 1 — the source of convergence of ingredients
3. Outline of the mechanism 2 — densities, observables, calibration
4. Outline of the mechanism 3 — the step that removes the step index
5. The limiting objects are not constructed — what can and cannot be said about the step-0 theory.
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## Statement — convergence of ingredients, uniform convergence of densities, the estimate that removes the step index, the ultraviolet limit

**The ultraviolet limit at fixed volume** (Theorem 4.1 of the committee's assessment) Assessment : the construction chooses a sequence of spacings $a_N \to 0$, a sequence of volumes $L_p$ and couplings $\beta_{N,p}$, and for each volume $p$, each sequence of labels and each test function (avoiding coincident points, with support inside the volume), the limit $S_p(v;f) = \lim_{N\to\infty} C_{N,p}(v;f)$ of the correlation $C_{N,p}(v;f)$ exists **along the whole chosen sequence of spacings**, and is represented by a $C^\infty$ kernel away from coincident points (assessment § 4.3). It also states that there is no universality with respect to the sequence of spacings Assessment . On the Lean side this is `Campaign.realUV : PositionsUV R.family Sequence.full` (`YM/FullVolume/Fields/Filled.lean:45`) Source . The public statement, too, requires convergence along the whole sequence in the cutoff $N$ for each volume $p$; only the sequence of volumes is a subsequence (`Converges`, `PublicWilson.lean:85-90`) Source . The strong form with kernels and differences (`PositionsUV … Sequence.full`) appears only in the conclusion of the construction theorem `constructed` Source .

The components of that line are as follows Source . All of them are reached in the record from `continuum_with_position_tests` (called by the existence theorem).

| Order | Statement | Location |
|---|---|---|
| 1 | Bundle of convergence of ingredients: from step $k$ up, $\beta(N,p,i)$ is Cauchy, the activities are Cauchy in the norm `Show`, and the path data converge | `Helmet` (`YM/FullVolume/Gables/Honors.lean:30`), supply `helmed` (:62) |
| 2 | Source: a sequence of trajectories whose step-0 couplings converge has ingredients at all steps converging along the whole sequence | `Window.allB` (`YM/FullVolume/Glide/Triumph.lean:37`) |
| 3 | The effective density at step $m$ (with the vacuum constant removed) is uniformly Cauchy in $N$ in the sup norm over configuration space, and so is its total mass | `Flux.fanned` (`YM/FullVolume/Fields/Cement.lean:50`), `Flux.lacks` (:56) |

| Order | Statement | Location |
|---|---|---|
| 4 | Images of observables at step $m$ are uniformly Cauchy | `Rising.hail` (`YM/FullVolume/Fields/Capital.lean:57`) |
| 5 | Difference between the true lattice correlation and the readout at step $j \le \text{refreeze}/(e^{Bj/64}/2)$ (independent of $N$) | `Sever.echoes` (`YM/FullVolume/Fields/Peat.lean:42`) |
| 6 | The true lattice correlation is Cauchy in $N$ | `Rising.arrives` (`YM/FullVolume/Fields/Timber.lean:74`) |
| 7 | Uniform boundedness + pointwise Cauchy at general positions ⇒ locally uniform convergence with derivatives | `positioned` (`YM/FullVolume/Profiles.lean:42`) |

"Uniformly Cauchy" is the type `Confluence` (`YM/FullVolume/Fields/Basin.lean:10`), consisting of eventual uniform boundedness and $\forall \varepsilon\ \exists N_0\ \forall n, n' \ge N_0\ \forall V :\ \mathrm{dist}(f_n(V), f_{n'}(V)) \le \varepsilon$ Source . The hypothesis `Flux` of item 3 is given by `Rising.flooded` (`Capital.lean:31`) at steps $k \le m$, and in the main theorem $k = 0$ (`sees`, `Filled.lean:30`). So it holds for every step $m \ge 0$, every $p$, along the whole sequence of cutoffs Source .

## 02 Outline of the mechanism 1 — the source of convergence of ingredients

**Coordinates.** The coupling at step $m$ is measured in the ratio coordinate $\lambda_m$ relative to the reference value $\hat{\beta}_m$ (the allowed band is a bounded interval), and the activities in the norm `Show` on the scale of step $m$ (`YM/FullVolume/Table/Tabs.lean:66`). `Show` is the supremum, over points, of the sum over anchored polygons of the suprema of derivatives weighted by an exponential of the covering number, divided by the step's ledgers ($\hat{g}_m^{11/5}$ and $e^{-P_m^{1/5}}$); it is independent of the volume Source . On every trajectory the norm of the activities is at most $1/8$ Source .

**Contraction of the one-step map** `Muse.ticket` (`YM/FullVolume/Gain/Rate.lean:94`) Source : with the max norm on $(\lambda, K)$,

$$\|\Delta K_m\| \le \tfrac{1}{16} \max\big(|\Delta\lambda_{m+1}|, \|\Delta K_{m+1}\|\big), \qquad |\Delta g_m| \le 8 f_m \max(\cdots). \quad (1)$$

$g_m$ is the one-step deviation of the coupling (the remainder after removing the one-loop part), and $f_m$ is a step-dependent constant. The proof is a Cauchy estimate from analyticity Source (statement read) / Inferred (reading of its meaning). `Window.curls` (`Glide/Duty.lean:45`) writes the same estimate in the metric of the completion.

**Four components** Source .

1. **Compactness** (`thinning`, `Triumph.lean:28`): $\lambda_m$ at each step lies in a bounded interval, so by compactness of a countable product a subsequence converging simultaneously at all steps can be extracted.
2. **If the couplings converge, so do the activities** (`bankA`, :25): the difference from the trajectory restarted with activities 0 at step $b$ is at most $(1/16)^{b-m}$ at step $m$ (`Window.culprit`, `Glide/Wings.lean:59`, forgetting of the top end). Restarted trajectories are compared with each other by continuity over finitely many steps. It closes with $\varepsilon/3$.
3. **Uniqueness** (`Window.accord`, `Glide/Coincide.lean:58`): two limiting trajectories whose step-0 coupling limits are equal agree at all steps.
4. **Subsequence principle** (`travels`, `YM/FullVolume/Artes/Events.lean:11`): if every subsequence has a further subsequence converging to the same limit, the whole sequence converges.

**The self-estimate for uniqueness.** Let the difference at step $i$ between two trajectories be $y_i = \max(|\Delta\lambda_i|, \|\Delta K_i\|)$, and $Y = \sup_i y_i$. Activities contract from above (the fine side), giving $\|\Delta K_i\| \le Y/16$. The couplings are equal at step 0 (the anchor), and propagate towards the fine side by the sum of one-step deviations (`arrest`, `Coincide.lean:43`):

$$|\Delta\beta_i| \le \sum_{j<i} 4\,\mathrm{stow}(j)\,Y \le 4Y\cdot\mathrm{glint}\cdot\mathrm{brace}(i). \tag{2}$$

The last inequality is the calendar condition `Search.radiusB` (`YM/FullVolume/Gain/Motion.lean:48`: $\sum_{i<m}\mathrm{stow}(i) \le \mathrm{glint}\cdot\mathrm{brace}(m)$). Changing coordinates gives $|\Delta\lambda_i| \le Y/32$. Together, $Y \le Y/16$, hence $Y = 0$ Source .

Meaning Inferred : this is uniqueness of the solution of a two-point boundary value problem on steps $[0,\infty)$, with the coupling at step 0 and "bounded activities" at step $\infty$ as boundary conditions. The marginal direction (the coupling) is anchored at step 0, and shooting (10-04) aligns it to $\beta = \bar\beta$ for all $N, p$. The irrelevant directions (the activities) contract from above. The difference at the top end (the bare Wilson step) need not be small; being bounded is enough.

The chain in the source is `helmed` ← `Winter.convergence` ← `allB` (`Winter`, `Glide/Vac.lean:31`) Source . No rate comes out, because it goes through compactness Source .

03

## Outline of the mechanism 2 — densities, observables, calibration

**Densities.** The representation at step $m$ is a continuous function of its ingredients: the path action is uniformly continuous in the coupling and the path data, the sum of activities is continuous, and the gas is a finite sum and product over boxes. Since `Confluence` is closed under sums, products and composition with continuous maps, the density is uniformly Cauchy in the sup norm (`fanned`), and since configuration space is compact and the reference measure is a probability measure, the total mass is Cauchy too (`lacks`) Source (statements read) / Inferred (reading of the line).

Uniform convergence of the normalised density $\mathrm{eval}/Z_N$ is not stated as a named statement. It follows in one line from `fanned`, `lacks` and a positive lower bound on the denominator independent of $N$ (`lingers`, `YM/FullVolume/Fields/Till.lean:20`). Convergence in sup norm includes convergence in total variation Inferred.

**Calibration of observables.** The coefficients of local fields of degree $\le d$ live in a finite-dimensional space; one step maps them by a linear isomorphism `leanings` (`Club` in `YM/FullVolume/Mark/Nursery.lean:11`) whose inverse is uniformly bounded, and the remainder contracts by $1/2$ per step. The coefficients are solved backwards so that reading at the reference step $r = \mathrm{ref}(d)$ gives exactly $v$ (8-04), and carrying them from step $r$ by the inverse maps makes the coefficients at steps $m \ge r$ Cauchy in $N$ (`Gear.spends`, `YM/FullVolume/Mark/Sages.lean:66`; `Branch.winds`, `stays`, `YM/FullVolume/Gables/Master.lean:44`, :50) Source. The structure is the same "anchored marginal direction + contracting remainder" as in § 02, and since it is linear and finite-dimensional, no compactness is needed Inferred.

04

## Outline of the mechanism 3 — the step that removes the step index

With step $j$ fixed, the readout on the representation at step $j$ using the step-$j$ images of observables is Cauchy in $N$ (`latter`, `Timber.lean:58`) Source. On the other hand, `echoes` states

$$\left\| \text{true lattice correlation} - \text{readout at step } j \right\| \;\le\; \frac{\text{refreeze}}{e^{B_j/64}/2}, \qquad B_j = \lceil (\log \hat{\beta}_j)^2 \rceil \tag{3}$$

under the hypothesis (`Drake`) that the separation of points is at least $B_j^2$ in units of step $j$ Source. The right-hand side does not depend on $N$. refreeze

(`YM/FullVolume/Fields/Flight.lean:28`) contains an exponential of the number of lattice points at the reference step and is not uniform in the volume $p$ Source . The smallness $e^{-B_j/64}$ is slower than any power of $a_j$ Inferred . In physical length $B_j^2 a_j \to 0$, so fixed separated points satisfy the hypothesis at sufficiently fine steps Inferred .

`arrives` takes $\varepsilon/4$, chooses a step $j$ at which the right-hand side of (3) is at most $\varepsilon/4$, and by `latter` and the triangle inequality shows that the true lattice correlation is Cauchy in $N$ Source . Finally `positioned` extends from dense points to locally uniform convergence with derivatives, using uniform boundedness (`FixedBounds`) and equicontinuity Source .

Here is the picture in words. Put the step on the vertical axis (0 at the bottom, the fine side at the top) and the cutoff $N$ on the horizontal axis; the trajectory for each $N$ is a vertical line from step $N$ (bare Wilson) down to step 0 (coupling $\bar{\beta}$). On a horizontal line at fixed height $m$, the cross-section (the representation at step $m$) converges as $N$ grows: that is § 02 and § 03. `arrives` says that if the height $j$ is taken high enough, the difference between the top (the true correlation) and the cross-section at height $j$ is small independently of $N$. Step 0 is where the units are fixed; it is not used in estimating correlations.

05

## The limiting objects are not constructed

- There are no named definitions of the limiting measure $\mu_{\infty,p,m}$ at step $m$, the limiting density, or the limiting representation Source (search scope: FullVolume, RG, Extraction). What exists goes only as far as convergence of the ingredients (coupling Cauchy, activities Cauchy, existence of some limiting path data).
- It is not stated that the limiting density has a representation of the same form, or that it is consistent across steps, $\mu_{\infty,p,m} = \mathrm{push}(\mu_{\infty,p,m+1})$ Source .
- Because the one-step kernel depends on $\beta(N, p, m+1)$, the push-forward measure at step $m$ is not "the Wilson measure pushed forward by a fixed map"; it is an object whose map itself moves with $N$ Source (the form of `Transfer`).
- The line of proof does not construct limiting objects; from Cauchy at a fixed step it removes the step index and goes directly to Cauchy for the true lattice correlations.
- Reflection positivity of the measures at renormalised steps (including step 0) is not stated Source (within the scope of the name search). The statement of reflection positivity is `Slice.reflected_positive` (`YM/Slice/Reflection.lean:98`) for the bare Wilson measure. Since the kernel's patches cross the plane, we read that the push-forward does not inherit it automatically Inferred .

- There is no renormalisation to scales coarser than step 0 (spacing 1). Step indices are natural numbers, and step 0 is the coarsest step Source (within the scope of the name search).
- Volume: the contraction and Lipschitz constants do not depend on $p$, but the convergence is a qualitative statement for each $p$. Estimates at the level of the density (`refreeze`, the lower bound on the denominator) include an exponential in the volume. The one-step ledgers (`EBound`, `KBound`, `Show`), on the other hand, are pointwise sums and do not depend on the volume Source .

### What can and cannot be said about the step-0 theory

When the remaining question is restated as "the long-distance question for the effective lattice theory at step 0 (spacing 1)", here is what can and cannot be said from the source.

| Item | In the source | Not in the source |
|---|---|---|
| Coupling | $\beta(N, p, 0) = \bar{\beta}$ for all $N, p$ (`Selected.reaches`, `YM/RG/History.lean:239`) Source | — |
| Limiting objects | The effective density at step 0 is uniformly Cauchy in $N$ ($p$ fixed) Source | A definition of the limiting measure and its representation Source (within the scope of the name search) |
| Images of observables | Images at step $m$ exist for observables with reference step $\psi(d) \leq m$ (hypothesis of `hail`, `YM/FullVolume/Fields/Capital.lean:57`) Source . At step 0, only for observables whose reference step is 0 or below | Images at step 0 of observables whose reference step is above 0 |
| Volume | What is independent of the volume is the one-step ledgers (pointwise sums) Source | A form of the density-level estimates uniform in the volume (`refreeze` and others include an exponential in the volume) |
| Reflection positivity | For the bare Wilson measure (`Slice.reflected_positive`) Source | For the measure at step 0 |

| Item | In the source | Not in the source |
|---|---|---|
| Entry for the one-step estimate | For a smoothing index $j$ and a coupling value $x$, of the form "holds if $\mathrm{betaEntry}(j) \le x$" (`Delays.betaEntry`, `YM/Price/Intent.lean:116`; `Opened`, `YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:45`) Source | A formula or number for the entry value. The value is extracted as an existence from a bundle of properties that "hold for sufficiently large $x$" |
| Reference coupling $\bar\beta$ | $\bar\beta > 1$, at least the first entry (`Bar.ready`), and $x^{-1/2} e^{(\log\log x)^2} \le 1$ for all $x$ at least $\bar\beta$ (`Bar.threshold`, `YM/Price/Intent.lean:126-127`) Source . From the last condition alone, $\bar\beta \gtrsim 9 \times 10^5$ Reading (root found numerically) | An explicit value of $\bar\beta$ |
| Renormalisation towards the coarse side | — | Steps coarser than step 0 (steps flowing towards strong coupling) |
| Relation to coarse Wilson loops | — (no statement) | The relation between Wilson loops of step-0 configurations and Wilson loops of the fine field (or the continuum) Source (within the scope of the name search; 16-03 § 04) |

06

## What this book's study had seen

The ultraviolet chain of this book's study took **subsequential limits** of the family of push-forward measures $\nu_{n,m}$ on a fixed coarse lattice and transferred properties to them (28-01 Implication of subsequential limits).

- Convergence at a fixed step in the published construction sits in the position of this book's "subsequential limit", and it comes out without subsequences Source (statement read) / Inferred (reading of the correspondence).
- The objects are not the same, however. The push-forward of the published construction is a composition of probability kernels depending on $\beta(N, p, m+1)$; this book's push-forward is a deterministic block map independent of $N$. Convergence of this book's $\nu_{n,m}$ does not follow from the convergence in the published construction Inferred .

- Patterns that can be borrowed as they are Inferred : the skeleton that eliminates subsequences (compactness, uniqueness, subsequence principle), forgetting of the top end, and the self-estimate pattern $Y \leq Y/16$. None of these depends on whether the map is deterministic.
- What cannot be borrowed Inferred : the representation itself keeping the same form at every step (the same point as the handling of large fields), and the **anchor**. This book's design supplies $\beta_N$ from outside, so the marginal direction has no anchor. The subsequence ambiguity in this book may come from here.

07

## What has not been read

- The source of the analytic estimates of the one-step map (`Muse.income`, `Pass.income`). This is where the contraction constant $1/16$ comes from, and it is the body of the small-field and large-field analysis.
- The continuity comparing one-step maps with different cutoffs (`Pith.shall` inside `Bill.taxes`).
- Uniqueness of shooting at finite $N$ (the content of the supply of `Jewel.unique`).
- The matrix form of the coefficient map `leanings` (whether mixing with the identity field and lower-degree fields is triangular).
- Completeness of `Show`, and existence of the limiting activities.
- The meaning of `Next.Free.Amplify` (field `rises` of `Helmet`).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Assessment The ultraviolet limit at fixed volume exists along the whole chosen sequence of spacings (Theorem 4.1 of the assessment). The public statement (`Converges`) also requires the whole sequence in the cutoff; only the volumes are a subsequence Source | Universality with respect to the sequence of spacings, convergence uniform in the volume, uniqueness of the infinite-volume limit |
| Source The source of convergence of ingredients is compactness + uniqueness by the step-0 anchor and contraction from above + the subsequence principle (`allB`) | The rate of convergence (not in the statements) |
| Source The effective density at step $m$ is uniformly Cauchy in $N$ in the sup norm (all $m \geq 0$, $p$, whole sequence) | A named statement of convergence of the normalised density (it follows in one line but is absent) |

| Can say | Cannot say |
|---|---|
| Source The difference from removing the step index is at most $\text{refreeze} \cdot 2e^{-B_j/64}$, independent of $N$ (point separation $\geq B_j^2$) | An estimate of this difference uniform in $p$ |
| Source For the limiting measure, the limiting representation and consistency across steps there is neither definition nor statement | Properties of the limiting objects (reflection positivity and so on) |
| Source Reflection positivity is stated only for the bare Wilson measure, and there is no renormalisation coarser than step 0 | Reflection positivity of the measures at renormalised steps (including step 0) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Ultraviolet limit at fixed volume | Assessment | Assessment § 4.3 (Theorem 4.1) |
| Convergence of ingredients and its source | Source | `YM/FullVolume/Gables/Honors.lean:30`, :62, `Glide/Vac.lean:31`, `Glide/Triumph.lean:25`, :28, :37, `Glide/Coincide.lean:43`, :58, `Glide/Wings.lean:59`, `Glide/Duty.lean:45`, `Gain/Rate.lean:94`, `Gain/Motion.lean:48`, `Artes/Events.lean:11`, `Table/Tabs.lean:66` |
| Densities, observables | Source | `YM/FullVolume/Fields/Basin.lean:10`, `Cement.lean:50`, :56, `Capital.lean:31`, :57, `Till.lean:20`, `Filled.lean:30`, :45 |
| Calibration | Source | `YM/FullVolume/Mark/Nursery.lean:11`, `Mark/Sages.lean:66`, `Gables/Master.lean:44`, :50 |
| The step that removes the step index | Source | `YM/FullVolume/Fields/Peat.lean:42`, `Flight.lean:28`, `Timber.lean:58`, :74, `YM/FullVolume/Profiles.lean:42` |
| Dependency record | Source | `verification/declaration-usage.tsv` (the declarations of this chapter are used, witness `continuum_with_position_tests`) |

**Words used in this chapter** — convergence of ingredients, uniformly Cauchy, forgetting of the top end, anchor, two-point boundary value problem, subsequence principle, calibration, the step that removes the step index. Their meanings are in the A-02 glossary.

Next chapter to read: 11-11 What this part has not yet read — the list for the whole part. After that, on to Part 12 Infinite volume and reconstruction.

# What this part has not yet read

This chapter is a map of open questions. § 01 is guide material.

**What this chapter shows** — The "what has not been read" items written in each chapter of Part 11, collected into one table, with their locations (relative file paths and declarations) and which claim of the part each one balances.

**Chapters assumed** — 11-00 (labels and levels of citation). The subject of each row is in the chapter in the right-hand column of the table.

**Said first** — What is listed here are places where this book has not read the source. It does not mean there are gaps in the proofs of the published construction. The results of the published construction remain at the level at which the committee assessed them as proved in Lean.

The order of this chapter

1. How to read this list
2. The large rows — places that balance the part's claims
3. List by chapter
4. Declarations whose target has not been checked
5. Cautions for reading

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Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00).

01

## How to read this list

Each row gives what has not been read, its location (path relative to `YM/` and the declaration name), and which claim in the part that place bears on. The locations were checked by this book in the source as far as names and types. The content of the proofs has not been read. Where a location is only a candidate, it is marked "candidate".

02

## The large rows — places that balance the part's claims

| Not read | Location | Claim it be |
|---|---|---|
| **Where the Haar-loss lemma is.** Which lemma puts, and where, the loss relative to Gaussian normalisation for links integrated with the Haar measure near the seeds. `Marginal` (45 declarations), `Haar` (3), `Leases` (26) and `Grant` (10) in the large-field location are all outside the dependencies | Candidates: `YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Haar.lean`, `Counting.lean`, `YM/AnalyticBridge/SELECTED_ORDERED_KERNELS`, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/IssuePlan.lean:34` (`Expense`) | The readi that it ent the field f logarithm cost per s (11-05 § 0 |

| Not read | Location | Claim it be |
|---|---|---|
| **The whole of the upper bounds on the remaining integrals after the price is extracted.** `peel` does not say how large the remaining (outgoing) integral is, and the docstring of its location also says "conditional on the cost of the exit multiplier" | `YM/Next/FLUCTUATIONS/Sector.lean:229` (`peel`), `YM/Push/ABSOLUTE_SUMS/Grants.lean` (docstring), `YM/SectorNext/LARGE_FIELD_REMAINDERS` (`Annulus.lean:47`, `CorePacket.lean:62`) → `YM/AnalyticBridge/ANNULUS_RATIO_INSERTIONS` → `YM/PhysicalNext/V9_JOINT_ENDPOINT_LOAD` | That the p seed facto actually reaches th one-step e (11-04 § 0. |
| **The content of how the one-step theorem is assembled.** Where the Move is produced from the three inputs (identity of the list, ledger, continuity), and how it ties to the large-field ledger | `YM/AnalyticBridge/VERIFIED_RESET_STEP/Downstream.lean:65` (`Use.Machine.run`), `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Venue.lean:47` (field `machine`) | The one-s certificate 00 § 03) |
| **The reorganisation lemmas.** The reorganisation from the early ledger $\beta^{-7/16}$ to the exit ledger $\hat{g}^{11/5}$ (cancellation of the cubic atoms), and the lemmas inside the part that builds the instance of the list | `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Marlin` (`carries` and others), `Trio`, `Meadow`, `Beam` (`travels`), `Rain` (`innings`) | The regul activity le (11-09 § 0. |

| Not read | Location | Claim it be |
|---|---|---|
| **The whole construction that rebuilds new forced activities after one step and fits them into the ledger** | `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP` (`Roost`, `Lotus`, `Sward`), `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6` (from `ratesCombine` in `OptionCost.lean:39` onward) | The balan of the rea of persist across ste (11-07 § 0 |
| **The source of the analytic estimates of the one-step map.** Where the contraction constant 1/16 comes from | `YM/FullVolume/Gain/Rate.lean` (`Muse.income`), `Pass.income` | Converge ingredien 10 § 02) |

03

## List by chapter

| Chapter | Not read | Location |
|---|---|---|
| 11-01 | The meaning of all the fields of `Bounds` for the path data | `YM/RG/PathActions.lean:173` |
| 11-01 | The role of the additional condition on forced activities (`extraTest`) | `YM/RG/Activities.lean` (`ForcedAddress`) |
| 11-02 | The construction that arranges the first forced activities into polymers, and their entry into the first ledger | `YM/Extra/WILSON_STARTUP/Packets.lean`, `Ink.lean` |
| 11-02 | The correspondence between due read in the orientation frame (`owed`) and due | `YM/Extra/WILSON_STARTUP/Ownership.lean:193` |
| 11-03 | Assembling the smooth partition of unity into a partition over seed types, and how the transition width is assigned | `YM/Push/SECTOR_MEASURE/Profile.lean` (`Chi`) |

| Chapter | Not read | Location |
|---|---|---|
| 11-03 | The description of overlaps between regions; how the blocking kernel $\Phi$ and the distribution to the noise are built | `YM/Push/SECTOR_MEASURE/Overlap.lean`, `YM/Pressure/Rows.lean` (`leap`, `kick`) |
| 11-04 | The proofs of the three inequalities: coarse plaquettes, fine plaquettes and penalty, payment for large coarse plaquettes | `YM/Work/E05`–`YM/Work/E07` |
| 11-04 | The meaning of the tube fields (`Deep`, `Roll`) of the fitting condition `Funds` | `YM/Pressure/Seeds.lean:101` |
| 11-05 | The proof of the sum of polymers containing a seed (Ursell coefficients, Penrose-type tree inequality, dictionary of connected activities) | `YM/Push/ABSOLUTE_SUMS/Ursell.lean`, `Graph.lean`, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6` |
| 11-05 | All the fields of the combinatorial condition `Buy`; the definition of the polymer-scale quantity | `YM/Push/ABSOLUTE_SUMS/Mixed.lean`, `Bulkhead.lean` |
| 11-06 | The map by which values of unflagged polymers enter the addresses of regular activities | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Colony.lean` (from `entries` onward) |
| 11-06 | The handling of incoming activities (`Sales`, `ratesCombine`) and what follows | `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/OptionCost.lean:39` |
| 11-07 | Where the strong ledger of the one-step certificate, `Move.forced`, is used at later steps | Unconfirmed |

| Chapter | Not read | Location |
|---|---|---|
| 11-07 | That action differences are not counted twice across steps (in the form including the difference between the orientation-averaged action and the action of a single orientation) | `Testing.ships` and others (unconfirmed) |
| 11-08 | The content of the price condition `Ringing`, the explicit form of the condition that $\bar{\beta}$ be taken large, the chain constructing a calendar that satisfies `Fits` | `YM/Pressure/Seeds.lean:155`, :161, `YM/Price/Intent.lean:123` |
| 11-09 | The content of the proof supplying the stationary-point conditions | `YM/Next/STATIONARY` → `YM/FullVolume/Roll/Fiscal.lean:97` (`sailsW`) |
| 11-09 | The Gram gap on non-flat backgrounds, and how it connects to the gap in the flat case | `YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Gap.lean` |
| 11-09 | The estimate of the normalising determinant (content of the vacuum term); the relation between the linearisation of the predictor and the kinetic term of the coarse path data | Unconfirmed |
| 11-10 | The continuity comparing one-step maps with different cutoffs; uniqueness of shooting at finite $N$ | `YM/FullVolume/Glide/Bill.lean` (`Bill.taxes`), `YM/FullVolume/Finale/Program.lean:133` (`Jewel`) |
| 11-10 | The matrix form of the coefficient map (mixing with the identity field and lower-degree fields); completeness of `Show` and the limiting activities | `YM/FullVolume/Mark/Hinge.lean` (`Gram.reads`), `YM/FullVolume/Table/Tabs.lean:66` |

### What the source lacks about the step-0 theory

For details see the table in 11-10 § 05. Here we list only what is not in the source `Source` (within the scope of the name search).

- A definition of the limiting measure at step $0$, and its representation.
- Images at step $0$ of observables whose reference step is above $0$ (`hail` assumes $\psi(d) \le m$).
- A form of the density-level estimates uniform in the volume (only the one-step ledgers are independent of the volume).
- Reflection positivity of the measures at renormalised steps (including step $0$). Renormalisation coarser than step $0$.
- The relation between Wilson loops of step-$0$ configurations and Wilson loops of the fine field (or the continuum) (relation to coarse Wilson loops; 16-03 § 04).
- Explicit values of the entry for the one-step estimate $\mathrm{betaEntry}(j)$ (`YM/Price/Intent.lean:116`) and of the reference coupling $\bar\beta$. The source contains only existence and the conditions `Bar.ready` and `Bar.threshold` (:126-127); from the latter alone, $\bar\beta \gtrsim 9 \times 10^5$ Reading .

04

## Declarations whose target has not been checked

For declarations that are "used" in the verification record (in the dependencies of one of the 17 headline theorems) and whose witness in the record is `SharpGram.habitat`, this book has not checked whether they lie on the chain of the existence theorem. The witness in the record is only the first one found by depth-first search, so this does not mean they are not on it either Source (how the record is made). Those cited in this part are as follows.

- Large fields: `WilsonStartup.due_eq`, `owed_zero`, `Tube.locates`, `Stores.Plot`, `Allocation.flagged`, `Chi.share_sum`, `Bid.Crowds`, `Seals`, `Lapse`, `Teller.Grants`, `Sector.peel`.
- Prices and conversion: `Buy.quote`, `Cash.armK`, `dictionary_fit`, `eventual_resize`, `Time.late_force`, `Time.now`, `Postage.saved`, `Collection.vac`.
- Lower-level small-field lemmas: `tree_gap`, `resolve_point`, `late_cubic`, `Rain.sprint`, `Song.forward`. The road up to the one-step theorem `Return.reset`, however, has been traced literally from the existence-theorem side (11-09 § 01).

Checked to be outside the dependencies (outside): `Remainder.hole_price`, `bulk_price`, `Marginal`, `Haar`, `Leases` and `Grant` in the large-field location, `Allocation.hard`, `Directed.holding` Source .

05

## Cautions for reading

- Many declaration names and file names in the source do not describe their content. Following words in the docstrings ("freeze", "hole", "delay" and so on) as clues can lead to reading paths outside the dependencies. It is safer to check in the verification record whether something is used before reading it.
- The source's docstrings contain many notes to the proof writer (of the form "do not ..."). This book treats them as cautions to the writer and does not cite them as mathematical claims.
- The equation numbers of the paper that the source cites (numbers for the row identity, the body of the truncated action and so on) could not be traced, because the paper has not been published.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The location of each row (file and declaration name) was checked in the source as far as names and types (except those marked as candidates) | The content of the proofs in each row |
| Source Declarations outside the dependencies (`hole_price` and others) are "outside" in the verification record | That paths outside the dependencies are mathematically wrong (nothing is said) |
| Assessment The dependencies of the 17 headline theorems were rechecked in a freshly initialised kernel environment (assessment §9.1) | That this book has checked that the places in this list contain no errors |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Lean source of the published construction | Source | `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0). Locations as in each row |
| Dependency record | Source | `verification/declaration-usage.tsv` (status, headline witness) |
| The committee's assessment | Assessment | `docs/assessment-1.pdf` in `yang-mills-group/YM` (2026-11-10), §9.1 |

**Words used in this chapter** — dependency record, headline theorem, outside the dependencies, chain of the existence theorem. Their meanings are in the A-02 glossary.

Next chapter to read: 12-01 Lattice reflection positivity — two kinds of plane — on to Part 12, infinite volume and reconstruction.

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction Reflection positivity · Kiichi, Shiori and Rin

# Lattice reflection positivity — two kinds of plane

This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — For the Wilson measure on a periodic lattice, the published construction states that the expectation of a function times its mirror image is never negative (reflection positivity), for both planes through lattice sites and planes through link midpoints, along all four coordinate axes, for $\beta \geq 0$. This chapter reads the form of that statement (the class of functions), the skeleton of the proof (split into slices, fix what lies on the plane, turn the interaction crossing the plane into a positive kernel), and how it is used in later chapters of this part.

**Chapters assumed** — 5-03 (Reflection positivity and the transfer matrix), 4-10 (the OS axioms), 8-01 (conventions for the lattice and the Wilson measure in the published construction). The same subject on this book's side is 20-01.

**Said first** — The results of the published construction written here are at the level "the committee's assessment is that they are proved in Lean". What this book has checked goes only as far as reading the form of the definitions and statements in the source; the correctness of the proofs has not been checked. Reflection positivity is a property on a single finite lattice, and the statement is only about the original Wilson measure. Reflection positivity for the measures at renormalisation steps (push-forward measures) is not in what we have read.

The order of this chapter

1. Legend of labels
2. Statement — planes, class of functions, positivity
3. Two kinds of plane and four axes
4. Outline of the mechanism — split into slices, fix what lies on the plane, turn the crossing interaction into a positive kernel
5. How it is used in this part
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

## Statement — planes, class of functions, positivity

The introduction is 5-03. The setting is the periodic lattice $(\mathbb{Z}/n)^4$ of side $n$ (even, $n = 2h$) and the Wilson measure on it for a gauge group $G$ (a closed matrix group $G \subset U(m)$). To the holonomy $U_p$ of a plaquette $p$ assign the "price"

$$\operatorname{price}(U_p) \;=\; 1 - \frac{\operatorname{Re}\operatorname{tr} U_p}{m} \qquad (1)$$

and put the density $Z^{-1}\exp\bigl(-\beta\sum_p \operatorname{price}(U_p)\bigr)$ on the product of Haar measures; this is the Wilson measure $\mu_\beta^{\mathrm{W}}$, and its expectation is $\langle\cdot\rangle\beta$ Source (construction source `YM/Gauge/MatrixGroup.lean:112` `Gauge.price`, `YM/Wilson/Law.lean:87` `law`, :99 `moment`). It differs from this book's convention only by a positive constant multiple of the weight. In the two-dimensional representation of SU(2), $\operatorname{Re}\operatorname{tr} U_p/2$ is this book's $\operatorname{Re} U_p$ (the real part of the quaternion), so the scale of $\beta$ is the same Proved here (for the full conventions see 8-01).

A plane $C$ is determined by four data: direction (coordinate axis) $a$, kind (a plane through lattice sites or a plane through link midpoints), base position $b$, and half-length $h$ ($n = 2h$) Source (`YM/Cells/Cut.lean:40` `Cut`). A plane comes with a "one-sided band" $\Lambda C$ (`sector`). The class of functions is given by the following four conditions.

Definition (the class of functions local to one side of a plane)

A function of configurations $F\colon (\text{link configurations}) \to \mathbb{C}$ is an **inside function** for the band $\Lambda C$ if it satisfies four conditions: (i) gauge invariant, (ii) measurable, (iii) depends only on links with both endpoints in $\Lambda C$ (local), (iv) bounded.

Source Construction source `YM/Cells/Cut.lean:109` `Interior` (fields `gauge`, `borel`, `localSites`, `cost`). Locality is `YM/Lattice/Path.lean:196` `LocalOn` (the

values are equal on two configurations that agree on links with both endpoints in the set).

Statement (lattice reflection positivity)

Let $\beta \geq 0$. For every plane $C$ (both kinds, all four axes, every base position) and every inside function $F$ of $\Lambda_C$,

$$0 \;\leq\; \langle F \cdot F \circ \Theta_C \rangle_\beta \,. \quad (2)$$

Here $\Theta_C$ is the map that pulls back the reflection in the plane $C$ to configurations, and the inequality is in the order on complex numbers (the left-hand side is a non-negative real number).

Source A theorem of the published construction (reflection positivity of the Osterwalder–Seiler type). Lean: `YM.Slice.reflected_positive` (construction source `YM/Slice/Reflection.lean:98`; the only hypothesis is $0 \leq \beta$). The definition of positivity is `YM/Cells/Cut.lean:116` `Positive`.

Source The form of the statement and the four conditions of the class of functions were read in the source. The bundle saying that inequality (2) holds for every $C$ is collected in the field `positive` of `YM/Cells/Certificate.lean:14` `Seam`, and `laws` (same file, :20) produces the whole bundle from $\beta \geq 0$.

**Which target it enters.** In the record of declaration usage (`verification/declaration-usage.tsv`) the status of `reflected_positive` is `used`. In this record, `used` means "reached from one of the 17 headline theorems listed by the verification record"; it does not mean the chain to the public theorem `JointExistence` has been checked (the first headline the record found is `SharpGram.habitat`). We infer that the road to the public theorem goes through `seamless` and `crops` in § 04 Inferred.

02

## Two kinds of plane and four axes

Let $x_a \in \mathbb{Z}/n$ be the coordinate in direction $a$, and $b$ the base position. For each kind of plane, the reflection and band are as follows Source (`Kind.shift`, `Kind.Strict`, `reflect_here` in `YM/Cells/Cut.lean`).

| Kind | Position of the plane | Reflection $x_a \mapsto$ | Band $\Lambda C$ ($r := x_a - b \bmod n$) |
|---|---|---|---|
| Plane through lattice sites (site) | $x_a = b$ and $x_a = b + h$ | $2b - x_a$ | $0 < r < h$ (does not include points on the plane) |
| Plane through link midpoints (link) | $x_a = b - \frac{1}{2}$ and $x_a = b + h - \frac{1}{2}$ | $2b - 1 - x_a$ | $0 \le r < h$ |

The other three coordinates do not move. The direction $a$ may be any of the four axes, and the base position $b$ is also arbitrary.

**Planes through lattice sites.** The mirror fixes the layer of sites $x_a = b$ (and, by periodicity, the opposite layer $x_a = b + h$). The band is the **open** part between the two mirrors and does not include the points on the mirrors. Since the function depends only on links with both endpoints in the band, it depends neither on links on the mirror nor on links in direction $a$ leaving the mirror Source (the note at the top of the source, "site tilings do not read transverse links at seams").

**Planes through link midpoints.** The mirror passes through the middle of links in direction $a$ and fixes no lattice site. The band is the $h$ layers $x_a = b, \dots, b + h - 1$, and reflection moves them to the opposite $h$ layers (`Cut.link_switch`: the image of a band point is not in the band). The links crossing the mirror (from $x_a = b - 1$ to $b$, and from $b + h - 1$ to $b + h$) are not inside links of either band. The source's note says this includes the degenerate case $h = 1$ Source (top of `YM/Slice/Gate.lean`).

**The functions are complex-valued and gauge invariant.** The class of the published construction allows complex-valued functions, and in exchange imposes gauge invariance for both kinds. The conjugate on the left-hand side of (2) is there to make pairs of complex-valued functions positive semi-definite Proved here (setting $F = \sum_i c_i F_i$, (2) says that the matrix $\left(\langle F_j F_i \circ \Theta \rangle\right)$ is positive semi-definite).

> 03
>
> ## Outline of the mechanism — split into slices, fix what lies on the plane, turn the crossing interaction into a positive kernel

The idea is the same as in 5-03 § 04. We write how the source assembles it in three stages Source (as far as the statements and lemma names; the content of each lemma's proof has not been read).

### Stage 1: split the configuration into slices

Cut the lattice into layers by the coordinate in direction $a$. The configuration of links within one layer (links of the three-dimensional lattice perpendicular to direction $a$) is

called a **slice** (`Leaf`), and the set of links in direction $a$ joining adjacent layers a **thread** (`Thread`). The whole configuration is split into four parts.

- Slices on the mirror (`Posts`, for planes through lattice sites)
- Threads crossing the mirror (`Doors`, for planes through link midpoints)
- Slices and threads inside the band (`Open`)
- Their mirror images (the second copy of `Open`; threads read by their inverses)

The lemma is that this split is an isomorphism preserving the product Haar measure Source (`YM/Slice/Independent.lean:165` `board`; that the copy reading threads by inverses preserves the measure is `twisted`, same file :149). It uses the invariance of Haar measure under inversion.

### Stage 2: fixing what lies on the mirror makes the two copies independent and identically distributed

Fix the slices on the mirror and the crossing threads. Under the product Haar measure, the inside of the band and its mirror image are independent and identically distributed. The logarithm of the Wilson weight (the action) splits into plaquettes inside the band, plaquettes on the mirror-image side, and plaquettes touching the mirror. The part not touching the mirror has the form $A(U_>) + A(U_<)$ and can be collected on the same side as $F$.

### Stage 3: the interaction crossing the mirror is a positive kernel

For planes through lattice sites, the function reads neither the slices on the mirror nor the threads leaving it, so for each fixed value on the mirror the integrand takes the form $g(U_>)\,g(U_<)$, and integrating over the two copies gives $\left|\int g\right|^2 \geq 0$ (the kernel is the constant 1).

For planes through link midpoints, the threads crossing the mirror join the slices on both sides of the band. Integrating out the crossing threads gives, as a function of the end slices $X$ and $Y$ on the two sides, the transfer-operator kernel

$$K_\beta(X,Y) \;=\; \int \mathrm{weight}_\beta(X,Y,u)\,du \qquad (3)$$

($u$ is the crossing thread, and weight is the weight of the plaquettes having that thread as an edge) Source (`YM/Slice/Positive.lean:54` `heat`). If $\beta \geq 0$, this kernel is a **positive kernel** (for any finitely many points $X_1, \ldots, X_k$, the matrix $K_\beta(X_i, X_j)$ is positive semi-definite) Source (Lean: `YM.Slice.step_positive`, `YM/Slice/Positive.lean:81`; the definition of a positive kernel is `YM/Analysis/Nonnegative.lean:18` `NNKernel`). For a positive kernel, the paired integral

$$\iint g(X)\, K_\beta(X,Y)\, g(Y)\, d\nu(X)\, d\nu(Y) \;\geq\; 0 \qquad (4)$$

holds; this is the last stage Source (`NNKernel.integrated`, used in `YM/Slice/Reflection.lean:72`; `portPos`, `YM/Slice/Gate.lean:94`, makes the kernels of both kinds of plane positive together: for site planes the kernel is 1, for link-midpoint planes a product of $K_\beta$).

**The roles of gauge invariance and $\beta \geq 0$.** $\beta \geq 0$ is used for the positivity of the kernel in stage 3. Gauge invariance is part of the definition of the class, but where in stages 1–3 it is needed for planes through lattice sites has not been read Inferred (for link-midpoint planes it is needed in handling the crossing threads). In this book's own proof (20-01 § 05), gauge invariance was used for link-midpoint planes to remove the crossing links by setting them to 1 by a gauge. The published construction does not remove the crossing threads; it integrates them into a kernel.

04

## How it is used in this part

Within what we have read, reflection positivity is used in the construction only for **the Wilson measure on the original lattice** Source (within the scope of the name search). The family of the construction has, for each cutoff $N$ and volume $p$, a lattice of side $n = L_p/a_N$ and a coupling $\beta_{N,p}$ (8-02), and the bundle `Seam` is set up from $\beta_{N,p} \geq 0$ for all $(N,p)$ Source (Lean: `YM.Cells.seamless`, `YM/Cells/Packing.lean:81`; $\beta_{N,p}$ is non-negative because the coupling lies in the band $[\hat\beta/2,\, 2\hat\beta]$ and $\hat\beta > 1$).

| Use | What it is applied to | Chapter |
|---|---|---|
| The chessboard inequality and the root triangle inequality | Positivity on all band planes (`bandCut`) | 12-02 |
| Positivity of the transfer-operator kernel | The kernel (3) of planes through link midpoints | 12-04 |
| Reflection positivity of the continuum limit | Non-negativity of the Gram matrix for the plane through lattice sites (coordinate 0) | 12-05 |
| The OS Hilbert space | Reflection positivity of the limiting table (continuum side) | 12-07, 12-08 |

**Not used for push-forward measures.** A statement of reflection positivity for the measures at renormalisation step $m$ (the fine-lattice measure pushed forward by a probability kernel, 9-01) is not in what we have read Source (within the scope of the name search). Since the kernel's patches are designed to overlap across planes, we infer that the push-forward does not inherit reflection positivity automatically Inferred.

The estimates on the construction's observables (12-03) are also stated as expectations under the Wilson measure $\mu^{\mathrm{W}}_{N,p}$ at step $N$ Source .

**How the assessment puts it.** § 5 of the assessment writes that local boundedness and equicontinuity in the infinite-volume limit come from "a general argument based on reflection positivity, the Chessboard Estimate (Fröhlich–Israel–Lieb–Simon), the transfer-operator representation and Cauchy estimates" Assessment .

05

## What this book's study had seen

This book's Lean had three forms of reflection positivity for the Wilson measure on the SU(2) lattice $(\mathbb{Z}/2M)^4$ (20-01).

| | **Published construction (`reflected_positive`)** | **This book** |
|---|---|---|
| Group | All closed matrix groups $G$ | SU(2) (unit quaternions) |
| Planes through lattice sites | $\beta \geq 0$, four axes, gauge invariant, complex-valued, local to the band (not including the mirror) | All real $\beta$, four axes, no gauge invariance imposed, real-valued, local to the closed positive half (including the mirror) (`TorusRP.wilson_rp`, `PeierlsPerm.wilson_rp_dir`) |
| Planes through link midpoints | $\beta \geq 0$, four axes, gauge invariant | $\beta \geq 0$, time direction only, gauge invariant, $M \geq 2$ (`MidRP.wilson_rp_mid`) |
| Handling of crossing links | Integrated into the transfer-operator kernel (a positive kernel) | Removed by setting them to 1 by a gauge; the remaining crossing plaquettes written as sums of products of coordinates |

Lean This book's three theorems are formalisations of results of the Osterwalder–Seiler type (Lean: `TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir`; scope as in the table of 20-01).

There are two differences. First, the published construction has link-midpoint planes along all four axes, while this book had only the time direction. Second, the published construction imposes gauge invariance even for site planes and is limited to $\beta \geq 0$, while this book, for site planes, asked neither the sign of $\beta$ nor gauge invariance. Which class is broader does not go in one direction.

For push-forward measures, this book had, with a different design (side 3, centred blocks, a block map with uniform weights), reflection positivity for site planes in Lean (Lean: `Block3.rp_push`, `Block3.rp_push_prob`; any real $\beta$, bounded measurable functions depending only on links strictly between two coarse site planes). The way

steps are made differs from the design of the published construction (the published construction uses probability kernels, this book deterministic maps), so this is not a comparison of the same object. This design is not this book's main line (28-05).

06

## What has not been read

- **Where gauge invariance is needed for site planes.** We have read only as far as the names of the lemmas of stages 1–3 (`link_positive`, `insidePos` in `YM/Slice/Reflection.lean`).
- **The content of the proof of kernel positivity.** The proof of `step_positive` (`YM/Slice/Positive.lean:81`) has not been opened. Since the weight is a product over plaquettes of $\exp(\beta \operatorname{Re}\operatorname{tr}(\cdot)/m)$, we infer that it expands this into products of positive kernels Inferred .
- **The chain to the public theorem.** `used` in the record means reached from one of the headline theorems; the chain up to `JointExistence` has not been traced step by step.
- **That reflection positivity of push-forward measures is absent.** This judgement is within the scope of a search by names (reflection, RP, positive); we have not read the whole repository.
- **Rotated planes.** `YM/Euclid/OtherPlanes.lean` seems to treat other planes on the continuum side, but it is a location of declarations not used according to the record Source (record). Its content has not been read.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source For $\beta \geq 0$, the Wilson measure on a periodic lattice is reflection positive for both kinds of plane and all four axes, on the class of gauge-invariant, measurable, band-local, bounded complex-valued functions (`YM.Slice.reflected_positive`) | $\beta < 0$. Reflection positivity of the measures at renormalisation steps (push-forward). The continuum limit itself |
| Source The form of the statement, the four conditions of the class of functions, the definitions of plane and band, and the names of the lemmas in the stages of the proof were read in the source | The content of the lemmas' proofs. The role of gauge invariance for site planes |
| Source The bundle `Seam` is set up from $\beta_{N,p} \geq 0$ for all $(N, p)$ of the construction (`seamless`) | That the chain to the public theorem `JointExistence` has been checked declaration by declaration |

| Can say | Cannot say |
|---|---|
| `Lean` Differences from this book's three forms (20-01): the published construction has link-midpoint planes along four axes, and this book, for site planes, does not ask the sign of $\beta$ or gauge invariance | That either class contains the other (neither does) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Reflection positivity in lattice gauge theory | Cited | K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. Introduction in 5-03, references in A-03 |
| (2) and the definitions of plane and class | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Slice/Reflection.lean:98` `reflected_positive`, `YM/Cells/Cut.lean:40` `Cut`, :109 `Interior`, :116 `Positive`, `YM/Lattice/Path.lean:196` `LocalOn`, `YM/Wilson/Law.lean:87` `law`, `YM/Gauge/MatrixGroup.lean:112` `price` |
| Lemmas of the stages of the proof | Source | `YM/Slice/Independent.lean:165` `board`, `YM/Slice/Positive.lean:54` `heat`, :81 `step_positive`, `YM/Slice/Gate.lean:94` `portPos`, `YM/Analysis/Nonnegative.lean:18` `NNKernel` |
| The bundle, and its application to the construction | Source | `YM/Cells/Certificate.lean:14` `Seam`, :20 `laws`, `YM/Cells/Packing.lean:81` `seamless`. Usage in `verification/declaration-usage.tsv` |
| Description in the assessment | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10) § 5 |
| This book's corresponding theorems | Lean | `TorusRP.wilson_rp`, `MidRP.wilson_rp_mid`, `PeierlsPerm.wilson_rp_dir` (20-01), `Block3.rp_push`, `Block3.rp_push_prob`. sorry 0, axioms the standard three |

**Words used in this chapter** — reflection positivity, plane through lattice sites, plane through link midpoints, band, inside function, gauge invariance, slice, thread, transfer-operator kernel, positive kernel, push-forward measure. Their meanings are in the A-02 glossary.

Next chapter to read: 12-02 The chessboard, and the root seminorm — stacking reflection positivity over all band planes to build a seminorm that measures size box by box.

# The chessboard, and the root seminorm

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation and § 07 is a map of open questions.

**What this chapter shows** — Tile the torus with boxes of equal size, carry a function local to one box to every box by reflections, and multiply: the size of the expectation is then bounded by the product of "the expectation when each function is copied to every box, raised to the power one over the number of boxes (the root)" (the chessboard inequality). The published construction also has that the root satisfies the triangle inequality for sums of functions, that is, the root is a seminorm. This chapter reads those two statements, the skeleton of the proof, and their uses in the construction.

**Chapters assumed** — 12-01 (lattice reflection positivity and the class of functions). The idea of the Chessboard Estimate (Fröhlich–Israel–Lieb–Simon) is also in 22-01.

**Said first** — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean". What this book has checked goes as far as the form of the definitions and statements. The chessboard inequality is an inequality on a single finite lattice, and its only hypothesis is "reflection positivity on all band planes". A quantity uniform in the volume comes out because the root is a per-box quantity; bounding the size of the root itself is separate work (12-03).

The order of this chapter

1. Legend of labels
2. Before the statements — tiling, copy, root
3. Statement (1) — the chessboard inequality
4. Statement (2) — the root triangle inequality (the root is a seminorm)
5. Outline of the mechanism — maximisation and reflection, expansion and root
6. Uses in the construction
7. What this book's study had seen
8. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

## Before the statements — tiling, copy, root

Tile the periodic lattice $(\mathbb{Z}/n)^4$ of side $n$ with boxes of side $s$. The boxes are indexed by $(\mathbb{Z}/P)^4$ ($P$ even, $n = Ps$), and the kind of box matches the kind of reflection plane (through lattice sites or through link midpoints) Source (construction source `YM/Cells/Tiling.lean:19` `Tiling`: fields `period` = $P$, `units`, `widthHalf` = $P/2$, `kind`, `base` = the corner of the base box). The **inside** of box $v$ (`chamber v`) is, by the same rule as the bands of 12-01, the set of points that excludes the box's boundary (site boxes) or includes one side of the boundary (link-midpoint boxes).

**Copies.** A function $F$ local to the inside of the base box 0 is carried to box $v$. The way to carry it is: for each axis $j$, if $v_j$ is odd insert one reflection in direction $j$, then translate Source (same file, :58 `omega`, :63 `translateBy`, :65 `copyMap`). In boxes with an odd number of reflections, complex conjugation is applied:

$$\operatorname{copy}_v F(U) \;=\; \begin{cases} F(\vartheta_v^* U) & \text{(odd number of reflections)} \\ F(\vartheta_v^* U) & \text{(even)} \end{cases} \tag{1}$$

Here $\vartheta_v^*$ is the pull-back to configurations of the map taking box 0 to box $v$ Source (same file, :220 `hand`, :222 `copy`). The copies in two adjacent boxes are mirror images of each other across the plane between them. Like the white and black squares of a chessboard, the orientation flips every other box.

**Root.** The size of the expectation of the product of copies of a single function $F$ over all boxes, raised to the power one over the number of boxes, is written

$$\|F\|_{\square} \;:=\; \Big|\Big\langle \prod_v \operatorname{copy}_v F \Big\rangle_\beta\Big|^{1/P^4} \tag{2}$$

and called the **chessboard root** Source (same file, :226 `rootTest`). The subscript $\square$ indicates which tiling it is measured with. The root is a per-box quantity. Since the same function is placed in every box, even when the volume grows and the number of boxes increases, it can be compared as a per-box contribution.

**Example** Proved here . If $F \equiv c$ (a constant), then $\|F\|_\square = |c|$. If $F$ is the indicator $\mathbf{1}_E$ of some event in box 0, then $\|F\|_\square^{P^4}$ is "the probability that the (copied) event occurs in all boxes simultaneously", and $\|F\|_\square$ is the per-box rate of that probability.

02

## Statement (1) — the chessboard inequality

Statement (the chessboard inequality)

Suppose reflection positivity holds on every band plane of the tiling (planes through box boundaries, one for each direction $a$ and position $p$). Choosing for each box an inside function $F_v$ of box 0 (the class of 12-01: gauge invariant, measurable, local, bounded),

$$\left|\left\langle \prod_v \mathrm{copy}_v\, F_v \right\rangle_\beta\right| \;\le\; \prod_v \|F_v\|_\square. \quad (3)$$

Source A theorem of the published construction (the Chessboard Estimate of the Fröhlich–Israel–Lieb–Simon type). Lean: `YM.Cells.Tiling.tiles` (construction source `YM/Cells/Chess.lean:80`; its name in the source's docstring is WFS1; the hypotheses are `Positive` on every `bandCut`, and that each $F_v$ is an inside function of box 0).

Source The band planes are `YM/Cells/Tiling.lean:40` `bandCut` (half-length $s \cdot P/2$, direction $a$, base at the corner of box $p$). The size of the expectation on the left-hand side is written as `YM/Cells/Polar.lean:91` `rootPower` ($\|\langle \prod_v \mathrm{copy}_v\, F_v \rangle\|$).

**Meaning.** The right-hand side is the product of "the per-box size when each function is replicated in every box". A correlation with different functions in different boxes is bounded by correlations with the same function in all boxes. This is the result of stacking the Cauchy–Schwarz inequality that comes from reflection positivity many times, over all planes ( § 04).

03

## Statement (2) — the root triangle inequality (the root is a seminorm)

Statement (the root triangle inequality)

Under the same hypotheses, for two inside functions $F, F'$ of box 0,

$$\|F + F'\|_\square \;\le\; \|F\|_\square + \|F'\|_\square. \quad (4)$$

Source A theorem of the published construction. Lean: `YM.Cells.Tiling.adds` (construction source `YM/Cells/Chess.lean:108`; its name in the docstring is WFS2).

The root satisfies $\|cF\|_\square = |c|\,\|F\|_\square$ for scalar multiples Proved here (the source's `rootPower_scale`, `YM/Cells/Chess.lean:70`, states only non-negative real multiples; it extends to complex $c$ because, by the definition (1) of copies taking conjugates in boxes with an odd number of reflections, the product is multiplied by a factor of modulus $|c|^{P^4}$). Together with (4), the root is a **seminorm** on the space of inside functions of box 0 (it need not be a norm, since $F \neq 0$ is possible with $\|F\|_\square = 0$) Proved here .

**The bundle.** Reflection positivity, the chessboard inequality and the root triangle inequality are collected in one bundle `Seam` (`YM/Cells/Certificate.lean:14`: fields `positive`, `inequality`, `semi`), and `laws` produces all three at once from $\beta \geq 0$ Source . This bundle holds for every cutoff $N$ and volume $p$ of the construction (`seamless`, `YM/Cells/Packing.lean:81`; 12-01 § 04).

**Why a seminorm is useful.** When an observable splits into "a main part + a small remainder", (4) lets the roots be added part by part. In the ultraviolet estimates, when a renormalised observable is held as a sum of many terms, bounding the root of each term and adding them is enough to bound the root of the whole (12-03) Inferred (a reading from the names of the places where it is used).

04

## Outline of the mechanism — maximisation and reflection, expansion and root

We write the source's line as an argument of the Fröhlich–Israel–Lieb–Simon type Source (lemma names and statements; details are inference).

### Stage 1: Cauchy–Schwarz on a single plane

A single band plane $C$ divides the boxes into two halves. Splitting the product of an assignment of boxes $F_\bullet$ into the part $\Pi_+$ on the positive side of the plane and the part on the negative side, the negative side becomes the mirror image of another assignment $\Pi'_+$ on the positive side Source (`YM/Cells/Copies.lean:124` `reflect_equation`: the product with the assignment rearranged into "half $F$, the other half the mirror image of $Q$" becomes $\Pi_+(F)\cdot\Pi_+(Q)\circ\Theta_C$). Reflection positivity makes the pairing $(X,Y)\mapsto\langle X\cdot Y\circ\Theta_C\rangle$ positive semi-definite, so by Cauchy–Schwarz

$$\left|\langle \Pi_+(F)\,\Pi_+(Q)\circ\Theta_C\rangle\right|^2 \;\leq\; \langle \Pi_+(F)\,\Pi_+(F)\circ\Theta_C\rangle\,\langle \Pi_+(Q)\,\Pi_+(Q)\circ\Theta_C\rangle. \quad (5)$$

Both factors on the right-hand side are expectations of assignments in which "one side's assignment is replicated to the other side by the mirror".

#### Stage 2: a maximising assignment stays maximal when symmetrised by reflection

Consider assignments $A$ that choose one of finitely many candidate functions $f_1, \dots, f_k$ for each box, and take an $A$ maximising $\|\langle \prod_v \mathrm{copy}_v\, f_{A(v)} \rangle\|$. By (5), replicating a maximal assignment from one side across any plane keeps it maximal. Repeating this over planes of every direction and position, the assignment that places the same candidate $f_c$ in every box becomes one of the maxima Source (`YM/Cells/Chess.lean:32` `maximize`: there is a $c$ such that, for every assignment $A$, the value is at most the value with $f_c$ placed in every box).

#### Stage 3: normalise to get (3)

Applying stage 2 to the functions $F_v$ divided by $\|F_v\|_\square + \varepsilon$ makes the left-hand side $\le 1$, and $\varepsilon \to 0$ gives (3) Source (the shape of the proof of `tiles`: add $\varepsilon$, divide, take the limit).

#### Stage 4: expand and take the root

Expanding $\prod_v \mathrm{copy}_v (F + F')$ gives a sum over the $2^{P^4}$ assignments choosing $F$ or $F'$ for each box. Applying (3) to each term,

$$\|F + F'\|_\square^{P^4} \;\le\; \sum_{\text{assignments}} \prod_v \|F_{\text{chosen}}\|_\square \;=\; \left(\|F\|_\square + \|F'\|_\square\right)^{P^4} \tag{6}$$

and taking the $P^4$-th root gives (4) Proved here (the source's `adds` has as a lemma the expansion into a sum over assignments $i : \mathrm{boxes} \to \{\mathrm{true}, \mathrm{false}\}$).

05

## Uses in the construction

**The centred tiling.** The tiling the construction uses has boxes of side equal to the lattice spacing $a_m$ at step $m$, with the base box placed with the observation point $o$ at its **centre** Source (`YM/Cells/Packing.lean:34` `center`: the kind is link-midpoint boxes, a box side is an odd number of fine lattice points, the physical side is $a_m$ (`trueLength`), and the reciprocal of the number of boxes is $(a_m / L_p)^4$ (`root_fraction`)). The root for this tiling is written $\|\cdot\|_{m,o}$.

| Use | What it does | Location | Chapter |
|---|---|---|---|
| Form of the input from the ultraviolet | Imposes upper bounds on the roots $\|\cdot\|_{m,o}$ of renormalised observables | `YM/Extraction/Center.lean:21` `centeredBudget`, :38 `Wcmain` | 12-03 |
| Local boundedness | Encloses observables at separated points in separate boxes, and bounds the correlation by a product of roots via (3) | `YM/Extraction/Held.lean:120` `Around.size_done` | 12-03, 12-05 |
| Equicontinuity | Bounds the norm of the "column" of the transfer-operator representation by roots | `YM/Extraction/Difference.lean:51` `reading` | 12-04 |
| Estimates on the ultraviolet side | Adds the roots of terms by the triangle inequality (field `semi`) | `YM/FullVolume/Rent.lean` and other places using `.semi` Source (search) | Part 11 |

**How the assessment puts it.** § 5 of the assessment writes that boundedness and equicontinuity in the infinite-volume limit come from "general mechanisms not specific to Yang–Mills" — reflection positivity, the Chessboard Estimate, the transfer-operator representation and Cauchy estimates — and that the input specific to the construction is control of renormalised lattice observables Assessment .

06

## What this book's study had seen

This book's Lean also had a Chessboard Estimate for the Wilson measure on the SU(2) lattice $(\mathbb{Z}/2m)^4$.

Lean For the expectation $V$ of the product of a family of bounded measurable functions depending only on the links of a unit cube, carried to each block: $|V(F)|^{N^4} \le \prod_w V(\text{constant family } F_w)$. All even sides, all real $\beta$. A formalisation of the Fröhlich–Israel–Lieb–Simon type. Lean: `FoldWilson.wilson_chessboard_even` (22-01).

There are three differences.

- **The use was different.** This book used the chessboard for **the rarity of large plaquettes** (the probability that $\mathrm{Re}\,U_p < 0$ on every plaquette of a set $X$ is at most $q^{\lfloor |X|/6 \rfloor}$, with $q$ independent of the volume; Lean: `FoldCap.rarity_explicit`, $\beta \ge 81$). The published construction uses the chessboard only for estimating the size of observables, and a statement about rarity of plaquettes is not in what we have read Inferred (within the scope of the search).

- **The root triangle inequality was not in this book's study.** This book applied its estimate only to one form, products of indicator functions, and did not use the root as a seminorm on a space of functions.
- **How the boxes are made.** This book's boxes are unit cubes (boxes of lattice spacing 1); the published construction places boxes of the step-$m$ spacing with the observation point at the centre.

This book's statement of rarity has a form that can be restated in the language of roots Proved here : setting $F = \mathbf{1}\{\text{at a chosen plaquette in the box},\ \operatorname{Re} U_p < 0\}$, the rarity estimate states an upper bound on the root $\|F\|_{\square}$. The details of this restatement (box size and choice of plaquettes) have not been checked. The restatement is treated in Part 22 of Volume 3.

07

## What has not been read

- **Details of the maximisation argument.** We read the statement of the type "stays maximal under reflection" ( `T.Reflects` ) inside `maximize` ( `Chess.lean:32` ), but not how the induction that walks over all band planes is organised.
- **The exact shape of the inside of link-midpoint boxes.** We read the definition of `chamber` ( `Tiling.lean:36` ), but have not checked the treatment of the boundaries of both kinds of box in every case.
- **How `semi` is used on the ultraviolet side.** We only found the places using it by name; we have not read in which stage of which estimate it is used.
- **The chain to the public theorem.** `tiles` and `adds` are `used` in the record (reached from one of the headline theorems). The chain to the public theorem has not been traced declaration by declaration.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The chessboard inequality (3), from reflection positivity on all band planes ( `Tiling.tiles` ) | Application to measures without reflection positivity (push-forward measures and so on) |
| Source The root triangle inequality (4); the root is a seminorm ( `Tiling.adds` ) | That the root is a norm (the root of a non-zero function can be 0) |
| Source The definitions of tiling, copy, root and centred tiling were read in the source | Details of the induction in the maximisation |

| Can say | Cannot say |
|---|---|
| Proved here (4) follows from expanding (3) and taking roots ((6)) | — |
| Lean This book's Chessboard Estimate (`FoldWilson.wilson_chessboard_even`) and rarity of large plaquettes (`FoldCap.rarity_explicit`) | Details of the form restating rarity in the language of roots. A root triangle inequality on this book's side |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| The Chessboard Estimate (from reflection positivity) | Cited | Reflection positivity and the Chessboard Estimate of J. Fröhlich, R. Israel, E. H. Lieb, B. Simon (bibliographic details as in 22-01 and A-03) |
| (3), (4) and definitions | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Cells/Chess.lean:80` `tiles`, :108 `adds`, :23 `powering`, :32 `maximize`, `YM/Cells/Tiling.lean:19` `Tiling`, :40 `bandCut`, :222 `copy`, :226 `rootTest`, `YM/Cells/Polar.lean:91` `rootPower`, `YM/Cells/Copies.lean:124` `reflect_equation` |
| The bundle and the centred tiling | Source | `YM/Cells/Certificate.lean:14` `Seam`, `YM/Cells/Packing.lean:34` `center`, :81 `seamless` |
| Uses | Source | `YM/Extraction/Center.lean:38` `Wcmain`, `YM/Extraction/Held.lean:120` `size_done`, `YM/Extraction/Difference.lean:51` `reading` |
| Description in the assessment | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` § 5 |
| This book's chessboard and rarity | Lean | `FoldWilson.wilson_chessboard_even`, `FoldCap.rarity_explicit` (22-01). sorry 0, axioms the standard three |

**Words used in this chapter** — Chessboard Estimate, tiling, inside of a box, copy, chessboard root, seminorm, band plane, centred tiling, rarity of large plaquettes. Their meanings are in the A-02 glossary.

Next chapter to read: 12-03 The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm — we read the form of the input that imposes volume-uniform upper bounds on the roots of renormalised observables.

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction From the ultraviolet to infinite volume · Kiichi, Shiori and Rin

# The estimate passed from the ultraviolet to infinite volume — the size of observables measured by the root seminorm

> This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — The published construction copies renormalised local observables with a tiling of boxes centred on the observation point, and has, in a form uniform over all volumes, the estimate that their chessboard root (12-02) is at most $A\,a^{d-\varepsilon}_{m}$ (the Lean name is `Wcmain`). This estimate is the input to the compactness argument that takes the infinite-volume limit. This chapter reads the form of that statement, the mechanism by which it turns into local boundedness, and the fact that "this is one of the inputs received from the ultraviolet, and there are others".

**Chapters assumed** — 12-02 (roots and centred tilings), 8-04 (renormalised insertions), 8-02 (the sequence of spacings $a_m$, the sequence of volumes $L_p$).

**Said first** — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean"; what this book has checked goes only as far as reading the form of the statements. `Wcmain` is the input to infinite-volume compactness (`crops`), not **all** of the inputs passed from the ultraviolet to the infrared side. Other estimates (`Scrim` and so on) are passed at the same time from the same run of the construction. How `Wcmain` itself is proved (the content of the estimates on the ultraviolet side) is not read in this chapter.

The order of this chapter

1. Legend of labels
2. Statement — the size of renormalised observables measured by the root
3. How to read the statement — what it is uniform in, and what it depends on
4. Outline of the mechanism (1) — from upper bounds on roots to local boundedness

5. Outline of the mechanism (2) — the hand-over to compactness, and other inputs
6. What this book's study had seen
7. What has not been read

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Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

## Statement — the size of renormalised observables measured by the root

We fix notation (8-02). Let $a_N$ be the lattice spacing at cutoff $N$, $L_p$ the physical side of the box with volume index $p$, and $\mu^{\mathrm{W}}_{N,p}$ the Wilson measure on the lattice with cutoff $N$ and volume $p$ (coupling $\beta_{N,p}$). The spacing $a_m$ at step $m \le N$ is the scale on the coarse side. To a label $v$ of a local field of degree $d$ (a vector of coefficients, 8-03) corresponds a renormalised insertion $O_{N,p}(v;o)$ (a polynomial in finitely many links placed at the lattice point $o$; 8-04) Source ( `C.seen v o` , `C.O N p v x` in the construction source).

Write $\|\cdot\|_{m,o}$ for the root of the tiling by boxes of physical side $a_m$ with the lattice point $o$ at the centre (12-02 § 05, `YM/Cells/Packing.lean:34` `center` ).

Form of the statement (estimate of renormalised observables measured by the root)

For every degree $d$, radius $R > 0$ and $\varepsilon > 0$, there is a constant $A \ge 0$ such that for every step $m$ at or above the reference step $r(d)$, for all sufficiently large cutoffs $N$, for every volume $p$, every label $v$ with $\|v\| \le R$ and every lattice point $o$,

$O_{N,p}(v;o)$ is an inside function of the centre box, and $\quad \big\|O_{N,p}(v;o)\big\|_{m,o} \;\le\; A\, a_m^{-d-\varepsilon}.$

The root is taken with $\mu^{\mathrm{W}}_{N,p}$ (coupling $\beta_{N,p}$).

Source This is the content of the predicate `YM.Extraction.Wcmain` in the construction source ( `YM/Extraction/Center.lean:38` (lines 38–40)). The estimate

for one box is `centeredBudget`, same file :21 ($m \le N$, $r(d) \le N$, and for every $o$, "is an inside function" and "root ≤ bound").

`Wcmain` is a predicate (the name of a property); that the construction's family satisfies it is given by other theorems. For example, `Wcmain C` appears in the conclusions of `oboes` in `YM/FullVolume/Dusk/Graff.lean:20` and `noel` in `YM/MergeProof/Noel.lean:24` Source . Which theorem lies on the road to the public theorem `JointExistence` has not been checked.

02

## How to read the statement — what it is uniform in, and what it depends on

| Quantity | Treatment in (1) | Meaning |
|---|---|---|
| Volume $p$ | Uniform ($\forall$ p inside $\exists$ A) | The key uniformity for passing to infinite volume |
| Observation point $o$ | Uniform | The same bound under translation |
| Cutoff $N$ | For sufficiently large $N$ (with a threshold for each $m$) | A form that can be stated on the side of the ultraviolet limit |
| Step $m$ | Depends through the power $a_m^{-d-\varepsilon}$ | The naive dimension $a^{-d}$ of a field of degree $d$, with a loss of $\varepsilon$ |
| Label $v$ | Uniform over $\|v\| \le R$ | On a bounded set of coefficients |

**Why measuring by the root can give a quantity uniform in the volume.** The root is the per-box quantity of "the expectation with the same function copied to every box" (12-02 (2)). As the volume grows the number of boxes increases, but the per-box quantity does not. That is why a volume-uniform upper bound like (1) is meaningful Proved here . However, that the bound actually holds uniformly in the volume is the **content** of (1), and the work of proving it lies on the ultraviolet side (Part 11). The one-step ledgers on the ultraviolet side (norms of activities, contraction constants) are written as pointwise sums and do not depend on the volume, but estimates at the level of densities and partition functions include an exponential in the volume Source (the activity ledgers `EBound`, `KBound` in `YM/RG/Activities.lean:98-104` are sums per lattice point; `refreeze` in `YM/FullVolume/Fields/Flight.lean:28` contains the number of lattice points in an exponent; 11-10). The chain showing from which estimate the volume uniformity of (1) comes has not been read ( § 06).

**Reading $a_m^{-d}$.** A field of degree $d$ (weight 2 for curvature, weight 1 for derivatives) has the dimension of length to the power $-d$. The right-hand side of (1) is the dimensional

count that "the typical size of a field placed in a box of side $a_m$ is $a_m^{-d}$", with a margin of $\varepsilon$ added Inferred .

**How the assessment puts it.** § 5 of the assessment writes, about the argument for boundedness and equicontinuity in infinite volume, that "the input specific to the construction is control of renormalised lattice observables uniform in the spacing" Assessment . (1) is one form of that input.

03

## Outline of the mechanism (1) — from upper bounds on roots to local boundedness

Boundedness of $n$-point functions at separated points follows from (1). The line is as follows Source (statements and proof stages of `YM/Extraction/Held.lean:74` `fits_budget` and :120 `Around.size_done`).

1. Suppose the continuum points $x_1, \dots, x_n$ are pairwise at distance at least $r$. Choose a step $m$ with $a_m < r$, and tile with boxes of spacing $a_m$.
2. Shift the position of the tiling so that each point falls inside a different box (sufficiently far from its boundary). Since the points are separated, no two points fall in the same box (the injection $D$ in `fits_budget`).
3. Each observable $O(v_i; x_i)$ is the copy to box $D(i)$ of an inside function $f_i$ of the base box. Boxes without an observable get the constant 1.
4. By the chessboard inequality ((3) of 12-02), the size of the $n$-point function is at most the product of the roots of the $f_i$. The root of the constant 1 is 1.

$$\left| \left\langle \prod_{i=1}^{n} O_{N,p}(v_i; x_i) \right\rangle_{\mu_{N,p}} \right| \;\le\; \prod_{i=1}^{n} \|f_i\|_{\square} \;\le\; A^n. \quad (2)$$

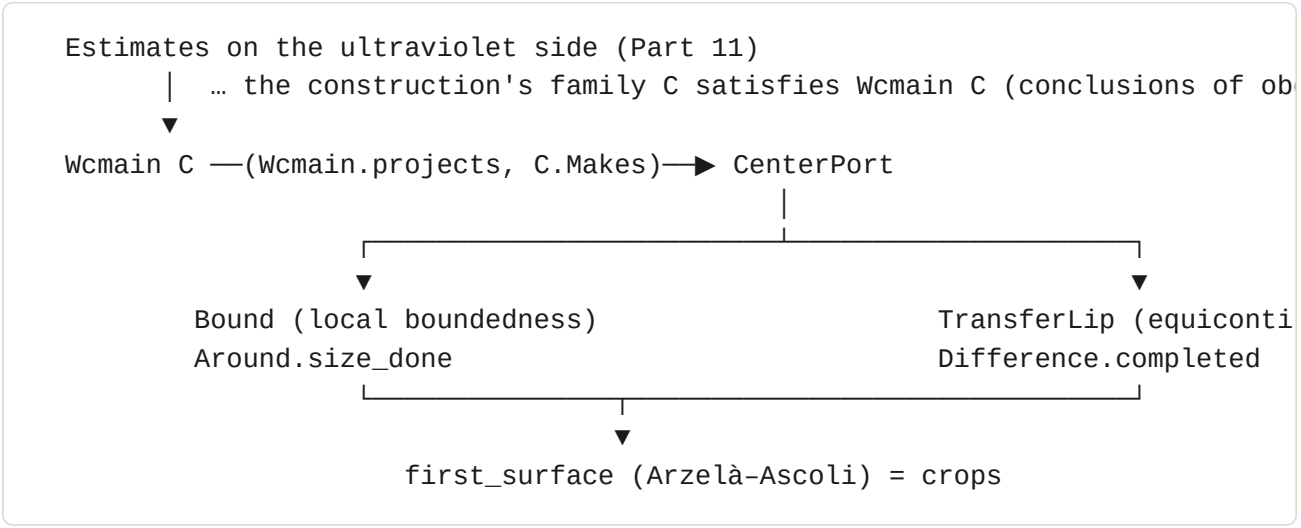
Here $A$ is the per-box bound coming from (1) Source (the conclusion $\|\cdot\| \le A^n$ of `fits_budget`). This bound depends neither on the volume $p$ nor on the positions of the points. This is `Extraction.Bound` (`YM/Extraction/Integrate.lean:25`), corresponding to the first part of equation (11) of the assessment (uniform boundedness on compact sets) Source . `Bound` has two fields: `cyclic` (volume fixed, uniform in the cutoff) and `volume` (uniform in both volume and cutoff, for sufficiently late volumes).

Equicontinuity (the second part of equation (11) of the assessment) is obtained by combining (1) with the transfer-operator representation. That is read in 12-04.

04

### Outline of the mechanism (2) — the hand-over to compactness, and other inputs

**The form of the hand-over.** (1) is converted into the form `CenterPort` along an extraction sequence (a pair of a subsequence of cutoffs and a subsequence of volumes) (`Center.lean:30`: fields `per` and `cells`) Source (`Wcmain.projects`, same file :42; it takes the construction's certificate `C.Makes` as a hypothesis). The field `cells`, `CellHyp` (`YM/Cells/Packing.lean:68`), holds the bundle `Seam` for all $(N, p)$ (12-01, 12-02) and a statement of the root bound with a margin $\eta$ for sufficiently late volumes. From `CenterPort` come the boundedness `Bound` and the equicontinuity `TransferLip`, which are passed to the Arzelà–Ascoli argument `first_surface` (`crops`, `YM/Extraction/Held.lean:136`; 12-05).




**This is not the only input from the ultraviolet.** The lemma using `Wcmain`, `MergeProof.Accent.rated` (`YM/MergeProof/Powder.lean:48`), takes, together with `Wcmain C`, a bundle of estimates `Scrim` and hypotheses about rank and metric Source. This is the main basis for "there are others". As supporting evidence, the bundle `Run.Ties` (`YM/FullVolume/Cross.lean:65-71`) that the construction's run carries out of the ultraviolet side at once also has, alongside `control : Wcmain`, the fields `tabs`, `scrim`, `fridge`, `receipts` and `detectors` Source (in the record of declaration usage, `Run.Ties` and the field `detectors` are `used`, and the field `control` is `outside`). So, precisely, we read that **`Wcmain` is the input to compactness (`crops`), and there are other inputs from the ultraviolet**. What the other inputs are used for (quotients of the space of fields, the stress tensor, the operator product expansion and so on) concerns chapters outside this part (Part 13) and Part 11 Inferred.

`Wcmain` also appears as a hypothesis of lemmas on the ultraviolet side (for example `YM/FullVolume/Rent.lean:120` `primaries`) Source. It is not a one-way "output" but an estimate referred to many times within the construction.

05

## What this book's study had seen

**A statement of this form did not exist in this book's study.** This book's results on the ultraviolet side (Part 15) were statements at a single scale, such as two-sided bounds on the covariance of one step of the free field and the linearisation of the block map. There is no estimate stating the size of renormalised local observables in a volume-uniform seminorm. This book's conditional chain theorem (28-01, Lean: `BoxWalk.chain_end_to_end_box`) was an implication that put rarity for the push-forward measures, the splitting of observables, reflection positivity and so on in its hypothesis fields, and concluded the existence and properties of subsequential limits. The quantities placed in the hypotheses there are not defined from the lattice.

(1) in the published construction is one answer to "in what form is it enough to hand ultraviolet results over to the infrared argument". It can be read as the position where this book's ultraviolet output should be placed Inferred . This book's rarity of large plaquettes (22-01, Lean: `FoldCap.rarity_explicit`) can be restated as bounding the root of an indicator function of a unit-size box under the original Wilson measure (12-02 § 06). That, however, is not an estimate of renormalised observables, and cannot replace (1).

**On the source of volume-uniform estimates.** This book's infinite-volume limit inside the strong-coupling window (20-03) obtained volume-uniform decay from Dobrushin's uniqueness argument. (1) in the published construction expresses uniformity by imposing an upper bound on a per-box quantity, the chessboard root. The ways of expressing uniformity differ (from which estimate the volume uniformity of (1) comes has not been read, as in § 06).

06

## What has not been read

- **The proof of (1).** The content of the theorems having `Wcmain C` as conclusion (and how `oboes`, `noel`, `gardena` are used) has not been read. The chain showing from which ultraviolet estimate the volume uniformity comes has not been checked either.
- **Which theorem lies on the road to the public theorem.** There are several theorems with `Wcmain` as conclusion, and which one the proof of `JointExistence` goes through has not been traced. The headline for `used` in the record is `SharpGram.habitat`.
- **The content of the other inputs.** What the fields `Scrim`, `tabs`, `fridge` and `receipts` say, we have only seen by name.

- **Where the $\varepsilon$ loss comes from.** Where the $\varepsilon$ in $a_{\overline{m}}^{d-\varepsilon}$ comes from has not been read.
- **How the reference step $r(d)$ is fixed.** The definition of the reference step for each degree (`C.ref d`) has not been traced (material for 8-04).

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The form of the estimate (1) of renormalised observables measured by the root: uniform in volume and observation point, $a_{\overline{m}}^{d-\varepsilon}$ in the step $m$, for sufficiently large cutoffs (`Wcmain`) | The content of the proof of (1). From which estimate the volume uniformity comes |
| Source From (1) and the chessboard inequality, $n$-point functions at separated points are bounded uniformly in the volume (2) (`fits_budget`, `size_done`) | Estimates where points coincide (not treated) |
| Source `Wcmain` is the input to compactness (`crops`). Other inputs such as `Scrim` are passed from the same run (`rated`, `Run.Ties`) | It cannot be said that "there is a single input from the ultraviolet". The content of the other inputs |
| Inferred $a_{\overline{m}}^{d}$ is the dimensional count for a field of degree $d$ | Where the $\varepsilon$ comes from |
| This book has no estimate of the same form | Restating this book's ultraviolet results in the form of (1) |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| The predicate of (1) and the box estimate | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Extraction/Center.lean:21` `centeredBudget`, :30 `CenterPort`, :38 `Wcmain`, :42 `Wcmain.projects`, `YM/Cells/Packing.lean:34` `center`, :68 `CellHyp` |
| Local boundedness | Source | `YM/Extraction/Held.lean:74` `fits_budget`, :120 `Around.size_done`, :136 `crops`, `YM/Extraction/Integrate.lean:25` `Bound` |
| Other inputs | Source | `YM/MergeProof/Powder.lean:48` `Accent.rated`, `YM/FullVolume/Cross.lean:65-71` `Run.Ties`, `YM/FullVolume/Rent.lean:120` `primaries` |
| Theorems with (1) as conclusion | Source (statements only) | `YM/FullVolume/Dusk/Graff.lean:20` `oboes`, `YM/MergeProof/Noel.lean:24` `noel` |
| Description in the assessment | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` § 5 and equation (11) |
| Related results in this book | Lean | `BoxWalk.chain_end_to_end_box` (implication, 28-01), `FoldCap.rarity_explicit` (22-01) |

**Words used in this chapter** — renormalised insertion, chessboard root, centred tiling, uniform in the volume, local boundedness, equicontinuity, extraction sequence, compactness. Their meanings are in the A-02 glossary.

Next chapter to read: 12-04 The transfer operator and equicontinuity — the difference of correlations when an insertion is moved is bounded by the trace representation of the transfer operator and Hölder's inequality.

# The transfer operator and equicontinuity

This chapter is established knowledge (a reading of the published construction). § 05 is this book's own investigation and § 06 is a map of open questions.

**What this chapter shows** — When one insertion is moved $\ell$ steps on the lattice, how much does a correlation at separated points change? The published construction bounds this in the form $\left(\prod_i c_i\right) \ell / r$ ($r$ is the margin between points, $c_i$ the upper bound on the chessboard root of each insertion). The tools are positivity of the lattice transfer operator, the representation of expectations as traces of products of powers of the transfer operator and insertion operators, re-assigning gaps, and Hölder's inequality. This chapter reads the form of that statement and the mechanism. Volume-uniform equicontinuity becomes the second input to the Arzelà–Ascoli argument (12-05).

**Chapters assumed** — 12-01 (positivity of the transfer-operator kernel), 12-02 (roots), 12-03 (upper bounds on roots). The introduction to the transfer matrix is 5-03.

**Said first** — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean", and what this book has read goes as far as the form of the statements. The proof of the body of the gap re-assignment estimate (`TimeKernel.passage`) has not been read. In the construction, the lattice transfer operator is used only for equicontinuity; analytic continuation in the time direction and the spectral condition are obtained separately from the semigroup on the continuum side of the limit (12-07).

The order of this chapter

1. Legend of labels
2. Statement (1) — the transfer-operator kernel and its positivity
3. Statement (2) — estimate of the difference when gaps are re-assigned
4. Outline of the mechanism — trace representation, re-assignment, Hölder
5. The form that moves one insertion, and assembling equicontinuity
6. What this book's study had seen
7. What has not been read

Labels: Source = read in the source of the published construction / Inferred = this book's inference / Assessment = according to the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (for details see "Labels" in 6-00). The level at which theorems of the published construction are "proved in Lean" rests on the committee's assessment Assessment .

01

## Statement (1) — the transfer-operator kernel and its positivity

The introduction is 5-03. Choose one time direction $a$ (any of the four axes) and cut the lattice into layers perpendicular to direction $a$. The configuration of links within one layer is called a **slice** $X$, and the set of links in direction $a$ joining adjacent layers a **thread** $u$ (12-01 § 03). Let $\mathrm{weight}_\beta(X, Y, u)$ be the plaquette weight of two adjacent slices $X$, $Y$ and the thread $u$ between them; integrating the thread over Haar measure,

$$K_\beta(X, Y) \;=\; \int \mathrm{weight}_\beta(X, Y, u)\, du \qquad (1)$$

is the kernel of the one-step transfer operator Source (construction source `YM/Slice/Positive.lean:54` `heat`). The weight is positive (`weight_pos` in `YM/Slice/Hamilton.lean`).

Statement (properties of the transfer-operator kernel)

(i) If $\beta \geq 0$, the kernel $K_\beta$ is a positive kernel: for any finitely many slices $X_1, \ldots, X_k$, the matrix $\big(K_\beta(X_i, X_j)\big)$ is positive semi-definite. (ii) Reversing the direction of the thread swaps the arguments of the kernel (adjoint under time reversal).

Source Theorems of the published construction. Lean: `YM.Slice.step_positive` (`YM/Slice/Positive.lean:81`, hypothesis $0 \leq \beta$), `YM.Slice.transition_reflect` (`YM/Slice/Hamilton.lean:121`).

From (i) and (ii), the operator $T$ that integrates against the kernel is a positive self-adjoint operator on $L^2$ of the space of slices Proved here (symmetric by (ii), positive by (i)). The operators of insertions (observables) can also be written as multiplication by functions of a slice, or as "blocks" spanning several layers (`Brick`: an operator that inserts a function local to a band of layers of width $w$ between products of transfer operators) Source (the name and usage of `Brick`; its full definition has not been read).

02

## Statement (2) — estimate of the difference when gaps are re-assigned

On the time circle (length $n$), place blocks $B_1, \dots, B_k$ of width $w$ in this order, and let the gaps between blocks be $Y_1, \dots, Y_k$ (numbers of layers) ($kw + \sum_i Y_i = n$). Write the expectation of the product for this arrangement as $\langle B; Y \rangle_\beta$.

The **column** of a block $B_i$ is the arrangement in which $B_i$ and its mirror image $\vartheta B_i$ alternate, $p$ pairs, filling the time circle ($n = 2pw$) Source (`YM/Slice/Calendar.lean:60` `column`). A column is the chessboard copy in one direction, the time direction.

Statement (Lipschitz estimate for re-assigning gaps)

Let $\beta \geq 0$, $n = 2pw$, all gaps larger than $2r$ ($Y_i > 2r$), and let the gap tuple $Z$ be obtained from $Y$ by moving $\ell$ layers between two gaps $i \neq j$ ($Z_i = Y_i + \ell$, $Z_j = Y_j - \ell$, others unchanged), with $\ell \leq r$. If the power of the size of the column expectation $\| \langle \mathrm{column}(B_i) \rangle \|^{\rho_p}$ ($\rho_p = 1/(2p)$; `Holder.rate p` in the source, `YM/Complex/Gaps.lean:105`) is at most $K_i$, then

$$\left| \langle B; Z \rangle_\beta - \langle B; Y \rangle_\beta \right| \;\leq\; \frac{\prod_i K_i}{r}\, \ell. \quad (2)$$

Where on the time circle the sequence of blocks starts to be read (the base position) may differ between the two sides (the source reads the two words with separate base positions `b`, `c`).

Source A theorem of the published construction. Lean: `YM.Slice.Time.powering` (`YM/Slice/Calendar.lean:72`). The body of the estimate is `YM.TimeKernel.passage` (`YM/TimeKernel/Unit.lean:76`).

**Meaning.** Shortening the distance between two insertions by $\ell$ and widening the distance between two other insertions by the same amount changes the correlation only by an amount proportional to $\ell/r$. The constant of proportionality is the product of the per-box sizes when each insertion is replicated in the time direction (powers of columns). $r$ is the margin between insertions; the larger the margin, the smaller the difference.

03

## Outline of the mechanism — trace representation, re-assignment, Hölder

### Stage 1: write the expectation as a trace

The expectation on a periodic lattice is the trace of the product of operators going once around the time circle, divided by the partition function Source (the source's docstring and the definitions of `closed` and `along`; `YM/TimeKernel/Unit.lean:17`):

$$\langle B;Y\rangle_\beta \;=\; \frac{\operatorname{Tr}\bigl(B_1 T^{Y1} B_2 T^{Y2}\cdots B_k T^{Yk}\bigr)}{\operatorname{Tr} T^{\,n}}. \tag{3}$$

Numerator and denominator change in the same ratio when $T$ is multiplied by a positive constant, so one may normalise $\|T\| = 1$ Proved here .

### Stage 2: the difference from re-assigning gaps is of order $\ell/r$

Since $T$ is positive and $\|T\| = 1$, for an eigenvalue $\lambda \in [0,1]$ and $s \geq Y > 2r$,

$$\Bigl|\frac{d}{ds}\lambda^s\Bigr| \;=\; \lambda^s\,|\log\lambda| \;\leq\; \frac{1}{e\,s} \;\leq\; \frac{1}{2e\,r} \tag{4}$$

holds Proved here (the maximum of $t \mapsto te^{-t}$ is $1/e$). Moving a gap by $\ell$ replaces the two powers $T^{Yi}$, $T^{Yj}$ by $T^{Yi+\ell}$, $T^{Yj-\ell}$, and by (4) the change per power is of order $\ell/r$. We infer this is where the $\ell/r$ of (2) comes from Inferred .

### Stage 3: split the remaining factors into columns by Hölder

The difference in the numerator of (3) is still a trace of the product of all blocks and powers. The trace Hölder inequality splits it into a product of "Schatten-type norms" of each block. Raising the Schatten norm of block $B_i$ together with the gap powers to the power $2p$ gives the trace of the product with $B_i$ and its adjoint alternating, that is, the column expectation (times the partition function) Inferred (from the source's docstring, "Read replica column strengths $\operatorname{Tr}((B^*B)^p)$"). So the constant becomes the product of the powers of the column expectations, $K_i$.

According to the source's docstring, the kernel is approximated by finite matrices (`Quad.shrink`), the estimate is done with the approximating matrices, and then the limit is taken. The partition function is bounded below by its actual value, and no assumption about the ground state is made Source (docstring).

### Stage 4: columns are bounded by roots

Since a column is a copy in one direction, the time direction, the column expectation is bounded by the chessboard root (12-02) Source (`YM/Extraction/Difference.lean:51`

`reading`: if the root of an observable and its mirror image in direction $a$ is at most $c$, a block whose column power is at most $c$ can be built). In this way the $K_i$ of (2) can be replaced by the root bound $A\,a_m^{-d-\varepsilon}$ of 12-03.

04

## The form that moves one insertion, and assembling equicontinuity

Statement (estimate for moving one insertion in one direction)

On the lattice of the construction's family with cutoff $N$ and volume $p$, place $m+1$ renormalised insertions at points $x_0,\dots,x_m$, whose coordinates in direction $a$ are pairwise more than $2h_\square + 2r$ apart ($h_\square$ is the number of lattice points along a side of the centred box; the source's letter is `d`, a different quantity from the degree of a field). If the root of each insertion and its mirror image in direction $a$ is at most $c_i$, then the difference of the $(m+1)$-point function when one point $x_u$ is moved $\ell \le r$ steps in direction $a$ is at most $\big(\prod_i c_i\big)\,\ell/r$.

Source A theorem of the published construction. Lean: `YM.Extraction.Difference.primitive` (`YM/Extraction/Difference.lean:90`). An intermediate form is `YM.Slice.Time.organized` (`YM/Slice/Ordered.lean:68`).

**Why the coordinates in that direction must be separated.** The representation (3) uses the fact that the insertions are lined up in order in the time direction $a$. If two insertions have close coordinates in direction $a$, they fall in the same band of layers and cannot be split into blocks. So the equicontinuity statement is made on configurations whose coordinates differ pairwise along every coordinate axis (`generic`)Source. By the hypercubic symmetry of the lattice, the time direction can be taken along any of the four axes, so combining moves in the four directions gives a Lipschitz estimate in all directionsInferred.

Definition (the equicontinuity input `TransferLip`)

(i) With the volume $p$ fixed: on a compact set $K$ of configurations with no coincident points, including periodic images, for every $\varepsilon > 0$ there is $\delta > 0$ such that, for sufficiently fine cutoffs, the difference of lattice correlations at two configurations in $K$ at distance less than $\delta$ is less than $\varepsilon$. (ii) With the volume also moving: on a compact set $K$ of configurations whose coordinates differ along every axis, the same holds uniformly for sufficiently late volumes and cutoffs.

Source Construction source `YM/Extraction/Lipschitz.lean:26` `TransferLip` (fields `grid`, `expanding`).

Source That `TransferLip` follows from the bundle of root bounds (`CenteredBook`) and the construction's certificate is a theorem of the published construction (Lean: `YM.Extraction.Difference.completed`, `YM/Extraction/Equicontinuity.lean:181`). This corresponds to the second part of equation (11) of the assessment (a uniform modulus of continuity $\omega_K$ on compact sets)Assessment .

**The role of the lattice transfer operator ends here.** In the construction, the lattice transfer operator is used for equicontinuity. The time evolution $e^{-tH}$ of the limiting theory, the spectral condition and analytic continuation are handled in the continuum-side OS space built from reflection positivity of the limiting table (12-07, 12-08)Inferred (a reading from names and usage).

05

## What this book's study had seen

**Equicontinuity via the transfer-operator representation did not exist in this book's study.** This book's transfer operator is the abstract operator $T$ in the vessel of reconstruction (the vessel of 19-03), built by OS reconstruction from the infinite-volume limiting measure inside the strong-coupling window (20-04). What was shown there is $T \geq 0$ and $\|T - P_\Omega\| \leq e^{-m}$ (the form of the mass gap); it was never used for the modulus of continuity of correlations.

Lean This book's $T \geq 0$: for $\beta \geq 0$ and in the Dobrushin window, the transfer operator of the vessel at a limit point is positive (Lean: `FullT.wilson_Tpos_full`; its source is `FullT.full_rp_torus`, which extends reflection positivity for link-midpoint planes by averaging over gauge transformations). It is a formalisation of the Osterwalder–Seiler type, with the lattice spacing fixed at 1 and only inside the strong-coupling window.

(2) in the published construction is an estimate for the continuum-limit problem in which point positions move continuously; in this book's setting with lattice spacing 1, point positions are discrete and the question of equicontinuity did not even arise. In this book's conditional chain theorem (28-01), the modulus of continuity of observables was placed as an **assumption** called "the splitting of observables" (Lean fields `split`, `lattice_vanishing`, `shift_vanishing`). The published construction **derives** it from the transfer operator and the root bounds.

06

## What has not been read

- **The proof of `TimeKernel.passage`** (`YM/TimeKernel/Unit.lean:76`). Stages 2 and 3 of § 03 are inferences from the docstrings and the form of the statements. We checked as far as the Hölder exponent `Holder.rate p` = $1/(2p)$ (`YM/Complex/Gaps.lean:105`).
- **The full definition of the block `Brick`**, and how block operators enter the trace.
- **Approximation by finite matrices** (`Quad.shrink`): how it is organised, and how it passes to the limit.
- **How the four directions are combined.** The assembly from `primitive` to `completed` (`stepTotal` in `Equicontinuity.lean` and so on) has been read only as far as the statement names.
- **The chain to the public theorem.** In the record, the direct user of `completed` is `crops`. The road from `crops` to the public theorem has been traced only as far as written in 12-05.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The lattice transfer-operator kernel is positive for $\beta \geq 0$ and adjoint under time reversal (`step_positive`, `transition_reflect`) | $\beta < 0$. Quantitative estimates on the spectrum of the kernel |
| Source Difference from re-assigning gaps by $\ell \leq (\prod K_i)\,\ell/r$ (`Time.powering`) | When there is no margin $r$ (points close together) |
| Source The estimate for moving one insertion in one direction (`Difference.primitive`) and the equicontinuity input `TransferLip` (`Difference.completed`) | Estimates for configurations whose coordinates in a direction coincide |
| Proved here Normalisation, and $\lvert d\lambda^{s}/ds\rvert \leq 1/(es)$ | — |
| Inferred The difference estimate has two stages: "derivative of a power $\sim 1/r$" and "trace Hölder down to columns" | How the proof of `passage` is actually organised |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Transfer matrix and reflection positivity | Cited | K. Osterwalder, E. Seiler, Ann. Phys. 110 (1978) 440. 5-03 |

| Item | Kind | Source / tool |
|---|---|---|
| Kernel and positivity | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Slice/Positive.lean:54` `heat`, :81 `step_positive`, `YM/Slice/Hamilton.lean:121` `transition_reflect` |
| (2) and its body | Source | `YM/Slice/Calendar.lean:60` `column`, :72 `Time.powering`, `YM/TimeKernel/Unit.lean:17` `along`, :76 `passage` (statements and docstrings only) |
| Estimate for moving one insertion, and assembly | Source | `YM/Slice/Ordered.lean:68` `organized`, `YM/Extraction/Difference.lean:51` `reading`, :90 `primitive`, `YM/Extraction/Equicontinuity.lean:181` `completed`, `YM/Extraction/Lipschitz.lean:26` `TransferLip` |
| Description in the assessment | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` § 5, equation (11) |
| This book's $T \geq 0$ | Lean | `FullT.wilson_Tpos_full`, `FullT.full_rp_torus` (20-04) |

**Words used in this chapter** — transfer operator, slice, thread, positive kernel, column, trace representation, re-assigning gaps, Hölder's inequality, equicontinuity, configurations whose coordinates differ along every axis. Their meanings are in the A-02 glossary.

Next chapter to read: 12-05 Arzelà–Ascoli, and the properties that pass to the limit — from boundedness and equicontinuity, limiting correlations are extracted by a two-stage subsequence.

# Arzelà–Ascoli, and the properties that pass to the limit

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 05 and § 07 are a map of open questions.

**What this chapter shows** — From local boundedness (12-03) and equicontinuity (12-04), the published construction extracts limiting correlations by a two-stage subsequence: the first step is a subsequence of cutoffs common to all volumes, the second step a subsequence of volumes. The cutoff subsequence is finally returned to the whole sequence using convergence on the ultraviolet side. Reflection positivity, the symmetry of the lattice hyperoctahedral group and translation invariance pass to the limit. This chapter reads that assembly, and why uniqueness of the limit and convergence along the whole sequence cannot be stated.

**Chapters assumed** — 12-03, 12-04, 8-05 (Correlation functions and the order of limits). The Arzelà–Ascoli theorem is functional analysis within the scope of 1-09.

**Said first** — The results of the published construction are at the level "the committee's assessment is that they are proved in Lean", and what this book has checked goes only as far as reading the form of the statements. The infinite-volume limit is taken **only along a subsequence of volumes**; it has not been shown that the limit is independent of the choice of subsequence (uniqueness), nor that it converges along the whole sequence of volumes. The assessment also lists this as the first of its open questions.

The order of this chapter

1. Legend of labels
2. Statement — the form of the subsequential limit
3. Outline of the mechanism (1) — two-stage Arzelà–Ascoli
4. Outline of the mechanism (2) — returning the cutoff subsequence to the whole sequence
5. Properties that pass to the limit
6. Why uniqueness and convergence cannot be stated
7. What this book's study had seen
8. What has not been read

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01

## Statement — the form of the subsequential limit

Write $S_p(v;x)$ for the kernel of correlations with the volume $p$ fixed and the cutoff $N \to \infty$ (8-05). $x = (x_1, \dots, x_n)$ is a configuration of mutually distinct points, and $v$ a sequence of field labels.

Statement (infinite-volume limit along a subsequence, Theorem 5.1 of the assessment)

For any strictly increasing sequence of volume indices $p(k)$, there are a strictly increasing $q(k)$ and a kernel $S^{\mathrm{mix}}(v;x)$, smooth on configurations with no coincident points, such that for every sequence of labels $v$ and every smooth compactly supported test function $f$ supported away from coincident points,

$$\lim_{k\to\infty} S_{p(q(k))}(v;f) \;=\; \int f(x)\, S^{\mathrm{mix}}(v;x)\, dx. \quad (1)$$

A single subsequence $q$ works for all correlations in common.

Assessment Theorem 5.1 in § 5 of the assessment. Source This corresponds to the field `wilson` (`C.Converges (p∘q) …`) of the public theorem `JointExistence` (construction source `ChallengeSpec.lean:186`), "for any strictly increasing $p$ there are a strictly increasing $q$ and a table $T$ with $T.\mathrm{Properties}(p \circ q)$". `Converges` (`PublicWilson.lean:85-90`) has the form "for every volume in which the test function fits, the limit $\mathrm{row}(p)$ along the **whole sequence** of cutoffs exists, and $\mathrm{row}(p(q(k)))$ converges to $\int f\, S^{\mathrm{mix}}$".

The "mix" in the limit symbol $S^{\mathrm{mix}}$ is a name anticipating that it will later be decomposed as a mixture of phases (12-07) Assessment .

02

## Outline of the mechanism (1) — two-stage Arzelà–Ascoli

**Extraction sequences.** The construction holds a pair of a cutoff subsequence $\mathrm{mesh}$ and a volume subsequence $\mathrm{per}$ as an **extraction sequence** (`Extraction.Order`, `YM/Extraction/Orders.lean:34`), and refinement (`refine`, same file :40) composes each of the two subsequences Source . The meaning of the limit (`limits`, same file :61) is two ordinary limits: "for sufficiently late volumes, with each volume fixed, the limit $v(p)$ along $\mathrm{mesh}$ exists, and $v(\mathrm{per}(k))$ converges to the limit".

Statement (limit along a common subsequence)

For an extraction sequence $f$, if local boundedness (`Bound`) and equicontinuity (`TransferLip`) hold, then there are a refinement $g$, a limiting table $S$ and per-volume tables $P_p$ such that: (i) the readings of $S$ are continuous on configurations whose coordinates differ along every axis; (ii) for each volume, the lattice correlations approach $P_p$ along the cutoff subsequence of $f \circ g$, uniformly on compact sets with no coincident points (including periodic images); (iii) $P_{\mathrm{per}(k)}$ approaches $S$ uniformly on compact sets; (iv) the smeared readings of $S$ are double limits of the smeared lattice correlations.

Source A theorem of the published construction (of Arzelà–Ascoli type). Lean: `YM.Extraction.first_surface` (`YM/Extraction/Lipschitz.lean:166`; conclusion `RawLimits`, same file :156). What calls it from the construction's inputs is `YM.Extraction.crops` (`YM/Extraction/Held.lean:136`).

The proof has two stages Source (statements of the lemmas and stages of the proof).

1. **Cutoff subsequence (common to all volumes).** The pairs of a volume $p$ and a sequence of labels are countably many. For each pair, the family of lattice correlations in volume $p$ is bounded (`Bound.cyclic`) and equicontinuous (`TransferLip.grid`) on compact sets with no coincident points. Applying Arzelà–Ascoli simultaneously to the countably many families gives a cutoff subsequence $t$ common to all volumes and per-volume limits $P_p$ (`setMeshes`, `Lipschitz.lean:78`).
2. **Volume subsequence.** Now consider the family of $P_p$ (with $p$ varying) on configurations whose coordinates differ along every axis. Boundedness (`Bound.volume`) and equicontinuity (`TransferLip.expanding`) are uniform for sufficiently late volumes, so applying Arzelà–Ascoli once more gives a volume subsequence $u$ and a limit $S$ (`localAscoli`, `YM/Extraction/Tame.lean:100`).

`localAscoli` is a lemma that, for a family of functions with countably many indices, chooses a subsequence and limit common to all indices from pointwise boundedness and control of "oscillation" (the local form of equicontinuity, `oscillation`). Its content is the classical line of taking a diagonal subsequence on a countable dense set and extending to everything by equicontinuity Cited (the Arzelà–Ascoli theorem). What we read in the source is the statement of the lemma and the shape of the proof using `simultaneous` and `growFrom` Source .

Note that the limiting kernel is taken **as a table of correlation functions, not as a measure**. The assessment lists defining a continuum probability measure as open question 2 Assessment .

03

## Outline of the mechanism (2) — returning the cutoff subsequence to the whole sequence

A cutoff subsequence was taken in the first step of § 02, but the public statement's `Converges` requires, for each volume, a limit along the **whole sequence** of cutoffs (§ 01). The construction pays for this with convergence on the ultraviolet side Source .

- The ultraviolet side shows, with the volume fixed, that the lattice kernels converge along the whole sequence of cutoffs (including the sequences of differences), from the Cauchy property of the kernels (`PositionsUV C e`, `YM/FullVolume/Profiles.lean:25`; `Sequence.full` when the sequence $e$ is the whole sequence; Part 11, 11-10).
- From the final certificate built with the subsequence $f$ and `PositionsUV C full`, it follows that, for the extraction sequence "cutoffs: the whole sequence; volumes: the volume subsequence of $f$", the limiting table can be read as a double limit (`from_spec`, `YM/FullVolume/Finale/Program.lean:117`).

For a sequence with a unique limit, a limit taken along a subsequence agrees with the limit of the whole sequence. Since the ultraviolet limit at fixed volume exists along the whole sequence, the cutoff subsequence can be removed at the end Proved here . What cannot be removed is the volume subsequence.

**Which target it enters.** In the field `wilson` of the public theorem too, the ultraviolet (cutoff) is the whole sequence and infinite volume is the subsequence $p \circ q$. The strong form with kernels and differences, `PositionsUV … Sequence.full`, is only in the conclusion of the construction theorem `constructed`, not in the public theorem Source (6-02).

04

## Properties that pass to the limit

Properties that hold exactly on finite lattices are preserved in the limit as non-negativity or equalities. The construction transfers the following four Source .

| Property | Lattice side | Limit side | Lean (construction source) |
|---|---|---|---|
| Reflection positivity | Reflection positivity for planes through lattice sites (coordinate 0, four axes) (12-01) | The Gram matrix of smeared words supported at positive time is positive semi-definite (`RPSmear`) | `Order.realPositive` (`YM/Extraction/Harvest.lean:46`). The lattice plane is `Banks.plane` (`Banks.lean:19`: kind site, base 0) |
| Discrete symmetry | Covariance and reality under the lattice hyperoctahedral group (permutations and signs of coordinates) | `RealSpin` | `Order.spin` (`YM/Extraction/Inherited.lean:124`) |
| Translation invariance | Invariance under lattice translations | Invariance under continuum translations (`Homogeneous`) | `Order.homogeneous` (`Inherited.lean:83`) |
| Basic properties | Symmetry, the 0-point function is 1, the unit field can be removed | `Basic` | `Order.limits_basic` (`Inherited.lean:151`) |

All four take smoothness of the limiting table (`S.Smooth`) as a hypothesis (`realPositive` also symmetry, `S.Symmetric`) Source .

**How reflection positivity is transferred.** Reflection positivity on the lattice side is the same as positive semi-definiteness of the Gram matrix $(\langle F_j F_i \circ \Theta \rangle)$ for linear combinations $\sum_i c_i F_i$ of finitely many words (12-01 § 02). Since each entry converges under the double limit in cutoff and volume, the limiting Gram matrix is also positive semi-definite, as a limit of positive semi-definite matrices Proved here (the source's `posVectors`, and the limit of sums). The continuum-side definition of reflection positivity, `Table.RP` (`YM/Euclid/Words.lean:123`), is that for every coordinate axis $a$, the Gram matrix of families of positive-time words is positive semi-definite; together with translation invariance it holds for every plane parallel to a coordinate plane Source .

**The bundle of the limiting table.** Adding smoothness away from coincident points (`Smooth`) and a growth condition (`Capped`) to the four properties gives the published construction's version of the bundle of Euclidean axioms, `Table.Hierarchy` (`Words.lean:141`) (`Order.assembles`, `Harvest.lean:62`) Source . `assembles` takes smoothness and the growth condition as **hypotheses**. Where they come from has been

read only as far as written in 12-08. Invariance under continuum rotations (O(4)) is not in this bundle (12-06).

05

## Why uniqueness and convergence cannot be stated

The inputs of § 02 are **only upper bounds**: boundedness and equicontinuity. Compactness gives existence of subsequential limits, but not that the limit is unique Proved here . To obtain uniqueness of the limit or convergence along the whole sequence, statements of kinds such as the following are needed: the difference of correlations when the volume changes becomes small (a Cauchy property in the volume), volume-uniform decay of correlations, disappearance of dependence on boundary conditions, uniqueness of DLR type. Within what we read, none of these is among the inputs Source (the types of the inputs, and within the scope of the name search).

The source's `Order.unique` (`YM/Extraction/Orders.lean:70`) says only that "two limits along the same extraction sequence are equal" (uniqueness of limits in a Hausdorff space), not independence from the choice of subsequence. In the record of declaration usage, the status of this declaration is `outside` (not reached from any of the 17 headline theorems) Source .

In § 5 the assessment states that "whether the subsequential limit depends on the chosen subsequence is unresolved", and lists as open question 1 in § 8 "convergence rather than a subsequence; showing uniqueness of the infinite-volume limit" Assessment . The same § 5 also writes that the Jaffe–Witten problem statement regards a subsequential limit obtained by compactness as insufficient unless there is a separate argument establishing the properties of the limit Assessment .

06

## What this book's study had seen

**(1) The implication from subsequential limits and transfer of properties.** This book's conditional chain theorem (28-01) included a part that takes subsequential limits by compactness for a family of push-forward measures and transfers to the limit reflection positivity, invariance, lower bounds on the mass of closed sets, and the modulus of continuity of observables.

Lean Machine-checked as an implication (hypothesis fields as in the table of 28-01). Lean: `BoxWalk.chain_end_to_end_box`, `ChainBlockMap.chain_end_to_end_blockMap`.

The **skeleton points in the same direction**: subsequential limits and transfer of properties. The difference is that this book took the limit **as measures** (a family of

probability measures on a compact configuration space always has a convergent subsequence) and put boundedness and the modulus of continuity in **hypothesis fields**, whereas the published construction takes the limit **as tables of correlation functions** and **derives** boundedness and equicontinuity from root bounds and the transfer operator (12-03, 12-04).

**(2) A limit without subsequences inside the strong-coupling window.** This book has the infinite-volume limit without subsequences inside the strong-coupling window.

Lean For $0 < \beta$, $18 \tanh \beta < 1$, the sequence of periodic lifts of the SU(2) Wilson measures on $(\mathbb{Z}/2(n+1))^4$ converges weakly, without subsequences, to a single probability measure $\nu_\beta$ on $\mathbb{Z}^4$, and the sequence over all sizes has the same limit. A formalisation of the Osterwalder–Seiler strong-coupling and Dobrushin-uniqueness type. Lean: `InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN` (20-03). The statement bundling translation invariance, reflection positivity and the mass gap into one is `InfMain2.wilson_infvol_main2` (20-04).

This is a stronger form than the published construction's subsequential limit, in that it has convergence and uniqueness. But it concerns **a different regime**: the lattice spacing is fixed at 1, the coupling is only inside the strong-coupling window (about $|\beta| < 0.0556$), and no continuum limit is taken. The published construction carries every step at weak coupling, takes the continuum limit with lattice spacing $a_N \to 0$, and then goes to infinite volume. The two differ in object and in order of limits; what can be compared is only the pattern of "what kind of input gives uniqueness" (in this book, volume-uniform decay from the Dobrushin condition).

**(3) Twisted boundary conditions.** The lattices of the published construction have only periodic boundary conditions Source . This book has partition functions with twisted boundary conditions and upper and lower bounds on the vortex free energy (Part 21). These are statements on lattices with fixed spacing, and do not overlap with the infrared components of the published construction.

07

## What has not been read

- **Details of the proof of the Arzelà–Ascoli lemmas** (`simultaneous`, `growFrom`, `components`).
- **The chain to the public theorem.** In the record, the direct user of `crops` is `MergeProof.Accent.rated`, and the record's headline is `SharpGram.habitat`. Literally, `fromFunded` on the construction's chain (`YM/R2I/Campaign/Accepted.lean:150`) calls `Run.extractible17` (`YM/RG17/Joint.lean:61`) at :157, which uses the field `reception` of the structure `Chateau` (`YM/MergeProof/Almandine.lean:31`), and `rated` is called

inside the proof (same file :79) of the lemma `grantor` (same file :40) that fills that field; so it is likely on the road to the public theorem Inferred . The chain has not been checked declaration by declaration.

- **Where smoothness and the growth condition come from.** Who pays for the hypotheses `Smooth` and `Capped` of `assembles` has not been traced.
- **That there is no Cauchy property in the volume.** This is a judgement from the types of the inputs and a name search; we have not read the whole repository.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source From boundedness and equicontinuity, a limit along a common subsequence by two-stage Arzelà–Ascoli (`first_surface`, `crops`) | Uniqueness of the limit. Convergence along the whole sequence of volumes |
| Source The cutoff subsequence is removed at the end by convergence along the whole sequence on the ultraviolet side (`from_spec`, `PositionsUV … full`) | Removing the volume subsequence |
| Source Reflection positivity, hyperoctahedral symmetry, translation invariance and the basic properties pass to the limit (`realPositive`, `spin`, `homogeneous`, `limits_basic`) | Invariance under continuum rotations (separately in 12-06). A limiting probability measure |
| Assessment The assessment: subsequential limit; dependence on the choice is unresolved ( § 5, item 1 of § 8) | — |
| Lean This book's limit without subsequences inside the strong-coupling window (`wilson_infvol_unique`, `wilson_infvol_allN`) | Varying the lattice spacing. Weak coupling. The continuum limit |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| The Arzelà–Ascoli theorem | Cited | A standard theorem of functional analysis (1-09) |
| Extraction sequences and the two-stage limit | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Extraction/Orders.lean:34` `Order`, :40 `refine`, :61 `limits`, :70 `unique`, `YM/Extraction/Lipschitz.lean:78` `setMeshes`, :156 `RawLimits`, :166 `first_surface`, `YM/Extraction/Tame.lean:100` `localAscoli`, `YM/Extraction/Held.lean:136` `crops` |
| Returning to the whole sequence | Source | `YM/FullVolume/Profiles.lean:25` `PositionsUV`, `YM/FullVolume/Finale/Program.lean:117` `from_spec`, `PublicWilson.lean:85-90` `Converges`, `ChallengeSpec.lean:186` `JointExistence` |

| Item | Kind | Source / tool |
|---|---|---|
| Properties that pass to the limit | Source | `YM/Extraction/Harvest.lean:46` `realPositive`, :62 `assembles`, `YM/Extraction/Banks.lean:19` `plane`, `YM/Extraction/Inherited.lean:83` `homogeneous`, :124 `spin`, :151 `limits_basic`, `YM/Euclid/Words.lean:123` `Table.RP`, :141 `Table.Hierarchy` |
| Description in the assessment | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` § 5 (Theorem 5.1, equations (10) and (11)), § 8 |
| This book's counterparts | Lean | `BoxWalk.chain_end_to_end_box`, `ChainBlockMap.chain_end_to_end_blockMap` (implication, 28-01), `InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN` (20-03), `InfMain2.wilson_infvol_main2` (20-04) |

**Words used in this chapter** — extraction sequence, cutoff subsequence, volume subsequence, Arzelà–Ascoli theorem, equicontinuity, subsequential limit, reflection positivity, hyperoctahedral group, translation invariance, uniqueness. Their meanings are in the A-02 glossary.

Next chapter to read: 12-06 Euclidean invariance — how the discrete symmetries of the lattice are lifted to invariance under continuum rotations.

# Euclidean invariance

> This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 07 is a map of open questions.

**What this chapter shows** — The lattice breaks rotational symmetry and keeps only a finite group made of coordinate permutations and sign flips (the hyperoctahedral group, with 384 elements). The committee's assessment is that the published construction shows, with three components, that the limiting theory is invariant under O(4), the group of continuous rotations and reflections: (1) keep the discrete symmetries of the lattice, gauge fixing included, and carry them to the limit; (2) build a divergence-free stress tensor and set up the Ward identities for translations and rotations; (3) show that the rotational flux vanishes far away, so that rotational invariance is not spontaneously broken. This chapter reads the **statements** of the three components and writes out why O(4) invariance follows from them. We have not read the **proof** of (3).

**Prerequisite chapters** — 12-05 § 04 (properties that pass to the limit), 13-01 (uniqueness and normalisation of the stress tensor), 9-05 (symmetries of the single step).

**Said first** — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean". This book has read the **definitions and the form of the statements** for the stress tensor, the Ward identities and rotational invariance. The proof of the vanishing of the flux (`Outer.outer`) is split across about 40 files, one group per coordinate axis, and we have not read its content. The steps of the argument in § 05 that cannot be read directly off the statements carry Inferred .

The order of this chapter

1. Key to the labels
2. What the problem is — the symmetry left on the lattice is a finite group
3. Component (1) — keep the discrete symmetries and carry them to the limit
4. Component (2) — the stress tensor and the Ward identities
5. Component (3) — the local rotational flux vanishes for configurations on a line

6. Outline of the mechanism — from the three components to O(4) invariance
7. What this book's investigation had seen
8. What we have not read

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

## What the problem is — the symmetry left on the lattice is a finite group

The OS axioms (4-10) require the Euclidean correlation functions to be invariant under the group of translations, rotations and reflections $E(4) = \mathbb{R}^4 \rtimes O(4)$. The lattice $a\mathbb{Z}^4$ preserves only the lattice translations and the hyperoctahedral group of coordinate permutations and sign flips (of order $2^4 \cdot 4! = 384$). Translation invariance returns in the continuum limit (12-05 § 04), but rotational invariance does not return automatically. That the limit is rotation invariant is a property that has to be proved, and one must also rule out that rotational invariance is **spontaneously broken** in the limiting state.

Assessment Assessment § 6.3: the proof has three main parts. The first is that the gauge fixing preserves the lattice symmetries (axial gauge fixing, for example, does not), which is achieved by averaging over the symmetry-breaking choices. The second is to show that there is a divergence-free stress tensor field and to use it to set up local Ward identities. Finally, one shows that Euclidean invariance is not spontaneously broken in the limiting state. This follows if the stress-energy flux falls off fast enough at infinity, and it is shown by an involved argument using reflection positivity in several orthogonal planes.

02

## Component (1) — keep the discrete symmetries and carry them to the limit

**Symmetries of the single step.** The kernel of one renormalisation step is invariant (covariant) under the signed permutations of the hyperoctahedral group (384 of them), the translations of the coarse lattice and the gauge transformations of the fine lattice.

The choice of gauge for large fields breaks the symmetry, but by drawing the choice at random and averaging, the symmetry of the kernel as a whole is preserved (9-03, 9-05). This part reads on that basis.

**Carrying to the limit.** The hyperoctahedral covariance and the reality of the lattice pass to the property `RealSpin` of the limiting table Source (construction source `YM/Extraction/Inherited.lean:124` `Order.spin`). `RealSpin` is the field stating that the limiting correlations do not change when an element of the hyperoctahedral group moves the points and the field labels together (a representation of the group acts on the field labels), and stating reality Source (from its name and how it is used; it is defined as the structure `RealSpin` in `YM/Euclid/Table.lean:80`, which we have not read in full).

03

## Component (2) — the stress tensor and the Ward identities

We use the field spaces $F_d$ (local fields of degree $d$, 8-03). A **stress tensor** is a family that assigns a field $T_{ab} \in F_4$ of degree 4 to each pair of indices $a, b \in \{0,1,2,3\}$ Source (`Stress S` in `YM/Euclid/Stress.lean` $= \mathrm{Axis} \to \mathrm{Axis} \to F_4$).

Definition (conserved stress tensor)

(i) symmetric, $T_{ab} = T_{ba}$; (ii) real (invariant under conjugation of fields); (iii) divergence zero: $\sum_a \partial_a T_{ab} = 0$ ($\partial_a$ is the derivative on the field spaces, $F_4 \to F_5$). In addition, it transforms as a rank-two tensor under the rotation action $R$ (`Tensorial`).

Source Construction source `YM/Euclid/Stress.lean:59` `Conserved` (fields `symmetric`, `real`, `divergence`), :64 `Tensorial`. The action of rotations on fields, `Rotation` (:22, fields `act`, `lie`), and its laws `Rotation.Laws` (:31: continuity, group action, agreement with the lattice reflections for signed permutations, reality, preserving the unit field, compatibility with derivatives, the infinitesimal generator `lie`).

The uniqueness of the stress tensor (the space of conserved symmetric tensors is one-dimensional over the reals) and its normalisation are treated in 13-01 (Lean: `YM.Pert.one_line`, `YM/Pert/Weak.lean:68`) Source.

**Windows.** For a configuration of points $x = (x_0, x_1, \ldots, x_n)$ (pairwise distinct), a function $\chi$ is a **window** if it is smooth with compact support, equal to 1 near $x_0$, and its support contains none of the other points $x_1, \ldots, x_n$ Source (`Window` in `Stress.lean`).

$\partial\chi$ is supported away from all the insertion points. A window is a smoothed version of a surface enclosing $x_0$.

Form of the statement (Ward identities for translations and rotations)

Put a central field $B \in F_d$ at $x_0$ and a list $q$ of spectator fields at $x_1, \dots, x_n$, and let $\chi$ be a window.

$$-\sum_a \int \partial_a \chi(y)\, \langle T_{ab}(y)\, B(x_0)\, q \rangle \, dy \;=\; \langle (\partial_b B)(x_0)\, q \rangle, \quad (1)$$

$$-\sum_{a,b} \int \partial_a \chi(y)\, (g\,(y - x_0))_b \, \langle T_{ab}(y)\, B(x_0)\, q \rangle \, dy \;=\; \langle (\mathrm{lie}_g B)(x_0)\, q \rangle. \quad (2)$$

(1) is the identity for translations and (2) the identity for an infinitesimal rotation $g$ (an antisymmetric matrix). $\mathrm{lie}_g$ is the derivative of the rotation action on fields.

Source Construction source `YM/Euclid/Stress.lean:90` `StressLaws` (fields `translation`, `rotation`); the left-hand sides are `shellGrad` and `shellSpin` in the same file. As properties of the limiting table $S$ they enter the construction's bundle of certificates `Spec.Locals` (`YM/Spec/Closed.lean`, fields `conserved`, `spin`, `charges`), and for the tables of all phases, including the chosen one, they enter the certificate `OS.Selection.Certificate` (`YM/Phase/Mixture.lean:136`, field `flux`).

(1) is a form of Gauss's theorem in which the step of $\chi$ plays the role of a surface enclosing $x_0$. Since $\partial\chi$ vanishes near the insertion points, the integral does not touch coinciding points. The committee's example check confirms that the right-hand side of (1) agrees with the ordinary coordinate derivative, that it holds in every phase and for every field, that the integrand is integrable, that it does not depend on the choice of $\chi$, and that admissible $\chi$ actually exist Assessment (on the assessment side, `submissions/openai-20261028/docs/examples/TranslationWard-20261028.lean`). The same check states that it does not confirm that all improvement terms vanish, nor the contact terms, nor the dilation Ward identity Source .

04

## Component (3) — the local rotational flux vanishes for configurations on a line

The predicate stating rotational invariance says that, for every $K \in O(4)$, the correlations do not change when the points are moved by $K$ and the field labels by the

rotation action Source (`YM/Euclid/Stress.lean:46` `Rotation.Oriented`: $S(Kx;\, R_K v) = S(x; v)$ for every configuration of non-coinciding points).

Form of the statement (for configurations on a line, the local rotational flux vanishes)

Suppose that the limiting table satisfies the Euclidean bundle (`Table.Hierarchy`), and that a derivative plan and an orientation condition, a stress tensor $J$, and further hypotheses `Pre` hold. Then the following two statements hold. (i) The sum condition `Bid.LineSum` (`YM/Bid/Angular.lean:35`: for configurations on a line, the value of a certain diagram is zero; we have not read its content). (ii) For a configuration $x$ whose points all lie on one line parallel to some coordinate axis $a$ (points pairwise distinct), the local rotational flux `localTurn` is zero for every rotation $K$ and every list of fields $q$.

Source A theorem of the published construction. Lean: `YM.Outer.outer` (construction source `YM/Next/ROTATIONS/Route.lean:114`). Its form applied to every object of the construction is `YM.Outer.miles` (same file, :123), which satisfies the predicate `YM.Bid.Miles` (`YM/Bid/Town.lean:13`).

The link that (ii) corresponds to "the stress-energy flux falls off fast enough at infinity" in Assessment § 6.3 is Inferred (because we have not read the definition of `localTurn`).

Source We have read the form of the statement. We have not read the content of the hypothesis `Pre` (`Climb.Pre`) of `outer`, nor the definition of the local rotational flux `localTurn`. The notes in the source say "the analytic estimates come before the continuous-rotation bootstrap and do not use it" and "include every spectator on the line and every empty tuple" Source (docstrings).

**Where rotational invariance becomes a conclusion.** Rotational invariance of the limiting table appears as the field `motion` (`Rotation.Oriented S`) of the construction's bundle of certificates `Spec.Locals`, and rotational invariance of each phase's table appears as the field `sym` of the certificate `Certificate` (`YM/Phase/Mixture.lean:142`) Source. We have not traced the chain from `outer` to the field `motion`. O(4) invariance is required separately, **outside** the published construction's bundle of Euclidean axioms `Table.Hierarchy` (12-08). This invariance is then used as a hypothesis in the proof of commutativity in the phase decomposition (12-07). The order is "Euclidean invariance → phase decomposition".

```
05
```

## Outline of the mechanism — from the three components to O(4) invariance

We write out, as the standard argument, how O(4) invariance follows from the three components. We have not checked whether the source assembles it in this order Inferred .

1. **Enlarge the window.** In (2) one cannot take the window $\chi$ equal to 1 on a large ball containing all the points $x_0, \dots, x_n$ (a window does not contain the spectators in its support). Instead, add up (2) with a small window around each point, over all the points, and compare the sum with the formula for a window on a large ball. For the large-ball window, $\partial\chi$ is supported on a spherical shell far from all the points, and the left-hand side is **the rotational flux through the distant shell** Inferred .
2. **If the flux vanishes, infinitesimal invariance.** If the flux tends to zero as the shell is moved far away, then the sum over the points of the action of the rotation generator ($\mathrm{lie}_g$ on the field labels, together with the derivative terms from the translation Ward identity (1) for rotating the positions of the points) is zero. This says that the correlations do not change under infinitesimal rotations Inferred .
3. **From infinitesimal to the group.** Since $SO(4)$ is connected, integrating along one-parameter subgroups gives $SO(4)$ invariance. Reflections (determinant $-1$) are contained in the hyperoctahedral group of component (1), so together this gives $O(4)$ Proved here (using that the correlations are smooth in the points, 12-05).

The conclusion of `outer` in the source is "the local flux is zero for configurations on a line", not for general configurations. How to extend from configurations on a line to general configurations, and how to derive the vanishing of the flux far away from reflection positivity (the assessment's "reflection positivity in several orthogonal planes"), is the content of about 40 files in `Next/ROTATIONS/` (with names such as spectral kernels per coordinate axis and Cauchy estimates), which we have not read.

**Why the flux is the issue.** The infinite-volume limit is taken along a subsequence, and decay of correlations at infinity has not been shown uniformly (12-05 § 05). Without decay, the flux through distant shells need not vanish, and there remains room for rotational invariance to be spontaneously broken. The vanishing of the flux is a statement that, in place of decay, controls only the quantity needed for rotational invariance Inferred .

06

## What this book's investigation had seen

**Nothing.** This book's Lean has none of the stress tensor, the Ward identities or invariance under continuous rotations. This book's lattice results are statements at

fixed lattice spacing, and the only symmetries are those of the lattice.

- The infinite-volume limit $\nu\beta$ inside the strong-coupling window is invariant under unit translations in the four directions (Lean: `TransI.wilson_infvol_translation`, 20-03). Rotations are not treated.
- The measure of the vessel with fermions is invariant under part of the hypercubic symmetry (90° rotations in spatial planes, axis reflections, time reflection) (20-05).
- Among the symmetries of the pushforward measure, commutation with the coarse translations of the side-3 design and with reflection in a boundary face is in Lean, but that stays within the lattice symmetries (`Block3.down_trans`, `Block3.down_reflB`, 28-05). Commutation with exchange of axes was only checked numerically (14-02).

Lifting Euclidean invariance from the Ward identities of the stress tensor and the vanishing of the flux is an idea that was not in this book's plan (the step in the table of construction steps that heads towards reconstruction) (14-05).

07

## What we have not read

- **The content of the proof of `outer`** (the 43 files in `Next/ROTATIONS/`). The content of the hypothesis `Pre` and the definition of `localTurn`.
- **The chain from `outer` and `miles` to rotational invariance (`Locals.motion`, `Certificate.sym`).** According to the records, the direct user of `miles` is `MergeProof.Acme.glistens`, and the headline is `SharpGram.habitat`.
- **The assessment's "reflection positivity in several orthogonal planes"**: which lemma in the source it corresponds to.
- **Where the symmetry-preserving gauge fixing lives** (the material of 9-03), and how it reaches `RealSpin` in the limit.
- **The full definition of `RealSpin`** (`YM/Euclid/Table.lean:80`).
- **Reflection positivity in rotated planes** (`YM/Euclid/OtherPlanes.lean`): it is imported, but according to the records it is a place for declarations that are not used Source (records).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The hyperoctahedral covariance and reality of the lattice pass to the limit (`Order.spin`) | The full definition of `RealSpin` (`Euclid/Table.lean:80`) |
| Source The definition of a conserved stress tensor, and the form (1)(2) of the translation and rotation Ward identities using windows (`Conserved`, `StressLaws`). They enter the construction's certificates and the certificates of each phase | That all improvement terms vanish; contact terms; the dilation Ward identity |
| Source The local rotational flux is zero for configurations on a line (`Outer.outer`) | The content of the proof; how to extend to general configurations |
| Assessment The three parts of Assessment § 6.3 and "reflection positivity in several orthogonal planes" | The correspondence with lemmas in the source |
| Inferred The standard argument: vanishing flux + Ward ⇒ infinitesimal invariance ⇒ O(4) invariance | That the source assembles it in this order |
| This book has no stress tensor, Ward identities or rotational invariance | — |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Euclidean invariance in the OS axioms | Cited | K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-10 |
| Stress tensor, Ward, rotations | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Euclid/Stress.lean:22` `Rotation`, :31 `Rotation.Laws`, :46 `Rotation.Oriented`, :59 `Conserved`, :64 `Tensorial`, :90 `StressLaws`; `YM/Spec/Closed.lean` `Locals`; `YM/Phase/Mixture.lean:136` `Selection.Certificate` |
| Carrying the discrete symmetries over | Source | `YM/Extraction/Inherited.lean:124` `Order.spin` |
| Vanishing of the flux | Source (statement only) | `YM/Next/ROTATIONS/Route.lean:114` `outer`, :123 `miles`; `YM/Bid/Town.lean:13` `Miles` |
| Statement of uniqueness | Source | `YM/Pert/Weak.lean:68` `YM.Pert.one_line` (13-01) |
| The assessment and the committee's examples | Assessment | `yang-mills-group/YM` `docs/assessment-1.pdf` § 6.3; `submissions/openai-20261028/docs/examples/TranslationWard-20261028.lean` |

| Item | Kind | Source / tool |
|---|---|---|
| This book's lattice symmetries | Lean | `TransI.wilson_infvol_translation` (20-03), `Block3.down_trans`, `Block3.down_reflB` |

**Terms used in this chapter** — hyperoctahedral group, Euclidean invariance, stress tensor, conservation, Ward identity, window, rotational flux, spontaneous breaking, infinitesimal generator. Their meanings are in the A-02 glossary.

Next chapter: 12-07 Phase decomposition — using Euclidean invariance, we make the algebra of operators on the zero-energy space commutative and split a mixed theory into phases.

# Phase decomposition

This chapter is established knowledge (a reading of the published construction). § 06 is this book's own investigation, and § 07 is a map of open questions.

**What this chapter shows** — The limiting theory $S^{\mathrm{mix}}$, taken along a subsequence, has not been shown to have clustering (correlations of two groups moved far apart factorise into a product). Without clustering, the vacuum produced by OS reconstruction need not be unique. The published construction goes around this by building a commutative C*-algebra of operators compressed to the zero-energy space and taking its Gelfand spectrum as the space of "phases": it decomposes $S^{\mathrm{mix}} = \int S^{\xi}\, d\mu(\xi)$, and in each component $S^{\xi}$ the vacuum is unique. This chapter reads how this is assembled, and what it says and does not say about the mixed theory and the components.

**Prerequisite chapters** — 12-05 (the limiting table and the properties that pass to it), 12-06 (Euclidean invariance), 4-10 (OS reconstruction), 1-10 (Hilbert spaces and spectra).

**Said first** — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean"; this book has read the form of the statements and the names of the lemmas in the steps of the proofs. Uniqueness of the vacuum in each component **comes with no rate** (it is not exponential decay). Clustering of the mixed theory itself, that the space of phases is a single point, and that the components are equal to one another have none of them been shown.

The order of this chapter

1. Key to the labels
2. Why a decomposition is needed
3. Statement (1) — the algebra of operators compressed to the zero-energy space
4. Statement (2) — commutativity, the Gelfand representation, a probability measure
5. Statement (3) — correlations of the components, and uniqueness of the vacuum in each component
6. What is said and not said about the mixed theory and the components

7. What this book's investigation had seen
8. What we have not read

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

## Why a decomposition is needed

Since the limiting table $S^{\mathrm{mix}}$ is reflection positive, the OS Hilbert space $\mathcal{H}$ and the time semigroup $E(t) = e^{-tH}$ (self-adjoint, contractive) are available Source (construction source `YM/OS/Dynamics.lean:57` `Chamber`: the GNS space built from positive-time words, the time semigroup `flow.E`, the spatial translations `flow.U`). The subspace fixed by $E(t)$ for all $t$ is called the **zero-energy space** $\mathcal{H}_0 = \ker H$ (`YM/OS/Core.lean:14` `ground`), and its orthogonal projection is $P_0$. As $t \to \infty$, $E(t)x \to P_0 x$ (same file, :38 `emerges`) Source .

In standard OS reconstruction, clustering makes $\mathcal{H}_0$ consist only of multiples of the vacuum $\Omega$, and the vacuum is unique (4-10). Clustering has not been shown for a limit taken along a subsequence, so $\mathcal{H}_0$ need not be one-dimensional. In physical terms, the possibility remains that the limit is a mixture of two or more "pure phases".

Assessment Assessment § 6: rather than applying OS reconstruction directly to this Euclidean theory, it takes the detour of decomposing it into components that satisfy clustering and applying reconstruction to each component. § 6.2: OS reconstruction can also be applied to the mixed limiting theory, but the vacuum of the resulting Wightman theory has not been shown to be unique.

02

## Statement (1) — the algebra of operators compressed to the zero-energy space

For a word $A$ (a finite list of field labels and insertion points) with points in the time slab $0 < t < s$, let $B_{s,A}$ be the operator that shifts the insertions of a vector

represented by positive-time words by $s$ into the future and puts the insertions of $A$ in front, in the slab thus freed Assessment (the assessment's glossary, `YangMills.Hierarchy.Character`) Source (`place` of `Spread`). Consider its compressions to the zero-energy space,

$$M_A \;=\; P_0\, B_{s,A}\big|_{\mathcal{H}_0} :\; \mathcal{H}_0 \to \mathcal{H}_0 \quad (1)$$

and let $\mathcal{A}$ be the norm-closed unital *-algebra that they generate.

Source Construction source `YM/Phase/Mixture.lean`: `Spread.atom` ($P_0 \circ \mathrm{place}\, d\,w$ restricted to $\mathcal{H}_0$), `generators`, :54 `Spread.algebra` (the closure of the generated *-subalgebra).

**Meaning.** $\langle \Omega_0, M_A \Omega_0 \rangle = \langle \Omega_0, B_{s,A} \Omega_0 \rangle = S^{\mathrm{mix}}(A)$ ($\Omega_0$ is the vacuum vector of the mixed theory) Source (`insertValue`). $M_A$ is the operator that looks, on the zero-energy space, at "what remains after inserting the word $A$ and waiting an infinitely long time". If the mixed theory is a mixture of pure phases, then in each phase $M_A$ should act as a scalar (the expectation of $A$ in that phase), and $\mathcal{A}$ should be an algebra of functions on the space of phases Inferred (the motivation).

03

## Statement (2) — commutativity, the Gelfand representation, a probability measure

Statement (commutativity of the zero-energy algebra)

If the limiting table satisfies the Euclidean bundle (`Table.Hierarchy`), is O(4) invariant under the rotation action $R$ (`R.Oriented S`, 12-06 § 04), and $\mathcal{H}_0$ is invariant under spatial translations (`Steady`), then $\mathcal{A}$ is commutative: $xy = yx$ for all $x, y \in \mathcal{A}$.

Source A theorem of the published construction. Lean: `YM.OS.Ground.adjunction` (`YM/Ground/Permute.lean:149`; it uses the section variables `Steady V`, `S.Hierarchy`, `Rotation S`, `R.Oriented S`: same file, :12-13).

**Why rotational invariance is needed.** To swap the order of the product of two generators $M_A$ and $M_{A'}$, one has to swap the time order of two words placed in the time slab. We infer the argument to be that, using O(4) rotations (for example a rotation exchanging the time axis with a space axis, together with spatial translations), one can lay the two words that are lined up in time out in space instead and bring them back into time, thereby swapping their order Inferred (read from the lemma

names `orbit` and `gens`). Either way, the statement in the source takes O(4) invariance as a **hypothesis**. Hence the order is "Euclidean invariance (12-06) → phase decomposition".

Statement (Gelfand representation and the measure on phases)

Under the hypotheses above and given a dense sequence in $\mathcal{A}$, there are a compact metric space $X$, a Borel measure $\mu$ on it, and for each point $\xi \in X$ a character (*-homomorphism) $\xi\colon \mathcal{A} \to \mathbb{C}$, such that: $\mathcal{A}$ is commutative; $\mu$ is a probability measure with full support (every non-empty open set has positive measure); the map from $X$ to the set of all characters is a bijection; $\xi \mapsto \xi(a)$ is continuous; and

$$\int_X \xi(a)\, d\mu(\xi) \;=\; \langle \Omega_0,\, a\,\Omega_0 \rangle \qquad (a \in \mathcal{A}). \tag{2}$$

Source A theorem of the published construction (an application of the Gelfand representation of commutative C*-algebras and the Riesz–Markov representation theorem; the measure on phases is written $\nu$ in the assessment and $\mu$ in this part). Lean: `YM.OS.Ground.resolution` (`YM/Ground/Spectrum.lean:127`), with the conclusion type `Spread.Resolution.Out` (`YM/Phase/Mixture.lean:78`: fields `abelian`, `normalized`, `support`, `onto`, `values`, `weak`, `ground`).

The tools are Mathlib's Gelfand transform for commutative C*-algebras and the Riesz–Markov–Kakutani representation theorem Source (the assessment's glossary and the imports of the source). $\mathcal{A} \cong C(X)$, and the vacuum state $a \mapsto \langle \Omega_0, a\Omega_0 \rangle$ is represented as a probability measure $\mu$ on $X$. Full support follows from the fact that no non-zero element of $\mathcal{A}$ annihilates $\Omega_0$ Assessment (the assessment's glossary, `fullSupport`).

04

## Statement (3) — correlations of the components, and uniqueness of the vacuum in each component

**Correlations of the components.** The correlations of the **phase** at a point $\xi \in X$ are defined, after translating the word $A$ into a positive-time slab, by

$$S^{\xi}(A) \;=\; \xi(M_A) \tag{3}$$

The value does not depend on the translation or on the width of the slab Assessment (the assessment's glossary, `Character`) Source (`Spread.Resolution.receives`, `Mixture.lean:89`). Applying (2) to $a = M_A$, the mixed theory is the average of the phases:

$$\int_X S^{\xi}(A)\,d\mu(\xi) \;=\; S^{\mathrm{mix}}(A). \tag{4}$$

Source Lean: `YM.OS.Spread.Resolution.barycenter` (`Mixture.lean:113`; translation invariance `S.Homogeneous` among the hypotheses). It corresponds to the field `Reconstructed.mixture` of the published statement (`ChallengeSpec.lean:133`).

**Each component satisfies the OS bundle.** The fields of the phase-selection certificate `OS.Selection.Certificate` (`Mixture.lean:136`) include: the table of each phase satisfies the Euclidean bundle (`states`), it is O(4) invariant (`sym`), its null space of fields is the same as for the mixed theory (`retains`), the laws of the stress tensor (`flux`), and the vacuum is unique in the chosen phase (`heat`) Source .

Statement (uniqueness of the vacuum in each component)

Write `UniqueHeat` (`Mixture.lean:35`) for the statement that the zero-energy space $\mathcal{H}_0^{\xi}$ of the OS space of the phase $\xi$ is one-dimensional (only multiples of the vacuum $\Omega^{\xi}$). (i) `UniqueHeat` holds for $\mu$-almost every $\xi$. (ii) In the construction's bundle of phases, `UniqueHeat` holds for every $\xi$.

Source Lean: (i) `YM.OS.Cluster.lifts` (`YM/Phase/Unique.lean:91`), (ii) `YM.EveryPhase.every` (`YM/EveryPhase/Vacuum.lean:146`). (ii) enters the field `euclid` of the construction's conclusion `EveryPhase.Completed` (a field of the structure in `YM/EveryPhase/Extension.lean:51`).

**The argument for (i)** Source (lemma names and statements). For a positive-time word $b$, the integral over $\xi$ of $\|P_0^{\xi} b\|^2$ and $\int |\xi(M_b)|^2\,d\mu$ are both equal to $\|P_0 b\|^2$ (`trail`, `at_M`, `almost`). Since $\|P_0^{\xi} b\|^2 \geq |S^{\xi}(b)|^2 = |\xi(M_b)|^2$ (the vacuum component is part of $P_0^{\xi} b$), the integral of the non-negative difference $\|P_0^{\xi} b - S^{\xi}(b)\Omega^{\xi}\|^2$ is 0, so this difference is 0 for almost every $\xi$ (`exp_real`). Combining over a countable family of words spanning the whole space, $\mathcal{H}_0^{\xi}$ consists only of multiples of $\Omega^{\xi}$ for almost every $\xi$ Inferred (the details of the direction of the inequality and the choice of the countable family). We have not read how (ii) extends "almost every" to "every" Inferred .

**No rate is attached.** Uniqueness of the vacuum is connected with $E^{\xi}(t)x \to \langle \Omega^{\xi}, x\rangle \Omega^{\xi}$ under time evolution (the committee's check derives clustering of words separated in the time direction without any assumption on the rate: on the assessment side, `submissions/openai-20261028/checks/ClusterCore.lean` `chamber_word_cluster`), but the rate of approach is not stated Source . An exponential rate would be a mass gap, but that is not in the statement; what is there is a polynomial estimate (12-08 § 05).

**The chosen phase.** The construction picks one phase $\xi^*$ from a set $E$ with $\mu(E) = 1$ (the set of phases whose zero-energy space is one-dimensional and in which the short-distance limit of the two-point function of the trace holds), and builds the Wightman theory from that phase Assessment (the assessment's glossary, `selected`).

05

## What is said and not said about the mixed theory and the components

| What is said (at the level of the assessment) | What is not said |
|---|---|
| $S^{\mathrm{mix}} = \int S^{\xi}\, d\mu$ (where points do not coincide) | Clustering of $S^{\mathrm{mix}}$ itself |
| $\mu$ is a probability measure with full support | That $X$ is a single point (one phase) |
| For each $x$, $\xi \mapsto S^{\xi}(x)$ is continuous | That the components are equal to one another |
| The vacuum is unique for every $\xi$ (no rate) | The rate of clustering (the decay rate) |
| Each component satisfies the OS bundle and O(4) invariance | Comparison of limits along different subsequences |
| — | Approximating the components directly by lattice measures |

Source The left column is in the fields of the published statement (`Reconstructed` in `ChallengeSpec.lean:117-135`: `normalized`, `fullSupport`, `characterContinuity`, `mixture`) and in the fields of the construction's conclusion `Completed`.

**If the mixed theory had clustering.** Assessment § 6.2 writes that if $S^{\mathrm{mix}}$ satisfies clustering, a short calculation gives $\int |S^{\xi}(A) - S^{\mathrm{mix}}(A)|^2\, d\mu = 0$, and continuity and full support then make all the components agree. It also writes that even with clustering, this would not amount to the infrared control needed for uniqueness of the infinite-volume limit Assessment . That all components are equal means "one phase within the limit along one subsequence", which is a different question from whether the limits along different subsequences are equal Proved here .

06

## What this book's investigation had seen

**The phase decomposition existed only in the plan of this book's investigation.** This book's table of construction steps (16-09) places reconstruction and uniqueness of the vacuum as plans for later steps, and there are no Lean statements for them. Separating phases with a commutative algebra compressed to the zero-energy space is an idea that was not in this book's investigation (14-05).

**Inside the strong-coupling window, we had it in a form that needs no decomposition.** The limit $\nu_\beta$ inside the window is a single measure without subsequences, and for the transfer operator $T$ of the reconstruction vessel it has $\|T - P_\Omega\| \le (18 \tanh \beta)^{1/2}$ and clustering at an exponential rate in the four directions.

Lean `InfMain2.wilson_infvol_main2` (the form of the mass gap and non-degeneracy, 20-04), `WilsonLimit.cluster_limit` (clustering in the four directions at limit points). A formalisation of the Osterwalder–Seiler strong-coupling, Dobrushin-type argument, with the lattice spacing fixed at 1, only inside the strong-coupling window, and with no continuum limit.

If clustering comes with a rate, the vacuum is unique and the decomposition is a single point Proved here . This book's results inside the window have that form, but the regime is different (fixed lattice spacing, strong coupling). The phase decomposition of the published construction is a tool for the weak-coupling continuum limit, where clustering cannot be shown.

One can ask whether the "phases" of the published construction can be distinguished by this book's twisted boundary conditions or vortex free energies (Part 21). That is a question for Volume 3 (16-03).

07

## What we have not read

- **How O(4) is used in the proof of commutativity.** We have read only as far as `orbit` and `gens` inside `adjunction`.
- **The proof of `EveryPhase.every`.** How it extends from almost every phase to every phase.
- **That $\mathcal{H}_0$ is invariant under spatial translations** (`Steady`) is derived inside `resolution` from `inert`, but we have not read its content.
- **The construction of the set $E$ of chosen phases** (`YM.Trace.submits` and others listed in the assessment's glossary).
- **The chain to the published theorem.** We read that `EveryPhase.all_characters` is called directly, at :158, inside `R2IGoals.fromFunded` (`YM/R2I/Campaign/Accepted.lean:150`) in the construction's chain Source . We have not traced, declaration by declaration, the chain from `resolution` and `adjunction` to `all_characters` (the headline in the records is `SharpGram.habitat`).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The definition of the algebra $\mathcal{A}$ of the operators $M_A$ compressed to the zero-energy space (`Spread.algebra`) | — |
| Source Assuming O(4) invariance and more, $\mathcal{A}$ is commutative (`Ground.adjunction`) | How O(4) is used in the proof of commutativity |
| Source The Gelfand representation and a probability measure with full support, $S^{\mathrm{mix}} = \int S^{\xi} d\mu$ (`Ground.resolution`, `barycenter`) | That $X$ is a single point; agreement of the components |
| Source The vacuum is unique in almost every phase, and moreover in every phase (`Cluster.lifts`, `EveryPhase.every`) | The rate of clustering; the proof of `every` |
| Assessment If the mixed theory clusters, all components agree, but even then uniqueness of the infinite-volume limit does not follow (Assessment § 6.2) | Clustering of the mixed theory |
| Lean This book's limit inside the window needs no subsequence and has clustering at an exponential rate (`wilson_infvol_main2`, `cluster_limit`) | Varying the lattice spacing; weak coupling |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Gelfand representation, Riesz–Markov representation theorem | Cited | Standard theorems on commutative C*-algebras and measure theory (1-10). In Lean, Mathlib |
| The OS space and zero energy | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/OS/Dynamics.lean:57` `Chamber`, `YM/OS/Core.lean:14` `ground`, :38 `emerges` |
| Algebra, commutativity, decomposition | Source | `YM/Phase/Mixture.lean:35` `UniqueHeat`, :54 `algebra`, :78 `Resolution.Out`, :89 `receives`, :113 `barycenter`, :127 `Selection`, :136 `Selection.Certificate`; `YM/Ground/Permute.lean:149` `adjunction`; `YM/Ground/Spectrum.lean:127` `resolution` |
| Uniqueness of the vacuum | Source | `YM/Phase/Unique.lean:91` `lifts`; `YM/EveryPhase/Vacuum.lean:146` `every`; `YM/EveryPhase/Extension.lean:51` `Completed`, :75 `all_characters`; `ChallengeSpec.lean:117-135` `Reconstructed` |
| The committee's checks | Source | `yang-mills-group/YM` `submissions/openai-20261028/checks/ClusterCore.lean` `chamber_word_cluster` |
| Text of the assessment | Assessment | `docs/assessment-1.pdf` § 6, § 6.1, § 6.2; glossary `Character`, `phaseMeasure`, `selected`, `fullSupport` |
| Clustering inside this book's window | Lean | `InfMain2.wilson_infvol_main2`, `WilsonLimit.cluster_limit` (20-03, 20-04) |

**Terms used in this chapter** — clustering, zero-energy space, compression, commutative C*-algebra, Gelfand representation, character, phase, measure on phases, full support, uniqueness of the vacuum. Their meanings are in the A-02 glossary.

Next chapter: 12-08 Reconstruction and the Wightman axioms — we build the Wightman theory from the chosen phase and read which axioms come out and what is not said (the mass gap).

2026-11-13 · chapter Yang–Mills and the mass gap Part 12 Infinite volume and reconstruction OS reconstruction · Kiichi, Shiori and Rin

# Reconstruction and the Wightman axioms

This chapter is established knowledge (a reading of the published construction). § 05 is a map of open questions (the bridge to Volume 3), § 06 is this book's own investigation, and § 07 is a map of open questions.

**What this chapter shows** — The published construction defines its own bundle of the Euclidean axioms that the limiting table satisfies (with a growth condition of the form "$r^{-A}$ for point separation $r \leq 1$"), builds a relativistic field theory that satisfies the Wightman axioms in every phase (translation spectrum and uniqueness of the vacuum, a common domain and cyclicity, Poincaré covariance, locality), and states the additional properties — analytic continuation, agreement at purely imaginary times, boundary values, faithfulness — for the chosen phase. The committee checked the standard, independently formulated OS axioms and growth condition for the correlations of **a single scalar field** in the chosen phase, the centred trace of the stress tensor. This chapter reads the bundle of axioms, the list of axioms that come out, and the scope of the check, and at the end confirms that there is no statement of a mass gap, only a polynomial estimate uniform over the phases.

**Prerequisite chapters** — 4-09 (the Wightman axioms), 4-10 (the OS axioms and reconstruction), 12-07 (phase decomposition).

**Said first** — The results of the published construction stand at the level "the committee's assessment is that it is proved in Lean". What the committee's independent check confirmed against the standard OS axioms concerns the correlations of **the centred trace scalar in the chosen phase**; it did not separately translate and check all the tensor fields. The construction says nothing about the mass gap (Assessment § 2, item 4).

The order of this chapter

1. Key to the labels
2. Statement (1) — the published construction's version of the bundle of Euclidean axioms
3. Statement (2) — reconstruction from phases — every phase, and the chosen phase

4. List of the Wightman axioms and additional properties that come out
5. The scope of the committee's independent check
6. There is no statement of a mass gap — a polynomial estimate uniform over the phases (the bridge to Volume 3)
7. What this book's investigation had seen
8. What we have not read

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). The level "proved in Lean" for the theorems of the published construction rests on the committee's assessment Assessment .

01

## Statement (1) — the published construction's version of the bundle of Euclidean axioms

The introduction is in 4-10. The construction defines the properties satisfied by the table $S$ of limiting correlations (functions on configurations of non-coinciding points) as a bundle of the following seven fields Source (construction source `YM/Euclid/Words.lean:141` `Table.Hierarchy`).

| Field | Content | Correspondence with the standard OS axioms |
|---|---|---|
| `laws` (`Basic`) | Symmetric under simultaneous permutation of points and labels; the 0-point correlation is 1; the unit field can be removed | Symmetry (E3) and normalisation |
| `spin` (`RealSpin`) | Covariance under the hyperoctahedral group (coordinate permutations and signs) and reality | The discrete part of Euclidean covariance (E1), and reality |
| `translations` | Invariant under continuous translations | The translations of E1 |
| `positive`, `onTests` | Reflection positivity for each coordinate axis (in the form with words and in the form smeared with test functions) | Reflection positivity (E2) |
| `smooth` | $C^\infty$ where points do not coincide | (regularity) |

| Field | Content | Correspondence with the standard OS axioms |
|---|---|---|
| `order` (`Capped`) | Growth condition ((1) below) | (one form of a growth condition) |

**The form of the growth condition.** When a configuration of points $x$ has "separation $r$" (every pair of points at distance at least $r$, in the cube norm) with $0 < r \leq 1$, then for the $j$-th derivative of the correlations

$$\left\| \nabla^j S(v; x) \right\| \;\leq\; C\, r^{-A} \quad (1)$$

holds for some $A$ and $C \geq 0$ (for each label and order) Source (`YM/Euclid/Table.lean:97` `Capped`). Because $r$ is capped at 1, the right-hand side stays $C$ even when the points are far apart, so **no growth at infinity is allowed**. Only the singularity as points approach each other is allowed, as a power.

**Properties outside the bundle.** Invariance under continuous rotations (O(4)) is not in the bundle and is required separately in the field `sym` of the phase certificate (12-06 § 04). Clustering (uniqueness of the vacuum) is also outside the bundle and is required separately as `UniqueHeat` for each phase (12-07 § 04) Source . That the limiting table satisfies the bundle is assembled from the properties that pass to the limit (12-05 § 04) plus smoothness and the growth condition (`Order.assembles`, `YM/Extraction/Harvest.lean:62`).

02

## Statement (2) — reconstruction from phases — every phase, and the chosen phase

The correlations $S^\xi$ of a phase $\xi$ (12-07) are continued analytically to complex times, and the real-time Wightman distributions $W^\xi$ are obtained as boundary values as the imaginary parts tend to 0 (after smearing with Schwartz test functions). On the space $\mathcal{P}$ of linear combinations of formal products of field labels $v_i$ and Schwartz test functions $f_i$ one puts $\langle u, w \rangle_\xi = W^\xi(u^* w)$; quotienting by the null-norm part and completing gives the Hilbert space $\mathcal{H}^\xi$, the empty word is the vacuum $\Omega^\xi$, and the operation of putting $(v, f)$ in front of a word is the smeared field operator $\Phi^\xi_v(f)$ Assessment (Assessment § 6.1).

Statement (reconstruction of the component theories, Theorem 6.1 of the assessment)

In the assessment's words: for every character $\xi \in X$, the component correlations $S^\xi$ reconstruct a Wightman theory on $\mathcal{H}^\xi$ (vacuum $\Omega^\xi$, fields $\Phi^\xi$ on a domain $\mathcal{D}^\xi$) satisfying: 1. translation spectrum, positive energy, uniqueness of the vacuum; 2. common domain, regularity, adjoints, cyclicity; 3. Poincaré covariance; 4. locality;

5. analytic continuation and recovery of the correlations; 6. faithfulness of the fields.

Assessment Theorem 6.1 in Assessment § 6.1 (attaching items 1–6 to every $\xi$ is the assessment's wording).

Source Split according to the Lean statements, it is as follows. The Wightman axioms hold **in every phase** (the field `characters` of the construction's conclusion `Headline.Reconstruction`, `YM/Nonperturbative/HeadlineSpec.lean:75`; the underlying structure `WightmanAxioms`, same file :52-71, covers normalisation of the vacuum, density and cyclicity of the domain, continuity in the test functions, adjoints, agreement of correlations, positivity, locality, symmetry, strong continuity, the spectrum, the forward light cone and uniqueness of the vacuum, and has no fields for faithfulness or analytic continuation). The **additional properties** (faithfulness `faithful`, analytic continuation `analytic`, agreement at purely imaginary times `root`, boundary values `boundary`) are stated only for **the chosen phase**, inside the field `Hierarchy.Reconstructed` of the published statement (`ChallengeSpec.lean:117-143`), and `YangMills.publicReconstruction` on the construction's side (`HierarchyBridge.lean:79`) satisfies it.

**Which goal it is in.** The conclusion field `reconstruction : T.Reconstructed` of the published theorem `JointExistence` is a statement about the Wightman theory of **the chosen phase** $\xi^*$ (the fields `axioms`, `analytic`, `root`, `boundary` concern the continuation $T.\mathrm{continuation}$ and $T.\mathrm{theory}$ of the chosen phase). The Wightman axioms for every phase are in the conclusion of the construction's theorem `constructed` (`Headline.Reconstruction.characters`) Source . In the records `publicReconstruction` is `used` (reached from the headline theorem `su2Theory`), and its direct user is `constructed` Source .

03

## List of the Wightman axioms and additional properties that come out

We list the content of the field `Reconstructed` of the published statement (the Wightman-axiom field `axioms` and the additional fields are statements about the chosen phase; the Wightman axioms for every phase are a separate conclusion, as in § 02), following the assessment's glossary Source (field names and types) Assessment (explanations of meaning).

| Field | Content |
| --- | --- |
| `axioms`: translation spectrum and uniqueness of the vacuum | A strongly continuous unitary representation of translations and a projection-valued energy–momentum measure; the support of the measure is the closed forward light cone; every vector invariant under all translations is a multiple of the vacuum |
| `axioms`: common domain, regularity, adjoints, cyclicity | The domain is dense, contains the vacuum and is preserved by the fields; $f \mapsto \Phi_v(f)\psi$ is continuous in the Schwartz topology; the adjoint is given by conjugating the label; the domain is the span of field words applied to the vacuum |
| `axioms`: Poincaré covariance | A strongly continuous unitary representation of the proper orthochronous Poincaré group (with the same translations as above), an action $\rho(g)$ on labels, and the field transformation law $U(g)\Phi_v(f)\psi = \Phi$ $\rho(g)v(fg)U(g)\psi$ |
| `axioms`: locality | Two fields whose supports are spacelike separated commute |
| `faithful`, `acts`, `injective`, `covers`, `represents` | A non-zero label gives a non-zero field; at least one field is non-zero (faithfulness is not vacuous); the map to labels is injective and surjective; every field is the image of a lattice label |
| `normalized`, `fullSupport`, `characterContinuity`, `mixture` | The measure on phases is a probability measure with full support; the correlations are continuous in the phase; the average over phases is the mixed correlation (12-07 § 05) |
| `permutation`, `smooth` | Permutation symmetry and smoothness of the correlations of the mixed phase and of each phase |
| `analytic`, `root`, `boundary` | The analytic continuation of the chosen phase is holomorphic on $\mathrm{Im}\, z_1 < \cdots < \mathrm{Im}\, z_n$; at purely imaginary times it agrees with the Euclidean correlations; its boundary values at real times agree with the vacuum expectation values (the Wightman distributions) |

According to the assessment's glossary, uniqueness of the vacuum goes as follows: an isometry from the Hilbert space on the Wightman side to the OS space of the chosen phase maps the vacuum to the vacuum and commutes with translations; non-negative energy on the OS space and Lorentz covariance give the spectral condition on the forward light cone; a translation-invariant vector is mapped by the isometry into the zero-energy space on the OS side, which is one-dimensional (`UniqueHeat`), so it is a multiple of the vacuum[Assessment] .

04

## The scope of the committee's independent check

The committee checks the output of the construction against OS axioms formulated independently of the construction's definitions (reference [DMY] of the assessment, repository `xiyin137/OSreconstruction`)[Assessment] (Assessment § 10).

- **Object.** The hierarchy of $n$-point functions of **a single scalar field**, the centred trace $\Theta$ of the stress tensor, in the chosen phase $\xi^*$. It does not separately translate all the tensor fields into the upstream formulation Assessment ( § 10.1).
- **Test functions.** `ZeroDiagonalSchwartz`: Schwartz functions all of whose derivatives of all orders vanish on every diagonal ($x_i = x_j$). They need not be 0 near the diagonals, nor have compact support. They include the compactly supported test functions supported away from coinciding points that are used in the statements of convergence from the lattice Assessment ( § 10, equation (21)).
- **Axioms checked.** E0 (temperedness, linearity), E1 (translation and rotation invariance), E2 (reflection positivity), E3 (symmetry), E4 (clustering in spatial directions, in the chosen phase), and the linear growth condition E0′ of OS-II type Source (on the assessment side, `verifiedOS`, `verifiedOS_growth`, `constructed_OS_family` in `submissions/openai-20261028/checks/CompleteOS.lean`; `completeOS`, `E0_prime` in `checks/RemainingObligations.lean`; `E2_reflection_positive` in `checks/ReflectionPositivity.lean`).
- **What is not said.** Test functions that are flat on the diagonals do not see contact terms on the diagonals, so uniqueness of the extension of contact terms and OS positivity on the unrestricted space of test functions are not checked Assessment ( § 10). E4 is a statement about the chosen phase, not a claim that the mixed table clusters Source (the docstring of `submissions/openai-20261028/docs/examples/OSAxioms-20261028.lean` on the assessment side).

Source On the assessment side, `constructed_OS_family` has the form: for a run of the construction (penalty coefficient $b > 0$) and any extraction sequence $p$, there are a refinement $q$, a table $S$ and a final certificate $F$ such that the limit can be read off along the extraction "cutoffs along the whole sequence, volumes along the subsequence of volumes of $(p \circ q)$", and E4 clustering and E0′ hold.

05

## There is no statement of a mass gap — a polynomial estimate uniform over the phases (the bridge to Volume 3)

A mass gap means that the spectrum of the Hamiltonian $H \geq 0$ lies in $\{0\} \cup [\Delta, \infty)$ (with $\Delta > 0$) (4-11). The assessment writes that "the construction gives no evidence either for or against a mass gap" ( § 2, item 4), and lists as open question 3 "a uniform mass gap (exponential decay of the corresponding correlations and a spectral gap of the OS Hamiltonian)" Assessment .

What the construction has is **a polynomial estimate uniform over all phases** Source .

Statement (low-energy and heat estimates uniform over the phases)

For a finite sum of words $w$, there is for each of (i) and (ii) a constant $M \geq 0$ (taking the larger makes it one constant Proved here ) such that in every phase $i$: (i) if $0 < u \leq 1$, the spectral weight of the state of $w$ at energies in $(0, u]$ is $\leq M u^2$; (ii) if $\tau \geq 1$, then

$$0 \;\leq\; \langle A,\; (e^{-\tau H} - P_0)\, A \rangle \;\leq\; M\, \tau^{-2} \qquad (2)$$

($A$ is the state of $w$, and $P_0$ is the projection onto energy 0).

Source Theorems of the published construction. Lean: `YM.EveryPhase.light_bound` (`YM/EveryPhase/Light.lean:221`), `YM.EveryPhase.inverse_square` (`YM/EveryPhase/Decay.lean:173`). They enter the fields `light` and `heat` of the construction's conclusion `EveryPhase.Completed`.

**How to read it.** (ii) follows from (i) by elementary means Proved here : $\langle A, (e^{-\tau H} - P_0)A \rangle = \int_{(0,\infty)} e^{-\tau E}\, d\sigma_A(E)$; the part on $(0, 1]$ is split into dyadic intervals and gives $O(\tau^{-2})$ from $\sigma_A((0, u]) \leq M u^2$, and the part on $(1, \infty)$ is bounded by $e^{-\tau}$. Here $\sigma_A$ is the spectral measure of the state $A$ (not the measure on phases $\mu$). What can be read in the source is that the proof of `light_bound` uses `late` (the average of the commutator is 0 far away in space, `Light.lean:208`) and the dyadic splitting `dyadic` Source . The reading that this is a general consequence of locality and support in the light cone is Inferred . This form is also compatible with a massless theory Inferred . It is a different thing from an exponential rate $e^{-\Delta\tau}$.

**The bridge to Volume 3.** The mass-gap question can be restated in these terms: "strengthen the low-energy weight estimate $\sigma_A((0, u]) \leq M u^2$ to $\sigma_A((0, \Delta)) = 0$ below some $\Delta > 0$". This is the same as strengthening the right-hand side of the heat estimate (2) from $M\tau^{-2}$ to $M e^{-\Delta\tau}$ Proved here . This is open question 3 of the assessment, treated in Volume 3 in 16-02 (Five open questions) and 16-03 (The infrared edge of the published construction).

06

## What this book's investigation had seen

**Reconstruction of a Wightman theory was not in this book's investigation.** This book's Lean reconstruction goes only as far as the vessel on the Euclidean side (the reconstruction vessel in 19-03: a positive semidefinite bilinear form on reflection pairs, time translations, a vacuum, a transfer operator $T$), and treats neither analytic continuation nor Wightman distributions.

**Inside the strong-coupling window, we had the form of a mass gap.** In the limiting vessel inside the window, $\|T - P_\Omega\| \le e^{-m}$ with $m = -\log(18 \tanh \beta)/2$, the complement of the vacuum is non-zero, and $T \ge 0$.

Lean `InfMain2.wilson_infvol_main2` (20-04). A formalisation of the Osterwalder–Seiler strong-coupling argument. The lattice spacing is fixed at 1, only inside a window of about $|\beta| < 0.0556$, only a lower bound on the mass, and no continuum limit.

The published construction reaches a Wightman theory in the weak-coupling continuum limit and has no statement of a mass gap. This book has the form of a mass gap in a strong-coupling window at fixed lattice spacing, and has no continuum limit. The two results lie in different regimes and are not connected (14-06).

07

## What we have not read

- **The proof of the published construction's version of OS reconstruction** (`YM/Boundary/*`, `YM/EveryPhase/*`). On how the analytic continuation is built and how the boundary values are taken, we have only the explanations in the assessment's glossary.
- **Where the growth condition (1) comes from.** We have not traced who discharges the hypothesis `Capped` of `assembles`.
- **The inputs from the construction's estimates in the committee's check.** Assessment § 10.1 writes that "the analytic inputs come from the submitted proof". We have not read which estimates were used.
- **The proof of `light_bound`.** We have only the names `late`, `annulus`, `dyadic`, `fee`.
- **Continuum Wilson loops.** Continuum Wilson-loop observables and an area law are not among the statements in the range we read (the range of a search by name) Source . Parallel transport along lattice paths, `travel` (`YM/Lattice/Path.lean:105`), is there. Assessment § 8 lists as open question 4 constructing Wilson loops, relating them to local observables and proving an area law Assessment .
- **Reconstruction of the mixed theory.** Assessment § 6.2 writes that OS reconstruction can also be applied to the mixed limiting theory (uniqueness of the vacuum is not shown). We have not checked which declaration in the source this is.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The published construction's version of the bundle of Euclidean axioms (seven fields) and the form of the growth condition (1) ($r^{-A}$ for separation $r \leq 1$, no growth at infinity) | Where (1) comes from. O(4) and clustering are outside the bundle |
| Source The Wightman axioms in every phase (`Headline.Reconstruction.characters`); the additional properties (faithfulness, analytic continuation, agreement at purely imaginary times, boundary values) for the chosen phase (`publicReconstruction`). At the level of the committee's assessment | Stating the additional properties in every phase |
| Assessment The committee's check: the standard OS axioms E0–E4 and E0′ (with Schwartz test functions flat on the diagonals) for the centred trace scalar in the chosen phase | A check of all the tensor fields. Uniqueness of the extension of contact terms |
| Source Polynomial estimates uniform over the phases: low-energy weight $\leq Mu^2$, heat $\leq M\tau^{-2}$ (`light_bound`, `inverse_square`) | A mass gap. Exponential decay. One-particle states |
| Proved here The heat estimate follows from the low-energy estimate. The restatement of the mass gap | — |
| Source (within the range of a search by name) Parallel transport along lattice paths, `travel` (`YM/Lattice/Path.lean:105`), is there | Continuum Wilson-loop observables and an area law (not among the statements; open question 4 in Assessment § 8) |
| Lean The form of a mass gap inside this book's window (`wilson_infvol_main2`) | The continuum limit. A Wightman theory |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| The Wightman axioms and OS reconstruction | Cited | K. Osterwalder, R. Schrader, Comm. Math. Phys. 31 (1973) 83; 42 (1975) 281. 4-09, 4-10; references in A-03 |
| Bundle of axioms and growth condition | Source | Construction source `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/Euclid/Words.lean:123` `Table.RP`, :141 `Table.Hierarchy`; `YM/Euclid/Table.lean:97` `Capped`; `YM/Extraction/Harvest.lean:62` `assembles` |
| Statements of reconstruction | Source | `ChallengeSpec.lean:117-143` `Hierarchy.Reconstructed`, :186 `JointExistence`; `HierarchyBridge.lean:79` `publicReconstruction`; `PublicView.lean:102` `view_axioms`; `YM/Nonperturbative/HeadlineSpec.lean:75` `Headline.Reconstruction` |
| Polynomial estimates | Source | `YM/EveryPhase/Light.lean:208` `late`, :221 `light_bound`; `YM/EveryPhase/Decay.lean:173` `inverse_square`; `YM/EveryPhase/Extension.lean:51` `Completed` |

| Item | Kind | Source / tool |
|---|---|---|
| The committee's checks | Source, Assessment | `yang-mills-group/YM` `submissions/openai-20261028/checks/CompleteOS.lean`, `RemainingObligations.lean`, `ReflectionPositivity.lean`, `ClusterCore.lean`; `submissions/openai-20261028/docs/examples/OSAxioms-20261028.lean`. Assessment § 6.1, § 10, § 10.1; glossary `Reconstructed.*` |
| Open questions of the assessment | Assessment | `docs/assessment-1.pdf` § 2, § 8 |
| This book's vessel and the form of a mass gap | Lean | `InfMain2.wilson_infvol_main2` (20-04) |

**Terms used in this chapter** — OS axioms, Wightman axioms, growth condition, analytic continuation, boundary value, spectral condition, locality, phase, centred trace, test function flat on the diagonals, mass gap, polynomial estimate. Their meanings are in the A-02 glossary.

Next chapter: 13-01 The stress tensor lies on a single line — we begin reading the evidence that the reconstructed theory is Yang–Mills, starting from the uniqueness of the stress tensor.

# The stress tensor lies on a single line

This chapter is established knowledge (a reading of the published construction). § 07 is this book's own investigation, and § 08 is a map of open questions.

**What this chapter shows** — That the published construction has a stress tensor $T_{ab}$ with values in the field space $F_4$; that the candidates for such a tensor form a one-dimensional real line; and that the translation Ward identity picks out one point on it. The Ward identity says that "the flux of stress through a surface enclosing a point generates coordinate translations", in a form that does not touch coinciding points.

**Prerequisite chapters** — 8-03 (the field spaces $F_d$), 8-05 (correlations and phases). The classical action and conservation laws are in 2-01 and 3-03.

**Said first** — The stress tensor **does not appear in the published statement**. It is on the side of the construction's theorem and on the side of the committee's joint goal. How it is used to obtain Euclidean invariance is in 12-06. This chapter reads only the definitions and the form of the statements.

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Statement — the conditions on the stress tensor and the one-dimensional line
2. The translation Ward identity: window functions and surfaces enclosing a point
3. The rotation Ward identity: flux weighted by the lever arm
4. Outline of the mechanism — why a line, and what picks out one point
5. What the committee's check confirmed and what it does not say
6. Which goal the statements belong to
7. What this book's investigation had seen

8. What we have not read

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01

## Statement — the conditions on the stress tensor and the one-dimensional line

In classical Yang–Mills theory the stress tensor (energy–momentum tensor) is obtained by varying the action with respect to the metric (3-03). In quantum theory, pointwise products of fields have no meaning, so it cannot be defined that way. Assessment § 7.1 writes that the construction's stress tensor is **uniquely specified** by the following properties Assessment .

> The symmetric tensors $J_{ab}$ with components in the field space $F_4$ that are unchanged by conjugation of fields, centred with respect to $S^{\mathrm{mix}}$ (one-point function 0), transform as rank-two tensors under coordinate permutations and sign flips, and have divergence 0 in $F_5$, form a one-dimensional real vector space. The stress tensor is a vector in this one-dimensional space, and its normalisation is fixed by the Ward identity.

The definitions on the construction's side are as follows Source (`YM/Euclid/Stress.lean` in `yang-mills-group/OpenAI-YM-20261028`).

- Type of a stress: $J : \{0,1,2,3\}^2 \to F_4$ (`Stress`, line 57).
- Conservation (`Conserved`, lines 59–62): symmetric $J_{ab} = J_{ba}$, real (invariant under conjugation), divergence zero $\sum_a \partial_a J_{ab} = 0$. Here $\partial_a : F_d \to F_{d+1}$ is the derivative map on fields, and the divergence is an element of $F_5$.
- Transformation as a tensor: a rank-two tensor under the hypercubic group (signed permutations) (`RawTensor`, line 68). The construction's stress is also a rank-two tensor under the action of the four-dimensional orthogonal group (`Tensorial`, lines 64–66).

The construction's stress tensor is the witness field `energy` (`YM/Spec/Closed.lean` line 31), and within the bundle of local laws `Locals` it has three properties, `conserved`, `spin` and `charges` (lines 41–43) Source .

**One-dimensionality.** The unconditional form is the field `ranks` of the structure `Pert.FirstExtension`, which enters the conclusion of the construction's theorem Source (`YM/Pert/FirstTarget.lean` lines 35–36, `Construction.lean` line 31): for the witness's derivative and rotation data, the real dimension of the kernel of the

divergence map is 1. The same fact also exists as the theorem `YM.Pert.one_line` (line 68) under the hypothesis `Supplier` (a bundle of ultraviolet data, Maxwell-field data, a sequence of scales and so on; `YM/Pert/Weak.lean` lines 22–32) Source . The kernel space (`KernelSpace`, `YM/Next/UV_STRESS/Tensor.lean` lines 75–82) is the space of the symmetry, reality and tensor conditions with "expectation 0 (centred)" added. Under the same hypothesis, `centered_line` (line 82) says that any $J$ that is conserved, transforms as a tensor and satisfies the Ward laws becomes, once centred, a **non-zero real multiple** of the stress `I.J` in the field of `Supplier` Source . We have not read that `I.J` is the same as the witness's stress `energy`.

$$\Big\{J : \text{ symmetric, real, centred, tensor, } \textstyle\sum_a \partial_a J_{ab} = 0\Big\} = \mathbb{R} \cdot T \quad (1)$$

02

## The translation Ward identity

What picks out one point on the line (1) is the translation Ward identity. Take pairwise distinct points $x_0, x_1, \dots, x_n$, a central field $B \in F_d$ and spectator fields $q_1, \dots, q_n$, and choose a **window function** $\chi$ as follows Source (`Window`, `Stress.lean` lines 72–73): smooth with compact support, equal to 1 near $x_0$, and with none of the other points $x_1, \dots, x_n$ in its support. Then

$$-\sum_{a=0}^{3} \int_{\mathbb{R}^4} \partial_a \chi(y)\, \big\langle T_{ab}(y)\, B(x_0)\, q_1(x_1) \cdots q_n(x_n) \big\rangle \, d^4y = \frac{\partial}{\partial x_0^b} \big\langle B(x_0)\, q_1(x_1) \cdots q_n(x_n) \big\rangle \quad (2)$$

holds Assessment (equation (13) in Assessment § 7.1) Source (`StressLaws.translation`, lines 92–94; the right-hand side is first written as "the correlation of the differentiated field $\partial_b B \in F_{d+1}$", and that this equals the coordinate derivative is a separate law).

**How to read it.** $\partial_a \chi$ is 0 near $x_0$ ($\chi \equiv 1$) and also 0 near the other points ($\chi \equiv 0$), so the integrand on the left-hand side of (2) does not touch coinciding points. The "step" of $\chi$ is a thick shell enclosing $x_0$, and $-\int \partial_a \chi\, T_{ab}$ is the flux of stress through that shell. In classical terms, by Gauss's theorem

$$-\int \partial_a \chi\, T_{ab}\, d^4y = \int \chi\, \partial_a T_{ab}\, d^4y \quad (3)$$

and since $\partial_a T_{ab} = 0$ holds away from the insertion points, the right-hand side picks up only the "contact contribution" at $x_0$ Inferred . In quantum theory the right-hand side touches coinciding points, so it cannot be written. (2) is the form that fixes that contribution as the translation generator $\partial_b$, using only correlations at separated points. In the assessment's words, "the flux of stress around a point generates coordinate translations" Assessment .

Since (2) is linear, replacing $T$ by $cT$ multiplies the left-hand side by $c$. The right-hand side of (2) is not identically 0 (in some configuration the derivative of the correlation is non-zero), so on the line (1) only the single point $c = 1$ satisfies (2), and this is the normalisation Proved here .

03

## The rotation Ward identity

An identity of the same form for rotations is also in the construction's bundle of laws Source (`StressLaws.rotation`, lines 95–97; the shell integral `shellSpin`, lines 84–86). For a four-dimensional antisymmetric matrix $g$ (a rotation generator),

$$-\sum_{a,b}\int \partial_a \chi(y)\,(g\,(y-x_0))_b\,\langle T_{ab}(y)\,B(x_0)\cdots\rangle\,d^4y = \langle (A_g B)(x_0)\cdots\rangle \quad (4)$$

where $A_g$ is the derivative of the rotation action on the field spaces (`Rotation.lie`, line 24). The flux weighted by the lever arm from $x_0$, as a flux of angular momentum, generates the rotation of the field. The main text of the assessment does not mention the rotation identity, but it is in the field `charges` of the construction's statement `Locals` Source .

The docstring of the definition says that this law "does not test contact terms at the point of $B$" (lines 88–89) Source .

04

## Outline of the mechanism — why a line, and what picks out one point

We write down the intuition for why the candidates form a one-dimensional space Inferred . If one builds symmetric rank-two tensors of weight 4 from the curvature, the main ones are $\sum_c \mathrm{Tr}(F_{ac}F_{bc})$, $\delta_{ab}\sum_{c,e}\mathrm{Tr}(F_{ce}F_{ce})$ and $\delta_{ab}\mathbf{1}$. Centring fixes the $\mathbf{1}$ part, and the conservation condition constrains the remaining combination. For the classical Maxwell field, the only conserved one is the energy–momentum tensor (up to a constant multiple).

The Lean proof does not take the form of proving this intuition directly in the interacting theory. The proof of `one_line` calls `differential_rank`, whose arguments are the Maxwell-field data, the sequence of scales, and its limit and fit (`limit_`, `fit`) Source . It appears to count the dimension of the kernel through a comparison with the free Maxwell field along a sequence of short-distance scales (13-03 § 05) Inferred . We have not read the content.

To summarise the flow:

1. Define the field spaces $F_4$, $F_5$ and the derivative maps $\partial_a$ via correlations (8-03).
2. Show that the space of $J$ satisfying the symmetry, reality, tensor and conservation conditions is one-dimensional over the reals (`ranks`; the form with hypotheses is `one_line`).
3. Have the construction's stress tensor $T$ as the point on that line satisfying the translation Ward identity (2) (`energy` and `charges`).
4. Use $T$ to lift Euclidean invariance in the continuum limit (12-06). Take the trace and study the trace anomaly (13-02).

05

## What the committee's check confirmed and what it does not say

The committee's annotated example `TranslationWard-20261028.lean` (in `submissions/openai-20261028/docs/examples/` of `yang-mills-group/YM` on the assessment side, revision 55ba9fc) proves four things in Lean about the translation Ward identity Assessment .

| What it confirmed | Where |
|---|---|
| Admissible windows $\chi$ actually exist (the statement is not vacuous). The size of the window may depend on the configuration | `admissible_shell_exists`, line 40 |
| The right-hand side of (2) equals the ordinary coordinate derivative, in every phase $\xi$, for central fields $B$ of every degree and every list of spectators | `tensor_generates_translations`, line 64 |
| The integrand of (2) is actually integrable (the integral does not rely on Lean's default value) | `shell_integrand_is_integrable`, line 85 |
| The left-hand side of (2) does not depend on the choice of window | `shell_charge_is_window_independent`, line 102 |

The same example stresses that the same tensor is used for every central field, spectator and phase, without readjusting coefficients for each correlation (lines 12–14). Assessment § 11 also writes that the Ward identity translator covers all local fields and all components and checks the normalisation Assessment .

**What it does not say** Assessment (lines 111–122 of the same example):

- That all **improvement terms** of the stress tensor (additions that do not change the translation charge) vanish.
- A construction of contact terms extended to coinciding points.
- The dilatation Ward identity.
- A comparison with an independently chosen Minkowski sign convention.

Also, (2) is a normalisation relative to the Euclidean conventions for coordinates and derivatives; it is not an independent check of any further physical normalisation.

06

## Which goal the statements belong to

| Goal | What is there about the stress tensor |
|---|---|
| The published theorem `JointExistence` | **Nothing.** The published hierarchy has only the trace field `centeredTrace` (an element of $F_4$), and it does not even say that it is the trace of the stress (13-02 § 01) Source |
| The construction's theorem `constructed` | Its conclusion contains `Pert.FirstExtension F X` (`Construction.lean` line 31), whose field `ranks` (`YM/Pert/FirstTarget.lean` lines 35–36) states the dimension 1 in (1). The stress tensor and the Ward laws are in the fields of the witness $F$ Source |
| The committee's joint goal | It includes the translation Ward check ([A6] of the assessment). It is also confirmed that the Ward identities, the scalar OS check and the trace law concern the same stress tensor and the same phase Assessment (Assessment § 9.3, § 9.4) |
| Headline theorems in the verification records | The record for the declaration `YM.Pert.one_line` is used, and the first headline found is `YM.Pert.Parity.parityEndpoint`. We have not checked the chain to the existence theorem Source (`verification/declaration-usage.tsv`) |

"used" in the verification records means "reached from one of the seventeen headline theorems", not that the existence theorem uses it.

07

## What this book's investigation had seen

This book's research and Lean have **neither** a stress tensor nor Ward identities. This book's objects are the Wilson measure on the lattice and coarse-step observables; it has no space of local fields (8-03), so there was nowhere to put a stress tensor.

On Euclidean invariance, this book went only as far as carrying the symmetry of the lattice's hypercubic group over to the pushforward measure (symmetric averaging of the block map), and placed invariance under continuous rotations among the statements to be shown (19-07). The idea of the published construction, lifting rotational invariance from "the Ward identities of the stress tensor and the vanishing of the flux at infinity", was not in this book (14-05).

08

## What we have not read

- The definitions of the kernel space of `one_line` (`KernelSpace`) and of the divergence map `currentDiv`, and `differential_rank` in the proof (beyond lines 68–88 of `YM/Pert/Weak.lean`).
- The source of the "law of derivatives" that ties the right-hand side of Ward to the coordinate derivative (`F.phase.phase.calculus` in the committee's example).
- Where in the proof of Euclidean invariance the rotation Ward identity is used (within the scope of 12-06).
- From which observables on the lattice side the construction's stress tensor `energy` is built (the construction of the stress tensor itself).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The space of $F_4$-valued tensors satisfying the symmetry, reality, tensor and conservation conditions is one-dimensional over the reals (`ranks` in the conclusion of the construction's theorem; `one_line` under the `Supplier` hypothesis) | A stress tensor in the published statement (there is none) |
| Assessment The translation Ward identity (2) holds in every phase and for every field, does not depend on the window, and its integrand is integrable | That all improvement terms vanish; the dilatation Ward identity |
| Source The rotation Ward identity (4) is in the construction's bundle of laws | Contact terms (coinciding points) |
| Proved here By the linearity of (2), the normalisation on the line is fixed to one point | An independent comparison with the Minkowski sign convention |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Conditions on the stress, windows, Ward laws | Source | `YM/Euclid/Stress.lean` lines 22–24 and 57–97 and `YM/Spec/Closed.lean` lines 27–43 of `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e) |
| One-dimensionality | Source | `YM/Pert/Weak.lean` lines 68 and 82, `YM/Pert/FirstTarget.lean` lines 26–39, `Construction.lean` lines 15–31, `verification/declaration-usage.tsv` |

| Item | Kind | Source / tool |
|---|---|---|
| Specification of the stress tensor, equation (13), the joint goal | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 7.1, § 9.3, § 9.4, § 11 |
| The committee's Ward check | Assessment | `submissions/openai-20261028/docs/examples/TranslationWard-20261028.lean` on the assessment side, lines 12–14, 40, 64, 85, 102, 111–122 |

**Terms used in this chapter** — stress tensor, Ward identity, window function, flux, improvement term, contact term. Their meanings are in the A-02 glossary.

Next chapter: 13-02 The trace anomaly, and not being free — how the trace of the stress tensor behaves at short distances.

# The trace anomaly, and not being free

This chapter is established knowledge (a reading of the published construction). § 07 is this book's own investigation, and § 08 is a map of open questions.

**What this chapter shows** — That in the chosen phase $\xi^*$, the two-point function of the centred trace field $\Theta$ behaves as $r^8 \log^2(1/r)\,\langle \Theta(-\frac{r}{2}e_0)\Theta(\frac{r}{2}e_0)\rangle \to 3\dim_{\mathbb{R}}\mathfrak{g}/\pi^4$; that the coefficient comes from the free two-point function of the curvature and the one-loop running; and that this logarithm excludes descriptions of the type "polynomials in free fields with finitely many masses".

**Prerequisite chapters** — 13-01 the stress tensor, 8-05 correlations and phases, 8-01 conventions ($b_0$ and $C_{\mathfrak{g}}$). Asymptotic freedom is in 4-06, dimensional transmutation in 4-07.

**Said first — All the published statement says about the trace field $\Theta$ is that it is an element of $F_4$.** $\Theta$ is one element inside an existential quantifier ($\exists T$), and what field it is is not written. That it is the trace of the stress tensor is written on the side of the construction's theorem and on the side of the committee's joint goal ( § 05). Also, what this law excludes is only one type of free-field description; it does not exclude all descriptions built from Gaussian fields.

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Statement — the four fields of the trace test
2. Where the coefficient 3 dim g/π⁴ comes from: the free two-point function and the trace-anomaly formula
3. Outline of the proof of the short-distance law: approach through lattice points and squeeze by monotonicity
4. Not being free — what is excluded and what is not

5. The Θ of the published statement is one element of F_4: where being the trace of the stress is stated
6. Which goal the statements belong to
7. What this book's investigation had seen
8. What we have not read

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01

## Statement — the four fields of the trace test

The published hierarchy `Hierarchy` has one element `centeredTrace` of $F_4$ (below, $\Theta$) Source (`ChallengeSpec.lean` line 48 in `yang-mills-group/OpenAI-YM-20261028`). In the chosen phase $\xi^*$ (`selected`), put $\Theta$ at the two points $\mp\frac{r}{2}e_0$ on the time axis and let the two-point function be

$$G_\Theta(r) = S^{\xi^*}\left(\Theta(-\tfrac{r}{2}e_0)\,\Theta(\tfrac{r}{2}e_0)\right) \quad (1)$$

(`timePair`, `traceTwo`, lines 145–149). The published property `nonfree` consists of the four fields of the structure `TraceTest` Source (lines 168–175).

1. `centered`: in the reconstructed vacuum, $\langle\Omega, \Phi_\Theta(f)\Omega\rangle = 0$ for every Schwartz test function $f$.
2. `amplitudePositive`: $0 < 3\dim_{\mathbb{R}}\mathfrak{g}/\pi^4$. **An inequality between constants that does not involve the theory**, placed there so that it can be used in downstream proofs Assessment (Appendix A). It says nothing about the theory.
3. `shortDistance`:

$$\lim_{r\downarrow 0} r^8 \log^2(1/r)\, G_\Theta(r) = \frac{3\dim_{\mathbb{R}}\mathfrak{g}}{\pi^4} \quad (2)$$

4. `excludes`: it does not agree near 0 with any function $W$ of the type in § 04.

(2) is the **full limit** as $r \downarrow 0$ (the filter $\mathcal{N}[>]0$), not a limit along a subsequence. Lengths are measured in the construction's fixed unit (the spacing at step 0, 8-02) Assessment (equation (16) in Theorem 7.1 of the assessment). For $SU(2)$, $\dim\mathfrak{g} = 3$ and the right-hand side is $9/\pi^4 \approx 0.0924$ Proved here .

The assessment also states, from Euclidean covariance, the form in all directions, $|x|^8 \log^2(1/|x|)\, K_\Theta(x) \to 3\dim\mathfrak{g}/\pi^4$ ($x \to 0$, $K_\Theta(x) = \mathrm{Re}\, S^{\xi^*}_{\Theta,\Theta}(x,0)$), and writes that the

committee's short-distance translator proved it Assessment (equation (17), § 11).

02

## Where the coefficient 3 dim g/π⁴ comes from

Classical Yang–Mills theory is conformally invariant in four dimensions, and the trace of the stress tensor is 0. In quantum theory a trace remains because of renormalisation; this is called the trace anomaly. Assessment § 7.2 writes that "since the trace vanishes for smooth connections, this result may look strange, but Θ is in general non-zero as a result of renormalisation" Assessment .

The value of the coefficient is computed, as an informal comparison with perturbation theory, in § 5 (lines 195–236) of the committee's annotated example `LeadingTrace-20261028.lean` Assessment . We follow the argument.

**(a) The free two-point function of the curvature.** With the gauge field canonically normalised, set $Q = \sum_a \sum_{\mu,\nu} F^a_{\mu\nu} F^a_{\mu\nu}$ (summing over both orders of $\mu, \nu$). The free propagator of the field strength is

$$\langle F^a_{\mu\nu}(x) F^b_{\rho\sigma}(0)\rangle = \delta^{ab}\,\frac{I_{\mu\rho}I_{\nu\sigma} - I_{\mu\sigma}I_{\nu\rho}}{\pi^2|x|^4}, \qquad I_{\mu\nu} = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{|x|^2} \quad (3)$$

(equation (5.15) of Osborn–Petkou, cited by the example). By Wick's theorem the connected part of $\langle QQ\rangle$ is the sum of the two crossed contractions, and using the squared norm of the antisymmetric tensor, $\sum_{\mu\nu\rho\sigma}(I_{\mu\rho}I_{\nu\sigma} - I_{\mu\sigma}I_{\nu\rho})^2 = 2 \cdot 4 \cdot 3 = 24$, one gets

$$\langle Q(x)Q(0)\rangle_{\text{free},c} = 2 \cdot 24 \cdot \frac{\dim \mathfrak{g}}{\pi^4|x|^8} = \frac{48\,\dim \mathfrak{g}}{\pi^4|x|^8} \quad (4)$$

Proved here (since $I$ is an orthogonal reflection matrix, $\sum_{\mu\rho} I^2_{\mu\rho} = 4$; the sum of squares is $16 + 16 - 2 \cdot 4 = 24$; we also checked this numerically).

**(b) The trace-anomaly formula.** With $\beta(g) = dg/d\log M$ ($M$ the renormalisation scale), $\sum_a T_{aa} = \frac{\beta(g)}{2g}[Q]$ (equation (4.3) of Zoller–Chetyrkin translated to this $\beta(g)$; cited by the example). At one loop $\beta(g) = -b_0 g^3$, and the running coupling is $g^2(1/|x|) \sim 1/(2b_0\ell)$ ($\ell = \log(1/(\Lambda|x|))$), so

$$\frac{\beta(g)}{2g} = -\frac{b_0 g^2}{2} \sim -\frac{1}{4\ell} \quad (5)$$

Proved here .

**(c) Combining.** Multiplying (4) by the square of (5), $1/(16\ell^2)$, gives

$$\langle \Theta(x)\Theta(0)\rangle_c \sim \frac{48}{16}\cdot\frac{\dim \mathfrak{g}}{\pi^4|x|^8\ell^2} = \frac{3\dim\mathfrak{g}}{\pi^4|x|^8\ell^2} \quad (6)$$

Proved here . The finite correction from subtracting the one-point function does not change the leading singularity, and changing $\Lambda$ shifts $\ell$ only by a constant.

To summarise how to read it:

- $|x|^{-8}$ is the naive power for the two-point function of a field of dimension 4.
- $\log^{-2}$ is the trace of the fact that the trace has the form $g^2\cdot F^2$, and $g^2$ becomes small like $1/\log$ by asymptotic freedom.
- **The numerical value of $b_0$ drops out of the coefficient.** This is because the $b_0$ in (5) cancels the $b_0$ in the running. So this law alone cannot confirm the value $b_0 = 11C_{\mathfrak{g}}/(48\pi^2)$. That value is fixed by other results (the effective coupling in 13-03, the running in 10-06) Assessment (Assessment § 7.2, lines 230–236 of the example).

On the Lean side, the coefficient is first written in the form $12\,(d_2^2/b_0^2)\dim\mathfrak{g}/\pi^4$ and simplified to $3\dim\mathfrak{g}/\pi^4$ using the identity $d_2 = -b_0/2$ (`Trace.Over.force_b`) Assessment (lines 170–177 of the example, `coefficient_is_fixed` line 261). $12\cdot\frac{1}{4} = 3$ Proved here . The example notes that this algebraic simplification is only the last step of the substantive short-distance analysis.

03

## Outline of the proof of the short-distance law

Appendix A of the assessment (the entry for `TraceTest.shortDistance`) describes the skeleton of the proof as follows Assessment . Put $c(r) = r^8\log^2(1/r)$.

1. **Approach through lattice points.** For some sequence $r_n\downarrow 0$, at distances $r_n\delta_n(s)$ ($\delta_n(s)\to s$) that approximate the physical distance $r_n s$ ($s$ a constant near 1) by nearby lattice points, show $c(r_n)\,G_\Theta(r_n\delta_n(s))\to A/s^8$ ($A = 3\dim\mathfrak{g}/\pi^4$) from the ultraviolet comparison, the logarithmic normalisation and the computation of the free two-point function.
2. **Monotonicity.** By the positive spectral representation (which comes from reflection positivity), $G_\Theta$ on the time axis is real and non-increasing.
3. **Squeeze.** Take $s_- < 1 < s_+$; eventually $G_\Theta(r_n\delta_n(s_+)) \le G_\Theta(r_n) \le G_\Theta(r_n\delta_n(s_-))$. Multiply by $c(r_n)$ and let $n\to\infty$, then $s_\pm\to 1$.

The committee's example notes that the subsequence used inside the proof does not weaken the conclusion to convergence along a subsequence Assessment (`LeadingTrace-20261028.lean` lines 186–193; `YM.Trace.Verse.outside` lifts the analysis along the controlled subsequence to the full limit over positive distances). The declarations the example lists as a reading path are `YM.Trace.wick_norm` (the Wick computation of the free two-point coefficient), `YM.Trace.Over.Scribe` (the scale comparison between the free field and the logarithmic clock), `YM.Trace.Verse.tagline` and `YM.Trace.Verse.border`. This book has not read their content.

04

## Not being free — what is excluded and what is not

**The excluded type.** Take a mass $m \geq 0$, a time order $q \geq 0$ and a continuous homogeneous (of degree $j \geq 0$) real function $h$, and consider the kernel

$$K_{m,q,h}(r) = \int_{\mathbb{R}^3} \frac{E_m(p)^q\, h(p)}{2E_m(p)}\, e^{-rE_m(p)}\, d^3p, \qquad E_m(p) = \sqrt{m^2 + |p|^2} \tag{7}$$

Source (`Mode`, `Mode.kernel`, `ChallengeSpec.lean` lines 152–161) Assessment (equation (18) in Assessment § 7.3). This is the time-direction representation of the two-point function (and its derivatives) of a free field of mass $m$. For finite complex linear combinations of finite products of such kernels,

$$W(r) = \sum_{a=1}^{N} c_a \prod_{b=1}^{n_a} K_{m_{ab},q_{ab},h_{ab}}(r) \tag{8}$$

(`wickProfile`, lines 165–166), `excludes` says "for every $W$ and $\delta > 0$, there is $r \in (0, \delta)$ with $G_\Theta(r) \neq W(r)$" Source (line 175; equivalent to the assessment's wording Assessment). (8) includes the two-point functions of polynomial observables (including derivatives) of free fields with finitely many masses.

**Why they are excluded** Assessment (the entry for `excludes` in Appendix A). The construction shows that the real part of any $W$ can be expanded near 0 as

$$\operatorname{Re} W(r) = \sum_{\nu} a_\nu\, r^{\alpha_\nu} (\log r)^{k_\nu} + O(1), \qquad \alpha_\nu \in \mathbb{Z},\ k_\nu \in \mathbb{Z}_{\geq 0} \tag{9}$$

and shows that the only possible finite limit of $r^8 \log^2(1/r) \operatorname{Re} W(r)$ is 0. If $W = G_\Theta$, this contradicts the positive limit in (2). The intuition is that no term of (9) behaves like $r^{-8}(\log r)^{-2}$ Inferred : the powers of the logarithm are non-negative, so each term multiplied by $r^8 \log^2$ either goes to 0 or diverges, and if the limit of the sum is finite it is 0. Terms of the form $\log^{-2}$ do not appear in (9).

**What is not excluded** Assessment (Assessment § 7.3, `LeadingTrace-20261028.lean` lines 337–341):

- All descriptions built from Gaussian fields (the assessment states explicitly that "it does not exclude all of them").
- All generalised free fields.
- Determination of the intrinsic mass scale $\Lambda$. Agreement with perturbation theory to all orders. The structure of the vacuum.

The positive leading coefficient excludes that the trace correlations are identically 0. The assessment derives in particular that the two-point function is non-zero for sufficiently small $r$, and that some smeared $\Phi\Theta(f)$ is non-zero Assessment .

The stronger form on the side of the construction's theorem, `Headline.TraceExclusions` (`YM/Nonperturbative/HeadlineSpec.lean` lines 84–120), additionally has the fields `identified` ( § 05), `wick`, `maxwell` and `invariant` Source (as far as field names and types). We have not read the meaning of the fields from `wick` on.

05

## The Θ of the published statement is one element of F_4

This is the point in this chapter that needs the most care.

**Reading only the published statement**, `centeredTrace` is one of the fields of the hierarchy $T$ given by the existential quantifier $\exists T$ of the published theorem, and all the statement says about it is that its type is $F_4$ (`Field 4`) Source (`ChallengeSpec.lean` lines 48 and 186–188). What the published property `nonfree` says is that "there is a field Θ of degree 4, centred in the chosen phase, whose two-point function satisfies (2) and `excludes`". That it is the trace of the stress tensor is not written in the published statement.

**Being the trace of the stress** is stated in two places.

- **The construction's side.** In the definition that builds the published hierarchy, `centeredTrace := TraceExtension.center F` (`HierarchyBridge.lean` line 42) Source . The field `identified` (`HeadlineSpec.lean` lines 87–91) of the conclusion `Headline.TraceExclusions` of the construction's theorem says that the operator of the trace field equals "the trace field of the stress tensor `F.sector.energy` minus its one-point function in the chosen phase times the identity" Source . This is equation (14) of the assessment, $\Theta = \sum_\mu T_{\mu\mu} - S^{\xi^*}\left(\sum_\mu T_{\mu\mu}\right)\mathbf{1}$ Assessment .

- **The committee's joint goal.** The short-distance check ([A4] of the assessment) confirms that it uses the trace of the same stress tensor whose normalisation was checked by the Ward identity, in the same chosen phase Assessment (Assessment § 9.4, § 11; in the example `LeadingTrace-20261028.lean`, `centering_uses_the_phase_mean` line 61 and `source_and_OS_fields_agree` line 97).

Assessment § 2 writes that "the construction defines the trace of the stress tensor, `centeredTrace`, and proves its properties in `nonfree`". That is a reading that includes the construction's side; this book keeps the distinction that **"the trace of the stress" does not follow from the wording of the published statement alone**. In short, what the published theorem says is that "(2) holds for some field of degree 4", and it is the construction's theorem and the committee's joint goal that identify that field with the trace of the stress tensor.

Conditions also enter the choice of the chosen phase $\xi^*$. Appendix A of the assessment writes that the set of phases whose ground-state space is one-dimensional and that satisfy a certain limiting property of the trace field is shown to have measure 1 for the measure on phases (the probability measure (4) in 8-05), and that $\xi^*$ is chosen from it Assessment (the entry for `Hierarchy.selected`).

06

## Which goal the statements belong to

| Goal | What is there |
|---|---|
| The published theorem `JointExistence` | The four fields of `TraceTest`. About $\Theta$ it says only that it is an element of $F_4$ Source |
| The construction's theorem `constructed` | `Headline.TraceExclusions` (including `identified`, `shortDistance`, `positive`, `wick`, `maxwell`, `invariant`) Source (`Construction.lean` line 28) |
| The committee's joint goal | The short-distance check [A4] (the all-directions form (17)), the independent check of the OS axioms for the hierarchy of the trace field [A1], the non-zero lattice limit [A5] ((5) in 8-05) Assessment |

The committee's independent check of the OS axioms is carried out for **the hierarchy of $\Theta$ alone** (Assessment § 10, § 10.1), not for all fields Assessment .

07

## What this book's investigation had seen

This book's research and Lean have **none** of the trace anomaly, the short-distance law or the exclusion of free fields.

- This book's viewpoint on non-triviality was on the infrared side (the sign and floor of the vortex ratio $\theta_M$, pairs of Polyakov loops, non-degeneracy of face averages) (Part 21). The non-triviality of the published construction is on the ultraviolet side (the logarithm of asymptotic freedom). The two lie on complementary sides Inferred .
- The $\theta$ in this book's vortex ratio $\theta_M$ is a symbol unrelated to the trace field $\Theta$.
- The idea of "stating non-freeness through the logarithm of the trace anomaly" was not in this book (14-05).

08

## What we have not read

- The main body of the proof of (2) (`wick_norm`, `Scribe`, `tagline`, `border`, `outside`).
- The proof of the expansion (9) (`YM.TraceExtension.Wick.real_profile` and others).
- The meaning of the fields `wick`, `maxwell` and `invariant` of `TraceExclusions` (`HeadlineSpec.lean` lines 96–120).
- The proof that the set of chosen phases has measure 1 (`YM.OS.Cluster.lifts` and others).
- The propagator (3) and the trace-anomaly formula in (b) are formulas from the literature cited by the committee's example; this book has not checked the original sources (bibliographic entries only in the table of sources).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The published `nonfree` states, for an element $\Theta$ of $F_4$ centred in the chosen phase, the full limit (2) and the exclusion of the free-field type | "$\Theta$ is the trace of the stress" from the published statement alone |
| Source Being the trace of the stress is in the construction's theorem (`identified`); Assessment that it is the same stress and the same phase is in the committee's joint goal | That the independent OS check exists for all fields (only for the hierarchy of $\Theta$) |
| Proved here The count of the coefficient $48/16 = 3$ (free two-point function $\times$ the square of the one-loop coefficient of the trace anomaly). $b_0$ drops out | Confirmation of the value of $b_0$ by this law |

| Can say | Cannot say |
|---|---|
| `Assessment` The type of two-point functions of polynomials in free fields with finitely many masses is excluded | Exclusion of all descriptions built from Gaussian fields or of all generalised free fields; the intrinsic scale; agreement with perturbation theory to all orders |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Trace test, kernels, `wickProfile` | Source | `ChallengeSpec.lean` lines 48 and 145–175 of `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e) |
| Identification with the trace of the stress | Source | `HierarchyBridge.lean` line 42, `YM/Nonperturbative/HeadlineSpec.lean` lines 84–120, `Construction.lean` line 28 |
| Theorem 7.1, equations (14)–(18), exclusion of free fields, outline of the proof, the chosen phase | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 2, § 7.2, § 7.3, § 9.4, § 10, § 11, Appendix A |
| Computation of the coefficient and the reading path | Assessment | `submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` in `yang-mills-group/YM` on the assessment side, revision 55ba9fc, lines 61, 97, 170–236, 261, 337–341 |
| Free propagator of the field strength (equation (5.15)) | Cited | H. Osborn, A. C. Petkou, Ann. Phys. 231 (1994) 311, arXiv:hep-th/9307010 — bibliographic entry only (not checked; cited by the committee's example) |
| Trace-anomaly formula (equation (4.3)) | Cited | M. F. Zoller, K. G. Chetyrkin, JHEP 12 (2012) 119, arXiv:1209.1516 — bibliographic entry only (not checked; cited by the committee's example) |
| Computation improved by the position-space renormalisation group | Cited | M. Bochicchio, S. P. Muscinelli, JHEP 08 (2013) 064, arXiv:1304.6409 — bibliographic entry only (not checked; cited by the committee's example) |
| One-loop coefficient | Cited | D. J. Gross, F. Wilczek, Phys. Rev. Lett. 30 (1973) 1343; H. D. Politzer, Phys. Rev. Lett. 30 (1973) 1346 |
| The counts 24 and 48/16 | Proved here | Numerical record of this book's investigation (`numcheck.py`; counted the sum of squares in (4) with random reflection matrices) |

**Terms used in this chapter** — trace anomaly, centred trace, short-distance law, chosen phase, exclusion of free fields, Wick product. Their meanings are in the A-02 glossary.

Next chapter: 13-03 Free Maxwell fields at short distance — how the theory approaches free Maxwell fields at short distances.

# Free Maxwell fields at short distance

This chapter is established knowledge (a reading of the published construction). § 07 is this book's own investigation, and § 08 is a map of open questions.

**What this chapter shows** — That asymptotic freedom appears in two forms. One is that the coupling $g_W(r)$ read from the effective action behaves as $\log(1/r)\, g_W(r)^2 \to 1/(2b_0)$ (Theorem 7.2 of the assessment). The other is that, rescaling the correlations along a certain common sequence $r_k \downarrow 0$, they approach the correlations of $\dim \mathfrak{g}$ independent free Maxwell fields, with derivatives of every order.

**Prerequisite chapters** — 8-02 (steps and spacings), 13-01 (the stress tensor). The definition of "the coupling at step m" is in 10-01, and the running of the bare coupling in 10-06. The introduction to asymptotic freedom is 4-06.

**Said first — The assessment does not give a Lean name for Theorem 7.2 (the effective coupling).** Statements with similar content exist in three places ( § 03): `Running.ratio`, outside the dependencies; `wilson_link`, inside the dependencies and used as an intermediate step in the trace analysis (its wording is close to the assessment's statement); and the headline theorem `YM.Pert.Asymptotics.comparison` (the running of the coupling of the cut-off steps, a different quantity from $g_W$). A statement in the form of $g_W$ does not survive into the conclusions of the published theorem or of the construction's theorem. It is not in the list of checks of the committee's joint goal either. The comparison with Maxwell fields is in the conclusion of the construction's theorem. Neither claims agreement with perturbation theory to all orders.

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Statement — the three results of Assessment § 7.4

2. Definition of the effective coupling, and how the running is derived
3. Where the effective coupling is in Lean, and the goals: outside and inside the dependencies
4. The second coupling — measured by continuum correlations
5. Comparison with free Maxwell fields: along a common sequence, with derivatives of every order
6. How the three short-distance results differ
7. What this book's investigation had seen
8. What we have not read

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01

## Statement — the three results of Assessment §7.4

Assessment § 7.4, "Asymptotic freedom and agreement with perturbation theory", lists three results Assessment .

1. **Effective coupling (Theorem 7.2).** Integrating out fluctuations finer than a length $r$ changes the coefficient of the curvature-squared term in the effective action. Its inverse square root, in a fixed normalisation, is the running coupling. The submission starts from the family of Wilson theories with the bare coupling of Theorem 4.1, blocks down to the first lattice spacing below $r$, sets the initial cutoff to 0, and after taking the limit along a subsequence of volumes defines $g_W(r)$ as the inverse square root of the normalised coefficient of the action. For small $r$, and for the limit along any chosen subsequence of volumes, $g_W(r) > 0$ and

$$\log(1/r)\, g_W(r)^2 \longrightarrow \frac{1}{2b_0} \qquad (r \downarrow 0), \qquad b_0 = \frac{11 C_{\mathfrak{g}}}{48\pi^2} = \frac{s}{2} \quad (1)$$

   (In this chapter $b_0$ is the one-loop coefficient, and $2b_0 = s$ is the one-loop slope. It is not the block side $b_i$ at $i = 0$.) It has not been shown that $g_W(r)$ does not depend on the choice of subsequence of volumes.
2. **The second coupling.** A coupling is defined from continuum correlations at separated points, and its ratio to $g_W$ tends to 1 as $r \downarrow 0$. The running law is visible not only in the effective action but also in continuum observables.
3. **Comparison with free Maxwell fields.** Along one common sequence $r_k \downarrow 0$, the averaged $n$-point correlations of the stress tensor, multiplied by $r_k^{4n}$ and evaluated at $r_k x_1, \dots, r_k x_n$, approach the correlations of $\dim \mathfrak{g}$ independent free Maxwell fields, with derivatives up to any fixed order, uniformly on compact sets avoiding

coinciding points. Here a Maxwell field is the Gaussian 2-form curvature obtained as the exterior derivative of a Gaussian free $\mathfrak{g}$-valued 1-form.

The assessment distinguishes that the comparison with Maxwell fields is for the mixed theory along a particular sequence, that Theorems 7.1 (the trace law, 13-02) and 7.2 hold for any way of letting $r \to 0$, and that 7.1 concerns the chosen phase; and writes that together they form "the main short-distance evidence" Assessment .

02

## Definition of the effective coupling, and how the running is derived

Here we read the definition in the form that is inside the dependencies (`Pilot.wilson_link` in § 03) Source (`YM/TraceExtension/Atlas/Sink.lean`, `YM/TraceExtension/Terrace/Dusk.lean` in `yang-mills-group/OpenAI-YM-20261028`).

**The limiting coupling at step $i$.** $\beta(N,p,i)$ is the coupling at step $i$ of the trajectory with cutoff $N$ and volume $p$ (the coefficient of $F^2$ in the representation of the effective action; 10-01). Take one ultrafilter $\mathcal{L}$ on $(N,p)$ (matched to the way the limit along a subsequence of volumes is taken), and take its limit

$$Z_\beta(i) = \lim_{\mathcal{L}} \beta(N,p,i) \quad (2)$$

at each step (`Arch`, `steep`, `Dusk.lean` lines 65–72). At each step $Z_\beta(i) \in [\hat\beta_i/2,\, 2\hat\beta_i]$ (`holds`, line 79), where $\hat\beta_i = \bar\beta + 2b_0 \log(1/a_i)$ is the one-loop running (8-02), and moreover $Z_\beta(i)/\hat\beta_i \to 1$ (`gamut`, line 89).

**The step just below $r$.** Let $i(r)$ be the first step with $a_i < r$ (`glen`, `Sink.lean` line 14). The effective coupling is

$$g_W(r) = Z_\beta\big(i(r)\big)^{-1/2} \quad (3)$$

(`wilson`, line 44).

**Running.** Multiply two limits together Source (statements) Proved here (combination).

- `pace_log` (line 21): $\log(1/r)/\hat\beta_{i(r)} \to 1/(2b_0)$. The spacing satisfies $a_{i(r)} < r \le a_{i(r)-1}$, and the proof uses the field `bloom` of the additional calendar condition `PostT` ($b_i \le \log\hat\beta_i$, `YM/TraceExtension/Terrace/Base.lean` lines 29–34) and the monotonicity of $\hat\beta$ to bound the difference between $\log(1/r)$ and $\log(1/a_{i(r)})$ at the order of $\log\hat\beta$ Source (`H.bloom` and `monoBeta` in `Sink.lean` lines 21–41).
- `gamut`: $Z_\beta(i)/\hat\beta_i \to 1$.

Dividing, $\log(1/r)\, g_W(r)^2 = \log(1/r)/Z_\beta(i(r)) \to 1/(2b_0)$ (`wilson_link`, line 50; the conclusion type `Coupling`, lines 45–47, has two parts: "eventually positive" and "$\log(1/r)\, u(r)^2 \to 1/(\text{slope})$"). Moreover $g_W(r) \to 0$ (`Coupling.conforms`, line 55). The hypotheses of `wilson_link` are that $G$ is simple, the calendar condition `PostT`, the ultrafilter limit `steep`, and that the ultrafilter follows the order of extraction ($\mathcal{L} \leq X.\text{fuse}$) Source . `PostT` is a condition imposed on the side of the trace analysis.

Two cautions on how to read this.

- $Z_\beta(i)$ depends on the ultrafilter $\mathcal{L}$. This is what the assessment means by "it has not been shown not to depend on the choice of subsequence".
- $\beta(N, p, i)$ is a **coefficient in a representation** of the effective action, not an invariant that can be read from the measure alone (10-01). So $g_W$ too is a quantity defined through the choice of representation in the construction.

03

## Where the effective coupling is in Lean, and the goals

The assessment does not give a Lean name for Theorem 7.2. Statements with similar content exist in three places Source . The one whose wording is close to the assessment's statement ($r \downarrow 0$, the first spacing below $r$, eventually positive) is `wilson_link` Inferred .

| Declaration | Content | Verification record (`verification/declaration-us` |
|---|---|---|
| `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean` line 145) | The ratio form $g(\mu) \cdot (2b_0 \log \mu)^{1/2} \to 1$ ($\mu \to \infty$). $g$ is `gLedger` (line 130), the inverse square root of the value of the step limit $B$ at the first step with $\log \mu \leq \log(1/a_m)$ ($a_m \leq 1/\mu$). Positivity holds for every $\mu$ (`ledger_positive`, line 139) | **outside** (outside the dependencies). The coupli (`isLimit` 59, `discrete` 66, `choose` 86, `band` 1 `gLedger` 130, `ledger_positive` 139, `ratio`, `va` 165 and so on) are all outside. What is used in t module is the extraction `extraction` (line 40) a `clock` (line 20), and the private helpers `cut` (l and `compact_diag` (line 27) |

| Declaration | Content | Verification record (`verification/declaration-us` |
|---|---|---|
| `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean` line 50) | $\log(1/r)\, gW(r)^2 \to 1/(2b_0)$ $(r \downarrow 0)$, eventually positive. $gW$ is (3). `PostT` ($b_i \le \log \hat{\beta}_i$ and so on) among the hypotheses | used. The first headline found is `continuum_with_position_tests`, and the direct `crest_ready`. An intermediate step in the trace |
| `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean` line 36) | For $N_i \ge i$ (eventually) and any sequence of volumes $k_i$, the coupling of the cut-off step $i$, $\beta(N_i, k_i, i)^{-1/2}$ (`regulatorRun`, line 35), multiplied by $\sqrt{\log(1/a_i)}$, tends to $(2b_0)^{-1/2}$. This is the running on the scale of steps before any limit is taken, a different quantity from $gW(r)$ | **headline** (a headline theorem itself) |

The docstring at the top of `Running.lean` says that the coupling in this module is "an auxiliary observation read from the represented action", and that the main equivalence "does not identify a position-space coupling" and "does not estimate the diagrammatic remainder" (lines 6–10).

The conclusion of `wilson_link` appears in the field `charts` of the intermediate structure `Crest` of the short-distance trace analysis (`YM/TraceExtension/Belfry.lean` lines 20–22), and in the field `wilson` of the comparison structure `FieldComparison` (`YM/TraceExtension/Endpoint.lean` line 50) Source . One can read this as the proof of the trace law (13-02) using the running of the coupling as an intermediate step Inferred . However, when passing from these structures to the published trace test (`Integration.ManuscriptTraceTests.of_result`), the comparison fields are dropped Source (`YM/Integration/ManuscriptTraceTests.lean` lines 62–89).

### Summary with respect to the goals.

- The published theorem `JointExistence`: the effective coupling is not in it. What is there is only the running of the bare coupling, `WilsonRegularity.running` ((1) in 8-02) Source .

- The construction's theorem `constructed`: there is no statement in the form of $g_W$ among the fields of its conclusion Source (`Construction.lean` lines 15–31). The conclusion's `Independent.Core` has the trace tests `ManuscriptTraceTests.Tests` in its field `trace` (`YM/Independent/Core.lean` lines 18–22), and `Trace.Target.Result`, which contains `FieldComparison` (`YM/TraceExtension/Endpoint.lean` lines 99–120), is dropped by `of_result`. The conclusion of `continuum_with_position_tests` has the same `Core` (`YM/Nonperturbative/Continuum.lean` lines 28–41) Source .
- The committee's joint goal: there is no coupling check in the list of checks in Assessment § 9.3 Assessment . The committee's example writes that the trace check and its example do not add an independent check of Theorem 7.2 Assessment (`LeadingTrace-20261028.lean` lines 233–236).
- Headline theorems: the running of the step coupling, `YM.Pert.Asymptotics.comparison`, is a headline theorem itself. In the form of $g_W$, `Running.ratio` is outside the dependencies and `wilson_link` is an intermediate step inside them Source .

04

## The second coupling — measured by continuum correlations

According to the assessment, the second coupling is defined from continuum correlations at separated points Assessment : the derivative of the scalar two-point function and the integrated tensor–scalar–scalar correlation are normalised to their free values, and the coupling is measured by a derivative of the tensor–scalar correlation. The submission shows that the ratio of this coupling to $g_W$ tends to 1 as $r \downarrow 0$.

On the Lean side this coupling is `Sundial.glo` (`YM/TraceExtension/Sundial/Noon.lean` line 28, used), and that the ratio tends to 1 is the field `calibrationX` of `FieldComparison` (`Endpoint.lean` line 49) Source . `FieldComparison` also has among its fields that the one-loop part of the coefficient is $-b_0/2$ (`coefficient_O`, `traced_O`: of the form $d(r) - d_2\, g^2 = O(g^4)$). We have not read the content.

05

## Comparison with free Maxwell fields

The conclusion of the construction's theorem contains `Pert.FirstExtension F X` Source (`Construction.lean` line 31). $X$ is a sequence of scales on the correlation table $S$ (`Climb.Scaling`), with the common sequence $r_k = X.r\, k \downarrow 0$. The fields of the structure `FirstExtension` (`YM/Pert/FirstTarget.lean` lines 26–39) are as follows.

- `jets` (lines 29–34): for every number of points $n$, derivative order $j$, list of labels of the free table, marks of which positions are stresses together with their indices, compact set $K$ avoiding coinciding points, and $\varepsilon > 0$, for all sufficiently large $k$ and at every point of $K$,

$$\left\| D^j \left[ z \mapsto \widetilde{S}_k(r_k z) \right](x) - D^j S_{\mathrm{free}}(x) \right\| \leq \varepsilon \quad (4)$$

  Here $\widetilde{S}_k$ is the rescaled correlation of the construction (`scaledPhysical`), and $S_{\mathrm{free}}$ is the correlation of the free table, with the stress of the free Maxwell field `freeStress` $= -$ `Free.original` (line 16) placed at the stress positions.
- `ranks` (lines 35–36): the line of candidates for the stress is one-dimensional (13-01).
- `derivatives`, `rotations` (lines 37–39): the derivatives and the rotation action on fields approach those of the free field along the sequence of scales.

(4) is the form, with derivatives of any order, of "at short distances the construction's theory looks like $\dim \mathfrak{g}$ independent free Maxwell fields". The docstring says that this is a first milestone concerning only the correlations of the mixed theory (the hierarchy), and that it "does not yet include estimates on the amplification of fixed labels or on the matched coupling" (lines 4–5, 21–25) Source . Fields at non-stress positions are measured in the frame `X.U k` for each element of the sequence of scales. The assessment's "including comparison with fields other than the stress tensor" can be read as this part Inferred .

We infer that the scale factor $r_k^{4n}$ (in the assessment) enters Lean inside `scaledPhysical` (`YM/Pert/Lifted.lean` line 81), but we have not checked.

The verification record for the declaration `FirstExtension.jets` is used (first headline `parityEndpoint`), and the projection `ranks` is outside, but both are fields of the structure in the conclusion of the construction's theorem, so as statements they are in the construction's theorem Source .

06

## How the three short-distance results differ

| | Trace law (13-02) | Effective coupling (1) | Comparison with Maxwell fields (4) |
|---|---|---|---|
| How $r \to 0$ is taken | Full limit | Full limit | One common sequence $r_k$ |
| State | Chosen phase $\xi^*$ | Limit along a subsequence of volumes (ultrafilter) | Mixed theory |

| | Trace law (13-02) | Effective coupling (1) | Comparison with Maxwell fields (4) |
|---|---|---|---|
| What is measured | Two-point function of the centred trace | Coefficient in the representation of the effective action | Correlations (stress and other fields), derivatives of every order |
| Value of $b_0$ | Drops out of the coefficient | Appears in $1/(2b_0)$ | Does not appear (comparison with free fields) |
| Goal | Published (with $\Theta$ only as one element of $F_4$), construction, committee | The $gW$ form is in no conclusion (the headline theorem is the running of the step coupling, `comparison`) | Conclusion of the construction's theorem |

None of the three **claims agreement with perturbation theory to all orders** Assessment (Assessment § 7.5: "agreement with perturbation theory to all orders is unproved"). The two-loop coefficient and the $\log\log$ correction are not treated either (10-06).

07

## What this book's investigation had seen

This book has **neither** a theorem on the effective coupling nor a comparison with free Maxwell fields.

- This book measured numerically how the coupling moves under one block step, and recorded that its direction agrees with asymptotic freedom and its size is 2.9–4.4 times the one-loop value in units of $\beta$, $\frac{11}{3\pi^2}\ln 2 \approx 0.2575$ (15-01, label Reading). The published construction's slope $11C_{\mathfrak{g}}/(24\pi^2)$ multiplied by $\ln 2$ is this one-loop value (8-01 § 04).
- This book's "matching" took the form of matching observables definable from the measure alone (expectations on coarse faces, the vortex ratio) to fixed values. The published construction's coupling is a coefficient in a representation, and (3) reads it as an effective coupling. The comparison of which of the two is meaningful in the continuum limit is in 14-03.

08

## What we have not read

- The whole chain from `wilson_link` to the existence theorem (beyond `crest_ready`). The meaning of the other fields of `PostT` (`cropK`, `share`, `purse`, `jig`).
- The definition of the second coupling `glo` (`lampX`) and the proof of `calibrationX`.

- `scaledPhysical` and the scale factor, how the free table `freeTable` is built, and the meaning of the frame `X.U k`.
- Whether the limit $B$ in `Running.lean` (the ordered extraction `f.fuse` of `isLimit`) includes the limit in volume.

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source Defining $g_W(r)$ by (2)(3), $\log(1/r)\, g_W(r)^2 \to 1/(2b_0)$ (`wilson_link`) | That $g_W$ does not depend on the subsequence of volumes, or is determined by the measure alone |
| Source `Running.ratio` is outside the dependencies, and `wilson_link` is an intermediate step inside them. A statement in the form of $g_W$ is not in the published or construction conclusions (the headline `comparison` is the running of the coupling of the cut-off steps, a different quantity) | The whole chain from `wilson_link` to the existence theorem |
| Source The conclusion of the construction's theorem contains the comparison with free Maxwell fields along a common sequence (derivatives of every order, uniformly on compact sets) | A comparison for every $r \to 0$; a comparison in the chosen phase |
| Assessment The ratio of the second coupling to $g_W$ tends to 1 | Agreement with perturbation theory to all orders; the two-loop coefficient |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Effective coupling (inside the dependencies) | Source | `YM/TraceExtension/Atlas/Sink.lean` lines 14, 21, 44–55, `YM/TraceExtension/Terrace/Dusk.lean` lines 65–89, `YM/TraceExtension/Belfry.lean` lines 16–22 of `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e) |
| Effective coupling (outside the dependencies) | Source | `YM/PertExtension/Running.lean` lines 6–10, 40, 59, 66, 130, 145, 165; `verification/declaration-usage.tsv` |
| Second coupling, comparison structure | Source | `YM/TraceExtension/Sundial/Noon.lean` line 28, `YM/TraceExtension/Endpoint.lean` lines 36–64, `YM/Integration/ManuscriptTraceTests.lean` lines 62–89 |
| Comparison with Maxwell fields | Source | `YM/Pert/FirstTarget.lean` lines 4–39, `Construction.lean` lines 15–31 |

| Item | Kind | Source / tool |
| --- | --- | --- |
| Theorem 7.2, the second coupling, the Maxwell comparison, agreement with perturbation theory | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 7.4, § 7.5; on the assessment side, `submissions/openai-20261028/docs/examples/LeadingTrace-20261028.lean` in `yang-mills-group/YM`, revision 55ba9fc, lines 230–236 |

**Terms used in this chapter** — effective coupling, running coupling, ultrafilter limit, free Maxwell field, sequence of scales, derivatives of every order. Their meanings are in the A-02 glossary.

Next chapter: 13-04 The operator product expansion and contact terms — the expansion of correlations as points approach each other, and the extension to coinciding points.

# The operator product expansion and contact terms

This chapter is established knowledge (a reading of the published construction). § 07 is this book's own investigation, and § 08 is a map of open questions.

**What this chapter shows** — That an expansion approximating correlations of products of fields gathered near a point $o$ by linear combinations of correlations of single fields placed at $o$ — the operator product expansion — is in the published statement, in the form of power bounds on the coefficients and on the remainder. The remainder bound is uniform over the mixed theory and all phases, and the coefficients do not depend on the spectators, the state or the truncation degree. For the distribution extended to coinciding points, only that it "exists" is stated.

**Prerequisite chapters** — 8-03 (field spaces and degrees), 8-05 (correlations and phases), 13-02 (the short-distance logarithm).

**Said first** — $D$ is a degree of fields, not a loop order. Convergence as $D \to \infty$, agreement of the coefficients with perturbative values, and independence of the coefficients from the choice of subsequence of volumes are not stated. Nor does equality of phases follow from the common coefficients.

Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00).

The order of this chapter

1. Statement — conditions on the configuration and the two bounds
2. How to read the powers: dimension counting and the loss ε
3. How the coefficients and channels are built
4. Uniformity over states, and what the common coefficients do not give
5. Extension of contact terms — existence only
6. Which goal the statements belong to, and what is not said
7. What this book's investigation had seen

8. What we have not read

01

## Statement — conditions on the configuration and the two bounds

**Configuration.** Take a centre of expansion $o \in \mathbb{R}^4$, a scale $r > 0$, a constant $M \geq 0$, and $\eta, c > 0$. Place input fields $A_1, \ldots, A_n$ (degrees $d_i$, total degree $w = \sum d_i$) at points $x_i$ and spectator fields $B_1, \ldots, B_\ell$ at points $y_\ell$, and impose the following Source (`ChallengeSpec.lean` in `yang-mills-group/OpenAI-YM-20261028`: `Within` lines 65–66, `Clear` lines 67–68) Assessment (`Properties.ope` in Appendix A).

$$\|x_i - o\| \leq Mr, \qquad \|x_i - x_j\| \geq \eta r \ (i \neq j), \qquad \|y_\ell - o\| \geq c, \qquad \|y_\ell - y_m\| \geq c \ (\ell \neq m$$

The input fields are gathered around $o$ at size $r$ without their shape collapsing (points are at least $\eta r$ apart), and the spectators are at least $c$ away from $o$ and from each other.

**Output fields.** For each degree $j$ there is a finite set of channels, and for each channel $\alpha = (j, i)$ an output field $O_\alpha \in F_j$ (`actor`) and a coefficient $c_\alpha(A; x, o) \in \mathbb{C}$ (`coefficient`), as fields of the hierarchy $T$ (lines 43–47).

**The two bounds.** The structure `OperatorProductExpansion` (lines 70–80) says the following Source . For every $\varepsilon > 0$ and all sufficiently small $r > 0$,

$$|c_\alpha(A; x, o)| \leq r^{\,d_\alpha - w - \varepsilon} \qquad \text{(configuration: the first two conditions of (1))} \ (2)$$

$$\Big| S^s\Big(\prod_i A_i(x_i)\, B\Big) - \sum_{d_\alpha \leq D} c_\alpha(A; x, o)\, S^s\big(O_\alpha(o)\, B\big) \Big| \leq r^{\,D+1-w-\varepsilon} \qquad \text{(configuration: all}$$

(3) holds for every truncation degree $D$, where $s$ is the mixed theory $S^{\mathrm{mix}}$ or any phase $S^\xi$, and $B = \prod_\ell B_\ell(y_\ell)$. The assessment writes it in the form (20) with $o = 0$ and $x_i = r\,\tilde{x}_i$ Assessment (§ 7.5).

The field `ope` of the published statement is this structure (`Properties`, line 180). The construction's theorem additionally has in its conclusion a nested OPE (`Headline.NestedOPE`, `YM/Nonperturbative/HeadlineSpec.lean` lines 128–138; estimates on coefficients and remainders for clusters split in the shape of a tree) Source (`Construction.lean` line 28). The assessment writes that "under additional separation and scale conditions, nested expansions hold" Assessment .

02

## How to read the powers

The powers in (2)(3) have the same form as engineering dimension counting Inferred . If a field of degree $d$ scaled exactly as length to the power $-d$, then $\prod A_i(x_i)$ would scale as length to the $-w$, $O_\alpha(o)$ as length to the $-d_\alpha$, and the coefficient would have size $r^{d_\alpha - w}$. The remainder after summing up to degree $D$ has the size of the term of the next degree $D+1$, $r^{D+1-w}$.

The arbitrarily small loss $\varepsilon > 0$ is there to absorb logarithm-like behaviour Assessment (the committee's example `CommonOPE-20261028.lean` lines 48–52). In an asymptotically free theory, the actual scaling of fields deviates from the engineering dimension by powers of $\log$ (the $\log^{-2}$ in 13-02 is an example). A power bound does not distinguish powers of $\log$, so a loss of $\varepsilon$ is enough. The committee's example notes that the prefactor 1 is obtained by shrinking the radius, and is not an explicit value of the radius nor a computation of perturbative coefficients.

For channels whose degree $j$ is smaller than the total input degree $w$ (for example the identity field, $j = 0$), the power in (2) is negative, so the coefficient is allowed to grow as $r \to 0$. This is how the expansion expresses the fact that correlations become singular as two fields approach each other.

03

## How the coefficients and channels are built

Appendix A of the assessment describes the construction as follows Assessment .

- **Channels** (`Channel`): split $F_d = F_{d-1} \oplus E_d$ and number a basis of $E_d$ (each element invariant under conjugation). The number of channels of degree $d$ is $\dim E_d$. The output field `actor` is an element of that basis.
- **Coefficients** (`coefficient`): from finitely many correlations of the input fields with additional test insertions, build a field in $F_d$, average it over the four-dimensional orthogonal group, symmetrise it with respect to conjugation, remove the lower-degree components, and take its coordinates in the basis $O_{(d,i)}$.

The coefficients are built only from the input fields and positions, and do not take the spectators, the state or the truncation degree as arguments. This can be read off the type Source (the type of `coefficient`, `ChallengeSpec.lean` lines 46–47: the arguments are the list of input labels, the output degree and channel, the input positions and the centre).

The committee's example shows, as checks of the dictionary, that adding the channel coordinates back recovers the coefficient field (`channel_assembly`, line 128), and that

the expansion of a single insertion returns that field itself (`single_insertion_recovered`, line 133) Assessment . The example says that this is not a computation of the values of an interacting multi-point OPE.

04

## Uniformity over states, and what the common coefficients do not give

**Uniformity.** In the Lean form of (3), the universal quantifier over states $s$ sits **inside** the quantifier "for sufficiently small $r$" ($\forall^{f}\ r$) Source (`ChallengeSpec.lean` line 76). That is, the radius is fixed before the state is asked for, and the remainder bound is uniform over the mixed theory and all phases. The committee's example makes this order explicit in terms of a radius $\rho$ Assessment (`remainder_uniform_in_state`, line 79: "$\rho$ is chosen before $s$ and all admissible positions. It may depend on the labels, $D$ and the geometric constants, but not on the phase").

**Common coefficients do not give agreement of phases.** Subtracting the expansions for two states $s, t$, since the coefficients are the same,

$$\left|(S^{s}-S^{t})\Big(\prod A_i\, B\Big)-\sum_{d_\alpha\le D} c_\alpha\,(S^{s}-S^{t})\big(O_\alpha(o)\,B\big)\right|\le 2\,r^{\,D+1-w-\varepsilon} \tag{4}$$

Proved here (triangle inequality) Assessment (`difference_of_states`, line 99). The difference is expanded in differences of correlations of the output fields, and there is no reason for those to be 0. The assessment writes that the case without spectators makes this clear: even with common coefficients, the one-point functions $S^{s}(O_\alpha(o))$ may differ from phase to phase Assessment (§ 7.5).

05

## Extension of contact terms — existence only

The correlations of 8-05 are functions defined only where the points are pairwise distinct. Using the OPE, they can be extended to distributions on test functions that include coinciding points. The definition `Hierarchy.Contacts`, placed in the public namespace (it is not in the conclusion of the published theorem), has the following form Source (`ContactSpec.lean` lines 12–15).

$$\forall s,\ n,\ v\quad \exists\, D:\ C_c^{\infty}\big((\mathbb{R}^4)^n\big)\to\mathbb{C}\ \text{continuous linear},\quad \operatorname{supp} f\subset\{x_i\ /=x_j\}\ \Rightarrow\ D(f)=\int$$

That is, it says only that there **exists** a distribution that agrees with the original correlations where the points are separated. The construction provides it with `contactDistribution` (`ContactBridge.lean` line 14), and the conclusion of the

construction's theorem contains `(hierarchy F Q).Contacts` Source (`Construction.lean` line 22).

The extension cannot be unique: adding distributions supported on the diagonals (delta functions and their derivatives) does not change the values where the points are separated (4-04). The assessment writes that uniqueness of this extension, the full OS axioms (positivity) on the unrestricted space of test functions, and convergence from the lattice have not been shown Assessment (§ 7.5, § 10). The committee's independent OS check uses test functions all of whose derivatives vanish on the diagonals, so contact terms are invisible to it (8-05 § 04).

06

## Which goal the statements belong to, and what is not said

| Goal | What is there |
|---|---|
| The published theorem `JointExistence` | The two OPE bounds (2)(3) (field `ope`) Source . The extension of contact terms is **not included** |
| The construction's theorem `constructed` | In addition to (2)(3), the OPE at separated points `SeparatedOPE`, the nested OPE `NestedOPE`, and the extension of contact terms `Contacts` Source (`Construction.lean` lines 22, 28) |
| The committee's goals | The consistency check of the OPE statement [A8] is a **separate endpoint** from the joint goal, and passed in a separate Comparator run Assessment (Assessment § 9.3, § 11). The statement `CommonOPE` states the two bounds directly, without using the source's predicates |

**What is not said** Assessment (Assessment § 7.5, `CommonOPE-20261028.lean` lines 159–165):

- Convergence of the expansion as $D \to \infty$.
- Uniqueness of the coefficients when the normalisation of fields or the OPE prescription is changed.
- The numerical values of the perturbative Yang–Mills coefficients (agreement with perturbation theory to all orders is unproved).
- That the coefficients do not depend on the choice of subsequence of volumes.
- Identifying phases from agreement of coefficients.
- Uniqueness of the extension of contact terms, OS positivity, convergence from the lattice.

$D$ is a truncation by degree of fields, not a loop order Assessment .

07

## What this book's investigation had seen

This book has **neither** an OPE nor contact terms. This book had no space of local fields (8-03 § 06), so there was nothing on which to set up an expansion as points approach each other.

Among this book's observables, the leading term of a small Wilson loop is an insertion of $F^2$ Inferred . In the language of the published construction, the expansion as the loop is shrunk to zero size is a question similar to the OPE, but loops are not in the space of local labels, so it cannot be discussed in the form (2)(3) as it stands. The fourth open question of the assessment, "construct Wilson-loop observables and relate them to local observables", points to this gap (16-02, 14-06).

08

## What we have not read

- The content of the construction of the coefficients (`YM.OPE.Prescription.ope`, `YM.OPE.Orbits.averageV` and others), and the proofs of (2)(3) (`coefficient_separated`, `quantitative_separated`).
- The meaning of the tree conditions of the nested OPE (`Accept`, `Tapered`).
- How `contactDistribution` is built (whether it subtracts the singular part using the OPE and then extends).
- How the complement is chosen in the splitting by degree of output fields, $F_d = F_{d-1} \oplus E_d$.

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The published statement contains the bound (2) on the OPE coefficients and the bound (3) on the remainder truncated at degree $D$ | Convergence as $D \to \infty$; agreement with perturbative coefficients |
| Source The remainder bound is uniform over the mixed theory and all phases (the radius is fixed before the state) | Independence of the coefficients from the subsequence of volumes |
| Source The coefficients do not take the spectators, the state or $D$ as arguments | Uniqueness of the coefficients when the normalisation or prescription is changed |
| Proved here From the common coefficients only the expansion (4) of the difference of two states follows, not agreement of phases | Identification of phases |

| Can say | Cannot say |
|---|---|
| Source The construction's theorem contains the existence (5) of an extension of contact terms | Uniqueness of the extension, OS positivity, convergence from the lattice; a statement of the extension in the published statement |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| OPE statement, configuration, type of the coefficients | Source | `ChallengeSpec.lean` lines 43–80 and 177–181 of `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e) |
| Contact terms, nested OPE, the construction's conclusion | Source | `ContactSpec.lean` lines 12–15, `ContactBridge.lean` line 14, `YM/Nonperturbative/HeadlineSpec.lean` lines 128–138, `Construction.lean` lines 15–31 |
| Equation (20), how channels and coefficients are built, what is not said, the separate endpoint | Assessment | The assessment, `yang-mills-group/YM` `docs/assessment-1.pdf` (2026-11-10), § 7.5, § 9.3, § 10, § 11, Appendix A (`Channel`, `actor`, `coefficient`, `Properties.ope`) |
| The committee's OPE example | Assessment | On the assessment side, `submissions/openai-20261028/docs/examples/CommonOPE-20261028.lean` in `yang-mills-group/YM`, revision 55ba9fc, lines 48–52, 79, 99, 128, 133, 159–165 |
| Arbitrariness of contact terms | Cited | Introduction in 4-04 |

**Terms used in this chapter** — operator product expansion, channel, spectator, remainder bound, uniformity over states, extension of contact terms. Their meanings are in the A-02 glossary.

Next chapter: 14-01 Comparing skeletons — in Part 14 we compare the published construction with this book's Lean, stage by stage.

# Comparing skeletons — a table stage by stage

> This chapter is this book's own investigation. Descriptions of the published construction are established knowledge (a reading of the published construction).

**What this chapter shows** — We split the proof of the published construction into eleven stages, from the lattice and reflection positivity to the mass gap and Wilson loops, and lay out in one table, stage by stage, "what the published construction does" and "what the Lean of this book's study had". We then separate what this book's study had seen (fix the volume → compactness → carry properties to the limit; fix the coupling by matching; blocks of odd side placed at the centre) from what it had not seen (the core of the multi-step estimates; everything from local fields onward).

**Prerequisite chapters** — 6-03 the skeleton of the proof, 28-01 implication of subsequential limits, 19-07 statements to be shown.

**Said first** — The results of the published construction are written at the level "the committee's assessment is that it is proved in Lean". What this book's study checked in the source goes only as far as the definitions and the form of the statements; there are many places where we have not read the content of the proofs. "Present" on this book's side means either a theorem at one scale with fixed lattice spacing, or an implication with all its hypothesis fields shown. It does not mean that this book's study showed anything about the continuum limit.

The order of this chapter

1. Conventions of the comparison — two levels, and five words for status
2. Table by stage (Table 1) — eleven stages
3. Lattice, reflection positivity, chessboard — present; the differences are the root seminorm and rarity
4. One renormalisation step and matching the coupling — partial; design and definitions differ
5. Multi-step estimates and large fields — absent; the part that is about 70% of the code
6. The ultraviolet limit at fixed volume — skeleton only; subsequences and the full sequence

7. Infinite volume, reconstruction, local fields — what is present in another regime, and what is absent
8. What was seen and what was not — three and two
9. What we have not read — with locations

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## Conventions of the comparison

This part sets side by side this book's study and the existence construction for pure Yang–Mills that was submitted on 28 October 2026 and published on 10 November together with the committee's assessment (below, "the published construction"). The code of the published construction is in the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), and the committee's assessment is `docs/assessment-1.pdf` (2026-11-10) in `yang-mills-group/YM`.

The two sides stand at different levels of certainty.

- **The published construction**: the results stand at the level "the committee's assessment is that it is proved in Lean" Assessment . What this book's study checked itself goes only as far as reading the definitions and the form of the statements in the source Source .
- **This book's study**: theorems machine-checked in Lean, or implications with all hypothesis fields shown. Every theorem is either a statement at one scale with fixed lattice spacing or an abstract implication, and none says anything about the continuum limit.

The "Status" column of Table 1 uses the following five words.

| Word | Meaning |
|---|---|
| Present | A statement of the same kind is also a theorem in the Lean of this book's study |

| Word | Meaning |
|---|---|
| Partial | The objects or the shape of the skeleton are there, but definitions, design or strength differ; or it is there only in the form of an implication |
| Absent | Not in this book's study, neither as a theorem nor as an implication |
| Present in another regime | A statement of the same kind is in this book's study, in a different coupling regime (the strong-coupling window) |
| Only in this book | A kind of statement not among the claims of the published construction is in this book's study |

02

## Table by stage

The proof of the published construction is assembled in the following order: starting from the calendar (the choice of constants), the one-step kernel, the representation of the density at each step and the induction, the choice of coupling by shooting, the ultraviolet limit at fixed volume, infinite volume, Euclidean invariance and phase decomposition, reconstruction, and the evidence that it is Yang–Mills (details in 6-03) Source . We split this into eleven stages.

**Table 1 Comparison by stage** (as of 2026-11-13)

| Stage | The published construction | This book's study | Status |
|---|---|---|---|
| Lattice and reflection positivity | Planes through lattice points and planes through midpoints of edges, all four axes, the Wilson measure with $\beta \geq 0$. Functions are gauge invariant and local to an open slab not containing points on the mirror (for midpoint planes, $r <$ half) Source | The same two kinds of plane. For lattice-point planes, all real $\beta$, all four directions, functions on the closed positive half (gauge invariance not imposed). For midpoint planes, $\beta \geq 0$ and gauge-invariant functions (20-01) | Present (different class of functions) |
| Chessboard | The chessboard inequality and the triangle inequality for its root. Used as a seminorm measuring the size of observables Source | The Chessboard Estimate, and rarity of large plaquettes at weak coupling (uniform in volume, 22-01). No form as a seminorm | Present |
| One renormalisation step | A probability kernel. Blocks of odd side placed at the centre of a coarse point, the predictor projected onto the group, gauge fixing in two strands Source | A deterministic push-forward map (side 2, overlapping weights), and the side-3, centred, uniform design (14-02) | Partial |

| Stage | The published construction | This book's study | Status |
|---|---|---|---|
| Matching the coupling | Match a coefficient in the representation to a reference value. Shooting method. A running law Source | A definition that matches with a quantity fixed by the measure alone. Existence of a matched sequence holds unconditionally only for the side-3 design (`Block3.tuned3_exists`) and for the vortex ratio on each lattice (`ThetaLimit.tuning_all`). In the current design (side 2) existence itself is conditional (if the value at the lower end is at most the target), and the chain theorems that use a matched sequence are conditional on "if it exists". No running law (14-03) | Partial |
| Multi-step small-field estimates | Induction hypothesis and the one-step theorem Source . About 70% of the code Inferred | One step of the free field (15-12), and implications with the estimates as hypotheses (28-01, 28-02) | Absent |
| Large fields | Re-count seeds at each step and set up an inequality per seed for every configuration satisfying the gauge section and the cell conditions Source (14-04) | Only rarity at one scale. Treatment across steps remains a premise (the large-field estimate in 28-04) | Absent |
| Ultraviolet limit at fixed volume | Correlations of local fields converge along the full sequence of spacings Assessment . In the published theorem too, the full sequence is used for the cutoff Source | Subsequential limits of the push-forward measures on the coarse lattice. The estimates are hypothesis fields (28-01) | Partial (skeleton only) |
| Infinite volume | Subsequence (Arzelà–Ascoli) Assessment | Inside the strong-coupling window, a limit without subsequences (20-03) | Present in another regime |
| Euclidean invariance, phase decomposition, reconstruction | Present Assessment | The lattice reconstruction vessel, and the mass gap at strong coupling (20-04). No Euclidean invariance or phase decomposition | Absent (the vessel is present in another regime) |
| Local fields, stress tensor, operator product expansion | Present Assessment | Not treated | Absent |
| Mass gap, Wilson loops, vortices | No statement of exponential decay. No continuum Wilson loops. Only periodic boundary conditions Source | Mass gap and area law at strong coupling, the vortex floor, pairs of Polyakov loops (14-06) | Only in this book |

By line count, the code of the published construction is roughly 70% multi-step induction and estimates, roughly 10% local fields and expansions, roughly 10% the ultraviolet at fixed volume, and about 4% reconstruction, with the definitional layers together under 4% Inferred . Of the 250,319 declarations, 100,168, including the headline theorems themselves, are reached from one of the seventeen headline theorems listed in the verification records Source (the same number as in Assessment § 9.1). The two rows this book's study marked "absent" (multi-step small-field estimates and large fields) also make up most of the construction by volume.

**What Table 1 does not say.** "Present" means that a statement of the same kind is also in this book's Lean, not that a statement of the same strength, the same class of functions and the same coupling regime is there. The "only in this book" row, too, consists of statements at fixed lattice spacing and says nothing about a continuum theory.

03

## Lattice, reflection positivity, chessboard

**Reflection positivity.** The published construction has reflection positivity with $\beta \geq 0$ for the Wilson measure on the periodic torus, for both planes through lattice points and planes through midpoints of edges, along all four axes (`reflected_positive` in `YM/Slice/Reflection.lean:98`) Source . The class of functions is gauge-invariant, measurable and bounded, local to the open slab $0 < r <$ half not containing points on the mirror (or the opposite face) for planes through lattice points, and to $r <$ half for planes through midpoints (`Strict` in `YM/Cells/Cut.lean:30`–:33, `Interior` and `Positive` at :109–:118) Source . The Lean of this book's study has the lattice-point-plane form `TorusRP.wilson_rp` (all real β, bounded measurable functions on the closed positive half, gauge invariance not imposed), the four-direction form `PeierlsPerm.wilson_rp_dir`, and the middle-plane form `MidRP.wilson_rp_mid` ($\beta \geq 0$, gauge-invariant functions) (20-01). Both sides are formalisations of known theorems Cited . Since both have the same two kinds of plane we mark it "present", but the classes of functions differ.

**Chessboard.** The published construction adds to the chessboard inequality (`tiles` in `YM/Cells/Chess.lean:80`) the triangle inequality for its root (`adds`, same file :108) Source . The quantity obtained by raising the expectation of the product of box-wise copies to the power of the reciprocal of the number of boxes (`rootTest` in `YM/Cells/Tiling.lean:226`) is used as a seminorm that measures the size of observables independently of the volume Source . The statement `Wcmain` (`YM/Extraction/Center.lean:38`), which states the ultraviolet estimates in this seminorm, is one of the inputs to compactness in infinite volume Source . It is not the

only input from the ultraviolet (for example `MergeProof.Accent.rated`, `YM/MergeProof/Powder.lean:48`, takes `Wcmain` and `Scrim` together)Source .

The Lean of this book's study has the Chessboard Estimate `FoldWilson.wilson_chessboard_even` (even side, all real β) and, using it, rarity of large plaquettes at weak coupling `FoldCap.rarity_explicit` (uniform in volume, explicit threshold) (22-01). There are two differences. This book's side lacks the triangle inequality for the root (the form as a seminorm), and on the published construction's side, within what we read, we did not find a rarity statement that uses the chessboard for probabilities of eventsSource (within the range searched).

04

## One renormalisation step and matching the coupling

One step of the published construction is a probability kernel that builds a "prediction" of the coarse links from the fine configuration and draws the coarse links with Gaussian-type fluctuations around it ( `density` in `YM/RG/Block.lean:71` )Source . One step of this book's study is a deterministic map that projects a weighted average of path products back onto the sphere, and the measure is pushed forward by that map (19-04). The side-3, centred, uniform design is close in form to the published construction's "odd side, placed at the centre, weights from convolving uniform distributions", and reflection positivity of the measure pushed forward with that design is in Lean. Details are in 14-02.

For the coupling, both are of the type "fix the scale by matching a quantity determined at unit length to a fixed value"Inferred . The differences are the quantity matched (in this book, a quantity determined by the measure alone; in the published construction, a coefficient in the representation of the density) and how existence is derived (in this book, global endpoints and the intermediate value theorem; in the published construction, nested intervals of bands at each step). Details are in 14-03. In both rows the objects or skeleton are there but the definitions and design differ, so we mark them "partial".

05

## Multi-step estimates and large fields

The published construction represents the effective density at each step $m$ in a fixed form (truncated action, small-field activities, a gas of large-field activities), and carries that form and its estimates through all steps as the induction hypothesis `History.Conv` ( `YM/RG/History.lean:213` )Source . Steps are linked by push-forward identities ( `History.Transfer` , same file :223), and the statement for all steps is `History.Selected`

(same file :234) Source . The form of the one-step theorem is the field `advances` of `Directed.Program` (a structure in `YM/PCalc/Band.lean:65`), and the one-step certificate is `Move` (same file :49) Source .

This book's study has nothing corresponding to this as a theorem. What it has is one step of the free field (the covariance of the linearised block map, 15-12, 15-13) and implications with the estimates as hypothesis fields (28-01, 28-02). Large fields went only as far as rarity at one scale (22-01), and estimates across steps remained, without a settled form, as the "large-field estimate" and "large-field gas" in the list of premises (28-04) (28-06). These two rows are "absent". How the published construction handles them is compared in 14-04.

06

## The ultraviolet limit at fixed volume

The published construction fixes the physical volume and shows that correlations of local fields converge along the full sequence of spacings Assessment (Assessment § 4.3). The location in Lean is `Campaign.realUV` (`YM/FullVolume/Fields/Filled.lean:45`), and the chain to the main theorem has been traced Source . In the convergence field `Converges` (`PublicWilson.lean:85-90`) of the published theorem `JointExistence` (`ChallengeSpec.lean:186`) too, convergence along the full sequence is required for the cutoff at each volume, and only the volume side is a subsequence Source . The stronger form with kernels and differences (`PositionsUV … Sequence.full`) is only in the conclusion of the construction's theorem `constructed` (`Construction.lean:15`) Source .

The skeleton is this Source . Show that the effective density at a fixed coarse step $j$ is Cauchy uniformly in the cutoff $N$ (`Flux.fanned`, `YM/FullVolume/Fields/Cement.lean:50`); bound the difference between the true correlations and the read-out at step $j$ by a quantity independent of $N$ (`Sever.echoes`, `YM/FullVolume/Fields/Peat.lean:42`); and from this, convergence at the fixed step and the triangle inequality, conclude that the true lattice correlations form a Cauchy sequence in $N$ (`Rising.arrives`, `YM/FullVolume/Fields/Timber.lean:74`).

On the side of this book's study, Lean has the implication that subsequential limits of the push-forward measures $\nu_{n,m}$ exist and that properties such as reflection positivity pass to those limits (`BoxWalk.chain_end_to_end_box`, 28-01). Subsequential limits exist for any family of probability measures, so the content of this implication is "carrying properties over". The estimates on the limit are all hypothesis fields, and include quantities not defined from the lattice. The shape of the skeleton is close, but the content of the estimates and full-sequence convergence are missing, so we mark it "partial (skeleton only)".

07

## Infinite volume, reconstruction, local fields

**Infinite volume.** Taking the volume-uniform ultraviolet estimates as input, the published construction derives volume-uniform boundedness and equicontinuity from reflection positivity on the original lattice, the chessboard and the trace representation of the transfer operator, and takes a subsequence by Arzelà–Ascoli (`first_surface`, `YM/Extraction/Lipschitz.lean:166`; `crops`, `YM/Extraction/Held.lean:136`) Source . No material for uniqueness is included Assessment (Assessment § 5, § 8). The Lean of this book's study has, inside the strong-coupling window ($|\beta|$ small), an infinite-volume limit that converges to a single measure along the sequence of sizes itself (`InfVol.wilson_infvol_unique`, 20-03). The coupling regime is different, so it is "present in another regime".

**Euclidean invariance, phase decomposition, reconstruction.** The published construction lifts rotational invariance from the Ward identities of the stress tensor, splits into phases by the Gelfand representation of a commutative algebra of operators compressed to the vacuum space (`OS.Selection`, `YM/Phase/Mixture.lean:127`), and reconstructs phase by phase Assessment Source . This book's study has the reconstruction vessel of the lattice transfer operator and the mass gap in the strong-coupling window (`InfMain2.wilson_infvol_main2` and others, 20-04). It has no continuum Euclidean invariance and no phase decomposition.

**Local fields.** The published construction has labels for local fields, renormalised insertions, the stress tensor, the trace anomaly and the operator product expansion (Parts 8 and 13) Assessment . This book's study does not treat these.

08

## What was seen and what was not

We reread Table 1 in terms of whether the shape of the skeleton had been seen Inferred .

**What was seen (three things).**

1. **Fix the volume → compactness → carry properties to the limit.** The implication of subsequential limits of this book's study (28-01) corresponds to the published construction's type "put the estimates as hypotheses and conclude a limit" (`FullVolume.positioned`, `YM/FullVolume/Profiles.lean:42`) and to its type "carry properties to the limiting table under conditions" (`realPositive`, `YM/Extraction/Harvest.lean:46`) Source . The order — fix the volume first, then go to infinite volume — is also the same.
2. **Fix the coupling by matching.** The type "choose the bare coupling so that a quantity at some coarse step takes a fixed value" is the same. This book's side has a

definition that matches with a quantity determined by the measure alone. Existence of a matched sequence is in Lean unconditionally only for the side-3 design (`Block3.tuned3_exists`) and for the vortex ratio on each lattice (`ThetaLimit.tuning_all`); in the current design (side 2) existence itself is conditional, and the chain theorems that use a matched sequence are conditional on "if it exists" (14-02 § 06, 14-03).

3. **Blocks of odd side placed at the centre.** The side-3, centred, uniform design points in the same direction as the way the published construction places its blocks (14-02).

**What was not seen (two things).**

1. **The core of the multi-step estimates.** Carrying the representation of the density at each step by induction; re-counting large fields at each step and setting up an inequality per seed for every configuration satisfying the gauge section and the cell conditions; putting a weak ledger in the induction hypothesis while having the one-step certificate produce strong bounds; and discharging the step-uniform conditions on the calendar side (14-04, 14-05).
2. **Everything from local fields onward.** Calibration of observables across steps, Cauchy at a fixed step and the one step that removes the step, the stress tensor and Euclidean invariance, phase decomposition. The endpoint of this book's study was the push-forward measure on the coarse lattice, not correlations of local fields. With a different endpoint, the skeleton needs two extra layers (calibration and the step-removing step)Inferred .

Conversely, this book's study has kinds of statements not among the claims of the published construction (the mass gap, Wilson loops and the area law, the vortex free energy). All are formalisations of known types at fixed lattice spacing, statements at the opposite end (strong coupling) from the questions the published construction left open (14-06).

09

## What we have not read

- The content of `Mallow.faces`, where the proof of the one-step theorem lives (`YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/Mallow/Pediment.lean`).
- The record of how declarations are used lists only the first headline found by depth-first search. The chains to the main theorem (`jointExistence`, `Solution.lean:4`) that this book's study checked are limited to the paths `rodB` → `constructionRun` and `realUV` → `fromFunded`, the path `allB` for convergence in the cutoff, `sailsW`,

and a few others. We have not checked whether the other declarations in Table 1 are among the dependencies of the main theorem.

- The theorem that builds the phase decomposition, and how the selected phase is chosen.
- We have not matched the Lean of this book's study against the Lean of the published construction, statement by statement in wording, for the same objects (because the types of the objects differ).

---

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The locations of the declarations for each stage of the published construction (in the text and the table of sources), checked against lines of the source | The content of their proofs. The dependency chain of the main theorem is traced only for some declarations |
| Assessment The published construction takes the ultraviolet limit at fixed volume along the full sequence of spacings, and the infinite-volume limit along a subsequence (Assessment § 4.3, § 5) | Uniqueness of the infinite-volume limit, the mass gap (listed as open questions in Assessment § 8) |
| The Lean of this book's study: reflection positivity for lattice-point planes and middle planes, the Chessboard Estimate, the infinite-volume limit without subsequences and the mass gap in the strong-coupling window (Part 20) | Varying the lattice spacing, the continuum limit, outside the window |
| The Lean of this book's study: the implication of subsequential limits (28-01). A conditional theorem with all hypothesis fields shown | None of the hypothesis fields is shown on the four-dimensional lattice |
| Inferred What this book's study had seen were three corners; what it had not seen were the core of the multi-step estimates and everything from local fields onward | "Seen" means that the shape of the skeleton is close, not that we had the same estimates |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Code of the published construction | Source | Repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0, revision 3456e5e). Paths are relative to the repository root. Declaration names and lines are given throughout the text |
| The committee's assessment | Assessment | `docs/assessment-1.pdf` (2026-11-10) in the repository `yang-mills-group/YM`. § 2, § 4.3, § 5, § 6, § 8 |
| How declarations are used | Source | `verification/declaration-usage.tsv` in the same repository. "used" means reached from one of the seventeen headline theorems listed in the verification records, not used by the published theorem |

| Item | Kind | Source / tool |
|---|---|---|
| Distribution of line counts | Inferred | A rough count by directory (this book's study's own tally) |
| Theorems of this book's study | Lean | The tables of sources in Parts 19–22 and Part 28 |

**Terms used in this chapter** — published construction, the committee's assessment, probability kernel, push-forward measure, reflection positivity, root seminorm of the chessboard, matching, shooting method, induction hypothesis, ultraviolet limit at fixed volume, implication of subsequential limits, phase decomposition. Their meanings are in the A-02 glossary.

Next chapter: 14-02 Comparing one-step designs — we compare the "one renormalisation step" row of Table 1 in detail across three designs.

2026-11-13 · chapter Yang–Mills and the mass gap Part 14 Comparison — this book's Lean, and ideas this book's study lacked one-step design · Kiichi, Shiori and Rin

# Comparing one-step designs — the push-forward map and the probability kernel

> This chapter is this book's own investigation. The description of the published construction ( § 04) is established knowledge (a reading of the published construction).

**What this chapter shows** — We lay out in one table three designs for one renormalisation step: the current design of this book's study (side 2, overlapping weights, a deterministic push-forward map), the side-3, centred, uniform design, and the design of the published construction (a probability kernel that draws the coarse links with Gaussian-type fluctuations around a predictor). We state what was closed in Lean for the side-3 design (reflection positivity of the pushed-forward measure, existence of a matched sequence), and explain why this is needed nowhere in the skeleton of the published construction (what is taken to the limit are correlations under the measure of the original lattice, and reflection positivity is used only on the original lattice).

**Prerequisite chapters** — 19-04 The block map and one step of renormalisation, 9-01 the block kernel, 14-01 Comparing skeletons.

**Said first** — We read the definition of the published construction's one-step kernel in the source `Source` , but not the content of the proofs of gauge fixing ( `Branch.align` , `Axial.existence` ). The theorems for the side-3 design are statements on a finite lattice with fixed lattice spacing, and this design is not the main line of this book's study. This chapter does not claim which design is better suited to the continuum limit.

The order of this chapter

1. What one step has to decide — the rule, and how the measure is carried
2. This book's current design — side 2, overlapping weights — a deterministic map and push-forward
3. This book's side-3 design — centred, uniform — commutes with reflections without gauge transformations
4. The published construction's design — a probability kernel — predictor, projection onto the group, gauge fixing in two strands, growing sides

5. Table of the three designs (Table 1) — nine items
6. What was closed in Lean for the side-3 design — reflection positivity of the pushed-forward measure, and existence of a matched sequence
7. Why it is not needed in the skeleton of the published construction — a difference in what is taken to the limit
8. What the choice of design affected — this book's skeleton and the published construction's skeleton
9. What we have not read

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## What one step has to decide

One renormalisation step is a pair: a rule that produces a coarse-lattice configuration $V$ from a fine-lattice configuration $U$, and a way of carrying the fine-lattice measure to the coarse lattice (introduction in 7-05). There are two types of rule.

- **Deterministic map**: $V = B(U)$. The measure is carried by push-forward $B_*\mu$ ($B_*\mu(E) = \mu(B^{-1}E)$).
- **Probability kernel**: the conditional distribution $T(U, \mathrm{d}V)$ of $V$ given $U$. The measure is carried by $\int T(U,\cdot)\,\mu(\mathrm{d}U)$. A deterministic map is the case of a kernel concentrated on a point mass, $T(U,\cdot) = \delta_{B(U)}$.

Either way, five things must be decided: the side of the block (by what factor the lattice spacing is reduced), where on the fine lattice the coarse points sit, the fine paths that build a coarse link and their weights, the procedure that brings an average that has left the group back onto the group, and compatibility with the lattice symmetries and gauge transformations.

02

## This book's current design — side 2, overlapping weights

The map on the main line of this book's study was defined in 19-04. The side is 2, and coarse points sit on the fine lattice points $2x$. The value of a coarse link $b$ is obtained by choosing, for each transverse direction, a shift $-1, 0, +1$ with weights $1/8,\ 3/4,\ 1/8$, averaging the products over the family of paths split equally among the six orders,

$$q_b(U) = \sum_{o,A} w(o,A)\, U(\gamma_{o,A}(b)) \tag{1}$$

and bringing it back onto the unit sphere by the retract with threshold $r_0$, $\rho_{r_0}(q) = q/\max(|q|, r_0)$. The measure is carried by the deterministic push-forward $\nu = B_*\mu$. The map does not depend on the coupling $\beta$.

Because paths shifted sideways are averaged, neighbouring blocks share fine links (an overlapping average). Reflection positivity of the pushed-forward measure is a theorem in this design for functions in the class of slabs of restricted width (`BoxWalk.wilson_h0p`, 28-05).

03

## This book's side-3 design — centred, uniform

In the side-3 design, on the lattice $(\mathbb{Z}/3^n L)^4$, coarse points are placed at the centres of blocks, and a coarse link is built as the uniform average over a family of paths of the form "outline + three straight steps in the direction of the coarse link + return along the outline" (`Block3.paths`, `Block3.wts`; 1944 indices, 633 distinct paths, weights summing to 1). The procedure for bringing the average back onto the sphere is the same as in the current design.

With centring and uniform averaging, the map commutes directly with reflection in a lattice-point plane, without any gauge transformation (`Block3.blockMap_refl_cfg`; for all steps, `Block3.down_refl`; with sphere values). Flipping the sign of the fine links that cross a cross-section of a block flips the sign of exactly one coarse link (one step: `Block3.blockMap_flip`; the set form for all steps: `Block3.down_flipSet`). The commutation properties were also checked numerically, with code that copies the Lean definition of the map literally, in the non-abelian, four-dimensional, two-step case Reading . This design is not the main line of this book's study; it was examined as a parallel track.

04

## The published construction's design — a probability kernel

One step of the published construction is a probability kernel (`StepData.density` in `YM/RG/Block.lean:71`) Source . As a formula, for a fine configuration $U$ and a coarse configuration $V$,

$$T_\beta(U,V)=\sum_t \ell(U,t)\prod_b \frac{1}{Z_\beta}\exp\!\big(-\beta\,c_{\mathrm{pen}}\,\mathrm{price}\big(V_b\,Q_b(U,t)^{-1}\big)\big) \tag{2}$$

which is a density in $V$ with respect to the product of Haar measures Source . Here $t$ is the label of the choice (the gauge frame and the predictor label), $\ell(U,t)$ its probability (`likelihood`), $Q_b(U,t)$ the predictor for the coarse link $b$, $c_{\mathrm{pen}}>0$ the penalty coefficient (`kernel.penalty` in the source's wording), and price a function measuring the distance of a group element from the identity. The coarse links are not constrained to the predictor but drawn with Gaussian-type fluctuations around it. $c_{\mathrm{pen}}$ may be any positive number, and the kernel depends on $\beta$ Source . Densities at successive steps are linked by push-forward with this kernel (`push`, same file :77) Source .

**Side and placement.** The side $b_m$ is an odd number at least 3 chosen at each step (`Schedule` in `YM/Lattice/Scale.lean:28`; only oddness and being at least 3 are here) Source . Non-decrease is the calendar field `Fits.monotoneL`, and $\log b_m/\log D_m\to 0$ is the field `Fits.mesh` (= `Schedule.Gentle`; `YM/RG/History.lean:97`–:100, `YM/Lattice/Scale.lean:129`) Source . Together with the calendar conditions, $b_m\to\infty$ follows Inferred (derived from the definitions; growing sides; the reason is in 10-05). A coarse point $x$ is placed at the fine point $b\cdot x$ (`embed` in `YM/Lattice/Subdivision.lean:29`), and the block is centred on the coarse point Source .

**Weights.** The weights on the points of a patch are the $J$-fold convolution of the centred uniform distribution of side $b$ (`weight` in `YM/RG/Patch.lean:87`, summing to 1 at :90), with $J\ge 4$ growing with the step Source . The support radius is $J(b-1)/2$, overlapping the neighbouring blocks Source .

**Predictor and projection onto the group.** Corresponding points of two neighbouring patches are joined by parallel straight lines of length $b$, the transports along them are taken, and they are gauge-fixed by the frames at both ends. The group element nearest to their weighted average is the predictor (`bondCost`, `best`, `YM/RG/Patch.lean:159`, :169) Source . That the nearest element coincides with the minimiser of the weighted sum of prices follows from the identity

$$2\,d\sum_i w_i\,\mathrm{price}(f_i u^{-1})=\sum_i w_i\|f_i\|^2-\|m\|^2+\|m-u\|^2,\qquad m=\sum_i w_i f_i \tag{3}$$

($d$ the dimension of the representation) (`min_iff` in `YM/Branch/Center.lean:72`) Source . When the minimum (the weighted spread plus the squared distance between the average and the group, divided by $2d$) is at least the threshold `predictor_cut`, one straight line is drawn with probability given by the weights and its transport is used (`usesBest`, same file :171) Source .

**Gauge fixing in two strands.** The frame of a patch (the gauge fixing) is, for small fields, a symmetric choice minimising the average price over all pairs of points in the patch along coordinate paths in axis order, and for large fields, an axial gauge with the root and axis order drawn uniformly; the kernel sums over the labels satisfying the conditions (`YM/RG/Patch.lean:102`–:155) Source .

**Symmetry.** The kernel built this way is exactly invariant, through the sum over choice labels, under the hypercubic group (the 384 signed permutations) and coarse translations, and covariant under gauge transformations (`StepData.channel` in `YM/Branch/Covariant.lean:50` and others) Source . Lattice symmetries pull back as $x \mapsto s \cdot x + b$ (`affine`, `YM/Lattice/Path.lean:145`), not only as rotations about a coarse point. Combining the flip of axis $i$ with $b = e_i$ gives $x_i \mapsto 1 - x_i$, so along with reflections in planes through lattice points, reflections in planes through the midpoints of coarse links are also included Source . This is covariance as a symmetry, not a statement of reflection positivity. The assessment states that gauge fixing invariant under the discrete lattice symmetries (§ 4.3) is built by "averaging over symmetry-breaking choices" Assessment (Assessment § 6.3).

05

## Table of the three designs

### Table 1 One-step designs

| Item | This book's current design | This book's side-3 design | The published construction |
|---|---|---|---|
| Type of rule | Deterministic map, push-forward | Deterministic map, push-forward | Probability kernel (depends on $\beta$) |
| Side | 2 | 3 | Odd, at least 3, non-decreasing, growing with the step |
| Placement of coarse points | Fine lattice points $2x$ (corners of blocks) | Centres of blocks | Centres of blocks ($b \cdot x$) |
| Weights | Transverse shifts $1/8$, $3/4$, $1/8$, split equally among six orders | Uniform | $J$-fold convolution of a centred uniform distribution ($J \geq 4$, growing with the step) |

| Item | This book's current design | This book's side-3 design | The published construction |
|---|---|---|---|
| Paths | A family of paths that shift sideways, go two steps and come back | Outline + three straight steps + return along the outline | Parallel straight lines joining corresponding points of two patches, gauge-fixed by frames at both ends |
| After averaging | Back onto the sphere (retract, threshold $r_0$) | Same | Projection onto the group; if the spread is large, one line drawn at random |
| Constraint | Push-forward (the coarse link is the value of the map itself) | Same | Gaussian-type penalty around the predictor, with arbitrary coefficient $c_{\mathrm{pen}} > 0$ |
| How symmetry is kept | Symmetric averaging over the family of paths | Centred and uniform; commutes with reflections without gauge transformations | Exactly invariant under the hypercubic group and coarse translations, through the sum over choice labels |
| Large fields | Outside the map (treated separately as events below the threshold) | Same | Enter as branches inside the definition of the kernel (axial gauge, one line drawn at random) |

The side-3 design is close to the published construction in three respects — odd side, centring, uniform (weights with $J = 1$) — and differs in four: being a probability kernel, projection onto the group, gauge fixing in two strands, and growing sides.

06

## What was closed in Lean for the side-3 design

**Reflection positivity of the pushed-forward measure.** Let $\nu$ be the measure obtained by pushing forward the SU(2) Wilson measure on the periodic lattice $(\mathbb{Z}/3^n L)^4$ with $L$ even (any real $\beta$, normalised or not) by the composition of $m \le n$ steps of the side-3 map. Then for the reflection $\Theta$ in a lattice-point plane ($y_0 \mapsto -y_0$) and every bounded measurable function $F$ depending only on coarse links lying strictly between the two planes 0 and $M_c = 3^m L/2$ whose ends are coarse lattice points (gauge invariance not imposed),

$$\int F(\Theta V)\, F(V)\, \nu(\mathrm{d}V) \;\ge\; 0 \qquad (4)$$

holds. Lean: `Block3.rp_push`, `Block3.rp_push_prob`. The proof is a short argument that reduces to reflection positivity of the fine lattice (`TorusRP.wilson_rp`, 20-01), using that the map commutes with the reflection without gauge transformations ( § 03) and the range of fine links that a coarse link uses (`Block3.down_local`). What it does not say: the version for planes through link midpoints, the closed class including links on the plane, infinite volume, the mass gap.

**Existence of a matched sequence.** In the side-3 design, under the push-forward at $\beta = 0$ (independent Haar), the expectation of the average real part of coarse plaquettes is 0 at every step (`Block3.Y3_zero`; because flipping the sign on a cross-section turns over exactly one coarse link and preserves Haar). If $0 < r_0 \le 1$, this expectation tends to 1 as $\beta \to \infty$ (`Block3.tendsto_Y3_one`). By the intermediate value theorem, under $0 < r_0 \le 1$, $3^{m_0} L \ge 2$ and $0 \le \kappa_0 < 1$, there is a sequence of non-negative couplings such that for every $k$ "the expectation of the average real part of coarse plaquettes pushed forward to step $m_0$ is $\kappa_0$" (`Block3.tuned3_exists`; for each $k$ one can take the largest solution). What it does not say: monotonicity, divergence or boundedness of the sequence of couplings, uniqueness of the solution. At $r_0 = 1$ the pull-back is the identity and a coarse plaquette is a product of sphere values, not an SU(2) Wilson loop. In the current design (side 2) the value at $\beta = 0$ is not 0 (numerically 0.006–0.024 Reading), and this existence was conditional on "if the value at the lower end is at most the target". The comparison of matchings is in 14-03.

07

## Why it is not needed in the skeleton of the published construction

What the published construction takes to the limit are correlations of renormalised local-field insertions under the Wilson measure $\mu^{\mathrm{W}}_{N,p}$ of the original lattice. In the assessment's formula,

$$C_{N,p}(v;f) = a_N^{4p} \sum_{x_1,\dots,x_n} f(a_N x_1,\dots,a_N x_n)\, \mathbb{E}_{\mu^{\mathrm{W}}_{N,p}}\Big[\prod_{j=1}^{n} I_{N,p}(v_j, x_j; U)\Big] \quad (5)$$

and the claim is that the limit $N \to \infty$ exists along the full sequence of spacings (Theorem 4.1) Assessment (Assessment § 4.3). The renormalisation step (equation (2)) is used as a tool for estimating this expectation, and does not appear as an object of the limit Source. The pushed-forward densities at a fixed step converge (`Flux.fanned`), but the limiting measure is not defined as a named object Source (within the range of a search by name). That measure would presumably follow from the uniform convergence of the densities at a fixed step Inferred.

Reflection positivity, too, is stated only for the Wilson measure of the original lattice (`law` in `YM/Wilson/Law.lean:87`) (`reflected_positive`) Source. Reflection positivity on the continuum side is obtained by keeping non-negativity of the lattice Gram matrices in the limit of correlations (`realPositive`, `YM/Extraction/Harvest.lean:46`) Source. Within what we read, there is no statement of reflection positivity for the one-step kernel Source (within the range searched). Since the patches of the kernel straddle the reflection plane, we infer that such a statement would not be trivial Inferred.

So reflection positivity of the pushed-forward measure (the type of (4)) has no place to enter at any stage of the published construction's skeleton Inferred . In the skeleton of this book's study the situation was different. The implication of subsequential limits (28-01) takes the limit of the push-forward measures $\nu_{n,m}$ themselves and carries reflection positivity over to that limit, so "$\nu_{n,m}$ is reflection positive" is among its hypothesis fields. The theorem for the side-3 design was meant to discharge that field. With a different endpoint (what one takes the limit of), whether the same property is needed or not switches around.

08

## What the choice of design affected

**This book's side.** What depended on the choices of the current design (weights, factor 2, threshold $r_0$) were the constants of the free-field single step (15-12) and the linearisation (15-13) (Table 2 in 28-05). The theorems of Parts 20–22 use only the fine-lattice measure, so they do not depend on the design. Switching to the side-3 design makes reflection positivity of the pushed-forward measure a theorem on the whole open positive side, and reduces the hypotheses for existence of a matched sequence to the design values and arithmetic ($0 < r_0 \le 1$, $3^{m_0} L \ge 2$, $0 \le \kappa_0 < 1$). The cost is rebuilding the free-field layer.

**The published construction's side.** With a probability kernel, the penalty coefficient $c_{\mathrm{pen}}$ enters the precision of the free field of one step in the form $K + c_{\mathrm{pen}} Q^{\mathsf{T}} Q$, and the constraint is no longer hard Source (11-09). Since the kernel depends on $\beta(N, p, m+1)$, the published construction's "push-forward measure" is an object whose map changes with the cutoff $N$, and is not the same object as the push-forward by a deterministic map in this book's study Source . Convergence of the published construction's densities at a fixed step does not give convergence of this book's push-forward measures.

**What they share.** Both designs aim to keep the discrete symmetries of the lattice intact after one step. This book does it by symmetric averaging over a family of paths, and the published construction by drawing symmetry-breaking choices at random and summing over them Inferred .

09

## What we have not read

- The content of the proof of existence and uniqueness of small-field gauge fixing, `Branch.align` ( `YM/Branch/Synchronize.lean:232` ), and of existence of the large-field axial gauge, `Axial.existence` ( `YM/Branch/Pick.lean:80` ). The value of the threshold.

- Where the body of the explicit renormalisation map for the main terms lives. As a one-step statement, we could read only as far as `History.Transfer` (`YM/RG/History.lean:223`).
- The concrete formula for the side $b_m$.
- Whether a statement of reflection positivity for the one-step kernel exists anywhere in the repository (we have not searched all of it).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source One step of the published construction is a probability kernel, and coarse links are drawn with the penalty $e^{-\beta\, c_{\mathrm{pen}}\, \mathrm{price}}$ around the predictor. The side is an odd number at least 3, and blocks are centred on coarse points | The content of the proofs of existence and uniqueness of gauge fixing |
| The Lean of this book's study: reflection positivity of the Wilson measure pushed forward by the side-3, centred, uniform map (`Block3.rp_push`). L even, m ≤ n steps, all real β, bounded measurable functions on the open positive side | The version for planes through link midpoints, infinite volume, the mass gap, varying the lattice spacing |
| The Lean of this book's study: a matched sequence exists for the side-3 design (`Block3.tuned3_exists`) | Monotonicity or divergence of the sequence of couplings, uniqueness of the solution |
| Source The published construction's statement of reflection positivity concerns the Wilson measure of the original lattice. A statement of reflection positivity for the one-step kernel is not within what we read | We have not searched the whole repository |
| Inferred Reflection positivity of the pushed-forward measure is not needed in the published construction's skeleton because what is taken to the limit are correlations under the measure of the original lattice | Whether the measure pushed forward with the published construction's design is reflection positive |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| One-step kernel, predictor, gauge fixing, symmetry | Source | In the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/RG/Block.lean` (:71 `density`, :77 `push`), `YM/RG/Patch.lean` (:87 `weight`, :159 `bondCost`, :171 `usesBest`), `YM/Branch/Center.lean:72` (`min_iff`), `YM/Branch/Covariant.lean:50` (`channel`), `YM/Lattice/Subdivision.lean:29` (`embed`), `YM/Lattice/Scale.lean:28` (`Schedule`) |

| Item | Kind | Source / tool |
|---|---|---|
| Definition of the correlations and the limit | Assessment | The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), equation (6) and Theorem 4.1 in § 4.3 |
| Reflection positivity of the original lattice | Source | `YM/Slice/Reflection.lean:98` (`reflected_positive`), `YM/Wilson/Law.lean:87` (`law`), `YM/Extraction/Harvest.lean:46` (`realPositive`) |
| This book's current design | Lean | Table of sources in 19-04 |
| Theorems for the side-3 design | Lean | The Lean bundle of this book's study (`Block3.paths`, `wts`, `blockMap_refl_cfg`, `down_refl`, `rp_push`, `rp_push_prob`, `down_flipSet`, `Y3_zero`, `tendsto_Y3_one`, `tuned3_exists`). Passed the kernel recheck and checked in the source |
| Numerics for commutation of the side-3 map | Reading | Numerical record of this book's study (non-abelian, four-dimensional, sphere values, two steps: differences in commutation with reflection, exchange of axes and coarse translations $\leq 1.1 \cdot 10^{-14}$; control maps show breaking of 0.26–1.8) |

**Terms used in this chapter** — renormalisation step, block map, push-forward measure, probability kernel, predictor, projection onto the group, penalty coefficient, gauge fixing in two strands, growing sides, reflection positivity, matching. Their meanings are in the A-02 glossary.

Next chapter: 14-03 Comparing matchings — we compare how the coupling is fixed after one step.

2026-11-13 · chapter Yang–Mills and the mass gap Part 14 Comparison — this book's Lean, and ideas this book's study lacked matching · Kiichi, Shiori and Rin

# Comparing matchings — a quantity fixed by the measure, and a coefficient in the representation

> This chapter is this book's own investigation. The descriptions of the published construction ( § 03– § 05) are established knowledge (a reading of the published construction). § 06 is inference.

**What this chapter shows** — The type "choose the sequence of bare couplings so that a quantity determined at some coarse step takes a fixed value" is the same on both sides. What differs is the quantity matched (in this book's study, quantities determined by the measure alone, such as the expectation of the average of coarse plaquettes or the vortex ratio; in the published construction, the coefficient of $F^2$ in the representation of the density at step 0) and how existence is derived (the intermediate value theorem from global endpoints / shooting with nested bands at each step). In the published construction, this matching plays the role of an "anchor" in the uniqueness argument for convergence in the cutoff. From the presence or absence of this anchor, we infer why the limits of this book's study could be taken only along subsequences.

**Prerequisite chapters** — 19-07 § 02 matching, 10-01 what the coupling at step m is, 10-04 the shooting method, 28-02 implication of full-sequence convergence.

**Said first** — The matching theorems of this book's study state only the existence of a solution on each lattice at fixed lattice spacing; they say nothing about monotonicity, divergence or uniqueness of the sequence of couplings, or about a running law (the coupling growing in proportion to $\log(1/a)$). § 06 is inference, not a confirmed cause of the subsequence ambiguity. The agreement of one-loop coefficients is a hand computation with matched conventions.

The order of this chapter

1. Matching as a type — the coupling is not an input but a number for fixing the scale
2. This book's study's definition — a quantity determined by the measure alone — the expectation of the average of coarse plaquettes, the vortex ratio
3. The published construction's definition — a coefficient in the representation — two normalisations, and rereading

4. How existence is derived — global endpoints and the intermediate value theorem / shooting with nested bands
5. Its role as an anchor — uniqueness for a two-point boundary value problem, and convergence of the full sequence
6. Why this book's limits could be taken only along subsequences — inference
7. Comparing one-loop coefficients — they agree once conventions are matched
8. What is not said, and what we have not read

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## Matching as a type

In the continuum limit, the lattice coupling $\beta$ is not an input but a number adjusted to calibrate the lattice spacing (introduction in 5-05). If, on a lattice with spacing $a_N$, one chooses the bare coupling $\beta_N$ so that "a quantity $Y$ determined by a single length independent of the spacing $a_N$ takes a fixed value $\kappa_0$", that length becomes the unit of scale. We call this matching.

The two sides share this type Inferred . They differ in three respects.

1. What the matched quantity $Y$ is ( § 02, § 03).
2. How existence of a solution is derived ( § 04).
3. Where the matched coupling is used in the argument for the limit ( § 05).

02

## This book's study's definition — a quantity determined by the measure alone

This book's study defines the matched quantity from the measure alone (19-07 § 02). Two examples are in Lean.

**The expectation of the average of coarse plaquettes.** Push the Wilson measure $\mu^\beta$ on the fine lattice at step $m_0 + k$ forward to step $m_0$ by the composition of $k$ steps of the

block map, and take the expectation of the average real part $Y_0$ of coarse plaquettes.

$$Y(m_0, k, \beta) = \int Y_0 \, \mathrm{d}\big(\mathrm{down}_* \mu^{\beta}\big) \quad (1)$$

In the side-3 design (14-02 § 03), $Y(m_0, k, 0) = 0$ (`Block3.Y3_zero`), $Y \to 1$ as $\beta \to \infty$ when $0 < r_0 \le 1$ (`Block3.tendsto_Y3_one`), and under $0 \le \kappa_0 < 1$ and $3^{m_0} L \ge 2$, for every $k$ there is $\beta_k \ge 0$ with $Y(m_0, k, \beta_k) = \kappa_0$ (`Block3.tuned3_exists`).

**The vortex ratio.** $\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$, built from the twisted partition functions $Z_0, Z_1$ of a torus of width $N$ and height $M$ (19-02), is a quantity that does not use the block map. $\theta_M$ is continuous in $\beta$ with $\theta_M(0) = 0$ (`ThetaTuning.thetaM_continuous`, `ThetaTuning.thetaM_zero`), tends to 1 as $\beta \to \infty$ with the lattice fixed (`ThetaLimit.tendsto_theta_one`), and for every $\kappa_0 \in [0, 1)$ there is $\beta \ge 0$ with $\theta_M(\beta) = \kappa_0$, the set of solutions having a largest element (`ThetaLimit.tuning_all`). Details are in 21-07.

The advantage of this definition is that it can be written without first fixing a representation or an approximation. In exchange, the relation between "the value of the observable and the coefficient of $F^2$ in the effective action" has to be shown separately.

## 03 The published construction's definition — a coefficient in the representation

In the published construction, the effective density at step $m$ of the theory with cutoff $N$ and volume $p$ is written in a fixed form of representation

$$\mathrm{eval}_m = \exp\big(v|\Lambda| - \beta K - B - A\big) \cdot \mathrm{gas} \quad (2)$$

($K$ the main action, $B$ the one-loop part independent of $\beta$, $A$ a local remainder, gas the gas of large-field activities), and the coupling at step $m$ is defined as the coefficient $\beta$ there Source (details in 10-01). So that the $F^2$ term cannot be moved between $\beta K$ and $B$, two normalisations are imposed: the zero-momentum coefficient matrix of the quadratic part of $K$ is the identity, and the zeroth- and first-order Fourier coefficients of $B$ are 0 Source .

The docstring in the source describes this $\beta$ as "an actual number in the representation, not a number tied to the measure independently" (top of `YM/RG/History.lean`) Source . The number itself is reread from the log-ratio of the density at the flat configuration 1 and at a test configuration $V$,

$$\log H(1) - \log H(V) = \beta K(V) + B(V) + A(V) \quad (3)$$

(`contrast` in `YM/UVNext/COUPLING_RESET/Reset.lean:50`) Source . The right-hand side increases in $\beta$ with slope $K(V) > 0$, so the root is unique and depends continuously on the input Source (10-03).

As for the direction of steps, $m = N$ is the finest, where the representation is the bare Wilson weight itself, and $m = 0$ has spacing 1 Source . The matching is "the coupling at step 0 equals a fixed reference value $\bar{\beta}$" Assessment (the reference conventions in Assessment § 11 and Appendix A). The reference value is taken large, keeping weak coupling $\beta_m \in [\hat{\beta}_m/2,\ 2\hat{\beta}_m]$ at every step (field `band` of `History.Conv`) Source .

| | This book's study | The published construction |
|---|---|---|
| Matched quantity | Expectation of the average of coarse plaquettes under the push-forward measure; the vortex ratio | The coefficient of $F^2$ in the representation of the density at step 0 |
| What the definition needs | The measure and the block map (the vortex ratio does not even need the map) | A representation preserved at every step, and two normalisations |
| Where it is read | An integral over all configurations of the coarse lattice | A log-ratio at a small-field test configuration |
| What must be shown separately | The relation between the value of the observable and the coefficient of $F^2$ | That the representation is preserved at every step (induction) |

04

## How existence is derived

**This book's study: global endpoints.** If the matched quantity $Y(\beta)$ is continuous in $\beta$ and its values at $\beta = 0$ and as $\beta \to \infty$ are known, the intermediate value theorem gives a solution for any target between them. In the side-3 design the value at $\beta = 0$ is 0 by symmetry, and the value as $\beta \to \infty$ is 1 because flat configurations are pushed forward to flat ones. Only the values of one continuous function at its two ends are used.

**The published construction: nested bands at each step.** Take the bare coupling as a parameter $x$, and view the coupling at step $m$ as a continuous function $f_m(x)$ of $x$. On the existence path (the path whose chain to the main theorem we checked), one uses the band at each step

$$\mathrm{low}_m = \tfrac{3}{4}\,\bar{\beta} + \tfrac{5}{8}\,s\,\log D_m, \qquad \mathrm{high}_m = \tfrac{5}{4}\,\bar{\beta} + \tfrac{3}{2}\,s\,\log D_m \quad (4)$$

($s$ the one-loop slope, $D_m$ the product of the sides up to step $m$) (`YM/FullVolume/Arc.lean:21`, :22) Source . From the one-step error inequality, inside an interval that reaches both ends of the band at step $m+1$ one can take a subinterval that reaches both ends of the band at step $m$ too, and nested intervals are obtained

going downward from step $N$ (`Arc.bought`, same file :95) Source . At step 0, $\bar{\beta}$ lies inside the interval $[\frac{3}{4}\bar{\beta}, \frac{5}{4}\bar{\beta}]$, so the intermediate value theorem (`covers`, same file :71) gives a bare coupling for which the coupling at step 0 is $\bar{\beta}$ Source . On the existence path, continuity of $f_m$ is supplied by `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean:76`) Source .

The repository also has shooting with a different band $[\bar{\beta} + \frac{s}{2}\log D_m,\ \bar{\beta} + 2s\log D_m]$ (under `YM/Shoot/`), but the path to the main theorem (`rodB` in `FullVolume/Stage/Rods.lean:26`) does not use it Source . The detailed form of the shooting method is in 10-04.

To summarise the difference: this book's study uses the global endpoints of one function, while the published construction uses local bands at each step and the one-step error inequality. The published construction's method works in the same form regardless of the number of steps, given the condition that the sum of one-step errors stays within a small fraction of the band width (the calendar condition). This book's method can be used only for quantities whose endpoint values are known from symmetry or limits Inferred .

05

## Its role as an anchor

In the published construction, matching does more than "fix the unit": it enters the argument that the ultraviolet limit at fixed volume can be taken along the full sequence Source .

The source of convergence in the cutoff is not a quantitative estimate comparing $N$ and $N'$, but three things: compactness, uniqueness of the limit, and the subsequence principle (`Window.allB`, `YM/FullVolume/Glide/Triumph.lean:37`) Source . The sequence of ingredients of the representation (couplings and activities) lies in a compact set, so it has subsequential limits. Every subsequential limit satisfies the "ladder equations" across all steps $m \in [0, \infty)$. If these have only one solution, all subsequential limits coincide and the full sequence converges.

Uniqueness (`accord`, `YM/FullVolume/Glide/Coincide.lean:58`) has the form of a two-point boundary value problem Source .

- **The marginal direction (the coupling)** is anchored at step 0. By shooting, the coupling at step 0 is $\bar{\beta}$ for every $N, p$, so the difference between the couplings of two solutions is 0 at step 0, and propagates towards the fine side through the sum of one-step deviations. The size of that sum is bounded by the calendar condition to a small fraction of the band width Source .

- **The irrelevant directions (the activities)** shrink by $1/16$ per step from the fine side to the coarse side (`Window.curls`, `YM/FullVolume/Glide/Duty.lean:45`; `Muse.ticket`, `YM/FullVolume/Gain/Rate.lean:94`) Source .

If $b$ is the supremum of the difference between two solutions, these two facts give $b \leq b/16$, so $b = 0$ Source . Differences in the initial values at the fine end are forgotten by $(1/16)^k$ as one goes down the steps, so they need not be small, only bounded (`culprit`, `YM/FullVolume/Glide/Wings.lean:59`) Source . Estimates of this amount of forgetting are also used for convergence of the activities Source . There is no rate of convergence in the statement (qualitative convergence) Source .

Without the anchor, the coupling direction does not contract, so this argument does not close. The matching that sets the coupling at step 0 to the same $\bar{\beta}$ for every cutoff is exactly that anchor Inferred . Details are in 11-10.

06

## Why this book's limits could be taken only along subsequences

This section is inference Inferred .

In the implication of subsequential limits of this book's study (28-01), the first conclusion states that "the matching point lies in the bracket" (a point where $u_n(x) = t$ for the family of functions $u_n$), but there was not a single field linking $u_n$ to the fine measure, so the sequence of bare couplings and the matching were not tied together. The implication of full-sequence convergence (28-02) had in Lean, for orbits of an abstract skew-product map (marginal coordinates and contracting irrelevant coordinates), the form "if the marginal coordinate at the coarse end is matched to the same value for every $n$, and the irrelevant coordinates contract per step, then the coarse end is Cauchy", and also showed by example that the matching at the endpoint cannot be removed. This is the same type as the two-point boundary value problem of § 05. However, that map was not defined from the lattice.

It is therefore possible that the limits of this book's study could be taken only along subsequences because there was no anchor in the coupling direction (the matching lay outside the chain). The existence of a matched sequence for the side-3 design ( § 02) is a concrete object that could be a candidate for an anchor. However, it has not been shown that it is a matching of the marginal coordinate of a one-step map. There are also two differences from 28-02. The marginal coordinate in the 28-02 theorem is not set equal to either $u_n$ of 28-01 or the matched quantity $Y$ of the side-3 design. And convergence in 28-02 is a quantitative Cauchy property (with a rate) obtained from Lipschitz contraction, derived differently from the published construction's "compactness and uniqueness" (without a rate).

The anchor alone is not enough, however. The uniqueness argument of the published construction rests on a representation preserved at every step (the ingredients splitting into couplings and activities) and on the per-step contraction of the activities. It has not been shown that, with this book's deterministic block map, the coarse density keeps a representation of this form at every step. This is where it overlaps with the treatment of large fields (14-04).

07

## Comparing one-loop coefficients

The slope of the one-step shift of the coupling in the published construction is the constant set by definition

$$s = \frac{11\, C_g}{24\pi^2} \quad (5)$$

(`slopeConstant` in `YM/Gauge/Colors.lean:184`) Source . The assessment states that the bare coupling satisfies $\beta_{N,p}/\log(1/aN) \to 11C_g/(24\pi^2)$ uniformly in the volume Assessment (equation (7) in Assessment § 4.3). What the published statement `WilsonRegularity.running` (`ChallengeSpec.lean:13`) requires is only "there is some positive slope"; $11C_g/(24\pi^2)$ is the value that the construction satisfies Source . $C_g$ is the Casimir constant for the inner product $\langle X, Y\rangle = -\mathrm{Re}\,\mathrm{Tr}(XY)/d$ ($d$ the dimension of the representation) Source .

For the two-dimensional representation of SU(2), $C_g = 8$ and $s = 11/(3\pi^2)$, and the shift for one step with side 2 is $s \ln 2 = 11 \ln 2/(3\pi^2) \approx 0.2575$ Proved here (hand computation; $C_g = 8$ also checked numerically). This agrees with the one-loop value that this book's study wrote in units of $\beta$ (15-01) Proved here . The premise is that this book's $\beta$ is the coefficient of $\sum_p (1 - \frac{1}{2}\mathrm{Re}\,\mathrm{tr}\, U_p)$ (the standard $\beta = 4/g^2$). The published construction uses only odd sides of at least 3, so it does not treat the side-2 step itself. Only the coefficient of $\log b$ agrees; the $O(1)$ constants of one step cannot be compared.

The published construction leaves the one-step error at $O(1)$, lets the side $b_m$ grow with the step and removes it by dividing by $\log b_m$ (10-05) Source . This book's study kept the side fixed, so it had a separate problem: the $O(1)$ error of one step accumulating over the number of steps (the deviation of the coefficient at the first step measured in 15-01).

08

## What is not said, and what we have not read

**What this book's study does not say.** Uniqueness of the matching solution, monotonicity or divergence of the sequence of couplings, a running law, estimates on $\theta_M$ uniform in $N$. The matching of the vortex ratio is a statement with the lattice fixed, and says nothing about sequences in which the lattice and the coupling move together.

**Where the short-distance effective coupling lives.** For Theorem 7.2 of the assessment (the short-distance effective coupling $\log(1/r)\, g_W(r)^2 \to 1/(2b_0)$; in the assessment's notation, $2b_0$ corresponds to the slope s in (5)), there are three declarations with similar content Source . The assessment does not name a location Assessment .

| Declaration | Content | Dependency |
|---|---|---|
| `YM.PertExtension.Running.ratio` (`YM/PertExtension/Running.lean:145`) | Ratio form ($\mu \to \infty$, $a_m \le 1/\mu$) | Outside the dependencies (outside) Source |
| `YM.Trace.Pilot.wilson_link` (`YM/TraceExtension/Atlas/Sink.lean:50`) | Wording close to the statement of Theorem 7.2 ($r \downarrow 0$, the first spacing below $r$) Inferred . `PostT` ($b_i \le \log \hat\beta_i$) among the hypotheses | Inside the dependencies. First headline `continuum_with_position_tests`, direct user `crest_ready`. Used as an intermediate step in the trace analysis; its conclusion enters `Crest.charts` and `FieldComparison.wilson` and is dropped by `of_result` Source |
| `YM.Pert.Asymptotics.comparison` (`YM/Pert/Asymptotics/RawAxis.lean:36`) | The running of the coupling of the cut-off step $i$, $\beta(N_i, k_i, i)$ $-1/2$; a different quantity from $g_W$ | A headline theorem itself (headline) Source |

A statement in the form of $g_W$ does not survive into any conclusion of the published theorem, the construction's theorem or the committee's joint goal Source . The headline theorem `comparison`, too, is one of the 17 headlines listed in the verification records and is not in the conclusions of the three goals (it is not referenced from the construction's theorem, the published theorem or the committee's joint goal) Source . Universality with respect to the choice of sequence of spacings, and the dependence on changing $\bar\beta$, are not stated Assessment Source .

**What we have not read.** The proof that connects the value of the one-loop shift with the kinetic term of the one-step theorem (beyond `under_contract`), the concrete form of the test configuration $V$, and how `Roof.Lyrics.supplier` produces continuity.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| The Lean of this book's study: the vortex ratio $\theta_M$ is continuous in $\beta$ with $\theta_M(0)=0$, tends to 1 as $\beta\to\infty$, and for every $\kappa_0\in[0,1)$ there is a largest solution of $\theta_M(\beta)=\kappa_0$ (`ThetaTuning.thetaM_continuous`, `thetaM_zero`, `ThetaLimit.tendsto_theta_one`, `tuning_all`; lattice fixed) | Uniqueness of the solution, continuity in $\kappa_0$, estimates uniform in $N$ |
| The Lean of this book's study: in the side-3 design there is a sequence matching the expectation of the average of coarse plaquettes (`Block3.tuned3_exists`) | Monotonicity, divergence or boundedness of the sequence; a running law |
| Source The docstring in the source says that the step coupling of the published construction is a coefficient in the representation of the density, not a number determined by the measure alone | The dependence when the representation is changed |
| Source Shooting on the existence path uses nested intervals of the band $\frac{3}{4}\bar\beta+\frac{5}{8}s\log D_m$ to $\frac{5}{4}\bar\beta+\frac{3}{2}s\log D_m$, and takes $\beta=\bar\beta$ at step 0 | Uniqueness of the bare coupling chosen by shooting |
| Source Convergence in the cutoff follows from compactness, uniqueness and the subsequence principle. Uniqueness comes from the anchor at step 0 and the contraction of the activities by 1/16 per step | The rate of convergence (not in the statement) |
| Inferred The subsequences in this book's study may come from the absence of an anchor in the coupling direction | Not checked. Even with an anchor, the same argument does not stand without the representation and the contraction |
| Proved here With conventions matched, the one-loop coefficient agrees at $\frac{11}{3}\frac{\ln 2}{\pi^2}$ for one step with side 2 (hand computation; $C_g=8$ also checked numerically) | Comparison of the $O(1)$ constants of one step |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Docstring on the definition of the step coupling | Source | In the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): the docstring at the top of `YM/RG/History.lean` (around :24), `Conv` (:213), `Transfer` (:223), `Selected` (:234) |
| Rereading (log-ratio of two configurations) | Source | `YM/UVNext/COUPLING_RESET/Reset.lean:50` (`contrast`) |
| Shooting method (existence path) | Source | `YM/FullVolume/Arc.lean` (:21 `low`, :22 `high`, :71 `covers`, :95 `bought`), `YM/FullVolume/C/Songbook.lean:76` (`Roof.Lyrics.supplier`), `YM/FullVolume/Stage/Rods.lean:26` (`rodB`) |

| Item | Kind | Source / tool |
|---|---|---|
| Anchor and contraction | Source | `YM/FullVolume/Glide/Triumph.lean:37` (`allB`), `YM/FullVolume/Glide/Coincide.lean:58` (`accord`), `YM/FullVolume/Glide/Duty.lean:45` (`curls`), `YM/FullVolume/Gain/Rate.lean:94` (`Muse.ticket`), `YM/FullVolume/Glide/Wings.lean:59` (`culprit`) |
| One-loop slope | Source | `YM/Gauge/Colors.lean:184` (`slopeConstant`). Equation (7) in Assessment § 4.3 |
| Short-distance effective coupling (three locations with similar content) | Source | `YM/PertExtension/Running.lean:145` (`ratio`, outside), `YM/TraceExtension/Atlas/Sink.lean:50` (`wilson_link`, used), `YM/Pert/Asymptotics/RawAxis.lean:36` (`comparison`, headline). `verification/declaration-usage.tsv`, `verification/joint-headlines.txt`; on the assessment side, `submissions/openai-20261028/checks/CombinedAcceptance.lean` |
| Matching, bare coupling, running | Assessment | The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 4.3, § 7.4 |
| This book's matching theorems | Lean | The Lean bundle of this book's study (`ThetaTuning.*`, `ThetaLimit.*`, `Block3.*`). Passed the kernel recheck and checked in the source |
| Comparison of one-loop coefficients | Proved here | Hand computation of this book's study (the value of $C_g$ also checked numerically). The value on this book's side is in 15-01 |

**Terms used in this chapter** — matching, bare coupling, expectation of the average of coarse plaquettes, vortex ratio, coefficient in the representation, rereading, shooting method, band, anchor, two-point boundary value problem, forgetting of the upper end, running law, one-loop coefficient. Their meanings are in the A-02 glossary.

Next chapter: 14-04 Large fields — we compare the treatment of large fields, which was open in this book's study, with that of the published construction.

# Large fields — where this book's study got stuck, and how the published construction handles them

> This chapter is this book's own investigation. Descriptions of the published construction are established knowledge (a reading of the published construction), and § 01 is a map of open questions. The judgements in § 06 and § 07 are inference.

**What this chapter shows** — On the ultraviolet side of this book's study, large fields were what remained without a known way forward. In a form that stacks up the large-field regions step by step and fixes them, the treatment of defects that persist across steps near configurations whose shape the block map does not change was left open. The published construction keeps no history of regions and re-counts seeds at each step, clips the main part of the representation to linear growth, sets up at every step an "energy per seed" inequality for every configuration satisfying the gauge section and cell conditions, puts a weak ledger in the induction hypothesis and strong bounds in the one-step certificate, and discharges the step-uniform conditions on the calendar side. We read this as "answering in a different form" the point where this book got stuck.

**Prerequisite chapters** — 11-00 the induction hypothesis and the one-step certificate, 9-07 the small-field threshold and the clipped action, 28-04 the list of premises (large-field estimate, large-field gas), 22-01 rarity of large plaquettes.

**Said first** — For many of the published construction's large-field lemmas, tracing them through the record of how declarations are used leads to a headline theorem other than the main theorem. We have not checked whether they are among the dependencies of the main theorem (this does not mean that they are not). We have not read the whole construction that re-files the coarse-step activities into the ledger after one step, the lemma that pays for the Haar loss, or the bound on the remaining integral. The judgement in § 06 is an application to the vocabulary of this book's study; it does not mean that the proofs of the published construction were written with this book's questions in mind.

The order of this chapter

1. Where this book's study got stuck — regions stacked up and fixed, and persistent defects
2. The published construction's representation at each step — the clipped action, and a gas of activities only on flags
3. Inside one step — seeds and cells — seed types, cells, partitions of unity
4. Energy per seed — an inequality for configurations satisfying the section and cell conditions
5. Two ledgers and the calendar — a weak ledger and strong bounds, the stamp, step-uniform conditions
6. Judgement — answered in a different form — what can and cannot be said (Table 1)
7. Configurations that keep exceeding the threshold — how they are handled across steps
8. Clusters that return to small fields — the test of whether they contain a coarse flag
9. What we have not read, and checking dependencies

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## Where this book's study got stuck

When a lattice field is coarsened step by step, at any step there may remain places where the field deviates strongly (where the holonomy of a plaquette is far from the identity). There the small-field expansion (Gaussian integrals and the expansion of their corrections) cannot be used, so a different treatment is needed. This is the large-field problem (7-05, 15-01).

Following the pattern of Bałaban's series, this book's study thought in terms of defining the large-field regions at each step and stacking them up, fixing them for later steps. In this form, defects that persist across steps may remain near configurations whose shape the block map does not change (configurations mapped to the same shape at

every step). For such defects, how to show that the gain in action and the loss from fluctuations do not balance out, so that the rarity at each step acts uniformly in the number of steps — this remained unknown.

In the list of premises (28-04), this part was placed in two rows, the "large-field estimate" (at each step and each hole, activities are bounded by the exponential of the lower bound of the action, with constants uniform in the step and the volume) and the "large-field gas" (an estimate of the gas of large-field activities, and its equality with the corresponding ratio), and neither had a settled form Assumption (28-06). Rarity at one scale (under the fine-lattice Wilson measure, the probability of the set of large plaquettes is small uniformly in the volume) is a Lean theorem (`FoldCap.rarity_explicit`, 22-01), but it does not reach a form that spans steps.

02

## The published construction's representation at each step

The published construction keeps the effective density at each step $m$ in the following form Source (details in 9-06, 11-01).

$$\mathrm{eval}_m = \exp\Big(v|\Lambda| - S_m^{\mathrm{clip}} - \sum_i 1[\mathrm{healthy}]\, E_i\Big) \cdot \mathrm{gas}_m(K) \quad (1)$$

$S^{\mathrm{clip}}$ is the clipped action, $E_i$ the small-field (regular) activities, and $\mathrm{gas}(K)$ the gas of large-field (forced) activities $K$.

**Threshold.** From the reference coupling $\hat{\beta}_m$ at step $m$, set $\hat{g}_m = \hat{\beta}_m^{-1/2}$ and $p_m = \exp\big((\log\log\hat{\beta}_m)^2\big)$, and let the threshold be $t_m = \hat{g}_m\, p_m$ (`YM/RG/History.lean:81`, :83) Source . For sufficiently large $\hat{\beta}$ (in the asymptotic sense), $p$ grows faster than any power of $\log\hat{\beta}$ and slower than any positive power of $\hat{\beta}$ Proved here . For example, at $\hat{\beta} = 10^{10}$, $p \approx 1.9 \cdot 10^4$, which is larger than $\hat{\beta}^{0.1} = 10$, so the slower growth becomes visible only for much larger $\hat{\beta}$ Reading .

**Clipped action.** For each plaquette, in terms of its deviation $s$ (the distance of the plaquette holonomy from the identity),

$$\mathrm{clipped}(s) = \begin{cases} \dfrac{c_{\mathrm{dir}}}{2}\, s^2 & (s \le t) \\ c_{\mathrm{lin}}\, t\, s & (s > t) \end{cases}, \qquad c_{\mathrm{lin}} < \frac{c_{\mathrm{dir}}}{16} \quad (2)$$

(`clipped` at `YM/RG/PathActions.lean:218`, `kink` at :145) Source . It is quadratic up to the threshold and linear beyond it, jumping down at $s = t$.

**Activities only on flags.** The large-field activities $K$ live only on clusters that exactly cover the places exceeding the threshold at the current step (the flags), and if there are

no flags the gas is 1 (`gas_one` in `YM/RG/Activities.lean:135`) Source . What the induction hypothesis `History.Conv` (`YM/RG/History.lean:213`) holds is the ledger of small-field activities $\hat{g}^{11/5}$ (:84), the ledger of large-field activities

$$\sum(\text{cover-weighted } K) \;\le\; \exp\!\left(-p_m^{1/5}\right) \quad (3)$$

(`forcedAllowance` at :85), and the coupling band Source . **There is no field representing a history of regions** Source .

**Between steps, an identity.** The densities at steps $m$ and $m+1$ are linked by the push-forward identity with the one-step kernel, $\mathrm{eval}_m = \mathrm{push}(\mathrm{eval}_{m+1})$ (`History.Transfer`, :223) Source . Clipping the main part of the representation does not change the measure; the clipped difference goes into the small-field and large-field activities Source .

At the finest step, the Wilson price of each plaquette is split into the clipped action, the quadratic part on healthy places, the remainder on healthy places, and a balance (due), and the balance is charged to the first large-field activities (`due_eq` in `YM/Extra/WILSON_STARTUP/Ownership.lean:68`) Source . On flagged plaquettes the balance is at least $t^2/4$ Source (11-02).

## 03 Inside one step — seeds and cells

Inside the integral of one step, configurations are not cut into "small fields and large fields" by characteristic functions Source . Three kinds of "seed" are counted: fine plaquettes exceeding the threshold, coarse links deviating strongly from the predictor, and points touching a coarse flag Source . A cell is built for each seed type (a triple of finite sets), and the cells are glued together by a smooth partition of unity Source (11-03).

The cell condition itself (`Seeded`, `YM/Push/SECTOR_MEASURE/Cells.lean:51`) is imposed on all plaquettes and links Source . What the estimates use is a form localised around the seeds, `Plot` (`YM/Pressure/Seeds.lean:51`), with localisation radius $5\lceil(\log\hat{\beta})^2\rceil$ in terms of the reference coupling $\hat{\beta}$ on the coarse side Source . The boundaries of cells are written with two values, $t/4$ and $t/2$, for the plaquette condition, giving a band of offset (hysteresis) Source .

## 04 Energy per seed

The key inequality of one step is, on a cell,

$$c_E\, t_{\mathrm{f}}^2 \cdot \#(\text{seeds}) \;\le\; \big(\text{fine action} + \text{deviation penalty} - \text{coarse clipped action}\big)_{\text{around seeds}}$$ (4)

(`cash`, `YM/Pressure/Seeds.lean:124`; `Grants`, same file :129) Source . $t_{\mathrm{f}}$ is the threshold on the fine side Source . The order of quantifiers is "there is $i$ such that for every side $L$ at least $i$ and every sufficiently large $W$ there is $c_E > 0$ such that for every sufficiently large reference coupling, ...", and $c_E$ depends on $L$ and $W$ Source ($\kappa$ in the source's wording). The statement is made, with patch side $L$, lattice side $c$ at least $100(1 + \lceil(\log \hat{\beta})^2\rceil)$, and the hypothesis `Funds` on the choice of kernel and so on, for **every configuration** satisfying the section with the gauge fixed on a tree (`SliceSet`, :48: $U = 1$ on the trunks of the forest) and the cell condition `Plot` (`live`, :113–:118) Source . This structure holds for every choice without hypotheses (`grants`, `YM/Next/FLUCTUATIONS/Grants.lean:24`) Source .

Multiplying by $\beta$, since $\beta t^2 \approx p^2$, each seed carries a factor of about $\exp(-c_E p^2)$ Inferred . Since $p^2$ grows faster than any power of $\log \beta$, polylogarithmic losses such as the volume around a seed, the Haar loss and combinatorial counts can be paid with this factor Inferred .

One of the positive remainders in inequality (4) comes from the clipped action. For coarse plaquettes of intermediate size (larger than the coarse threshold $t_{\mathrm{c}}$ and at most some $\delta$), a positive term $\frac{c_{\mathrm{dir}}}{4}\, t_{\mathrm{c}} \cdot (\text{deviation})$ appears in the lower bound for the work on the coarse side (`Crowds` at `YM/Bid/Energy.lean:94`–:97, `E05` at :101) Source . The shape is that, since the action handed to the coarse step is clipped to linear growth, a positive difference from the quadratic action on the fine side remains Inferred .

## 05 Two ledgers and the calendar

**A weak ledger and strong bounds.** The large-field ledger held by the induction hypothesis `Conv` is only the weak form (3) Source . The one-step certificate `Move` (`YM/PCalc/Band.lean:49`) states, for the exit activities, the much stronger

$$K_{\text{exit}} \;\lesssim\; \exp\!\big(-g \cdot p_{m+1}^2\big)$$ (5)

(field `forced`; $p_{m+1}$ is the value at the fine step) Source . Dropping the strong bound to the weak ledger is a short inequality: for all sufficiently large $y$, if $x$ and $y$ are "alike" (six quantities — threshold, $P$, gap, number of clusters, ledger $w$ and so on — are within a factor of 2 of each other; `Alike`, `YM/Next/SCHEDULE/Dues.lean:118`), then $\exp(-d\,P(y)^2) \le w(x) = \exp(-P(x)^{1/5})$ (`saved`, `YM/Next/SCHEDULE/Salary.lean:178`; `late_force`, `YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:19`) Source . The proof of `saved` handles the exponent $1/5 \le 1$ (using that the ratio of the $P$ values is

within 2), and the form stating the same for any exponent below 2 is `dictionary_fit` (`YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean:78`) Source . The strong estimate is not put into the induction hypothesis; it is the weak ledger that is rebuilt at every step. The step-uniform conditions are discharged on the calendar side, as in "Calendar" below Inferred .

**The stamp.** Of the income from one seed, half of the guaranteed lower bound $c_E P^2/2$, namely $c_E P^2/4$, is left at the exit as a "stamp", and the rest pays for the losses of the tree expansion and the combinatorics of covers (`Target.costs`, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/OptionCost.lean:83`, $u = 1/2$) Source (11-05).

**Incoming activities.** The formula for the exit rate (the stamp and the target rate) is a function of $c_E$, $p$, $\beta$ only, and estimates of the incoming activities do not enter it Source . Estimates of the incoming activities are used on the side of the one-step hypotheses (a field of `Target.Invoices`, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Issue.lean:73`) to make the sums inside one step converge Source . Rarity is rebuilt at every step from the action difference of that step Inferred .

**Calendar.** The weak ledger is also a function of the step $m$. The step-uniform conditions are discharged on the calendar side. The lower bound on sides, the entry point of the reference coupling, the allocation of rates and so on are fixed in advance, in the order "step constants → sides → widths → ... → entry point of the coupling" Source (11-08). On the existence path, carrying the ledger from step to step is done by `Roof.Lyrics.supplier` (`YM/FullVolume/C/Songbook.lean:76`); the similarly named `Directed.holding` (`YM/PCalc/Band.lean:92`) is not used Source .

06

## Judgement — answered in a different form

We read the published construction as **answering in a different form** the point where this book got stuck (§ 01) Inferred .

### Table 1 Where this book's study got stuck, and how the published construction handles it

| Point | This book's study | The published construction |
|---|---|---|
| Large-field regions | Defined at each step, stacked up and fixed thereafter | No history; at each step seeds are re-counted from that step's flags Source |

| Point | This book's study | The published construction |
|---|---|---|
| How to split | Small-field characteristic functions | Cells by seed type, and a smooth partition of unity Source |
| Main part of the representation | Quadratic action as the main part | Action clipped to linear beyond the threshold; the difference goes to the activities Source |
| Threshold | A polylogarithmic multiple of the coupling | $\hat{g} \cdot \exp((\log\log\hat{\beta})^2)$ Source |
| Factor per seed | Of type $e^{-c/g^2}$ | Of type $\exp(-cEp^2)$. Much weaker, but the ledger $\exp(-p^{1/5})$ is weaker still Source Inferred |
| Persistent defects | How to bound them uniformly in the number of steps was open | An inequality for every configuration satisfying the section and cell conditions is set up at every step, producing the factor at every step Source Inferred |
| Step-uniform conditions | Placed in the premise fields Assumption | Discharged on the calendar side (lower bound on sides, entry point of the coupling and so on) Source |
| Instantons, vortices, topological sectors | Listed as candidates for persistent defects | No argument treating them by name was found on the path to the main theorem Source (within the range searched) |

**What can be said.** The published construction keeps no history of regions, re-counts seeds at each step, and produces the factor per seed at every step through inequality (4), stated for every configuration satisfying the gauge section and cell conditions. Clipping the main part of the representation to linear growth is consistent with the positive remainder of the inequality (intermediate coarse plaquettes).

**What cannot be said.** One cannot say that "the premise of the configurations this book feared no longer applies". What is clipped is only the main part of the representation; steps are linked by identities, and the measure does not change. The clipped difference goes to the activities. Nor can one say that "the problem of estimates uniform in the number of steps does not arise". The weak ledger is also a function of the step, and the step-uniform conditions are discharged on the calendar side. What can be written is only that it answers in the form "clip the main part of the representation, put the difference into the activities, set up an inequality for every configuration satisfying the section and cell conditions, and discharge the step-uniform conditions with the calendar".

07

## Configurations that keep exceeding the threshold

How does the published construction handle configurations of the type this book feared, for example a configuration that just exceeds the threshold of every step at every step?

The energy-per-seed inequality (4) is stated for every configuration satisfying the section and cell conditions, with no hypothesis excluding types of configuration Source . For such a configuration too, at each step the fine flags become seeds, and since there is at least one seed, the right-hand side of (4) is at least $c_E t^2$ Source (quantifiers) Inferred (application). At each step a stamp is attached, and the exit activities fit the strong form (5) and then the weak ledger (3) Inferred .

The rarity of continuing over many steps is not multiplied across steps. The rarity paid at a previous step is dropped to the weak ledger, and new rarity is extracted from the action difference at each step Inferred . Since the ledger has the same form at every step, the induction runs on that. The step-uniform conditions do not disappear; they are discharged on the calendar side ( § 05). On the action side, the coarse action term used at step $m$ is the same as the fine action term at step $m-1$, so each step only extracts the action difference of that step and nothing is counted twice — this is how we read it Inferred . We have not checked the exact form including the symmetric averaging.

08

## Clusters that return to small fields

After one step, a defect may return to small fields at the coarse step. The published construction separates clusters that contain a coarse flag ( `ThickP` ) from those that do not ( `ThinP` ) by a single local test ( `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Colony.lean:26` , :27) Source . Only clusters containing a coarse flag become large-field activities $K$ at the next step; the contribution of clusters without one goes into the terms of the small-field expansion (Ursell terms), whose value at flat configurations flows into the vacuum term $v$ and whose quadratic part flows into the correction of the coupling Source (classification and the formula for the vacuum term) Inferred (the flow) (11-06).

In this book's study, "what to do if a defect returns to small fields" was a question of case analysis; in the published construction it is settled by one test inside the procedure that rebuilds the representation at the coarse step Inferred .

09

## What we have not read, and checking dependencies

- **Dependencies.** What we confirmed by chain to lie on the path of the main theorem ( `jointExistence` ) is limited to the path of shooting and the one-step theorem (from `rodB` to the rereading), convergence in the cutoff ( `allB` and others), `sailsW` , which closes the small-field conditions, and the ultraviolet limit `realUV` . For many of the large-field lemmas cited in this chapter ( `late_force` , `due_eq` and others),

tracing them through the record of how declarations are used leads to a headline theorem other than the main theorem. This does not mean they are not on the main theorem's path; it means we have not checked.

- The whole construction that rebuilds the coarse-step activities after one step and files them into the ledger (3) (under `YM/PhysicalNext/RG_NEXT_COMPLETE_REPEATABLE_NONLINEAR_STEP/`, `V9_JOINT_ENDPOINT_LOAD`, `ANNULUS_RATIO_INSERTIONS`).
- Where the lemma lives that pays for the loss where integration stays with Haar. We read it as entering the field for the logarithmic cost per seed, but have not identified the concrete lemma.
- The whole bound on the remaining integral after the prices are extracted.
- The proofs of inequality (4) and of the positive remainder (we read only the statements).
- The map by which the values of clusters without a coarse flag enter the slots of the small-field activity ledger.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The induction hypothesis `History.Conv` has no field for a history of regions, and large-field activities live only on the flags of the current step | The whole construction that re-files activities into the ledger after one step |
| Source The main part of the representation is quadratic when the plaquette deviation is at most the threshold and linear beyond it, with the linear coefficient smaller than 1/16 of the quadratic one. Steps are linked by push-forward identities, and the measure does not change | All of how the clipped difference is distributed to the activities |
| Source The energy-per-seed inequality is stated for every configuration satisfying the gauge section and cell conditions (with hypotheses `Funds`, `SliceSet`, `Plot`) | The content of its proof (we read only the statement) |
| Source The induction hypothesis holds only the weak ledger $\exp(-p^{1/5})$, and the one-step certificate produces the strong bound $\exp(-g\,p^2)$. It is weakened by a one-line inequality | The whole part that rebuilds the gas of activities from the strong bound |
| Source Estimates of the incoming activities do not enter the formula for the exit rate. They are used on the side of the one-step hypotheses | — |
| Inferred The published construction "answers in a different form" the point where this book got stuck | One cannot say "the configurations this book feared do not occur" or "the premise no longer applies" |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Threshold, ledgers, induction hypothesis | Source | In the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0), `YM/RG/History.lean` (:81 `monitor`, :83 `threshold`, :84 `allowance`, :85 `forcedAllowance`, :213 `Conv`, :223 `Transfer`) |
| Clipped action | Source | `YM/RG/PathActions.lean` (:145 `kink`, :218 `clipped`) |
| Gas of activities | Source | `YM/RG/Activities.lean` (:98 `EBound`, :135 `gas_one`) |
| Cells and seeds | Source | `YM/Pressure/Seeds.lean` (:48 `SliceSet`, :51 `Plot`, :120 `Serves`, :124 `cash`, :129 `Grants`), `YM/Push/SECTOR_MEASURE/Cells.lean:51` (`Seeded`), `YM/Next/FLUCTUATIONS/Grants.lean:24` (`grants`) |
| Positive remainder for intermediate coarse plaquettes | Source | `YM/Bid/Energy.lean` (`Crowds` at :94–:97, :101 `E05`) |
| Conversion between the two ledgers | Source | `YM/PCalc/Band.lean` (:49 `Move`, :92 `holding`), `YM/AnalyticBridge/VERIFIED_RESET_STEP/Opening.lean:19` (`late_force`), `YM/Next/SCHEDULE/Salary.lean:178` (`saved`), `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Bulkhead.lean:78` (`dictionary_fit`), `OptionCost.lean:83` in the same directory (`costs`), `Issue.lean:73` in the same directory (`Invoices`), `YM/FullVolume/C/Songbook.lean:76` (`Roof.Lyrics.supplier`) |
| The finest step | Source | `YM/Extra/WILSON_STARTUP/Ownership.lean:68` (`due_eq`) |
| That the assessment contains no description of the large-field treatment | Assessment | Full-text search of the assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10) (the words large field, small field and polymer do not occur) |
| This book's study's side | Assumption | The "large-field estimate" and "large-field gas" in the list of premises (28-04), and the open places (28-06) |

**Terms used in this chapter** — large field, small field, threshold, clipped action, activity, gas of forced activities, flag, seed, cell, partition of unity, energy per seed, ledger, one-step certificate, stamp, calendar, persistent defect. Their meanings are in the A-02 glossary.

Next chapter: 14-05 Ideas this book's study lacked — we list 35 ideas, including the treatment in this chapter, divided into groups.

2026-11-13 · chapter Yang–Mills and the mass gap Part 14 Comparison — this book's Lean, and ideas this book's study lacked list of ideas · Kiichi, Shiori and Rin

# Ideas this book's study lacked — a list in seven groups

> This chapter is this book's own investigation. The "What it does" column of each item is established knowledge (a reading of the published construction), and the "What changes" column is inference.

**What this chapter shows** — Reading the published construction, we pick out 35 ideas that this book's study lacked and list them in seven groups. Each item is written in three columns: what it does (with its location in the source), what this book's study did instead, and what changes. At the end we place the five things that pointed in the same direction, and cautions for reading this list (what cannot be said).

**Prerequisite chapters** — 14-01 Comparing skeletons to 14-04 Large fields. Detailed explanations of each item are in the chapters of Parts 9–13.

**Said first** — The "What changes" column is an inference applied to this book's study; it has not been checked that the idea can be used as is in this book's design. Every idea works in the published construction in combination with other components, and taken out alone it need not have the same effect. Whether each item is among the dependencies of the main theorem has been checked only to the extent written in § 09.

The order of this chapter

1. The one-step design — probability kernel, projected predictor, gauge fixing in two strands, growing sides
2. The small-field step — exact identities, finite-range covariance, derivative norms, redefining the ledger
3. Large fields — clipping the representation, re-counting, two ledgers, rarity on the action side
4. The coupling — a coefficient in the representation, the shooting method, staying at weak coupling
5. How limits are taken — renormalisation as a tool for estimates, Cauchy at a fixed step, uniqueness, the root seminorm

6. The infrared and reconstruction — the trace representation, phase decomposition, Ward identities, the trace anomaly, fields as a quotient
7. How the Lean is organised — request propositions, bundling over one construction
8. What pointed in the same direction — five things
9. Cautions for reading this list — what cannot be said

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## The one-step design

Details are in 14-02 and Part 9.

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Make one step a probability kernel rather than a map | Draw the coarse links with Gaussian-type fluctuations $e^{-\beta c_{\mathrm{pen}}\,\mathrm{price}}$ around the predictor. The penalty coefficient $c_{\mathrm{pen}}$ is any positive number (`kernel.penalty` in the source's wording) (`YM/RG/Block.lean:71` `density`) | Push forward by a deterministic block map (19-04) | The constraint is no longer hard, and $c_{\mathrm{pen}}$ enters the precision of the free field of one step. In exchange, the push-forward measure becomes an object that depends on $\beta$ |
| The predictor is an explicitly built central value projected onto the group | The group element nearest to the weighted average of transports along parallel straight lines; if the spread is large, one line is drawn at random (`YM/RG/Patch.lean:159`, :171, `YM/Branch/Center.lean:72`) | Bring the average back onto the sphere (retract). The background field is the minimiser of a constrained action (28-05) | The background minimisation problem need not be solved inside the definition of one step, and the value is always in the group |

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Gauge fixing in two strands, drawn at random and summed over | Symmetric minimisation for small fields; for large fields, the root and order of an axial gauge drawn uniformly. Through the sum over labels, exactly invariant under the hypercubic group and coarse translations (`YM/RG/Patch.lean:102`–:155, `YM/Branch/Covariant.lean:50`) | Symmetric averaging over a family of paths. Gauge at each step chosen as Landau, axial or tree (28-05) | Gauge fixing is determined even for large fields, and the lattice symmetry is not broken |
| Let the side grow with the step | The side is an odd number at least 3 (`YM/Lattice/Scale.lean:28`), non-decreasing (`Fits.monotoneL`), with $\log b\, m / \log D_m \to 0$ (`Fits.mesh`, `YM/RG/History.lean:97`–:100). The one-step error is left at $O(1)$ and removed by dividing by $\log b$ | Side fixed at 2 (and also in the side-3 design) | Precise two-loop estimates become unnecessary. In exchange, the sequence of lattice spacings becomes a special one, and there is no universality with respect to the choice of sequence Assessment |

02

## The small-field step

Details are in 11-09.

| Idea | What it does (source) | In this book's study | What ch |
|---|---|---|---|
| Link steps by exact identities without approximation | The field of the one-step certificate is the push-forward identity (`ident` at `YM/PCalc/Band.lean:50`). Both the error of the Gaussian approximation and the non-Gaussian remainder are carried inside the integrand | The loss of margin in handing over was carried on paper as a loss in the estimates | They ca carried terms o integra rather t losses. T ledger s consists inequal so this c not mea estimat no approxi |

| Idea | What it does (source) | In this book's study | What ch |
|---|---|---|---|
| Replace the covariance by an approximate inverse cut off at finite distance | Invert the Gram matrix only inside a window and symmetrise; the difference from the true inverse is exponentially small in the window radius (`YM/SectorNext/GAUSSIAN_SECTOR_CHANGE/Windows.lean:184` `windows`). The difference is kept inside the identity | A covariance decomposition using contraction uniform in the step (15-12) | Indepe at a dist holds e giving a foundat expansi |
| Measure the size of activities by norms of finite-order derivatives | The supremum of finite-order derivatives in directions of size $t$ (`YM/RG/Masks.lean:164` `supBounds`) | Estimates on paper measured by an analyticity radius | Losses type wh radius s at every do not c |
| Carry agreement of the main action up to fourth order as a condition | The condition that the coarse main action and the stationary value of the fine main action plus the constraint penalty agree, at flat configurations, in their jets up to fourth order (`YM/PCalc/Chart.lean:108` `ClassicalMatch`). It is closed by `sailsW` (`YM/FullVolume/Roll/Fiscal.lean:97`). The coarse main action is data given separately | The expansion of the background field was placed on the premise side (28-04) | The remain become small e to fit in ledger Inferred |
| Redefine the ledger at every step from that step's coupling | The threshold and ledger are functions of the reference coupling $\hat{\beta}_m$ at step $m$ (`YM/RG/History.lean:83`, :84). One shows only that the exit fits into the fixed ledger of the next step | The next step's estimates were derived from the previous step's, with the risk that losses accumulate step by step | The ind hypothe carries losses. I from rereadi prepaid step co |
| Do not leave marginal terms in the activities; move them to the coupling and the vacuum term | Flat values go to the vacuum term and quadratic parts to the coupling; the small-field activity ledger is $\hat{g}^{11/5}$ (`YM/RG/History.lean:84`). The exponent lies between the marginal order 2 and the first non-marginal order 3 | — | How m orders be matc for the remain fit in th ledger c counte order |

03

## Large fields

Details are in 14-04 and Part 11.

| Idea | What it does (source) | In this book's study | Wha |
|---|---|---|---|
| Clip the main part of the representation | Represent the action of plaquettes beyond the threshold weakened to linear growth, and put the difference into the activities (`YM/RG/PathActions.lean:218`). Steps are linked by identities, and the measure does not change | Hand over the quadratic action as the main part | It b foot sett ever inec con sati sect con |
| Keep no history of regions; re-count at each step | The induction hypothesis has no history field, and seeds are re-counted at each step (`YM/RG/History.lean:213`) | Large-field regions stacked up and fixed thereafter | Pers defe trea step step |
| Keep two ledgers | A weak ledger $\exp(-p^{1/5})$ in the induction hypothesis and a strong bound $\exp(-gp^2)$ in the one-step certificate (`YM/PCalc/Band.lean:49`), linked by a one-line inequality (`YM/Next/SCHEDULE/Salary.lean:178`) | Thought in terms of one ledger (of type $e^{-c/g^2}$) | The esti put indu hyp wea rebu step unif con disc the side |
| Put rarity on the action side rather than in the activities | Keep the clipped action explicitly in the representation (`YM/RG/PathActions.lean:218`), and make the activities factors carrying the difference. Rarity is extracted from the action difference at each step Inferred (line of reading in 11-06, 14-04 § 07) | Rarity carried over as estimates on activities | No a for is n |
| Split clusters by whether they contain a coarse flag | Clusters containing one become large-field activities at the next step; those without go to the small-field side (`ThickP`, `ThinP`, `YM/AnalyticNext/CONNECTED_ACTIVITY_JETS_E6/Colony.lean:26`, :27) | A separate case analysis for defects returning to small fields | One suff |
| Use $\exp((\log\log\beta)^2)$ in the threshold | Threshold $t = \hat{g}\,p$, $p = \exp((\log\log\hat{\beta})^2)$ (`YM/RG/History.lean:81`, :83) | A threshold that is a polylogarithmic multiple of the coupling | Poly loss over with cou pow |

| Idea | What it does (source) | In this book's study | Wha |
|---|---|---|---|
| Cells with hysteresis and a smooth partition of unity | Cells for each seed type glued by a partition of unity; the conditions are imposed globally while the estimates are localised around seeds (`YM/Push/SECTOR_MEASURE/Cells.lean:51`, `YM/Pressure/Seeds.lean:51`) | Cut with small-field characteristic functions | Disc at th bou avoi eacl esti inde |
| Locate the point of taking large sides in the coercivity constant | First-order quantities of fine plaquettes shrink onto coarse plaquettes with weight $L^{-2}$, so taking the side large lets the fine side win Inferred (`E03`, `E06`, `E07` in `YM/Bid/Energy.lean`, :67, :110, :130). The lower bound on sides is a calendar field (`YM/Price/Intent.lean:39` `Large`) | The point of taking large sides was the gain in coupling | The side proc con larg inec Infe |

04

## The coupling

Details are in 14-03 and Part 10.

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Define the coupling as a coefficient in the representation, and reread it | The step coupling is the coefficient of $F^2$ in the representation, with two normalisations forbidding reassignment (docstring at the top of `YM/RG/History.lean`). It is reread from the log-ratio of the density at two configurations and monotonicity (`YM/UVNext/COUPLING_RESET/Reset.lean:50`) | Match with a quantity determined by the measure alone (19-07). No representation | One can follow directly in the representation how the coupling moves by its one-loop value at each step |
| The shooting method | Build nested intervals of bands at each step and choose the bare coupling by the intermediate value theorem (`YM/FullVolume/Arc.lean:95`). Only continuity and the one-step error inequality are needed | The intermediate value theorem from global endpoints ($\beta = 0$ and $\beta \to \infty$) | One can match even quantities whose endpoint values are unknown. Under the calendar condition that the sum of one-step errors fits within a small fraction of the band width, it works in the same form regardless of the number of steps |

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Carry every step at weak coupling | $\beta_m \in [\hat{\beta}_m/2, 2\hat{\beta}_m]$ at every step (field `band` at `YM/RG/History.lean:213`). The reference coupling is taken large and one stops at the step with spacing 1 | Thought of weak to strong coupling as one flow | The strong-coupling regime is not passed through by renormalisation. Beyond that (long distances) is handled by compactness |

05

## How limits are taken

Details are in the second half of Part 11 and in Part 12.

| Idea | What it does (source) | In this book's study | What chang |
|---|---|---|---|
| Renormalisation is a tool for estimates, not the object of the limit | What is taken to the limit are correlations of renormalised local fields under the measure of the original lattice Assessment (Assessment § 4.3). Reflection positivity is used only on the original lattice (`YM/Slice/Reflection.lean:98`) | Take the limit of the push-forward measures themselves (28-01) | Reflection positivity c the pushec forward measure is needed (14 § 07) |
| Show Cauchy at a fixed coarse step first, then remove the step | The effective density at a fixed step is Cauchy uniformly in the cutoff (`YM/FullVolume/Fields/Cement.lean:50`); bound the difference between the true correlations and the read-out at step $j$ by a quantity independent of $N$ (`Sever.echoes`, `YM/FullVolume/Fields/Peat.lean:42`); conclude that the true lattice correlations form a Cauchy sequence in $N$ (`Rising.arrives`, `YM/FullVolume/Fields/Timber.lean:74`). The limiting object itself is not built | Subsequential limits | Subsequen become unnecessa Convergen follows by continuou operations from the ingredient the representa being Cauc |
| Derive convergence of the ingredients from uniqueness rather than quantitative estimates | The coupling is anchored at step 0, and the activities shrink by 1/16 per step. There is only one orbit satisfying the conditions in both directions. Compactness and the subsequence principle give convergence of the full sequence (`YM/FullVolume/Glide/Triumph.lean:37`, `YM/FullVolume/Glide/Coincide.lean:58`). No rate comes out | Had an implication of the same type for orbits of an abstract skew product (28-02), but the map was not defined from the lattice | No quantitativ estimate comparing cutoffs $N$ a $N'$ is need |

| Idea | What it does (source) | In this book's study | What chang |
|---|---|---|---|
| Differences at the fine end need only be bounded | Differences in the initial values of the activities are forgotten by $(1/16)^k$ as one goes down the steps (`YM/FullVolume/Glide/Wings.lean:59` `culprit`) | It existed in abstract form: in the implication for skew-product orbits, the limit does not depend on the choice of bare irrelevant coordinates (`SkewOrbit.limit_independent`, 28-02). The map was not defined from the lattice | The representa at the fines step need r be made sr |
| Make the one-step ledger independent of the volume | The ledger is a sum per point (`YM/RG/Activities.lean:98` `EBound`), and the contraction constants do not depend on the volume. Estimates at the level of densities and partition functions contain exponentials of the volume; the uniform estimates handed to infinite volume are built, with the volume-uniform ultraviolet estimates (`Wcmain` and others) as input, from reflection positivity on the original lattice, the chessboard and the transfer-operator representation Assessment (Assessment § 5) | Forms uniform in volume and step were placed inside the premise statements (28-04) | The divisic of labour between estimates becomes c |
| Give the renormalisation of composite fields by the inverse of the renormalisation transport | Have at each step a linear isomorphism on a finite-dimensional space of coefficients (with uniformly bounded inverse), and carry from the reference step by the inverse (`Club` at `YM/FullVolume/Mark/Nursery.lean:11`, `YM/FullVolume/Mark/Sages.lean:66` `spends`) | The endpoint is the coarse-lattice measure; no calibration of observables | Correlation of local fie can be the endpoint |
| State the estimates handed to compactness in the root seminorm of the chessboard | The size of renormalised observables in the root seminorm is at most $A\, a_m^{-d-\varepsilon}$, uniformly in the volume (`YM/Extraction/Center.lean:38` `Wcmain`, `YM/Cells/Tiling.lean:226` `rootTest`) | The chessboard is used for rarity of large plaquettes (22-01) | One of the estimates handed fro the ultravi to the infra can be wri as a single volume-uniform statement (there are other inpu from the ultraviolet 03) |

```
06
```

## The infrared and reconstruction

Details are in Parts 12 and 13.

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Obtain equicontinuity from the trace of the transfer operator | Write expectations as traces of the transfer operator, and bound the movement of insertions as reassigning gaps, by Hölder. Arzelà–Ascoli applied to correlation functions rather than measures (`YM/Extraction/Lipschitz.lean:166` `first_surface`) | Compactness in weak convergence of measures | Uniform estimates with derivatives on correlations at separated points are obtained |
| Without proving clustering, split into phases and then reconstruct | Split into phases by the Gelfand representation of a commutative algebra of operators compressed to the vacuum space, and obtain uniqueness of the vacuum in each component (`YM/Phase/Mixture.lean:127`, `YM/EveryPhase/Vacuum.lean:146`). Commutativity comes from Euclidean invariance | Prove clustering in the strong-coupling window (20-03) | One can get as far as reconstruction without proving uniqueness or decay |
| Lift rotational invariance from the Ward identities of the stress tensor | Ward identities for translations and rotations (`translation`, `rotation` at `YM/Euclid/Stress.lean:90`–:97). That rotational invariance is not spontaneously broken in the limiting state is derived, from the stress flux vanishing fast far away, by an argument using reflection positivity in several orthogonal planes Assessment (Assessment § 6.3) | Not treated | One route from the discrete lattice symmetry to continuous rotational invariance is fixed |
| State non-freeness by the logarithm of the trace anomaly | The short-distance law for the two-point function of the trace, the limit of $r^8 \log^2(1/r)$ (`ChallengeSpec.lean` :145–149 and :173–174) | Not treated | "Non-trivial" can be stated by the value of a single limit. The types of free field excluded are limited (13-02) |
| Define the field spaces as a quotient by the null space of correlations | Quotient by the part where all correlations at separated points vanish (`YM/Euclid/Null.lean:34` `null`) | Not treated | A form that could also be used to define continuum objects for non-local observables (loops, twists) Inferred |

07

## How the Lean is organised

| Idea | What it does (source) | In this book's study | What changes |
|---|---|---|---|
| Conditional theorems with request propositions as hypotheses, all closed at the end | Intermediate theorems are written in conditional form taking a request proposition (`Bid`) as hypothesis, the proofs (`Work`) are placed separately, and the theorem above hands over all the answers (`YM/RG17/Joint.lean:83`–:85) | Implications with hypothesis fields (28-01, 28-02). The same type | The difference is that at the end all requests are closed |
| Bundle all conclusions over one construction chosen once | To avoid the accident that objects whose existence was shown separately are incompatible, the choices are gathered into one run (docstring at the top of `YM/Org/Input.lean`, around :14) | Objects were taken separately for each theorem | The types guarantee that the conclusions are about the same objects |

08

## What pointed in the same direction

1. Fixing the scale by matching a quantity determined at unit length to a fixed value (14-03).
2. Odd sides, with blocks centred on coarse points. Symmetric averaging (14-02).
3. Fixing the volume first, then going to infinite volume by compactness (14-01).
4. Using reflection positivity and the chessboard as the source of volume-uniform estimates (14-01).
5. The one-loop coefficient. With conventions matched, it agrees with the value for one side-2 step in this book Inferred (14-03 § 07).

09

## Cautions for reading this list

Please do not reread the items of the list in the following ways. Each would say more than the statements in the source.

- One cannot say "because the representation is clipped, the premise of persistent defects no longer applies". Steps are linked by identities, and the measure does not change. What can be said is only that the main part of the representation is clipped, the difference is put into the activities, and an inequality is set up for every configuration satisfying the section and cell conditions.
- One cannot say "the problem of estimates uniform in the number of steps does not arise". The weak ledger is also a function of the step, and the step-uniform conditions are discharged on the calendar side.

- One cannot say "the smallness of incoming activities is not used". It does not enter the formula for the exit rate, but estimates of incoming activities are used on the side of the one-step hypotheses to make the sums converge.
- One cannot say "the coarse main action is determined as the expansion of a stationary value". Agreement up to fourth order is carried as a condition, and the coarse main action is data given separately.
- One cannot say "volume-uniform estimates come only from reflection positivity on the original lattice". The one-step ledger has a volume-independent form, and the input from the ultraviolet to infinite volume is not just one.
- One can say "used by the main theorem" only for declarations whose chain to the main theorem has been checked. What this book's study checked are the path of shooting and the one-step theorem (from `rodB` at `FullVolume/Stage/Rods.lean:26`), convergence in the cutoff (`allB` and others), `sailsW` and `realUV`. Many of the large-field lemmas are tied in the record of how declarations are used to other headline theorems, and we have not checked whether they lie on the path of the main theorem.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Source The "What it does" of each item comes from reading the definition or statement of the declaration at its location | The content of the proofs. Whether they are among the dependencies of the main theorem has been checked only in part |
| The columns on this book's study's side are based on the theorems and the list of premises of Parts 15, 19, 22 and 28 | — |
| Inferred The "What changes" column | That the idea can be used as is in this book's design |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Locations of each item | Source | Repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0). Paths and lines are in each row of the tables; the lines were checked by this book's study by opening the source |
| Definition of the correlations, order of limits, absence of universality | Assessment | The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), §4.3, §5, §6 |

| Item | Kind | Source / tool |
|---|---|---|
| This book's study's side | — | The chapters cited in each row (19-04, 15-12, 22-01, 28-01, 28-02, 28-04, 28-05, 28-06 and others) |

**Terms used in this chapter** — probability kernel, predictor, projection onto the group, gauge fixing in two strands, growing sides, finite-range approximate inverse, finite-order derivative norm, ledger, clipped action, seed, stamp, calendar, shooting method, anchor, forgetting of the upper end, root seminorm, phase decomposition, request proposition. Their meanings are in the A-02 glossary.

Next chapter: 14-06 What this book has and the published construction does not — the comparison in the other direction, and a bridge to Volume 3.

2026-11-13 · chapter Yang–Mills and the mass gap Part 14 Comparison — this book's Lean, and ideas this book's study lacked only in this book · Kiichi, Shiori and Rin

# What this book has and the published construction does not — a bridge to Volume 3

> This chapter is this book's own investigation. Descriptions of the published construction are established knowledge (a reading of the published construction), and § 09 is a map of open questions.

**What this chapter shows** — We list statements of kinds not among the claims of the published construction that are closed in the Lean of this book's study, in four groups (the strong-coupling lattice, centre vortices, rarity of large plaquettes at weak coupling, the fermion weight). We list separately the three things that can be called specific to this book's renormalisation design. We tabulate where each stands relative to the open questions of the published construction (convergence and uniqueness, the mass gap, Wilson loops and the area law), and at the end give an outline for going from where the published construction stops (the step with spacing 1, weak coupling) on to Volume 3.

**Prerequisite chapters** — 14-01 Comparing skeletons, 6-04 what has not been shown, the chapters of Parts 20–22.

**Said first** — Every theorem in this chapter is a formalisation of a result of a known type. The lattice spacing is fixed at 1, and nothing is said about the continuum limit. Most are statements inside a strong-coupling window: $18 \tanh |\beta| < 1$ ($|\beta| < 0.0556$ Reading ), the 441 version of the area law and vortex free energy ($\beta < 1.135 \cdot 10^{-3}$ Reading ), and the 4096 version ($\beta < 1.22 \cdot 10^{-4}$ Reading ). They are statements at the opposite end (strong coupling, fixed lattice spacing) from the questions the published construction left open; they neither answer those questions nor point towards them. What can be said is only that we have statements of the same kind of question, at a different place on the lattice.

The order of this chapter

1. Conventions of this chapter — known types, fixed lattice spacing, nothing said about the continuum limit
2. The strong-coupling lattice — infinite-volume limit, mass gap, clustering

3. The area law at strong coupling, and an upper bound on the vortex free energy — inside the window, the rate per area does not depend on the lattice size (the prefactor remains)
4. Centre vortices — the thin-vortex floor, the thick-vortex floor, pairs of Polyakov loops, the vortex ratio
5. Rarity of large plaquettes at weak coupling — uniform in volume, explicit threshold
6. The fermion weight — reflection positivity, positive partition function, symmetries
7. What is specific to this book's design — at most three. What neither has
8. Relation to the open questions (Table 1) — the same kind of question, at the opposite end
9. A bridge to Volume 3 — how to turn what lies beyond the step with spacing 1 into questions

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Labels: Source = we read the source of the published construction; Inferred = this book's inference; Assessment = from the committee's assessment. The other labels (Proved here, Cited, Reading, Lean and so on) mean the same as in the rest of the book (details in "Labels" in 6-00). For Lean theorems of this book's study, the name and scope are written on the same line, outside the labels.

01

## Conventions of this chapter

The assessment lists five open questions as the main points the published construction does not treat Assessment (Assessment § 8).

1. Prove convergence rather than subsequences, and uniqueness of the infinite-volume limit.
2. Define the probability measure of the continuum theory.
3. Prove a uniform mass gap (exponential decay of correlations and a spectral gap of the Osterwalder–Schrader Hamiltonian).
4. Construct Wilson-loop observables and relate them to local observables. Prove the area law for Wilson loops.
5. Construct one-particle states and study the energy–momentum spectrum.

The Lean of this book's study has statements of the same kind as 1, 3 and 4 of these. But their place differs in three respects.

- **Known types.** Every theorem is a formalisation of a result of a type known in lattice gauge theory (strong-coupling expansion, the Osterwalder–Seiler area law, 't Hooft's twists and the vortex free energy, the chessboard Peierls argument)Cited .
- **Fixed lattice spacing.** Each is a statement on a single lattice with lattice spacing fixed at 1, or about its infinite-volume limit. Nothing is said about the continuum limit.
- **Coupling regime.** Most are statements inside a strong-coupling window. The window differs from theorem to theorem: $18\tanh|\beta| < 1$ ($|\beta| < 0.0556$Reading ), $24(1 - e^{-4|\beta|}) < 1$ ($|\beta| < 0.0106$Reading ), the 441 version of the area law and vortex free energy $441(1 - e^{-2\beta}) < 1$ ($\beta < 1.135 \cdot 10^{-3}$Reading ), and the 4096 version ($\beta < 1.22 \cdot 10^{-4}$Reading ). The published construction carries every step at weak coupling, so the two do not overlap in the coupling. These are statements at the opposite end from the questions the published construction left open.

02

## The strong-coupling lattice

For the SU(2) Wilson measure on the periodic torus $(\mathbb{Z}/N)^4$, the following are in Lean in strong-coupling windows (Part 20).

- **Infinite-volume limit (without subsequences).** For $18\tanh|\beta| < 1$, the sequence of torus measures lifted periodically converges weakly, without taking subsequences, to a single probability measure $\nu_\beta$. The limit is the same for every sequence of sizes. Lean: `InfVol.wilson_infvol_unique`, `AllN.wilson_infvol_allN` (20-03).
- **Translation invariance and clustering.** The limit is invariant under unit translations in the four directions, and correlations of distant observables decay exponentially. Lean: `TransI.wilson_infvol_translation`, `WilsonLimit.cluster_limit`.
- **Mass gap.** The transfer operator $T$ on the reconstructed Hilbert space satisfies $\|T\| \le e^{-m}$ ($m > 0$) on the orthogonal complement of the vacuum, and that complement is non-zero (non-degeneracy). The form for some limit point is in the Dobrushin window (`WilsonLimit.wilson_massGap_dob_exists`), and the form for all limit points is in the window $24(1 - e^{-4|\beta|}) < 1$ (`WilsonLimit.wilson_massGap_nondegenerate`). There are also a form for the gauge-invariant sector and a statement combining everything (`GaugeZ.wilson_massGap_gauge_exists`, `InfMain2.wilson_infvol_main2`, 20-04).

What is not said: outside the window, other boundary conditions, finiteness of the mass (the bound $\|T\| \le e^{-m}$ is only a lower bound on the mass), a mass gap in the continuum limit.

03

## The area law at strong coupling, and an upper bound on the vortex free energy

**Area law (finite box).** In SU(2) Wilson lattice gauge theory on $(\mathbb{Z}/N)^4$, when $0 \le \beta$, $q = 1 - e^{-2\beta}$, $4096q < 1$ and $1 \le R, T < N$, the Wilson loop $W$ on the boundary of the $R \times T$ rectangle in the $(0,1)$ plane with a corner at the origin satisfies

$$\left|\langle \tfrac{1}{2}\operatorname{tr} W\rangle_\beta\right| \;\le\; \frac{16\,(4096\,q)^{R\cdot\min(T,N-T)}}{4096\,(1-4096\,q)} \quad (1)$$

The constants do not depend on $N, R, T$. Lean: `AreaLaw.area_law_wilson` (the form with exponent $RT$ for $2T \le N$ is `area_law_wilson_area`). Rectangles with any corner and orientation are `area_law_wilson_general`. The versions with constants tightened to 6 and 441 (`area_law_wilson_sharp`, `area_law_wilson_max_sharp`) hold in the window $441q < 1$ ($\beta < 1.135 \cdot 10^{-3}$ Reading), with exponent $\max(R \cdot \min(T, N-T),\, T \cdot \min(R, N-R))$.

**Area law (infinite volume).** Under the three conditions $\beta > 0$, $18 \tanh \beta < 1$ (the Dobrushin condition) and $4096q < 1$, in the strong-coupling infinite-volume limit $\nu\beta$, for rectangles in the $(0,1)$ plane with a corner at the origin and all $R, T \ge 1$,

$$\left|\langle \tfrac{1}{2}\operatorname{tr} W(\partial(R\times T))\rangle_{\nu\beta}\right| \;\le\; \frac{16\,(4096\,q)^{RT}}{4096\,(1-4096\,q)} \quad (2)$$

holds. Lean: `AreaLaw.area_law_infvol` (at every limit point, `area_law_limit_sub`, 20-06). Both are formalisations of a known type (Osterwalder–Seiler) Cited .

**Upper bound on the vortex free energy.** On $(\mathbb{Z}/N)^2 \times (\mathbb{Z}/M)^2$, for the twisted partition functions $Z_0, Z_1$ (19-02), for every background, when $\beta \ge 0$ and $441q < 1$,

$$\frac{|Z_0 - Z_1|}{Z_0 + Z_1} \;\le\; \frac{N^2\,(441\,q)^{M^2}}{441\,(1-441\,q)} \quad (3)$$

holds. Lean: `VortexStrongMN.vortex_floor_MN_sharp` (the only hypotheses are $\beta \ge 0$, $441q < 1$ and the choice of background). The forms for the isotropic torus $(\mathbb{Z}/N)^4$ are `VortexStrong.vortex_floor` (hypotheses $\beta \ge 0$, $4096q < 1$, constant 4096 on the right-hand side, exponent $N^2$) and `VortexStrong.vortex_floor_sharp` (the 441 version).

Inside the strong-coupling window, the rate per area $-\log(441q)$ does not depend on $N, M$. The prefactor $N^2$ (the transverse area) remains. Outside the window there is no form uniform in the transverse size. The sign side, $Z_1 \le Z_0$, is not stated.

04

## Centre vortices

For the vortex ratio $\theta_M = (Z_0 - Z_1)/(Z_0 + Z_1)$ and its relatives, the following are in Lean (Part 21). None uses the block map; they use only the fine-lattice Wilson measure and the twisted partition functions.

| Item | Content | Lean and scope |
|---|---|---|
| Thin-vortex floor | $\theta_M \le \tanh(N^2\lvert\beta\rvert)^{M^2}$ (also the two-sided form) | `TwistedWilson.twisted_vortex_bound`, `_abs`. All real $\beta$, no hypothesis fields (21-02). The rate depends on $N$ |
| Thick-vortex floor | For each sufficiently large $\beta$ and sufficiently large $M$, a floor with the minimal vortex action $S_N$ in the exponent | `TwistBlocks.thick_vortex_floor` (no hypothesis fields, 21-03). The value of $S_N$ and the thresholds are not explicit |
| Minimal vortex action | $0 < S_N \le 2N^2$. $S_N < 2N^2$ for $N = 2, 3, 4$ | `VortexPos.SN_pos`, `TwistBlocks.SN_bounds`; for $N = 2, 3, 4$ the computer certificates `VortexCert.SN_two_lt`, `SN_three_lt`, `SN_four_lt` (21-05) |
| Sign of $\theta$ | If the vertical size $M$ is even, $Z_1 \le Z_0$ for all real $\beta$ | `HalfSplit.Z1_le_Z0`, `TwistBlocks.theta_nonneg` (hypothesis: $M$ even, 21-06) |
| Pairs of Polyakov loops | $\lvert\langle P_b(0)P_{b'}(T)\rangle_0\rvert \le 2\tanh(N^2\lvert\beta\rvert)^{M\min(T,M-T)}$ $(0 \le T \le M)$ and others | `PolyakovT.polyakov_pair_all` (hypothesis $M \equiv 0 \pmod 4$, all real $\beta$), `HalfSplit.polyakov_pair_theta`. No tension uniform in $N$ is stated |
| The vortex ratio — continuity, limit, matching | $\theta_M$ is continuous, $\theta_M(0) = 0$, tends to 1 as $\beta \to \infty$, and has a largest solution for every $\kappa_0 \in [0, 1)$ | `ThetaTuning.thetaM_continuous`, `ThetaLimit.tendsto_theta_one`, `ThetaLimit.tuning_all` (21-07) |
| Lower bound on the defect of a twisted box | A box with twists in two orthogonal planes has no flat configuration, and the defect of a symmetric box is at least $1/2$ independently of $N$ | `TwistFlat.no_flat_twist11`, `TwistFloor.defect11_sym_half` (21-07) |

What is not said: the continuum limit, rates uniform in $N$ outside the strong-coupling window, confinement itself. That a twisted box has no flat connection is of a known type Cited .

05

### Rarity of large plaquettes at weak coupling

For the Wilson measure on a torus of even side, when $\beta$ is large (weak coupling), the probability that $\operatorname{Re} U_p < 0$ on every plaquette of a set $X$ of plaquettes is at most $q_1^{\lfloor |X|/6 \rfloor}$, where $q_1 < 1$ does not depend on $X$ or the size of the torus. Lean: `FoldCap.rarity_explicit` (explicit threshold); the form with rate $e^{-c\beta}$ is `FoldQuarter.rarity_uniform4` (22-01). A formalisation of the chessboard Peierls argument from reflection positivity Cited .

This is an estimate of the type that controls large plaquettes (large fields) at one scale, and it stands in a position facing this book's study's large-field question (14-04, the large-field estimate in 28-04) Inferred . It is not a statement of the type that gives uniqueness. What is not said: measures at coarse scales, a form spanning steps, a form uniform in the lattice spacing. The threshold is a value convenient for the proof, far from the band of $\beta$ of physical interest.

06

## The fermion weight

For the lattice measure with fermions, weighted by the determinant of the Wilson–Dirac operator, Lean has reflection positivity in lattice-point planes (`WilsonGauge.site_reflection_positive_wilson_full`), positivity of the partition function (`WilsonGauge.partition_pos_wilson_full`), and invariance under generators of the lattice symmetries (`CubicSym.wfProb_lattice_generators` and others) (20-05). The published construction is pure Yang–Mills (without fermions) Assessment . What is not said: the continuum limit, chiral symmetry, mass renormalisation, a mass gap with fermions.

07

## What is specific to this book's design

On the renormalisation side, at most the following three can be called specific to this book's design. All are statements at the level of design.

1. **A fixed block map independent of the cutoff.** This book's block map is a single map depending on neither the coupling nor the cutoff, and consistency between steps is a push-forward identity (for any family of fine measures, pushing one step at a time or all at once gives the same measure). Lean: `ChainBlockMap.hcons_blockMap` (19-04). It is an identity coming from composition of maps, a statement of thin content. The published construction's kernel depends on $\beta(N, p, m+1)$, and the map changes with the cutoff Source (14-02).

2. **Reflection positivity of the pushed-forward measure.** Reflection positivity of the Wilson measure pushed forward by the side-3, centred, uniform map, on periodic lattices with $L$ even, for reflection in a lattice-point plane, in the class of bounded measurable functions depending only on links strictly between two planes of coarse lattice points (gauge invariance not imposed; all real $\beta$; no version for planes through link midpoints). Lean: `Block3.rp_push` (14-02 § 06). The published construction's statement of reflection positivity concerns only the original Wilson measure Source (`YM/Slice/Reflection.lean:98`, within the range of a search by name).
3. **Matching by quantities determined by the measure.** Existence of sequences of couplings matching to a fixed value the expectation of the average of coarse plaquettes under the push-forward measure (`Block3.tuned3_exists`) or the vortex ratio (`ThetaLimit.tuning_all`) (14-03). The hypotheses of `tuned3_exists` are only design values and arithmetic ($0 < r_0 \le 1$, $3^{m0} L \ge 2$, target $\kappa_0 \in [0, 1)$); `tuning_all` speaks lattice by lattice (with $N, M$ fixed) about $\kappa_0 \in [0, 1)$. Neither states monotonicity, divergence or boundedness of the sequence of couplings, or uniqueness of the solution. In the current design (side 2), existence of a matched sequence is conditional. The published construction's matching uses a coefficient in the representation, so they differ already at the level of design choices.

**What both have, or neither has.** We do not write the following three as "only in this book".

- A limiting measure on a fixed coarse lattice. In this book's study it exists as a subsequential limit (28-01), and for the published construction's kernel it would presumably follow from the uniform convergence of densities at a fixed step (`Flux.fanned`, `YM/FullVolume/Fields/Cement.lean:50`) Inferred. The published construction merely has no named statement for it Source (within the range of a search by name).
- The second open question of the assessment is the probability measure of the continuum theory, and a family of measures on a fixed coarse lattice is not that Assessment.
- Neither has continuum Wilson loops. This book's area law is a statement about lattice Wilson loops.

08

## Relation to the open questions

The infrared side of the published construction is closed with upper bounds only Source. Infinite volume is taken along subsequences, and no statement of exponential

decay of long-distance correlations was found Source (within the range of a search by name). What there is at long distances are polynomial upper bounds uniform over the phases (spectral weight at energies $(0, u]$ of order $u^2$, `YM/EveryPhase/Light.lean:221`; an estimate $\tau^{-2}$ in heat time $\tau$, `YM/EveryPhase/Decay.lean:173`) Source . This form is also compatible with a massless theory Inferred . The continuum correlations themselves exist at all separated points Assessment (Assessment § 5). Parallel transport along lattice paths (`travel` at `YM/Lattice/Path.lean:105`) exists, but no continuum Wilson-loop observables were found Source (within the range of a search by name). The boundary conditions are periodic only (`YM/Wilson/Law.lean:87`) Source .

**Table 1 Items of this book's study and their relation to the open questions of the published construction**

| Item of this book's study | Open question of the same kind | What is the same | What differs |
|---|---|---|---|
| Strong-coupling infinite-volume limit (without subsequences) | 1. Convergence and uniqueness | The sequence of periodic lifts has a single limit without subsequences | Opposite end (strong-coupling window, lattice spacing 1). The published construction's limit is of continuum correlations coming from the weak-coupling ultraviolet |
| Strong-coupling mass gap, clustering | 3. Uniform mass gap | A gap of the transfer operator, and exponential decay of correlations | As above. Nothing about a gap in the continuum limit |
| Strong-coupling area law | 4. Wilson loops and the area law | Expectations of lattice Wilson loops are exponentially small in the area | Lattice Wilson loops; the relation to local observables is not treated |
| Upper bound on the vortex free energy, thin-vortex floor, thick-vortex floor, pairs of Polyakov loops, the vortex ratio | 4 (another way of stating confinement) | Twisted boundary conditions and centre vortices Source (not found in the published construction, within the range of a search by name) | The rate is independent of the transverse size only inside the strong-coupling window (the prefactor remains) |

| Item of this book's study | Open question of the same kind | What is the same | What differs |
|---|---|---|---|
| Rarity of large plaquettes | No direct relation | — | A one-scale estimate of the type that controls large fields |
| Fermion weight | (Outside pure Yang–Mills) | — | Only properties of the vessel. Nothing about a mass gap or the continuum limit |

"Of the same kind" in Table 1 means that we hold the same kind of question at the opposite end of the lattice. It does not mean that a theorem of this book's study answers an open question of the published construction, or points towards one.

09

## A bridge to Volume 3

This section is an outline, and includes inference.

**Where the published construction stops.** The published construction carries every step at weak coupling, $\beta_m \in [\hat{\beta}_m/2, 2\hat{\beta}_m]$, and stops at the step with lattice spacing 1 (step 0) Source . The coupling at step 0 is the large reference value $\bar{\beta}$, and the volume $L_p$ is measured in that unit Source . For distances longer than spacing 1, no statements of exponential decay of correlations, an area law or uniqueness of the infinite-volume limit were found within what we read Source (within the range of a search by name). What there is are the polynomial upper bounds of § 08. The lengths at which confinement and a mass gap become visible lie further beyond Inferred .

**A candidate restatement of the remaining questions.** One conceivable route is to restate the remaining questions (mass gap, area law, uniqueness) as the question of "following the effective theory at step 0 (with large coupling $\bar{\beta}$) to long distances, from weak coupling to strong coupling" Inferred . What this book's study has called "the crossover band" (Part 26) lies in between. However, this cannot be asserted. The theory at step 0 is not a lattice theory with the Wilson action but an effective action with a representation (clipped action, activities, gas), and the continuum correlations are not a function of the step-0 measure alone Source .

**Statements missing for the restatement.** Putting this restatement in the form of statements needs at least five statements not found in the published construction Source (within the range of a search by name).

1. Existence and a name for the limiting object, as the cutoff $N \to \infty$, of the push-forward densities at step 0 (uniform Cauchy for the densities exists: `Flux.fanned`;

the limit would presumably follow Inferred , but there is no named statement).

2. Reflection positivity of the push-forward measure at step 0 (or at any step) (not trivial because the patches of the kernel straddle the plane Inferred ). In this book's design, `Block3.rp_push` has a corresponding statement for this book's map, but not for the published construction's kernel.
3. The relation between Wilson loops of coarse configurations and Wilson loops of the fine (or continuum) field.
4. Local estimates at step 0 uniform in the volume (estimates at the level of densities for step-0 objects contain exponentials of the volume Source ).
5. The images of observables at step 0, and their limits.

**Where this book's holdings stand.** Many of the Lean theorems of this book's study lie at the strong-coupling end of this gap (the mass gap, clustering, the area law, and the vortex free energy whose rate is independent of the transverse size inside the strong-coupling window). What holds at every coupling is the thin-vortex floor, whose rate depends on the transverse size. The gap in between remains open. Volume 3 starts from the infrared edge of the published construction (16-03) and this book's position (16-04), and writes about this gap in the form of conditional statements and the tree of premises (Parts 27 and 28).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| The Lean of this book's study: in strong-coupling windows, an infinite-volume limit without subsequences, a mass gap (with non-degeneracy), clustering (Part 20). Formalisations of known types | Outside the windows, varying the lattice spacing, a mass gap in the continuum limit |
| The Lean of this book's study: the strong-coupling area law (finite box and infinite volume), an upper bound on the vortex free energy (inside the strong-coupling window, the rate per area does not depend on $N$, $M$; the prefactor $N^2$ remains). Formalisations of known types | Uniform forms outside the window, the sign side $Z_1 \leq Z_0$, weak coupling |
| The Lean of this book's study: the thin-vortex floor (all real $\beta$), the thick-vortex floor, pairs of Polyakov loops, continuity and matching of the vortex ratio (Part 21) | Rates uniform in the transverse size outside the strong-coupling window, confinement, the continuum limit |
| The Lean of this book's study: rarity of large plaquettes at weak coupling ($\beta$ large), uniform in volume, explicit threshold (22-01) | Measures at coarse scales, a form spanning steps, a form uniform in the lattice spacing, uniqueness |

| Can say | Cannot say |
|---|---|
| The Lean of this book's study: three items specific to the design (consistency of push-forward by a fixed block map, reflection positivity of the pushed-forward measure, matching by quantities determined by the measure) | All are statements at the level of design and say nothing about the content of the continuum limit |
| Source (within the range of a search by name) No statement of exponential decay of long-distance correlations, no continuum Wilson loops and no boundary conditions other than periodic were found in the published construction. What there is at long distances are polynomial upper bounds uniform over the phases | We have not searched the whole repository other than by name |
| Inferred One conceivable route is to restate the remaining questions as "long-distance questions for the effective theory at step 0" | At least five statements needed for that restatement are missing (§ 09). The theory at step 0 is not the Wilson action |

## Sources and reproduction

| Item | Kind | Source / tool |
|---|---|---|
| Theorems for the strong-coupling lattice | Lean | The tables of sources in 20-02, 20-03, 20-04 (`InfVol.wilson_infvol_unique`, `WilsonLimit.cluster_limit`, `WilsonLimit.wilson_massGap_nondegenerate`, `WilsonLimit.wilson_massGap_dob_exists`, `InfMain2.wilson_infvol_main2` and others) |
| Strong-coupling area law | Lean | The Lean bundle of this book's study (`AreaLaw.area_law_wilson`, `area_law_wilson_area`, `area_law_wilson_general`, `area_law_wilson_sharp`, `area_law_wilson_max_sharp`, `area_law_infvol`, `area_law_limit_sub`). Passed the kernel recheck and checked in the source. Details in 20-06 |
| Upper bound on the vortex free energy | Lean | `VortexStrong.vortex_floor`, `VortexStrong.vortex_floor_sharp` (isotropic torus), `VortexStrongMN.vortex_floor_MN_sharp` (width $N$, height $M$). 20-06 |
| Theorems on centre vortices | Lean | The tables of sources in 21-02, 21-03, 21-05, 21-06 and 21-07 (`VortexCert.SN_two_lt`, `SN_three_lt`, `SN_four_lt` are computer certificates) |
| Rarity of large plaquettes | Lean | The table of sources in 22-01 |
| Fermion weight | Lean | The table of sources in 20-05 |
| Three items specific to the design | Lean | `ChainBlockMap.hcons_blockMap` (19-04), `Block3.rp_push`, `Block3.tuned3_exists` (14-02, 14-03), `ThetaLimit.tuning_all` (21-07) |

| Item | Kind | Source / tool |
|---|---|---|
| Known types of the area law, strong coupling and vortex free energy | Cited | Osterwalder–Seiler (strong-coupling area law), the Dobrushin condition, 't Hooft's twisted boundary conditions and vortex free energy. Bibliographic details in the tables of sources of each chapter and in A-03 |
| Open questions | Assessment | The assessment (`docs/assessment-1.pdf` in `yang-mills-group/YM`, 2026-11-10), § 8 |
| The infrared side of the published construction, and step 0 | Source | In the repository `yang-mills-group/OpenAI-YM-20261028` (Apache-2.0): `YM/EveryPhase/Light.lean:221` (`light_bound`), `YM/EveryPhase/Decay.lean:173` (`inverse_square`), `YM/Lattice/Path.lean:105` (`travel`), `YM/Wilson/Law.lean:87` (`law`, periodic torus), `YM/Slice/Reflection.lean:98` (`reflected_positive`), `YM/RG/History.lean:213` (field `band` of `Conv`), `YM/FullVolume/Fields/Cement.lean:50` (`fanned`) |

**Terms used in this chapter** — strong-coupling window, infinite-volume limit, mass gap, clustering, area law, Wilson loop, twisted partition function, vortex free energy, thin-vortex floor, thick-vortex floor, Polyakov loop, vortex ratio, rarity of large plaquettes, fermion weight, push-forward measure, crossover band. Their meanings are in the A-02 glossary.

Next chapter: 15-00 Entrance — what ultraviolet control is — the entrance to the last part of Volume 2 (Part 15, Record of the attempts at ultraviolet control). To go on to Volume 3 first, 16-02 Five open questions — we read the open questions of the published construction one by one.

# Entrance — what ultraviolet control is

> This chapter is a guide. § 01 is established knowledge, § 03 is this book's own investigation, and § 08 is a record of this book's study.

**Where this part stands** — This part is a record. It keeps the traces of this book's study's attempt at ultraviolet control (existence of the continuum limit) with a design different from that of the published construction — statements, line-by-line computations, elementary lemmas, numbers measured on minimal systems, and rows that do not close — as they are. Existence itself was shown by the construction published on 2026-11-10 (at the level "the committee's assessment is that it is proved in Lean"). How to read it is in Parts 6–13 of Volume 2, and the differences between the two designs are in the immediately preceding Part 14, as a stage-by-stage comparison.

The premises that appear in this part and in the list of premises (28-04) — the small-field step, the large-field estimate and gas, induction uniform in the step, the coupling bracket, full-sequence convergence — are statements about this book's design (measures pushed forward by a fixed block map independent of the cutoff, and matching by quantities determined by the measure). The results of the published construction are statements about correlations of local fields and about that construction's kernel and representation, so these premises do not follow from them. The premises on large fields and the gas are answered by the published construction in a different form (no history of regions, re-counting seeds at each step) (14-04). What can be called specific to this book's design is at most three things: a fixed block map independent of the cutoff (consistency between steps can be written as a push-forward identity), reflection positivity of the pushed-forward measure (Lean), and matching by quantities determined by the measure (at the level of design choices) (14-06).

The theorems machine-checked all the way are in Parts 20–22, and the machine-checked form of the free-field single step is in 15-12 in the second half of this part. The conditional implication that this book's study assembled beyond the components of this part, and its known flaws, are written in § 08.

**What this chapter shows** — That sending the lattice spacing $a$ to 0 makes the number of renormalisation steps $\log_L(L_0/a)$ grow without bound. That what is then needed is to show "one step returns to the same form" at every step without letting the constants degrade. The form the effective density must keep and the three conditions (the

content of ultraviolet control). Seeing the difference between a one-step estimate and an all-step proof through one number (the product of losses), and the ten patterns of misreading that difference. The distinction between the words of this part (renormalisation step, the staircase of Part 23, the band of Part 26), and the map of chapters, reading order and symbols of the five grades. The conditional implication assembled by this book's study, and its known flaws.

**Prerequisite chapters** — 7-05 § 01 (the sequence of renormalisation group steps), 7-05 § 04 (small fields and large fields), 7-07 § 03 (four dimensions: Bałaban's main theorem), 26-00 (confluence and the band), 27-02 (the ledger of conditions).

**Said first** — What this part treats is the ultraviolet-control row of the ledger of conditions, the foundation at the bottom of 16-09 Figure 1, underneath both of the two roads. It is a task separate from the crossover band (Part 26), and deals only with scales finer than the band. Within this part, ultraviolet control is not closed (existence was shown by the published construction with a different design). What this part has are statements of what needs to be shown, line-by-line computations, a few elementary lemmas, and numbers measured on minimal systems; each restates a part of ultraviolet control or splits it into several conditions. This part expands 26-11 (ultraviolet control) by splitting it, and 26-11 keeps only a summary and pointers to each chapter. A note on symbols: in this part $L$ is the blocking factor (usually 2), and $L_0$ is the side of a fixed box (as in 7-05 § 01).

The order of this chapter

1. What the problem is to show — the number of steps, the same form, the three conditions (the content of ultraviolet control), the form that closes in one step
2. A note on words — the renormalisation step, the staircase of Part 23, the band, the window (Table 0)
3. What it means to close at infinitely many steps — the product of losses, and the ten patterns of misreading (Table 1)
4. The map of this part and the reading order (Table 2)
5. The symbols of the five grades (Table 3)
6. What cannot be said
7. Table of the five grades and the dependency graph (Table 4)
8. The conditional implication assembled by this book's study, and its known flaws — as a record

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00

## What this chapter shows

- Ultraviolet control means showing "one step returns to the same form" at infinitely many steps without the constants degrading ( § 01).
- If the constant shrinks by a factor $1-\delta_k$ at each step, what remains after $K$ steps is $\prod_k(1-\delta_k)$, and everything is decided by whether the sum of losses $\sum_k \delta_k$ is finite uniformly in the number of steps ( § 03).
- The misreading of writing a one-step estimate and reading it as closing the induction happens again and again. We divide it into ten patterns and set each beside the correct treatment ( § 03 Table 1).
- The "step" of this part is one renormalisation step (step $k$); it is different both from the steps of the "staircase" of Part 23 and from the "band" of Part 26 ( § 02).
- The chapters of the first half are arranged in the order: the difficulty (15-01), the sources (15-02), the decomposition statement (15-03), numbers on minimal systems (15-04 to 15-06), background dependence (15-07), the four rows that do not close (15-08), the index (15-09), and the systematic attempts (15-10). The four chapters of the second half (15-11 to 15-14) are theorems completed in Lean for one step of the free field and one scale ( § 04).
- The conditional implication assembled by this book's study has three known flaws ( § 08).

01

## What problem is to be shown — the number of steps, the same form, the three conditions

The object is Wilson's lattice for $SU(2)$ on a four-dimensional torus of side $L_0$, with lattice spacing $a$. $L_0$ is fixed at a physical scale, and only $a$ is varied. As in 7-05 § 01, we repeat blocking with factor $L$ and integrate from the finest scale upwards. The number of steps is

$$K = \log_L \frac{L_0}{a} \longrightarrow \infty \qquad (a \to 0) \quad (1)$$

Proved here (from the definition). Example: if $L=2$ and $L_0/a = 2^{20}$, then $K=20$, and each halving of $a$ adds one step. Writing $\rho_k$ for the weight (effective density) after step $k$, we have $Z = \int \rho_0 = \int \rho_1 = \cdots = \int \rho_K$.

**The same form.** In Bałaban's method, at each step the lattice is divided into a small-field region $\Omega$ and its outside, and one sums over all the ways of dividing it (7-05 § 04). The form the effective density must keep is, in abbreviated notation,

$$\rho_k(V) = \sum_{\Omega} \chi(\Omega, V)\, \zeta(\Omega^c, V)\, \exp\Big(-\frac{1}{g_k^2} A_\Omega(V) - \sum_{X \subset \Omega} E_k(X, V)\Big)\, R_k(\Omega^c, V) \tag{2}$$

Cited (the form follows the exposition for scalar fields, Dimock II § 1, and the wording of the three-dimensional theorem; 7-05 § 07). $A_\Omega$ is Wilson's action on the small-field region (written in terms of the background field), $E_k(X, \cdot)$ is a correction localised on a connected lump $X$, $R_k$ is the factor carried by the large-field region, and $g_k$ is the effective coupling at step $k$.

**The three conditions.** Ultraviolet control means keeping the form (2) at every step while satisfying the following three. The constants $r, C, \kappa, c, \gamma$ depend neither on the step $k$ nor on the lattice spacing $a$.

- **Small-field analyticity condition** (locality and analyticity for small fields). Each $E_k(X, \cdot)$ is analytic in a complex neighbourhood of radius $r$ in the background-field variables, and $|E_k(X)| \le C\, e^{-\kappa\, d(X)}$ ($d(X)$ is the length of a tree joining $X$).
- **Large-field rarity condition.** $|R_k(\Omega^c)| \le e^{-c\, p(g_k)^2\, |\Omega^c|}$ ($|\Omega^c|$ is the number of large-field blocks, and $p(g)$ is the threshold factor of equation (7) in 7-05).
- **Condition that the coupling stays small** (below, "the coupling stays small"). The sequence of effective couplings $g_0, g_1, \dots, g_K$ lies entirely in $(0, \gamma]$.

The three put the content of ultraviolet control into the language of this book. This book's study has not shown any of the three in four dimensions Assumption (the ultraviolet-control row of 27-02). Existence of the four-dimensional continuum limit was shown by the published construction with a different design (one that carries the representation of the density at each step through all steps as the induction hypothesis) (6-03, 14-01).

**The form of closing in one step.** Let the class $\mathcal{K}(g;\, r, \kappa, c)$ be "the collection of effective densities that have the form (2) and satisfy the small-field analyticity condition and the large-field rarity condition at coupling $g$ with constants $r, \kappa, c$". What has to be shown is

$$\rho_k \in \mathcal{K}(g_k;\, r, \kappa, c) \implies \rho_{k+1} \in \mathcal{K}(g_{k+1};\, r, \kappa, c), \qquad \frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - 2b_0 \ln L + r_k \tag{3}$$

The point is that **the constants $r, \kappa, c$ are the same on both sides, and only the coupling may move**. $b_0 = 22/(48\pi^2)$ is the one-loop coefficient for $SU(2)$ (7-07 § 01), and $r_k$ is the remainder in the flow of the coupling. 15-03 restates (3) by splitting it into "a class that closes with finitely many parameters" and "the one-step remainder".

02

## A note on words — the renormalisation step, the staircase of Part 23, the band, the window

In this part we use the word "step" only in the sense of one renormalisation step. This book has other words built on "step", "band" and "window", and when words coincide they read as if they referred to the same thing, so Table 0 separates them.

Table 0 The words of this part, and similar words in other parts

| Word | Meaning | Chapters that use it | Relation to this part |
|---|---|---|---|
| The renormalisation step (step $k$) | One blocking with factor $L$. The effective density $\rho_k$ after step $k$, the effective coupling $g_k$, the number of steps $K = \log_L(L_0/a)$ | 7-05, Part 15 | This is the only "step" of this part |
| The steps of the staircase of Part 23 (the first step, the second step, ...) | An arrangement that divides the road towards a proof into four steps. The first step is a study on the double well in 0+1 dimensions | Part 23, 26-00 | A different word. Unrelated to the number of steps $K$ of this part |
| The crossover band (the band) | The interval around the scale $L \sim 1/\Lambda$ at which weak coupling passes over to strong coupling | Part 26 | This part deals only with scales finer than the band. Inside the band the coupling is not small |
| The window (the $A_W$ window, the $\hbar$ window, the time window) | Test functions restricted to an interval of finitely many sites, an interval of parameters, or an interval of time | Parts 23 and 24 | Not used in this part. A single plaquette of the coarse lattice is written "a coarse plaquette" ($2\times2$, $4\times4$, side $L$) |
| The ultraviolet | All the steps from the lattice spacing $a$ up to the scale $L_0$ at which the coupling is still small | 26-11, Part 15 | The ultraviolet-control row of the ledger of conditions. The coupling is small at every step |

One more caution: in the smallest systems of this part (15-04 to 15-06), we sometimes call "the marginal distribution of one coarse plaquette" a one-step map. This map looks only at the distribution of the holonomy of one coarse plaquette; it is not a true blocking step, which would produce the joint distribution of all the plaquettes of the coarse lattice. Each chapter says this once where it first appears.

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03
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## What it means to close at infinitely many steps — the product of losses, and the ten patterns of misreading

Let us see the difference between a one-step estimate and an all-step proof in a single number. If a one-step estimate comes only in the form "constant $r_{k+1} \geq r_k(1-\delta_k)$", then what is left after $K$ steps is

$$r_K \;\geq\; r_0 \prod_{k=0}^{K-1} (1-\delta_k), \qquad \prod_{k=0}^{K-1} (1-\delta_k) \;\geq\; \exp\Bigl(-2 \sum_{k=0}^{K-1} \delta_k\Bigr) \quad (0 \leq \delta_k \leq \tfrac{1}{2}) \tag{4}$$

Proved here (the inequality on the right follows from $\ln(1-x) \geq -2x$ for $0 \leq x \leq \frac{1}{2}$). If the sum of losses $\sum_k \delta_k$ is finite uniformly in $K$, the constant stays positive uniformly in $K$. If the sum grows with $K$, the constant degrades with the number of steps. **The content of ultraviolet control is making this sum finite.** Bałaban's three-dimensional theorem (equation (10) in 7-05 § 07) and the four-dimensional theorem of 1989 (7-07 § 03) carry out an induction of this form in finite volume Cited. Why in four dimensions this sum is not naively finite ($\sum_k g_k^2$ grows like $\ln K$) is calculated in 15-01 § 01.

In examining ultraviolet control, the same kind of misreading happens again and again. The common pattern is **writing a one-step estimate and reading it as closing the induction**, which amounts to looking at just one $\delta_k$ without looking at the product $\prod(1-\delta_k)$. Table 1 divides it into ten logical patterns. Every row can be avoided by putting the one-step claim and the all-step claim on separate lines.

Table 1 Patterns of misreading one step as all steps (logical cautions; labels only on the rows with numbers)

| Pattern | The misreading (one line) | The correct treatment |
|---|---|---|
| Circularity | Taking the inputs needed to close the expansion (the lower bounds on the Hessian at each step) from the outputs of that same expansion | Put both inputs and outputs on the side of the induction hypothesis, and bring nothing in from outside |
| Mistaking the component | Reading the replacement of a one-step estimate and a structural lemma as replacing the whole induction | Write the components replaced and the parts of the induction not replaced on separate lines |
| Accumulation of constants | Writing "the constants are fixed afresh at each step, so they do not accumulate" | That is a reduction to the radius of analyticity being uniform in the step, and depends on the small-field analyticity condition |
| Worst-case geometry | Running the large-field induction as "if it is large at the next step, it was large at one of the earlier steps" | A constant is lost at each step, and by type (4) of 15-01 it collapses to 0 over $k$ steps Proved here. What is needed is one-step stability that passes the stiffness on through the integral over the fluctuations |

| Pattern | The misreading (one line) | The correct treatment |
|---|---|---|
| Goodness on the fine lattice | Defining a "good frame" by all plaquettes of the shell being small | With probability $1-q$ at each point ($q>0$ fixed), at $n$ points this gives $(1-q)^n \to 0$ Proved here . Define goodness at the renormalised scale |
| Direction of the inequality | Using the variational principle for the partition function (which gives only lower bounds) for an upper bound on a difference of free energies | An upper bound needs an upper bound on the partition function, or control of the conditional measure |
| The cost at the bare coupling | Deriving $e^{-c/g^2(L)}$ at the running coupling from the existence of localised configurations of finite action | Mere existence gives a cost at the bare coupling, which vanishes as $a \to 0$. A cost at the running coupling needs uniform control |
| Scheme-dependent numbers | Testing the coefficient of $F^2$ in one step against the universal one-loop coefficient | The one-step coefficient depends on the blocking scheme. What is universal is the coefficient of the logarithm accumulated over many steps |
| Dropping the assumption | Citing four-dimensional ultraviolet stability as an unconditional theorem | It is conditional on the coupling staying small, and the basic inequality that would remove that condition (Theorem 2 of Bałaban 1989) is unpublished Cited (15-02 § 01) |
| Overlooking the domain | Settling the continuity of the map from the bare coupling to the coupling at the last step with "expected to follow from the construction" | The effective coupling is defined only while the previous step is small. Write continuity within the domain and the intermediate-value argument separately (15-03 Coupling-Smallness Induction Lemma) |

The top four rows of the table have the same form. Getting the all-step version from the one-step version needs the assumption that "the constants do not degrade over the steps", and unless this is made explicit on the side of the rules, a one-step proof looks like an all-step proof. The bundle of rules in 15-03 puts this caution into the way the rules are written.

04

## The map of this part and the reading order

Table 2 lists the chapters of Part 15. The first half (15-00 to 15-10) is the record of the study of ultraviolet control, and the second half (15-11 to 15-14) consists of theorems completed in Lean for one step of the free field and one scale. The level marks are: [2nd year] = readable by second-year students of mathematics or physics; [3rd year+] = third year and above (uses estimates of measures, operators and norms); [research] = research level (follows the construction of the study as it stands).

Table 2 The chapters of Part 15

| Chapter | Title | One line | Premise | Level |
|---|---|---|---|---|
| 15-00 | Entrance — what ultraviolet control is | The number of steps, the same form, the three conditions, the patterns of misreading, a note on words | 7-05, 7-07 | [2nd year] |
| 15-01 | The difficulty of the ultraviolet | With a dimensionless coupling the sum of losses diverges logarithmically; properties of the weight that group-valued fields lack; re-fixing the gauge | 15-00 | [2nd year]–[3rd year+] |
| 15-02 | A map of the sources | The five stages of Bałaban's series, the BBS chain, the ultraviolet statement for a small twisted four-dimensional torus, two rows absent from the sources | 15-01 | [3rd year+] |
| 15-03 | The decomposition statement | A class that closes with finitely many parameters and three remainders, the Coupling-Smallness Induction Lemma, the ledger of the transverse contraction, questions | 15-01, 15-02 | [3rd year+] |
| 15-04 | The one-link heat-kernel class and the hierarchical model | A class that closes with one parameter, the invariant curve of Migdal–Kadanoff and the contraction 1/16 | 15-03 | [2nd year]–[3rd year+] |
| 15-05 | The flux-network theorem | The transverse contraction and the parameter flow in one projection, the box table, two steps | 15-03, 15-04 | [2nd year]–[3rd year+] |
| 15-06 | Counting the parameters | Couplings by loop shape, the criterion, the attracting class | 15-04, 15-05 | [3rd year+] |
| 15-07 | Background-dependent fluctuations η | The Background–Fluctuation Decomposition Lemma, the group version of closure under the fluctuation expectation ((iii) of BBS), the Lemma on One Step with Background-Independent Fluctuations, shifting the local couplings | 15-02, 15-03 | [3rd year+] |
| 15-08 | The four rows that do not close, and proposals for them | The remainder including large fields, the source of the observables, η over many steps, gauge fixing | 15-03, 15-07 | [3rd year+]–[research] |
| 15-09 | What cannot be said, the index of questions, the dependency diagram | What cannot be said for the part as a whole, the ultraviolet questions, the table of Lean correspondences | All chapters | [2nd year] |
| 15-10 | Systematic attempts | Dominant matrices, the loss from discarding signs, the geometry of representatives, the double valuation, the cone calculus (a study; not the main line) | 15-05, 15-08 | [3rd year]–[research] |

| Chapter | Title | One line | Premise | Level |
|---|---|---|---|---|
| 15-11 | Entrance — one step and one scale | The scope of the chapters of the second half (linear, vacuum, one scale) | 19-04 | [2nd year] |
| 15-12 | One step and many steps of the free field | The recursion for the symbol obtained by pushing the free field with the path kernel; contraction uniform in the step (Lean) | 15-11, 19-04 | [3rd year+] |
| 15-13 | Linearisation of the block map and the covariance of the push-forward | Locality, gauge covariance, linearisation around the identity, the covariance of the push-forward (Lean) | 15-12 | [3rd year+] |
| 15-14 | What the second half (15-11 to 15-13) says, and what it does not | The theorems of the second half in one table, with their scope and Lean declarations | 15-11 to 15-13 | [2nd year] |

There are three reading orders. **Shortest**: 15-00 → 15-03 → 15-09 (what has to be shown, and what remains). **The computational line**: 15-04 → 15-05 → 15-06 (how the numbers that appear in the decomposition can be measured on the smallest systems). **The line of the wall**: 15-02 → 15-07 → 15-08 (where the sources stop, and what does not close). 15-01 can be read before any of the lines.

**Relation to the wall of Part 23.** For the core Poincaré uniformity assumption of the first step of Part 23 (the study on the double well in 0+1 dimensions), besides the two existing ways of supporting it, there is a candidate for a third route that supports it by the influence of a coarse chain (the Dobrushin condition), and the column sums of that influence are bounded by $\rho_{\mathrm{col}} \le 0.176 < 1$ at 23 points of $\hbar \in [10^{-5}, 1]$ [computer-assisted (double precision, not interval arithmetic)] (on the assumptions of an outward margin and of the precision of the floating-point functions; between the points and for $\hbar < 10^{-5}$ it is Reading ). This is a part of the route on the time side of Part 23 and is independent of the renormalisation steps of this part (for the distinction between the words, see Table 0).

05

## The symbols of the five grades

Every statement in this part carries one of the following five labels (the same five grades as the whole book). A label marks "how it was checked", not how weighty the statement is.

Table 3 The labels of the five grades (how they are used in this part)

| Label | Meaning | Examples in this part |
|---|---|---|

| Label | Meaning | Examples in this part |
|---|---|---|
| Proved here | A proof placed on paper (an elementary proof) | The Coupling-Smallness Induction Lemma, the Background–Fluctuation Decomposition Lemma, the rational numbers of the bundle network, the algebra of the one-loop leading term (written "Proved here, at the one-loop level") |
| Cited | Known results checked in the sources | Bałaban's Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989), Lemma A.6.1 of BBS |
| Computer-assisted | Computer-assisted (interval arithmetic) | The tube and chain of the hierarchical model in 15-04. An estimate that only adds an outward margin to double-precision floating point is written separately as [computer-assisted (double precision, not interval arithmetic)] |
| Reading | Numerical readings, physical estimates | Values of the source, the contraction at finite $\beta$, agreement with Monte Carlo |
| Assumption | Assumptions, statements not yet reached | Ultraviolet control, the decomposition statement, the twisted-torus ultraviolet statement |

Components checked in Lean are written in a separate column rather than as a label, and are collected in the correspondence table of 15-09. The Lean components of this part are all lemmas about finite sums of real numbers or algebra; the lattice, the group and the measure are not formalised.

06

## What cannot be said

(a) Within this part, ultraviolet control is not closed. Neither this chapter nor any chapter of this part shows the three conditions of ultraviolet control, in four dimensions or in two. Existence of the four-dimensional continuum limit was shown by the published construction with a different design (Parts 6–13).

(b) (4) is arithmetic for when the one-step losses act multiplicatively; it is not a theorem about an actual sequence of effective couplings or about actual constants.

(c) The ten patterns of Table 1 are logical cautions; they are not a claim that any existing argument is wrong.

(d) The level marks and reading orders of Table 2 are the book's arrangement; they are not a judgement of how hard the contents of the chapters are.

07

## Table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [01] | The number of steps $K = \log_L(L_0/a) \to \infty$ ((1)) | Proved here | § 01 (definition) | $L_0$ fixed |
| [02] | The form (2) of the effective density (a sum over small fields and large fields) | Cited | § 01, 7-05 § 04, § 07 | The level of the form (Dimock II, the wording of the three-dimensional theorem) |
| [03] | The three conditions of ultraviolet control and the form (3) that closes in one step | Assumption | § 01, the ultraviolet-control row of 27-02 | From $a$ to $L_0$; constants independent of $k, a$ |
| [04] | The lower bound $\prod(1-\delta_k) \geq e^{-2\sum \delta_k}$ on the product of losses ((4)) | Proved here | § 03 | $0 \leq \delta_k \leq \frac{1}{2}$ |
| [05] | Bałaban's theorems in three and four dimensions (1989) are inductions of the type (3) | Cited | § 03, 7-05 § 07, 7-07 § 03 | Finite volume; in four dimensions, conditional on the coupling staying small |
| [06] | The ten patterns of misreading, and the two pieces of arithmetic for worst-case geometry and goodness on the fine lattice | Proved here | § 03 Table 1 | Logical patterns. The rows with numbers are for fixed $q > 0$ and one-step losses that act multiplicatively |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV STEP   number of steps K → ∞                                  (L₀ fix
[02] CIT FORM   the form (2) of the effective density
[03] ASM UV     the three conditions of UV control and the form (3) that close
[04] PRV PROD   lower bound (4) on the product of losses                (0 ≤ δ
[05] CIT BAL    Bałaban's theorems are inductions of type (3) ⟸ [02]    (fini
[06] PRV MIS    the ten patterns of misreading ⟸ [04]                   (logi
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. The row of the three conditions of ultraviolet control is an assumption row, a restatement of ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

```
08
```

### The conditional implication assembled by this book's study, and its known flaws

Beyond the components of this part, this book's study assembled a conditional implication (a chain) about families of pushed-forward measures (28-01 implication of subsequential limits, 28-02 implication of full-sequence convergence). Its form is: push the fine-lattice measure forward to the coarse lattice by a fixed block map, and under the premises of this implication (from the list of premises in 28-04: the small-field step, the large-field estimate and gas, induction uniform in the step, the coupling bracket and so on), subsequential limits of the pushed-forward measures exist, and properties of the finite lattice (rarity, reflection positivity, symmetries, how observables factorise) pass to those limits Assumption . Since four-dimensional existence was shown by the published construction with a different design, we read this implication not as a statement of existence but as a record of what this book's design took as premises. As a record, we write down three known flaws.

(a) **The two parts are not connected inside the theorem.** The Lean theorem of the chain is a juxtaposition of two things: a lemma about the function of the effective coupling (a point matching any target value can be taken inside the bracket), and compactness and passage to the limit for families of measures. Inside the theorem, both the family of fine measures and the sequence of bare couplings can be chosen arbitrarily, and there is no field linking the two. What would link them is the condition "match the sequence of bare couplings with a quantity determined by the measure", and existence of a matched sequence remains a condition inside this implication. That a sequence matching the expectation of the average of coarse plaquettes exists for the side-3 design (14-03) is a separate theorem, and there is no statement connecting it to the chain theorem.

(b) **Without matching, the convergence conclusion has no content.** Subsequential convergence of pushed-forward measures holds for any family of probability measures, by compactness alone. The content of this implication lies in carrying properties of the finite lattice over to the limit. Full-sequence convergence was placed not among the conclusions but on the premise side.

(c) **There are two ways of measuring compatibility in the large-field gas.** In the premise of the large-field gas there is a form that measures compatibility of polymers by "not adjacent" and a form that measures it by "supports do not intersect", and the prose description and the Lean statement took different forms. They are different gases. Which form the premise should be written in has not been decided.

All three are flaws on the side of this book's study that remain regardless of the results of the published construction.

## What this chapter can say

| Said | Not said |
|---|---|
| Proved here That the number of steps grows like $\ln(1/a)$, the lower bound (4) on the product of losses, and the arithmetic of the rows with numbers among the patterns of misreading | That the sum of losses is finite for an actual sequence of effective densities |
| Cited The form (2) of the effective density, and that Bałaban's theorems are inductions of the type (3) | An induction in four dimensions without the coupling-staying-small condition |
| Assumption The form that states ultraviolet control as three conditions | Any of the three conditions (four-dimensional existence was shown by the published construction with a different design) |
| Assumption The form of the conditional implication assembled by this book's study, and its three known flaws (§ 08) | Any of the premises of the implication. A statement connecting the two parts of the chain |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The sequence of steps, the form of the effective density, small fields and large fields | Cited | The source table of 7-05 (J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705. T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275) |
| The Conditional Ultraviolet Stability Theorem in four dimensions | Cited (abstract and introduction, pp. 355–356) | T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (the source table of 7-07) |
| (1)(4), the rows with numbers in Table 1 | Proved here | The arithmetic in the text of this chapter |
| The ten patterns of Table 1, Table 0, Table 2 | This book's study | The record of this book's study (the arrangement of the study of ultraviolet control) |

**Words used in this chapter** — ultraviolet control, the number of steps, the same form, a class that closes in one step, the small-field analyticity condition, the large-field rarity condition, the condition that the coupling stays small, the product of losses, conditional implication, the renormalisation step, the coarse plaquette, the five grades. Their meanings are in the A-02 glossary.

Next chapter: 15-01 The difficulty of the ultraviolet — why in four dimensions the sum of losses is not naively finite.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control dimensionless coupling, group-valued fields, gauge · Kiichi, Shiori and Rin

# Why the ultraviolet is hard — dimensionless coupling, group-valued fields, gauge

This chapter organises established knowledge (sources carry the “Cited” label in the text). It is placed in this part as background for this book's own investigation.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — In four dimensions the coupling is dimensionless, so the sum of the one-step losses of size $g_k^2$ diverges logarithmically in the number of steps, and a class that absorbs the losses exactly into the flow of the coupling has to be built (computed line by line). In three dimensions this sum is finite and there is no need to follow the flow of the coupling. For group-valued fields, among the tools for the weight of flat fields, the property of "closing in one step under the expectation of the fluctuations" is missing. Gauge-invariant functions have no convexity in the orbit directions, the gauge can only be re-fixed scale by scale, and the seams also enter the estimates. The three sources of difficulty, and that even the two-dimensional model without gauge is stuck.

**Prerequisite chapters** — 15-00 § 01 (the three conditions and the form that closes in one step), 7-05 § 06 (the running of the coupling), 7-05 § 05 (local gauges), 24-07 § 05 (the three sources of difficulty), 24-06 § 07 (the four properties of the weight).

**Said first** — The computations in this chapter are arithmetic that treats the one-loop flow as an exact equation; they are not theorems about the actual sequence of effective couplings. We split the difficulty into three (the dimensionless coupling, group-valued fields, gauge), but the three are not independent; in 15-07 and 15-08 they gather in one place (the fact that the measure of the fluctuations depends on the background).

The order of this chapter

1. Difficulty 1: the dimensionless coupling — the sum of losses diverges logarithmically (equations (1)–(5)), and the contrast with three dimensions
2. Difficulty 2: a group-valued field — the four properties of the weight, and closure under the fluctuation expectation ((iii) of BBS), which falls short for groups
3. Difficulty 3: gauge symmetry — convexity in the orbit directions, re-fixing at each scale
4. The sources of difficulty, and models without gauge — the three sources, the two-dimensional $O(N)$ (Table 1)
5. What cannot be said
6. Table of the five grades and the dependency graph (Table 2)

---

00

## What this chapter shows

- In four dimensions the sum of the one-step losses $g_k^2$ diverges like $\ln K$, so that part has to be absorbed exactly into the flow of the coupling, and the rest confined to sizes whose sum is finite (of type $g_k^{2+\delta}$, or a geometric series) ( § 01).
- In three dimensions $\sum_k g_k^2$ is finite, and the one-step losses can be carried as errors ( § 01).
- For a group-valued field, among the four properties of the weight, "closes in one step under the expectation of the fluctuations" (closure under the fluctuation expectation) is missing, and the measure of the fluctuations depends on the background ( § 02).
- At a minimum of a gauge-invariant function the Hessian form in the orbit directions is 0, and the tools of convexity cannot be used unless the gauge is removed. The gauge fixing works only at the size of a block, and the seams where it is re-fixed scale by scale also enter the estimates ( § 03).
- The core of the wall lies in "group-valued field × dimensionless coupling × large fields"; even for the two-dimensional $O(N)$ σ model without gauge ($N \geq 3$), the ultraviolet construction is stuck ( § 04).

01

## Difficulty 1: the dimensionless coupling — the sum of losses diverges logarithmically

This section summarises existing knowledge (the arithmetic of the one-loop flow and the contrast with three dimensions). The introduction is in 7-06 § 02.

In four dimensions the coupling $g$ has no dimension of length, and the only thing that weakens the coupling at fine scales is asymptotic freedom (24-07 § 01). Cutting the one-loop flow into steps, with $b := 2b_0 \ln L > 0$,

$$\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - b \quad (1)$$

Cited (equation (9) of 7-05 § 06, keeping only one loop). Below, we treat (1) as an exact equation and count the sum of losses line by line.

**(a) The sequence of couplings.** Fix the coupling at the coarsest step at $g_K = g_*$, and write in terms of the step $j = K - k$ counted from the coarse end ($j = 1, \dots, K$). Using (1) $j$ times,

$$\frac{1}{g_{K-j}^2} = \frac{1}{g_*^2} + bj, \qquad \frac{1}{(b + g_*^{-2})\, j} \;\le\; g_{K-j}^2 \;\le\; \frac{1}{bj} \quad (j \ge 1) \quad (2)$$

Proved here. The inequality on the left comes from $1/g_*^2 + bj \le (1/g_*^2 + b)j$ ($j \ge 1$), and the one on the right just drops $1/g_*^2 > 0$. The coupling is smaller on the finer side, but only like $1/j$.

**(b) The sum.** Comparing with the integral of $1/x$, $\ln(K+1) \le \sum_{j=1}^{K} \frac{1}{j} \le 1 + \ln K$, so adding up (2),

$$\frac{\ln(K+1)}{b + g_*^{-2}} \;\le\; \sum_{k=0}^{K-1} g_k^2 \;\le\; \frac{1 + \ln K}{b} \quad (3)$$

Proved here. **If there is a loss of size $g_k^2$ at each step, its sum keeps growing like the logarithm of the number of steps.** Example: for $SU(3)$ and $L = 2$, $1/b = 1/(2b_0 \ln 2) \approx 10.4$, and each doubling of $K$ raises the upper bound by $(\ln 2)/b \approx 7.2$. This matches the partial sums in 7-05 Table 4, which grow by about 7.2 with each doubling of $K$ (arithmetic).

**The measured one-step coefficient (note).** The $b$ of (1) treats the one-loop coefficient as an exact formula; if one fixes a blocking rule and measures one step, the decrease at the first step does not agree with the one-loop coefficient. On a periodic lattice that fixes every coarse link (the product along a line of $L$ fine links), running the long-

wavelength coefficient of the coarse action of $SU(2)$ and $\beta = 4/g^2$ by one step ($L = 2$) gives $\beta' - \beta = -0.745$ (the Manton action), $-1.078$ (heat kernel) and $-1.136$ (Wilson): the direction is that of asymptotic freedom, and the size is 2.9–4.4 times the one-loop value $8b_0 \ln 2 = \frac{11}{3\pi^2} \ln 2 \approx 0.2575$ in units of $\beta$ Reading (one loop, $O(1)$ of $\beta \to \infty$, a commuting background, extrapolated to wavenumber 0 from the series of transverse coarse tori $6^3$ and coarse lengths $\leq 32$ along the fourth axis). Of this, the constants that depend on the action ($-0.31$ for Manton, $-0.70$ for Wilson, and for the heat kernel additionally $-\frac{1}{3}$ from the Van Vleck prefactor) take other values for the actions of the second and later steps (which are no longer Wilson or Manton). The change when the blocking factor goes from 2 to 3 is $-0.254$ in the part that does not depend on the action, 1.7–1.9 times $-\frac{11}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ Reading (wavenumber 0 for factor 3 is a two-point extrapolation). Therefore, if the arithmetic of (1)–(5) is read with the numbers of the first step, the one-step decrease 0.2575 of $\beta$ must be read as 0.75–1.14, and the coefficient when steps are stacked has not been measured. That the coefficient of $\ln L$ does not match at small blocking factors can be read as due to the rule that fixes the products along lines integrating fluctuations of wavelengths longer than the coarse scale Reading (15-03 § 02). The difference $-\frac{1}{3}$ between heat kernel and Manton and the difference $\frac{5}{3}t$ between Wilson and Manton ($t$ is the Wilson value of $U(1)$) are identities Proved here (a count of the coefficients of $s^2$; the arithmetic of the coefficients is in Lean: `tadpole_weights`, `vanvleck_ratio`, `shift_bounds` ($0.2575 < \frac{11 \ln 2}{3\pi^2} < 0.2576$), `beta_shift_of_inv_sq` (assumption $1/g'^2 = 1/g^2 - 2b_0\ell$)).

**(c) The product.** If at each step a constant (the radius of analyticity, the rate of decay, the lower bound of the Hessian, and so on) shrinks by a factor $1 - cg_k^2$, then from $1 - x \leq e^{-x}$ and the left of (3),

$$\prod_{k=0}^{K-1} \left(1 - c\, g_k^2\right) \;\leq\; \exp\!\Big(-c \sum_{k=0}^{K-1} g_k^2\Big) \;\leq\; (K+1)^{-c/(b+g_*^{-2})} \;\longrightarrow\; 0 \qquad (K \to \infty) \tag{4}$$

Proved here (in the range $cg_k^2 \leq 1$). Since $K = \log_L(L_0/a)$, the constant goes to 0 like a power of $\ln(1/a)$. Slowly, but not uniformly.

**(d) The size at which the sum is finite.** If the loss is of size $g_k^{2+\delta}$ ($\delta > 0$), then from the right of (2)

$$\sum_{k=0}^{K-1} g_k^{2+\delta} \;\leq\; \sum_{j=1}^{\infty} (bj)^{-1-\delta/2} \;=\; b^{-1-\delta/2}\, \zeta\big(1 + \tfrac{\delta}{2}\big) \;<\; \infty \tag{5}$$

and this is finite uniformly in $K$ Proved here. For $\delta = 2$ (a loss of $g_k^4$, the size next to one loop) it is $\pi^2/(6b^2)$.

**Hence what is needed.** Putting (4) and (5) together, the part that moves by size $g_k^2$ after one step cannot be carried as an "error". That part must be made **the motion of the parameter of the class itself**. Concretely, one defines the class $\mathcal{K}$ in a form that has the coupling $g$ as its parameter, reads off the coefficient of the leading term (the Wilson term, the direction of $F^2$) from the effective action after one step, and sets that to be $g_{k+1}$. In this way the part of size $g_k^2$ is absorbed exactly into the flow of the coupling (3) of 15-00. The losses left after absorption (such as the degradation of the constants of the localised corrections $E_k$) must be confined to sizes whose sum is finite, of the type (5) — typically $O(g_k^4)$, or irrelevant terms that shrink geometrically in the scale Proved here (a requirement that follows from the arithmetic of (4)(5), in the case where the one-step losses act multiplicatively). Rewriting this requirement in the form of a statement gives the decomposition of 15-03.

On the side of the flow of the coupling things are different. The remainder $r_k$ in (3) of 15-00 may be allowed, in the variable $1/g^2$, to be of size $|r_k| \le C g_k^2$ (the size of two loops); its sum grows like $\ln K$ as in (3), but since $1/g_k^2$ itself grows like $bK$, the sequence of couplings does not collapse. What is needed is that the sequence stays in $(0, \gamma]$, and that is handled by the Coupling-Smallness Induction Lemma of 15-03.

**Contrast with three dimensions.** In three dimensions $g^2$ has the dimension of length to the power $-1$, the dimensionless coupling at step $k$ has the form $g^2 L^k a$ and is geometrically small on the fine side, and $\sum_k g_k^2$ is finite (it stays within 2 in 7-05 Table 4) Cited . Bałaban 1985 writes that the types that can diverge are finitely many, that the counterterm needed is a single constant, and that by semisimplicity there are no variables that grow in the effective action (relevant variables) (7-06 § 05) Cited . In three dimensions the one-step loss $g_k^2$ can be carried as an error, and there is no need to follow the flow of the coupling. In four dimensions the $F^2$ term is exactly marginal and must be followed. This is the precise form, on the ultraviolet side, of the source: dimensionless coupling, among the sources of difficulty in 24-07 Table 3.

How "absorption" looks in the smallest setting — that for the one-link heat-kernel class the losses form a telescoping sum — is seen in 15-04 § 01.

02

## Difficulty 2: a group-valued field — the four properties of the weight, and "closure under the fluctuation expectation", which falls short for groups

The edge variables $U_l$ of the lattice are elements of the group $G$. Group elements cannot be scaled, nor added and divided (7-05 § 05 (i)(ii)) Cited . Since the group is compact the field is bounded, but a "large field" is the far side of the group (far from the identity), where the curvature of Wilson's one-link weight is negative (the One-Link

Wilson Weight Proposition of 24-06 § 03) Proved here (proved in 24-06). Large fields can only be paid for with rarity, not with convexity.

**The tools for flat fields.** In the renormalisation of scalar fields, the rarity of large fields is put into the norm of a weight (regulator) $\tilde{G}(X, \phi)$, and closing under the expectation of one step is derived from the following four properties Cited (Lemma A.6.1 (One-Step Bound on the Activity Norm) in Appendix A.6 of the book by Bauerschmidt–Brydges–Slade; the four properties are those listed in § A.6.2 as the desired properties of a regulator; 24-06 § 07; the chain of the original papers is in 15-02 § 04).

1. It factorises into a product over disjoint sets.
2. $\tilde{G}(X, \phi + \phi') \leq \tilde{G}^2(X, \phi)\, \tilde{G}^2(X, \phi')$ (for a sum of fields).
3. $\mathbb{E}_+ \tilde{G}^t(X) \leq 2^{|B(X)|}$ (closes in one step under the expectation of the fluctuations).
4. $\tilde{G}^t \leq \tilde{G}_+$ (bounded by the weight of the next step).

From these follows the one-step estimate $\|\mathbb{E}_+ \theta K(X)\|_{\tilde{G}_+} \leq 2^{|B(X)|} \|K(X)\|_{\tilde{G}}$ (One-Step Bound on the Activity Norm). Submultiplicativity in sums of fields uses the addition of fields $\phi + \phi'$, and closure under the fluctuation expectation uses the fact that the fluctuations are Gaussian variables independent of the background. In the proof of the lemma, closure under the fluctuation expectation is used as the expectation of a function of the fluctuations alone, and that measure does not depend on the background $\phi$ Reading (from reading the chain of the proof).

**What survives on the group, and what falls short.** Submultiplicativity in sums of fields holds in the version where the sum of fields $\phi + \phi'$ is replaced by "background $\times$ fluctuation" $Ul = Wle^{Al}$, using only the triangle inequality and conjugation invariance (the Background–Fluctuation Decomposition Lemma and its corollary, the Splitting of the Weight; 15-07 § 02) Proved here . What falls short is closure under the fluctuation expectation. For a group, the measure of the fluctuations depends on the background (the propagator in a background field, the title of the foundational paper in 7-07 Table 1). So the estimate of $\mathbb{E}_+ G^t$ needs the regularity of the background, and in the region of large fields, where the background is on the far side of the group, the fluctuations are not a small perturbation of the background. Writing the part due to the background dependence of the measure as $\eta_k$, stating the group versions of closure under the fluctuation expectation and of monotonicity to the next step in the form of a statement, and seeing how $\eta_k$ is absorbed is the job of 15-07. For scalar fields the measure does not depend on the background, so $\eta_k = 0$, and the problem itself does not exist.

03

### Difficulty 3: gauge symmetry — convexity in the orbit directions, and re-fixing at each scale

**There is no convexity in the orbit directions.** On a finite lattice the group of gauge transformations is a product of compact groups, and the action $V$ is gauge invariant, that is, constant along orbits.

**Orbit-Direction Flatness Proposition.** Let $V$ be a smooth gauge-invariant function on the configuration space (a compact manifold), and $x_0$ a minimum point of $V$. For a tangent vector $\xi \neq 0$ to the orbit through $x_0$, $\operatorname{Hess} V(x_0)(\xi,\xi) = 0$. In particular, if there is even one such $\xi$, then $\operatorname{Hess} V \geq m\, g$ $(m > 0)$ does not hold.

Proved here Proof. Let $\gamma(t) = e^{t\eta} \cdot x_0$ ($\eta$ an element of the Lie algebra of gauge transformations with $\gamma'(0) = \xi$). Then $V(\gamma(t)) = V(x_0)$ holds for all $t$. Since $dV(x_0) = 0$, $\frac{d^2}{dt^2} V(\gamma(t))|_{t=0} = \operatorname{Hess} V(x_0)(\xi,\xi) + dV(x_0)(\nabla_{\gamma'}\gamma') = \operatorname{Hess} V(x_0)(\xi,\xi)$, and the left-hand side is 0. ∎

Tools that rely on convexity (Bakry–Émery, Brascamp–Lieb) cannot be used unless the orbit directions are removed Reading (a physical estimate; the same as the "theory" paragraph of 24-07 § 05). To remove them one fixes the gauge, but there is no continuous global gauge fixing that works for all configurations at once (Gribov 1978, Singer 1978) Cited (bibliographic only; 7-05 § 05).

**A gauge works only at the size of a block.** Taking the complete axial gauge on a two-dimensional block of side $\ell$, the plaquette condition $\|U_{\partial p} - 1\| < \varepsilon$ gives for edges only $\|U_b - 1\| < \ell\,\varepsilon$ (7-05 § 05) Proved here (proved in 7-05). Once $\ell\varepsilon$ exceeds 2 this says nothing, so the gauge is chosen anew at each scale, within the size of a block. Bałaban's three-dimensional theorem introduces the axial gauge block by block by the Faddeev–Popov method (7-06 § 05) Cited . We have not checked in the text how the gauge is chosen concretely in the four-dimensional papers (7-07 Table 3).

**The seams of re-fixing.** The gauge at step $k$ and the gauge at step $k+1$ are different, and the re-fixing is a field-dependent gauge transformation. Its Jacobian (the Faddeev–Popov determinant) and the terms that appear at the seams between neighbouring blocks also enter the effective action, and within the class of (3) of 15-00 they must be estimated as losses whose sum is finite in the sense of § 01 Reading (a physical estimate; it states the form of the estimate that is needed). MRS, constructed with a continuum cutoff, also joins the axial gauge of the large-field region with the background-field-dependent gauge of the small-field region, and has seams of the same type (7-07 § 06) Cited . The organisation in which, in a tree gauge, the Jacobian of re-fixing becomes a constant and the cost moves elsewhere (to the ghost determinant and the coupling at the seams) is in the row of the boundary of gauge re-fixing in 15-08.

04

## The sources of difficulty, and models without gauge

Transcribing the three difficulties into the language of the sources of difficulty of 24-07 Table 3 gives Table 1. In this part, which deals only with the ultraviolet, the coupling is small at every step, so the source: no expansion linking scales does not appear Reading (a physical estimate).

Table 1 The difficulties and the three sources of difficulty (a reading; the names of the sources are from 24-07 Table 3)

| Source of difficulty | Content (one line) | Form on the ultraviolet side | Chapters in this part |
|---|---|---|---|
| Source: dimensionless coupling | The coupling is dimensionless | The $F^2$ term is exactly marginal, and the sum of the losses $g_k^2$ diverges like $\ln K$ ((3)(4) of § 01). A class that absorbs the losses into the flow of the coupling is needed | 15-03, 15-04–15-06 |
| Source: no expansion linking scales | There is no expansion connecting the ultraviolet and the infrared | Does not appear (the coupling is small at every step). The task of the band in Part 26 | — |
| Source: no global gauge fixing | There is no global gauge fixing, and large fields cannot be controlled by convexity | Broken down within one step: closure under the fluctuation expectation for group-valued fields ( § 02), convexity in the orbit directions ( § 03, Orbit-Direction Flatness Proposition), and the seams of re-fixing ( § 03) | 15-07, 15-08 |

**Models without gauge are also stuck.** The two-dimensional $O(N)$ sigma model ($N \geq 3$) has no gauge and is a model with a dimensionless coupling and asymptotic freedom, yet the construction of its ultraviolet limit is listed as an open problem Cited ( § II.1.1 of the text of Rivasseau's 2000 review, which says that many experts have tried and not succeeded). The same place says that a hierarchical version of this model has been constructed (Gawędzki–Kupiainen, Comm. Math. Phys. 106 (1986) 533) Cited (the review's bibliography). That there is no construction even in finite volume, and that the point is restricted to asymptotically free $N \geq 3$, are this book's reading, not the words of the original Reading . The case $O(4)$ is the principal chiral model, whose field takes values in $S^3 = SU(2)$. As of 2024, transplanting Bałaban's method has reached the variational problem and the background-field stage (Goswami, a family of two-dimensional nonlinear sigma models; Dybalski–Stottmeister–Tanimoto, two-dimensional $O(4)$: the equation for the critical point is derived by the calculus of variations on the Lie group and solved by a contraction mapping) Cited (both from abstracts). On the other hand, the fermionic two-dimensional Gross–Neveu model (the

massive case) has been constructed (24-07 Table 3) Cited . The proposal to use this model as a test bench is in 15-02 § 05.

Reading from the contrast Reading (a physical estimate): the core of the wall can be read as lying in the combination "a field that takes values in a group (a nonlinear manifold)", "a dimensionless coupling (source: dimensionless coupling)" and "large fields", with gauge invariance a layer sitting on top of it. We do not say that gauge invariance is not an essential difficulty.

05

## What cannot be said

(a) (1)–(5) of § 01 are arithmetic that treats the one-loop flow (1) as an exact equation, not theorems about the actual sequence of effective couplings. "A class that absorbs the losses into the flow is needed" is a requirement for the case where the losses act multiplicatively at each step.

(b) § 02 only states that, among the four properties of the weight, closure under the fluctuation expectation falls short for groups; it does not prove the group version of closure under the fluctuation expectation.

(c) "The core of the wall is group-valued field × dimensionless coupling × large fields" is an estimate from the known situation of two-dimensional models; we do not say that gauge invariance is not an essential difficulty.

(d) We do not show that the seams of re-fixing can be estimated as losses whose sum is finite (we only state the form of the estimate that is needed).

(e) The measured one-step coefficient of § 01 is a one-loop number at the first step of a single rule (a periodic lattice that fixes every coarse link by the product along a line); the coefficient when steps are stacked, finite $\beta$ and non-commuting backgrounds have not been measured. The value for factor 3 is a two-point extrapolation.

06

## Table of the five grades and the dependency graph

Table 2 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [01] | The one-loop flow (1) | Cited | § 01, 7-05 § 06 | The equation keeping only one loop |
| [02] | Upper and lower bounds (2) on the sequence of couplings, and the logarithmic growth of the sum (3) | Proved here | § 01 | Under the one-loop flow (1) |

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [03] | The product of the losses $c g_k^2$ is $(K+1)^{-c'} \to 0$ ((4)); the sum of the losses $g_k^{2+\delta}$ is finite ((5)) | Proved here | § 01 | Under (1), $c g_k^2 \le 1$, $\delta > 0$ |
| [04] | In three dimensions $\sum g_k^2$ is finite and the counterterm is a single constant | Cited | § 01, 7-05 § 06, 7-06 § 05 | Three dimensions |
| [05] | The four properties of the weight and the one-step estimate (One-Step Bound on the Activity Norm) | Cited | § 02, 24-06 § 07 (the text of Appendix A.6.2) | Flat fields |
| [06] | On the far side of the group, the curvature of the one-link Wilson weight is negative | Proved here | § 02, 24-06 § 03 (One-Link Wilson Weight Proposition) | One link |
| [07] | Orbit-Direction Flatness Proposition: at a minimum point of a gauge-invariant function, the Hessian in the orbit directions is 0 | Proved here | § 03 | Tangent vectors to the orbit $\xi \neq 0$ |
| [08] | There is no continuous global gauge fixing | Cited | § 03 (bibliography only) | — |
| [09] | The terms at the seams of re-fixing at each scale also need estimating as losses whose sum is finite | Reading | § 03 (a physical estimate) | — |
| [10] | The ultraviolet construction of the two-dimensional $O(N)$ sigma model is listed as an open problem; the hierarchical version has been constructed. Transplantation has reached the background-field stage | Cited | § 04 (the text of the review § II.1.1, its bibliography and abstracts) | Within our search |
| [11] | The core of the wall is group-valued field × dimensionless coupling × large fields | Reading | § 04 (a physical estimate) | — |
| [12] | The measured one-step coefficient: $\beta' - \beta = -0.745 -- -1.136$ at the first step on the periodic lattice (2.9–4.4 times the one-loop value); the change for factor 2→3 is 1.7–1.9 times. The differences of the actions $-\frac{1}{3}$ and $\frac{5}{3}t$ are identities | Reading, Proved here (identities) | § 01 | $SU(2)$, one loop, commuting background, transverse $6^3$, wavenumber 0 extrapolated, factor 3 from two points, first step only |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] CIT FLOW   the one-loop flow (1)
[02] PRV RUN    couplings (2), log sum (3) ⟸ [01]  (one loop)
[03] PRV LOSS   product → 0 (4), sum finite (5) ⟸ [02]  (cg² ≤ 1, δ > 0)
[04] CIT 3D     in three dimensions the sum is finite
[05] CIT BBS    the four properties of the weight
[06] PRV CURV   negative curvature on the far side  (one link)
[07] PRV ORB    Orbit-Direction Flatness Proposition
[08] CIT GRIB   no global gauge fixing
[09] RDG SEAM   the seams of re-fixing ⟸ [03][07][08]
[10] CIT SIGMA  two-dimensional O(N) not constructed
[11] RDG CORE   the core of the wall ⟸ [03][04][06][10]
[12] RDG MEAS   the measured one-step coefficient ⟸ [01][02]  (one rule, firs
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

## What this chapter can say

| Said | Not said |
|---|---|
| Proved here Under the one-loop flow, the sum of the one-step losses $g_k^2$ diverges like $\ln K$, and the sum of losses $g_k^{2+\delta}$ is finite ((2)–(5)). Hence a class that absorbs the losses into the flow is needed. Orbit-Direction Flatness Proposition | Theorems about the actual sequence of effective couplings. The construction of the absorbing class |
| Cited The finiteness of the sum in three dimensions, the four properties of the weight, the absence of a global gauge fixing, that the construction of the two-dimensional sigma model is listed as an open problem, and the stage its transplantation has reached | The group version of closure under the fluctuation expectation. The choice of gauge in the four-dimensional papers |
| Reading The form of the estimate of the seams of re-fixing; the reading of the core of the wall | The estimate of the seams itself |
| Reading The measured one-step coefficient (one loop at the first step on the periodic lattice; direction and order of magnitude agree with the one-loop value, the coefficient is 2.9–4.4 times it) | The coefficient when steps are stacked; comparison with averaging-type rules |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The one-loop flow (1); the sum of the couplings in three dimensions | Cited | The source table of 7-05 (J. Dimock, The renormalization group according to Balaban I, II, III, arXiv:1108.1335, 1212.5562, 1304.0705. T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275) |

| Item | Kind | Source / tools |
|---|---|---|
| The four properties of the weight, One-Step Bound on the Activity Norm | Cited (the text of Appendix A.6.2 and of the One-Step Bound on the Activity Norm) | R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474 (the source table of 24-06) |
| The situation of the two-dimensional sigma model | Cited (the text of the review § II.1 and § II.1.1 and its bibliography; abstracts) | V. Rivasseau, Constructive field theory and applications: perspectives and open problems, arXiv:math-ph/0006017 (the source table of 24-07). P. Goswami, arXiv:2204.08252, Ann. Henri Poincaré 25 (2024) 2065–2085. W. Dybalski, A. Stottmeister, Y. Tanimoto, arXiv:2403.09800. Hierarchical version: K. Gawędzki, A. Kupiainen, Comm. Math. Phys. 106 (1986) 533 (according to the review's bibliography) |
| Global gauge fixing | Cited (bibliographic data only) | I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12 (the source table of 7-05) |
| (2)–(5), Orbit-Direction Flatness Proposition | Proved here | Proofs in the text of this chapter. The only numbers are the arithmetic of $1/(2b_0 \ln 2)$ and $\pi^2/6$ |
| The measured one-step coefficient | Reading (the identities for the differences of the actions are Proved here) | Numerical records of this book's study (the two routes, the Hessian of the real-space analysis and the transverse Bloch sum, agree to $10^{-10}$; the tree eigenvalues are reproduced; a negative control giving 0 for Manton at $U(1)$). The arithmetic of the coefficients is in the Lean bundle (15-09 Table 3, re-verified) |

**Words used in this chapter** — the dimensionless coupling, the sum of losses, absorption, marginal, the four properties of the weight, closure under the fluctuation expectation, Orbit-Direction Flatness Proposition, the seams of re-fixing, the three sources of difficulty. Their meanings are in A-02 Glossary.

Next chapter to read: 15-02 A map of the sources — how far Bałaban's series went, and where it stopped.

# A map of the sources — Bałaban's programme, the BBS chain, two rows absent from the sources

This chapter organises established knowledge. § 05 and § 06 are a map of open questions.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — A map dividing Bałaban's series on gauge theories into five stages by the work each takes on. That the Conditional Ultraviolet Stability Theorem of the last four-dimensional paper (Theorem 1 of Bałaban 1989) assumes "the effective coupling stays small", and that the basic inequality which removes it (Theorem 2 of Bałaban 1989) is unpublished (from the text of the source). How far MRS and the rewriting in an easier model (three-dimensional QED) have come. By what chain the one-step estimate for flat fields (Lemma A.6.1 in Appendix A.6 of Bauerschmidt–Brydges–Slade, the One-Step Bound on the Activity Norm) is obtained. A proposal for stating ultraviolet control on a small twisted $T^4$ with the zero modes removed, and where each of its rows falls in the sources. That what is absent from the sources are two rows — the one-step remainder including large fields, and the source for observables.

**Prerequisite chapters** — 15-01 § 02 (the four properties of the weight), 7-07 § 03 (the main theorem), 7-07 Table 1 (the bibliography of the series), 25-01 (the small twisted $T^2$).

**Said first** — The texts we have checked are the opening of the three-dimensional paper, the abstract and introduction of the last paper, and the text of Appendix A.6.2 of BBS and of the One-Step Bound on the Activity Norm. For the other papers of the series we have gone only as far as the abstracts or the titles, and the label column states the range that was checked. The ultraviolet statement for a small twisted four-dimensional torus is a proposal and has not been proved.

The order of this chapter

1. Bałaban's programme — the five stages of the series (Table 1)
2. Where it stopped — the assumption of the Conditional Ultraviolet Stability Theorem, and the unpublished basic inequality
3. MRS, and rewriting in easier models
4. The chain of Appendix A.6 of BBS — from submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step ((ii)–(iv) of BBS) to the One-Step Bound on the Activity Norm
5. The twisted-torus ultraviolet statement and the test bench — the small twisted $T^4$ and two-dimensional $O(4)$
6. Correspondence with the sources — two rows absent from the sources (Table 2)
7. What cannot be said
8. The table of five grades and the dependency graph (Table 3)

00

## What this chapter shows

- Bałaban's four-dimensional theorem assumes "the effective coupling stays in $(0, \gamma]$", and the theorem that removes this is unpublished. What was reached is: on a finite-volume lattice, upper and lower bounds on the effective densities and the partition function uniform in the lattice spacing, assuming the condition that the coupling stays small (§ 01, § 02).
- Rewriting in easier models has reached three-dimensional $\phi^4$ and three-dimensional QED, but the later stages for three-dimensional Yang–Mills and the two-dimensional sigma model are not to be found (§ 03).
- The one-step estimate for flat fields follows from the chain: interchange of expectation and seminorm → the definition of the norm → submultiplicativity in sums of fields, closure under the fluctuation expectation and monotonicity to the next step; and the measure of the expectation taken in closure under the fluctuation expectation does not depend on the background (§ 04).
- On a twisted $T^4$ with the zero modes removed, ultraviolet control can be stated in five rows, and only two rows are absent from the sources: the content of the coupling staying small in the twisted box (the one-step remainder including large fields, that is, the net one-step bracket of the coupling) and the observable-source condition (§ 05, § 06).

01

## Bałaban's programme — the five stages of the series

This section summarises existing knowledge (Bałaban's series). The introduction is in 7-07 § 02.

Dividing Bałaban's series on gauge theories (Commun. Math. Phys., 1983–1989) into five stages by the work each takes on gives Table 1. The bibliography is the same as in 7-07 Table 1. The only text we have checked is the opening of the three-dimensional paper and the abstract and introduction of the last paper; stages 4 and 5 and the row for 119 were confirmed from abstracts. The "work it takes on" for stages 1–3 is read from the titles, not checked against the content. The "three conditions" are the ultraviolet control of 15-00 § 01.

Table 1 The five stages of Bałaban's series (labels are for the range checked; the one-line content is from the titles for stages 1–3, and from abstracts for stages 4 and 5 and 119)

| Step | Papers (CMP volume, year, page) | Work it takes on | Correspondence with the three conditions | Label and range checked |
|---|---|---|---|---|
| 1 Foundation | 89 (1983) 571, 95 (1984) 17, 96 (1984) 223 | Regularity and decay of lattice Green functions. The block-averaging transformation and the propagators under it | Components for every step | Cited Bibliography and titles only |
| 2 Averaging of group-valued fields, and gauge | 98 (1985) 17, 99 (1985) 75 | Gauge-covariant averaging operations. The space of regular (small-field) configurations, and gauge fixing that works locally on it | The handling of the source: no global gauge fixing in 15-01 § 02, § 03 | Cited Bibliography and titles only |
| 3 Background field | 99 (1985) 389, 102 (1985) 277 | Propagators in a background field. The variational problem with the averages fixed, and the background field | The entrance to the small-field analyticity condition | Cited Bibliography and titles only |

| Step | Papers (CMP volume, year, page) | Work it takes on | Correspondence with the three conditions | Label and range checked |
|---|---|---|---|---|
| 4 Small-field expansion and the coupling | 109 (1987) 249, 116 (1988) 1 | In the small-field approximation, a sequence of localised effective actions by a cluster expansion at each step. Constructs the β function and defines the renormalisation of the coupling by recursive renormalisation-group equations (109). The exponentiated cluster expansion satisfies the induction hypotheses, and the construction in the small-field approximation is completed (116) | The small-field analyticity condition, and the perturbative part of the coupling staying small | Cited Abstracts |
| (Induction including large fields) | 119 (1988) 243 | A description of the complete induction for the effective densities including large-field regions, and the renormalisation transformation preserving that form. Completes here the analysis of superrenormalisable models (three dimensions) and gives a convergent expansion | Completes the small-field analyticity condition and the large-field rarity condition in three dimensions. In four dimensions, the foundation of stage 5 | Cited Abstracts |
| 5 Large fields | 122 (1989) 175, 122 (1989) 355 | The basic step of the R operation. Localisation, exponentiation, estimates. The Conditional Ultraviolet Stability Theorem | The large-field rarity condition. The coupling staying small is assumed | Cited 175 from the abstract (described as an operation that removes the main obstacle to the ultraviolet stability of four-dimensional gauge theories). 355 from the abstract and introduction, pp. 355–356 |
| (Three dimensions) | 102 (1985) 255 | Ultraviolet stability in three dimensions (equation (10) of 7-05). The author writes 'the proof is very sketchy' | The small-field analyticity condition and the large-field rarity condition in three dimensions. The coupling staying small is not needed (15-01 § 01). The complete induction including large fields is completed in 119 | Cited Text pp. 255–257 (the relation to 119 from the abstract of 119) |

02

### Where it stopped — the assumption of the Conditional Ultraviolet Stability Theorem, and the unpublished basic inequality

The theorem in the introduction of the last paper is as follows Cited (7-07 § 03).

'Theorem 1. If the sequence of the effective coupling constants is contained in an interval ]0, γ] with a sufficiently small positive γ, then the effective densities ρk have the form, and satisfy all the conditions and bounds, described in Sect. 2.'

The assumption is the condition that the coupling stays small itself, and the conclusion is the form of the effective densities ((2) of 15-00) and the estimates of the small-field analyticity condition and the large-field rarity condition. The author says that the assumption can be removed because it follows from a basic inequality derived from Theorem 2 of an earlier paper, and continues: 'The proof of Theorem 2, which is based on second order perturbative calculations, is very awkward and long ... and has not been published yet' Cited . Within our search of the bibliography of the author's papers after 1989, we found none that contains the proof of the basic inequality (unpublished) (the source table of 7-07). So the four-dimensional achievement is: **on a finite-volume lattice, upper and lower bounds on the effective densities and the partition function uniform in the lattice spacing, assuming that the coupling stays small** Cited (the level of uniform estimates in 7-07 Table 4). No source includes estimates of correlation functions, uniqueness of the limit, the OS axioms, infinite volume or the gap in what was reached. The review (Rivasseau 2000 § II.1) summarises Bałaban's result as the ultraviolet stability of the effective action after arbitrarily many steps in finite volume, and does not mention the assumption that the coupling stays small Cited (the text of the review). This is a difference in the level of summary; the statement in this section of what was reached rests not on the review but on the introduction of the last paper.

In one line, where it stopped is **the estimate of the one-step remainder that removes the assumption that the coupling stays small**. In the Coupling-Smallness Induction Lemma of 15-03 we see that the coupling staying small splits into that estimate (the net one-step bracket of the coupling) and an elementary part, continuity within the domain (definition and continuity of the effective coupling).

03

### MRS, and rewriting in easier models

Magnen–Rivasseau–Sénéor (1993), with a continuum momentum cutoff, stated the ultraviolet limit of the Schwinger functions for four-dimensional $SU(2)$ in the trivial topological sector, with a fixed infrared cutoff and a regularised axial gauge, but wrote 'we do not provide a detailed proof' and said the details would run to 'several hundred pages' Cited (7-07 § 06).

Work rewriting the method in easier models has reached three-dimensional $\phi^4$ (Dimock I–III, up to convergence of the expansion and the stability estimates) and three-dimensional QED (Dimock's three papers arXiv:1512.04373, 1712.10029, 2009.01156: the stage of tuning the bare quantities in the bounded-field approximation to fix the endpoint of the flow; multiscale propagators and the local regularity of minimisers; and upper and lower bounds for ultraviolet stability in a fixed finite volume, with the flow fully controlled at weak coupling) Cited (the three QED papers from abstracts; the source table of 7-05, 24-07 § 05). For three-dimensional Yang–Mills, no series of the same form was found within our search by author name. Transplanting it to the two-dimensional sigma model has reached the point corresponding to stage 3 of Table 1 (15-01 § 04), and work corresponding to stages 4 and 5 has not yet been written even for easier models (within our search) Cited (abstracts). For fields valued in a sphere, Bałaban's series on the low-temperature expansion of the classical $N$ vector model gives a description of the complete induction including large-field regions (abstract of Comm. Math. Phys. 196 (1998) 485) Cited (abstract). This is an infrared (low-temperature) problem, not the two-dimensional ultraviolet Reading .

04

### The chain of Appendix A.6 of BBS — from three properties of the weight to the one-step estimate

In the renormalisation of flat fields (scalar fields), the estimate of the expectation in one step follows by a short chain from the properties of the weight (regulator) $\tilde{G}$ alone. Of the four properties of 15-01 § 02, the lemma uses

three: submultiplicativity in sums of fields, closure under the fluctuation expectation (closing one step with the expectation over the fluctuation) and monotonicity to the next step (bounding by the weight of the next step). The text of the source is as follows Cited (Bauerschmidt–Brydges–Slade, arXiv:1907.05474v2, Appendix A § A.6.2, the text on pp. 214–215).

'Lemma A.6.1. Suppose that $\tilde{G}$ obeys (ii), (iii) and (iv) above. Then $\|\mathbb{E}_+\theta K(X)\|_{\tilde{G}_+} \le 2^{|B(X)|}\|K(X)\|_{\tilde{G}}$.'

The proof is a chain of three steps Cited (the text). $\theta$ is the map that shifts the field by $\phi \mapsto \phi + \zeta$, $\mathbb{E}_+$ is the expectation over the fluctuation $\zeta$, and $T_\phi(h)$ is the Taylor seminorm at scale $h$ at the point $\phi$.

$$\|\mathbb{E}_+\theta F\|_{T_\phi(h_+)} \;\le\; \mathbb{E}_+\|F\|_{T_{\phi+\zeta}(h)} \;(1)$$

$$\|K(X)\|_{T_\phi(h)} \;\le\; \|K(X)\|_{\tilde{G}}\,\tilde{G}(X,\phi) \;(2)$$

$$\mathbb{E}_+\tilde{G}(X,\phi+\zeta) \overset{\text{(ii)}}{\le} \tilde{G}^2(X,\phi)\,\mathbb{E}_+\tilde{G}^2(X,\zeta) \overset{\text{(iii)}}{\le} 2^{|B(X)|}\,\tilde{G}^2(X,\phi) \overset{\text{(iv)}}{\le} 2^{|B(X)|}\,\tilde{G}_+(X,\phi) \;(3)$$

(1) is the interchange of expectation and seminorm ((A.6.5) of the source, which takes it from Prop. 3.19 of reference [54]), (2) is the definition of the norm ((A.6.7) of the source), and (3) is the application of the properties ((A.6.8) of the source; the power convention in (3) is written with the notation abbreviated, and in the source it is in the form of the power $t$). Putting (2) into (1) and paying for the expectation with (3) gives the lemma.

One thing is learned from reading this chain Reading (a reading of the chain of the proof). The expectation in closure under the fluctuation expectation is the expectation of a function of the fluctuation $\zeta$ alone, and **its measure does not depend on the background** $\phi$. The interchange in (1) also holds with factor 1 because the law of the fluctuation does not depend on the background. For group-valued fields, a background-dependent factor $1 + \eta_k$ enters here. What becomes of each step of this chain on the group — that submultiplicativity in sums of fields holds by the Background–Fluctuation Decomposition Lemma, and that if the law of the fluctuation does not depend on the background every step goes through and gives $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations) — is in 15-07 § 02 and 15-07 § 05.

05

## The twisted-torus ultraviolet statement and the test bench — the small twisted $T^4$ and two-dimensional $O(4)$

On a periodic $T^4$ the classical minima at the last step (flat connections) form a continuous family, and the integrals of the last few steps degenerate. We put forward, as a proposal, a statement in which this incidental difficulty is removed by 't Hooft's twist. It moves the small twisted $T^2$ of the spatial road (25-01) to the whole four-dimensional torus.

**Setting.** $G = SU(2)$, a four-dimensional torus of side $L_0$, blocking factor 2, the number of points on a side of the last lattice $M$ (fixed), the number of steps $n$, and lattice spacing $a_n = L_0/(M \cdot 2^n)$. Put the twist $n_{12} = 1$ into Wilson's action (multiply one stack of plaquettes in the $(1,2)$ plane by the central element $z = -1$):

$$S_{\mathrm{tw}}(U) = \beta \sum_p \Big(1 - \tfrac{1}{2}\operatorname{tr}\big(z_p U_{\partial p}\big)\Big), \qquad \beta = \frac{4}{g_0^2} \;(4)$$

Define the blocking transformation $T$ by an averaging that is gauge covariant and commutes with central shifts, and set $\rho_0 = e^{-S_{\mathrm{tw}}}$, $\rho_{k+1} = T\rho_k$.

**The twisted-torus ultraviolet statement (proposal, unproved).** There are a $\gamma > 0$ and constants $C, \kappa, c, r > 0$ depending only on $M$ such that, for every $\gamma_0 \in (0, \gamma)$ and every $n$, choosing the bare coupling $g_0 = g_0(n, \gamma_0)$, the following hold Assumption.

- **The coupling stays small (twisted box)**: the sequence of effective couplings $g_0 < g_1 < \cdots < g_n = \gamma_0$ is defined throughout.

- **Small-field analyticity (twisted box)**: $\rho_n$ has the form (2) of 15-00, and each $E_n(X,\cdot)$ is analytic in a complex neighbourhood of radius $r$ in the background-field variables, with $|E_n(X)| \le C\gamma_0\, e^{-\kappa d(X)}$.
- **Large-field rarity (twisted box)**: $|R_n(\Omega^c)| \le \exp\bigl(-c\,p(\gamma_0)^2|\Omega^c|\bigr)$ ($p$ is the same threshold factor as in the large-field rarity condition of 15-00).
- **Gaussian comparison condition for the partition function**: $e^{-CM^4} \le Z_n/Z_n^{\mathrm{G}} \le e^{CM^4}$ ($Z_n^{\mathrm{G}}$ is the sum of the Gaussian integrals around the isolated flat connections).
- **Observable-source condition**: the three above hold with the same constants even when sources (of strength $\le s_0$) are inserted for smoothed observables whose support is larger than the scale of the last lattice $L_0/M$.

None of the constants depends on $n$ or on $a_n$. The coupling staying small in the twisted box has the form of the conclusion of the Coupling-Smallness Induction Lemma.

**Where the twist works.** The flat connections on the last lattice are only those whose holonomies in directions 1 and 2 anticommute and whose holonomies in directions 3 and 4 are central elements, and up to gauge equivalence there are finitely many of them. The adjoint fluctuations around them are antiperiodic in direction 1 or 2, and there are no zero modes Assumption (a reading that extends to four dimensions the facts about the twisted part of $T^2$ in 25-01; the four-dimensional classification has not been checked against the sources). At fine steps, on the other hand, the twist has no effect, because the twist's plane can be moved by central gauge transformations and is not visible locally Reading (a physical estimate). The Background–Fluctuation Decomposition Lemma (15-07 § 02) holds as it is for the twisted plaquette $z_p U_{\partial p}$: since $z_p$ is a central element, one only replaces $W_{\partial p}$ in the proof by $z_p W_{\partial p}$ Proved here.

**What the statement says and does not say.** If shown, what it says is an estimate of the effective action uniform in the lattice spacing, and non-degeneracy at the last step; this amounts to Bałaban's Conditional Ultraviolet Stability Theorem + the unpublished basic inequality (the coupling staying small in the twisted box) + an extension to observables (the observable-source condition) + non-degeneracy at the last step from the twist. Subsequential limits of the smoothed observables follow from it by compactness. What it does not say is uniqueness of the limit, the OS axioms, recovery of rotations, non-triviality, or a gap uniform in the volume. **The difficulty of ultraviolet control at the fine steps is exactly the same as for the periodic $T^4$**; what the twist removes is only the incidental difficulty of the degeneracy of the last few steps Reading (a physical estimate). The role of the twisted-torus ultraviolet statement is not to reduce the difficulty but to state ultraviolet control in a pure form without zero modes.

**A test bench with the gauge taken out.** The two-dimensional $O(4)$ sigma model (the principal chiral model of $SU(2)$) has a field taking values in $SU(2)$, a dimensionless and asymptotically free coupling, and large fields, and no gauge. Writing ultraviolet control on the twisted cylinder $\mathbb{R}\times S^1_L$ (no zero modes because of the twist) gives the four-dimensional ultraviolet control with the gauge half of the source of difficulty "source: no global gauge fixing" taken out Reading (a physical estimate). As in 15-01 § 04, the ultraviolet limit has not been constructed for this model either, and transplantation has reached the variational problem and the background-field stage Cited. Writing stage 4 of Table 1 (one step of the small-field expansion) for this model is cheaper than rereading the second half of the four-dimensional series, and is a test that cuts out the "group-valued field × dimensionless coupling" part of ultraviolet control. Two-dimensional results imply nothing about four dimensions.

06

## Correspondence with the sources — two rows absent from the sources

Table 2 sets out, for each row of the twisted-torus ultraviolet statement, where it falls in Bałaban's series, whether it changes with the twist, and whether it is absent from the sources. The last column gives the correspondence in three-dimensional QED, where the method has been rewritten for an easier model (Dimock's three papers in § 03).

Table 2 Each row of the twisted-torus ultraviolet statement and its correspondence with the sources (labels per cell; the labels for the series are at the level of abstracts, or of the introduction via 7-07 § 03)

| Row | Where it falls in the series | With the twist ($n_{12}=1$) | Absent from the original sources? | Correspondence in three-dimensional QED |
|---|---|---|---|---|

| Row | Where it falls in the series | With the twist ($n_{12} = 1$) | Absent from the original sources? | Correspondence in three-dimensional QED |
|---|---|---|---|---|
| The coupling stays small (twisted box) | 109: constructs the β function and defines the renormalisation of the coupling by recursive renormalisation-group equations (within the small-field approximation) Cited (abstract). 122, 355: with large fields it is an assumption (Conditional Ultraviolet Stability Theorem), and the basic inequality, which removes it, is unpublished Cited (7-07 § 03) | Unchanged Reading (a local quantity of the fine steps; the twist's plane is moved by central gauge transformations) | **Absent**: the one-step remainder including large fields (the net one-step bracket of the coupling). The rest is elementary by the Coupling-Smallness Induction Lemma | 1512.04373: shows, in the bounded-field approximation, the pattern of making the bare quantities depend on the lattice spacing so that they take predetermined values at the unit scale Cited (abstract). What is tuned are relevant quantities (the vacuum energy density, the fermion mass), not a marginal coupling |
| Small-field analyticity (twisted box) | 109 (localised effective actions by a cluster expansion at each step) and 116 (completion of the construction in the small-field approximation) Cited (abstracts). The abstracts contain no wording that the radius of analyticity is uniform in the step | The fine steps are unchanged. In the last few steps, the centre of the expansion becomes an isolated twisted flat connection Reading | The form exists (in four dimensions, under the assumption that the coupling stays small). The expansion around the twisted configuration at the last step is absent from the sources Reading | 1712.10029: multiscale propagators and the local regularity of minimisers Cited (abstract) |
| Large-field rarity (twisted box) | 119 (a description of the complete induction including large-field regions), 122, 175 (the basic step of the R operation), 122, 355 (localisation, exponentiation, estimates) Cited (abstracts) | Unchanged Reading | The form exists (in four dimensions, under the assumption that the coupling stays small) | 2009.01156: closes including large fields Reading (from "the flow is fully controlled at weak coupling" in the abstract) |
| Gaussian comparison condition for the partition function | Upper and lower bounds on the partition function from the Conditional Ultraviolet Stability Theorem. The constants do not depend on the step, the lattice or the field Cited (7-07 § 03) | **Changes**: the integral at the last step becomes non-degenerate around the isolated flat connections Assumption (a reading that extends the fact about $T^2$ in 25-01 to four dimensions) | The form of the upper and lower bounds exists. The normalisation that makes the denominator a sum of Gaussian integrals around the isolated flat connections is absent from the sources, but it is a problem of finite-dimensional integrals Reading | 2009.01156: upper and lower bounds on the renormalised partition function, uniform in the lattice spacing Cited (abstract) |
| Observable-source condition | The introduction of 122, 355 leaves the analysis of the expectations of loop variables as future work Cited (7-07 § 03). MRS: Schwinger functions in the axial gauge (with an infrared cutoff, no detailed proof) Cited | The fine steps are unchanged. The last step becomes non-degenerate, as in the Gaussian comparison condition for the partition function Reading | **Absent** | Correlation functions are left as the task of the next stage Cited (abstract of the overview arXiv:2204.07201) |

What to read from Table 2 Reading . What is absent from the sources are two rows, the content of the coupling staying small in the twisted box (the one-step remainder including large fields, that is, the net one-step bracket of the coupling) and the observable-source condition; these two are where new mathematics is needed. The

normalisation of the Gaussian comparison condition for the partition function and the non-degeneracy of the last step are specific to the twist, but they are problems of finite-dimensional integrals, and the difficulty is of a different quality. Three-dimensional QED in 1512.04373 has, in the bounded-field approximation, a statement of the type that tunes the bare relevant quantities to fix the endpoint of the flow, and this is the form of the conclusion of the Coupling-Smallness Induction Lemma. Writing the exponent of the three ultraviolet conditions (twisted-box version) with $p(\gamma_0)^2$ matches the form of the large-field factor that comes from the original threshold $\varepsilon = g\,p(g)$ (equation (7) of 7-05); writing it with a negative power of $\gamma_0$ would give a statement stronger than the original threshold. A one-line statement of what the two rows must show to close, and proposals for them, are in 15-08.

07

## What cannot be said

(a) The content of the series in Table 1, apart from the opening of the three-dimensional paper and the abstract and introduction of the last paper, is from abstracts for stages 4 and 5 and 119, and a reading from the titles for stages 1–3.

(b) We do not know the content of the basic inequality (unpublished). The judgement that it is unpublished rests on the sentence in the introduction and on the bibliography within the range we searched.

(c) The chain of § 04 is the text of a source on flat fields and says nothing about the group. "The measure of the expectation taken in closure under the fluctuation expectation does not depend on the background" is a reading of the chain of the proof.

(d) The twisted-torus ultraviolet statement is a proposal and has not been proved. Even if proved, it says nothing about uniqueness, the OS axioms, non-triviality, infinite volume, or a gap uniform in the volume. The classification of twisted flat connections in four dimensions has not been checked against the sources.

(e) The correspondence in Table 2 is at the level of abstracts and of introductions (via 7-07); it does not reach the section and theorem numbers in the text of the series.

08

## Table of the five grades and the dependency graph

Table 3 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [01] | Abstracts of 109, 116, 119, 122 (175) of the series, and of the three three-dimensional QED papers | Cited | § 01, § 03, Table 1 (abstracts) | Section and theorem numbers in the text not checked |
| [02] | The Conditional Ultraviolet Stability Theorem assumes that the coupling stays small; the basic inequality is unpublished | Cited | § 02 (abstract and introduction) | Finite-volume lattice |
| [03] | MRS has an infrared cutoff and gives no detailed proof | Cited | § 03 (introduction) | $SU(2)$, trivial sector, axial gauge |
| [04] | The One-Step Bound on the Activity Norm and the chain (1)–(3) | Cited | § 04 (the text of Appendix A.6.2, pp. 214–215) | Flat fields. The source takes the inequality exchanging the expectation and the seminorm from reference [54] |
| [05] | In the chain, the measure of the expectation taken in closure under the fluctuation expectation does not depend on the background | Reading | § 04 (a reading of the chain of the proof) | Flat fields |
| [06] | Ultraviolet statement for a small twisted four-dimensional torus | Assumption | § 05 (proposal, unproved) | $SU(2)$, $n_{12} = 1$, finite volume; the three ultraviolet conditions (twisted-box version) in the form of the original threshold |

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [07] | The Background–Fluctuation Decomposition Lemma also holds for the twisted plaquette $z_p U_{\partial p}$ | Proved here | § 05 ($W_{\partial p}$ replaced by $z_p W_{\partial p}$ in the proof of 15-07 § 02) | $z_p$ is a central element |
| [08] | What the twist removes is only the degeneracy of the last few steps | Reading | § 05 (a physical estimate) | — |
| [09] | The twisted cylinder for two-dimensional $O(4)$ is a test bench that takes the gauge out of ultraviolet control | Reading | § 05 (a physical estimate) | Two-dimensional results imply nothing for four dimensions |
| [10] | Correspondence of each row of the twisted-torus ultraviolet statement with the sources. Absent from them: the net one-step bracket of the coupling and the observable-source condition; the Gaussian comparison condition for the partition function is a finite-dimensional problem | Reading | § 06 Table 2 | Level of abstracts and introductions |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] CIT ABS    abstracts of the series and of three-dimensional QED
[02] CIT THM    Conditional Ultraviolet Stability Theorem, basic inequality unpublished
[03] CIT MRS    what MRS reached
[04] CIT BBS    One-Step Bound on the Activity Norm and the chain           flat fields
[05] RDG FLAT   measure of closure under fluctuation expectation independent of background ⟸ [04]
[06] ASM T4     twisted-torus ultraviolet statement ⟸ [02]                  proposal
[07] PRV RTW    Background–Fluctuation Decomposition Lemma unchanged by the twist   z_p central
[08] RDG TW     the twist only at the last step ⟸ [06]
[09] RDG O4     the two-dimensional O(4) test bench ⟸ [06]
[10] RDG MAP    twisted-torus ultraviolet statement and the sources ⟸ [01][02][03][06][07]   level of a
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. The row of the twisted-torus ultraviolet statement is an assumption row, in the form of a question that restates ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

## What this chapter can say

| Said | Not said |
|---|---|
| Cited The assumption of Bałaban's Conditional Ultraviolet Stability Theorem and the unpublished basic inequality, the abstracts of stages 4 and 5 and 119 of the series and of the three three-dimensional QED papers, the reservations of MRS, the One-Step Bound on the Activity Norm of BBS and its chain | Section and theorem numbers in the text of the series (stages 4 and 5 and 119 only up to abstracts, stages 1–3 titles only). The content of the basic inequality (unpublished) |
| Proved here That the Background–Fluctuation Decomposition Lemma is unchanged for the twisted plaquette | The non-degeneracy of the last step around the twist |
| Reading The correspondence of each row of the twisted-torus ultraviolet statement with the sources (Table 2), and that the measure in the chain does not depend on the background | Checking against the text of the series |
| Assumption The twisted-torus ultraviolet statement | Its proof. Any prospect that ultraviolet control closes |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Conditional Ultraviolet Stability Theorem, basic inequality (unpublished) | Cited (abstract and introduction, pp. 355–356) | T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (the source table of 7-07) |

| Item | Kind | Source / tools |
|---|---|---|
| Stages 1–3 of the series (Table 1) | Cited (bibliography and titles only) | Comm. Math. Phys. 89 (1983); 95, 96 (1984); 98, 99 (two papers), 102 (277–309) (1985) |
| Stages 4 and 5 of the series and 119 (Tables 1 and 2) | Cited (abstracts) | T. Bałaban, Comm. Math. Phys. 109 (1987) 249–301; 116 (1988) 1–22; 119 (1988) 243–285; 122 (1989) 175–202 |
| Three-dimensional ultraviolet stability | Cited (the text, pp. 255–257) | T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275 (the source table of 7-05) |
| Three-dimensional QED ( § 03, Table 2) | Cited (abstracts) | J. Dimock, Ultraviolet regularity for QED in d=3, arXiv:1512.04373; Multiscale block averaging for QED in d=3, arXiv:1712.10029; Ultraviolet stability for QED in d=3, arXiv:2009.01156; overview arXiv:2204.07201 |
| Low-temperature expansion of the $N$ vector model ( § 03) | Cited (abstracts) | T. Bałaban, Comm. Math. Phys. 196 (1998) 485 |
| What MRS reached, and its reservations | Cited (pp. 325–328) | J. Magnen, V. Rivasseau, R. Sénéor, Construction of YM4 with an infrared cutoff, Comm. Math. Phys. 155 (1993) 325–383 |
| The One-Step Bound on the Activity Norm and the chain ( § 04) | Cited (the text of Appendix A.6.2 and of the One-Step Bound on the Activity Norm, pp. 214–215) | R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474v2 (the source table of 24-06) |
| Table 2, the twisted-torus ultraviolet statement | This book's study | The record of this book's study (organising the study of ultraviolet control) |

**Terms used in this chapter** — the five stages of Bałaban's series, the Conditional Ultraviolet Stability Theorem, the basic inequality (unpublished), MRS, the One-Step Bound on the Activity Norm, the interchange of expectation and seminorm, the twisted-torus ultraviolet statement, 't Hooft's twist, the test bench, the two rows absent from the sources. Their meanings are in the A-02 glossary.

Next chapter: 15-03 The decomposition statement — splitting ultraviolet control into a class that closes with finitely many parameters and three remainders.

# The decomposition statement — a class that closes with finitely many parameters, and three remainders

This chapter is a map of open questions. § 03 and § 04 are this book's own investigation.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — That ultraviolet control can be stated in a form that splits it into "a class that closes with finitely many parameters" and three remainders — the coupling-flow remainder condition, the shape remainder condition, and continuity within the domain (continuity of the coupling) (a statement only). That, among these, the condition that the coupling stays small follows in an elementary way from the estimate of the one-step remainder (the net one-step bracket of the coupling) and the definition and continuity of the effective coupling (the Coupling-Smallness Induction Lemma, with proof). That if the one-step map contracts transversely around the class, the departure from the class is bounded by a supremum rather than by a sum over steps, uniformly in the number of steps (the bookkeeping of the transverse contraction). That the class is better taken not as a family of constant multiples of the heat kernel or the Wilson action, but as the one-parameter class into which the one-step map attracts. That compared with the decomposition of the band, the form of the split is the same, and the difference lies in whether there is something to pay the one-step loss against. The three ultraviolet questions that come out of this decomposition (transverse contraction on a one-parameter class, minimal model of the remainder including large fields, smallest adjoint eigenvalue on the twisted lattice). In a revision we added a proposal for the final form of the coordinates of the class in the class-closure condition, and how the shape remainder condition closes when the background is given to the class as a parameter (conditionally). In the third revision we added the final form of the four-dimensional class coordinates (one parameter and ratio coordinates whose values are fixed by the flow), the blocking conditions needed for the shape remainder condition over many steps, the one-loop measurement of the flow of the coupling, and a one-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling.

**Prerequisite chapters** — 15-00 § 01 (ultraviolet control and the form that closes in one step), 15-01 § 01 (that the sum of losses diverges logarithmically), 15-02 § 01 (the assumption of Bałaban's Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989)), 26-06 § 01 (the decomposition of the band).

**Said first** — This chapter does not show ultraviolet control. What it shows are a statement of what would need to be shown, two elementary components (the Coupling-Smallness Induction Lemma and the bookkeeping of the transverse contraction), candidate rules, and questions. Of the implications of the decomposition, what we have checked in an elementary way is only the Coupling-Smallness Induction Lemma and the arithmetic of products and geometric series; none of the class-closure condition, the coupling-flow remainder condition or the shape remainder condition is shown in four dimensions. Unless stated otherwise, a "step" in this chapter means one step of the real blocking (with

many links, gauge re-fixing and large fields), and the numbers for the smallest systems (15-04 to 15-06) are not a substitute for it.

The order of this chapter

1. The class and the one-step map — the flow of the parameters $\Phi$ and the remainder $r_\theta$
2. The decomposition statement — the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling
3. The Coupling-Smallness Induction Lemma — splitting the coupling staying small into the net one-step bracket of the coupling and continuity of the coupling (with proof)
4. The bookkeeping of the transverse contraction — the departure by a supremum, not a sum
5. The bundle of rules (Table 1)
6. Contrast with the decomposition of the band (Table 2)
7. Three ultraviolet questions (Table 3)
8. What cannot be said
9. The table of the five grades and the dependency graph (Table 4)

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00

## What this chapter shows

- The ultraviolet wall can be stated in a form that splits it into "a class that closes with finitely many parameters" (the class-closure condition) and "the remainder of the flow, the remainder of the shape, and continuity" ( § 02, a statement only).
- The condition that the coupling stays small, "the effective coupling stays in $(0, \gamma]$", follows from the estimate of the one-step remainder (the net one-step bracket of the coupling) and continuity within the domain (continuity of the coupling) by monotonicity and an intermediate-value argument alone (the Coupling-Smallness Induction Lemma, § 03). The real content is the net one-step bracket of the coupling.
- If the one-step map contracts transversely around the class by $\lambda < 1$, the departure from the class is bounded by $\sup_j \varepsilon(\beta_j)/(1-\lambda)$, uniformly in the number of steps ( § 04).
- The class is taken as "the one-parameter attracting class", and the deviation of the form $O(1/B)$ is carried by the shape remainder condition. We do not state it as a class that keeps the ratio of the quadratic couplings at 2 ( § 02; grounds in 15-06).
- The proposal is that the coordinates of the class in the class-closure condition can be written out as the coupling matrix, the shape, the background, the one-link coupling, the corner triple product and the multi-face quartic. If the background is given to the class as a parameter, $\sum_k \eta_k$ of the shape remainder condition becomes finite under conditions, and the difficulty gathers in the class-closure condition ( § 02).
- The four-dimensional class has the form "one parameter $B$ + ratio coordinates whose values are fixed by the flow (2 quadratic, 3 cubic, 65 quartic)"; the coordinate of the skew neighbour is nonzero whatever the rule, but its value depends on the rule. The "local" of the class-closure condition is written with the rule attached. The shape remainder condition over many steps needs blocking

conditions, and the flow of the coupling agrees in direction and order of magnitude at one loop ( § 02; all at tree level or within the range measured at one loop).

- The way of splitting is the same as in the decomposition of the band, with the difference that in the ultraviolet there is nothing to pay the one-step loss with ( § 06).

01

## The class and the one-step map

The goal of ultraviolet control was to show at every step the formula (3) of 15-00 § 01 — if the effective density at step $k$ lies in the class $\mathcal{K}(g_k; r, \kappa, c)$, then the effective density at step $k+1$ lies in the class $\mathcal{K}(g_{k+1}; r, \kappa, c)$ with the same constants. As we saw in 15-01 § 01, in four dimensions the part that moves by an amount of size $g_k^2$ in one step cannot be carried as an error, and must be made a motion of the parameters of the class. This section sets up the frame in which that "motion of the parameters" and "the rest" are written separately.

Take a family of weights $\mathcal{C} = \{w_\theta : \theta \in \Theta\}$ indexed by finitely many parameters $\theta = (\theta^1, \dots, \theta^m) \in \Theta \subset \mathbb{R}^m$, and suppose that one step of blocking $T$ takes the form

$$T\big(w_\theta\,(1+\rho)\big) = w_{\Phi(\theta)}\,\big(1 + r_\theta[\rho]\big) \quad (1)$$

Here $\Phi : \Theta \to \Theta$ is the **flow of the parameters** and $r_\theta[\rho]$ the **remainder** (gathering the localised irrelevant terms and the large-field factors); the size of the remainder is measured by the norm of the weight $G_k$ of 15-07. (1) is a form of definition: for any $T$ it can be written once the choice of $\Phi$ and $r$ is fixed. The content lies in "whether $\Phi$ and $r$ can be chosen so as to have estimates uniform in the step".

**A rule of thumb for counting the parameters.** For pure gauge theory in four dimensions, the local terms that remain under power counting, given gauge invariance and the lattice symmetries, are two, the vacuum energy (a constant) and the coefficient of $F^2$, and we read $m = 2$, $\theta = (E, \beta)$, $\beta = 4/g^2$ Reading (power counting; the four-dimensional version of the sentence in 15-01 § 01 that in three dimensions "the counterterm is a single constant"). In the smallest systems, however, the action after one step does not fit into a constant multiple of a single quadratic form, and "shape coordinates" such as the ratio of the couplings by loop shape move (15-06). Under power counting these coordinates correspond to irrelevant directions that act at the square of the lattice spacing; if the one-step map contracts them, then there is effectively one parameter, $\beta$, and the shape deviation is paid for by the transverse contraction $\lambda$ — that is the reading Reading . The class-closure condition of § 02 puts this reading into the form of a statement.

02

## The decomposition statement — the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling

From the following four, we set down a statement of the form that ultraviolet control follows uniformly in the number of steps Assumption (the form of the statement only; none of the four is proved in four dimensions).

- **Class-closure condition.** The number of parameters $m$ is finite independently of the step, and the leading part of $T$ returns into the family (every change of the leading part goes into $\Phi$). The class is taken not as the family of constant multiples of the heat kernel or the Wilson action, but as **the one-parameter attracting class of the one-step map** — a single coupling $B$ (in the direction of $\beta$) and a

fixed shape of $O(1/B)$ determined by the curvature of the group. The shape coordinates (the coefficients of the action of order $\psi^4$ and higher, the ratio of the couplings by loop shape, and so on) are directions that contract at each step, and the shape deviation from the class is carried by the shape remainder condition.

- **Coupling-flow remainder condition.** The $\beta$ component of $\Phi$ is $\beta_{k+1} = \beta_k - 8b_0 \ln L + O(1/\beta_k)$. This is the net one-step bracket of the coupling of § 03 written in the variable $\beta$. The values measured at one loop for one step with one rule are in "the measured flow of the coupling" below: the direction and order of magnitude agree, and the coefficient of $\ln L$ does not agree at small blocking factors Reading .
- **Shape remainder condition.** $\sum_k \eta_k < \infty$ of the type $\|r\theta_k[\rho_k]\|_{G_{k+1}} \le (1+\eta_k)\, 2^{|B|}\, \|\rho_k\|_{G_k}$, and $\rho_k$ itself is either bounded by a supremum through the transverse contraction ( § 04) or stays at a finite size of the sum. $\sum_k \eta_k < \infty$ can be expected only for the background-dependent **shape part** (the part whose coefficients are proportional to the values of the shape coordinates and which enters a geometric series); the **group-geometry part** of $SU(2)$ (the BCH cubic) has a divergent sum of the type $g_k^2 p$, so the field strength of the background is given to the class as a parameter and moved to the side of the parameters Reading (15-07 § 07). If the logarithm $a$ of the background holonomy is given to the class as a parameter, in the smallest system the rest drops to $\theta^0 \beta^{-3/2}$, and $\sum_k \eta_k$ is bounded uniformly in the number of steps (conditionally, 15-07 § 06). The difficulty has not disappeared, however, but moved to the class-closure condition — whether a class carrying $a$ closes in one step of faces and multiple loops Reading . The contraction of the cone $\ell^4$ at tree level for the shape remainder is at most $0.823$ in the first step (blocking that takes a coarse edge as the sum of two fine ones, with the normalisation that divides by the long-wavelength coupling), but **it is not uniform in the number of steps, and blocking conditions are needed** Reading : the covariance-flow fixed-point condition: the flow of the covariance of the blocked field has a scale-invariant fixed point (with blocking by edge sums the coarse action becomes half as soft at each step and there is no fixed point, the factor being 1.26 or more at the third step; with gauge-covariant transverse averaging the direction is towards a fixed point). The cone-contraction condition at the fixed point: $\sup \mu_\infty < 1$ at that fixed point (with transverse averaging the lower bound 0.57–0.64 does not grow with the step; there is no upper bound). Conditions: restricted to the abelian case, tree level, periodic, $2^4$ blocks, linearised; the numbers come from the symbol flow ($k \le 5$) and local optimisation over periodic configurations on the torus, with no interval estimates (15-05 § 07).
- **Continuity of the coupling** (continuity within the domain; § 03).

**The coordinates of the class in the class-closure condition (a proposal for the final form).** Putting the results for the smallest systems side by side, we read that the class of the class-closure condition is best stated as a family with the following coordinates Assumption (only the names and origins of the coordinates; we do not show that it closes in four dimensions).

- **The coupling matrix** $B_{pq}$: the coupling $B_p$ per face, and the coupling of pairs of adjacent faces (placed at the ends of the shared edge). In one step of three-dimensional $2 \times 2 \times 2$, the quadratic part closes exactly with these two by the symmetry of the coarse box, and the tree-level ratio is $1/10$ (15-07 § 07).
- **The shape** $q_p$: the shape coordinates of $O(1/B)$ per face. They contract at each step (at most $17/125$ for four in a row, $1/25$ for the cross).
- **The background** $a_p$: the logarithm of the background holonomy per face. In one step of a single loop it is updated exactly by the ordered product in the group (non-abelian Stokes), and the BCH

cubic enters this coordinate at all orders (15-07 § 07).

- **The one-link coupling** $b$: the size of the sum of staples $\times\ \beta$. Re-taking the centre moves it only by $cg^2$ $p^2$, and it is carried as an update of the parameters (15-07 § 05).
- **The corner triple product** $\kappa$: $\kappa \sum_c a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of the cube. It is the only cubic non-abelian term in $2 \times 2 \times 2$ ($3/1024$ at tree level), is larger than the shape truncation in the range $\theta \sim \beta^{-1/2}$, and cannot be omitted. In genuine multi-step blocking that fixes only the edges of the coarse cube (three dimensions, free boundary $L^3$, $L \leq 20$, tree level), the ratio $\kappa/c_2$ does not contract at the second step but grows as 0.019, 0.033, 0.050, while the step-to-step ratios 1.78, 1.30, 1.13, 1.10 approach 1, so it looks like a shape approaching a value of 0.055–0.066 Reading (in the form of a fit; the existence of the limit is not shown; one loop ($\beta = 32$–$128$, $L \leq 4$) moves the ratio by no more than 3%). The first-step ratio 1.78 is not a wall but the shape of an approach to the fixed point from below (15-07 § 07).
- **The multi-face quartic**: couplings of three and four faces (counted by orbits of the symmetry of the coarse box).

This writes out the above "one-parameter attracting class + a contracting shape" down to the coordinates of the background and of multiple loops. The view is that there is still a single marginal direction (the direction of the coupling $\beta$), and that the other coordinates either contract like the shape coordinates or are carried by the group product as the background itself Reading .

**The final form of the coordinates of the four-dimensional class.** Measuring the blocking that fixes only the 32 edges of the coarse hypercube ($32L$ fine links) in the four-dimensional free-boundary box $\{0, \ldots, L\}^4$, from one step ($L = 2$) up to genuine multi-step blocking ($L \leq 10$ at tree level), the class of coarse weights takes the form of **one parameter** $B$ ($\beta' = \beta b$) **and ratio coordinates whose values are fixed by the flow** Reading ($SU(2)$, heat kernel, the leading saddle-point term and one loop ($L \leq 4$, $\beta = 32$–$128$, $O(1/\beta)$ not included), the face angle $\lesssim 0.5$). The number of ratio coordinates is counted by the representation theory of the hyperoctahedral group $B_4$: 2 quadratic (pairs of neighbours sharing an edge, and pairs of "skew neighbours" sharing no vertex), 3 cubic (the corner, path and parallel types of a cell), and 65 quartic Proved here (the space of linearised coarse face angles is 17-dimensional and splits into $3 + 6 + 8$; the counts are computed by two routes, the average of characters and the numerical rank; the Bianchi rank 7 of a cell is in Lean: `relation_is_multiple_of_sigma`, assuming 24 equalities per face). Unlike in three dimensions, the per-face class and the neighbour pairs alone do not close even at quadratic order (0.6% remains). The quadratic and cubic ratios do not go to 0 at the second step; the step-to-step ratios approach 1 (1.55, 1.12, 1.08 for the skew neighbour), and they approach values around $-0.145$ for the neighbour, $-0.021$ for the skew neighbour, and $0.064$, $0.013$, $-0.021$ for the cubic ones Reading (fit $r_\infty - A/L - B/L^2$ over $L = 4$–$10$; the existence of the limit is not shown; one loop moves the two-step ratios by less than 1%). Many of the first-step values lie on the 0 side of the fixed point, and a ratio exceeding 1 at the second step is the shape of an approach Proved here (Lean: `ratio_gt_one_below_fixpt`, assuming $0 \leq \lambda < 1$ and $0 < r < s/(1 - \lambda)$). For the quartic, the size of the multi-face part at the second step is 1.6 times its ratio to $c_2$, the direction is nearly kept (cosine 0.99), and we have not measured whether it is a fixed point.

**The skew neighbour is not 0 — but its value and "local" depend on the rule.** Within one coarse hypercube, the rule that fixes edges and the rule that fixes all faces give the same quadratic form Proved here ($32 - 15 = 24 - 7 = 17$ and (iii) of the Bundle Network Theorem; the count of degrees of freedom is in Lean: `edge_face_dof_agree`, assuming exactness of the augmented cochain complex). A genuinely different rule fixes all coarse faces on a periodic network in which coarse cells share fine fields; there too the skew neighbour approaches a nonzero negative value, but the value is about twice

as large ($-0.045$), or $-0.06$–$-0.10$ depending on the convention for projecting onto cell types, and the part that does not fit in one cell grows with the step up to about 20% of the quadratic form Reading (tree level of the abelian linearisation, quadratic, infinite coarse lattice, $L \leq 16$). **The sign and the non-vanishing depend neither on the rule nor on the projection convention; the value does.** The "local" of the class-closure condition is therefore written with the rule attached: for rules that separate cells, "closes within one coarse hypercube (reach one side of the coarse lattice)"; for rules that share fields, "quasi-local with a tail" Reading . The only independent parameter is $B$, and the ratio coordinates are quantities that carry the values of the fixed point. Note that at tree level, with either rule, $b$ of the cell part falls off as $1/L$ ($bL \to 0.39, 0.20$), the long-wavelength eigenvalue stays at 1, and **there is no running in** $\ln L$ Reading . The logarithmic running comes only from one loop (next paragraph).

**The measured flow of the coupling (one loop).** Measuring the flow of the class parameter $B$ at one loop, on the periodic network that fixes all coarse links (products along lines of $b$ fine links), the long-wavelength coefficient of the coarse action for $SU(2)$ goes down in one step (block 2) in the direction of asymptotic freedom for every action: $\beta' - \beta = -0.745$ (Manton), $-1.078$ (heat kernel), $-1.136$ (Wilson) Reading ($\beta = 4/g^2$, one loop of $\beta \to \infty$, abelian background, transverse coarse torus $6^3$, extrapolated to wavenumber 0 from length $\leq 32$). For $U(1)$ it is 0 for Manton and only the constant $-0.234$ of the non-quadratic nature of the action for Wilson, depending hardly at all on the blocking factor. The non-abelian part with the action-dependent constants removed (the tadpole; for the heat kernel also $-\frac{1}{3}$ of the Van Vleck prefactor) is $-0.43$, and its change $-0.25$ from blocking factor 2 to 3 is 1.7–1.9 times the one-loop coefficient $-\frac{11}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ Reading (wavenumber 0 for blocking factor 3 is a two-point extrapolation). With this rule one can say as far as "the direction and order of magnitude of the flow agree with asymptotic freedom, but the coefficient of $\ln b$ does not agree at small blocking factors". The difference $-\frac{1}{3}$ between heat kernel and Manton, and the difference $\frac{5}{3}t$ between Wilson and Manton ($t$ being the Wilson value of $U(1)$), are identities Proved here (bookkeeping of the coefficients of $s^2$; the arithmetic is in Lean: `tadpole_weights`, `vanvleck_ratio`). A rule fixing the products along lines is likely to integrate fluctuations up to wavelengths longer than the coarse scale (the $1/L$ of the tree level above); comparing the coefficients needs an averaging rule or larger blocking factors (the question "one-loop coefficient of the long-wavelength network"). The coefficient $b_0$ follows the convention of equation (9) of 7-05.

Of the implications, that the coupling stays small follows from the coupling-flow remainder condition and continuity of the coupling is the Coupling-Smallness Induction Lemma Proved here ; that the product of constants $\prod_k (1 + \eta_k)$ is bounded by the shape remainder condition is the arithmetic of products Proved here ($1 + x \leq e^x$); that the departure is uniform in the number of steps under the transverse contraction is a geometric series Proved here ( § 04, assuming the one-step inequality); and the part in which localisation and rarity follow from the norm of the weight is a reading from the framework of Appendix A for flat fields Reading (15-02 § 04). As for the correspondence between components and conditions: the parameter $\beta$ of the class-closure condition is the "coupling" of the coupling staying small itself (the rule of projection onto the leading term in § 05), the coupling-flow remainder condition is the coupling staying small (the rule of the flow of the coupling), the localised part of the shape remainder condition is the small-field analyticity condition (the rule of the activity norm), and its large-field part is the large-field rarity condition (the rule of large-field rarity).

**Why we do not state it as "a class that keeps the ratio at 2".** In the smallest system (four plaquettes in a row, 15-06) one finds a curve of actions that keeps the ratio of the two quadratic couplings at its tree value 2 (actions mixing the fundamental and adjoint representations, with adjoint coupling $\rho \approx -1/4$). But the one-step map acts in the direction that contracts the shape coordinates, and sends that curve in

one step to a place $O(1)$ away from it, so the curve does not close under the flow (15-06; SU(2), four in a row, the one step defined on the diagonal, at the level of the leading term) Reading . On the attracting class of the flow, the ratio is $2 + \frac{2}{3B} + O(B^{-2})$, and the leading term of the deviation is a one-loop quantity coming from the curvature of the group Proved here (at the one-loop level, the same system). So the ratio 2 is not a coordinate that can be tuned by a parameter, but a quantity that remains in the attracting class. Stating the class-closure condition not as "a class that keeps the ratio at 2" but as "the one-parameter attracting class + a contracting shape" is consistent with the smallest systems we measured Reading (limited to this system).

**An example with remainder 0.** In the one-link heat-kernel class $\{p_\tau\}$ of 15-04 § 01 (one parameter), $T(p_\tau) = p_{\tau+t}$ exactly and the remainder is 0 Proved here (the semigroup property). That the losses form a telescoping sum is a restatement of this "remainder 0", and can be read as another face of the same fact as the background-independent fluctuations (the Lemma on One Step with Background-Independent Fluctuations of 15-07) Reading .

03

## The Coupling-Smallness Induction Lemma — splitting the condition that the coupling stays small in two

Bałaban's assumption "the effective coupling stays in $(0, \gamma]$" (15-02 § 02) is a sentence of the form "it can be satisfied by choosing the bare coupling $g_0$ well". But the effective coupling $g_{k+1}$ is defined only while the earlier steps are small (since it is a coefficient read off from the effective action). Making this explicit through the domain, the coupling staying small splits into two conditions.

**Coupling-Smallness Induction Lemma.** Suppose $b > 0$, $C \geq 0$, $\gamma_1 > 0$ satisfy $C\gamma_1^2 \leq b/2$ and $\gamma_1^2 < 2/(3b)$, and fix $n \geq 1$. Suppose that for each $g_0 \in (0, \gamma_1]$ the sequence $g_0, g_1, \dots$ is defined inductively in the form "if $g_k \in (0, \gamma_1]$ then $g_{k+1}$ is determined", and satisfies the following two.

- **Net one-step bracket of the coupling** (the remainder of one step). $\frac{1}{g_{k+1}^2} = \frac{1}{g_k^2} - b + r_k$, $|r_k| \leq C\, g_k^2$.
- **Continuity of the coupling** (continuity within the domain). On $D_k := \{g_0 :\ g_0, \dots, g_k \in (0, \gamma_1]\}$, $g_0 \mapsto g_{k+1}(g_0)$ is continuous.

Then for every $\gamma_0 \in (0, \gamma_1)$ there is a $g_0$ with $g_0 < g_1 < \cdots < g_n = \gamma_0$ (all defined).

Proved here Proof. (i) If $g_k \leq \gamma_1$, then from the net one-step bracket of the coupling and $Cg_k^2 \leq b/2$

$$\frac{1}{g_k^2} - \frac{3b}{2} \;\leq\; \frac{1}{g_{k+1}^2} \;\leq\; \frac{1}{g_k^2} - \frac{b}{2} \quad (2)$$

The left-hand side is at least $1/\gamma_1^2 - 3b/2 > 0$, so $g_{k+1}$ is a positive finite number, and from the right, $g_{k+1} > g_k$. (ii) Set $J := \{g_0 \in (0, \gamma_0] :\ g_0, \dots, g_n$ all defined and $\leq \gamma_0\}$. By the monotonicity in (i), $J = \{g_0 \leq \gamma_0 :\ g_n$ is defined and $g_n \leq \gamma_0\}$. If $g_0^2 \leq (1/\gamma_0^2 + 3bn/2)^{-1}$, then repeating the left of (2), $1/g_k^2 \geq 1/g_0^2 - 3bk/2 \geq 1/\gamma_0^2$ $(k \leq n)$, so $J$ is non-empty. (iii) Let $s := \sup J$ and take a sequence $g_0^{(j)} \to s$ of points of $J$. Since $s \leq \gamma_0 < \gamma_1$, $g_1(s)$ is determined, and by continuity of the coupling $g_1(s) = \lim_j g_1(g_0^{(j)}) \leq \gamma_0$. Repeating the same for $k = 2, \dots, n$ gives $s \in J$. (iv) If $g_n(s) < \gamma_0$, then from $n \geq 1$ and (i), all $g_k(s) < \gamma_0 < \gamma_1$. By continuity of the coupling, $g_0, \dots, g_n$ are also defined in a right neighbourhood of $s$ and are $< \gamma_0$, contradicting $s = \sup J$. Hence $g_n(s) = \gamma_0$, and by monotonicity $g_0 < \cdots < g_n = \gamma_0$. ∎

A variant: if the net one-step bracket of the coupling is replaced by $|r_k| \le C g_k^2 + \eta_k$ ($0 \le \eta_k \le b/4$), the same proof goes through with the constants of (2) replaced by $7b/4$ and $b/4$ and the positivity condition by $\gamma_1^2 < 4/(7b)$ Proved here . Non-universal scale-dependent terms are absorbed as a constant shift in $1/g^2$ .

**What the Coupling-Smallness Induction Lemma says.** The coupling staying small splits into **the estimate of the one-step remainder (the net one-step bracket of the coupling)** and **continuity within the domain (continuity of the coupling)**, and the rest (monotonicity and the intermediate-value argument) is elementary. The net one-step bracket of the coupling is the statement, on the side of the flow of the coupling, of "absorb the part of size $g_k^2$ into the flow, and confine the remainder to $O(g_k^2)$ in the variable $1/g^2$".

**What the Coupling-Smallness Induction Lemma does not say.** The net one-step bracket of the coupling has not been shown in this book's design for four dimensions with the non-abelian large-field contributions included (the published construction, by a different design, carries the coupling through nested bands step by step; 14-03). This corresponds to the content of the unpublished basic inequality (Theorem 2 of Bałaban 1989) Assumption (since the text of the basic inequality (unpublished) does not exist, the correspondence is a reading of one sentence in the introduction). Continuity of the coupling is the statement that "the effective action at each step depends continuously on the bare coupling, and the effective coupling is extracted as a continuous functional of it". Since these are integrals over finitely many compact groups, the line of argument goes through by the dominated convergence theorem, but since the way the effective coupling is extracted is fixed inside the small-field expansion, it depends on the analyticity of that expansion (the small-field analyticity condition) Assumption (outline only). We do not say that the Coupling-Smallness Induction Lemma has made the coupling staying small easier. It only separates the bookkeeping part from the main part. A proposal for estimating the net one-step bracket of the coupling with large fields included is in 15-08.

**Splitting the net one-step bracket of the coupling further.** By an identity the remainder of the net one-step bracket of the coupling splits into $r = r^{\mathrm{small}} + r^{\mathrm{large}}$, and if the fluctuation is a product of heat kernels link by link, the large-field part $r^{\mathrm{large}}$ is bounded by the Lipschitz constant of the read-out $\times$ the rarity Proved here (one link, one step, background-independent fluctuations; the Large-Field Remainder Bound Lemma and the Heat-Kernel Fluctuation Rarity Lemma of 15-08 § 02). The small-field part (the small-field contribution to the coupling) splits into three: the Taylor truncation ($(h_0/r_0)^{2k+2}$, of the same form as Lemma 7.5.3 of the source for flat fields), the small-field cut-off ($e^{-c(1-q)^2 h^2 \beta}$), and the explicit quadratic terms between blocks together with their local cancellation; what is absent from the sources is the group version of the third — the place the source itself calls a sketch for flat fields Reading (15-08 § 06; the first two are Proved here under the analyticity condition at complex backgrounds and the margin $q$). This writes, conditionally and on the group, the same form as the structure known in the sources; it does not show the net one-step bracket of the coupling. In the smallest system of two blocks, the explicit quadratic terms between blocks appear at one loop as $O(1)$ (the local part goes into $\beta_0'$, the rest into the shape coordinates), and the local cancellation is needed for the part $O(1/\beta)$. The type of its non-local tail (flat, exponential, power) is decided by the definition of the coarse variables, and a sufficient way to obtain an exponential one is to avoid sharing that creates conserved quantities, to promote local constraints to coarse variables, and to separate global constraints as finitely many tags Reading (numbers for one-, two- and three-dimensional models; four dimensions not measured). Of the rows of the net one-step bracket of the coupling, what has the same form as in the sources is the projection of $\mathbf{Loc}$, the way of writing $W$, and the form of the Taylor remainder; four points — taking $\mathbf{Loc}$ as the projection onto components, the reason for the zero sum,

how the cut is paid for, and how the sum over steps is taken — may be cut differently from the sources Reading (15-08 § 06).

**The recursion of the inter-block terms of the net one-step bracket of the coupling — its form in the one-row model.** Carrying the inter-block terms to the second step, in the one-row model it can be written in the form "the image of the first-step non-local term closes in the same type, contracts by the ratio of couplings to at most $5/8$ at quadratic and $5/48$ at quartic order, and the terms newly born at that step do not depend on the step", and the non-local term is drawn geometrically not to 0 but to a definite value Proved here (a first-order Taylor formula writing the image as a conditional expectation, and rational numbers at tree level), Reading (the numbers: $SU(2)$ heat kernel, $U(1)$ Wilson as a control, one row, the diagonal definition, one step of two blocks of seven plaquettes taken twice, four types of adjacent pairs, first order in the non-local terms, up to quartic in the coarse field, $\beta = 32$–$512$ and extrapolation). When three or more superblocks are lined up, the quartic image contains terms with three coarse fields at once. For coarse variables that give half of each boundary plaquette to each side (with no conserved quantity), their coefficient decreases with the width $d$ of the triple as $e^{-\kappa d}$, $\kappa = \ln(5 + 2\sqrt{6}) \approx 2.29$ (the rate of the background map), the sum over terms containing one block is uniform in the volume, and its own image contracts to at most $5 \times 10^{-3}$ Reading (one step, tree and one loop, $L \leq 8$, $d \leq 7$; the prefactor is constant at tree level and first order in $d$ at one loop). The three-body coefficient for $d = 2$, however, is of the same order as the coefficient of adjacent pairs; only the tail is small. Hence in the one-row model, the inter-block terms of the net one-step bracket of the coupling close up to quartic order as "the pair type (fixed-point value + contracting deviation) + a three-body type decreasing exponentially in the width" Reading (the bookkeeping is in Lean: `affine_bound_uniform` ($0 \leq \lambda < 1$, $0 \leq C$), `pair_tail_uniform` ($0 \leq \lambda, \mu < 1$ and two recursive inequalities), `three_body_sum_le` ($0 \leq \rho < 1$, $|q(d)| \leq C\rho^d$)). Going back to a definition with a conserved quantity, the three-body coefficient is flat in the distance. For continuum fields in the sources this is the part written by the additivity "the non-local terms of the previous step remain without changing the kernel, and the part of the new step is added" (BBS (A.1.4)–(A.1.5) and Exercise 2.2.4); in one row on the group it becomes "a contracting image + new terms independent of the step" Cited (the formulas of the sources), Reading (the correspondence). Four-body terms, the self-image of the three-body term at one-loop level, and the transfer to four dimensions remain (15-08 § 06).

04

## The bookkeeping of the transverse contraction — the departure by a supremum, not a sum

We write the requirement of the shape remainder condition that "$\rho_k$ stays uniform in the number of steps" in the language of measured quantities. Set up the space of weights $X$, one step $T : X \to X$, and a measured quantity $\Phi_{\mathrm{obs}} : X \to \mathbb{R}$ (for example the curvature outside the layer $\kappa/\beta$, or the total variation distance); take a class $\{w_\beta\}$ and a read-out $b : X \to \mathbb{R}$ ($b(w_\beta) = \beta$), and define the function of the class $F(\beta) := \Phi_{\mathrm{obs}}(w_\beta)$. With $u_k := T^k(u_0)$, $\beta_k := b(u_k)$, and the departure from the class $\delta_k := \Phi_{\mathrm{obs}}(u_k) - F(\beta_k)$, the identity

$$\Phi_{\mathrm{obs}}(u_K) - \Phi_{\mathrm{obs}}(u_0) = \big[F(\beta_K) - F(\beta_0)\big] + \big[\delta_K - \delta_0\big] \quad (3)$$

holds Proved here (from the definitions). The first term is the part absorbed into the movement of the parameters, and only the difference of the end values remains (the telescoping sum). The second term is the loss, and if $r_k \geq |\delta_{k+1} - \delta_k|$ then $|\delta_K - \delta_0| \leq \sum_{k<K} r_k$ Proved here .

**The case of transverse contraction.** Let $T(w_\beta) = w_{\tau(\beta)}(1 + r_\beta)$, $\|r_\beta\| \le \varepsilon(\beta)$, and assume transverse stability $T(w_\beta(1+\rho)) = T(w_\beta)(1+\rho')$, $\|\rho'\| \le \lambda\|\rho\|$ (a linear inequality with the quadratic terms dropped). Writing $u_k = w_{\beta_k}(1+\rho_k)$, we have $\|\rho_{k+1}\| \le \lambda\|\rho_k\| + \varepsilon(\beta_k)$, and

$$\|\rho_k\| \;\le\; \lambda^k\|\rho_0\| + \frac{\sup_j \varepsilon(\beta_j)}{1-\lambda} \qquad (0 \le \lambda < 1) \quad (4)$$

Proved here (a geometric series, under the linear inequality above). **Even if the remainder is $O(1/\beta)$, the departure is bounded by a supremum rather than a sum, uniformly in the number of steps.** If $\lambda = 1$ then $\|\rho_K\| \le \|\rho_0\| + \sum_{k<K}\varepsilon(\beta_k)$, and if $\varepsilon(\beta) = C/\beta^2$ and $\beta_k \ge \beta_* + ck$ (growing linearly at one loop towards the fine side, 15-01 § 01), then

$$\sum_{k<K} \frac{C}{(\beta_* + ck)^2} \;\le\; C\Big(\frac{1}{\beta_*^2} + \frac{1}{c\beta_*}\Big) \quad (5)$$

is finite Proved here (the telescoping sum of $1/(u+c)^2 \le 1/(cu) - 1/(c(u+c))$). For $\varepsilon(\beta) = C/\beta$ the sum grows like $\ln K$ and is not uniform (the same type as the logarithmic divergence of 15-01). So there are two routes: "uniform if the one-step loss is $O(1/\beta^2)$" and "uniform even with a remainder of $O(1/\beta)$ if things contract geometrically to a fixed shape", and the latter goes through on a weaker assumption.

**The shape deviation when the flow makes the coupling smaller.** In one step of the smallest system (15-06), the tree coupling changes in one step as $B_{n+1} = B_n/5$, and the departure $r_n$ of the shape coordinates (in units of $q/B$) satisfies

$$r_{n+1} \le \lambda\, r_n + \frac{C}{B_n}, \quad B_n = \frac{B_0}{5^n},\ 0 \le \lambda < 5 \qquad \Longrightarrow \qquad r_n \;\le\; \lambda^n r_0 + \frac{C}{(5-\lambda)\,B_n} \quad (6)$$

holds Proved here (by induction; the one-step inequality is assumed). For the projected model of four in a row there is a form that gives the constants of this inequality as numbers: at the scale of the actual steps, $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, $C = 0.0108$ (boxes of $B \ge 32$ and $r \le 0.05$) Proved here , Reading (the line of the estimate is on paper; the constants are numerical values of explicit integrals and series; the Shape-Remainder Theorem for Four Plaquettes in a Row of 15-06). If $\lambda < 1$, the initial deviation dies out geometrically, and what remains is always the $O(1/B_n)$ of that step. In the smallest system, $\lambda$ was at most $17/125$ in the shape coordinates (15-06, four in a row, leading term) Reading . That (6) is written with $B_n$ decreasing towards the fine side is a convention of that system; in the ultraviolet direction (where $B$ is larger on the finer side), $C/B_n$ becomes smaller at finer steps.

Implication for the decomposition: if the class of the class-closure condition is taken to be the attracting curve, then what must be asked of one step in four dimensions becomes two numbers, "the size of the source $\varepsilon$" and "the transverse contraction $\lambda$" Reading . How these two can be measured in the smallest systems is in 15-04 (the hierarchical model) and 15-05 (the bundle network).

**Contraction-map window for the fixed point of the class — the fixed point of the class in the class-closure condition in the form of a contraction map.** Under asymptotic freedom the only fixed point of the coupling $g$ is $g = 0$, and it repels towards the infrared. What should be stated is therefore not a fixed point of the pair $(g, W)$, but that when the one-step maps $W \mapsto T(g_k)W + R(g_k, W)$ are composed while $g$ flows, there is exactly one two-sided infinite orbit that stays in a certain closed ball. That orbit is the one-parameter attracting class $W = h(g)$. If the map of each step preserves the closed ball and contracts on it, this existence and uniqueness are the arithmetic of the contraction mapping principle Lean (ClassFixedPoint: `exists_unique_entire_orbit` , `tracking` , `norm_iterate_sub_le` . Even if one step

does not contract, it suffices that $\mu_n + (\tau+\ell)^n - \tau^n < 1$, with $\mu_n$ the linear part of $n$ steps, $\tau$ the linear part of one step and $\ell$ the Lipschitz constant of the remainder. The deviation from the fixed point at frozen $g$ stays within $\delta/(1-K)$, the same form as $\sup \varepsilon/(1-\lambda)$ in (4)). What remains, then, are the numbers on the side of the assumptions.

The remainder is measured relative to the action of the tree fixed point, with a weight of dimension 6. As the yardstick we use a gauge-invariant action $S^*$, obtained by turning the quadratic part of the fixed-point action back into a kernel on pairs of plaquettes and wrapping it with plaquette holonomies and parallel transport (its quadratic part is within 15% of the fixed point). In this yardstick the linear part of one step contracts by $\tau = 0.695$ at every momentum, and the nonlinear coefficient is $a = 4.335$. The one-loop source is assembled with the same action, and its $F^2$ part is moved into $\beta$ along the direction of the action itself. When the quartic vertex of the action is tuned by one number so that the shift of $\beta$ takes the universal value, the remaining source is $c_0 = 0.02$–$0.04$ per $g^2$ (on a $12^4$ lattice). With a three-step estimate ($\mu_3 = \tau^3$) the window of the contraction map opens from $\beta$ of about 28–54 Computed (the Gaussian part of the tree and the one-loop quadratic term). There are four conditions. (i) Taking the direction of the shift into $\beta$ as the direction of the action itself (the A-type **Loc**) is an argument on paper; a formal check has not been done Paper . (ii) $a$ is the agreement of upper and lower values at a point found by a search over momenta (the upper value from a semidefinite relaxation, the lower value from a concrete field), not the supremum itself over continuous momenta. (iii) $S^*$ is not a fixed point (its quadratic part moves by 23% in one step). (iv) The Lipschitz constant of the remainder is set to $\ell = 0$.

In the original yardstick ($A_4$), the maximum of the nonlinear coefficient over momenta is at most 4.3313 by a computation covering every box of momenta with the upper bound of the second-order expansion, and it agrees with the value 4.3308 from a concrete field at the maximising point to within $5 \times 10^{-4}$ Computed (a floating-point computation; it was confirmed computer-assisted with interval arithmetic only on the small box $|q - q^*|_\infty \le 3 \times 10^{-4}$ around the maximising point. On the other hand, the positive definiteness of the action of the step, needed to use this upper bound, was confirmed with interval arithmetic at every momentum except in a small box around the origin Computer-assisted ; around the origin it was looked at by sampling).

05

## The bundle of rules — what to make into rules so that one step closes

We rewrite the form that closes in one step of 15-00 § 01 as a bundle of rules. Each row of Table 1 is a candidate for a rule needed to build "a class that closes in one step", set against which of the rule candidates in 24-08 Table 2 (the rule of axiomatising scale, the rule of the rarity class, the rule of an asymptotic-freedom axiom, the curvature rule) it fills. Every row is a candidate; we have checked neither consistency nor that they can restate existing results Reading (a physical estimate).

Table 1 The bundle of rules for one step to close (candidates)

| Rule | Content (one line) | Candidate in 24-08 | Components at hand | What is missing |
|---|---|---|---|---|
| Gauge-covariant averaging | The averaging map is covariant under gauge transformations and central shifts; its linearisation is the scalar average, and the nonlinear part goes up to second order | Rule of axiomatising scale | Stage 2 of the series (titles) | Estimates of the constants of the second-order remainder |

| Rule | Content (one line) | Candidate in 24-08 | Components at hand | What is missing |
|---|---|---|---|---|
| The background-field map | The configuration that minimises the action for given averages is unique, analytic and local within a radius independent of scale | Rule of axiomatising scale | Stage 3 of the series (titles). The contraction mapping in two dimensions $O(4)$ | Writing it out in four dimensions, local gauge fixing included |
| The activity norm | The background × fluctuation weight has the group version of the four properties | Rule of the rarity class (a gauge-covariant activity norm) | The group version of submultiplicativity in sums of fields ((ii) of BBS) (the Background–Fluctuation Decomposition Lemma). One step with background-independent fluctuations (the Lemma on One Step with Background-Independent Fluctuations). A statement of the group versions of closure under the fluctuation expectation and monotonicity to the next step (15-07) | A proof of the group versions of closure under the fluctuation expectation and monotonicity to the next step, and the absorption of $\eta_k$ into the parameters |
| Projection onto the leading term | Split the effective action into "$\frac{1}{g_k^2}$ × Wilson's action + localised irrelevant terms", and send only the marginal $F^2$ to the leading term | Rule of axiomatising scale (contracting directions and marginal directions) | The telescoping-sum type of the heat kernel. The attracting class of the smallest system (15-06) | Defining this projection on the group (up to what order in gauge-invariant local functions) |
| Rarity of large fields | The weight of the large-field region is $z_k^{\lvert Z\rvert}$, $z_k \le e^{-cp(g_k)^2}$, and $c$ does not degrade over the steps | Rule of the rarity class | Stage 5 of the series (with assumptions). The negative one-link curvature (15-01) | One step of the induction (in a form that avoids "worst-case geometry") |
| The flow of the coupling | The remainder of one step $\lvert r_k\rvert \le Cg_k^2 + \eta_k$ | Rule of an asymptotic-freedom axiom | Coupling-Smallness Induction Lemma ( § 03) | The net one-step bracket of the coupling including large fields |
| Making the constants explicit | All the constants above as explicit numbers or explicit functions | New candidate | A record of floating-point numbers | Interval arithmetic, analytic estimates |

Correspondence with the sources of difficulty Reading (a physical estimate): the source: dimensionless coupling goes with the rules of projection onto the leading term and of the flow of the coupling, and the source: no global gauge fixing (large fields cannot be controlled by convexity) with the rules of gauge-covariant averaging, the background-field map, the activity norm and the rarity of large fields (15-01 § 03). In ultraviolet control, which deals with the ultraviolet alone, the source: no expansion linking scales (no expansion connecting the ultraviolet and the infrared) does not appear.

Cautions in turning these into rules (from the patterns of misreading in 15-00).

- Write a "one-step" version and an "all-step" version of each rule separately, and make explicit on the side of the rules the assumption that gets the all-step version from the one-step version (the constants do not degrade over the steps).
- Do not take the input of a rule from the output of the same rule (the "Circularity" pattern).
- Write one direction of inequality (upper bound or lower bound) for each rule.

06

## Contrast with the decomposition of the band — what pays the one-step loss

In 26-06 § 01 the crossover band was split into β-monotonicity (the vortex free energy is monotone in $\beta$) and the renormalisation comparison (the comparison with the doubled lattice) Assumption . Table 2 sets the two decompositions side by side.

Table 2 The decomposition of the band and the decomposition of the ultraviolet (a reading)

| | Band (26-06) | Ultraviolet (this chapter) |
|---|---|---|
| One-step comparison | The renormalisation comparison: compares the doubled lattice with the original lattice at a strengthened coupling, with constant $K$ | (1): $T(w\theta) = w\Phi(\theta)(1 + r)$ |
| Where asymptotic freedom enters | Only the direction of $\beta' < \beta$ | Only that the $\beta$ component of $\Phi$ points in the direction of $\beta_{k+1} < \beta_k$ (the main term of the coupling-flow remainder condition) |
| Components that do not distinguish the group | β-monotonicity: tuning (choosing $\beta'$ on the small side costs nothing) | Continuity of the coupling and monotonicity: the intermediate-value argument of the Coupling-Smallness Induction Lemma (tuning the bare coupling) |
| Handling the one-step loss | Pays the loss $K^k$ with the smallness of the area exponent at the strong-coupling end (doubly exponential as steps accumulate) | There is nothing to pay $\prod_k(1 + \eta_k)$ with (the weak-coupling end is Gaussian and has no smallness to cancel the loss). Hence $\sum_k \eta_k < \infty$ or a transverse contraction $\lambda < 1$ is needed |
| Allowed size of the loss | A constant $K$ per step (it may exceed 1) | $1 + \eta_k$ per step, with $\eta_k$ of type $g_k^{2+\delta}$ (type $g_k^2$ is ruled out because the product tends to 0). In a transversely contracting direction, even a remainder of $O(g_k^2)$ can be carried by a supremum ( § 04) |
| Undecided core | β-monotonicity carries the same weight as the $\beta$-monotonicity of the string tension. The renormalisation comparison is open within the band | The net one-step bracket of the coupling including large fields. Absorbing $\eta_k$ into the parameters (15-07) |

What to read from Table 2 Reading (a physical estimate). The two decompositions are of the same type (a one-step comparison that distinguishes the group, and a tuning that does not), and they split in the same way. The difference lies in **whether there is something to pay the one-step loss with**. The band has a doubly exponential smallness at the strong-coupling end, so the one-step constant $K$ may be crude; the ultraviolet has nothing to pay with, so the one-step loss itself can only be confined to the finite size of a sum, to the movement of the parameters, or to the supremum given by the transverse contraction.

07

## Three ultraviolet questions

Table 3 lists the questions that come out of this decomposition, in the form of 24-08 Table 1. The numbering carries over from Table 4″ of 26-11. The ultraviolet questions of the other chapters of Part 15 and the overall index are in 15-09.

Table 3 Questions arising from the decomposition of the ultraviolet wall (status: New = a question added in 26-11; Updated = rewritten in this chapter)

| Q. | Question | Size | Status | What moves if settled | What does not move when settled | From which chapter |
|---|---|---|---|---|---|---|

| Q. | Question | Size | Status | What moves if settled | What does not move when settled | From which chapter |
|---|---|---|---|---|---|---|
| 1 | Does it contract transversely on the one-parameter attracting class? In the smallest system there is an attracting class (a single coupling $B$ + a fixed shape of $O(1/B)$), and the shape coordinates contract in one step (at most $17/125$ for four in a row, leading term). Questions: (i) for a block embedded in the four-dimensional lattice (including the escape routes of the surrounding flux and the re-fixing of the gauge), do the transverse contraction $\lambda$ and the source stay finite? (In a four-dimensional box with a $2\times2$ coarse face inside, $\lambda \approx 0.027$ and source $\approx 5\times10^{-4}$; in the network with all coarse faces fixed, $\lambda$ is a matrix and exceeds $1/16$ in the coarse directions. 15-05.) (ii) Does the constant $C$ of the one-step inequality (6) exist independently of the step? | Paper and small computations | Updated | The number and coordinates of the parameters of the class-closure condition. The dimension of the attracting class | The class-closure condition and the net one-step bracket of the coupling on the four-dimensional lattice | 15-03 § 02, § 04, 15-05, 15-06 |

| Q. | Question | Size | Status | What moves if settled | What does not move when settled | From which chapter |
|---|---|---|---|---|---|---|
| | (For the projected model of four in a row, the Shape-Remainder Theorem for Four Plaquettes in a Row gave it as numbers; 15-06.) (iii) Is the weight of the sign of the central $\mathbb{Z}_2$ (the vortex weight) needed as a separate parameter? | | | | | |

| Q. | Question | Size | Status | What moves if settled | What does not move when settled | From which chapter |
|---|---|---|---|---|---|---|
| 2 | Which is the smallest model in which the net one-step bracket of the coupling can be shown including large fields? (The large-field part is settled by one link of heat-kernel fluctuations, and the core that remains is the small-field contribution to the coupling; 15-08 § 02, § 06.) Candidate: the hierarchical $SU(2)$ model: does $\lvert r_{\boldsymbol{k}}\rvert \le C g_k^2$ hold including the contribution of the far hemisphere of one link (of weight of order $e^{-\beta}$)? Also, can the absorption under a constraint $\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)}$ for one plaquette be restated at the true constrained minimum of many plaquettes? | Paper | New | One line of the net one-step bracket of the coupling in the rule of the flow of the coupling, at the hierarchical level | The basic inequality (unpublished) on the four-dimensional lattice | 15-03 § 03, 15-07 |

| Q. | Question | Size | Status | What moves if settled | What does not move when settled | From which chapter |
|---|---|---|---|---|---|---|
| 3 | The role of the twist: on the $M^4$ lattice of $n$ $12 = 1$, is the smallest eigenvalue of the adjoint fluctuation operator around the isolated twisted flat connections at least $c/M^2$ (finite-dimensional linear algebra)? And, beyond the Background–Fluctuation Decomposition Lemma being unchanged by the twist, is the group version of closure under the fluctuation expectation unchanged by the twist? | Paper (partly small computations) | New | The denominator of the partition function in the ultraviolet statement for a small twisted four-dimensional torus, and the non-degeneracy of the last step | Ultraviolet control at the fine steps. The net one-step bracket of the coupling | 15-02 § 05 |

08

## What cannot be said

(a) Within this part, ultraviolet control is not closed. This book's study has shown none of the pieces of the decomposition statement (the class-closure condition, the coupling-flow remainder condition, the shape remainder condition, continuity of the coupling) in four dimensions. Existence in four dimensions was shown by the published construction, with a different design (Parts 6–13).

(b) The Coupling-Smallness Induction Lemma only splits the coupling staying small into the net one-step bracket of the coupling and continuity of the coupling; it does not show the net one-step bracket of the coupling (the content of the unpublished basic inequality).

(c) The bookkeeping of § 04 is arithmetic under the assumption of a linear inequality with the quadratic terms dropped and of the one-step inequality. We do not show that $\lambda < 1$ and the estimate of the remainder hold for one step in four dimensions.

(d) $m = 2$ is a reading from power counting, not shown for a non-perturbative class. "The one-parameter attracting class" is a reading of the smallest system — four in a row, the diagonal definition, the leading term (the limit of large coupling) — and says nothing about blocking in four dimensions.

(e) The bundle of rules is a set of candidates; we have checked neither consistency nor that they can restate existing results. The contrast with the band is an estimate, and shows none of the band's β-

monotonicity, the renormalisation comparison, or the ultraviolet remainders.

(f) The splitting of the net one-step bracket of the coupling in § 03 is a summary of the statements and estimates of 15-08 § 06, and does not show the small-field contribution to the coupling. The same form as in the sources holds under the analyticity condition at complex backgrounds and the margin $q$, and we do not show the group version of the explicit quadratic terms between blocks. The "geometry part to the side of the parameters" of § 02 is a reading that carries the numbers of the two-link model over to the four-dimensional steps. The final form of the coordinates of the class in § 02 is a proposal from the saddle-point expansion of one three-dimensional step and the numbers for two links and the cross; we do not show that it closes in one four-dimensional step.

(g) The final form of the four-dimensional class in § 02 consists of tree-level fits for the rule that fixes only the edges in the free-boundary box (and for the rule that fixes all faces and the quadratic order of the periodic network), and one-loop numbers ($L \le 4$). We show neither the existence of fixed points of the ratio coordinates nor the values of the 65 quartic ones. On the periodic network the class does not fit within one cell. The measured flow of the coupling is a one-loop number for the first step of one rule; we have not measured the coefficient over repeated steps. Of the multi-step conditions for the shape remainder condition, the cone-contraction condition at the fixed point is at the level of a lower bound; there is no upper bound. The recursion of the net one-step bracket of the coupling in § 03 is a statement for one row, a definition with no conserved quantity, and first order in the non-local terms; we have not measured four dimensions, four-body terms, or the three-body image at one-loop level.

09

## The table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [01] | The class and the form (1) of the one-step map; $m = 2$ by power counting | Reading | § 01 (power counting) | Four-dimensional pure gauge, local terms |
| [02] | Coupling-Smallness Induction Lemma: net one-step bracket of the coupling + continuity of the coupling $\Rightarrow g_0 < \cdots < g_n = \gamma_0$ (with a variant) | Proved here | § 03 | $C\gamma_1^2 \le b/2$, $\gamma_1^2 < 2/(3b)$, $n$ fixed |
| [03] | The content of the net one-step bracket of the coupling is the one-step remainder including large fields, corresponding to the content of the basic inequality (unpublished) | Assumption | § 03 (a reading of one sentence in the introduction) | Four-dimensional non-abelian |
| [04] | The identity (3); the telescoping-sum upper bound on the loss | Proved here | § 04 | By definition |
| [05] | The bookkeeping of the transverse contraction (4); the sum of $\lambda = 1$ (5); the shape deviation (6) | Proved here | § 04 (geometric series, telescoping sum, induction) | Assuming the linear inequality and the one-step inequality |
| [06] | The curve that keeps the ratio at 2 does not close under the flow, and the ratio on the attracting class is $2 + \frac{2}{3B}$ | Reading , Proved here | § 02 (the results of 15-06) | SU(2), four in a row, the diagonal definition, leading term / one-loop level |

| Row | Statement (one line) | Grade | Grounds | Scope |
|---|---|---|---|---|
| [07] | The decomposition of the ultraviolet wall: class-closure condition, coupling-flow remainder condition, shape remainder condition, continuity of the coupling ⇒ ultraviolet control. The class-closure condition is the one-parameter attracting class + a contracting shape | Assumption | § 02 (the form of the statement) | Statement only |
| [08] | The bundle of rules and its correspondence with the roots | Reading | § 05 (physical estimate) | Candidate |
| [09] | Contrast with the decomposition of the band: they split in the same way, and differ in whether there is something to pay the one-step loss with | Reading | § 06 (physical estimate) | — |
| [10] | Three ultraviolet questions | Assumption | § 07 | The form of questions |
| [11] | The splitting of the net one-step bracket of the coupling: $r^{\mathrm{small}} + r^{\mathrm{large}}$; the small-field contribution to the coupling splits into the Taylor truncation (of the same form as in the sources), the cut-off, and the quadratic terms between blocks. The background-dependent geometry part goes to the side of the parameters | Assumption , Reading | § 02, § 03 (summaries of 15-07 and 15-08) | Analyticity condition at complex backgrounds, margin $q$, a reading of the two-link model |
| [12] | The final form of the coordinates of the class in the class-closure condition (the coupling matrix, the shape, the background, the one-link coupling, the corner triple product, the multi-face quartic) | Assumption | § 02 (summary of 15-07) | A reading of three-dimensional $2 \times 2 \times 2$, two links and the cross |
| [13] | $\sum_k \eta_k$ of the shape remainder condition is uniform in the number of steps if the background is given as a parameter; the difficulty moves to the class-closure condition | Reading , Proved here (arithmetic of the budget) | § 02 (15-07 § 06) | The smallest system, conditional |
| [14] | The final form of the four-dimensional class: one parameter $B$ + ratio coordinates whose values are fixed by the flow (2 quadratic, 3 cubic, 65 quartic). The skew neighbour is nonzero whatever the rule; the value and "local" depend on the rule. No running in $\ln L$ at tree level | Reading , Proved here (counting, agreement of degrees of freedom) | § 02 (15-07 § 07) | $SU(2)$, heat kernel, tree $L \leq 10$, one loop $L \leq 4$, periodic network at abelian tree level and quadratic order $L \leq 16$, extrapolation |
| [15] | The measured flow of the coupling: $\beta' - \beta = -0.745 -- -1.136$ in one step of the periodic network; the change from blocking factor 2 to 3 is 1.7–1.9 times the one-loop coefficient | Reading , Proved here (identities for the differences of actions) | § 02 | $SU(2)$, one loop, abelian background, transverse $6^3$, wavenumber 0 extrapolated |
| [16] | The contraction of the cone of the shape remainder condition is not uniform in the number of steps; blocking conditions (the covariance-flow fixed-point condition and the cone-contraction condition at the fixed point) are needed | Reading | § 02 (15-05 § 07) | Abelian, tree level, periodic, $2^4$, linearised, no interval estimates, the cone-contraction condition at the fixed point a lower bound only |
| [17] | One-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling: closes up to quartic order with a contracting image of the pair type + new terms independent of the step + three-body terms decreasing exponentially in the width | Reading , Proved here (first-order Taylor formula and bookkeeping) | § 03 (15-08 § 06) | One row, definition with no conserved quantity, heat kernel, one step, first order in the non-local terms, up to quartic |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] RDG CLS    the class and the one-step map, m = 2                  power counting
[02] PRV IND    Coupling-Smallness Induction Lemma                     under the net bracket a
[03] ASM S      the net one-step bracket of the coupling is the main part ⟸ [02]   four-dime
[04] PRV TEL    the identity and the telescoping sum
[05] PRV LEDG   bookkeeping of the transverse contraction ⟸ [04]     one-step inequality ass
[06] RDG ATTR   ratio-2 curve does not close, attracting ratio ⟸ [01]  four in a row, leadin
[07] ASM DEC    decomposition of the UV wall ⟸ [01][02][05][06]      form of the statement
[08] RDG RULE   bundle of rules ⟸ [02][07]
[09] RDG BAND   contrast with the band ⟸ [07]
[10] ASM Q      three ultraviolet questions ⟸ [03][05][07]
[11] ASM SPLIT  splitting of the net bracket and the small-field contribution, geometry part t
[12] ASM COORD  final form of the class coordinates ⟸ [07][11]       three-dimensional readi
[13] RDG MOVE   sum of η to parameters, difficulty to the class-closure condition ⟸ [07][11]
[14] RDG CLS4   final form of the 4D class, local with the rule ⟸ [07][12]  tree L ≤ 10, one
[15] RDG FLOWM  measured flow of the coupling ⟸ [01][07]            first step, one loop
[16] RDG CONE   multi-step conditions of the shape remainder (fixed point, cone contraction) ⟸
[17] RDG SREC   one-row version of the recursion of the net bracket ⟸ [03][11]   one row, up
```

The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. [03], [07], [10], [11] and [12] are assumption rows, and all are in the form of questions that split ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

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## What this chapter can say

| Said | Not said |
|---|---|
| Proved here The Coupling-Smallness Induction Lemma (the coupling stays small ⇐ net one-step bracket of the coupling + continuity of the coupling, with a variant), and the bookkeeping of the transverse contraction (3)–(6) (under the assumption of the linear inequality and the one-step inequality) | Neither the net one-step bracket of the coupling nor continuity of the coupling. $\lambda < 1$ and the estimate of the remainder for one step in four dimensions |
| Assumption The decomposition of the ultraviolet wall and the form of the class-closure condition (the one-parameter attracting class + a contracting shape), the three ultraviolet questions, the splitting of the net one-step bracket of the coupling and of the small-field contribution to the coupling (the part of the same form as in the sources, and the group version of the quadratic terms between blocks) | A proof of any of them. Any prospect of ultraviolet control closing |
| Assumption A proposal for the final form of the coordinates of the class in the class-closure condition. Reading How $\sum_k \eta_k$ closes when the background is given as a parameter (conditionally) | That the class closes in one four-dimensional step. The second step for faces and multiple loops |
| Reading Power counting for $m = 2$, the bundle of rules, the contrast with the band, the reason for not adopting a class that keeps the ratio at 2 (the smallest system) | The number of parameters for a non-perturbative class. The attracting class for blocking in four dimensions |
| Reading The final form of the four-dimensional class (one parameter + ratio coordinates; the skew neighbour is nonzero), the one-loop measurement of the flow of the coupling, the multi-step conditions for the shape remainder condition, the one-row version of the recursion of the net one-step bracket of the coupling | The existence of fixed points of the ratio coordinates. That the class closes within one cell on the periodic network. The coefficient of the coupling over repeated steps. An upper bound on $\sup \mu_\infty < 1$. The recursion of the net one-step bracket of the coupling in four dimensions and for four-body terms |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The assumption of the Conditional Ultraviolet Stability Theorem (the coupling stays small) and the unpublished basic inequality | Cited (abstract and introduction) | T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (7-07 § 03, 15-02) |
| The decomposition of the band | Assumption (statement) | 26-06 § 01 |
| Rule candidates (axiomatising scale, the rarity class, an asymptotic-freedom axiom, curvature) | Reading | 24-08 Table 2 |
| Coupling-Smallness Induction Lemma, bookkeeping (3)–(6) | Proved here | Proofs in the text of this chapter |
| The attracting class, the ratio-2 curve, the contraction **17/125** | Reading (the tree map and the one-loop formula are proved here) | This book's study (15-06; the four-in-a-row model, numbers by two routes) |
| Tables 1–3, the decomposition statement | This book's study | The record of this book's study (an organisation of the study of ultraviolet control) |
| The coordinates of the class, how $\sum_k \eta_k$ closes | Reading (the arithmetic of the budget is Proved here) | This book's study (the numbers of 15-07 § 06 and § 07) |
| The coordinates of the four-dimensional class, comparison of rules, the flow of the coupling, the cone over many steps, the one-row version of the recursion of the net one-step bracket of the coupling | Reading (counting, agreement of degrees of freedom, the first-order Taylor formula and the bookkeeping are Proved here) | The record of this book's study (the numbers of 15-07 § 07, 15-05 § 07 and 15-08 § 06). Additivity in the sources: (A.1.4)–(A.1.5) and Exercise 2.2.4 of BBS arXiv:1907.05474v2. The bookkeeping is in the Lean bundle (15-09 Table 3, re-verified) |

**Words used in this chapter** — class, the flow of the parameters $\Phi$, remainder, class-closure condition, coupling-flow remainder condition, shape remainder condition, continuity of the coupling, net one-step bracket of the coupling, Coupling-Smallness Induction Lemma, the bookkeeping of the transverse contraction, the transverse contraction $\lambda$, the source, the attracting class, the bundle of rules, something to pay against. Their meanings are in A-02 Glossary.

Next chapter to read: 15-04 The one-link heat-kernel class and the hierarchical model — how the decomposition of this chapter looks in the smallest setting.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control heat-kernel class, Migdal–Kadanoff, the invariant curve · Kiichi, Shiori and Rin

# The one-link heat-kernel class and the hierarchical model — the class that closes with one parameter, and the attracting curve

This chapter is this book's own investigation. The established knowledge it builds on is in 15-01 and 15-02.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — How the decomposition of 15-03 (a class that closes with finitely many parameters × a remainder) looks in the smallest setting. The one-link heat-kernel class is a semigroup and closes with one parameter, its remainder is 0, and the loss becomes a telescoping sum. In one step of the Migdal–Kadanoff hierarchical model the heat-kernel class does not close exactly, but the image is attracted, step by step at the rate $1/16$, to a one-parameter invariant curve, and the position of the curve is fixed by the balance $1/3600$ between "the source poured in at every step" and "the contraction" (a saddle-point estimate). In this model, over a finite range of couplings, the invariance of the tube and the attracting curve have been confirmed by interval arithmetic. One step on the two-dimensional open strip is exactly a one-link convolution and is no test of many links.

**Chapters assumed** — 15-03 § 04 (the ledger of the transverse contraction), 18-04 (the conventions of one Migdal–Kadanoff step, and $(1-b^{-2})/3$), 24-06 § 07 (the class of measures paid for by rarity), 15-01 § 01 (the logarithmic divergence of the sum of losses).

**Said first** — Every statement in this chapter is a statement about **a map applied to the weight of a single plaquette (a class function)**. The convolution of the one-link heat kernel and the map on the hierarchical lattice (Migdal–Kadanoff) are maps that return the weight of one plaquette to the weight of one plaquette in one step; structurally they have no multi-link couplings, no re-fixing of the gauge and no large-field remainder. So what closes here implies nothing about the ultraviolet control of one step on the four-dimensional hypercubic lattice. Moreover the hierarchical model confines even at $U(1)$, so it is no test of non-commutativity either.

The order of this chapter

1. The one-link heat-kernel class — semigroup, the telescoping sum, curvature $A + B/\beta$, three reservations
2. One Migdal–Kadanoff step — the invariant curve, contraction $1/16$, the flow of the coupling
3. The balance of source and contraction — a saddle-point estimate, $1/3600$, $b^{d-4m}$
4. The two-dimensional open strip — independent plaquettes, $\lambda = 1/4$, no source
5. The ultraviolet theorem for the hierarchical model — the interval-arithmetic tube, the chain from the ultraviolet to strong coupling, and the range not reached (Tables 1 and 2)
6. What can be said in this class (Table 3)
7. What cannot be said
8. The table of the five grades and the dependency graph (Table 4)

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00

## What this chapter shows

- The one-link heat-kernel class is a semigroup and closes with the single parameter $\tau$; the remainder of one step is 0, and the change of a quantity over the steps is a telescoping sum in which only the difference of the end values remains ( § 01).
- In one Migdal–Kadanoff step the image is attracted at $1/16$ per step to the curve of the form of the heat kernel $\times \exp(\theta^4/3600 + \cdots)$, and on the curve the coupling flows as $\beta \mapsto \beta - \frac{1}{4} - c/\beta$ ( § 02, § 03).
- The contraction of the coefficient of $\theta^{2m}$ is $b^{d-4m}$ (with $d = 4$, $b = 2$, $m = 2$ is $1/16$), the same form as dimensional counting ( § 03, the leading saddle-point term).
- In this model it has been confirmed by interval arithmetic that, for $\beta \in [2.46, 128]$, the tube around the heat-kernel class is invariant under the map, and that, for $\beta \in [15.26, 64]$, all orbits in the tube approach a single curve ( § 05). $\beta > 128$ (the limit of lattice spacing 0) is not closed.

01

## The one-link heat-kernel class — the loss becomes a telescoping sum

We look at how "absorbing the loss into the flow of the coupling" (15-01 § 01) appears in the smallest setting, taking as the weight a one-link class function of $SU(2)$ (a function only of the angle $\theta$ from the identity) and following the flow that convolves it with the heat kernel one step at a time (the candidate of 24-06 § 07, and the question "Hessian of the one-link weight" of 24-08). Every statement in this section is a statement about **one link, the heat-kernel class, and the range written below**; it says nothing about one four-dimensional step, which contains many links and couplings between plaquettes.

The heat kernel $p_\tau$ is a semigroup,

$$p_\tau * p_t = p_{\tau+t} \quad (1)$$

so within the family of heat kernels a step yields a heat kernel again, and the parameter $\tau$ (the inverse of the effective coupling) merely moves by $t$ Proved here (the semigroup property alone). Hence every quantity computed from a heat-kernel weight is **a function of the parameter at that moment only**, and when the changes over the steps are added up only the difference of the end values remains (the telescoping sum). It does not become a product Proved here . In terms of the identity of 15-03 § 04, this is the case where the deviation from the class $\delta_k$ is identically 0.

In the coordinates divided by the centre $\mathbb{Z}_2$ ($SO(3)$, $\theta \in [0, \pi/2]$, the branch set $\operatorname{tr} U = 0$ is $\theta = \pi/2$), write $\kappa$ for the minimum of the Bakry–Émery curvature outside the layer $|\theta - \pi/2| < h$. For weights obtained from the Wilson weight by convolving with the heat kernel, the curvature outside the layer was positive at every point of $\beta \in \{2, 3, 4, 6, 8\}$, $t \in \{0, 0.05, 0.1, 0.2, 0.5\}$, half-width of the layer $h = \beta'^{-1/2}$ (without dividing, $SU(2)$, it was negative at every point of the same range) Reading (two independent implementations, in the character series and in position space, agree). The curvature outside the layer takes the form

$$\frac{\kappa}{\beta} = A(h) + \frac{B(h)}{\beta} \qquad \text{(excluding exponentially small terms)} \quad (2)$$

and for the Wilson weight $A = \sin h$, $B = 2$ Reading (for the heat-kernel weight it agrees with the numerics).

Applying this form to one four-dimensional step $\beta' = \beta - 8b_0 \ln L$ (the one-loop flow written in $\beta = 4/g^2$), the leading term $A\beta$ moves with the coupling (the part that is absorbed), and the one-step change of $\kappa/\beta$ is $B\,(1/\beta' - 1/\beta) = O(1/\beta^2) = O(g^4)$, a loss with a finite sum Reading (a reading obtained by fitting the form above). The $\kappa_{j+1} \geq \kappa_j(1 - c/\beta_j)$ that 24-06 § 07 put forward as a candidate is of product type and goes to 0 as a power of the number of steps, but within this class it is merely pessimistic.

There are, however, three reservations. (i) The rarity rate of the layer is proportional to $\beta$, falls by $O(1)$ per step and runs out at $\beta = O(1)$ Reading . This is the one-link form of what 24-06 § 07 described as the limit of the range of couplings that can be reached, not a new obstacle on the ultraviolet side. (ii) The convexity of the divided coordinates can pay only for the part invariant under the centre (the adjoint part); the sign of $\mathbb{Z}_2$ on each link needs another currency. With many links this sign part corresponds to centre vortices Reading (a physical estimate). (iii) One step of four-dimensional renormalisation is not a convolution of the one-link heat kernel; multi-link couplings between plaquettes enter. Whether the form of $A + B/\beta$ is preserved there has not been examined.

This class has no direction leading off the curve, so neither the transverse contraction nor the source appears. In the language of the decomposition, the heat-kernel class is a class with "remainder 0", the case in which the shape remainder condition of 15-03 is empty.

02

## One Migdal–Kadanoff step — the invariant curve and the contraction 1/16

The definition of the single step in this section summarises existing knowledge. The introduction is in 18-04 § 04.

**The map.** To the weight of one plaquette of $SU(2)$ (a class function) $w = \sum_n c_n \chi_n$ ($\chi_n$ is the character of spin $n/2$, $a_n = c_n/(n+1)$) we apply one Migdal–Kadanoff step ($d = 4$, $b = 2$). The conventions are the same as in 18-04:

$$T = (\text{decimation}) \circ (\text{bond moving}) : \qquad w \;\mapsto\; \frac{w^4}{\int w^4} \;\; (\text{exponent } b^{d-2} = 4), \qquad a_n \;\mapsto\; a_n^4 \;\; (\text{convolu}$$

Bond moving is the operation that bundles four parallel plaquettes into one (the product of the weights), and decimation is the operation that convolves four plaquettes in series. On the hierarchical lattice this map is not an approximation but the map itself, yet it is not one step of renormalisation on the four-dimensional hypercubic lattice Cited (the definition of the Migdal–Kadanoff approximation; 18-04).

**The invariant curve.** The heat-kernel class does not close exactly under this step, but the image is attracted, step by step at the rate $1/16$, to a one-parameter curve (the heat kernel $\times \exp(\theta^4/3600 + \cdots)$). The ratio of the differences (total-variation distance) between weights that arrive at the same $\beta'$ after different numbers of steps was $16.06$, $17.00$ at $j = 1:2:3$ (the prediction of a contraction by $1/16$ each: $16.0$, $17.0$), and it was the same for every starting $\beta T \in \{3.5, 7, 15, 30\}$ (starting from Wilson, $15.8$, $16.7$) Reading (numerical values from this book's study; one class function, the map on the hierarchical lattice, $\beta \le 32$, half-width of the layer $h \in \{\pi/4, \pi/8\}$). On the curve, $\kappa/\beta$ outside the layer is a function only of $\beta$ at that moment, and the telescoping sum holds.

**The flow of the coupling.** In one step starting from the heat kernel,

$$\beta \;\mapsto\; \beta - \frac{1}{4} - \frac{0.0215}{\beta} \quad (4)$$

was found Reading (coefficients 0.022, 0.022, 0.021 at the three points $\beta = 4, 8, 16$). $\frac{1}{4}$ agrees with $(1 - b^{-2})/3$ of 18-04. Written in the form of the net one-step bracket of the coupling in the Coupling-Smallness Induction Lemma (15-03 § 03), in this model $b = 1/16$ (in units of $1/g^2$), $|rk| \approx 0.0054/\beta_k \approx 0.0013\, g_k^2$ Reading ($\beta \in [2, 32]$). The remainder relative to the heat-kernel class is $O(1/\beta)$ at $\kappa/\beta$ (a deviation of shape that does not depend on $\beta$), but it does not accumulate as the steps are repeated Reading. As a negative control, replacing the coefficients by Wilson's at each step keeps the deviation at $\delta \approx -0.04$ ($\beta \approx 16$) without decreasing it (the test has detecting power) Reading.

**Fitting into the ledger.** In the language of 15-03 § 04, transverse stability holds with $\lambda \approx 1/16$, and the remainder is a correction of the form $O(1)$ relative to the heat-kernel class. It is therefore uniform in the form of "the transversely contracting case", and the deviation from the heat-kernel class was $|\delta| \le 1.6 \times 10^{-3}$ ($\beta \ge 2$, $h = \pi/4$), and $\le 6.0 \times 10^{-5}$ at $\beta \ge 8$ Reading (over the range measured).

03

## The balance of source and contraction — why $1/16$, and why $1/3600$

At the saddle point for large $\beta$ we estimate one step of the map as a map of the coefficients of the action Proved here (the leading saddle-point term; it drops the higher orders of the fluctuations and the other terms of the sum over images, of order $e^{-\pi\beta h}$; the constants of the error are not given).

- Heat kernel (taking only one image): $\mathrm{HK}_\beta(\theta) \propto (\theta/\sin\theta)\, e^{-\beta\theta^2/2}$. $\log(\theta/\sin\theta) = \theta^2/6 + \theta^4/180 + \theta^6/2835 + \cdots$.
- Put the weight $w = \mathrm{HK}_\beta \cdot e^{R(\theta)}$, $R = \sigma\theta^4 + \tau\theta^6 + \cdots$. **Bond moving** gives

$$w^4 = \mathrm{HK}_{4\beta}\Big(\frac{\theta}{\sin\theta}\Big)^3 e^{4R} = \mathrm{HK}_{4\beta-1}\, e^{\rho+4R}, \qquad \rho := 3\log\frac{\theta}{\sin\theta} - \frac{\theta^2}{2} = \frac{\theta^4}{60} + \frac{\theta^6}{945} + \cdots \quad (5)$$

  where $\beta \mapsto 4\beta - 1$ is $b^2\beta - (b^2-1)/3$ of 18-04. $\rho$ is **the source**: what is left over from the prefactor $\theta/\sin\theta$ (the remnant of the Haar weight) when four plaquettes are bundled.
- **Decimation** (the convolution of four plaquettes): for large $\beta$ the dominant configuration is the one in which the four group elements are spaced equally along the geodesic from the identity to $U$ (each $\theta/4$), and the correction $S$ is mapped to $4S(\theta/4)$. The heat-kernel part is a semigroup: $\mathrm{HK}_{(4\beta-1)/4} = \mathrm{HK}_{\beta-1/4}$.

As a consequence, the map of the coefficients of $\theta^4$ and $\theta^6$ is

$$\sigma \;\mapsto\; \frac{1/60 + 4\sigma}{64} = \frac{\sigma}{16} + \frac{1}{3840}, \qquad \tau \;\mapsto\; \frac{1/945 + 4\tau}{1024} = \frac{\tau}{256} + \frac{1}{967680} \quad (6)$$

**The contraction is $1/16$, the source from the heat kernel in one step is $1/3840$, and the fixed point is fixed by the balance of source and contraction,**

$$\sigma_* = \frac{\sigma源}{1-\lambda_\sigma} = \frac{1/3840}{1-1/16} = \frac{1}{3600}, \qquad \tau_* = \frac{1/967680}{1-1/256} = \frac{1}{963900} \quad (7)$$

Proved here (the arithmetic of the geometric series on (6)). In general the contraction of the coefficient of $\theta^{2m}$ is $b^{d-2}\cdot b^2 \cdot b^{-4m} = b^{d-4m}$ (with $d=4$, $b=2$, $4^{2-2m}$): $m=1$ gives 1 (marginal, $\beta$ itself), $m=2$ gives $1/16$, and $m=3$ gives $1/256$ Proved here (the leading saddle-point term). Reading $\theta \sim a^2 F$, $\theta^{2m}$ corresponds to an operator of dimension $4m$, and $b^{d-4m}$ is the same as the dimensional counting for that dimension Reading (an estimate).

**Check.** The coefficient of $\theta^4$ in one step is $2.615\times10^{-4}$ ($\beta=60$) against $1/3840 = 2.604\times10^{-4}$, and on the attracting curve $2.789\times10^{-4}$ against $1/3600 = 2.778\times10^{-4}$. The difference has the form $6.5\times10^{-5}/\beta$ (the correction from the fluctuations). $\theta^6$ was $1.06\times10^{-6}$ against $1.04\times10^{-6}$ Reading (numerics for the same system). The ratios 16.06, 17.00 of § 02 are a direct measurement of the contraction $1/16$.

**Effect on the curvature.** Since $-\sigma_*\theta^4$ enters $V = -\log w$, the curvature in the tangential direction at the edge of the layer $\theta = \pi/2 - h$ moves by $-4\sigma_*\theta^3\cot\theta$. With $h = \pi/4$, $-4(\pi/4)^3$

$/3600 = -5.4 \times 10^{-4}$ Proved here (arithmetic). The measured deviation $\delta \cdot \beta$ of $\kappa/\beta$ approaches this value: $-2.6 \times 10^{-4}$ ($\beta = 16$), $-3.6 \times 10^{-4}$ (24), $-4.1 \times 10^{-4}$ (32) Reading .

Reading Reading : in this model the source and the contraction come from separate mechanisms. The source is the $\theta^4/60$ that **bond moving** (the operation of bundling in parallel) pours in from the prefactor, and it exists only at $d > 2$. The contraction is what **decimation** (the operation of combining in series) does in dividing the coefficient of $\theta^4$ by $1/64$, the contraction of the scale of the coupling, to give $1/16$. In the language of 15-03 § 04, the two numbers to ask about (the size of the source and $\lambda$) are $1/3840$ and $1/16$ in this model. Values of the same two numbers measured in a small four-dimensional box are in 15-05 ($\lambda$ in the band $0.026$–$0.049$, the source in the band $4$–$6 \times 10^{-4}$).

04

## The two-dimensional open strip — no test of many links

Consider on the two-dimensional open strip (a narrow strip-shaped lattice) one step that blocks plaquettes two at a time. In axial gauge the plaquette variables on the open strip are independent, so the step combining two plaquettes is **exactly the one-link convolution** $w * w$, and the two coarse plaquettes are also independent Proved here (a known structure; with 200,000 independent samples the averages of the characters agree to around $1\sigma$, correlation $0.003 \pm 0.002$ Reading ). So this setting is no test of many links (no new couplings are born).

On that basis, decimation combining $m$ plaquettes multiplies the coefficient of $\theta^4$ by $1/m^3$ and the deviation measured at $\beta' = \beta/m$ by $1/m^2$. For two plaquettes this is $\lambda = 1/4$; since there is no bond moving there is no source term either, and the fixed form is the heat kernel itself Proved here (the leading saddle-point term). The path dependence starting from Wilson has an order that differs with the observable: $O(1/\beta^3)$ for the coefficients of low representations, $O(1/\beta)$ for the total-variation distance, and $O(1)$ for $\kappa/\beta'$ outside the layer (all $1/m^2$ in $m$) Reading ($\beta T = 8$–$64$, $m = 1, 2, 4$). The heat kernel and the Villain weight at $U(1)$ do not depend on the path; the deviation is at most $10^{-8}$ Reading .

The smallest example with many links is the three-dimensional $2 \times 2 \times 1$ slab, in which the plaquettes are not independent because of the cube constraint. The numbers beyond that are in 15-05.

05

## The ultraviolet theorem for the hierarchical model — the interval-arithmetic tube, and the range it does not reach

§ 02 and § 03 were numerics and a saddle-point estimate. Within this model, we list, together with their conditions, the results of checking each row of the ultraviolet decomposition (15-03 § 02) over a finite range of couplings. All are statements about **the MK map $T$ ((3)), the weight of one plaquette of $SU(2)$, $d = 4$, $b = 2$.**

**Coordinates and the tube.** Let the reading of the coupling be $s(w) := 1/b(w) := -\frac{2}{3}\log a_1$ (for the heat kernel $\mathrm{HK}_\beta$, $a_n = e^{-q_n/(2\beta)}$ and $q_n := n(n+2)$, so $b(\mathrm{HK}_\beta) = \beta$). Put the transverse coordinate

$$e_n(w) := \log a_n - \frac{q_n}{3}\log a_1 \qquad (n \geq 2) \tag{8}$$

Then in the heat-kernel class $e \equiv 0$, and decimation gives $e \mapsto 4e$, $s \mapsto 4s$ exactly Proved here (from the definition). **The tube** $U$ is the set of weights such that "$b(w) \in [\beta_{\min}, \beta_{\max}]$, $|e_n(w)| \leq R\,(q_n^2 s^4 + 10\,q_n s^3)$ $(2 \leq n \leq N,\ R = 10^{-3})$, and the tail of high $n$ is bounded by a weighted sum of the coefficients". $q_n^2 s^4$ corresponds to the size of a shape of type $\theta^4$ (for $w = \mathrm{HK} \cdot e^{\sigma\theta^4}$, $e_n \approx \sigma q_n(q_n - 3)s^4$).

Table 1 Each ultraviolet row of the hierarchical model, and the range confirmed (the MK map, the weight of one plaquette of $SU(2)$, $d = 4$, $b = 2$, the tube is $R = 10^{-3}$)

| Row | Content | Scope | Label |
|---|---|---|---|
| Invariance of the tube | $T(U) \subset U$. At $[64, 128]$ the maximum of the ratio of the images goes from 0.494 to 0.492; no term gets tighter as $\beta$ is raised | All 6277 intervals of $\beta \in [2.46, 128]$ (the 3077 intervals of $[2.46, 64]$ and the 3200 intervals of $[64, 128]$; the 23 intervals of $\beta \in [2, 2.46)$ fail) | Computer-assisted (interval arithmetic, with positive and negative controls) |
| Pointwise transverse contraction | The norm of the transverse linearisation on the tube $\lambda_{\mathrm{tube}} < 1$ (0.81 at $\beta = 8$, 0.46 at 16, 0.27 at 64, 0.26 at 128) | $\beta \in [5.88, 128]$ | Computer-assisted |
| Attracting curve | All orbits in the tube approach a single curve with ratio $\Lambda \leq 0.735$ per step ($\Lambda \leq 0.384$ at $\beta \in [23.62, 64]$). The accuracy of the curve is $\lesssim 2.2 \times 10^{-4}$ in the normalisation, from the bound on the tail | $\beta \in [15.26, 64]$ | Computer-assisted (input) + Proved here (an elementary lemma on graph transforms) |
| The flow of the coupling (at heat-kernel points) | $(\beta - \beta' - \frac{1}{4})\beta \in [0.020928,\ 0.023690]$ ($[0.020876,\ 0.020929]$ at $[64, 128]$), monotone towards $1/48 = 0.020833$ (agreeing with $1/48 + 0.0062/\beta$) | $\beta \in [2.5, 128]$ | Computer-assisted |
| The flow of the shape coefficient | $\sigma_k \mapsto (16\sigma_k + 12\ell_k)/4^{2k}$ ($\ell_k$ is the coefficient of $\theta^{2k}$ in $\log(\theta/\sin\theta)$), contraction $16^{1-k}$, fixed point $\sigma_k^* = 12\ell_k/(4^{2k} - 16)$ $(1/3600,\ 1/963900)$ | Leading term for large $\beta$ | Proved here (the leading saddle-point term, no constants for the error) |
| The flow of the coupling on the curve | $\beta' = \beta - \frac{1}{4} - c(\sigma)/\beta$, $c(\sigma) = \frac{1}{48} - 15\sigma$, on the curve $c_* = \frac{1}{60}$ (on the heat-kernel class $\frac{1}{48}$) | Leading term for large $\beta$ | Proved here (leading term). In floating point $1/60 + 0.0078/\beta$ Reading |

| Row | Content | Scope | Label |
|---|---|---|---|
| The seam to the band and the chain | The tube $\hat{U}$, with radius and weights narrowed as functions of $\beta$, closes under $T$; the image of its lower end enters the tube $D_T$ on the band side, and from there the orbit goes to the strong-coupling criterion region. Points with $\beta \le 64$ reach the criterion region within 846 steps, points with $\beta \le 128$ within 1137 steps | The 755 intervals of $\beta \in [0.95, 2.46]$, the 27 intervals of the seam, and the 139 intervals of the exit; $[64, 128]$ for the tube above | Computer-assisted (interval arithmetic; the band side uses the earlier computer-assisted record as it is) + Proved here (assembling the step counts) |
| The envelope of $c_*$ (the recentred tube) | The tube $U_\sigma$, centred on the transverse leading term of $\mathrm{HK} \cdot e^{\theta^4/3600}$ and narrowed to radii $R = 10^{-4}$, $A = 0$ and interval width 0.002, is invariant, and over the whole tube $(\beta - \beta' - \frac{1}{4})\beta \in [0.0151,\ 0.0188]$: it contains $1/60$ and contains neither $1/48$ nor the value at the heat-kernel point | Only the 50 intervals at the five places $\beta \approx 16$, 32, 64, 96, 128. That the curve enters $U_\sigma$ is not shown | Computer-assisted (conditional) |
| The curvature outside the layer and the weight of the layer | On the curve $\kappa/\beta \ge A(h) - C/\beta > 0$, the weight of the layer $\le Ce^{-c\beta}$ | Leading term for large $\beta$ | Assumption (no interval arithmetic done) |

$\sigma \mapsto \sigma/16 + 1/3840$ in (6) is the case $k = 2$ of the flow of the shape coefficient in Table 1 ($\ell_2 = 1/180$, $(16\sigma + 12/180)/256 = \sigma/16 + 1/3840$).

**What ultraviolet means in this model.** Even if the starting $\beta$ is made large within a finite range and the flow is run for $K$ steps, the weight at the last scale lies within $\Lambda^K \times$ (the starting distance) of a point on the curve, and the error is uniform in $K$ Computer-assisted (a statement on the range $\beta \in [15.26, 64]$; $\beta$ falls by about $1/4$ per step, so for example from the final $\beta = 16$ up to 64 is $K \approx 190$ steps). That is, the form of (4) of 15-03 § 04 stands, over a finite range in this model, as a computer-assisted inequality.

**The range not reached (in the same rows).** $\beta > 128$ — the limit of lattice spacing 0 itself — has only the leading saddle-point term, with no estimate carrying the constants of the error Assumption. The attracting curve remains at $[15.26, 64]$ (the input of the graph transform has not been computed at $[64, 128]$). Interval arithmetic for the curvature outside the layer and the weight of the layer is not reached. Separating $c_* = 1/60$ from $1/48$ is impossible in principle over the whole of a tube centred on the heat kernel (a tube centred on the heat kernel, and also a recentred tube whose radius is $10^{-3}/3$ or more, contains the heat kernel itself, so the envelope of $\beta'$ over the whole tube always contains $1/48$ Proved here); it was reached only on intervals at five places, on a tube with radius narrowed to $10^{-4}$ (Table 1). That the curve enters that tube is not reached. $\beta < 2.46$ is not reached with the same tube, but if the tube is narrowed as a function of $\beta$, its lower end comes down to 0.95 and is joined to the band side (next paragraph). Above all, it closes here because this is **a model in which the weight of one plaquette returns to the weight of one plaquette in one step**. Structurally it has no multi-link couplings, no re-fixing of the gauge and no large-field remainder, and the "two rows absent from the sources" of 15-02 (the one-step remainder including large fields, and the source of

observables) are empty in this model. The hierarchical model confines even at $U(1)$, so it is no test of non-commutativity either. $\frac{1}{4}$ is close to the four-dimensional one-loop value ($8b_0 \ln 2 \approx 0.258$ for factor 2), but there is no basis for comparing $c_* = 1/60$ with the four-dimensional two-loop coefficient Reading .

**One chain from the ultraviolet to strong coupling (hierarchical model).** The earlier record for the same model contains the tube $D_T$ on the band side and a computer-assisted theorem that carries it in finitely many steps to the strong-coupling criterion region ($\kappa > 0$) ($a_1 \in [0.0144, 0.21)$, that is, $\beta \in [0.354, 0.961)$; the criterion region is $\beta \le 0.366$). With its coordinates aligned to those of the tube of Table 1 ($\beta \in [2.46, 64]$), it is $a_1 = e^{-3/(2\beta)}$, and as it stands a gap of $\beta \in [0.961, 2.46)$ remains Proved here (matching the coordinates). If we lay down the tube $\hat{U}$ with radius and weights narrowed piecewise as functions of $\beta$ — the same $|e_n| \le 10^{-3}(q_n^2 s^4 + 10 q_n s^3)$ as in Table 1 for $\beta \ge 2.16$, $4 \times 10^{-4} q_n^2 s^4$ for $\beta \le 1.16$, narrowed linearly on pieces of width 0.05 in between, with the tail parameters taken per cell of width 0.1 — the following hold Computer-assisted (interval arithmetic, interval width 0.002).

- On all of the 755 intervals of $\beta \in [0.95, 2.46]$ and the 27 intervals of the seam ($[2.46, 3.0]$), $T(\hat{U}) \subset \hat{U} \cup D_T$ (maximum ratio of the image 0.934). In one step $\beta$ drops by 0.2196–0.2990.
- The images leaving the lower end (the 139 intervals of $\beta \in [0.95, 1.228]$) all enter $D_T$ (maximum ratio 0.9888).
- Hence the points of $\hat{U}$ with $\beta \le 64$ enter $D_T$ within 289 steps, and then enter the criterion region within 557 steps: within 846 steps in all Proved here (assembling the step counts; Lean: `mk_full_chain_64`, `tube_exit`, `cell_descent`, `chain_compose`; the assumptions are the inputs from interval arithmetic and the one-step drop $\delta = 0.2196$). The orbit leaving the heat kernel at $\beta_0 = 64$ enters $D_T$ in 211–289 steps.
- Combined with the result of extending the tube above to $[64, 128]$ (Table 1; the one-step drop is at least 0.2296 on all intervals), the points of $\beta \le 128$ enter $D_T$ within 580 steps and then the criterion region within 557 steps: within 1137 steps in all Proved here (assembling the step counts; Lean: `mk_full_chain_128`, `tube_exit_top`, `chain_top`; the assumptions are the inputs from interval arithmetic, $b_{\mathrm{lo}} = 0.95$, $\delta = 0.2196$, and the cell index $\le 556$ on the band side).

What worked at the seam was not including the tube itself in $D_T$, but **comparing the image of the lower end of the tube with the boundary of $D_T$**. The upper boundary of $D_T$ lies only 1–7% above the image of the heat kernel in the coefficient of $n = 2$, so inclusion with the shape of the tube kept fails at every radius Reading . Conditions (in the same rows): (a) the interval arithmetic is our own, and the derivations of the Taylor remainder and of the upper bounds on the cumulants are on paper (not yet in Lean); (b) the band side uses the computer-assisted certificate of the earlier record as it is, and is not recomputed here; (c) the reference curve of $D_T$ was rebuilt following the construction in the record, and has been matched against the numbers of the record at one point; (d) the 557 steps are a coarse upper bound counted by the number of cells; (e) by convention, points on the boundaries of the pieces belong to the smaller tube. It passes positive controls (string-for-string agreement with the 27 intervals of Table 1;

agreement of a recomputation of the 200 intervals of $[60, 64]$) and negative controls (thinning the tube to 1/5, breaking the coefficients of the image, narrowing $D_T$ by 3%). $\beta > 128$ **(the limit of lattice spacing 0) is not closed.** The chain closes here because this is a model in which the weight of one plaquette returns to the weight of one plaquette in one step, and it implies nothing about how things close on the four-dimensional lattice. Even at $U(1)$ it flows to strong coupling in the same way, so it is no test of non-commutativity either.

**This chain does not carry over to four dimensions — its relation to the sandwich.** The line that starts from the four-dimensional Wilson action and, through a sequence of representatives in which the Migdal–Kadanoff image is shrunk uniformly (Tomboulis's sandwich), puts it onto the chain above, breaks at two places Reading . (i) What the sandwich proves is an upper and a lower inequality for one number, the partition function (the upper bound by Migdal–Kadanoff decimation, from translation invariance and convexity, using no property of the group; the lower bound from reflection positivity) Cited . The step that represents the twisted and the untwisted partition functions by the same representative (a common interpolation parameter) is not proved (as K. R. Ito and E. Seiler pointed out) Cited . This step has to be false at $U(1)$, and needs a non-commutative input (the same place as the middle of the renormalisation comparison in 27-02). (ii) The representative is the image shrunk uniformly, and shrinking pushes the transverse coordinate up by at least $(5/3)(1-\alpha)$ Proved here (Lean: `shrink_shift_lower`; assumptions $0 < \alpha$, $q \geq 3$); to stay in the tube above, $1-\alpha$ must be at most of the order of $10^{-2}$ in the tube on the band side and of $10^{-7}$ in the ultraviolet tube Reading . This is incompatible with the device of "keeping $\alpha$ away from 1" that the sandwich argument needs. When the one-step sandwich of the same observable is measured in small boxes ($4^4 \to 2^4$, $8^4 \to 4^4$, $\beta \in \{2, 4, 8\}$, Metropolis), it holds in all 12 rows for both $SU(2)$ and $U(1)$, and does not distinguish the groups Reading . Hence this chain is a theorem inside the hierarchical model, and not a seam to the four-dimensional lattice.

**A note on the proposal of a one-step inequality with the comparison fixed.** One of the candidates for the middle of the renormalisation comparison of the band is $\sigma[T_{\mathrm{MK}} w] \leq 4\sigma[w]$: the string tension of the one-step Migdal–Kadanoff image does not exceed the string tension of the true coarse theory Assumption (not shown for any $\beta$). Used at each step of the orbit $w_k = T_{\mathrm{MK}}^k w$, it gives $\sigma[w] \geq 4^{-k}\sigma[w_k]$ without monotonicity of the map Proved here (a finite induction; Lean: `orbit_lower`, `lower_bound_four`; the one-step inequality is assumed). Testing it in a finite box needs two notes. (i) The twist free energy $\Delta F = -\ln(Z^-/Z^+)$ is dual to the string tension, and goes to 0 under confinement. What corresponds to $\sigma \cdot A$ is $-\ln\tanh(\Delta F/2)$, and the true step keeps $\Delta F$ on the box (the parity of the twist is preserved by blocking), so the ratio 4 of the tensions and the ratio $1/4$ of the areas cancel, and the threshold of the one-step inequality is 1, not 4, that is, $R := \Delta F(2^4; T_{\mathrm{MK}} w)/\Delta F(4^4; w) \geq 1$ Proved here (an identity from the definitions), Cited (the dual reading). (ii) In periodic boxes of $4^4$ and $2^4$, for the Wilson-start step at $SU(2)$ ($\beta = 1.5$, 2.0, 2.3) and also for one step of an orbit lying on the Migdal–Kadanoff form (effective $\beta$ from 2.47 to 2.19), $R \geq 1$, with $R = 1.34(7)$ in the closest row (conservative lower bound 1.19, two independent seeds) Reading (Metropolis, multi-stage BAR, twist $n_{01} = 1$). It is, however, a weak test: the do-nothing map also passes it in the same box ($R = 2.46$ at $\beta = 2.3$). Raising the same orbit step to $6^4$ and $3^4$ gives $R = 1.17(20)$, and the margin converted to Wilson's $\beta$ shrinks

from 0.15 at $4^4/2^4$ to 0.02(2) ($R = 0.97(10)$ at the point of effective $\beta = 2.8$): **within the resolution of the measurement, it does not decide** Reading (by statistical error; the $\Delta F$ measurement has a systematic floor of 0.1–0.25 without a consistent sign, cause not identified). There is neither evidence that the inequality is broken nor numerical support for the side on which it holds inside the band. The 7% discretisation of the $2^4$ box may have made most of the first margin. At $U(1)$ too, one step goes the same way ($1.265(19)$ at $\beta = 1.3$), and a violation that distinguishes the groups does not show up in one step in a small box Reading (one Migdal–Kadanoff step from Wilson raises the effective $\beta$, so there is no step crossing the transition in the first place; by Bessel bookkeeping). This note does not change "does not carry over to four dimensions" above.

**Questions.** We set down three questions that come out of this chapter (numbered consecutively through Part 15; the index is in 15-09).

Table 2 Questions from the hierarchical model

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|
| 4 | Extend the invariance of the tube and the attracting curve of the hierarchical model to $\beta > 64$ (for $\beta \le 64$ and below the chain has been joined down to the band side; the only gap left is the upper end). **Answer (partial):** the tube up to $\beta \le 128$ (6277 intervals), the chain from $\beta \le 128$ in 1137 steps. The gaps left are $\beta > 128$ and the curve for $[64, 128]$: join the asymptotic estimate for large $\beta$ (with the constants of the error) seamlessly to the interval-arithmetic range, making a statement uniform up to $\beta \to \infty$ (there is a proposal to recentre the tube on $\mathrm{HK} \cdot e^{\theta^4/3600}$ instead of the heat kernel) | Paper and computer-assisted | The ultraviolet control of the hierarchical model closes down to lattice spacing 0 | Ultraviolet control for the four-dimensional lattice (because the model has no multi-link couplings, no re-fixing and no large fields) |
| 5 | On the curve of the hierarchical model, confirm by interval arithmetic the lower bound $\kappa/\beta \ge A(h) - C/\beta$ on the curvature outside the layer and the weight of the layer $\le Ce^{-c\beta}$, and at the same time separate $c_* = 1/60$ from $1/48$. **Answer (partial):** $(\beta - \beta' - \frac{1}{4})\beta \in [0.0151, 0.0188]$ on the tube $U_\sigma$ with radius narrowed to $10^{-4}$ (50 intervals at five places). That the curve enters $U_\sigma$, and the curvature and the weight of the layer, are still open | Computer-assisted | The "Assumption" row of Table 1, and the coefficients of the two terms of the flow of the coupling | Same as above |

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|
| 31 | Run the recentred tube $U_\sigma$ ($R = 10^{-4}$, $A = 0$) through the whole of $[15.26, 128]$ with the interval width kept at 0.02 (adding the cubic term of $\delta s$ to the test so that the remainder is only the transverse part), take the input of the graph transform at $[64, 128]$, and turn the curve entering $U_\sigma$ and the envelope of $c_*$ on the curve into an unconditional computer-assisted statement. At the same time, cover $\beta \geq 128$ by one asymptotic interval | Computer-assisted and paper | The conditions of the $c_*$ row of Table 1, and the gap at $\beta > 128$ | Same as above |

06

## What can be said in this class

Table 3 lines up how each component of the decomposition looks in three smallest systems.

Table 3 The three smallest systems and the components of the decomposition (all are maps applied to the weight of one plaquette, not one step on the four-dimensional hypercubic lattice)

| System | Class | Flow of the coupling | Transverse contraction $\lambda$ | Source | Handling of the remainder |
|---|---|---|---|---|---|
| Convolution of the one-link heat kernel | Heat kernel $\{p_\tau\}$ (one parameter) | $\tau \mapsto \tau + t$ Proved here | — (no direction leading off) | 0 Proved here | Remainder 0, the telescoping sum |
| Two-dimensional open strip (combining two plaquettes) | The heat-kernel class (the fixed form is the heat kernel) | $\beta \mapsto \beta/2$ | $1/4$ Proved here (leading saddle-point term) | 0 Proved here | The deviation starting from Wilson only contracts |
| Migdal–Kadanoff ($d = 4$, $b = 2$) | The invariant curve $\mathrm{HK} \cdot \theta^4/3600 + \theta^6/963900 + \cdots$ | $\beta \mapsto \beta - \frac{1}{4} - c/\beta$, on the curve $c_* = \frac{1}{60}$ Proved here (leading term) | $1/16$ ($\theta^4$), $1/256$ ($\theta^6$) Proved here (leading term); on the tube $< 1$ ($\beta \in [5.88, 64]$) Computer-assisted | $1/3840$ ($\theta^4$) Proved here (leading term) | Approaches the curve with $\Lambda \leq 0.735$ ($\beta \in [15.26, 64]$) Computer-assisted. The tube is invariant for $\beta \in [2.46, 128]$ Computer-assisted |

What to read from Table 3 Reading. The class of the class-closure condition of 15-03 is either the heat-kernel class itself (remainder 0) or the invariant curve that deviates from the heat kernel only by a shape of $O(1)$ (the remainder is bounded by a supremum through the transverse contraction); in both the only parameter is the coupling. The source comes from the operation of bundling in parallel, and the contraction from the operation of combining in series. The two-dimensional open strip has no bundling in parallel and so no source; Migdal–Kadanoff has

one. One step on the four-dimensional lattice contains both operations, and moreover multi-link couplings are born. The smallest system for that is the bundle network of 15-05.

07

## What cannot be said

(a) All results of this chapter are statements about maps applied to the weight of one plaquette (the convolution of the one-link heat kernel, the two-dimensional open strip, the map on the hierarchical lattice), and imply nothing about one four-dimensional step containing many links, couplings between plaquettes, re-fixing of the gauge and large fields.

(b) The form (2) of the curvature for the one-link heat-kernel class is a reading of numerics over the registered range of $\beta$ and $t$.

(c) (5)–(7) and $b^{d-4m}$ are estimates from the leading saddle-point term, dropping the higher orders of the fluctuations and the other terms of the sum over images. The constants of the error are not given.

(d) The tube of the hierarchical model and the chain to the band are computer-assisted statements over a finite range of $\beta \le 128$, and the attracting curve over a finite range of $\beta \le 64$ (the band side of the chain uses the certificate of the earlier record as it is); the limit of lattice spacing 0 ($\beta \to \infty$) is not closed. The curvature outside the layer and the weight of the layer have not been confirmed by interval arithmetic. The envelope of $c_*$ is a statement only on intervals at five places on the recentred tube, and it is not shown that the curve enters that tube.

(e) The hierarchical model confines even at $U(1)$, so what closed here is no test of non-commutativity. $c_* = 1/60$ has not been compared with the four-dimensional two-loop coefficient.

(f) That the chain of § 05 does not carry over to four dimensions is a reading from checking the sources of the sandwich (where the unproved step lies) and from numbers for observables in small boxes. The size of $1-\alpha$ for the representatives of the sandwich has not been measured.

(g) The note in § 05 on the proposal of a one-step inequality with the comparison fixed consists of Monte Carlo numbers in periodic boxes of $4^4/2^4$ and $6^4/3^4$ (in the latter the margin is within the resolution and does not decide; the cause of the systematic floor of the $\Delta F$ method is not identified) and the correction of the threshold (an identity from the definitions and the dual reading). The one-step inequality is not shown for any $\beta$. Since the do-nothing map also passes in the same box, the numbers say no more than "no violation was seen". The discretisation of $2^4$ alone carries a systematic error of about 7%.

08

## The table of the five grades and the dependency graph

Table 4 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [01] | The heat-kernel class closes as a semigroup; quantities are functions of the parameter at that moment, and the change over the steps is a telescoping sum | Proved here | § 01 (the semigroup property) | One link, within the heat-kernel family |
| [02] | The curvature outside the layer in divided coordinates is positive, with $\kappa/\beta = A + B/\beta$. In one four-dimensional step, $O(1/\beta^2)$ | Reading | § 01 (two independent implementations) | One link, $SU(2)/\mathbb{Z}_2$, outside the layer, $\beta \in \{2, \dots, 8\}$, $t \le 0.5$ |
| [03] | In one MK step the image approaches the invariant curve at $1/16$ per step, $\beta \mapsto \beta - \frac{1}{4} - 0.0215/\beta$ | Reading | § 02 (numerics for the same system, with a negative control) | One class function, the hierarchical lattice, $\beta \le 32$ |
| [04] | Saddle-point estimate (5)–(7): source $1/3840$, contraction $1/16$, fixed point $1/3600$, contraction $b_{d-4m}$ of $\theta^{2m}$ | Proved here | § 03 (leading saddle-point term) | Large $\beta$; higher orders of the fluctuations dropped |
| [05] | One step on the two-dimensional open strip is a one-link convolution, $\lambda = 1/4$, no source | Proved here, Reading | § 04 | Axial gauge, open strip |
| [06] | Hierarchical model: invariance of the tube, pointwise contraction, flow of the coupling at heat-kernel points | Computer-assisted | § 05 Table 1 (interval arithmetic, with controls) | $\beta \in [2.46, 128]$, $[5.88, 128]$, $[2.5, 128]$, $R = 10^{-3}$ |
| [07] | Hierarchical model: the attracting curve (ratio $\Lambda \le 0.735$) | Computer-assisted, Proved here | § 05 Table 1 (input by interval arithmetic; an elementary lemma on graph transforms) | $\beta \in [15.26, 64]$ |
| [08] | Flow of the shape coefficient, $c(\sigma) = \frac{1}{48} - 15\sigma$, $c_* = \frac{1}{60}$ | Proved here | § 05 Table 1 (leading term) | Large $\beta$, no constants for the error |
| [09] | Lower bound on the curvature outside the layer on the curve, and the weight of the layer. Extension to $\beta > 128$, and the curve for $[64, 128]$ | Assumption | § 05 (the questions "extending the tube of the hierarchical model" and "interval arithmetic for curvature outside the layer") | — |
| [10] | The chain of the hierarchical model: the narrowed tube $\hat{U}$ closes under $\beta \in [0.95, 2.46]$, the image of the lower end enters the band tube $D_T$, and points with $\beta \le 64$ reach the criterion region within 846 steps, points with $\beta \le 128$ within 1137 steps | Computer-assisted, Proved here (assembling the step counts) | § 05 (interval arithmetic, with controls) | The MK map, one plaquette of $SU(2)$; the band side from the earlier record; $\beta > 128$ not included |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [11] | This chain does not carry over to four dimensions: the unproved step of the sandwich (the common interpolation parameter), and the representatives not staying in the tube | Reading (the push-up by shrinking is Proved here) | § 05 | Checking the sources of the sandwich; observables for $4^4 \to 2^4$, $8^4 \to 4^4$; $\beta \in \{2, 4, 8\}$ |
| [12] | The envelope $[0.0151, 0.0188]$ of $c_*$ on the recentred tube $U_\sigma$ (contains $1/60$, does not contain $1/48$) | Computer-assisted | § 05 Table 1 (interval arithmetic; matching by independent quadrature in position space) | $R = 10^{-4}$, $A = 0$, width 0.002, 50 intervals at five places; that the curve enters $U_\sigma$ is open |
| [13] | Note on the proposal of a one-step inequality with the comparison fixed: the one-step threshold in a finite box is 1 ($R \geq 1$); one step of the orbit gives $R = 1.34(7)$ ($4^4/2^4$) and $1.17(20)$ ($6^4/3^4$, within the resolution); a weak test, and no distinction between the groups appears in a one-step box | Proved here (the identity for the threshold, the finite induction), Reading (the numbers) | § 05 | $4^4/2^4$ and $6^4/3^4$ periodic; Metropolis; BAR; $SU(2)$ and $U(1)$; at $6^4/3^4$ it does not decide within the resolution; the one-step inequality open for every $\beta$ |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV SEMI   telescoping sum in the heat-kernel class          (one link)
[02] RDG HK1    curvature outside the layer A + B/β ⟸ [01]          (one link, regist
[03] RDG MK     invariant curve, 1/16, flow of β ⟸ [01]             (hierarchical lat
[04] PRV SADDLE source 1/3840, contraction 1/16, 1/3600 ⟸ [01]     (leading saddle-p
[05] PRV STRIP  open strip is a one-link convolution ⟸ [01][04]
[06] CMP TUBE   tube invariance, contraction, flow of β ⟸ [03][04]  (β ∈ [2.46, 128]
[07] CMP CURVE  attracting curve ⟸ [06]                             (β ∈ [15.26, 64]
[08] PRV FLOW   flow of the shape coefficient, c* = 1/60 ⟸ [04]     (leading term)
[09] ASM MKUV   β > 128, curvature, weight of the layer ⟸ [02][07][08]  (question)
[10] CMP CHAIN  narrowed tube, seam to the band, 846 steps, 1137 steps ⟸ [06]  (β ∈ [
[11] RDG NO4D   does not carry over to four dimensions ⟸ [10]      (checking the sand
[12] CMP CSTAR  envelope of c* on the recentred tube ⟸ [06][08]    (five places, curv
[13] RDG SD     note on the one-step inequality with the comparison fixed (threshold 1
```

To check it, see by eye that in every "⟸" the numbers on the right are smaller than the number on the left. There are no cycles. [09] is an assumption row, in the form of a question. Among the nodes of this chapter there is no node saying that four-dimensional ultraviolet control has been shown.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Proved here The semigroup and the telescoping sum of the heat-kernel class; the saddle-point estimate for one MK step (source, contraction, fixed point, $b^{d-4m}$, $c_*$); that one step on the open strip is a convolution | The constants of the error of the saddle-point estimate |
| Computer-assisted For the hierarchical model: invariance of the tube ($\beta \in [2.46, 128]$), pointwise contraction ($[5.88, 128]$), the attracting curve ($[15.26, 64]$), the flow of the coupling at heat-kernel points, the chain from the narrowed tube via the band side to the strong-coupling criterion region ($\beta \in [0.95, 128]$; the band side from the earlier record), the envelope of $c_*$ on the recentred tube (five places) | $\beta > 128$ (the limit of lattice spacing 0). The curve for $[64, 128]$. Interval arithmetic for the curvature outside the layer and the weight of the layer. The separation of $c_*$ on the curve |
| Reading The form of the one-link curvature $A + B/\beta$; the MK numerics (ratios 16.06, 17.00, the flow of $\beta$); the order of the path dependence on the open strip | One step containing many links and couplings between plaquettes. The source and the contraction for one step on the four-dimensional lattice (15-05) |
| Reading That this chain does not carry over to four dimensions (the unproved step of the sandwich, the margin of the tube) | A non-commutative proof of the common interpolation parameter. The size of $1 - \alpha$ for the representatives |
| Proved here That the threshold of the proposal of a one-step inequality with the comparison fixed is 1 in a finite box; the finite induction from the one-step inequality. Reading The numbers in small boxes | The one-step inequality for any $\beta$. A step that distinguishes the groups |

## Sources and reproduction

| Item | Type | Source, tools |
|---|---|---|
| Conventions of one Migdal–Kadanoff step, and $(1 - b^{-2})/3$ | Cited | 18-04 |
| The heat-kernel semigroup, the telescoping sum, the saddle-point estimate (5)–(7) | Proved here | The text of this chapter |
| Curvature for the one-link heat-kernel class | Reading | Record of numerics from this book's study (convolving a one-link class function with the heat kernel for one step; two implementations, in the character series and in position space; as negative controls, the undivided $SU(2)$ and $U(1)$) |
| The invariant curve and the contraction of one Migdal–Kadanoff step; the open strip | Reading | Record of numerics from this book's study (applying one MK step to the class function of one plaquette; as a negative control, replacing the coefficients by Wilson's at each step; the open strip with a check against independent samples) |
| The tube, the attracting curve and the flow of the coupling for the hierarchical model | Computer-assisted | Record of computer-assisted work from this book's study (the interval-arithmetic tube, 3077 intervals, with positive and negative controls; the curve by an elementary lemma on graph transforms) |

| Item | Type | Source, tools |
|---|---|---|
| The narrowed tube, the seam to the band, the number of steps of the chain | Computer-assisted, Proved here | Record of computer-assisted work from this book's study (interval arithmetic; overlapping checks at interval widths 0.002 and 0.001; decisions on the small exit intervals; matching by independent quadrature in position space), and the earlier record for the band side. The assembly of the step counts is in the Lean bundle (15-09 Table 3, re-verification: done) |
| Tomboulis's sandwich and the unproved step | Cited | E. T. Tomboulis, arXiv:0707.2179. K. R. Ito, E. Seiler, arXiv:0711.4930, arXiv:0803.3019. T. Kanazawa, Phys. Lett. B 670 (2009) 421, arXiv:0805.2742 |
| The push-up of the transverse coordinate by shrinking | Proved here | The text of this chapter. The algebra is in the Lean bundle (15-09 Table 3) |
| Extension of the tube to $[64, 128]$; the recentred tube and the envelope of $c_*$; the 1137 steps of the chain | Computer-assisted, Proved here | Record of computer-assisted work from this book's study (the same test run by the main implementation and by our own interval arithmetic; agreement on the 3200 intervals of $[64, 128]$ and the 200 intervals of $[60, 64]$, cross-checked on two machines; negative controls; the recentred tube only on intervals at five places; independent cross-check by the trapezoidal rule in position space). The assembly of the step counts is in the Lean bundle (15-09 Table 3, re-verification: done) |
| The threshold of the proposal of a one-step inequality with the comparison fixed, and the numbers in small boxes | Proved here, Reading | Record of this book's study (our own Metropolis; multi-stage BAR and thermodynamic integration along the twist interpolation; reruns with independent seeds; positive controls (the classical flux of $\beta = 4$ at $U(1)$; 0 for a double twist); negative control (a map that is too fast)). The induction is in the Lean bundle (15-09 Table 3) |
| Numbers for the one-step sandwich of observables | Reading | Record of numerics from this book's study (Metropolis; periodic; $4^4 \to 2^4$ and $8^4 \to 4^4$; $\beta \in \{2, 4, 8\}$; $SU(2)$ and $U(1)$; errors from 20 bins) |

**Terms used in this chapter** — the heat-kernel class, semigroup, the telescoping sum, one Migdal–Kadanoff step (bond moving, decimation), the invariant curve, the transverse contraction $\lambda$, the source, balance, the tube, the attracting curve. Their meanings are in the A-02 Glossary.

Next chapter: 15-05 The flux-network theorem — how the two numbers, source and contraction, are fixed in the smallest system with many links.

# The flux network theorem — transverse contraction and the flow of the parameter from one projection

This chapter is this book's own investigation. § 08 is a map of open questions.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — If we fix the angle $\theta$ of the loop $W$ of the single coarse plaquette, restrict to commuting configurations and minimise the tree-level action $\sum_p \phi_p^2/2$, the minimiser is unique, $\phi = \theta f$, $f = Ps/\|Ps\|^2$ ($s$ is a surface with boundary $W$, and $P$ is the orthogonal projection onto the space of plaquette angles that come from links) (Unit Bundle Theorem). From this, the ratio of the coarse coupling $\beta'/\beta = \sum f^2$, the factor $\sum f^4$ by which the fine quartic term maps to the coarse quartic term, and the one-step factor of the deviation from the heat-kernel class (the transverse contraction) $\lambda = \sum f^4 / \sum f^2$ are all determined, and $\lambda \le \max f^2$ follows in one line. In a three-dimensional box with free boundary, $\sum f^2$ is the effective resistance of a resistor network. We set out, with their conditions, the table of numbers for varying box size, boundary and size of the coarse plaquette; that when the coarse plaquette is enlarged the action is paid along the perimeter; that the genuine two steps and the iterated two steps do not agree; and that in the network with all coarse plaquettes fixed at once (Bundle Network Theorem) the contraction becomes a matrix and exceeds $1/16$ in coarse directions.

**Chapters assumed** — 15-00 § 01 (steps, the same form), 15-03 § 02 (the statement of the decomposition), 15-03 § 04 (the ledger of the transverse contraction), 15-04 § 02 (one Migdal–Kadanoff step and $1/16$). We use linear algebra (orthogonal projections, Cauchy–Schwarz) and Kirchhoff's laws for electric circuits.

**Said first** — All numbers in this chapter are numbers for **a step that looks at the marginal distribution of one coarse plaquette (not a genuine blocking step)**; only § 07 is "a step with all coarse plaquettes fixed at once". The Unit Bundle Theorem and the Bundle Network Theorem are statements **at tree level restricted to commuting configurations (the main term as $\beta \to \infty$)**; they do not include the minimum over non-commuting configurations, the one-loop correction (the source), large fields, or the re-fixing of the gauge. The numbers for the source are one-loop values, started from the heat kernel, with a single image, at $\beta \to \infty$. This chapter does not show ultraviolet control.

The order of this chapter


1. Setting — commuting configurations, the projection $P$, the unit flux
2. Unit Bundle Theorem — the minimiser, $\beta'/\beta = \sum f^2$, the map of the quartic term, $\lambda$ (equations (2)–(6))
3. The three-dimensional free boundary is a resistor network — $2 \times 2 \times 1$ solved by hand
4. Inequalities — monotonicity under inclusion, $1/(16d-8) \le \lambda \le 1/16$
5. The table of boxes (Table 1)
6. Enlarging the coarse plaquette — the perimeter type; the genuine two steps and the iterated two steps (Tables 2 and 3)
7. The network with all coarse plaquettes fixed — the Bundle Network Theorem, its local form and the One-Step Contraction of the $\ell^4$ Cone, the tail type (Tables 4 and 5)
8. Implications for the decomposition, and questions (Table 6)
9. What cannot be said
10. The five grades table and the dependency graph (Table 7)


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## What this chapter shows

- At tree level restricted to commuting configurations, the minimal configuration with the angle of the single coarse plaquette fixed is determined by the projection $P$ and the unit flux $f$ alone, and $\beta'/\beta$, the map of the quartic term and the transverse contraction $\lambda$ all follow from it (Unit Bundle Theorem, § 02).

- At $\lambda \le \max f^2$, in a box where the 4 plaquettes of the coarse plaquette carry $\frac{1}{4}$ symmetrically, $1/(16d-8) \le \lambda \le 1/16$. The Migdal–Kadanoff $1/16$ is the upper end of this ( § 04).
- The more the box is widened, made periodic or embedded in the lattice, the more $\sum f^2$ rises and $\lambda$ falls. For a $2\times2$ coarse plaquette embedded in the four-dimensional lattice, $\sum f^2 \approx 0.73$, $\lambda \approx 0.026$ and the source is $\approx 5\times10^{-4}$ ( § 05).
- With a $4\times4$ coarse plaquette, $\sum f^2$ falls to 0.29–0.31, smaller than the iterated $(0.72)^2$. The action is paid along the perimeter, and the source is diluted as $\approx 8\times10^{-3}/L^4$ ( § 06).
- If all coarse plaquettes are fixed at once, the coarse action does not stay a plaquette action, the map of the quartic term becomes a matrix, and in coarse directions it exceeds $1/16$ (Bundle Network Theorem, § 07).
- With the local representative one obtains an upper bound uniform in the volume (Bundle Network Theorem (local form)), and in four dimensions, by interpolation that weights the rows by their $\ell^1$ norm, the factor in the $\beta_0' = 1$ normalisation is at most 0.823 (the number is computed, not an interval estimate). The factor in the normalisation per direction is not an amount of contraction, and the normalisation is fixed to the form divided by the long-wavelength coupling. The One-Step Contraction of the $\ell^4$ Cone is limited to the first step; several steps need conditions on the blocking ( § 07).

01

## Setting — commuting configurations, the projection, the unit flux

**Lattice.** Take the cubical complex of a $d$-dimensional box $[0,n_1]\times\cdots\times[0,n_d]$ (**free boundary**) or a $d$-dimensional torus (**periodic**). Let $C_k$ be the space of real-valued functions on the $k$-dimensional cells ($C_1$ are links, $C_2$ plaquettes, $C_3$ cubes), with inner product the sum of the products of the values on each cell, and let $d : C_1 \to C_2$ be the map that makes plaquette angles from link angles (the coboundary).

**Commuting configurations.** Restrict all links of $SU(2)$ to rotations about one direction ($\sigma_3$) and write them by link angles $A \in C_1$; then the plaquette angles are $\phi = dA$. The action of a single image of the heat kernel is $\beta\sum_p \phi_p^2/2$, and Wilson's action is $\beta\sum_p(\phi_p^2/2 - \phi_p^4/24 + \cdots)$ (the same conventions as 15-04 § 01).

**The single coarse plaquette.** Take a coarse loop $W$ (for example the 8 links of the outer boundary of a $2\times2$ square) and a surface $s \in C_2$ with boundary $W$ (the 4 plaquettes of that square, coefficients $\pm1$, support written $S$). The constraint that the angle of the holonomy of $W$ is $\theta$ is $\langle s,\phi\rangle = \theta$.

**Allowed fields and the projection.** Let $\mathcal{A} := \operatorname{im} d \subset C_2$ be the space of plaquette angles that come from links, $P$ the orthogonal projection onto $\mathcal{A}$, and $u := Ps \;/=0$. Since a box with free boundary is contractible, $\mathcal{A}$ coincides with the space in which "the oriented sum over each cube is 0"; in the periodic case the condition that the flux through each coordinate 2-torus is 0 is added Proved here .

$$f := \frac{Ps}{\|Ps\|^2}, \qquad \mathcal{A}_0 := \{\psi \in \mathcal{A} : \ \langle s,\psi\rangle = 0\} \ (1)$$

and call $f$ the **flux**. $f \in \mathcal{A}$, $\langle s,f\rangle = 1$ (the unit flux), $f \perp \mathcal{A}_0$.

The "one step" of this chapter means: given the angle $\theta$ of the single coarse plaquette, integrate out all the remaining degrees of freedom, and read the resulting distribution of $\theta$ (the marginal distribution) as the weight of a single plaquette at the next step. It is a step that looks at the marginal distribution of one coarse plaquette, not a genuine blocking step (the joint distribution of all plaquettes of the coarse lattice is treated only in part, in § 07).

02

## Unit Bundle Theorem — the minimiser and three ratios

**Unit Bundle Theorem (Theorem F of the paper that writes the tree-level blocking of the abelian linearised lattice gauge action as a flow network).** In the setting above, under the conditions (a) restriction to commuting configurations, (b) the main term in $\beta\to\infty$ (tree level; the one-loop correction and $O(1/\beta)$ are not included), (c) free boundary or periodic, fixed, the following holds Proved here .

- (i) For each $\theta$ the minimiser of $\min\{\frac{1}{2}\|\phi\|^2 : \ \phi\in\mathcal{A},\ \langle s,\phi\rangle=\theta\}$ is exactly one, $\phi^* = \theta f$, and the minimum is $\theta^2/(2\|u\|^2)$.
- (ii) The coarse action of the action $\beta\sum\phi^2/2$ is $\beta'\theta^2/2$ with

$$\frac{\beta'}{\beta} = \sum_p f_p^2 = \frac{1}{\|Ps\|^2}. \ (2)$$

  and the lower bound $\sum f^2 \ge 1/|S|$ holds. The upper bound "$\sum f^2 \le 1$" is not unconditional (at $|S|=1$, if that plaquette is a face of a cube, $\sum f^2 > 1$).

- (iii) Let the fine action be $V(\phi) = \sum_p (\beta\phi_p^2/2 - \sigma\phi_p^4)$ ($\sigma \geq 0$; for Wilson $\sigma = \beta/24$), and set the local coarse action $S'_r(\theta) := \inf\{V(\theta f + \psi) : \ \psi \in \mathcal{A}_0, \ \|\psi\| \leq r\}$. With $m := \max_p |f_p|$ and $\|f^3\| := (\sum_p f_p^6)^{1/2}$, if $\sigma(6\theta^2 m^2 + 4|\theta| m r + r^2) \leq \beta/4$ then

$$\frac{\beta \sum f^2}{2}\theta^2 - \sigma \sum f^4\, \theta^4 - \frac{16\sigma^2 \|f^3\|^2}{\beta}\theta^6 \ \leq \ S'_r(\theta) \ \leq \ \frac{\beta \sum f^2}{2}\theta^2 - \sigma \sum f^4\, \theta^4. \quad (3)$$

  that is, the coefficient of $\theta^4$ in the coarse action is **exactly** $\sigma' = \sigma \sum f^4$ (excluding $\theta^6$ and higher).
- (iv) The normalised quartic deviation $\kappa := \sigma/\beta$ maps in one step to

$$\kappa' = \kappa \cdot \lambda, \qquad \lambda := \frac{\sum_p f_p^4}{\sum_p f_p^2} \quad (4)$$

  and

$$\frac{1}{|S|^3 \sum_p f_p^2} \ \leq \ \lambda \ \leq \ \max_p f_p^2. \quad (5)$$

**Proof.** (i) If $\phi \in \mathcal{A}$, then from $P\phi = \phi$ and the self-adjointness of $P$, $\langle s, \phi\rangle = \langle Ps, \phi\rangle = \langle u, \phi\rangle$. By Cauchy–Schwarz $\theta^2 \leq \|u\|^2 \|\phi\|^2$, hence $\|\phi\|^2 \geq \theta^2/\|u\|^2$. $\phi = \theta u/\|u\|^2 = \theta f$ lies in $\mathcal{A}$, satisfies the constraint and gives equality. The constraint set is closed under midpoints, and $\|\phi\|^2$ on it is strictly convex by the parallelogram identity, so the minimiser is unique. If $\psi \in \mathcal{A}_0$ then $\langle f, \psi\rangle = \langle u, \psi\rangle/\|u\|^2 = \langle s, \psi\rangle/\|u\|^2 = 0$.

(ii) (2) follows from the minimum $\beta\theta^2/(2\|u\|^2)$ and $\sum f^2 = \|u\|^2/\|u\|^4$. The lower bound is Cauchy–Schwarz on $\langle s, f\rangle = \sum_{p\in S} s_p f_p = 1$. If $|S| = 1$, then $s$ itself is not closed (the sum over the boundary of a cube is not 0), so $u \ /= s$, $\|u\| < \|s\| = 1$, $\sum f^2 > 1$.

(iii) Every $\phi$ satisfying the constraint can be written uniquely as $\theta f + \psi$ ($\psi \in \mathcal{A}_0$). The upper bound is $\psi = 0$. Lower bound: from $f \perp \psi$, $\sum(\theta f + \psi)^2 = \theta^2 \sum f^2 + \|\psi\|^2$. Expanding the quartic term, $|\sum f^3 \psi| \leq \|f^3\| \|\psi\|$, $\sum f^2 \psi^2 \leq m^2 \|\psi\|^2$, $|\sum f \psi^3| \leq m r \|\psi\|^2$, and bounding with $\sum \psi^4 \leq r^2 \|\psi\|^2$, we get, with $x = \|\psi\|$ and $a = 4\sigma|\theta|^3 \|f^3\|$, $V(\theta f + \psi) \geq$ (right-hand side) $+ \frac{\beta}{4}x^2 - ax \geq$ (right-hand side) $- a^2/\beta$.

(iv) The coarse action has the same form as $\beta'\theta^2/2 - \sigma'\theta^4 + O(\theta^6)$, so $\sigma'/\beta' = (\sigma/\beta)\sum f^4 / \sum f^2$. The inequality on the right is $f_p^4 \leq (\max f^2) f_p^2$ added up. The one on the left follows from the power mean $(\sum_S g)^4 \leq |S|^3 \sum_S g^4$ ($g_p = s_p f_p$, Cauchy–Schwarz twice) and $\sum_S g = 1$. ∎

The meaning of (iv) of the Unit Bundle Theorem. For two weights that arrived at the same $\beta'$ ($\kappa = 1/24$ started from Wilson and $\kappa = 0$ started from the heat kernel), the difference of the coefficients of $\theta^4$, normalised by $\beta'$, is multiplied by $\lambda$ in one step. This is the main term of the transverse contraction in the ledger of 15-03 § 04. It agrees with the ratio of total-variation distances measured at finite $\beta$ (0.0443, 0.0451, 0.0463, 0.0469 for $\beta_T = 1, 2, 3, 4$ at $2 \times 2 \times 1$) approaching $7/144 = 0.0486$ from below, but we have not shown that the ratio of total-variation distances converges to $\lambda$ Reading .

03

## The three-dimensional free boundary is a resistor network

In a box with free boundary the orthogonal complement of $\mathcal{A}$ consists of 2-cycles, and since the box is contractible it coincides with sums of boundaries of cubes. Hence $\|Ps\|^2 = \min_c \|s - \partial c\|^2$ ($c$ is a function on cubes; Dirichlet's principle) Proved here . In three dimensions, consider the **dual resistor network** whose nodes are the cubes, whose edges of resistance 1 are the interior faces, with the faces of $S$ as edges to a source S and the other outer faces as edges to a sink T; then $\phi \in \mathcal{A}$ says that the oriented sum at each cube is 0 (Kirchhoff's current law), the constraint says that the amount of current from S to T is $\theta$, $\sum f^2$ is the effective resistance between S and T, and $f$ is the unit current (Thomson's principle) Cited (a restatement of standard facts about electric circuits).

**Example: the $2 \times 2 \times 1$ slab.** Each cube receives the current $\frac{1}{4}$ from the bottom face and passes it to the 2 side faces on the outer rim (in parallel, resistance $\frac{1}{2}$) and to the top face (resistance 1). The resistance of a column is $1 + (\frac{1}{2} \parallel 1) = \frac{4}{3}$; with four in parallel,

$$\sum f^2 = \frac{1}{3}, \qquad \sum f^4 = 4 \cdot \frac{1}{4^4} + 12 \cdot \frac{1}{12^4} = \frac{7}{432}, \qquad \lambda = \frac{7}{144} \quad (6)$$

Proved here . The flux is $\frac{1}{4}$ on the bottom faces, $\frac{1}{12}$ on the top faces and the outer side faces, and 0 on the inner walls. In four dimensions the network is a potential not on cubes but on 3-cells, and the picture is harder to draw, but the projection formula (1) can be used as it is, independently of dimension.

04

### Inequalities — monotonicity under inclusion, and both sides of the contraction

**(a) Monotonicity under inclusion.** For complexes $K \subset K'$ containing $W$, $\sum f^2(K) \le \sum f^2(K')$ Proved here . Proof: restricting a configuration on $K'$ to $K$ does not change the holonomy of $W$, and the action (a sum of non-negative terms) decreases, so the minimum is larger for the larger complex. ∎ In three dimensions this is the same content as Rayleigh's monotonicity (a box with free boundary is the result of shorting the surrounding cubes to the sink, and shorting lowers the effective resistance). So **a small box with free boundary makes $\beta'/\beta$ look small**.

**(b) Both sides of the contraction.** When the 4 plaquettes of the coarse plaquette carry $f = \frac{1}{4}$ symmetrically (when the box is invariant under the 90° rotation about the coarse plaquette; this holds for all boxes in Table 1), $\sum f^4 \ge 4 \cdot 4^{-4} = \frac{1}{64}$. Moreover, the trial configuration "$\theta/8$ on each of the 8 links of $W$, 0 elsewhere" gives $\theta/4$ to each plaquette of the coarse plaquette and $\theta/8$ to each of the other $2d-3$ plaquettes containing a link of $W$, so from Thomson's inequality ((i) of the Unit Bundle Theorem), **in any box (even in the infinite lattice)** $\sum f^2 \le \frac{1}{4} + \frac{2d-3}{8}$. Together,

$$\frac{1}{16d-8} \;\le\; \frac{1}{64\sum f^2} \;\le\; \lambda \;\le\; \max f^2 = \frac{1}{16} \quad (7)$$

Proved here (the equality on the right, $\max f^2 = \frac{1}{16}$, holds when the largest flux is on the coarse plaquette; we have confirmed this in all boxes computed, but there is no general proof Reading ). In four dimensions $1/56 \le \lambda \le 1/16$, in three $1/40 \le \lambda$. The equality $\lambda = 1/16$ holds when the flux sits only on the faces of $\frac{1}{4}$, and the 16 plaquettes of Migdal–Kadanoff (15-04 § 02) are exactly this. **For a step of the type in which the coarse plaquette is made of 4 fine plaquettes, the Migdal–Kadanoff $1/16$ is the upper end of the transverse contraction**, and the more parallel paths there are, the more thinly the flux is divided and the smaller $\lambda$ becomes.

In the boxes computed, $\lambda \cdot \sum f^2 = \sum f^4$ stays within 1.04–1.22 times $1/64$, and the movement of $\lambda$ is determined almost entirely by the reciprocal of the denominator $\sum f^2$ (the slope of the invariant curve) Reading (an observation).

05

### The table of boxes

Table 1 lists, box by box, the three numbers of one step for a 2×2 coarse plaquette. $\sum f^2$ and $\lambda$ are the main terms of the flux network (for the three-dimensional free boundary and the boxes free only on the outer rim they are rational, and the two formulations agree Proved here ; in four dimensions and the periodic case they are floating-point, and the two formulations agree to about $10^{-12}$ Reading ), and the source is the numerical value of the coefficient of $\theta^4$ at one loop, started from the heat kernel, with a single image, at $\beta \to \infty$ Reading . "Free only on the outer rim" means that only the outer faces on the rim of the slab are free boundary, with cubes left around the coarse plaquette; "interior" means the coarse plaquette is embedded in the interior of the box; "periodic" is a torus with only one coarse plaquette fixed and the other plaquettes free.

Table 1 The three numbers of one step for a 2×2 coarse plaquette ($SU(2)$; tree level restricted to commuting configurations = the main term in $\beta \to \infty$; the marginal distribution of one coarse plaquette. The source: started from the heat kernel, single image, $\beta \to \infty$, one loop)

| System | $\sum f^2 = \beta'/\beta$ | $\lambda = \sum f^4 / \sum f^2$ | Source $\sigma$ | Checks and notes |
|---|---|---|---|---|
| Two-dimensional open strip (two plaquettes combined in series) | $1/2$ | $1/4$ | $0$ Proved here | Only series (convolution), so it closes in the heat-kernel class |
| Migdal–Kadanoff ($d=4, b=2$) | $1$ | $1/16$ | $1/3840 \approx 2.60 \times 10^{-4}$ | The map on the hierarchical lattice. Flux $\frac{1}{4}$ on 16 plaquettes (15-04) |
| Three-dimensional $2\times2\times1$, free boundary | $1/3$ | $7/144 \approx 0.0486$ | $4.11 \times 10^{-4} \approx 1/2433$ | For the source, the extrapolation of finite sums of characters and the one-loop value agree to 4 digits. Heat-bath Monte Carlo on all links and finite sums of characters give $\le 1.7\sigma$ at $\beta = 4$–$32$ |
| Three-dimensional $2\times2\times2$, free boundary | $15/44$ | $5079/106480 \approx 0.0477$ | $4.65 \times 10^{-4} \approx 1/2149$ | One loop in two implementations (finite differences and the analytic Hessian). Monte Carlo gives $\le 0.96\sigma$ at $\beta = 16, 32$ |
| Three-dimensional $2\times2\times3$, free boundary | $14/41$ | $\approx 0.04765$ | $4.71 \times 10^{-4}$ | One loop in one implementation |
| Three-dimensional $2\times2\times\infty$ | $(1+\sqrt{3})/8 \approx 0.3415$ | $\approx 0.04764$ | $\approx 4.72 \times 10^{-4}$ | $\sum f^2$ from the ladder formula $R = \frac{1}{2} \parallel (1+R)$ Proved here . The source is an estimate from three points |
| Three-dimensional, the centre of $4\times4\times2$, free only on the outer rim | $1023/2815 \approx 0.3634$ | $\approx 0.04466$ | $6.13 \times 10^{-4} \approx 1/1630$ | The flux agrees across three implementations. At one loop the two implementations agree to relative $1.4 \times 10^{-5}$ |
| Three-dimensional slab of thickness 2, widened ($6\times6\times2$, $8\times8\times2$, ...) | $0.3664 \to 0.36680$ | $0.04430 \to 0.04425$ | $6.41 \to 6.45 \times 10^{-4}$ | At $4\times4\times2$ within 1% (flux) and 5% (source) of the limit. The limit of the source is an estimate |
| Three-dimensional $4^3$, interior of $6^3$ (coarse plaquette on a plane inside the box) | $0.4669, 0.4760$ | $0.0361, 0.0355$ | $1.04 \times 10^{-3}$, $9.92 \times 10^{-4}$ | One loop in one implementation. The source is not monotone in the box size |

| System | $\sum f^2 = \beta'/\beta$ | $\lambda = \sum f^4 / \sum f^2$ | Source $\sigma$ | Checks and notes |
|---|---|---|---|---|
| Three-dimensional, periodic $4^3$, $8^3$, $16^3$ (only one plaquette fixed) | $8/15$, $0.4859$, $0.4814$ | $0.0332$, $0.0349$, $0.0352$ | — | Flux in two formulations. $8/15$ for $4^3$ agrees to 15 digits |
| Four-dimensional $2 \times 2 \times 1 \times 1$, free boundary | $3/7$ | $\approx 0.0393$ | $4.68 \times 10^{-4}$ | One loop in two implementations |
| Four-dimensional $2 \times 2 \times 2 \times 1$, free boundary | $105/241$ | $\approx 0.0390$ | $4.80 \times 10^{-4}$ | One loop in one implementation |
| Four-dimensional $2^4$, free boundary | $180/407 \approx 0.4423$ | $\approx 0.0386$ | $4.86 \times 10^{-4} \approx 1/2058$ | At one loop the two implementations agree to relative $4.9 \times 10^{-5}$ |
| Four-dimensional, the centre of $4 \times 4 \times 2 \times 2$, free only on the outer rim | $\approx 0.4867$ | $\approx 0.0350$ | $5.46 \times 10^{-4} \approx 1/1833$ | One loop in one implementation |
| Four-dimensional, interior of $4^4$ (open box) | $0.7180$ | $0.0266$ | $4.96 \times 10^{-4} \approx 1/2017$ | One loop in two implementations, sparse and dense, agreeing in log det to $1.5 \times 10^{-11}$. In the interior of $12^4$, $\sum f^2 = 0.7301$, $\lambda = 0.02645$ (flux only) |
| Four-dimensional, periodic $4^4$, $8^4$, $12^4$ (only one plaquette fixed) | $16/21$, $0.7315$, $0.7305$ | $0.0264$, $0.02645$, $0.02645$ | — | Flux in two formulations. $16/21$ agrees to 13 digits |
| Villain for $U(1)$ (any box) | Same $\sum f^2$ | — | $0$ Proved here | Closes at $\beta' = \beta \sum f^2$, and the deviation is only $e^{-c\beta}$ (Poisson summation) |

What to read from Table 1 Reading .

- **Direction.** The more lattice is added around the coarse plaquette (free boundary → free only on the outer rim → interior → periodic), the more $\sum f^2$ rises (as in § 04 (a)) and the more $\lambda$ falls. In four dimensions $\sum f^2$ goes from 0.44 ($2^4$) to 0.73 (the limit embedded in the lattice), and $\lambda$ from 0.039 to 0.026. The free boundary created "a sink for the flux" and made $\beta'$ look small and $\lambda$ look large.
- **The flow of the parameter.** The main term of $\beta'$ is $\beta \sum f^2$, and the shift at one loop is a $O(1)$ independent of $\beta$ (started from the heat kernel): $\beta/3 - 0.146$ for $2 \times 2 \times 1$, $180\beta/407 - 0.258$ for $2^4$, $1023\beta/2815 - 0.220$ at the centre of $4 \times 4 \times 2$ (all $+O(1/\beta)$). It is the same type as the Migdal–Kadanoff $-\frac{1}{4}$.
- **The source.** It is positive and converges to a value independent of $\beta$: $4.7$–$5.5 \times 10^{-4}$ in four-dimensional boxes (1.8–2.1 times Migdal–Kadanoff), and $4.1$–$6.5 \times 10^{-4}$ in three-dimensional slabs. For the Villain action of $U(1)$ it is 0: this reads as saying that the source needs both "parallel", where the weights of two discs are multiplied on a closed surface, and the Haar weight $\theta/\sin\theta$ of a non-commutative group. An open two-dimensional region is series only, so it has no source.

06

## Enlarging the coarse plaquette — the perimeter type, and the genuine two steps

We widen the coarse plaquette to $L \times L$ (the outer boundary $W$ has $4L$ links). $L = 2$ is one step, and $L = 4$ is the "genuine two steps" that take the factor 4 in the plane directions at once. The "iterated two steps" compared with it are the approximation of applying the one-step map once more to the coarse lattice (feeding the marginal distribution of one coarse plaquette back in as the action of the next step); we count $\beta''/\beta = (\sum f_1^2)^2$, $\lambda$ as $\lambda_1^2$ and the source as $\sigma_1(1 + \sum f_1^4)$.

**(a) Upper bound.** The trial configuration "$\theta/(4L)$ on each of the $4L$ links of $W$, 0 elsewhere" gives $2\theta/(4L)$ to the 4 corner plaquettes of the coarse plaquette, $\theta/(4L)$ to the $4(L-2)$ edge plaquettes and 0 to the inner plaquettes (the sum over the coarse plaquette is 1, matching the holonomy of $W$), and the outer plaquettes containing each link of $W$ number at most $2d - 3$. Hence $L \geq 2$, and **in any box (even in the infinite lattice)**

$$\sum f^2(L) \;\leq\; \frac{d-1}{2L} + \frac{1}{2L^2}, \qquad \lambda(L) \;\geq\; \frac{L^{-6}}{\sum f^2(L)} \;\geq\; \frac{2}{L^4\big((d-1)L+1\big)} \quad (8)$$

Proved here (the right side from Stokes, i.e. the sum of $f$ over the $L^2$ plaquettes of the coarse plaquette is 1, and the power mean). In four dimensions $L = 2$ is $7/8$ and $1/56$ (the same as § 04), and $L = 4$ is $13/32$ and $1/1664$.

**(b) The genuine two steps lower $\beta$ much more than the iteration does.** By monotonicity under inclusion, one step embedded in the four-dimensional lattice has $\sum f_1^2 \geq 0.717961$ (the value in the interior of $4^4$; the two formulations agree Reading ), hence $(\sum f_1^2)^2 \geq 0.5155 > 13/32 \geq \sum f_2^2$ Proved here (the input 0.717961 is a computed number). The iteration falls only by about $L^{-0.48}$ in $L = 2^k$, while the genuine $L \times L$ is $O(1/L)$ by (8), so the ratio tends to 0 with the number of steps. **The $\beta'/\beta$ of the marginal distribution of one coarse plaquette falls in four dimensions with the perimeter type ($O(1/L)$), and cannot be expressed by repeating the square of one step.** In the two-dimensional plane alone the two agree (for a $4 \times 4$ coarse plaquette, $\sum f^2 = 1/16 = (1/4)^2$, $\lambda = 1/256 = (1/16)^2$).

**(c) The perimeter type.** In the four-dimensional boxes computed, 44–51% of $\sum f^2$ sits on "the plaquettes in transverse directions touching $W$" ($2(d-2)=4$ per link of $W$), and this proportion hardly changes when the coarse plaquette goes from $2\times2$ to $4\times4$ Reading . This reads as the action being paid not over the area of the coarse plaquette but along the perimeter of $W$.

Table 2 One step, the iterated two steps, the genuine two steps (four dimensions, open box with the coarse plaquette in the interior, tree level restricted to commuting configurations. The source: started from the heat kernel, single image, $\beta\to\infty$, one loop; for the $4\times4$ coarse plaquette, one implementation with the sparse analytic Hessian, whose path agrees with finite differences to relative $9\times10^{-4}$ in the small system $4\times4\times1\times1$) Reading

| Box | Quantity | One step (coarse plaquette 2×2) | Iterated two steps | Genuine two steps (coarse plaquette 4×4) | Genuine / iterated |
|---|---|---|---|---|---|
| $4^4$ (the $4\times4$ coarse plaquette reaches the edge of the box in the plane directions) | $\sum f^2$ | 0.7180 | 0.5155 | 0.2564 | 0.50 |
| | $\lambda$ | $2.66\times10^{-2}$ | $7.06\times10^{-4}$ | $2.62\times10^{-3}$ | 3.7 |
| | Source | $4.96\times10^{-4}$ | $5.05\times10^{-4}$ | $3.70\times10^{-5}$ | 0.073 |
| $6\times6\times4\times4$ (margin 1 in the plane directions) | $\sum f^2$ | 0.7218 | 0.5210 | 0.2895 | 0.56 |
| | $\lambda$ | $2.65\times10^{-2}$ | $7.02\times10^{-4}$ | $2.37\times10^{-3}$ | 3.4 |
| | Source | $4.89\times10^{-4}$ | $4.98\times10^{-4}$ | $3.30\times10^{-5}$ | 0.066 |
| $12^4$ (flux only) | $\sum f^2$ | 0.7301 | 0.5330 | 0.3060 | 0.57 |
| | $\lambda$ | $2.65\times10^{-2}$ | $7.00\times10^{-4}$ | $2.57\times10^{-3}$ | 3.7 |

Table 3 The sequence in the side $L$ of the coarse plaquette (four dimensions, open box of $(L+2)\times(L+2)\times4\times4$ with the coarse plaquette in the interior, transverse thickness 4. Flux in floating point, source at one loop in one implementation) Reading

| $L$ | $\sum f^2$ | Upper bound (8) | $\lambda$ | Coefficient $c_2$ of $\theta^2$ at one loop | Source | Source $\cdot L^4$ |
|---|---|---|---|---|---|---|
| 1 (a single plaquette) | 1.989 | 2 (the trial of $L=1$) | 0.521 | 0.499 | $7.66\times10^{-3}$ | $7.66\times10^{-3}$ |
| 2 | 0.7180 | 7/8 | 0.0266 | 0.238 | $4.96\times10^{-4}$ | $7.93\times10^{-3}$ |
| 3 | 0.4190 | 5/9 | 0.00598 | 0.156 | $9.85\times10^{-5}$ | $7.98\times10^{-3}$ |
| 4 | 0.2895 | 13/32 | 0.00237 | 0.116 | $3.30\times10^{-5}$ | $8.46\times10^{-3}$ |

What to read from Tables 2 and 3 Reading (an observation of four points, transverse thickness 4, one implementation). The source is $\approx 8\times10^{-3}/L^4$, and $c_2$ is $\approx 0.46$–$0.50/L$. If read as $\theta\sim L^2F$, $\theta^4$ is an operator of dimension 8, and the coefficient falling as $L^4$ fits the same counting as the Migdal–Kadanoff $b^{d-8}=1/16$. So the source of a genuine blocking looks not like "a number reborn in a fixed band at each step" but like the type in which "the shift of $\theta^4$ born at the finest scale is diluted by about $1/16$ at each step", and the fixed point $\sigma/(1-\sum f^4)\approx5\times10^{-4}$ of the iterated ledger lies, in this way of measuring, on the side of an upper bound. For $\lambda$ it is the other way round: the genuine two steps are 3.4–3.7 times the iterated $\lambda_1^2$, and the iteration makes the contraction look fast. But at $2.4$–$2.6\times10^{-3}\ll1$ the denominator $1-\lambda$ of the ledger stays almost 1.

07

## The network with all coarse plaquettes fixed at once — Bundle Network Theorem

The periodic rows of Table 1 were networks in which "only one coarse plaquette is fixed and the surrounding coarse plaquettes are free". The main term of a genuine blocking step is the network in which the lattice is divided into $2^d$ blocks and **the angles of all plaquettes of the coarse lattice are fixed at once**, and this is a different network. Conditions of this section: restriction to commuting configurations, tree level (the main term in $\beta\to\infty$), a periodic lattice $(\mathbb{Z}/L)^d$ ($L$ even), $2^d$ blocks, and the coarse angle $\theta_P:=\langle s_P,\phi\rangle$ is the linearisation of the angle around the $2\times2$ outer boundary spanning the coarse plaquette $P$.

**Notation.** Let $S$ be the matrix with the $s_P$ in rows, $D_c$ the coboundary of the coarse lattice, and $C$ the map to coarse links ($(CA)_\ell$ is the sum of the angles of the 2 fine links that make up the coarse link $\ell$); then $SD=D_cC$. Let $u_P:=Ps_P$, $G:=SPS^{\mathsf T}$ ($G_{PQ}=\langle u_P,u_Q\rangle$), $B:=G^{+}$ (the Moore–Penrose pseudoinverse), and $U$ the matrix with the $u_P$ as columns.

**Bundle Network Theorem (Theorem G of the same paper).** Under the conditions above Proved here :

- (i) The constraint $S\phi=\theta$ ($\phi\in\mathcal{A}$) is non-empty if and only if $\theta\in\operatorname{im}D_c$ (Bianchi sum 0 for each coarse cube, and coarse flux 0 through each coordinate 2-torus). In particular, $\theta$ **that twists only one coarse plaquette and keeps the others at 0 is not allowed**.
- (ii) For $\theta\in\operatorname{im}D_c$ the minimiser is unique, $\phi^*=F\theta$, $F:=UB$, and the minimum is $\frac12\theta^{\mathsf T}B\theta$. The coarse action is

$$S'(\theta)=\frac{\beta}{2}\,\theta^{\mathsf T}B\,\theta,\qquad \beta'_{PQ}=\beta B_{PQ}\quad(\operatorname{im}D_c:\text{ as a quadratic form on it}).\ (9)$$

- (iii) Written in the coarse link variables $a$, $\min\{\frac12\|dA\|^2:CA=a\}=\frac12a^{\mathsf T}Qa$, $Q=D_c^{\mathsf T}BD_c$. The Lagrange multipliers are exactly the gradient of the coarse action (the divergence of current at each coarse link).
- (iv) Since $\theta$ are not independent coordinates, the off-diagonal entries of $B$ and the "diagonal / cross" components of the quartic form depend on the representation. Quantities that do not depend on the representation are: the spectrum of $B$ on $\operatorname{im}D_c$, the

ratio by direction $\beta'(\theta) := \theta^{\mathsf{T}} B\theta / \sum_P \theta_P^2$, $\mu(\theta) := \sum_q (F\theta)_q^4 / \sum_P \theta_P^4$, $\lambda(\theta) := \mu(\theta)/\beta'(\theta)$ (an oriented version of the $\lambda$ of the Unit Bundle Theorem).

- (v) For all $\theta \in \operatorname{im} D_c$, $\beta'(\theta) \geq \frac{1}{4}$.

**Proof.** (i) $\theta = SDA = D_c(CA) \in \operatorname{im} D_c$. Conversely, putting $a_\ell$ on the first fine link of each coarse link and 0 elsewhere gives $CA = a$. (ii) The minimum-norm solution lies in $\operatorname{im} U$, and with $\phi = Uc$, $Gc = \theta$. Since $\operatorname{im} G = S(\operatorname{im} d) = \operatorname{im}(D_c C) = \operatorname{im} D_c$, a solution exists, and $c = B\theta + \ker G$, $U \ker G = 0$, $\|\phi^*\|^2 = \theta^{\mathsf{T}} BGB\theta = \theta^{\mathsf{T}} B\theta$. Uniqueness is the same strict convexity as in the Unit Bundle Theorem. (iii) The inner minimum is invariant under coarse gauge directions, so substitute $\theta = D_c a$ and read off the KKT conditions. (v) The sets of 4 plaquettes of the coarse plaquettes do not overlap one another, so $\sum_q \phi_q^2 \geq \sum_P \sum_{q \in sP} \phi_q^2 \geq \sum_P \theta_P^2/4$ (Cauchy–Schwarz on each coarse plaquette). ∎

**A smooth trial.** For a configuration in which the fine field has the same value $\theta/4$ on the $2^d$ plaquettes of each coarse plaquette, the energy ratios are $2^{d-4}$, $\lambda = 1/16$ (independent of $d$) Proved here (arithmetic). This configuration cannot be uniform on the torus, and has meaning only as a trial in the limit of long wavelength. In Migdal–Kadanoff (all coarse plaquettes independent, with $P$ taken as the identity) $\beta'/\beta = 2^{d-4}$, $\lambda = 1/16$.

Table 4 Numbers for the network with all coarse plaquettes fixed ($SU(2)$, tree level restricted to commuting configurations, periodic, $2^d$ blocks. Two implementations (proj and pseudoinverse / KKT on coarse links) agree to about $10^{-12}$. The sup is the value found by multi-start local optimisation = a lower bound on the true sup) Reading

| Box | Eigenvalues of $B$ (on $\operatorname{im} D_c$) | $\beta'$ at the longest wavelength | Deviation from the plaquette action | $\lambda$ of one coarse link | $\lambda$ in the smooth direction | Sup four |
|---|---|---|---|---|---|---|
| Three-dimensional $4^3$–$16^3$ | $[0.25,\ 0.375$–$0.474]$ | 0.4744 at $k = \pi/4$ | 15–17% | 0.066–0.068 | 0.055 at $k = \pi/4$ | 0.11–0.1 |
| Three-dimensional $48^3$ | $[0.25,\ 0.497]$ | 0.4968 at $k = \pi/12$ | 16% | 0.068 | 0.0615 at $k = \pi/12$ | (no optimis |
| Four-dimensional $4^4$–$12^4$ | $[0.25,\ 0.536$–$0.811]$ | 0.8108 at $k = \pi/3$ | 24–27% | 0.085–0.091 | 0.037–0.046 | 0.22–0.2 |
| Four-dimensional $16^4$ | $[0.25,\ 0.879]$ | 0.8786 at $k = \pi/4$ | 24% | 0.091 | 0.047 at $k = \pi/4$ | (no optimis |

Table 5 Verdict: the "true numbers" of one four-dimensional step (conditions as in Table 4; the rows with one plaquette fixed are from Table 1)

| Network | $\beta'/\beta$ | $\lambda$ (factor of $\theta^4$) | Label |
|---|---|---|---|
| One plaquette in $2^4$ with free boundary | 0.442 | 0.0386 | Proved here (rational) |
| One plaquette in periodic $8^4$ (others free) | 0.731 | 0.0264 | Reading |
| All coarse plaquettes fixed, long wavelength | $\to 1 = 2^{d-4}$ (0.879 at $k = \pi/4$) | $1/16$ in the smooth trial; 0.037–0.047 at the measured $k \geq \pi/4$ | Reading (the trial value is Proved here) |
| All coarse plaquettes fixed, one coarse link | 0.354 | 0.091 | Reading |
| All coarse plaquettes fixed, the coarsest direction | 0.25 (smallest eigenvalue) | Sup found $\geq 0.27$ | Proved here ($\geq \frac{1}{4}$), Reading (sup) |
| Migdal–Kadanoff | 1 | $1/16$ | Proved here |

What to read from Tables 4 and 5 Reading . (1) The coarse action does not stay a plaquette action. In four dimensions the eigenvalues spread from $\frac{1}{4}$ up to nearly 1 at long wavelength, and already at quadratic order terms of loops of length 2 or more are born (this seems to correspond to the quadratic part of the classical fixed-point action, but we have not checked against the original sources). (2) The long-wavelength $\beta'/\beta$ is larger than the 0.731 for one plaquette fixed, and moves towards the Migdal–Kadanoff side of 1. (3) The map of the quartic term is a matrix that mixes with neighbours, and **in coarse directions it exceeds** $1/16$ (0.091 for one coarse link, 0.27 for the sup found). In the smooth direction, in three dimensions it heads towards $1/16$ (0.0615 at $k = \pi/12$), while in four dimensions it is 0.047 up to $k = \pi/4$ and convergence is not yet visible. The Migdal–Kadanoff $\lambda = 1/16$ survives in a genuine step only as the value in the long-wavelength direction. (4) The values found are all below 0.3, but the pseudoinverse representative $F$ is non-local, and the Riesz–Thorin upper bound grows with the box (4.2 in four-dimensional $16^4$); **we have not obtained an upper bound uniform in the volume** (with the pseudoinverse representative; taking the local representative gives an upper bound uniform in the volume: the Bundle Network Theorem (local form) below). So we do not say "the transverse contraction is smaller than 1".

Candidate coordinates for writing it as a matrix. If the class of quartic deviations is taken to be "$|Q(\theta)| \leq c \sum_P \theta_P^4$" (quartic forms bounded by $\ell^4$), the tree-level step preserves this class with $c \mapsto c \cdot \sup \mu$ Proved here ($|Q(F\theta)| \leq c \sum (F\theta)^4 \leq c \sup \mu \sum \theta^4$). The diagonal class of plaquettes (the Migdal–Kadanoff class) is not preserved, but the cone of $\ell^4$ is.

**The local representative and an upper bound uniform in the volume — Bundle Network Theorem (local form; Theorem G′ of the same paper).** The conditions are the same as for the Bundle Network Theorem (restricted to commuting configurations, tree level, periodic $(\mathbb{Z}/L)^d$, $2^d$ blocks, linearisation of the coarse angle). $F$ is determined only on $\operatorname{im} D_c$, so how the action is chosen

outside it is the "representative". The pseudoinverse representative is the Hodge decomposition itself and is not local, and the axial gauge with its base point fixed at one point also grows with the volume. So we turn to the kernel $R := FD_c$ written in the coarse link variables.

- (a) $R$ is local (exponentially decaying, with a measured rate of about 2.2–2.5 per lattice step), and a representative that agrees with $F$ on $\operatorname{im} D_c$ is the same thing as a "filling" $\partial k = r$ of each row (cycle) of $R$. Assigning to each row **an axial cone (tree) whose base point moves with the row** gives a translation-covariant, exponentially decaying representative $K_\infty$; its periodisation $K_L$ agrees with $F_L$ on $\operatorname{im} D_c$ for every $L$, and $\|K_L\|_{p\to p} \le \|K_\infty\|_{p\to p}$ $(p = 1, \infty)$ Proved here (the exponential decay of $R$ is a sketch from the constancy of the Bloch rank, plus numerics).
- (b) Hence for all $L$ and $\theta \in \operatorname{im} D_c$

$$\sum_q (F_L\theta)_q^4 \;\le\; \Lambda_d \sum_P \theta_P^4, \qquad \Lambda_d := \|K_\infty\|_{2\to 2}^2 \, \|K_\infty\|_{\infty\to\infty}^2 \quad (10)$$

that is, $\sup\mu$ is bounded uniformly in the volume Proved here (Riesz–Thorin is known; the Schur form that passes from the row and column $\ell^1$ sums to the quartic sum is in Lean: `schur_quartic`, `representative_bound`; assumptions: upper bounds on the row and column $\ell^1$ sums, and agreement of the representative on $\operatorname{im} D_c$). (c) The $\ell^4$ cone of quartic deviations is mapped in one step with its constant multiplied by at most $\Lambda_d$ Proved here (Lean: `cone_step`; assumptions $c \ge 0$, $|Q(\phi)| \le c\sum\phi^4$, and (10) on $\operatorname{im} D_c$).

(d) Numbers Reading (two implementations agree row by row to within $3 \times 10^{-15}$; three dimensions $4^3$–$48^3$, four dimensions $4^4$–$20^4$): $\|K\|_{\infty\to\infty}$ of the local representative is uniform in the volume (converging to 0.685 symmetrised and 0.748 tree in three dimensions, and 1.341 symmetrised and 1.541 tree in four dimensions), while for the pseudoinverse representative it grows, 0.62→1.94 (three dimensions) and 0.95→2.39 (four dimensions), and for the axial gauge with a fixed base point it grows by 1.12 (three dimensions) and 3.08 (four dimensions) per lattice step. **In three dimensions** $\Lambda_3 = 0.245$ (symmetrised); with the long-wavelength normalisation $\beta_0' = \frac{1}{2}$ this gives $\sup\mu/\beta_0' \le 0.49$, and using $\beta' \ge \frac{1}{4}$ gives $\sup\lambda \le 0.98$, smaller than 1. **In four dimensions** $\Lambda_4 \approx 1.83$ (an estimate from the filling on $24^4$ and a line scan), which exceeds 1, and the lower bound on $\sup\mu$ is 0.148. Moreover, from the $\infty \to \infty$ norm $\nu_4 \ge 1.18$ restricted to $\operatorname{im} D_c$ (a linear program on $16^4$; a quantity that does not depend on the representation), **the Riesz–Thorin bound of no representative can go below 1.39**. Whether $\sup\mu < 1$ in four dimensions is undecided, and a different estimate that bounds $\ell^4$ directly is needed (the question "direct upper bound for the four-dimensional $\ell^4$"). The gap of about a factor of 12 between the true value ($\ge 0.148$) and the upper bound (1.83) is the loss in the interpolation between $\ell^2$ and $\ell^\infty$.

**One-Step Contraction of the $\ell^4$ Cone — the four-dimensional $\ell^4$ cone contracts by a factor of at most 0.823 in one step (in the $\beta_0' = 1$ normalisation).** The loss in Riesz–Thorin comes from the fact that the rows that determine $\infty \to \infty$ (the fine plaquettes lying on the coarse plaquette, 24 of the 96 types) and the directions that determine $\ell^2$ (close to the long wavelength) are different. So if we interpolate with the rows weighted by their $\ell^1$ norm $A_q$ (the weighted Riesz–Thorin via Stein's analytic family, used on a finite-dimensional torus), then for all $L$

$$\sup\mu \;\le\; \|K\|_{4\to 4}^4 \;\le\; \Lambda_w := \sup_P \lambda_{\max}\big(K(P)^{\mathrm{H}}\,\mathrm{diag}(A_q^2)\,K(P)\big) \quad (11)$$

holds Proved here (the interpolation theorem is a known tool Cited; the weight $A_q^2$ is the best within this family). Bounding $\Lambda_w$ from above with the kernel of the local representative of $20^4$, on the momentum grid $N = 64$ with a one-sided quadratic remainder, gives $\Lambda_w \le 0.823$ Reading (computed, not an interval estimate; the grid value at $20^4$ is 0.763, the real-space power method gives 0.758 from below, and a test on 60 small matrices does not break the upper bound). Combined with `cone_step`, under the conditions (restricted to commuting configurations, tree level, periodic, $2^4$ blocks, linearised coarse angle, $K_\infty$ replaced by the kernel of $20^4$), **the $\ell^4$ cone of the quartic deviation contracts by a factor of at most 0.823 in one step** (in the $\beta_0' = 1$ normalisation as it stands). The gap to the lower bound 0.148 on the true value has shrunk from about 12 times to 5.6 times. The estimate that splits by distance into a near part and a tail (the Minkowski form is in Lean: `quartic_triangle`, assumption $a, b > 0$, upper bounds on the sum of fourth powers of each part) and the estimates that ignore signs (the Schur and Hölder families; the fourth power of the $|K|$ norm is 10.76) go well above 1; what worked was the weight by row type, not by distance Reading. **The factor measured in the normalisation per direction is not an amount of contraction** (next paragraph).

**Note on normalisation — fix it to the form divided by the long-wavelength coupling.** In the coordinates (a), where the constant of the cone is divided by the scalar coupling of each step (the long-wavelength $\beta_0'$), the one-step factor is $\sup\mu/\beta_0' = \sup\mu$; in the cone (b), re-measured by the quadratic form of its own direction $\beta'(\theta)$, the factor is $\sup\lambda$, $\lambda := \mu/\beta'$. From $\frac{1}{4} \le \beta' \le \beta'_{\max} \le 1$ we get $\frac{1}{4}\lambda \le \mu \le \lambda$, and the two cones are equivalent within a constant 4 Proved here (a finite ledger; Lean: `ledger` (assumptions $b_{\min}\sum\theta^2 \le \sum\phi^2 \le b_{\max}\sum\theta^2$, $0 < b_{\min}$), `transfer`, `cone_b_to_a`; the $\beta'_{\max} \le 1$ above is a measured value). **If the normalisation is fixed to one for all steps, the constant 4 appears only once**, and the conclusion does not depend on the normalisation. If one switches between (a) and (b) at each step, a factor 4 is incurred at each step. In fact, in the volume limit $\sup\lambda = 4\sup\mu_{\mathrm{loc}}$ Proved here (Mixing-Limit Lemma; the coarse $L$ is even, $L \to \infty$. Mixing a direction $\theta_{\mathrm{loc}}$ of finite support with the coarsest direction $\theta_\pi$, which attains $\beta' = \frac{1}{4}$, by $\varepsilon = V^{-1/3}$, $\ell^4$ is carried by the localised direction and $\ell^2$ by the spread-out direction, so $\lambda \to 4\mu(\theta_{\mathrm{loc}})$; the estimate of the cross terms is a sketch using the exponential decay of the rows of the representative). Numerically too, the optimisation of $\lambda$ gives 0.269,

0.308, 0.338 at $12^4$, $16^4$, $20^4$, increasing with the volume; the extrapolation of the mixture is 0.395, and $4\mu_{\mathrm{loc}}$ is 0.3955 Reading . Hence $\lambda$ **is not an amount of contraction of directions**, and $\lambda < 1$ is equivalent to $\sup \mu < \frac{1}{4}$. Furthermore, the row-weighted Stein route cannot go below $\sum_s \nu_s^2/16 \geq 0.452 > \frac{1}{4}$ for any local representative or any weight Proved here (Floor Lemma for the Stein Route; the type in which 16 plaquettes per coarse plaquette receive $\theta/4$ at the long-wavelength minimiser differs between the paper sketch and $20^4$ by $3.5 \times 10^{-3}$; the finite part is in Lean: `stein_floor`, assumptions $A_i^2 \leq F v_i$, $0 \leq \nu_i \leq A_i$; $\nu$ is a linear-programming number). The quartic coefficient of the coarsest direction itself is multiplied by exactly $1/64$ in one step ($1/16$ in its own direction) Proved here (equality in Cauchy–Schwarz). The range of $\lambda$ is $[0.59, 3.29]$ Reading (the lower end goes through the sketch "$\sup \mu_{\mathrm{loc}}$ is at least $\sup \mu_L$ of every $L$"), and the normalisation is fixed to (a).

**The One-Step Contraction of the $\ell^4$ Cone is limited to the first step — several steps need conditions on the blocking.** A tree-level step is a Gaussian conditional mean, and the inverse $G = B^+$ of the coarse action (the covariance of the blocked flux) iterates linearly: $G_{k+1}(q) = \frac{1}{16} \sum_\eta T\, G_k\, T^{\mathrm{H}}$ ($T$ is the symbol of the coarse angle) Proved here (Conditional-Mean Covariance Lemma; it appears to be the standard form of blocking for Gaussian fields, but the sources have not been checked). In **blocking by edge sums**, where a coarse edge is the sum of two fine ones, the eigenvalues of $B_k$ escape to 0 at a rate $2^{-k}$ for fixed $q \neq 0$ (0.25, 0.090, 0.038, 0.017, 0.0083 in the direction $\pi^4$), only the long wavelength stays at 1, and there is no scale-invariant fixed point Reading (exact in symbols, $k \leq 5$; the reading is that the cause is the divergence of the perimeter length in the variance of the flux of the $2^k \times 2^k$ square). The minimiser escapes to those soft directions, and the lower bound on the factor of the cone is 0.699 at the second step ($12^4$) and goes above 1 **at the third step with 1.262** ($8^4$; the symbolic route and a separate nested implementation agree), 2.200 ($12^4$) and 2.746 ($16^4$) Reading (local optimisation over periodic configurations on the torus; no interval estimate). So **the One-Step Contraction of the $\ell^4$ Cone above (0.823) is limited to the first step**, and cannot be used for a multi-step geometric series that repeats the same blocking. If the coarse edge is defined by the gauge-covariant average of two parallel lines among $2^{d-1}$ transverse ones (the average of staples with transverse legs attached), the flow of the covariance goes to a fixed point (the eigenvalues of $B_k$ increase for all $q$ and the differences shrink with a ratio of 0.27–0.52, width $[1, \approx 4.1]$; in axial momentum the closed form $1/G_k = 3/(1 + 2 \cdot 4^{-k})$ goes to 3), and the lower bound on the factor of the cone does not grow with the step: **0.64, 0.57, 0.64, 0.62** ($j = 1$–4, $8^4$, $12^4$), and for a one-directional field it is exactly $\frac{1}{2}$ Reading (symbolic flow $k \leq 5$; the second step agrees with a separate nested real-space implementation from the fine step). **There is no upper bound** (the weighted Stein gives 4.5–8.1). The naive average without legs is not gauge-covariant: the minimum with links fixed and the minimum with flux fixed differ by 37–73%, and the lower bound at the second step also approaches 1 at 0.887. Hence the multi-step form is "for a blocking whose covariance flow has a scale-invariant fixed point, and with $\sup \mu_\infty < 1$ at the fixed point, the cone is a geometric series of $x_n \leq x_0 c^n$" Proved here (the ledger is in Lean: `cone_chain_uniform` (assumption $M_k \leq m$), `cone_series` (assumptions $0 \leq c < 1$, $x_{k+1} \leq c x_k$), `axis_flow_closed`), and the antecedent $\sup \mu_\infty < 1$ is supported only at the level of lower bounds (the question "upper bound on the fixed point of the lateral average"). Conditions: restricted to commuting configurations, tree level, periodic, $2^4$ blocks, linearised.

**Why the pseudoinverse representative grows with the volume — the tail type.** That the $\ell^1$ row sums of the pseudoinverse representative (the representation that uses the holonomies of the coarse plaquettes directly as coordinates) grow with the volume is consistent with a power-law kernel caused by the propagation of the Bianchi constraint being massless on the periodic lattice. Measuring the same step on a three-dimensional column $2 \times 2 \times L$ with the Bianchi constraint (which has open faces), the tail is exponential (rate $\ln(3 + 2\sqrt{2})$) and the row sums are uniform in the volume (saturating at 0.6035); on a slab without boundary it is $r^{-2}$, and with three-dimensional periodicity it is a power law (effective exponent around 3.3) Reading (commuting background, tree level, quadratic coefficients). The side of the local representative (the representation written in the variables of coarse links) was exponential in all three geometries. With four-dimensional periodicity the kernel of the pseudoinverse representative reads as $r^{-4}$, and the $\ell^1$ sums grow logarithmically (consistent with the numbers of Table 5; four dimensions has not been measured). The counting rules are in 15-08 § 06. The open four-dimensional point of the Bundle Network Theorem (local form) (the factor in the normalisation per direction) is a question separate from this power law, and this observation does not move it.

**The coordinates of coarse links plus global tags are exponential even for the four-dimensional tree.** If the same step is written by splitting the coarse variables into "coarse links (coordinates in which the Bianchi constraint is promoted to variables) plus global tags (the total flux through each plane; the harmonic part of the coarse plaquettes)", the tags have no cross terms with the coarse-link part Proved here (Lean: `norm_sq_add_harmonic`, assumption $D^\dagger b = 0$; uniqueness of the decomposition is `decomp_unique` and `range_sup_tags` (assumption $d_2 \circ D_C = 0$)), and the tree kernel with four-dimensional periodicity $n^4$ ($n \leq 16$) is exponential (prefactor around $r^{-1}$, rate $\kappa \approx 2.08$) and independent of the volume, with the distance-weighted row sums 15.62, 17.43, 17.50 at $n = 4, 8, 12$, tending to saturation Reading (commuting background, quadratic coefficients, one step, tree level). In the pseudoinverse representative with the tags removed (the coordinates of the coarse plaquettes), the projection of the tags produces a flat $-1/(2N)$, and together with the power law of the Bianchi propagation ($r^{-4.4}$ by fitting), the distance-weighted sums grow in proportion to $n$ Reading . In a three-dimensional periodic box without boundary (coarse $2 \times 2 \times L$) the one-loop kernel is also exponential ($\kappa = 3.094$, agreeing with the tree $\ln(11 + 2\sqrt{30}) = 3.089$), and the sums were uniform in the volume (15-08 § 06). The open point of the Bundle Network Theorem (local form) is separate from this question of coordinates, and this observation does not move it. The four-dimensional one loop has not been measured.

**The long-wavelength limit of the network and one loop.** If the network with all coarse plaquettes fixed is taken at once with block $L$, the long-wavelength eigenvalue of the coarse action stays at 1 for all $L$ (classically marginal), while at momenta on the scale of the

coarse lattice it becomes smaller as $1/L$ ($L \cdot B = 0.500$, 0.359, 0.305 in the direction $\pi^4$, $L = 2, 4, 8$; the same quantity as the iteration of block 2) Reading . The coefficient projected onto the cell type is $bL \to 0.20$, and the transition between the two scales, long and short wavelength, appears as the part that does not fit in the cell (20% at $L = 16$). **At tree level there is no running of** $\ln L$**.** At one loop this long-wavelength eigenvalue changes to $\beta'/\beta = 1 - 0.745/\beta$ (Manton, block 2), and most of the change comes from the fluctuations inside the block Reading ($SU(2)$, one loop, transverse coarse torus $6^3$, wave number 0 by extrapolation, 15-03 § 02). On a small coarse torus ($2^4$), the uniform direction of the coarse torus (the Polyakov direction) acquires a one-loop "mass" that contaminates the long-wavelength value ($\delta$ moves by 0.19), so the one loop of the network must be measured with the transverse coarse torus at 4 or more, or with a transverse Bloch sum Reading .

The axial cone, the Koszul contraction, linear-programming duality and Riesz–Thorin are all known tools, and the way of combining them may also be known (a view close to the construction of local minimisers; we have not checked against the original sources).

08

## Implications for the decomposition, and questions

The ledger of 15-03 § 04 said that if the one-step map contracts transversely around the class ($\|\rho_{k+1}\| \le \lambda\|\rho_k\| + \varepsilon(\beta_k)$, $\lambda < 1$), the departure is bounded not by a sum but by the supremum $\sup_j \varepsilon(\beta_j)/(1-\lambda)$ Proved here (under the linear inequality with quadratic terms dropped). This chapter showed that this $\lambda$ and the main term $\beta'/\beta$ of the flow of the parameters **can be counted beforehand on paper from the same projection** $P$ for the marginal distribution of one coarse plaquette (Unit Bundle Theorem). The numbers to ask about for one four-dimensional step are two, "the size of the source" and "$\lambda$"; the former is a one-loop quantity and the latter a tree-level one. However, when all coarse plaquettes are fixed, $\lambda$ becomes a matrix and exceeds $1/16$ in coarse directions ( § 07), so to show the $\lambda < 1$ of the ledger for a genuine step, an upper bound, uniform in the volume, on an operator norm that does not depend on the representation is needed Reading . The uniform upper bound itself is obtained with the local representative (Bundle Network Theorem (local form)), and in four dimensions row-weighted interpolation brought the factor in the $\beta_0' = 1$ normalisation to at most 0.823 (the number is computed). Since the factor in the normalisation per direction is not an amount of contraction, the normalisation is fixed to the form divided by the long-wavelength coupling (the note on normalisation in § 07), and this one-step number is limited to the first step. Several steps need a blocking whose covariance flow has a fixed point, and $\sup \mu_\infty < 1$ at that fixed point ( § 07).

Table 6 Questions arising from this chapter (status: new; answers have been added to questions 7, 20 and 25; questions 25, 28 and 30 were added in revisions)

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|
| 6 | In the sequence in the side $L$ of the coarse plaquette, does the source $\cdot L^4$ (now $7.7$–$8.5 \times 10^{-3}$ at $L = 1$–$4$ and transverse thickness 4) approach a constant when the transverse extent is widened? If so, the sum of the sources over steps is geometric and finite, and the ledger's assumption that "the source is in a fixed band at each step" can be strengthened | Small computation (the coefficient of $\theta^4$ at one loop by a perturbative formula) | The handling of the source in the shape remainder condition | Non-commuting configurations, large fields |
| 7 | In the tree-level network with all coarse plaquettes fixed at once, construct a **local representative** that agrees with $F$ on $\operatorname{im} D_c$, and see whether the operator norm of $\ell^4$ has an upper bound uniform in the volume. Also, whether the operator norm of $\lambda$ in coarse directions (the sup found is 0.27 in four dimensions) goes below 1. **Answer (partial):** the local representative and an upper bound uniform in the volume have been obtained (Bundle Network Theorem (local form), three dimensions $\Lambda_3 = 0.245$). In four dimensions it is $\Lambda_4 \approx 1.83$, and by this route it cannot go below 1.39. Row-weighted interpolation gives $\sup \mu \le 0.823$ (the $\beta_0' = 1$ normalisation, computed) | Paper and small computations | The main term of the transverse contraction of a genuine step | One loop, large fields, re-fixing of the gauge |
| 20 | Bound the four-dimensional $\ell^4$ directly: split the local representative as $K = K_0 + K_{\text{tail}}$ (a finite part near the block, and a tail), bound from above the $\ell^4 \to \ell^4$ norm of the small matrix of the near part on a finite network, apply Riesz–Thorin only to the tail, and see whether the four-dimensional $\sup \mu$ goes below 1. **Answer:** the split into a near part and a tail does not work; interpolation that weights the rows by their $\ell^1$ norm gives $\sup \mu \le 0.823$ (the $\beta_0' = 1$ normalisation; the number is computed, not an interval estimate). In the normalisation per direction it is $\le 3.29$ and undecided ( § 07). Resolved by composing three steps: weighted Stein 0.10 after three steps and $\le$0.20 at every step (computed, tree level, commuting; 15-08) | Paper and small computations | The transverse contraction of a tree-level step in four dimensions | One loop, non-commuting configurations, large fields |

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|
| 25 | In the normalisation per direction (the factor $\lambda$ divided by $\beta'(\theta)$ of its own direction), can one split the weighted symbol by momentum band, show that $\mu$ is small in the directions where $\beta'(\theta)$ is close to $\frac{1}{4}$, and bring $\sup \lambda \le 3.29$ below 1? **Answer:** it was a question in a direction that cannot be brought down. In the volume limit $\sup \lambda = 4$ $\sup \mu_{\mathrm{loc}} \ge 0.59$ (Mixing-Limit Lemma), so $\lambda$ is not an amount of contraction, and $\lambda < 1$ is equivalent to $\sup \mu < \frac{1}{4}$. The Stein route cannot go below the floor 0.452 (Floor Lemma for the Stein Route). The ratio per band is also 1.83 and gives no upper bound. The normalisation is fixed to the form divided by the long-wavelength coupling ( § 07) | Paper and small computations | Tree level of the shape remainder condition in the normalisation per direction | One loop, non-commuting configurations, large fields |
| 28 | On the fixed point $B_\infty$ of gauge-covariant blocking by transverse averaging, obtain an upper bound on $\sup \mu_\infty$ by a route other than Stein: separate the one-directional sector (exactly $\frac{1}{2}$) from the rest, which carries transverse momentum, and bound the rest by the decay of the rows (0.41, 0.05, 0.005) to get below 1. Also, take the existence of the fixed point from numbers ($k \le 5$) to paper | Paper and small computations | The multi-step condition of the shape remainder condition (the cone-contraction condition at the fixed point) | One loop, non-commuting configurations, large fields |
| 30 | With blocking by transverse averaging (or with larger blocking factors), measure the one-loop coefficient of the long-wavelength periodic network at blocking factors 2 and 3, and see whether the change approaches $-\frac{11 b_1}{3\pi^2} \ln \frac{3}{2} \approx -0.151$ (whether the factor 1.7–1.9 of the rule fixing the products along lines is an artefact of the rule) | Small computation | The one-step coefficient of the flow of the coupling (the coupling-flow remainder condition) | Coefficients from the second step on, large fields |

09

## What cannot be said

(a) The Unit Bundle Theorem and the Bundle Network Theorem are statements at tree level restricted to commuting configurations (the main term in $\beta \to \infty$). They do not include that the global minimum, including non-commuting configurations, is commuting; the one-loop correction (the source, the shift $O(1)$ of $\beta'$); or large fields (outside $\|\psi\| \le r$).

(b) The numbers in Tables 1–3 are numbers for a step that looks at the marginal distribution of one coarse plaquette, not for a genuine blocking step. The iterated two steps are the approximation of feeding back the marginal distribution of one plaquette.

(c) The source, $c_2$ and the ratios at finite $\beta$ are computer values, and the source is at one loop, started from the heat kernel, with a single image, at $\beta \to \infty$. The one-loop values for the $4 \times 4$ coarse plaquette and for large boxes are from one implementation. The source $\approx 8 \times 10^{-3}/L^4$ is an observation of four points; there is no limit or closed form.

(d) The equality on the right of (7), $\max f^2 = 1/16$, and that the largest flux is on the coarse plaquette in the boxes computed, are box-by-box confirmations; there is no general proof.

(e) The numbers of the Bundle Network Theorem (Tables 4 and 5) are values under periodic boundary, $2^d$ blocks and linearisation of the coarse angle, and the sup is a lower bound from local optimisation. The upper bound for the pseudoinverse representative grows with the volume. The upper bound uniform in the volume from the local representative (Bundle Network Theorem (local form)) is estimated at 1.83 in four dimensions (an estimate from below, with no interval estimate) and exceeds 1. Row-weighted interpolation brought it to at most 0.823 (the $\beta_0' = 1$ normalisation, $K_\infty$ replaced by the kernel of $20^4$, computed with no interval estimate), but in the normalisation per direction it does not go below 1, so we do not say "the transverse contraction is smaller than 1" in the normalisation per direction. Nor do we say "at most 1/16". The "smaller than 1" in three dimensions depends on the numbers for $\Lambda_3$. The Bundle Network Theorem (local form) does not include one loop, non-commuting configurations, large fields, or the second and later steps. The four-dimensional long-wavelength $\lambda \to 1/16$ is an estimate from the arithmetic of the smooth trial and the three-dimensional numbers. The Mixing-Limit Lemma and the Floor Lemma for the Stein Route of the note on normalisation are paper sketches (the estimate of the cross terms, the type of the long-wavelength minimiser) and numbers on finite tori; Lean covers only the finite ledger.

(f) This chapter does not show ultraviolet control. It implies nothing about the renormalisation of the four-dimensional lattice or the mass gap.

(g) The observation of the tail type in § 07 consists of tree-level numbers on the column, the slab and the three-dimensional periodic lattice (one loop only for the column). For the coordinates of coarse links plus global tags, four dimensions is tree level only ($n \le 16$), and the four-dimensional one loop has not been measured.

(h) The multi-step cone of § 07 goes only as far as this: with blocking by edge sums it exceeds 1 at the third step (a lower bound, periodic configurations on the torus), and with blocking by transverse averaging the lower bound does not grow with the step ($j \le 4$); there is no estimate of $\sup \mu_j < 1$ from above for either blocking. The existence of the fixed point rests only on the symbolic numbers for $k \le 5$ and a line of argument by smoothing the box. The one loop of the long-wavelength limit of the network is a first-step number for one rule.

10

## The five grades table and the dependency graph

Table 7 The rows of this chapter

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [01] | Unit Bundle Theorem (i)(ii): the minimiser $\theta f$, $f = Ps/\|Ps\|^2$, $\beta'/\beta = \sum f^2$. "$\sum f^2 \le 1$" is not unconditional | Proved here | § 02 | Restricted to commuting, tree level |
| [02] | Unit Bundle Theorem (iii)(iv): the coefficient of $\theta^4$ is exactly $\sigma \sum f^4$, $\lambda = \sum f^4 / \sum f^2$, (5) | Proved here | § 02 | Local minimum, $\sigma(6\theta^2 m^2 + 4\|\theta\| mr + r^2) \le \beta/4$ |
| [03] | For the three-dimensional free boundary, $\sum f^2$ is the effective resistance; at $2 \times 2 \times 1$, $1/3$ and $7/144$ | Proved here | § 03 | Three-dimensional, free boundary |
| [04] | Monotonicity under inclusion and $1/(16d - 8) \le \lambda \le 1/16$ | Proved here | § 04 | The 4 plaquettes of the coarse plaquette symmetric, $\frac{1}{4}$; the position of the largest flux confirmed box by box |
| [05] | The rational numbers of Table 1 (three-dimensional free boundary and free only on the outer rim) | Proved here | § 05 (two formulations) | Main term |
| [06] | The floating-point numbers and the source of Table 1; agreement with Monte Carlo | Reading | § 05 | $SU(2)$, started from the heat kernel, $\beta \to \infty$, one loop |
| [07] | (8): the upper bound on $\sum f^2(L)$ and the lower bound on $\lambda(L)$; genuine two steps < iterated two steps | Proved here | § 06 | $L \ge 2$, in any box (the comparison with the iteration takes the input 0.717961 as a computed number) |
| [08] | The perimeter type, Tables 2 and 3, the source $\approx 8 \times 10^{-3}/L$ 4 | Reading | § 06 | Four-dimensional, transverse thickness 4, one implementation, four points |
| [09] | Bundle Network Theorem (i)–(v); preservation of the cone of $\ell^4$ | Proved here | § 07 | Commuting, tree level, periodic, $2^d$ blocks, linearisation |
| [10] | Tables 4 and 5: $2^{d-4}$ at long wavelength, $\lambda > 1/16$ in coarse directions, no upper bound | Reading | § 07 | The boxes measured; the sup is a lower bound |
| [11] | The questions "lateral constancy of the coarse-plaquette source" and "local representative of the tree-level network" | Assumption | § 08 | Questions |
| [12] | Bundle Network Theorem (local form) (a)–(c): the local representative, the upper bound (10) uniform in the volume, the map of the cone of $\ell^4$ | Proved here | § 07 | Commuting, tree level, periodic, $2^d$ blocks, linearisation. The decay of $R$ is a sketch |
| [13] | $\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$, $\nu_4 \ge 1.18$; in four dimensions it cannot go below 1.39 | Reading | § 07 (d) | Three dimensions up to $48^3$; four dimensions an estimate at $24^4$; $\nu_4$ is $16^4$ |
| [14] | One-Step Contraction of the $\ell^4$ Cone: row-weighted interpolation (11), $\Lambda_w \le 0.823$; the $\ell^4$ cone contracts by a factor of at most 0.823 in one step | Proved here ((11)), Reading (numbers) | § 07 | Commuting, tree level, periodic, $2^4$ blocks, linearisation, the kernel of $20^4$, no interval estimate, the $\beta'_0 = 1$ normalisation (the normalisation per direction is undecided) |
| [15] | The growth of the pseudoinverse representative is consistent with the power-law kernel of the Bianchi constraint; the local representative is exponential | Reading | § 07 | Column, slab, three dimensions, tree level; four dimensions not yet |
| [16] | Note on normalisation: the two cones are equivalent within a constant 4 (only once if the normalisation is fixed); $\sup \lambda = 4 \sup \mu_{\mathrm{loc}}$ (Mixing-Limit Lemma), so $\lambda$ is not an amount of contraction; the floor of the Stein route is 0.452 (Floor Lemma for the Stein Route); the coarsest direction gives $1/64$ | Proved here (the ledger, sketches of the Mixing-Limit Lemma and the Floor Lemma for the Stein Route), Reading (numbers) | § 07 | Commuting, tree level, periodic, $2^4$, linearisation, coarse $L$ even with $L \to \infty$, $\nu$ is a linear-programming number |
| [17] | The One-Step Contraction of the $\ell^4$ Cone is limited to the first step: with blocking by edge sums the covariance has no fixed point ($2^{-k}$) and the factor at the third step is $\ge 1.26$. With transverse averaging it goes to a fixed point, lower bounds 0.64, 0.57, 0.64, 0.62, no upper bound | Proved here (Conditional-Mean Covariance Lemma, the ledger of the geometric series), Reading (numbers) | § 07 | Commuting, tree level, periodic, $2^4$, linearisation, symbolic flow $k \le 5$, lower bounds by local optimisation, no interval estimate |
| [18] | The coordinates of coarse links plus global tags are exponential for the four-dimensional periodic tree ($\kappa \approx 2.08$); without the tags, flat plus power law, with sums growing with the volume | Proved here (orthogonality of the tags), Reading (numbers) | § 07 | Commuting background, quadratic coefficients, one step, tree level, $n \le 16$ |
| [19] | The long-wavelength limit of the network: the tree gives 1 for all $L$, the cell part is $1/L$, the tree has no running of $\ln L$, and one loop gives $1 - 0.745/\beta$ | Reading | § 07 | Commuting, tree level and one loop (block 2, transverse $6^3$); beware of contamination by the Polyakov direction |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV F1     minimiser θf, β'/β = Σf²  (commuting, tree level)
[02] PRV F2     θ⁴ exactly σΣf⁴, λ, (5) ⟸ [01]  (local minimum)
[03] PRV NET    three-dimensional resistor network, 1/3, 7/144 ⟸ [01]  (free boundary)
[04] PRV BOUND  monotonicity under inclusion, 1/(16d−8) ≤ λ ≤ 1/16 ⟸ [01][02]  (coarse plaquette symmetric)
[05] PRV RAT    the rational numbers of Table 1 ⟸ [01][03]
[06] RDG BOX    the floating-point numbers and the source of Table 1 ⟸ [02][05]  (one loop, β→∞)
[07] PRV LBOUND (8), genuine two steps < iteration ⟸ [04]  (L ≥ 2)
[08] RDG PERIM  perimeter type, source ≈ 8e−3/L⁴ ⟸ [06][07]  (four points, one implementation)
[09] PRV G      Bundle Network Theorem ⟸ [01]  (periodic, 2^d blocks)
[10] RDG GNUM   long wavelength 2^{d−4}, coarse directions λ > 1/16 ⟸ [09]
[11] ASM Q      questions "lateral constancy of the coarse-plaquette source" and "local representative of the tree-level
[12] PRV G2     local representative, uniform bound (10), map of the cone ⟸ [09]  (periodic, 2^d blocks)
[13] RDG G2NUM  Λ₃ = 0.245, Λ₄ ≈ 1.83, ν₄ ≥ 1.18 ⟸ [12]
[14] PRV G2C    weighted interpolation (11), cone contracts by at most 0.823 ⟸ [12][13]  (β'₀ = 1 normalisation, number
[15] RDG TAIL   growth of the pseudoinverse representative is a power-law kernel ⟸ [10][12]  (up to three dimensions, t
[16] PRV NORM   note on normalisation: equivalence, Mixing-Limit Lemma, Floor Lemma for the Stein Route ⟸ [09][14]  (L
[17] RDG MULTI  One-Step Contraction of the ℓ⁴ Cone limited to the first step, multi-step averaged blocking ⟸ [14][16]
[18] RDG TAG    coarse links plus tags exponential for the four-dimensional tree ⟸ [12][15]  (tree level, n ≤ 16)
[19] RDG LONG   long-wavelength limit of the network and one loop ⟸ [09][10]  (block 2, one loop)
```

Every number to the right of "⟸" is smaller than the one on the left, and there are no cycles. The only assumption row is [11], and it is in the form of a question (the direct upper bound for the four-dimensional $\ell^4$, the normalisation factors by direction, the upper bound on the fixed point of the lateral average, and the one-loop coefficient of the long-wavelength network are in Table 6 of § 08, and continue [11]).

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Proved here At tree level restricted to commuting configurations: that $\beta'/\beta$, the map of the quartic term and $\lambda$ are determined by the projection (Unit Bundle Theorem); $\lambda \le \max f^2$; $1/(16d-8) \le \lambda \le 1/16$ if the coarse plaquette is symmetric; (8); the Bundle Network Theorem and $\beta'(\theta) \ge \frac{1}{4}$; the upper bound uniform in the volume from the local representative (Bundle Network Theorem (local form)); row-weighted interpolation (11) | The minimum including non-commuting configurations, one loop, large fields. A general proof of the position of the largest flux |
| Reading The numbers box by box (Tables 1–5), the perimeter type, the source $\approx 8 \times 10^{-3}/L^4$, and that with all coarse plaquettes fixed $\lambda > 1/16$ in coarse directions; $\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$; $\Lambda_w \le 0.823$ (four dimensions, the $\beta_0' = 1$ normalisation); the observation of the tail type | $\lambda$ and the source of a genuine blocking step. Interval estimates. A closed form for the source |
| Proved here The note on normalisation (equivalence of the two cones, the finite parts of the Mixing-Limit Lemma and the Floor Lemma for the Stein Route); that a tree-level step is a conditional mean and the covariance iterates linearly (Conditional-Mean Covariance Lemma). Reading Going above 1 at the third step of blocking by edge sums; the fixed point and lower bounds of transverse averaging; the four-dimensional tree of coarse links plus tags; the one loop of the long-wavelength limit of the network | An upper bound on the multi-step $\sup \mu_j < 1$ (for any blocking). A proof of the existence of the fixed point. The four-dimensional one-loop tail. The one loop of the network from the second step on |

## Sources and reproduction

| Item | Type | Source, tools |
|---|---|---|
| Thomson's principle, Kirchhoff's laws, Rayleigh's monotonicity, Dirichlet's principle | Cited | Standard facts about electric circuits and harmonic functions (of the kind in P. G. Doyle, J. L. Snell, Random Walks and Electric Networks, 1984) |
| Unit Bundle Theorem, Bundle Network Theorem, (5)(7)(8), the hand computation of $2 \times 2 \times 1$ | Proved here | The proofs in the text of this chapter. The parts that are inequalities on finite sums are in the Lean bundle (re-verification: done; 15-09 Table 3) |
| The numbers of Tables 1–5 | Reading (rational numbers: Proved here) | The numerical records of this book's study (two to three formulations of the flux network: Kirchhoff on the dual network, the normal equations in the tree gauge, the projection formula, the Hodge Laplacian of the Lagrange conditions. Two implementations of the one-loop Hessian: finite differences and analytic, and a sparse-matrix version. Finite sums of characters; heat-bath Monte Carlo on all links. For the Bundle Network Theorem, two implementations: projection and pseudoinverse, and KKT on coarse links. For the numbers of the Bundle Network Theorem (local form), two implementations of the local representative, and a linear program) |
| Weighted interpolation (Stein's analytic family, Hadamard's three lines) | Cited (a known tool; bibliographic details unchecked) | E. M. Stein's interpolation theorem for analytic families |

| Item | Type | Source, tools |
|---|---|---|
| The note on normalisation, the multi-step cone, the four-dimensional tree of coarse links plus tags, the long-wavelength limit of the network | Reading (the ledger and the Conditional-Mean Covariance Lemma: Proved here) | The numerical records of this book's study (the symbolic route and the Bloch rows of the first step, weighted least squares in real space, cross-checks against a separate nested implementation, negative controls with random small matrices, the negative control of the naive average, two implementations dense and Bloch, 0 for the $U(1)$ Villain). The ledger is in the Lean bundle (15-09 Table 3, re-verified) |
| $\Lambda_w \le 0.823$, the numbers of the tail type | Reading | The numerical records of this book's study (the symbol of the kernel of $20^4$ on the $N = 64$ grid with a one-sided quadratic remainder, the real-space power method, small-matrix tests and negative controls; for the column, slab and three-dimensional periodic cases, two implementations, projection and KKT) |

**Terms used in this chapter** — flux, the flux network, the projection $P$, the unit flux, the Unit Bundle Theorem, the transverse contraction $\lambda$, the source $\sigma$, monotonicity under inclusion, the perimeter type, the genuine two steps, the iterated two steps, the Bundle Network Theorem, quantities that do not depend on the representation, the local representative, the axial cone, the Bundle Network Theorem (local form), row-weighted interpolation, the tail type. Their meanings are in A-02 Glossary.

Next chapter to read: 15-06 Counting the parameters — how many coordinates are needed at quadratic order, and where the one-step map draws in.

# Counting the parameters — couplings by loop shape and the attracting class

This chapter is this book's own investigation. The established knowledge it builds on is in 15-01 and 15-02.

This part is a record of the attempts made in this book's study at ultraviolet control (the existence of the continuum limit) by a design different from that of the published construction. For the existence itself see Parts 6–13 of Volume 2; for the difference between the two designs see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — Using the smallest systems, with plaquettes added one at a time, we look at which coordinates should be used to count the parameters of the class-closure condition of 15-03 § 02, "the class closes with finitely many parameters". With one plaquette, the background dependence is absorbed into a single replacement of the coupling; with two or three, the commutator is merely the image of the coupling of the composite loop; with four, the ratio of the couplings by loop shape moves with $\beta$; and even if the heat kernel is used instead, the part $2/(3\beta)$ due to the curvature of the group remains. Searching for actions that keep the ratio at 2, the family that moves the power of the prefactor has no root, and the fundamental + adjoint family has a root, but that curve does not close under the one-step flow. Finally, since the one-step map contracts all the shape coordinates of the action and draws them to the unique fixed point of the power of the prefactor ($37/76$ at $\psi^4$), what is consistent with this model is to state the class-closure condition not as "a class that keeps the ratio at 2" but as **"the one-parameter attracting class + a contracting shape"**.

**Chapters assumed** — 15-03 § 02 (the statement of the decomposition), 15-03 § 04 (the ledger of the transverse contraction), 15-04 § 01 (the heat-kernel semigroup), 15-05 § 02 (the flux network), 15-07 § 02 (the background dependence $\eta$). We regard $SU(2)$ as the unit quaternions (the three-sphere $S^3$) and use the character expansion and the Laplace method (the saddle-point approximation).

**Said first** — Every system in this chapter is one plaquette up to four plaquettes in a row of $SU(2)$ (with $U(1)$ as a control), with the field strength of the background taken small (holonomy angle $\theta \le 0.3$). The one step is an even smaller version of **the one step seen through the marginal distribution of a single coarse plaquette (not the one step of a true blocking)**. The maps in § 07 and § 08 are maps for four plaquettes in a row, two loops, the diagonal definition and the main term ($B \to \infty$). None of the statements says anything about one step of the four-dimensional lattice, block averages, large fields or gauge re-fixing.

The order of this chapter

1. One plaquette — the type of the background dependence, and absorption under a constraint (equation (1))
2. Two plaquettes — the commutator is the image of BCH (equations (2)(3))
3. Three plaquettes — single and composite loops in 1:1:1 (equation (4))
4. Four plaquettes — the ratio of the couplings by loop shape moves with $\beta$ (equations (5)–(7))
5. The heat kernel and the criterion — the part due to the curvature of the group (equations (8) (9), Table 1)
6. Searching for an action that keeps the ratio — the general one-loop formula, the prefactor family, fundamental + adjoint (equations (10)(11))
7. The ratio-2 curve does not close under the flow — the tree map and $9/89$ (equations (12)(13))
8. The flow of the power of the prefactor and the attracting class — $37/76$, $c = 2/3$ (equations (14)–(16))
9. The form of the statement of the class-closure condition (Table 2) and questions (Table 3)
10. What cannot be said
11. The five grades table and the dependency graph (Table 4)

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00

## What this chapter shows

- The coordinates for counting the parameters of the class-closure condition are not "the coefficient of $F^2$ and the commutator coefficient" but the **couplings by loop shape**. The commutator coefficient follows afterwards as the BCH image of the coupling of the composite loop ( § 02, § 03).
- For four plaquettes in a row the second coordinate (the ratio $\kappa_{ab}/\kappa_a$) moves with $O(1/\beta)$, and part of it, $2/(3\beta)$, is due to the curvature of the group and does not vanish even if the action is replaced by the heat kernel ( § 04, § 05).
- Keeping the ratio at 2 for all $\beta$ requires an action whose quartic term in the exponent is steeper than Gaussian ($q/B = -1/12$); the family of powers of the prefactor has no root, and the fundamental + adjoint family has a root at $\beta \geq 12$ ( § 06).
- That root curve does not close under the one-step flow, and the deviation stays at $O(1)$ ($9/89$), because the one-step map contracts all the coordinates of the shape of the action ( § 07).
- The power of the prefactor follows a linear contraction map and is drawn to the unique fixed point $\gamma^* = 37/76$ ($\psi^4$), where the ratio is $2 + (2/3)/B$. The class-closure condition can be stated as "the one-parameter attracting class + a contracting shape" ( § 08, § 09).

01

## One plaquette — the type of the background dependence, and absorption under a constraint

Conventions: $SU(2)$ is written with unit quaternions, $\frac{1}{2}\operatorname{tr} U = \operatorname{Re} U$, background $W = \exp(i\,a\cdot\sigma/2)$ ($a \in \mathbb{R}^3$, angle $\theta = |a|$). The numbers were measured at $\beta = 4$–$32$, $\theta \le 0.3$.

On a single plaquette (four links), consider the measure $\mu_W$ of the fluctuations around the background holonomy $W$. It is the smallest model of the $\eta_k$ of 15-07. Of the 12 dimensions of the four links, the conditional law of the 9 gauge directions does not depend on $W$, and the background dependence gathers in the 3 dimensions of the holonomy Proved here (invariance of the Haar measure). Placing on the fluctuation $X = i\,x\cdot\sigma/2$ ($r = |x|$) the constraint $x_3 = 0$ that removes the direction of the generator of the background (the smallest substitute for the background-field constraint), we have $\frac{1}{2}\operatorname{tr}(W e^X) = \cos\frac{\theta}{2}\cos\frac{r}{2}$, so

$$\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)} \quad (1)$$

holds exactly Proved here . The background dependence goes entirely into a single replacement of the coupling $\beta \mapsto \beta\cos(\theta/2)$. The $\eta$ measured by the ratio of covariances is $\theta^2/8$, and the score is $\theta/4$; neither depends on $\beta$ (main term Proved here ; by two routes, quadrature and Monte Carlo, the ratio to the main term is 0.98–1.00 Reading ).

- **Without the constraint it is not small.** If the background is not taken at a critical point, the ratio of covariances grows with $\beta$ as $\beta\theta^2/12$ Reading . From this one plaquette we read that the minimal condition for the background dependence to be small uniformly in the background is "the background is a constrained critical point".
- **The one-step map of the coupling.** The weight of the background with the fluctuations integrated out is, under the constraint, $\beta_{\text{eff}} = \beta - d/2 + O(1/\beta)$ ($d$ is the dimension of the fluctuations; $\beta - \beta_{\text{eff}} = 1.005$ at $d = 2$, $\beta = 64$) Reading (Proved here as a count of the dimensions of the fluctuations, at the one-loop level).
- **The source of $\eta$ is not non-commutativity.** In the $U(1)$ control it is $\eta = 0$ only for the quadratic (Gaussian) action, and for the cos action a $\eta$ of the same $\theta^2$ type appears Reading . The source is the non-quadratic nature of the action (the compactness of the group). Measuring $\eta$ by a single number changes the type (the ratio of covariances is $\theta^2$, the Kullback–Leibler quantity is $\theta^4$, the score is $\theta$), so the statement must specify "which $\eta$" Reading .

02

## Two plaquettes — the commutator is the image of BCH

On a $2\times 1$ rectangle with one shared link, fix the outer loop $W = P_1P_2$ ($P_1, P_2$ are the holonomies of the two plaquettes, independent Haar after the tree gauge). The constrained minimum is $P_1 = P_2 = W^{1/2}$, in which the two plaquettes tilt by half each in the same direction; setting $P_1 = W^{1/2}e^X$, $P_2 = e^{-X}W^{1/2}$,

$$\operatorname{Re} P_1 + \operatorname{Re} P_2 = 2\cos\frac{\theta}{4}\cos\frac{r}{2} \quad (2)$$

holds exactly Proved here . The action is even in the fluctuations, and the odd cubic term that appeared when the substitute of "subtracting the first-order term" was used for one plaquette vanishes at the true constrained minimum (numerically the odd part of the score is $\leq 3 \times 10^{-12}$) Reading . The weight with the fluctuations integrated out is a function $Z_1(\beta|1+W|)$ of the class of $W$ alone, $Z_1(b) = 2I_1(b)/b$ (the same integral as the one-link distribution of the heat-bath method) Proved here , and the coupling after one step is $\beta_{\text{eff}} = \beta - \frac{3}{4} + O(1/\beta)$ (Proved here , at the one-loop level; $\beta - \beta_{\text{eff}} = 0.723 \to 0.747$ at $\beta = 4 \to 32$ Reading ). The commutator of the field strengths $a, b$ of the two background plaquettes enters only through the BCH of the group product $e^{F_1} e^{F_2}$:

$$\theta_W^2 = |a+b|^2 - \tfrac{1}{12}|a \times b|^2 + O(|F|^5) \quad (3)$$

Proved here (algebra under truncation of BCH at third order). So the coefficient $c$ of $|a \times b|^2$ in the weight is tied to $1/12$ of the coefficient $k$ of $F^2$, and $c/k = 1/12$ holds to $10^{-8}$ for $\beta = 4$–$32$ Reading . **The commutator is not a new parameter at this step.**

## 03 Three plaquettes — single and composite loops in 1:1:1

With three in a row, fix the two overlapping $2 \times 1$ loops $W_a = P_1 P_2$, $W_b = P_2 P_3$ and integrate over the central $P_2$; the weight is then, with $Z_1(\beta|Q|)$, $Q = 1 + W_a + W_b$,

$$|Q|^2 = 3 + 2\,\mathrm{Re}\,W_a + 2\,\mathrm{Re}\,W_b + 2\,\mathrm{Re}(W_a W_b^{-1}) \quad (4)$$

Proved here (agrees with direct quadrature to $3.5 \times 10^{-15}$ Reading ). It contains the two single loops and the composite loop $W_a W_b^{-1}$ in the ratio 1:1:1. The invariants of simultaneous conjugation of a pair of elements of $SU(2)$ are determined by the three traces $\mathrm{Re}\,W_a, \mathrm{Re}\,W_b, \mathrm{Re}(W_a W_b^{-1})$ Proved here (two imaginary-part vectors are determined up to rotation by their lengths and inner product), so every gauge-invariant two-loop weight is a function of the three class angles. Taking the class angle of the composite loop as the reference, the coefficient of the commutator is necessarily $1/12$ of the quadratic coupling $\kappa_{ab}$ of the composite loop Proved here ((3) enters as it is). The coupling after one step is $12k = \beta - \frac{1}{2} + O(1/\beta)$ Reading . The "commutator coefficient" measured in the basis of monomials appears to move with $\beta$, but it appears with the same size even for a composite function with no commutator, so it is not a quantity of the commutator Reading .

## 04 Four plaquettes — the ratio of the couplings by loop shape moves with $\beta$

With four in a row, fix the two overlapping $3 \times 1$ loops $W_a = P_1 P_2 P_3$, $W_b = P_2 P_3 P_4$ and integrate over the two plaquettes inside. Setting $U = P_2 P_3$,

$$w(W_a, W_b) = \int dU\; Z_1(\beta|1+U|)\, \exp\big(\beta \langle W_a + W_b,\, U\rangle\big), \qquad \langle p, q\rangle = \mathrm{Re}(p\bar{q}) \quad (5)$$

holds Proved here . $w$ is a function of the two variables $\mathrm{Re}\,W_a + \mathrm{Re}\,W_b$ and $\mathrm{Re}(W_a W_b^{-1})$ and does not collapse to the single variable $|Q|$ of the three-plaquette case (on the same $|Q|$ background, $\log w$ moves by $1.8 \times 10^{-2}$, $\beta = 4$) Reading . The quadratic couplings are defined by

$$-\log w = \text{const} + \kappa_a(\theta_a^2+\theta_b^2) + \kappa_{ab}\,\theta_{ab}^2 + O(F^4) \quad (6)$$

($\theta_{ab}$ is the class angle of the composite loop $W_a W_b^{-1}$). The tree (the commuting Gaussian minimisation) gives the ratio $\kappa_{ab}/\kappa_a = 2$ Proved here , and the main term at one loop is $-\frac{1}{40}$ for both $\kappa_a, \kappa_{ab}$, with ratio 1. Together,

$$40\kappa_a = \beta - 1, \qquad 20\kappa_{ab} = \beta - \tfrac{1}{2}, \qquad \frac{\kappa_{ab}}{\kappa_a} = \frac{2(\beta-\frac{1}{2})}{\beta-1} + O(\beta^{-2}) \quad (7)$$

(the first two are $+O(1/\beta)$) Proved here , at the one-loop level. Numerically $\kappa_{ab}/\kappa_a = 2.3211,\ 2.1409,\ 2.0662,\ 2.0322$ ($\beta = 4, 8, 16, 32$, the two routes agree to $7.8\times10^{-12}$) Reading . **The reason the ratio moves is "the tree shape ≠ the one-loop shape"**. The quadratic form of the tree is 1:2, fixed by the geometry of the lattice, and the quadratic form at one loop is 1:1, fixed by the measure; since the two are not parallel, the direction of the sum rotates with $\beta$. For the cos action of $U(1)$ too the ratio moves with $2 + 0.2/\beta$, while for the Villain action of $U(1)$ it is 2 for all $\beta$ Reading . That one step of blocking produces terms of larger loops whose coefficients depend on the coupling is, we think, a known type in the same direction as the idea of improved actions (the original sources have not been checked).

05

## The heat kernel and the criterion — the part due to the curvature of the group

If the weight of each plaquette is replaced by the heat kernel $p_t$ of $SU(2)$ (character coefficients $(n+1)e^{-tn(n+2)/2}$), the two inner plaquettes combine by the semigroup property $p_t * p_t = p_{2t}$ into

$$w(W_a, W_b) = \int dU\; p_{2t}(U)\, p_t(W_a U^{-1})\, p_t(U^{-1} W_b) \quad (8)$$

Proved here (the same formula for finite groups and families of semigroups is in Lean: `four_plaquette_of_semigroup`, `four_plaquette_convPow`, assumption $p_s * p_t = p_{s+t}$). Even so, the ratio is $2.2090,\ 2.0927,\ 2.0439,\ 2.0214$ ($t = 1/4, \dots, 1/32$; the two routes agree at $7.6\times10^{-12}$) Reading , and at one loop it is $2 + \frac{2t}{3} + O(t^2)$ Proved here , at the one-loop level. For the heat kernel (Villain) of $U(1)$ it is 2 for every $t$ Proved here .

The reason can be written as a one-line criterion. Let the outer kernel be $e^{F(\psi)}$ and the inner kernel $G(\psi)$ (both class functions; $\psi$ is the angle of a unit quaternion), and write the expectation with respect to the measure $\sin^2\psi\, G\, e^{2F}\, d\psi$ as $E$; then integration by parts gives

$$\kappa_a = \frac{E[F'\,(\log G)']}{24}, \qquad \kappa_{ab} = \frac{E[F'^2]}{24}, \qquad \frac{\kappa_{ab}}{\kappa_a} = 2 \iff D := E\big[F'\,(F' - 2(\log G)')\big] = 0 \quad (9)$$

Proved here (under the conditions that $F, G$ is a smooth class function for $[0,\pi]$ and that the boundary terms vanish). A pointwise sufficient condition is that **the inner kernel is the square root of the outer one**, $G \propto e^{F/2}$. For a flat Gaussian the semigroup $p_{2t}$ and the square root $p_t^{1/2}$ are the same thing, so the ratio stays at 2; for a curved group the two split. The heat kernel of $SU(2)$ is a semigroup, but the Van Vleck prefactor $P(\psi) := \psi/\sin\psi$ enters $p_t$ and $p_{2t}$ with the same power, so it is not a square root. The Gaussian in the geodesic distance $e^{-\psi^2/(2t)}$ satisfies the square-root relation but is not a semigroup Proved here (numerically too the ratio is 2 for all $t$, $3\times10^{-12}$ Reading ).

Matching $t$ and $\beta$ by the coefficient of the fundamental representation, the main term of the deviation of the ratio is $1/\beta$ for Wilson and $2/(3\beta)$ for the heat kernel. **The $2/(3\beta)$ that does not vanish even for the heat kernel is the part due to the curvature of the group, and the remaining $1/(3\beta)$ is the part due to the non-quadratic nature of the action**, we read Reading (main term, this system, quadratic order).

Table 1 Counting the parameters in the smallest systems ($SU(2)$, $\theta \le 0.3$, $\beta = 4$–$32$. The weight formulas are Proved here , the numbers are Reading )

| System (what is fixed) | Weight with the fluctuations integrated out | Quadratic coupling | Commutator coefficient | $U(1)$ control |
|---|---|---|---|---|
| One plaquette (background $W$, constraint $x3 = 0$) | $\mu_W^\beta = \mu_1^{\beta\cos(\theta/2)}$ ((1)) | $\beta_{\text{eff}} = \beta - d/2$ | — (no commutator can be formed) | $\eta$ of the same $\theta^2$ for the cos action; 0 for the Gaussian |
| Two plaquettes (outer loop $W$) | $Z_1(\beta\lvert 1 + W\rvert)$ | $\beta_{\text{eff}} = \beta - \frac{3}{4}$ | $k/12$ ((3)) | 0 |
| Three in a row (overlapping $W_a, W_b$) | $Z_1(\beta\lvert 1 + W_a + W_b\rvert)$ | single : single : composite = 1 : 1 : 1, $12k = \beta - \frac{1}{2}$ | $\kappa_{ab}/12$ | 0 |
| Four in a row, Wilson | (5) (a function of two variables) | $\kappa_{ab}/\kappa_a = 2(\beta - \frac{1}{2})/(\beta - 1)$ | The BCH image of the coupling of the composite loop | $2 + 0.2/\beta$ at cos; 2 for Villain |
| Four in a row, heat kernel | (8) | $\kappa_{ab}/\kappa_a = 2 + \frac{2t}{3}$ | Same as above | 2 for the heat kernel (Villain) |

06

## Searching for an action that keeps the ratio — the general one-loop formula

Is there an action that satisfies the criterion (9) for all $\beta$ (an action that closes with one parameter for four plaquettes in a row)? Writing an action concentrated at the identity as $\log f = \text{const} - B\psi^2/2 + q\psi^4 + \cdots$ ($B$ is the tree coupling, $q = O(B)$) and counting the correction of the inner kernel by the Brownian-bridge average (the flat approximation), we get

$$D = \frac{2}{5} + \frac{24}{5}\,\frac{q}{B} + O\Big(\frac{1}{B}\Big), \qquad E[F'(\log G)'] = \frac{3}{5}B + O(1), \qquad \frac{\kappa_{ab}}{\kappa_a} = 2 + \frac{D}{\frac{3}{5}B} + O(B^{-2}) \quad (10)$$

Proved here , at the one-loop level. $\frac{2}{5}$ comes from the curvature of the group (the Van Vleck prefactor $\log P = \psi^2/6 + \cdots$), and $\frac{24}{5}q/B$ from the quartic term of the exponent (non-Gaussianity). A multiplicative correction of $\psi^2$ is absorbed into a shift of the tree coupling $B$ and does not enter $D$ Proved here . Calibration: for Wilson ($q = \beta/24$) $D = \frac{3}{5}$ and ratio $2 + 1/\beta$; for the heat kernel ($q = 0$) $D = \frac{2}{5}$ and ratio $2 + 2t/3$; for $U(1)$, $D = \frac{24}{25}q/B$ ($\frac{1}{25}$ at cos, 0 for Villain). All the values of § 04 and § 05 are reproduced Reading .

- **The family of powers of the prefactor has no root.** For $f = e^{\beta\cos\psi}(\psi/\sin\psi)^\gamma$, in the range $\gamma < \frac{1}{2}$ where the quadratic coupling exists, $D \in [0.534, 0.600]$ at every point of the measured grid $\gamma \in [-2, 0.45]$, $\beta \in \{4, 8, 16, 32\}$, and $D \ge 0.343$ also on the extension $\gamma \ge -128$; it is monotonically

increasing in $\gamma$, with no change of sign Reading (the two routes agree to $\le 6.5\times10^{-9}$). At one loop too, $D>8/25$ for all $\gamma=c\beta$ ($c<3$) Proved here, at the one-loop level. At $\gamma\ge\frac{1}{2}$ the singularities of $\psi=\pi$ overlap and the quadratic coupling itself does not exist, and at $\gamma\ge\frac{3}{2}$ the weight itself diverges Proved here. The reason is that the power of the prefactor is absorbed into $B$ by the correction of $\psi^2$, and the curvature term $\frac{2}{5}$ does not move.

- **The fundamental + adjoint family has a root.** From (10), $D=0$ requires $q/B=-\frac{1}{12}$ (a quartic term steeper than Gaussian). $F=\beta\cos\psi+\rho\beta\cos^2\psi$ is $B=\beta(1+2\rho)$, and with $q=\beta(\frac{1}{24}+\frac{\rho}{3})$

$$D=\frac{2}{5}+\frac{1}{5}\frac{1+8\rho}{1+2\rho}+O\Big(\frac{1}{B}\Big),\qquad \text{one-loop root } \rho=-\tfrac{1}{4}\ (11)$$

Proved here, at the one-loop level. The numerical root is $\rho^*(\beta)=-0.318,\ -0.279,\ -0.264,\dots,\ -0.2516$ ($\beta=16,32,64,\dots,512$; ratio 2 to $3\times10^{-10}$ by bisection and two routes), $\rho^*=-\frac{1}{4}-0.84/\beta+O(\beta^{-2})$ Reading. For $\beta\le10$ there is no root on the measured grid (minimum $D=+0.029$), and the root appears between $\beta=10$ and 12 Reading.

- **One parameter for one step only.** In neither family does $f*f$ belong to the same family (the residual of the fit is at least $10^5$ times that of the heat-kernel control) Reading. Even if the root action is taken, the inner kernel after one step leaves the family, so the same criterion cannot be used at the next step. The pointwise square-root relation $f*f\propto f^{1/2}$ cannot be satisfied at the one-loop level by a concentrated class function Proved here (the coefficient $-\frac{1}{3}$ of $\psi$ remains).

07

## The ratio-2 curve does not close under the flow

We check whether the root curve $\rho^*(\beta)$ returns to itself under the one-step map. The outer kernel after one step is defined on the diagonal as $f'(\psi):=w(W,W)^{1/2}=[(f*f)*f^2]^{1/2}(W)$ (**the diagonal definition**: at $W_a=W_b$ the angle of the composite loop is 0), refitted to the fundamental + adjoint family by the Taylor coefficients of $\psi^2$ and $\psi^4$, and $(\beta',\rho')$ is read off.

**The tree map.** With the outer kernel $F=-Bx^2/2+qx^4+\cdots$, at the flat, collinear saddle point (on the geodesic joining 1 and $W$, $u_0=4w/5$),

$$B'=\frac{B}{5},\qquad \Big(\frac{q}{B}\Big)'=\frac{17}{125}\frac{q}{B}+O\Big(\frac{1}{B}\Big),\qquad \psi^{2k}\ \text{shape coordinate: }\frac{4^k+1}{5^{2k-1}}\ \text{times}\ (12)$$

holds Proved here, tree (main term of the saddle point). Because the saddle point is collinear, the commutator and the curvature of the group do not enter the tree, and enter only the one-loop $O(1/B)$. Combined with (10), the image of the ratio-2 curve $q/B=-\frac{1}{12}$ is $q/B=-\frac{17}{1500}$, and for fundamental + adjoint

$$\rho'=-\frac{53}{356},\qquad \rho'-\rho^*=\frac{9}{89}\approx0.101,\qquad D(\text{image})=\frac{216}{625}\ (13)$$

holds Proved here, tree (algebra of substitution). Numerically, $\rho'-\rho^*(\beta')=0.160,\ 0.127,\ 0.113$ (starting from $\beta=128,256,512$, $\beta'=17.8,36.1,72.5$) approaches $9/89$ as $O(1/\beta')$ Reading (the two routes agree to $\le6\times10^{-9}$ at $q'$). Adding a third coordinate $\sigma\beta\cos^3\psi$ does not bring $D$ back to 0 ($\sigma$

enters $D$ only at $O(1/B')$) Reading . **One step contracts the shape coordinate $q/B$ and draws the flow towards the Gaussian side. For $\mathrm{SU}(2)$ the ratio-2 condition lies at a point $O(1)$ away along the contracting direction, so the curve does not close**, we read. For $U(1)$ the ratio-2 line $q/B = 0$ is itself the fixed point of the tree map (Villain), so it closes Reading .

08

## The flow of the power of the prefactor, and the attracting class

Then where does the flow draw to? Write the class function as $f = \exp(-B\psi^2/2 + h(\psi))$ $(h = O(1))$ and follow one step by the main term of the Laplace method with the argument held at $B \to \infty$. Regarding $SU(2)$ as the $S^3$ of radius 1, the saddle point of the convolution of two kernels lies on the geodesic joining 1 and $W$, and the determinant of the Hessian in the two transverse directions gives the ratio of the prefactors $P$:

$$h_{f*g}(w) = h_1(a) + h_2(b) + \log P(w) - \log P(a) - \log P(b) + \mathrm{const} + O(1/B) \quad (14)$$

($a, b$ is the split at the saddle point) Proved here , main term of Laplace. Writing $g := h - \log P$, **$g$ is carried by convolution in the same way as the flat tree**. This is why the heat kernel ($g = 0$) is preserved; it is a restatement of the semigroup property. The product (the square at the same point) leaves one prefactor over, and the square root halves it. Substituting one step $f'^2 = (f * f) * f^2$, the power of the prefactor $\gamma_k := h_k/p_k$ read from $\psi^{2k}$ ($\log P = \sum p_k \psi^{2k}$, $p_2 = \frac{1}{180}$) follows

$$\gamma'_k = \frac{4^k + 1}{25^k}(\gamma_k - 1) + \frac{1 + 25^{-k}}{2}, \qquad \psi^4 \text{ gives } \gamma' = \frac{17\gamma + 296}{625} \quad (15)$$

a linear contraction map Proved here , main term of Laplace (algebra of the coefficients). The unique fixed point is

$$\gamma^*_k = \frac{25^k - 2\cdot 4^k - 1}{2(25^k - 4^k - 1)}, \qquad \gamma^*_2 = \frac{37}{76}, \quad \gamma^*_3 = \frac{1937}{3890}, \quad \gamma^*_k \to \frac{1}{2}, \qquad \left(\frac{q}{B}\right)^* = \frac{37}{13680\,B} \quad (16)$$

Proved here . The heat-kernel power 1 moves to $313/625$ in one step, which explains why the power of the prefactor appears to change from 1 to about $\frac{1}{2}$. The contraction rate is $\frac{17}{625}$ ($\psi^4$), not $\frac{1}{2}$, and $\frac{1}{2}$ is the value at the fixed point. Numerically: the extrapolation of $\gamma'_2, \gamma'_3$ from $B = 32$–$256$ agrees with (15) to $\le 10^{-7}$; in the iteration holding $B$, four starts ($\gamma = 0, \frac{1}{2}, 1, 2$) and the start of $q/B = -\frac{1}{12}$ gather to the same point with ratio $17/625$; and the one-step extrapolation including the $h^*$ of (16) gives $37/76$ and $1.4 \times 10^{-8}$ Reading (two routes: the character series and a one-dimensional multiple integral).

**The ratio at the attracting class.** Putting $(q/B)^* = O(1/B)$ into (10), on the attracting class

$$\kappa_{ab}/\kappa_a = 2 + c/B + O(B^{-2}), \qquad c = \frac{2/5}{3/5} = \frac{2}{3}$$

Proved here , at the one-loop level (numerical extrapolation 0.666667, two routes $1.8 \times 10^{-9}$ Reading ). $c$ does not depend on $h^*$ and is the same as for the heat kernel; in the main term of the quadratic

ratio the attracting class cannot be distinguished from the heat kernel. For $U(1)$ there is no inhomogeneous term, and the fixed point is Villain ($h = 0$), $c = 0$ Proved here .

09

## The form of the statement of the class-closure condition

Putting § 01– § 08 together Reading (limited to four plaquettes in a row, two loops, the diagonal definition and the main term): the coordinates of the parameters of the class-closure condition are the couplings by loop shape, and the commutator is their BCH image. One step contracts all the shape coordinates (Table 2), and the family is drawn to "one $B$ + the $O(1/B)$ shape $h^*$ fixed by the curvature of the group". The ratio 2 is not a coordinate that can be tuned with the parameters, but a one-loop quantity that becomes $2 + 2/(3B)$ at the attracting point. So what is consistent with this model is to state the class-closure condition not as "a class that keeps the ratio at 2" but as **"the one-parameter attracting class + a contracting shape carried by the shape remainder condition"**. If we assume that the shape deviation $r_n$ satisfies $r_{n+1} \le \lambda r_n + C/B_n$ ($B_n = B_0/5^n$) in one step, then $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, and it closes in the geometric-series pattern of 15-03 § 04 Proved here (assuming the one-step inequality). The Shape-Remainder Theorem for Four Plaquettes in a Row, next, gives the constants of this one-step inequality as numbers inside this model.

**Shape-Remainder Theorem for Four Plaquettes in a Row — the one-step constants as numbers (the projection model of four in a row).** Conditions (on the same line): $SU(2)$, four plaquettes in a row, the diagonal definition, the projection model that truncates at $\psi^8$ and re-inflates, coordinates taken as the analytic main part of $\log f'$ (the rest is at most $10^{-35}$), $B \ge 32$, a box of $O(1)$. If class functions are lifted to radial functions on $\mathbb{R}^3$ ($f = P \cdot \varphi$, $P = \psi/\sin\psi$; the known lift that sends convolution of adjoint-invariant functions to convolution on the Lie algebra), one step becomes a single Gaussian integral of three segments of stiffness $B$, $B$, $2B$ joined in series, and the curvature of the group remains only in "the $P$ left in the lift of the product $f^2$" and in "the final $P$ and the square root" Proved here (the lift is of a known type Cited ; the original source has not been checked). The next term of Laplace's method can be written in closed form by the second-order cumulants of the Gaussian; the $O(1/B)$ term of $\psi^4$ is $-\frac{8}{3125}h_2 + \frac{42}{125}h_3 - \frac{14}{140625}$, and on the attracting class $h^*$

$$\left(\gamma_2' - \frac{37}{76}\right)B \;\to\; -\frac{591986}{69290625} \approx -0.00854 \quad (17)$$

Proved here (numerical extrapolations $-0.0085438$ and $3 \times 10^{-7}$ Reading ). Writing $r := \max_{k=2,3,4} 4^k\,|h_k - h_k^*|$ for the distance in the coefficients of $\psi^4, \psi^6, \psi^8$, and with $\lambda = 17/625$, one step

$$r' \;\le\; \lambda\, r + \frac{C}{B}, \qquad C = 0.0108\ (r \le 0.05),\quad 0.0036\ (r \le 0.01),\quad B \to \infty \text{ sharp value } 0.00076 \quad (18)$$

satisfies Proved here , Reading (the line of the estimate is on paper; the constants are numbers from explicit integrals and series, and the monotonicity in $B$ was only checked on the grid $B = 32$–$10^6$), and for the actual step scales $B_0, B_1, \ldots$ ($B_{n+1} \le B_n/5$)

$$B_0, \ldots, B_{N-1} \ge 32,\ r_0 \le r_{\text{box}} \quad\Longrightarrow\quad r_n \le r_{\text{box}}, \qquad r_n \;\le\; \lambda^n r_0 + \frac{C}{(5-\lambda)B_n} \quad (n \le N) \quad (19)$$

Proved here (the assembly assuming (18) is in Lean: `uv_uniform` [$0 \le \lambda < 5$, $C \ge 0$, $B_n > 0$, $5B_{n+1} \le B_n$ and the one-step inequality], `stays_in_box`). The second term **is determined by the last scale alone, and does not depend on the number of steps** $N$: however many steps are run from the ultraviolet, the shape deviation at scale $B$ is at most $\lambda^N r_0 + C/((5-\lambda)B)$. In the shape coordinate $h/B$ the contraction is $17/125$ and the remainder is $1/B^2$. In numbers (one to three steps for $B_0 = 32$–$512$), starting points inside the box (the point of the class, a Gaussian, the heat kernel) stay below the upper bound of (18) at every step, and for $C/10$ the point of the class and the Gaussian break it at step 1 Reading (positive and negative controls).

**Outside the box.** A shape deviation $x = h_2/B$ of the $O(1)$ shape (for Wilson, $x = 1/24$) is outside the box. The negative side (fundamental + adjoint $x = -1/12$) lies within the upper bound of $\lambda$, but the positive side (Wilson) breaks the upper bound at step 1 ($\bar\lambda/\lambda = 1.018$ [$B_0 = 32$] $\to 1.001$ [512]); the cause is the one-loop quadratic term $c\,x^2$, $c = 11168/15625 \approx 0.715$ Proved here , Reading . For $U(1)$ the inhomogeneous term vanishes Reading .

**Why it closed in this model.** Four in a row is a system with only one block, and the explicit quadratic terms between blocks vanish by the zero-sum condition. It can be read that the Four-in-a-Row Theorem could become a complete example of the small-field contribution to the coupling because this row was empty Reading (15-08 § 06).

Table 2 Coordinates of the one-step map and their contraction ($SU(2)$, four in a row, the diagonal definition, main term of Laplace. On paper Proved here , measured values Reading )

| Coordinate | One-step factor in units of $O(1)$ | Factor in the shape coordinate ($\div B$, $B' = B/5$) | Measured value |
|---|---|---|---|
| $B$ (the one flowing parameter) | $1/5$ | — | $5B'/B = 0.998$ ($B = 256$) |
| $\psi^4$ ($q$, power $\gamma$ 2) | $17/625$ | $\lambda = 17/125 = 0.136$ | ratio of iterations 0.02720; contraction of $q/B$ 0.1223→0.1325 ($\beta$ = 128→512) |
| $\psi^6$ ($\gamma_3$) | $13/3125$ | $13/625$ | ratio of iterations 0.00414 |
| $\psi^8$ ($\gamma_4$) | $257/390625$ | $257/78125$ | — |
| General $\psi^{2k}$ | $(4^k+1)/25^k$ | $(4^k+1)/5^{2k-1}$ | — |

Table 3 Questions arising from this chapter (status: New; the question "one-step correction for four plaquettes in a row" has had its answer added)

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|

| Q. | Question | Size | What moves if settled | What does not move even if settled |
|---|---|---|---|---|
| 8 | Can the $O(1/B)$ correction of the one-step map of four in a row be derived on paper, giving the constant $C$ of the one-step inequality $r_{n+1} \le \lambda r_n + C/B_n$ of the shape remainder condition (turning the geometric series under an assumption into one without the assumption, inside this model)? **Answer (in this model):** the Four-in-a-Row Theorem gave $C = 0.0108$ ($B \ge 32$, $r \le 0.05$). The remaining assumptions are the projection (discarding $\psi^{10}$ and higher) and the size of the box; an estimate that includes shape deviations of the $O(1)$ type, such as Wilson's, is not yet done | Paper | The shape remainder condition of the attracting class | The class-closure condition on the four-dimensional lattice |
| 9 | Including loop shapes other than four in a row ($2\times 2$, L-shaped, mixed quartic terms), is the attracting class still determined as a single one ($B$ + a fixed shape)? Does the fixed point stay put if the diagonal definition is replaced by a fit to the six quartic coefficients? | Paper and small computations | The number of parameters of the class-closure condition | One step of a true blocking |

10

## What cannot be said

(a) The systems in this chapter are one plaquette up to four in a row of $SU(2)$, $\theta \le 0.3$, quadratic order in $F$ ( § 07 and § 08 go up to the quartic Taylor coefficients); these are not the numbers for one step including blocks embedded in the four-dimensional lattice, gauge re-fixing, or large fields.

(b) "Two coordinates are needed at the quadratic level" is a reading for four in a row with two loops fixed; we have not looked at the quartic order (the direction of the commutator) or at systems with three or more block variables.

(c) That the family of powers of the prefactor has no root is a statement of numbers on the measured grid and of an inequality at the one-loop level, not a statement about all actions. The fundamental + adjoint root is a number at $\beta \ge 12$, and the coefficient of $-0.84/\beta$ is an extrapolation.

(d) The maps of § 07 and § 08 are maps for the diagonal definition $f' = [(f * f) * f^2]^{1/2}$ and the main term ($B \to \infty$); mixed quartic terms do not enter, and they cannot be used in the strong-coupling regime of $B = O(1)$. The one-step inequality of the shape remainder condition is in general an assumption. The Four-in-a-Row Theorem is a statement about four in a row, the diagonal definition, the projection model truncated at $\psi^8$, coordinates given by the analytic main part, $B \ge 32$, and a box of $O(1)$; some of the constants are numerical values of explicit integrals, and the exponentially small parts are rough estimates. It says nothing about one step of the four-dimensional lattice, block averages, large fields or the strong-coupling regime.

(e) We say neither "there is no action that closes with one parameter" nor "the class-closure condition closes with finitely many parameters". What we say is a main-term statement: in this model all the shape coordinates contract, and the attracting point is determined as a single one.

(f) This chapter does not show ultraviolet control.

11

## The five grades table and the dependency graph

Table 4 The rows of this chapter

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [01] | One plaquette: under the constraint $\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)}$ ((1)); the gauge directions do not depend on the background | Proved here | § 01 | The constraint $x_3 = 0$ is a substitute for the background-field constraint |
| [02] | $\beta_{\mathrm{eff}} = \beta - d/2$; for cos of $U(1)$ too, a $\eta$ of the same type | Reading | § 01 | One plaquette, $\beta = 4$–$64$, $\theta \le 0.3$ |
| [03] | Two plaquettes (2)(3), three plaquettes (4): the commutator is the $1/12$ of the coupling of the composite loop | Proved here , Reading | § 02, § 03 | BCH up to third order, $\beta = 4$–$32$ |
| [04] | Four plaquettes: the ratio $\kappa_{ab}/\kappa_a = 2(\beta - \frac{1}{2})/(\beta - 1)$ moves ((5)–(7)) | Proved here , Reading | § 04 | Main term at the one-loop level; numbers by two routes |
| [05] | $2 + 2t/3$ for the heat kernel; the criterion (9) | Proved here , Reading | § 05 | Class functions, boundary terms 0 |
| [06] | General one-loop formula (10); one-loop root of fundamental + adjoint $-\frac{1}{4}$ ((11)); $D > 8/25$ at $\gamma = c\beta$ | Proved here | § 06 | One-loop level, concentrated actions |
| [07] | No root in the prefactor family; fundamental + adjoint root $\rho^* = -\frac{1}{4} - 0.84/\beta$; one parameter for one step only | Reading | § 06 | $\beta \in \{4, \dots, 512\}$, measured grid, two routes |
| [08] | Tree map (12), image (13): deviation $9/89$ | Proved here | § 07 | Tree (main term of the saddle point), the diagonal definition |
| [09] | Numbers for the deviation of the curve 0.160, 0.127, $0.113 \to 9/89$ | Reading | § 07 | Starts from $\beta = 128$–$512$ |
| [10] | (14)(15)(16): the contraction map of the power of the prefactor, fixed point $37/76$, $c = 2/3$ | Proved here | § 08 | Main term of Laplace, four in a row, the diagonal definition |
| [11] | Numerical extrapolation of (15)(16), convergence of the iteration | Reading | § 08 | $B = 32$–$256$, two routes |
| [12] | The form of stating the class-closure condition as "the one-parameter attracting class + a contracting shape"; the questions "one-step correction for four plaquettes in a row" and "uniqueness of the class for other loop shapes" | Assumption | § 09 | The form of the statement; the one-step inequality is an assumption |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [13] | Four-in-a-Row Theorem: by the lift, one step is a Gaussian integral of three segments in series, (17), the one-step inequality (18), and (19), independent of the number of steps | Proved here , Reading (numerical values of the constants) | § 09 | $SU(2)$, four in a row, the diagonal definition, the projection at $\psi^8$, $B \geq 32$, a box of $O(1)$ |
| [14] | Outside the box: Wilson's positive shape deviation breaks $\lambda$ at step 1; the cause is $c\,x^2$ ($c = 11168/15625$) | Proved here , Reading | § 09 | The same model, $B_0 = 32$–$512$ |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV ONE    one plaquette: (1)                              constraint x₃ = 0
[02] RDG BEFF   β_eff = β − d/2 ⟸ [01]
[03] PRV BCH    two, three plaquettes: commutator is the image of the composite loop ⟸ [0
[04] PRV FOUR   four plaquettes: the ratio moves with β ⟸ [03]   (one-loop level)
[05] PRV CRIT   heat kernel 2 + 2t/3, criterion (9) ⟸ [04]
[06] PRV D1     general one-loop formula (10)(11) ⟸ [05]   (concentrated actions)
[07] RDG ROOT   no root in prefactor family; fund. + adj. root ⟸ [06]
[08] PRV TREE   tree map (12), deviation 9/89 (13) ⟸ [06]   (diagonal definition)
[09] RDG GAP    numbers for the deviation ⟸ [07][08]
[10] PRV FIX    fixed point of the prefactor power 37/76, c = 2/3 ⟸ [06][08]   (main term
[11] RDG FIXN   numbers for the fixed point ⟸ [10]
[12] ASM CLS    statement of the attracting class, questions "one-step correction for four
[13] PRV UNIF   Four-in-a-Row Theorem: one-step constants and a bound independent of the nu
[14] PRV OUT    outside the box: c x² breaks λ ⟸ [13]
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Every number to the right of "⟸" is smaller than the one on the left, and there are no cycles. The only assumption row is [12], the form of the statement and the questions.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Proved here The weight formulas (1)–(8) from one to four plaquettes; the criterion (9); the general one-loop formula (10)(11); the tree map and $9/89$; the linear contraction map of the power of the prefactor and its fixed point $37/76$, $c = 2/3$; the Four-in-a-Row Theorem (the projection model, $B \geq 32$, a box of $O(1)$; constants numerical) | The same numbers for one step of the four-dimensional lattice. Quartic coefficients; systems with three or more block variables |
| Reading The numbers showing the ratio moves with $\beta$; no root in the prefactor family; the fundamental + adjoint root; the numbers for the deviation of the curve; convergence of the iteration to the fixed point | "No root" for all actions. A paper formula for the correction of $O(1/B)$ |
| Assumption The form of stating the class-closure condition as the one-parameter attracting class + a contracting shape | That class closing in four dimensions |

## Sources and reproduction

| Item | Type | Source, tools |
|---|---|---|
| The form of the heat kernel of $S^3$ ($e^{-\psi^2/2t}\cdot\psi/\sin\psi$ and its mirror images), the character expansion, the Laplace method, composition of Van Vleck prefactors | Cited | Standard facts of harmonic analysis and asymptotic expansions on Lie groups (bibliography as in the table of sources of 15-04) |
| Equations (1)–(16), the criterion, the general one-loop formula, the tree map, the map of the power of the prefactor | Proved here | The proofs and algebra in the main text of this chapter. The algebra of the coefficients and the trigonometric identities are in the Lean bundle, re-verified (15-09 Table 3) |
| Four-in-a-Row Theorem, (17)–(19), $c = 11168/15625$ | Proved here, Reading (numerical values of the constants) | The main text of this chapter. The lift is of a known type (the lift of adjoint-invariant functions; the original source has not been checked). The numbers are the record of this book's study (two implementations, by series and by rational functions of $B$; Gauss–Legendre quadrature; positive and negative controls). The assembly is in the Lean bundle (re-verified) |
| The numbers in Tables 1 and 2, and in § 06– § 08 | Reading | The numerical records of this book's study (two routes, quadrature and closed formulas; quaternion Monte Carlo; two routes, the character series and a one-dimensional multiple integral; the Wilson and Villain controls of $U(1)$; positive control by the heat kernel) |

**Words used in this chapter** — couplings by loop shape, the composite loop, the criterion, the curvature of the group, the Van Vleck prefactor, one parameter for one step only, the diagonal definition, the tree map, the power of the prefactor, the attracting class, the class-closure condition, the lift, the Four-in-a-Row Theorem, outside the box. Their meanings are in the A-02 glossary.

Next chapter: 15-07 Background-dependent fluctuations — the $\eta$ of one plaquette in § 01, as a problem of absorption at each step and of induction over many steps.

# Background-dependent fluctuations η — absorption step by step

This chapter is this book's own investigation. The established knowledge it builds on is in 15-02 and 15-01.

This part is a record of this book's study attempting ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is set out in 15-00.

**What this chapter shows** — In one step of renormalisation, the fine field is split into a "background" and a "fluctuation", and the fluctuation is integrated out. For a flat field (a scalar field) the law of the fluctuation does not depend on the background, and of the four properties of the weight, the property "closing in one step under the expectation of the fluctuations" (closure under the fluctuation expectation; (iii) of BBS) can be used as it is. For a group-valued field the law of the fluctuation depends on the background, and the size of that dependence, $\eta_k$, becomes the difficulty. In this chapter we set down on paper the group version (the Background–Fluctuation Decomposition Lemma) of submultiplicativity in sums of fields ((ii) of BBS), set down the group versions of the exponential moment of fluctuations and of background regularity in the form of statements, and write the one-step chain; and we state, conditionally, that with a naive estimate of $\eta_k$ the sum $\sum_k \eta_k$ diverges, that when the law of the fluctuation does not depend on the background the chain goes through as it stands with $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations), a proposal to absorb $\eta$ into the parameters as a shift of local couplings, and that the multi-step induction has not yet been written. Alongside, we state the version of the Lemma on One Step with Background-Independent Fluctuations for the one-link Wilson law (the One-Link Heat-Kernel Comparison Lemma), the order of the remainder that cannot be absorbed as measured in the smallest systems, the four-way splitting of the derivative in the background, the coefficient of the group-geometry part for $SU(2)$ (the cubic BCH term), and the divergence of its budget. In the revision we added the cost of re-taking the centre for the one-link Wilson law (the recentring

condition), the remainder and the budget when the class is given the background as a parameter, the update of the background in a one-step with a single loop as the coarse variable, and the coordinates of the multi-loop class in a three-dimensional box.

**Prerequisite chapters** — 15-01 § 02 (group-valued fields and the four properties of the weight), 15-02 § 04 (the chain of BBS Appendix A.6), 15-03 § 02 (the statement of the decomposition), 15-06 (the type of $\eta$ for a single plaquette), 7-05 § 05 (elements of a group cannot be added).

**Said first** — This chapter does not show ultraviolet control. What we set down on paper is the Background–Fluctuation Decomposition Lemma, the one-step chain (assuming the group versions of the exponential moment of fluctuations and of background regularity), the Lemma on One Step with Background-Independent Fluctuations (only the single step in which the law of the fluctuation does not depend on the background), and a quaternion identity; none of them is a statement about a genuine step of renormalisation (with the constraint of block averaging, the propagator in a background field, the Wilson action, and large fields included). The absorption of $\eta$ is a proposal, and the multi-step induction has not been written.

The order of this chapter

1. The measure of the fluctuations depends on the background — where $\eta k$ sits
2. The Background–Fluctuation Decomposition Lemma and the Splitting of the Weight — the group version of submultiplicativity in sums of fields (equations (1)(2))
3. The group versions of the exponential moment of fluctuations and of background regularity, and the interchange — statements and the one-step chain (equation (3))
4. A naive estimate of $\eta$ and its divergence
5. The Lemma on One Step with Background-Independent Fluctuations — with background-independent fluctuations the chain goes through with $\eta = 0$ (Table 1); the one-link Wilson law (the One-Link Heat-Kernel Comparison Lemma); the cost of re-taking the centre
6. $\eta$ = the proposal of shifting local couplings — the fact about the heat-bath method, the quaternion identity, screening by the order of $\theta$, the remainder measured in the smallest systems, the four-way splitting, giving the class the background as a parameter
7. The cubic BCH term, the class $(B, q, a)$, the budget, the one-face step, the multi-loop class, and the multi-step induction — question "splitting the density of the joint law", question "minimal multistep statement", ultraviolet questions

17–19, question "second step of the multi-loop class", question "class with the coupling as a parameter" (Table 2)

8. What cannot be said
9. The five grades table and the dependency graph (Table 3)

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00

## What this chapter shows

- For fields that take values in a group the measure of the fluctuations depends on the background, and $\eta_k$ accordingly puts a factor $1+\eta_k$ into the one-step estimate. Unless $\sum_k \eta_k < \infty$, the constants break down with the number of steps, exactly as in the arithmetic of 15-01 § 01 ( § 01).
- The group version of the submultiplicativity in sums of fields of the weight (it splits into a product over a sum of fields) holds on paper in the form background $\times$ fluctuation (the Background–Fluctuation Decomposition Lemma, § 02). The group versions of the exponential moment of fluctuations and of background regularity can be set down as statements, and assuming them the one-step chain closes in the same form as for flat fields ( § 03).
- Estimating $\eta_k$ naively from the curvature of the background, the sum diverges as a power of the number of steps or a power of its logarithm, and cannot be carried as an error ( § 04).
- When the law of the fluctuation does not depend on the background (a product of heat kernels link by link is an example), the chain goes through as it stands with $\eta = 0$ (the Lemma on One Step with Background-Independent Fluctuations, § 05). The fluctuation of a genuine step is not of this type.
- For the Wilson action of $SU(2)$, the conditional law of one link is a member of a one-parameter family that does not depend on the background, and the background enters only through the local coupling and the direction of the centre. Using this we set down a proposal to absorb $\eta$ as the shift of local couplings ( § 06, proposal). The multi-step induction has not been written ( § 07).
- For the one-link Wilson law, the law of the fluctuation is bounded inside the angle cutoff by a constant multiple of the heat kernel, and the chain of the Lemma on One Step with Background-Independent Fluctuations goes through conditionally (the One-Link Heat-Kernel Comparison Lemma, § 05).
- Measured in the smallest systems, the lowest orders of the non-absorbable remainder are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$. The derivative in the background splits into four:

coupling, rotation, shape and group geometry. The shape part enters a geometric series, but the group-geometry part (the cubic BCH term, coefficient $1/9$) of $SU(2)$ rides on the background itself and does not contract, and its budget diverges however the threshold is chosen. The improvement lies on the side of the class — letting the parameters carry the field strength of the background ( § 06, § 07).

- If the class is given the logarithm of the background holonomy $a$ as a parameter, the remainder in the smallest systems drops to $\theta^0 \beta^{-3/2}$ and the budget becomes uniform in the number of steps (conditionally, § 06). The divergence has not disappeared; it has moved to the question of whether the class closes in one step.
- In one step whose coarse variable is a single loop, the update of $a$ closes to all orders by the ordered product in the group. In the multi-loop step of a three-dimensional box, the class closes up to second order with per-face classes and neighbouring pairs; at third order the corner triple product is needed, and at fourth order multi-face couplings ( § 07). The corner triple product does not contract at the second step; it is a coordinate that approaches the fixed point of the flow from below. The four-dimensional class consists of one parameter and the ratio coordinates whose values are fixed by the flow (2 at second order, 3 at third, 65 at fourth), and the coordinate of "the skew neighbour", which shares no vertex, is not 0 whatever the rule ( § 07; within the measured range of tree level and one loop).
- Re-taking the centre for the one-link Wilson law produces a loss of about a factor 4 in the Taylor radius and a motion of the coupling $c g^2 p^2$. The latter is a term to be carried over to the side of the parameters of the class ( § 05).

01

## The measure of the fluctuations depends on the background — where $\eta_k$ sits

In one step at stage $k$ we split the edge variable into background and fluctuation as $U_l = W_l\, e^{A_l}$ ($W_l \in G$, and $A_l$ an element of the Lie algebra $\mathfrak{su}(N)$), hold the background $W$ fixed and integrate out the fluctuation $A$. We write the law of this fluctuation as $\mu_{k,W}$. The covariance of its Gaussian part is the propagator in a background field $g_k^2 C_W$ (the title of the foundational papers in 7-07 Table 1).

For flat fields (a scalar field $\phi = \phi_{\mathrm{bg}} + \zeta$) the law of the fluctuation $\zeta$ is a Gaussian law that does not depend on the background, and the closure of the weight under the fluctuation expectation (closing in one step under the expectation of the fluctuations) is used as the expectation of a function of the fluctuation alone Reading (from reading the chain of 15-02 § 04). On the group this is not so. When the background changes, $C_W$ changes, and the law of the fluctuation itself changes. We write the size of this dependence as $\eta_k$ and put it into the one-step estimate as a factor $1 + \eta_k$; this is how this chapter sets things up (equation (3)). From the arithmetic of 15-01 § 01, for the

product of factors $\prod_k(1+\eta_k)$ to be bounded uniformly in the number of steps $K$ one needs $\sum_k \eta_k < \infty$, and if $\eta_k$ is of size $g_k^2$ it is not enough Proved here (arithmetic).

02

## The Background–Fluctuation Decomposition Lemma — the group version of submultiplicativity in sums of fields

Let $l_1,\dots,l_4$ be the edges on the boundary of a plaquette $p$ and $s_i \in \{\pm 1\}$ their orientation signs, set $U_{\partial p} = \prod_i U_{l_i}^{s_i}$, and define $W_{\partial p}$ in the same way. The norm is the operator norm.

**Background–Fluctuation Decomposition Lemma.** For $G = SU(N)$,

$$\|U_{\partial p}-1\| \;\le\; \|W_{\partial p}-1\| + \sum_{i=1}^{4}\|A_{l_i}\|, \qquad \|U_{\partial p}-1\|^2 \;\le\; 2\|W_{\partial p}-1\|^2 + 8\sum_{i=1}^{4}\|A_{l_i}\|^2 \tag{1}$$

Proved here Proof. (1) Split each factor into a $W$ part and a fluctuation part. If $s=+1$, $We^{A} = W\cdot Y$ ($Y = e^{A}$); if $s=-1$, $(We^{A})^{-1} = W^{-1}\cdot Y$ ($Y = We^{-A}W^{-1}$). In both cases $Y$ is a conjugate of $e^{\pm A}$. (2) In the product $W_1'Y_1W_2'Y_2W_3'Y_3W_4'Y_4$ ($W_i' = W_{l_i}^{s_i}$), move each $Y_i$ to the right of the $W'$ to its right by $YW' = W'(W'^{-1}YW')$; then $U_{\partial p} = W_{\partial p}\,Y_1''Y_2''Y_3''Y_4''$, and each $Y_i''$ is a conjugate of $e^{\pm A_{l_i}}$. (3) For unitary matrices, repeating $\|XY-1\| \le \|X-1\| + \|Y-1\|$ and $\|hYh^{-1}-1\| = \|Y-1\|$ (equation (8) of 7-05) gives $\|U_{\partial p}-1\| \le \|W_{\partial p}-1\| + \sum_i \|e^{\pm A_{l_i}}-1\|$. (4) $A$ is anti-Hermitian and so unitarily diagonalisable, and for an eigenvalue $e^{i\vartheta}$, $|e^{i\vartheta}-1| = 2|\sin(\vartheta/2)| \le |\vartheta| \le \|A\|$. Hence $\|e^{\pm A}-1\| \le \|A\|$. (5) The squares follow from $(x+y)^2 \le 2x^2+2y^2$ and $(\sum_{i\le 4} y_i)^2 \le 4\sum_i y_i^2$. ∎

**Splitting of the Weight (a corollary of the Background–Fluctuation Decomposition Lemma).** Setting $G_c(X,U) := \exp\bigl(c\sum_{p\subset X}\|U_{\partial p}-1\|^2\bigr)$ ($c>0$), since on a four-dimensional lattice an edge lies on at most 6 plaquettes,

$$G_c(X, We^{A}) \;\le\; G_c(X,W)^2\cdot\exp\Bigl(48c\sum_{l\in X}\|A_l\|^2\Bigr) \tag{2}$$

Proved here (sum the second of (1) over $p\subset X$; $8\times 6 = 48$). This is the version of submultiplicativity in sums of fields in which the sum of fields is replaced by "background × fluctuation", and on the fluctuation side the weight is of Gaussian type. It may be a known elementary fact (an estimate of the same form may appear in the large-field papers of the series, but we have not checked the text).

**The form that writes the fluctuation with $Y$.** Steps (1)–(3) go through with $e^A$ replaced by any unitary $Y$. From $\|Y^{-1}-1\| = \|Y-1\|$ and conjugation invariance, $\|(WY)_{\partial p}-1\| \le \|W_{\partial p}-1\| + \sum_{i\le 4}\|Y_{l_i}-1\|$, and with $q$ the maximal number of plaquettes on which one link lies (6 in four dimensions), $G_c(X,WY) \le G_c(X,W)^2 \exp\big(8qc\sum_{l\in\bar X}\|Y_l-1\|^2\big)$ Proved here ($\bar X$ are the links on the boundary of the plaquettes of $X$). In this form no small-field condition on the fluctuation is needed (since $\|e^A-1\|\le\|A\|$ of step (4) is not used). The Lemma on One Step with Background-Independent Fluctuations in § 05 uses this form.

**What the Background–Fluctuation Decomposition Lemma does not say.** (2) speaks only of submultiplicativity in sums of fields and does not touch on closing in one step under the expectation of the fluctuations (closure under the fluctuation expectation).

03

## The group versions of the exponential moment of fluctuations and of background regularity, and the interchange — statements and the one-step chain

Let the weight at step $k$ be $G_k := G_{c/h_k^2}$ ($h_k$ is the field scale at step $k$, for example the small-field threshold $g_k p(g_k)$; $p(g)$ is the factor of the threshold in equation (7) of 7-05), let $\mathcal{W}_k(X)$ be the set of small-field (regular) backgrounds on $X$, and let $B(X)$ be the set of blocks in $X$. We set down the following three as statements Assumption (statements only; the constant $c, t_0$ and the constants in the definition of $\mathcal{W}_k$ depend neither on the step nor on the lattice spacing).

- **Group version of the exponential moment of fluctuations**: for all $W\in\mathcal{W}_k(X)$ and $0<t\le t_0$, $\mathbb{E}_{\mu_k,W}\exp\big(\frac{48tc}{h_k^2}\sum_{l\in X}\|A_l\|^2\big) \le 2^{|B(X)|}$. The expectation is with respect to the true measure of the fluctuations, not only its Gaussian part.
- **Background-regularity condition**: for $W$ in the image of the background map, $G_k(X,W)^{2t} \le G_{k+1}(X,W)$ ($0<t\le t_0$). Its content is the regularity of the background (the curvature of the fine plaquettes being bounded by the curvature of the coarse plaquettes).
- **Group version of exchanging the expectation and the seminorm**: let $\theta_W F(W) := F(We^A)$, and let $T_W(h)$ be the Taylor seminorm built from derivatives in the Lie-algebra directions of the background. For some $\eta_k\ge 0$, $\|\mathbb{E}_{\mu_k,W}\theta_W F\|_{T_W(h_{k+1})} \le (1+\eta_k)\,\mathbb{E}_{\mu_k,W}\|F\|_{T_{We^A}(h_k)}$, and $\sum_k \eta_k<\infty$. This is a version of the original inequality exchanging the expectation and the seminorm ((A.6.5) and others of BBS) that allows a factor $1+\eta_k$ for the dependence of the measure on the background.

Assuming these three and the Splitting of the Weight (2) (with $c$ replaced by $c/h_k^2$), the same chain as in the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) (interchange → definition of the norm → (2) → group version of the exponential moment of fluctuations → background-regularity condition) gives, on $W \in \mathcal{W}_k(X)$,

$$\left\| \mathbb{E}_{\mu_{k,W}} \theta_W K(X) \right\|_{G_{k+1}} \;\le\; (1+\eta_k)\, 2^{|B(X)|}\, \|K(X)\|_{G_k} \quad (3)$$

Proved here (only the chain, assuming the three). Outside $\mathcal{W}_k$, the large-field factor of the large-field rarity condition takes over (15-00 § 01). The group version of the closure under the fluctuation expectation itself has a form that closes with three things: an estimate of the propagator $C_W$ uniform over regular backgrounds, a uniform estimate of the deviation of $\mu_{k,W}$ from Gaussian, and $c g_k^2/h_k^2$ being small Reading (from the formula that writes Gaussian exponential moments as determinants).

**For a product of one-link heat kernels, the group version of the closure under the fluctuation expectation holds for free.** When $Y_l = W_l^{-1} U_l$ follows the heat kernel $p_\tau$ independently on each edge, the invariance of Haar measure (invariant on both left and right, since the group is compact) makes the law of $Y$ independent of $W$, and $\mathbb{E}\exp\big(s\sum_l \|Y_l - 1\|^2\big) = \big(\mathbb{E}\exp(s\|Y-1\|^2)\big)^{|X|}$ ($|X|$ is the number of edges of $X$) does not depend on the background Proved here. This is at most $2^{|B(X)|}$ when $s\tau$ is small Reading (from the Gaussian-type upper bound on the heat kernel; the numbers are in Table 1 of § 05). The telescoping sum of 15-04 § 01 can be read as another face of these "fluctuations independent of the background" Reading.

## 04 A naive estimate of $\eta$ and its divergence

**Estimating from the curvature.** A naive estimate of the part by which the measure depends on the background, through the curvature $\|W_{\partial p} - 1\| \lesssim g_k p(g_k)$, gives $\eta_k \sim g_k p(g_k)$, and for the sequence of one-loop couplings $g_{K-j}^2 \le 1/(bj)$ (15-01 § 01) $\sum_k \eta_k$ diverges at the order of $K^{1/2}$ (up to logarithmic factors) Proved here (arithmetic, under the naive estimate $\eta_k \sim g_k p(g_k)$).

**Measuring with a single plaquette.** In the smallest model of a single plaquette (15-06 § 01), if we impose a constraint that removes the directions of the generators of the background, the background dependence enters exactly through a single replacement of the coupling $\beta \mapsto \beta\cos(\theta/2)$ ($\mu_W^\beta = \mu_1^{\beta\cos(\theta/2)}$) Proved here. $\eta$ measured by the ratio of covariances is $\theta^2/8$; below the threshold $\theta \le \varepsilon_k = g_k p(g_k)$ it is $\eta_k \lesssim g_k^2 p(g_k)^2$, and under the one-loop flow $\sum_{k\le K}\eta_k$ diverges as a power of $\ln K$ Proved here (the arithmetic of the naive estimate). Without the constraint (not taking the background at

a critical point), the ratio of covariances is $\beta\theta^2/12$ and grows with $\beta$ Reading . Under either estimate, $\eta$ **is not of a size that can be carried as an error, and absorption into the parameters is needed**. The single-plaquette example is the smallest example in which that absorption happens exactly.

**Which $\eta$?** Measuring $\eta$ by a single number changes its type (the ratio of covariances is $\theta^2$, the Kullback–Leibler quantity is $\theta^4$, and the score, i.e. the first-order rate of change of the expectation when the background is moved, is $\theta$) Reading . The group version of the exponential moment of fluctuations uses an even quantity and the interchange inequality (the group version of exchanging the expectation and the seminorm) a first-order one, so the statements must specify "which $\eta$". For a scalar field the measure does not depend on the background, so $\eta_k = 0$ and the problem itself does not arise.

05

## The Lemma on One Step with Background-Independent Fluctuations — the chain goes through with $\eta = 0$

There is one case in which the chain of § 03 goes through without assumptions: when the law of the fluctuation does not depend on the background.

**Setting.** $G = SU(N)$, operator norm. A finite set of links $\Lambda$, a partition into blocks, $X$ a union of blocks, $B(X)$ the blocks in it, $\bar{X}$ the links on the boundary of the plaquettes of $X$. The weight $G_h(X,U) := \exp\big((c/h^2)\sum_{p\in X}\|U_{\partial p}-1\|^2\big)$. The Taylor seminorm is $\|F\|_{T_U}(h) := \sum_{n\geq 0}\frac{h^n}{n!}\|D^nF(U)\|$, where $\|D^nF(U)\|$ is the supremum over $|X_j|\leq 1$ of the partial derivative at $t=0$ of right multiplication $F(Ue^{t1X1}\dots e^{tnXn})$. The weighted norm is $\|K(X)\|_{G_h} := \sup_U \|K(X)\|_{T_U}(h)/G_h(X,U)$. The fluctuation is a probability measure $\nu$ on $G^\Lambda$, and one step is $\theta_\nu K(W) := \int K(WY)\,d\nu(Y)$. $M_\nu(s,X) := \mathbb{E}_\nu \exp\big(s\sum_{l\in\bar{X}}\|Yl-1\|^2\big)$.

**Lemma on One Step with Background-Independent Fluctuations.** Assume the following three. Background-independence condition for fluctuations: $\nu$ does not depend on the background $W$. Scale-contraction condition: $h_+^2 \leq h^2/2$. Exponential-moment condition for fluctuations: $M_\nu(8qc/h^2, X) \leq 2^{|B(X)|}$. Then for every background $W \in G^\Lambda$ and smooth $K(X)$,

$$\|\theta_\nu K(X)\|_{G_{h_+}} \;\leq\; 2^{|B(X)|}\,\|K(X)\|_{G_h} \quad (4)$$

Moreover the following hold Proved here (conditions: the three conditions of background independence of the fluctuations, scale contraction and the exponential moment of fluctuations, and one step only).

- (a) The interchange is $\eta = 0$: for all smooth $F$, $\|\theta_\nu F\|_{T_W}(h) \le \mathbb{E}_\nu \|F\|_{T_{WY}}(h)$.
- (b) Locality: if $\nu = \otimes_l \nu_l$ (independent link by link), then $\theta_\nu K(X)$ depends only on the restriction of $W$ to $\bar{X}$, and if $\bar{X} \cap \bar{X}' = \emptyset$ it splits into a product. The support does not spread in one step.
- (c) For a product of heat kernels link by link $\nu = p_\tau^{\otimes \Lambda}$, $M_\nu(s, X) = m(s,\tau)^{|\bar{X}|}$ and $m(s,\tau) := \mathbb{E}_{p_\tau} e^{s\|Y-1\|^2}$, independent of the background. An elementary upper bound is $m(s,\tau) \le 1 + (e^{4s} - 1)\,\mathbb{E}_{p_\tau}\|Y-1\|^2/4$. For $SU(2)$ (with the convention for the character coefficients $(n+1)e^{-n(n+2)/(2\beta)}$), $\mathbb{E}\|Y-1\|^2 = 2(1 - e^{-3/(2\beta)}) \le 3/\beta$.
- (d) Small fields: $\nu'$, obtained by conditioning $\nu = \otimes \nu_l$ on $\{\|Y_l - 1\| \le h' \ \forall l\}$, also satisfies the background-independence condition for fluctuations, and $M_{\nu'} \le M_\nu$. If the background is taken in $\mathcal{U}_\varepsilon := \{\max_{p \in X} \|W_{\partial p} - 1\| \le \varepsilon\}$, the supremum of the norm on the right-hand side need only be taken over $\mathcal{U}_{\varepsilon + 4h'}$.

Proved here Proof. **Step 1 (submultiplicativity in sums of fields)**: from "the form that writes the fluctuation with $Y$" of § 02, $G_h(X, WY) \le G_h(X,W)^2 \exp\big((8qc/h^2)\sum_{l \in \bar{X}} \|Y_l - 1\|^2\big)$. **Step 2 ((a))**: inserting $Y^{-1} e^{tX} Y = e^{t\,\mathrm{Ad}_{Y^{-1}} X}$ between the factors gives $F(We^{t_1 X_1} \dots e^{t_n X_n} Y) = F(WYe^{t_1 X'_1} \dots e^{t_n X'_n})$, $X'_j := \mathrm{Ad}_{Y^{-1}} X_j$, $|X'_j| = |X_j|$. Right multiplication only rotates the direction of differentiation by the adjoint, and the adjoint preserves the operator norm. $F$ is smooth on a product of compact groups, so we can differentiate under the integral (dominated convergence), and **since $\nu$ does not depend on $W$, the derivative falls only on the integrand**. Hence the derivative of order $n$ is at most $\int \|D^n F(WY)\| d\nu(Y)$; multiply by $h^n/n!$ and sum. **Step 3 (the chain)**: in the order (a) → definition of the norm → Step 1 → exponential-moment condition for fluctuations → scale-contraction condition, $\|\theta_\nu K(X)\|_{T_W}(h) \le \|K(X)\|_{G_h} G_h(X, W)^2\, 2^{|B(X)|} \le \|K(X)\|_{G_h} G_{h_+}(X,W)\, 2^{|B(X)|}$. The last follows from the scale-contraction condition: $G_h^2 \le G_{h_+}$. (b) follows from Fubini and independence; (c) from independence, the left and right invariance of Haar measure, the convexity of $e^{sz}$ on $z = \|Y-1\|^2 \in [0,4]$ (the chord inequality), $\|Y-1\|^2 = 2 - \mathrm{tr}\, Y$ and $\mathbb{E}\,\mathrm{tr}\, Y = 2e^{-3/(2\beta)}$. (d) follows from the fact that conditioning a function increasing in each coordinate on a lower set does not increase its expectation (the coordinates being independent), and from the first inequality of Step 1. ∎

Table 1 The numbers for the exponential-moment condition for fluctuations: the largest $s/\beta$ (in steps of $5 \times 10^{-4}$) with $m(s,\beta)^{96} \le 2$, for $SU(2)$, heat kernels edge by edge, $n_b = 96$ (a rough count of the links in the closure of a block in four dimensions with factor 2) Reading

| $\beta$ | 4 | 8 | 16 | 32 | 64 | 128 |
|---|---|---|---|---|---|---|
| Largest $s/\beta$ | 0.0025 | 0.0025 | 0.0025 | 0.0020 | 0.0020 | 0.0020 |

Table 1 is a numerical record of this book's study (trapezoidal quadrature in the angular direction and a sum over characters; as a positive control it agrees with the closed form for $\mathbb{E}\|Y-1\|^2$ to 9 digits). It agrees with $\ln 2/(3n_b) \approx 0.0024$ from the Gaussian estimate $(1-2s/\beta)^{-3/2}$. Reading $s = 8qc/h^2$, $\beta = 4/g^2$, the condition is $c\,g^2/h^2 \lesssim 1.7\times 10^{-4}$ ($q=6$, $n_b = 96$) Reading . If $h = g\,p(g)$ then $c/p(g)^2 \lesssim 1.7\times 10^{-4}$, and since $p(g)$ grows as a power of the logarithm, it is satisfied at fine steps.

**What is not in the Lemma on One Step with Background-Independent Fluctuations.** This is the precise edge of "the part that can be closed" Reading .

1. **All of the requirement that the law of the fluctuation not depend on the background (the background-independence condition for fluctuations).** The $\mu_{k,W}$ of a genuine step depends on $W$ through three things: the non-quadratic nature of the action (the Wilson action: $\mu_W^{\beta} = \mu_1^{\beta\cos(\theta/2)}$ for a single plaquette), the constraint of block averaging (the constraint surface depends on $W$), and the propagator in a background field $C_W$ (fluctuations of neighbouring links coupled by plaquette terms that depend on $W$). The place where "the derivative falls only on the integrand" of Step 2 breaks down is $\eta$ itself.
2. **Relation to the heat-kernel action.** The fluctuation of the Lemma on One Step with Background-Independent Fluctuations is "a convolution of heat kernels link by link", and the heat-kernel action plaquette by plaquette takes the form of a product over edges only when the plaquette variables become independent coordinates (two dimensions, axial gauge). In three or more dimensions the cube constraints tie plaquettes together (its manifestation is the source of 15-05).
3. **One step only.** Stacking the scale-contraction condition over $k$ steps, the weight collapses as $h_k \le 2^{-k/2}h_0$. Many steps need the blocking factor, the background-regularity condition, and a contraction of the many-body norm that cancels $2^{|B(X)|}$ (§ 07).
4. **The rarity of large fields does not come out.** Since $G_h(X,U) \le e^{4c|X|/h^2}$, this norm does not produce the $e^{-c\,p(g)^2|\Omega^c|}$ of the large-field rarity condition. (d) says only that "a small-field background spreads in one step only to $\varepsilon \to \varepsilon + 4h'$".
5. **The Wilson action is not treated here** (the $\eta$ of non-quadratic nature). Restricted to one link, it can be treated by the One-Link Heat-Kernel Comparison Lemma below.
6. **Correspondence with the small-field analyticity condition.** Of "the analyticity and decay of localised corrections" in the small-field analyticity condition, what the Lemma on One Step with Background-Independent Fluctuations contains is that the support does not spread in one step ((b)) and that the Taylor norm of radius $h$ is preserved up to a factor $2^{|B(X)|}$ ((4)); it does not contain the uniformity of the decay $e^{-\kappa d(X)}$ or constants uniform in the step.

**Towards the one-link Wilson law — the One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma.** If only one link is made to fluctuate, and the background is taken at its conditional maximiser (the direction of the sum of staples $\hat{q}(W)$, § 06), the law of the fluctuation $Y$ is $\mu^b$ ($b = \beta|Q(W)|$), and depends on the background only through $b$. Inside the angle cutoff $\theta_0 \in (0, \pi)$, $\mu^b$ is bounded from above by $K$ times the heat kernel $p_{1/b'}$ ($b' = b(1 - \theta_0^2/12)$), and

$$\mu^b(\theta) \;\le\; K(b, \theta_0)\, p_{1/b'}(\theta) \quad (\theta \le \theta_0), \qquad K(b, \theta_0) \;\to\; \Big(1 - \frac{\theta_0^2}{12}\Big)^{-3/2} \quad (b \to \infty)$$

$K$ is attained at $\theta = 0$ and is optimal, and the weight outside $\theta_0$ is at most $\sqrt{2/\pi}\, b^{1/2} e^{-b(1-\cos\theta_0)}(1 + O(1/b))$ Proved here (the One-Link Heat-Kernel Comparison Lemma; $SU(2)$, one link, class densities with respect to Haar). The proof is three one-liners: the Taylor remainder $1 - \cos\theta \ge (\theta^2/2)(1 - \theta_0^2/12)$ ($\theta \le \theta_0$, $\theta_0^2 \le 24$), the lower bound of the heat kernel (the leading term of the sum over images and the Van Vleck prefactor $\theta/\sin\theta \ge 1$), and a chain of constants. The Taylor remainder and the algebra of the constants are in Lean: `one_sub_cos_window`, `wilson_weight_window` (assumptions $0 \le \theta \le \theta_0$, $\theta_0^2 \le 24$, $b \ge 0$), `window_constant_chain`, `window_split_bound`. By this comparison the chain of the Lemma on One Step with Background-Independent Fluctuations goes through for the one-link Wilson law as well (the One-Link One-Step Fluctuation Lemma), and the loss splits in two: $K$, applied to the heat-kernel quantity (the side of the closure under the fluctuation expectation), and the factor $1 + \eta$, $\eta \approx \frac{3}{2}\,\delta_b/b$ coming from the movement of the coupling ($\delta_b$ is the Taylor width of $b(W)$; the side of exchanging the expectation and the seminorm) Proved here (conditions: one link, one step, the background at the conditional maximiser, the geometry of re-taking the centre (the recentring condition) assumed, $\delta_b < b$). The former can be made small by the choice of the angle cutoff, but the latter does not depend on $\theta_0$ and is $\delta_b/b \sim g^2 p^2$ around a flat background, so the sum diverges like a power of the logarithm Reading. Numbers ($b = 4$–$64$, $\theta_0 = 0.3, 0.5, 1$): the paper $K$ agrees with the true supremum to $10^{-10}$, and the negative control that drops $\theta_0^2/12$ breaks at the edge of $\theta_0$ in 7 of 15 cells at $SU(2)$ and in all 15 cells at $U(1)$ Reading ($U(1)$ has no prefactor). The remainder of many links with a shared plaquette (§ 06) lies outside this lemma.

**The cost of the recentring condition.** If all the angles of the background plaquettes are at most $\theta_0 < \pi/2$, the size of the sum of staples is $|Q| \ge \sum_p \cos\theta_p \ge n_s \cos\theta_0$ ($n_s = 6$ in four dimensions, with equality for the paired configuration), and for $\theta_0 = \pi/2$ it can be 0 Proved here ($SU(2)$ and $U(1)$, one link; Lean: `sum_norm_ge_of_inner_ge`, assumptions: real inner-product space, $\|u\| = 1$, $\langle x_i, u\rangle \ge c$). The first-order derivative of the direction of the centre $\hat{q}$ with respect to the background is at most $18/|Q| \le 3/\cos\theta_0$, and exactly 3 for a flat background (because one staple carries three links;

independent of the dimension) Proved here . Hence, if the chain of the Lemma on One Step with Background-Independent Fluctuations is run after re-taking the centre, the input Taylor radius has to be about 4 times the output one ($h_q \approx (1 + 3/\cos\theta_0)h$) Proved here (the composition step is a sketch, under the convergence domain of the majorant series $h < 0.075$), Reading (4.02–4.24 times measured in one direction). The coupling $b = \beta|Q|$ sits at a first-order stationary point for a flat background, and its relative motion is $\delta_b/b \le 6h\sin\theta_0 + 13.5h^2$ Proved here (leading term, $h < 0.11$), $\approx 3\tan\theta_0\, h + 4.5h^2$ measured in one direction Reading . At the threshold $h = \theta_0 = gp$ it is $\eta \approx \frac{3}{2}\delta_b/b \approx c\, g^2p^2$ (measured $c \approx 12$, on paper $c \le 29$), one power of $g^2$ short of the budget assumption of § 06, and the sum diverges as a power of the logarithm Proved here (arithmetic). The rotation of the centre can be moved from the measure to the loss of radius, but **the motion of the coupling has nowhere to go except into the parameters of the class (the coupling $b$)** Reading .

06

## $\eta$ = the proposal of shifting local couplings

The Lemma on One Step with Background-Independent Fluctuations cannot be used once the background-independence condition for fluctuations breaks down. So we consider the direction of splitting the genuine $\mu_{k,W}$ into "a law independent of the background $\nu_k$ (to which the Lemma on One Step with Background-Independent Fluctuations applies) $\times$ a density $\rho_{k,W}$", and extracting the part of the background dependence of the density that can be absorbed into the parameters. The statements of this section are proposals, except for one fact (the basis of the heat-bath method) and one identity.

**Fact: the conditional law of one link is a member of a one-parameter family.** For the Wilson action of $SU(2)$, let $Q(W)$ be the sum, as quaternions, of the products of each plaquette with the link $U$ removed (the staples), and write $Q = |Q|\hat{q}$ ($\hat{q} \in SU(2)$). The conditional law of $U$ with the other links held fixed is $\exp(\beta\langle Q, U\rangle)\, dU$, and setting $U = \hat{q}Y$, the left invariance of Haar measure makes the law of $Y$ equal to $\exp(\beta|Q|\,\mathrm{Re}\,Y)\, dY = \mu_1^{\beta|Q|}$. **The conditional law is a member of the one-parameter family $\{\mu_1^b\}$, which does not depend on the background, and the background enters only through the local coupling $b(W) = \beta|Q(W)|$ and the direction of the centre $\hat{q}(W)$** Cited (a basic fact of the heat-bath method for $SU(2)$; as bibliography we give Creutz 1980 and Kennedy–Pendleton 1985, but we have not checked the text, i.e. unconfirmed). The $\beta \mapsto \beta\cos(\theta/2)$ of a single plaquette ( § 04), and the closed forms of the weights for two and three plaquettes (15-06 § 02, § 03), can both be read as examples of this one sentence Reading .

**Identity: how to split a many-link action.** Applying the unit-quaternion identity $\mathrm{Re}(ab) = \mathrm{Re}\,a\,\mathrm{Re}\,b - \mathrm{Im}\,a \cdot \mathrm{Im}\,b$ to the rearrangement $U_{\partial p} = W_{\partial p} Y_p$ of the Background–Fluctuation Decomposition Lemma ($Y_p$ is the product of the fluctuation parts), with $\theta_p$ the rotation angle of $W_{\partial p}$ and $n_p$ its axis,

$$\beta\,\mathrm{Re}(W_{\partial p} Y_p) \;=\; b_p(W)\,\mathrm{Re}\,Y_p \;-\; \beta \sin\frac{\theta_p}{2}\,\langle n_p, \mathrm{Im}\,Y_p\rangle, \qquad b_p(W) := \beta\cos\frac{\theta_p}{2} \tag{5}$$

Proved here (convention for $\frac{1}{2}\,\mathrm{tr} = \mathrm{Re}$; quaternion algebra). The first term is **an action of background-independent form with local coupling** $b_p(W)$, and the second is a term whose leading part is first order in the Lie-algebra fluctuation. In addition there remains a "conjugation mismatch" term, coming from the same $Y_l$ being conjugated by a different partial product of the background in a neighbouring plaquette (of triple-product type).

**The proposal of screening by order: screen the score by the order in $\theta$.** We divide the background derivative (the score) of the density $\rho_{k,W}$ according to its order in the background angle $\theta$ Assumption (a reading of the screening; not yet checked around the true critical point for many plaquettes).

- The $\theta^0$ term (a first-order term independent of the background): it vanishes once the centre of the background is moved to the true constrained critical point. For two plaquettes the action becomes even in the fluctuation around the true critical point, and for three plaquettes the weight becomes a function of $|1 + W_a + W_b|$ (15-06 § 02, § 03, both [Proved here]).
- The $\theta^1$ term: the derivative of $b_p(W)$ (the coupling derivative), and the anisotropic part of the quadratic form of the fluctuation left after recentring. The former is absorbed by the proposal of extending the coupling to a variable, and the latter is expected to be absorbable in the same way, as the derivative of a local anisotropic coupling.
- The non-absorbable remainder: **in the smallest systems, the lowest orders of the remainder are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$ (not $\theta^2$ or higher), and the term multiplied by the width of the threshold $h_k \sim g_k p(g_k)$ is of type $g_k^2 p$. However the threshold is chosen, the sum diverges like a power of the logarithm** (the same type as the $\sum g_k^2$ of 15-01 § 01; see the paragraph below and § 07), and the improvement lies not in the choice of the threshold but on the side of the class — letting the parameters carry the field strength of the background Reading (a reading that carries the numbers for two links and for four in a row over to the four-dimensional step).

**The remainder measured in the smallest systems.** For two links sharing one plaquette and for four plaquettes in a row, the remainder after subtracting from the

score with respect to the background angle $\theta$ the "coupling derivative (the three $\operatorname{Re} Y_p$)" and the "rotation of the centre (the 12 generators of left and right translations)" is

$$\eta^{\mathrm{res}} \;\approx\; \sqrt{A^2/\beta + D^2\theta^2}, \qquad A = 0.04\text{–}0.14 \text{ (depending on the system)}, \quad D \approx 0.018 \; (SU(2$$

Reading ($SU(2)$, $\beta = 4$–$32$ [partly 128], $\theta = 0.05$–$0.3$ [partly 1.0], two routes). If the frame is taken at the true joint critical point, the action of two links splits exactly into $\sum_p b_p \operatorname{Re} Y_p - s\, n \cdot J$, $J = \operatorname{Im}(Z_1 Z_2) - \operatorname{Im} Z_1 - \operatorname{Im} Z_2$ Proved here (quaternion algebra; Lean: `two_link_split`, `re_mul_split` and others, ring identities under assumptions on the components), and if $J$ is added to the coordinates — making it the family of quaternion couplings per plaquette $(b_p,\ s_p n_p)$ — the remainder is at most $2.4 \times 10^{-10}$ Reading . $\theta^0 \beta^{-1/2}$ is the cubic term of $J$ (the non-quadratic nature of the action) and appears with the same size in $U(1)$ as well; $\theta^1 \beta^0$ is the anisotropic quadratic form left after absorbing the commutator by conjugation, and is present only in $SU(2)$ Reading .

**The derivative in the background splits into four.** Write the weight of the class function of one plaquette as $F(U) = G(|\log U|^2)$, $G(u) = \mathrm{const} - \frac{B}{2}u + qu^2 + \cdots$ ($B$ is the coupling, $q$ the shape coordinate; $|\log U|$ is the half angle); then for the background $W = e^a$, from $|\log(e^a e^y)|^2 = |y|^2 + 2a \cdot y + O(a^2)$ (to all orders in $y$)

$$F(e^a e^y) = G(|y|^2) - B\,(a \cdot y) + 4q\,(a \cdot y)|y|^2 + \cdots + O(a^2) \quad (6)$$

Proved here . The background score splits into (i) coupling (even, $O(a^2)$), (ii) rotation ($-B\, a \cdot y$), (iii) **shape** (odd cubic $4q(a \cdot y)|y|^2$: the coefficient is the product of the value of the shape coordinate $q$ and the background), and (iv) **group geometry** (only $SU(2)$: the cubic BCH term $n \cdot [y_1 \times (y_1 \times y_2) + y_2 \times (y_2 \times y_1)]$, independent of $q$) Proved here . For the two links of $U(1)$ the remainder is exhausted by the shape, and $\eta \approx \frac{8}{3}(q/B)(\beta B)^{-1/2}$ agrees with the measurement to within 0.7% Reading ($U(1)$, the Gaussian leading term, $\theta \le 0.3$, $\beta = 8$–$64$). In a chain in a row the convolution of class functions carries the background exactly to the coarse holonomy ($(f_{W_1} * f_{W_2})(x) = (f * f)(W_1 x W_2)$, of a known type), so the contraction of the shape coordinate $\lambda = 17/125$ does not change with the background, and **the shape part enters a geometric series** Proved here (a chain in a row, assuming the one-step inequality; the bookkeeping is in Lean: `residual_sum_uniform` [assuming $0 \le \lambda < 1$ and the partial sums $\sum_k u_k \le U$], `forced_uniform`). **The group-geometry part does not contain the shape coordinate, and does not contract** (§ 07).

**The proposal of extending the coupling to a variable: make the coupling $b$ the zeroth variable of the Taylor norm.** We regard the activity $K(X; W, b)$ as a function of both the background and the coupling, and extend the seminorm to $T_{W,b}(h, hb) := \sum$

$n, m$

$_{n,m}\,\frac{h'^n}{n!}\frac{h_b^m}{m!}\|D_W^n\partial_b^m K\|$ ($h_b \propto b$). If the law of the fluctuation has the form $\mu^{b(W)}$, the background derivative falls on the integrand and on $b(W)$, and the latter is $\partial_W b\cdot\partial_b\mathbb{E}_b F = \partial_W b\cdot \mathrm{Cov}_b(F,\mathrm{Re}\,Y)$; that is, **what remains of the background derivative is a coupling derivative**. If the coupling derivative is put into the norm, the group version of exchanging the expectation and the seminorm is expected to hold without the factor $1+\eta_k$ Assumption (assumption: the central moments of the statistic of the exponential family are bounded uniformly in $b$; the consistency of the norm for a multivariable set of local couplings $\{b_p\}$ with $2^{|B(X)|}$ is not yet settled). In this form, the definition and continuity of the effective coupling in the Coupling-Smallness Induction Lemma (15-03 § 03) would also follow from analyticity in $b$. Our reading is that the precise meaning of "absorbing $\eta$ into the parameters" is "counting the parameters as components of the field" Reading .

**Giving the class the background as a parameter — the Proposition on a Class with the Background as a Parameter (proposal).** If the conditional laws of two links sharing one plaquette and of four in a row are regarded as the class $\mathcal{C}_2=(B,q,a)$ consisting of class functions obtained by cutting the plaquette weight at second order in $u=|\log P|^2$ (coupling $B$ and shape $q$) plus the logarithm of the background holonomy $a$ (for each plaquette carrying background), then the remainder after subtracting the coupling, recentring, shape and $a$ directions from the score with respect to the background is of size $\theta^0\beta^{-3/2}$ and $\theta^1\beta^{-1}$ Reading (fits over $SU(2)$, $\beta=4$–$128$, $\theta=0.05$–$0.3$; consistent with the order count on paper). The cubic BCH term of § 07 (coefficient $1/9$) and the anisotropic second-order term enter the $a$ direction and vanish; what remains is only the contribution of the shape coefficients $\hat g_3=-(1+32\rho)/720$ truncated by the class, which for $\hat g_3=0$ ($\rho=-1/32$) drops to $\beta^{-5/2}$ Reading . Each shape coefficient added to the class lowers the remainder by $\beta^{-1}$ (for $\mathcal{C}_1\to\mathcal{C}_2\to\mathcal{C}_3$ the power goes 0.5→1.5→2.5). Since $\beta=4/g^2$, the one-step term is of size $h_k g_k^3$, and under the threshold $h_k\le Hg_k$ and the one-loop upper bound $g_k^2\le 1/(c_1(k_0+k))$

$$\sum_{k<K}\eta_k \;\le\; \frac{H(A+DH)}{c_1^2\,(k_0-1)} \qquad \text{(uniform in the number of steps } K\text{)}$$

holds Proved here (arithmetic; Lean: `class_parameter_budget_bounded`, where inside Lean $c_1,k_0$ is written as $a,b$; assumptions $A,D,H\ge 0$, $c_1>0$, $k_0>1$, $g_k^2\le 1/(c_1(k_0+k))$, $0\le h_k\le Hg_k$, $\eta_k\le h_k(Ag_k^3+Dh_kg_k^2)$). This pairs with `budget_unbounded_of_threshold` of § 07 (if the term is of type $h_kg_k$, there is no uniform upper bound). Conditions (on the same line): (1) the orders are those measured with the laws of two links and four in a row, and carrying them to the four-dimensional step is a reading; (2) small fields ($\theta\le 0.3$); (3) $a$ is per plaquette carrying background (in four in a row, two are needed if the axes differ); (4) the $(B,q)$ of the class is refitted at

every step, and the truncated shape coefficients are bounded uniformly in the step; (5) the one-step update of $a$ is exact (an identity in a single-loop step, § 07; not yet for the multi-loop step). **That the score enters the $a$ direction once $a$ is made a parameter is the chain rule; what we measured is only the order of what remains when the class is cut to a finite one.** The divergence of $\sum_k \eta_k$ in the shape remainder condition has not disappeared; it has moved to the class-closure condition — whether the class carrying $a$ closes in one four-dimensional step Reading . For $U(1)$ there is no cubic BCH term, and $a$ does not change the order. The motion of the coupling $cg^2p^2$ for the one-link Wilson law of § 05 is likewise a term moved to the update of the parameters by giving the class the coupling $b$ as a parameter Reading .

**Reservations.** (5) is specific to $SU(2)$ (only for $SU(2)$ is a sum of staples a constant multiple of a group element). For $SU(N)$, $\mathrm{tr}(WY)$ is a bilinear form in $W$ and $Y$, the local coupling is a matrix rather than a number, and the conditional law does not fall into a one-parameter family (it does for each $SU(2)$ subgroup). Also, a fundamental-plus-adjoint action breaks the form of the first term of (5) (the expansion of $\cos^2$ produces an anisotropic quadratic form that acts only on the components of the fluctuation in the directions of the background) Proved here (quaternion algebra).

07

## The multi-step induction has not been written

The Lemma on One Step with Background-Independent Fluctuations, the proposal of screening by order, and the proposal of extending the coupling to a variable are all one-step statements. Stacking steps needs the following three, none of which has been written Assumption .

- Background-regularity condition: for $W$ in the image of the background-field map, the curvature of the fine plaquettes is bounded by $L^{-2}$ times the curvature of the coarse plaquettes plus $O(\varepsilon_k^2)$ (the regularity of the background).
- Contraction of the many-body norm: the one-step loss $2^{|B(X)|}$ is cancelled by the contraction of the irrelevant part of the activity (the group version of the contraction estimate in the appendix for flat fields).
- The remainder after absorbing $\eta$ stays of type $g_k^{2+\delta}$ uniformly in the step. If it is put on the ledger of the transverse contraction of 15-03 § 04, the remainder is expected to be bounded by a supremum rather than a sum.

**The coefficient of the group-geometry part — the cubic BCH term is $1/9$.** For two links with outer coupling $m\beta$ and shared plaquette $\beta$ ($m = 1$ is the system of § 06), the remainder after projecting the background score onto the coupling and the 12

generators of left and right translations is, at the Gaussian leading term, $\eta = A\beta^{-1/2}$ (per background angle),

$$A\sqrt{B} = \sqrt{\frac{1}{3(m+2)^3} + \frac{240\,(m^2+m+1)}{(m+2)^5}\,r^2}, \qquad r = \frac{q}{B} \tag{7}$$

Proved here (the Gaussian leading term, the two-link model; $m = 1$–$5$ are exact by rational Wick computations, and the formula for general $m$ was read from five points). **The geometric part for $m = 1$ is exactly $1/9$, and does not depend on the shape coordinate $r$.** It is orthogonal to the shape part in the Gaussian $L^2$ (cross term 0), and $\eta^2$ splits into the sum of the two. It agrees with independent numbers for the true action and with the measurement of § 06 to within 0.6% Reading . The algebra by which the cubic generator term returns $+\frac{B}{9}$ to the cubic BCH coefficient $-\frac{B}{6}$ of the score, giving $-\frac{B}{18}$ (in general $-\frac{B}{6}\cdot\frac{m}{m+2}$), is in Lean: `bch3_coefficient` (assumption $m + 2 \neq 0$), `nT_sum_diff`, `shape_split` (polynomial identities without assumptions).

**The class $(B, q, a)$, and $s$ as the background itself.** The coefficient $s$ of $J$, which carries the cubic BCH term, does not contract in the one step that shrinks two links in series to one link: $s'/s = 1 - (1/3 + 10r)/B + O(B^{-2}, a^2)$ Proved here (the leading term and its $O(1/B)$; agrees with the measurements for $B = 16$–$128$, $s/B \le 0.1$ to three digits Reading ). $s$ is not a shape coordinate but **the background field itself** ($s/B$ is the half angle of the background plaquette); one step carries the background to the coarse holonomy as the group product $a' = \log(W_2 W_1)$ (BCH), and only the shape coordinate $q/B$ contracts (by a factor $1/4$ in this step). The class $(B, q, a)$ ($a$ is the logarithm of the background holonomy) closes with its three parameters in one step in series, with $B' = B/2 - 1/6 - 5r + O(1/B)$, $r' = r/4 + O(1/B)$, $a' = \log(W_2 W_1)$ Proved here (the model of two links in series, $SU(2)$; the residual of the fit is at most $3.5 \times 10^{-7}$ Reading ). The coefficient of the cubic BCH term to $\eta$ is a constant that depends neither on the class nor on $s$, so the geometric part remains as "a constant $\times$ the budget of background change at each step".

**The budget diverges.** The geometric-part term of one step is $\frac{1}{9}\,h_k\,(\kappa\beta_k)^{-1/2}$ ($h_k$ is the width of the background change, $\kappa \in [0.2, 1]$ the contraction of blocking), and since $\beta = 4/g^2$, it is of size $h_k g_k/(18\sqrt{\kappa})$ Proved here (arithmetic; carrying the two-link constant over to the four-dimensional step is Reading , and all that is used is that the constant does not depend on $\beta$). With the small-field threshold $h_k = g_k p(g_k)$ it is of size $g_k^2 p(g_k)$, and under the one-loop flow $\beta_{k+1} = \beta_k - c$ ($c = \frac{11}{3\pi^2}\ln 2 \approx 0.2575$), if the infrared end is held fixed and steps are added towards the ultraviolet, the sum diverges like a power of $\ln K$ Proved here (arithmetic; if $h = g$, the sum is $2/(9c) \approx 0.863$ times $\ln(\beta_0/\beta_*)$, and numerically too it is 0.865–0.87 times Reading ). Keeping the probability of large fields

small needs $h_k \geq g_k$, and then $\sum h_k g_k \geq \sum g_k^2 = \infty$, so **the sum cannot be made finite by the choice of the threshold** Proved here (Lean: `budget_unbounded_of_threshold`, assumptions $g_j^2 \geq 1/(A(j+1))$, $0 \leq g_j \leq h_j$). Therefore this part cannot be carried as an error factor $1+\eta_k$; the field strength of the background (the vector coupling per plaquette $s_p n_p$, statistic $J$) must be given to the class as a parameter and moved into the movement of the parameters Reading . For two links, adding $J$ to the coordinates made the remainder 0 (§ 06). This part is absent for $U(1)$ (the cubic BCH term is identically 0), and the parameter is the logarithm of the background holonomy $a$.

**The one-face step — for a single loop the update of $a$ closes by the group product.** In a cross (two-dimensional) made of a central plaquette whose four links all fluctuate and the four plaquettes outside it, integrating out the four links puts the coarse weight exactly into a class function of the outer holonomy $W$ (convolution of class functions; a restatement of gauge invariance); the centre of the coarse class is $a' = \log W$, $W$ is the ordered product $e^{b1}e^{b2}e^{b3}e^{b4}e^{a0}$ in the tree gauge (non-abelian Stokes), and the shape coordinates do not depend on the background Proved here (conditions: the coarse variable is a single loop, the weights are class functions, the other links are integrated with Haar, $a_p$ is held in the common frame of the tree gauge). If $a$ is updated by the commutative sum $\sum_p a_p$, the remainder is the sum of second-order commutators $\sum_{i<j} a_i \times a_j$ $(\theta^2\beta^0)$, larger than the shape truncation of the class $(\theta^0\beta^{-3/2})$. If it is updated by the group product, there is no remainder and no new coordinate is needed Proved here , Reading (for $SU(2)$, $\beta = 8$–$128$, $\theta \leq 0.3$, a quadrature that does not use the closed form reproduces the group product to $2.1\times10^{-4}$–$1.4\times10^{-6}$, and the difference falls with the quadrature nodes). That the quadratic term of the logarithm of the ordered product is half the sum of commutators is Lean: `log_quadratic_part` (assumption: ring), `commSum_eq_zero_of_comm` (commutative ring; for $U(1)$ the commutative sum is exact up to second order). It can be read as: **the cubic BCH term goes, to all orders, into the side of the background coordinate $a$** Reading . The shape coefficient after one step is $q'/B' \approx (q/B)/25$ (the tree value of the convolution of five plaquettes), $B' - \beta/5 \to -0.40$ for Wilson, and exactly $t' = t_0 + 4t_m$ for the heat kernel Reading .

**The multi-loop step — the class coordinates need the corner triple product.** In a three-dimensional $2\times2\times2$ step that fixes only the twelve edges of the coarse cube ($SU(2)$; heat kernel and Wilson; the leading saddle-point term and one loop; $O(1/\beta)$ not included), the coarse weight is a function of five independent loops and does not factor into a product of per-face class functions. Within second order, the symmetry of the coarse box ($O_h$) makes it close exactly with two coefficients: the per-face class (diagonal) and the coupling of pairs of neighbouring faces ($a_A \cdot a_B$ placed at the end of the shared edge) Proved here (symmetry; uniqueness of the representation is Lean:

`diag_adj_unique`, assumption: the two forms are equal on $\sum \theta = 0$). The tree-level values are diagonal $5/16$ and neighbour $1/32$ (ratio 1/10) Reading (rational fit; two routes). Since there is no non-abelian scalar of $\theta^2$, non-commutativity first appears with three faces (a corner) Proved here (counting the representations of $O_h$). At third order the triple product $\kappa \sum_c a_x \cdot (a_y \times a_z)$ of the three faces meeting at a corner of the cube is needed as the one and only new term, $\kappa = 3/1024$ at tree level (numerical fit) and about 0.006 at one loop; at fourth order couplings of three and four faces are needed Reading . In the range $\theta \sim \beta^{-1/2}$ the corner triple product is $\approx 2.2 \times 10^{-3} \beta^{-1/2}$ and the fourth-order multi-face coupling $\approx 3.3 \times 10^{-3} \beta^{-1}$, larger than the shape truncation of the class $\beta^{-3/2}$, so they cannot be omitted Reading . If $a_p$ is held in a common frame (the tree gauge), a frame error breaking $O_h$ remains at third order, so a definition is needed that places the neighbouring pairs at the end of the shared edge and the triple product at the corner. For a four-dimensional $2^4$ box one counts 24 coarse faces, 96 neighbouring pairs and 64 corners Reading (only counted, not computed). We therefore read the class coordinates as taking the form of **per-face** ($B_p$, $q_p, a_p$)**, the coupling of neighbouring pairs** $B_{pq}$**, the one-link coupling** $b$**, the corner triple product** $\kappa$**, and the fourth-order multi-face couplings** Assumption (a statement within the saddle-point expansion of one three-dimensional step; the second step, large fields and four dimensions are not shown; 15-03 § 02).

**The flow of the corner triple product — it does not contract, and approaches the fixed point from below.** In a three-dimensional free-boundary box $L \times L \times L$ fixing only the twelve edges of the coarse cube, measuring the genuine multi-step flow at tree level from $L = 2$ (one step) to $L = 20$, the ratio $r_\kappa = \kappa / c_2$ ($c_2$ is the coefficient of $\sum |a_p|^2$) increases as 0.019 ($L = 2$), 0.033 ($L = 4$), 0.050 ($L = 20$); the ratio per doubling approaches 1 as 1.78, 1.30, 1.13, 1.10; and the increment shrinks by about 0.55 per doubling Reading ($SU(2)$, the leading saddle-point term, up to third order in $\varepsilon$; the form of the fit approaches 0.055–0.066; the existence of the limit is not shown). One loop (heat kernel and Wilson, $\beta = 32$–$128$, $L \le 4$) moves the ratio by no more than 3%, and hardly changes the two-step ratio 1.78 Reading . **That the first-step ratio exceeds 1 is not a wall but the form of the approach**: terms absent from the fine action are born in the first step, and since they start below the fixed point, if $0 \le \lambda < 1$ and $0 < r < s/(1 - \lambda)$ the fixed point is not overshot even with a one-step ratio $(\lambda r + s)/r > 1$ Proved here (Lean: `ratio_gt_one_below_fixpt`, `abs_iterate_dev` ($0 \le \lambda < 1$)). In the iterated two steps, which feed the one-step coarse action back into the action on the intermediate lattice, the deviation of the ratio shrinks to 1/9–1/17 per step (the third-order output is exactly linear in the input); the iterated fixed point 0.032–0.045 is smaller than the limit of the genuine multi-step flow, and the approach is much faster Reading . Fixing only the edges, the second-order coefficient is nearly constant, $c_2 L \ln L \approx 0.27$–$0.29$ ($\ln L$ along the edges), and does not reach the classical three-dimensional scale $1/L$

Reading (interpretation: because the interiors of the faces are free and only the edges are fixed). The corner triple product is not a quantity that vanishes geometrically like the shape remainder, but **a coordinate with a value fixed by the flow**, and the class has to carry that value. It is not, however, an independent parameter.

**The four-dimensional class — one parameter and the ratio coordinates; the skew neighbour is not 0.** In one step of a four-dimensional $2^4$ box fixing the 32 edges of the coarse hypercube, the space of linearised angles of the coarse faces is 17-dimensional and splits under the hyperoctahedral group $B_4$ into $3 + 6 + 8$ (6 is the uniform field); the invariants number 3 at second order, 3 at third order (colour triple products) and 65 at fourth order Proved here (two routes: averaging of characters and numerical rank; reproduces $2 \cdot 1 \cdot 9$ in the three-dimensional version). The orbit sum of pairs of orthogonal faces (type $\operatorname{tr} F \wedge F$) is 0 by reflection. Unlike in three dimensions, **per-face classes and neighbouring pairs sharing an edge do not close already at second order** (remainder 0.57% at tree level, 0.79% at one loop), and one "skew neighbour" pair, shifted by one edge (sharing no vertex, averaged by transport along the two connecting edges), is needed: at tree level diagonal $5/26$, neighbour $-79/2080$, skew $-9/2080$ Proved here (linear expressions in the eigenvalues $1/4$, $5/13$, $3/10$; Lean: `diag_adj_skew_unique`, `diag_adj_not_enough` (assumptions: the computed componentwise values and eigenvalues), `coeffs_of_spectrum` ($k = (2e8 - e3 - e6)/8$), `skew_zero_iff`). The third order likewise does not close with the types gathered at a vertex (cell corners, paths) alone, and a parallel type is needed (the cell-corner coefficient is a fit of $5931/1406080$ at tree level). In the range $\theta \sim \beta^{-1/2}$ the skew neighbour is $\approx 0.02$ at $\beta^0$, the third order $\approx 0.023\beta^{-1/2}$ and the fourth order $\approx 0.046\beta^{-1}$, all larger than the shape truncation of the class $\beta^{-3/2}$, so none can be omitted Reading (saddle point plus one loop, no $O(1/\beta)$, 90 random samples of the face angles $\lesssim 0.5$).

In the genuine multi-step flow with the box enlarged to $\{0, \dots, L\}^4$ ($L \leq 10$, tree), none of the second- and third-order ratio coordinates head to 0 at the second step ($\lambda \geq 1.14$); the ratio per doubling approaches 1 and they approach the values of the fixed point (skew neighbour $r_k = k/b$: $-0.01125$, $-0.01744$, $-0.01988$, extrapolated $-0.0208$; neighbour $-0.145$, third order $0.064$, $0.013$, $-0.021$) Reading (the fit $r_\infty - A/L - B/L^2$ represents $L = 4$–$10$ to within $4 \times 10^{-5}$; the $\ln L$ form as in three dimensions is 30–300 times worse). The second-order class coefficients have no $\ln L$ at $b \approx 0.39/L$ and approach faster than in three dimensions. One loop moves the two-step ratio by less than 1% ($L \leq 4$). The fourth-order multi-face part has a ratio of 1.6 to $c_2$ at the second step, with direction cosine 0.99. **Dependence on the rule**: within one coarse hypercube, the rule fixing all faces gives the same quadratic form as the rule fixing the edges ($32 - 15 = 24 - 7 = 17$; Lean: `edge_face_dof_agree`, `euler_cube`). On a periodic network where the coarse cells share fine fields, the skew neighbour still approaches a

non-zero negative value, but the value is about twice as large ($-0.045$; $-0.06$–$-0.10$ depending on the projection convention), and the part not contained in a cell grows with the steps from 6% to 20% Reading (tree level of the commutative linearisation, second order, $L \le 16$). **The sign and non-vanishing of the skew neighbour depend neither on the rule nor on the projection convention; its value, and "closing within one cell", do.** Under either rule the $b$ of the cell part at tree level is $1/L$, and the running of $\ln L$ is not at tree level. The four-dimensional class is "one parameter $B$ plus the ratio coordinates whose values are fixed by the flow (2 at second order, 3 at third, 65 at fourth)", and the "local" in the class-closure condition is written with the rule attached (15-03 § 02) Reading .

**The minimal form in which to write the multi-step induction, and its cost.** Viewing the flow of the class coordinates as a sequence of linear maps, in the iterated two steps that feed the one-step class back into the per-hypercube action on the intermediate lattice, the output of the third-order coordinates is exactly linear in the input, $\kappa' = s + \Lambda\kappa$, the eigenvalues of the normalised $\Lambda$ are all below $0.1$, and the deviation of the second-order ratio also shrinks to 0.1–0.2 per step Reading (four dimensions, tree, versions with two weights bracketing the genuine two steps). The genuine multi-step flow, however, approaches the same coordinates only at a rate of $1/L$, and **the iterated fixed point is shifted towards 0 relative to the genuine limit** (for the skew neighbour, $-0.019$ against $-0.021$ for the rule that separates cells, $-0.035$ against $-0.045$ for the periodic network). On the periodic network the shift is larger, since the "part not contained in a cell" that the iteration discards at every step is 6–10%. The induction can be written as a contraction of the iterated map, but that its fixed point agrees with the value of the genuine flow must be shown separately; as long as the class coordinates are truncated to the type of a single cell, the induction has to be written in a form that carries the truncation error from step to step Reading (the bookkeeping is Lean: `tail_bound` (assumption $0 \le q < 1$ and the ratio of increments $\le q$), `affine_iter`). Under the rule fixing only the edges, $\beta' = \beta b$ becomes about 0.39 times per step, so the relative weight of one loop grows with each step, and the induction closes only within the weak-coupling range.

With this, splitting the density of the joint law in Table 2 has the form of its answer in the smallest systems. Ultraviolet questions 17–19 and the second step of the multi-loop class also have partial answers. The multi-step induction (the minimal multistep statement) has not been written.

Table 2 Questions arising from this chapter (status: New = a question added in this part; Updated = an answer has been added)

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 10 | For four in a row of $SU(2)$ (two overlapping loops fixed), split the joint law of the two inner plaquettes into the product of local couplings $\mu^{b2} \otimes \mu^{b3} \times$ a density, and derive the lowest order in $\theta$ of the score of the density, and the isotropy of the quadratic form of the fluctuation (the coefficient of the anisotropic term), as functions of $\beta$. Is the $\theta^1$ term only the coupling derivative? **Answer (smallest systems):** the lowest orders are $\theta^0 \beta^{-1/2}$ and $\theta^1 \beta^0$. $\theta^1$ contains not only the coupling derivative but also an anisotropic quadratic term (only for $SU(2)$). | Paper and small computations | Updated | Whether the screening of the proposal of screening by order is right. The type of $\eta$ in the group version of exchanging the expectation and the seminorm | $\mu_{k,W}$ on the four-dimensional lattice. The multi-step induction |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| | Adding $J$ to the coordinates makes the remainder $0$ ( § 06) | | | | |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 11 | The minimal statement of the multi-step induction: can the background-regularity condition and the contraction of the many-body norm be written with constants independent of the step, in a hierarchical model of single-plaquette weights or in a two-dimensional model? Together with this, using the norm of the proposal of extending the coupling to a variable, derive the width of the strip of analyticity of the coupling from the positions of the complex zeros of $Z_b = 2I_1(b)/b$ (the positions of the zeros are unconfirmed in the original sources) | Paper | New | A form of the Lemma on One Step with Background-Independent Fluctuations stacked over $k$ | The one-step remainder including large fields (the net one-step bracket of the coupling) |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 17 | For the central plaquette of a $2\times2$ plate (four links fluctuating), write out the remainder after dividing the score by the space of generators as odd orders of BCH symmetrised in cyclic order, and derive the geometric coefficient by rational Wick computations. Does it grow or shrink on a face compared with the $1/\sqrt{3(m+2)^{3}}$ of two links? Also, does the insertion of the background stick out of the family of quaternion couplings $A_{p}+J$? **Answer (partial):** in the cross (a two-dimensional step in which the four links of the central plaquette fluctuate), if $a$ is updated by the group product there | Paper and small computations | Updated | The geometric coefficient for the four-dimensional step | The divergence of the budget (the threshold offers no escape) |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| | is no remainder, and the odd BCH orders go into the coordinate $a$ to all orders. With the commutative sum, $\theta^{2} \beta^{0}$. In three dimensions $2 \times 2 \times 2$ the corner triple product $\kappa = 3 / 1024$ is needed (§ 07) | | | | |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 18 | For a class with the logarithm of the background holonomy $a$ as a parameter, measure in one step in series whether the geometric part of one step moves exactly into the flow $a' = \log(W_2 W_1)$ of $a$ (the remainder drops one order from $\theta^0 \beta^{-1/2}$). **Answer (smallest systems):** for the class $(B, q, a)$ with $a$ as a parameter, the remainder drops to $\theta^0 \beta^{-3/2}$ and $\theta^1 \beta^{-1}$, and the budget is uniform in the number of steps (conditionally, § 06). In a single-loop step the update of $a$ is exact (§ 07) | Small computations | Updated | The form of moving the geometric part into the movement of the parameters | The multi-step induction |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 19 | The condition of the One-Link One-Step Fluctuation Lemma (the recentring condition): for one link of the four-dimensional lattice, make the Taylor radius of the map of the sum of staples $W \mapsto \hat{q}(W)$ explicit in terms of the angles of the background plaquettes and the lower bound of $\lvert Q\rvert$, and make the $g^2p^2$ coefficient of $\delta_b/b$ a single number. **Answer:** $\lvert Q\rvert \geq 6\cos\theta_0$, $\lVert D\hat{q}\rVert \leq 3/\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$ (measured $c \approx 12$, on paper $c \leq 29$). The composition step is a sketch, and the motion of the coupling is a term carried to the side of the parameters of the class (§ 05) | Paper | Updated | The one-link Wilson chain without assumptions | The remainder of many links |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 23 | The second step of the multi-loop class: perform the second step with the class after one three-dimensional $2 \times 2 \times 2$ step (per face, neighbouring pairs, corner triple product, multi-face), and see whether $\kappa$ is a coordinate preserved from step to step (expected to be the lattice version of the dimension-6 $\operatorname{tr} F[F, F]$), and whether the same coordinates suffice in four dimensions $2^4$. **Answer (partial):** in three dimensions $\kappa/c_2$ does not contract at the second step and approaches the fixed point (0.055–0.066 by the fit) from below. In four dimensions the same coordinates do not suffice: | Small computations | Updated | The class coordinates of one four-dimensional step | Large fields, the small-field contribution to the coupling |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| | the skew neighbour is needed at second order and a parallel type at third, and all the ratio coordinates approach the fixed point (tree $L \leq 10$, § 07) | | | | |
| 26 | When the class is given the coupling $b$ as a parameter, does the motion of the coupling $c g^2 p^2$ for the one-link Wilson law move exactly into the update of the parameters, with the remainder dropping by one power of $g^2$? | Paper and small computations | New | The budget of the One-Link One-Step Fluctuation Lemma | The remainder of many links |

| Q. | Question | Size | Status | What moves if settled | What does not move even if settled |
|---|---|---|---|---|---|
| 27 | Can the fixed-point values of the ratio coordinates of the four-dimensional class (the skew neighbour, the 3 of third order, the 65 of fourth order) be obtained directly, instead of running many steps, from a half-infinite problem near a corner of the coarse hypercube (the Green function of a half-space with a bundle of edges fixed)? Also, under the rule in which coarse cells share fields, should the part not contained in a cell (up to 20% with the steps) be closed by putting it into finitely many pairs of class coordinates (orbits of shift $\leq 1$), or carried as a tail? | Paper and small computations | New | The values of the four-dimensional coordinates of the class-closure condition and the form of "local" | Large fields, the small-field contribution to the coupling, the second step at one loop |

08

## What cannot be said

(a) Within this part, ultraviolet control is not closed (existence was shown by the published construction, with a different design). Every statement of this chapter is a one-step statement, a statement set down as an assumption, or a proposal.

(b) The Background–Fluctuation Decomposition Lemma is only the group version of submultiplicativity in sums of fields among the four properties of the weight, and does not touch the closure under the fluctuation expectation, which is the main issue. It may be a known elementary fact.

(c) The group versions of the exponential moment of fluctuations and of background regularity, and the interchange inequality, are only statements and have not been shown. (3) is nothing more than the chain assuming them.

(d) $\eta_k \sim g_k p(g_k)$ and $\eta_k \lesssim g_k^2 p(g_k)^2$ are naive estimates, not estimates for the actual measure of the fluctuations.

(e) The Lemma on One Step with Background-Independent Fluctuations is a statement about a single step in the case where the law of the fluctuation does not depend on the background, and shows nothing about Bałaban's $\mu_k$,$W$, the Wilson action, many steps, or the rarity of large fields. The numbers of Table 1 are a numerical reading with the rough figures $SU(2)$, $n_b = 96$.

(f) The fact about the heat-bath method gives its bibliography from memory, and we have not checked the text. The proposal of screening by order and the proposal of extending the coupling to a variable are proposals. The orders of the remainder in the smallest systems (§ 06) are numbers for two links and four in a row, and carrying them over to the four-dimensional step is a reading. $SU(N)$ does not have the form of (5).

(g) The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma are statements for one link and one step, and assume the recentring condition. The coefficient (7) of the cubic BCH term is a value at the Gaussian leading term in the two-link model, and the formula for general $m$ was read from five points. The closing of the class $(B, q, a)$ is a statement about one step of two links in series, and configurations with faces have been examined only in the single-loop cross (§ 07). The divergence of the budget is arithmetic that treats the one-loop flow as an exact formula, and carrying the two-link constant over to the four-dimensional step is a reading.

(h) The numbers of the recentring condition in § 05 are statements for one link and one step; the composition step is a sketch, and the majorant series on paper converges only for $h < 0.075$–$0.078$. The Proposition on a Class with the Background as a

Parameter in § 06 is a proposal; what we measured is the order of the remainder when the class is cut to a finite one (two links, four in a row, $\beta = 4\text{–}128$, $\theta \leq 0.3$), and it rests on the chain rule. That the class carrying $a$ closes in a four-dimensional step is not shown. The one-face step of § 07 is a statement for a single loop, and the class coordinates of the multi-loop class are statements within the saddle-point expansion (leading term and one loop) of one three-dimensional $2 \times 2 \times 2$ step. The second step and four dimensions have not been measured.

(i) The flow of the corner triple product in § 07 consists of numbers at tree level ($L \leq 20$) and one loop ($L \leq 4$) in a three-dimensional free-boundary box (only the edges fixed), and the limit is in the form of a fit. The four-dimensional class consists of numbers at tree level ($L \leq 10$) and one loop ($L \leq 4$) in a free-boundary box, and the periodic network of numbers at second order in the tree of the commutative linearisation ($L \leq 16$); the existence of the fixed point of the ratio coordinates, the values of the 65 fourth-order ones, and the third order and one loop on the periodic network are not shown. That the sign and non-vanishing of the skew neighbour do not depend on the rule is a statement about the two rules and two projection conventions that were measured. The numbers for the contraction of the iterated map are at tree level only.

09

## The five grades table and the dependency graph

Table 3 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [01] | For $\prod_k (1+\eta_k)$ to be bounded uniformly in the number of steps, $\sum_k \eta_k < \infty$ is needed | Proved here | § 01 (arithmetic) | Under the one-loop flow |
| [02] | The Background–Fluctuation Decomposition Lemma (1) and the Splitting of the Weight (2), the form written with $Y$ | Proved here | § 02 | $SU(N)$, operator norm, four-dimensional lattice (Splitting of the Weight) |
| [03] | The group versions of the exponential moment of fluctuations and of background regularity (the group version of the exponential moment of fluctuations, the background-regularity condition), and the group version of exchanging the expectation and the seminorm | Assumption | § 03 (statements only) | Small-field backgrounds $\mathcal{W}_k$; constants independent of $k, a$ |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [04] | The chain (3), the group version of the One-Step Bound on the Activity Norm | Proved here | § 03 (assuming [03]) | On $\mathcal{W}_k$ |
| [05] | For a product of one-link heat kernels, the law of the fluctuation does not depend on the background | Proved here | § 03 (invariance of Haar measure) | Heat kernels independent edge by edge |
| [06] | With a naive $\eta_k$ the sum diverges ($K^{1/2}$ for $g\,p$, a power of $\ln K$ for $g^2p^2$) | Proved here | § 04 (arithmetic) | Naive estimate, under the one-loop flow |
| [07] | The type of $\eta$ changes with how it is measured ($\theta^2$, $\theta^4$, $\theta$). Without the constraint, $\beta\theta^2/12$ | Reading | § 04, 15-06 § 01 | Single plaquette, $\theta \le 0.3$, $\beta = 4$–$32$ |
| [08] | The Lemma on One Step with Background-Independent Fluctuations (4) and (a)–(d) ($\eta = 0$, the support does not spread, $M = m^{\lvert X\rvert}$ for heat kernels, small fields) | Proved here | § 05 | The three conditions of background independence of the fluctuations, scale contraction and the exponential moment of fluctuations; one step only |
| [09] | The numbers for the exponential-moment condition for fluctuations $s/\beta \le 0.0020$–$0.0025$, $c\,g^2/h^2 \lesssim 1.7 \times 10^{-4}$ | Reading | § 05 Table 1 | $SU(2)$, heat kernels edge by edge, $n_b = 96$, $q = 6$, $\beta = 4$–$128$ |
| [10] | The conditional law of one link for $SU(2)$ is $\mu_1^{\beta\lvert Q\rvert}$, and the background enters only through $b(W)$ and $\hat{q}(W)$ | Cited | § 06 (a basic fact of the heat-bath method) | $SU(2)$, Wilson action, bibliography unconfirmed |
| [11] | The quaternion identity (5): local coupling $b_p = \beta\cos(\theta_p/2)$ | Proved here | § 06 | $SU(2)$, $\frac{1}{2}\,\mathrm{tr} = \mathrm{Re}$ |
| [12] | The proposal of screening by order (screening by the order of $\theta$; the orders of the remainder are [15]) and the proposal of extending the coupling to a variable (the coupling as a variable of the norm) | Assumption | § 06 (proposal) | Small fields; not yet checked for many plaquettes |
| [13] | The three requirements of the multi-step induction (the background-regularity condition, the many-body contraction, uniformity of the remainder) | Assumption | § 07 | Statement only |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [14] | The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma: inside the angle cutoff the one-link Wilson law is $K$ times the heat kernel; the losses are $K$ and $1+\eta$ ($\eta \approx \frac{3}{2}\delta_b/b$) | Proved here | § 05 | $SU(2)$, one link, one step, the recentring condition assumed, $\delta b < b$ |
| [15] | The remainder in the smallest systems $\eta^{\mathrm{res}} \approx \sqrt{A^2/\beta + D^2\theta^2}$; adding $J$ gives $0$ | Reading (the splitting of the action is Proved here) | § 06 | Two links, four in a row, $\beta = 4$–$32$, $\theta \le 0.3$ |
| [16] | The four-way splitting (6) of the derivative in the background, the $\eta = \frac{8}{3}(q/B)(\beta B)^{-1/2}$ of $U(1)$, the shape part enters a geometric series | Proved here, Reading (numbers for the agreement) | § 06 | Small fields (first order in $a$), a chain in a row, assuming the one-step inequality |
| [17] | The coefficient (7) of the cubic BCH term: $1/9$ at $m = 1$ | Proved here | § 07 | Gaussian leading term, two links; the formula for general $m$ is read |
| [18] | The class $(B, q, a)$ closes in one step in series, and $s$ does not contract | Proved here, Reading | § 07 | Two links in series, $SU(2)$, $B = 16$–$128$ |
| [19] | The budget of the geometric part diverges however the threshold is chosen | Proved here (arithmetic), Reading (carrying over to four dimensions) | § 07 | One-loop flow, $h \ge g$ |
| [20] | The recentring condition: $\lvert Q\rvert \ge n_s \cos\theta_0$, $\lVert D\hat{q}\rVert \le 3/\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$ | Proved here, Reading (the measured $c$ and $h_q/h$) | § 05 | $SU(2)$, one link, $\theta_0 < \pi/2$, composition step sketched, $h < 0.075$ |
| [21] | The class $(B, q, a)$ with the background $a$ as a parameter: remainder $\theta^0\beta^{-3/2}$, $\theta^1\beta^{-1}$; the budget is uniform in the number of steps (the Proposition on a Class with the Background as a Parameter) | Reading (orders), Proved here (arithmetic of the budget), Assumption (the proposition) | § 06 | Two links, four in a row, $\beta = 4$–$128$, $\theta \le 0.3$, conditions (1)–(5) |
| [22] | In a single-loop step the update of $a$ closes to all orders by the group product; with the commutative sum, $\theta^2\beta^0$ | Proved here, Reading (numbers) | § 07 | Cross, class-function weights, common frame |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [23] | Multi-loop step: second order is per face plus neighbouring pairs; at third order the corner triple product $\kappa = 3/1024$; at fourth order multi-face | Proved here (second-order symmetry), Reading ($\kappa$ and fourth order) | § 07 | Three-dimensional $2 \times 2 \times 2$, one step, leading saddle-point term and one loop |
| [24] | The final form of the class coordinates (per face, neighbouring pairs, one-link coupling, corner triple product, multi-face) | Assumption | § 07 | A reading of one three-dimensional step; four dimensions not yet (four dimensions in [26]) |
| [25] | The ratio $\kappa/c_2$ of the corner triple product does not contract at the second step and approaches the fixed point from below (the first-step ratio 1.78 is the form of the approach) | Reading , Proved here (bookkeeping of the linear maps) | § 07 | Three dimensions, free boundary, only edges fixed, tree $L \leq 20$, one loop $L \leq 4$, form of the fit |
| [26] | Four-dimensional class: one parameter plus the ratio coordinates (2 at second order, 3 at third, 65 at fourth); the skew neighbour is non-zero whatever the rule, with a value that depends on the rule; no running of $\ln L$ at tree level | Proved here (counting, agreement of degrees of freedom, linear expressions for the coefficients), Reading (numbers for the flow) | § 07 | $SU(2)$, heat kernel, tree $L \leq 10$, one loop $L \leq 4$, periodic network at second order of the commutative tree $L \leq 16$ |
| [27] | The iterated map contracts fast, but its fixed point is shifted towards 0 relative to the genuine limit, and a form carrying the truncation error from step to step is needed | Reading | § 07 | Tree level, two rules |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

```
[01] PRV  product uniform ⟺ sum finite                                  one-
[02] PRV  Background–Fluctuation Decomposition Lemma, Splitting of the Weight
[03] ASM  group versions of the exponential moment and background regularity,
[04] PRV  one-step chain (3) ⟸ [02][03]                                 assu
[05] PRV  heat-kernel product: fluctuation law does not depend on the backgrou
[06] PRV  naive η: the sum diverges ⟸ [01]                              naiv
[07] RDG  the type of η depends on how it is measured ⟸ [06]            sing
[08] PRV  Lemma on One Step with Background-Independent Fluctuations ⟸ [02][0
[09] RDG  the numbers for the exponential-moment condition ⟸ [08]       SU(2
[10] CIT  one-link conditional law is μ₁^{β|Q|}                        SU(2)
[11] PRV  quaternion identity (5) ⟸ [02]                                SU(2
[12] ASM  proposals of screening by order and of extending the coupling to a v
[13] ASM  three requirements for many steps ⟸ [04][08][12]              stat
[14] PRV  One-Link Heat-Kernel Comparison Lemma, One-Link One-Step Fluctuation
[15] RDG  smallest-system remainder θ⁰β^{−1/2}, θ¹β⁰ ⟸ [11][12]         two
[16] PRV  four-way split of the background derivative; shape is geometric ⟸ [
[17] PRV  cubic BCH coefficient 1/9 ⟸ [16]                              Gaus
[18] PRV  class (B, q, a); s does not contract ⟸ [17]                   one
[19] PRV  divergence of the budget ⟸ [06][17][18]                       one
[20] PRV  cost of the recentring condition ⟸ [14]                       one
[21] RDG  class with a as parameter, budget uniform ⟸ [18][19]          two
[22] PRV  one-face step: a updated by group product ⟸ [18]              sing
[23] RDG  multi-loop class: neighbour pairs, corner triple product ⟸ [22]   2
[24] ASM  final form of the class coordinates ⟸ [21][22][23]            3D r
[25] RDG  corner triple product approaches fixed point from below ⟸ [23]   3D
[26] RDG  4D class, skew neighbour, dependence on the rule ⟸ [23][24]    tree
[27] RDG  shift of iterated fixed point, truncation error ⟸ [13][25][26]   tr
```

To check it, verify by eye that every number to the right of "⟸" is smaller than the number on the left. There are no cycles. [03], [12], [13] and [24] are assumption rows, and each is a restatement of ultraviolet control or a question that splits it. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Proved here The Background–Fluctuation Decomposition Lemma and the Splitting of the Weight (the group version of submultiplicativity in sums of fields); the one-step chain (assuming the group versions of the exponential moment of fluctuations and of background regularity); that for a product of heat kernels the law of the fluctuation does not depend on the background; the divergence of the sum for a naive $\eta_k$; the Lemma on One Step with Background-Independent Fluctuations ($\eta = 0$ in one step where the law of the fluctuation does not depend on the background, and the support does not spread); the quaternion identity (5) | A proof of the group versions of the exponential moment of fluctuations and of background regularity. An estimate of the actual $\eta_k$. A single step with a genuine $\mu_{k,W}$ |
| Reading The numbers for the exponential-moment condition for fluctuations ($SU(2)$, $n_b = 96$); that the type of $\eta$ changes with how it is measured | The constants of the exponential-moment condition for fluctuations on the closure of a four-dimensional block |
| Cited That the conditional law of one link for $SU(2)$ is a member of a one-parameter family (a fact of the heat-bath method) | Checking the text of the bibliography |
| Proved here The One-Link Heat-Kernel Comparison Lemma and the One-Link One-Step Fluctuation Lemma (the one-link Wilson law, assuming the recentring condition); the four-way splitting (6) of the derivative in the background; the coefficient (7) of the cubic BCH term (Gaussian leading term, two links); one step in series of the class $(B, q, a)$; the divergence of the budget (arithmetic) | The group version of the exponential moment of fluctuations for many links; the geometric coefficient for configurations with faces; the recentring condition; the budget for the four-dimensional step |
| Reading The orders of the remainder in the smallest systems ($\theta^0\beta^{-1/2}$, $\theta^1\beta^0$), and that adding $J$ gives 0; agreement of the values of the shape for $U(1)$ | Systems in which four or more links fluctuate around one plaquette |
| Proved here The explicit constants of the recentring condition; a uniform upper bound on the budget (with $a$ as a parameter, arithmetic); the background update in the one-face step (single loop); the second-order closing of the multi-loop class (symmetry) | That the class carrying $a$ closes in a four-dimensional step; the composition step of the recentring condition |
| Reading The order of the remainder of the class carrying $a$; the class coordinates of the multi-loop class (neighbour/diagonal 1/10, corner triple product 3/1024); the flow of the corner triple product (approaches the fixed point from below); the four-dimensional class (one parameter plus the ratio coordinates; the skew neighbour is non-zero); the shift of the iterated fixed point | The existence of the fixed point of the ratio coordinates; the values of the 65 fourth-order ones; the third order and one loop on the periodic network; large fields |
| Assumption The proposal to absorb $\eta$ as the shift of local couplings; the three requirements for many steps | A proof of any of them. Any prospect of ultraviolet control closing |

## Sources and reproduction

| Item | Type | Source, tools |
|---|---|---|
| The four properties of the weight, the One-Step Bound on the Activity Norm (Lemma A.6.1), the inequality exchanging the expectation and the seminorm ((A.6.5)) | Cited (the text of Appendix A.6.2 and of Lemma A.6.1) | R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, Lecture Notes in Math. 2242 (2019), arXiv:1907.05474 (15-02 § 04) |
| Propagators in a background field | Cited (bibliography and titles only) | The foundational papers of T. Bałaban's series (7-07 Table 1) |
| The basic fact of the heat-bath method for $SU(2)$ | Cited (bibliography unconfirmed) | M. Creutz, Phys. Rev. D 21 (1980) 2308; A. D. Kennedy, B. J. Pendleton, Phys. Lett. B 156 (1985) 393 |
| The Background–Fluctuation Decomposition Lemma, the Splitting of the Weight, the chain, the Lemma on One Step with Background-Independent Fluctuations, (5), the arithmetic of the sum of $\eta_k$ | Proved here | The proofs in the text of this chapter |
| The numbers for the exponential-moment condition for fluctuations (Table 1) | Reading | A numerical record of this book's study (the generating function of the heat kernel of $SU(2)$ by trapezoidal quadrature in the angular direction and a sum over characters; positive control: the closed form for $\mathbb{E}\|Y-1\|^2$ to 9 digits, normalisation to 10 digits) |
| The type of $\eta$ for a single plaquette | Reading (the form of (1) in 15-06 is Proved here) | The source table of 15-06 § 01 |
| The proposal of screening by order and the proposal of extending the coupling to a variable | This book's study | The record of this book's study (sorting out the rows of ultraviolet control that do not close) |
| The One-Link Heat-Kernel Comparison Lemma, the One-Link One-Step Fluctuation Lemma, (6), (7), the class $(B, q, a)$, the arithmetic of the budget | Proved here | The main text of this chapter. The algebraic parts are in the Lean bundle (15-09 Table 3, re-verified) |

| Item | Type | Source, tools |
|---|---|---|
| The remainder in the smallest systems, the agreement of $U(1)$, $s'/s$, the partial sums of the budget | Reading | The numerical records of this book's study (two routes for the projections of two links and four in a row; rational Wick computations and Gaussian samples of the true action; fits and two routes of moments for one step in series; controls of $U(1)$ and $\theta = 0$) |
| The one-loop coefficient $b_0$ | Cited (textbook formula) | Equation (9) of 7-05 |
| The recentring condition, the Proposition on a Class with the Background as a Parameter, the one-face step, the numbers for the multi-loop class | Reading | The numerical records of this book's study (random draws and sign patterns of the one-link sum of staples; fits and negative controls for two links and four in a row; quadrature and quasi-Monte Carlo for the cross; two routes for the heat-kernel and Wilson saddle points of $2 \times 2 \times 2$, and rational fits) |
| The flow of the corner triple product; the four-dimensional class and comparison of rules; the iterated two steps | Reading (counting, degrees of freedom and linear expressions; the bookkeeping is Proved here) | The numerical records of this book's study (at tree level, two routes: the tree gauge with sparse normal equations, and the minimum norm; a third route by Newton's method for the non-abelian saddle point. One loop by the analytic Hessian. Positive controls: reproducing $5/32$, $1/32$, $3/1024$, $5/13$, $-79/2080$, $-9/2080$ of one step. A negative control with third order 0 for a parallel background. The periodic network agrees to 10 digits between two routes, real space and Fourier). The bookkeeping is in the Lean bundle (15-09 Table 3, re-verified) |
| Lower bound of the sum of staples; arithmetic of the budget; the sum of commutators; uniqueness of diagonal and neighbour | Proved here | The main text of this chapter. The algebraic parts are in the Lean bundle (15-09 Table 3, re-verified) |

**Words used in this chapter** — the measure of the fluctuations $\mu_{k,W}$, background dependence $\eta_k$, the four properties of the weight, the Background–Fluctuation Decomposition Lemma, the group versions of the exponential moment of fluctuations and of background regularity, the interchange inequality, the Lemma on One Step with Background-Independent Fluctuations, fluctuations independent of the background, the shift of local couplings, the fact about the heat-bath method, the score, the multi-step induction, the One-Link Heat-Kernel Comparison Lemma, the shape part and the group-geometry part, the cubic BCH term, the class $(B, q, a)$, the budget of background change, the cost of the recentring condition, the background as a parameter of the class, the Proposition on a Class with the Background as a Parameter, the one-face step, the multi-loop class, the corner triple product, the ratio coordinates, the skew neighbour, dependence on the rule. Their meanings are in the A-02 glossary.

Next chapter to read: 15-08 The four rows that do not close, and proposals for them — which of the four rows the absorption of $\eta$ in this chapter makes lighter.

2026-10-15 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control
the four rows that do not close, proposals, dependence · Kiichi, Shiori and Rin

# The four rows that do not close, and proposals for them

This chapter is this book's own investigation. § 01, § 03, § 04, § 07 and § 08 are a map of open questions.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of the second volume; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**What this chapter tells you** — Of ultraviolet control, the part that lies outside the components closed in this part (the Coupling-Smallness Induction Lemma, the Bundle Network Theorem, the Lemma on One Step with Background-Independent Fluctuations, and the bookkeeping of the transverse contraction) is written as four rows. The first (the row of the remainder including large fields) is the one-step remainder including large fields (the net one-step bracket of the coupling); the second (the observable-source condition) is the source for observables; the third (the row of background-dependent absorption and many steps) is the absorption of the background dependence $\eta$ of the fluctuations and the multi-step induction; the fourth (the row of the boundary of gauge re-fixing) is the seam where gauge fixing is re-fixed scale by scale. For each row we write in one line "what would have to be shown for it to close", and then set out the boundary with the closed components, proposals that can be tried, and proposals that were discarded. The four rows are not independent: we put forward the reading that the row of background-dependent absorption and many steps is the core, and that what remains without being reduced is two things, the small-field contribution to the coupling and the multi-step regularity of the background. Split further, the small-field contribution to the coupling divides into a part of the same form as is known in the sources for flat fields, and the group version of the place that the sources themselves call a sketch (the explicit quadratic term between blocks and its local cancellation). Finally, we list in ten rows the directions being tried towards ultraviolet control. In the revision we added: that the explicit quadratic term between blocks appears at $O(1)$ from one loop in the smallest system of two blocks; that the type of its non-local tail is decided by the definition of the coarse variables; the places where, when the algebraic layer of the small-field contribution to the coupling was written in Lean, the cut may differ from the sources; and that the ghost determinant of the row of the boundary of gauge re-fixing agrees with the tree gauge on critical-point backgrounds.

**Reading level** — This chapter runs from third-year level up to research level (it uses measures, operators and norm estimates). Table 1 of § 01 at the start alone can be read by second-year students.

**Prerequisite chapters** — 15-03 § 02 (the decomposition statement), 15-03 § 03 (the Coupling-Smallness Induction Lemma), 15-07 § 05 (the Lemma on One Step with Background-Independent Fluctuations), 15-05 (the Bundle Network Theorem), 15-02 § 06 (the two rows absent from the sources).

**Said first** — None of the rows in this chapter closes. A proposal is a proposal, not a result. The label [Proved here] is attached only to elementary facts checked on paper as a step within a proposal.

The order of this chapter

1. Statement of the four rows — one line, the boundary with the closed components, beyond the boundary (Table 1)
2. Proposals for the row of the remainder including large fields — the one-step remainder including large fields
3. Proposals for the observable-source condition — the source for observables
4. Proposals for the row of background-dependent absorption and many steps — the absorption of $\eta$ and the multi-step induction
5. Proposals for the row of the boundary of gauge re-fixing — the seam of re-fixing of the gauge fixing
6. Dependence among the four rows — the row of background-dependent absorption and many steps is the core; splitting the small-field contribution to the coupling; $W^{\text{gauge}}$ for two blocks; the tail type; the algebraic layer and how it differs from the sources
7. Discarded proposals (Table 2)
8. Ten rows of directions being tried (Table 3); ultraviolet questions 12–16, the question "two definitions of Loc", the question "three-dimensional non-local weights", the question "gauge agreement on non-abelian backgrounds" (Table 4)
9. What cannot be said
10. The table of the five grades and the dependency graph (Table 5)

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00

## What this chapter shows

- The part that does not close can be written in four rows: the net one-step bracket of the coupling, the observable-source condition, the absorption of $\eta$ and many steps, and the seam of re-fixing ( § 01).
- The large-field rarity condition (the rarity of large fields) of the row of the remainder including large fields is a corollary of Chebyshev's inequality once the group version (the group version of the exponential moment of fluctuations) of the closure under the fluctuation expectation ((iii) of BBS) holds, and needs no independent estimate ( § 02, [Proved here], assuming the group version of the exponential moment of fluctuations).

- If the source of the observable-source condition is taken to be a function of the block variables, it commutes with the integration at the fine steps and produces no loss ( § 03, [Proved here], trivial).
- For the row of the boundary of gauge re-fixing, the reading is that the direct-product decomposition of the tree gauge leaves no Jacobian depending on the field of re-fixing, and the cost moves to the ghost determinant and the coupling at the seam ( § 05).
- The four rows gather at the row of background-dependent absorption and many steps, and the irreducible core is the small-field contribution to the coupling (the two-loop remainder uniform over small-field backgrounds, which we read as the content of Bałaban's unpublished basic inequality (Theorem 2 of Bałaban 1989)) and the multi-step background-regularity condition ( § 06).
- The row of the remainder including large fields, the observable-source condition and the row of the boundary of gauge re-fixing have been split. Net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling is an identity, and the latter is bounded under heat-kernel fluctuations (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma); a source in the block variables has no loss step by step; and that the tree gauge has no Jacobian is in Lean for finite groups and on paper for $SU(2)$ ( § 02, § 03, § 05).
- The small-field contribution to the coupling splits into "the local part + the Taylor truncation + the small-field cut + the explicit quadratic term between blocks", and for the first three the same form as in the sources for flat fields holds conditionally on the group too. The only thing absent from the sources is the group version of the last term, which the sources themselves call a sketch ( § 06).
- The explicit quadratic term between blocks $W^{\mathrm{gauge}}$ does not start at second order as in the flat-field model, but appears from one loop at $O(1)$ ($13/384$) in the smallest two-block system. Its origin is not the non-commutativity of the group but the non-locality of the background map; the one-loop formula can be written explicitly, and what needs local cancellation is the $O(1/\beta)$ that comes after it ( § 06; $SU(2)$, seven in a row, one step, quadratic coefficients).
- Its non-local tail type (flat, exponential, power) is decided by the definition of the coarse variables. We read that the coordinates of "coarse links plus global tags" fit the cubic-remainder estimate in four dimensions. In three dimensions without boundary it was exponential even at one loop, with the sum uniform in volume; on the four-dimensional periodic lattice it was exponential at tree level ( § 06; the four-dimensional one loop has not been measured).
- In the one-row model, the recursion of the core row closes up to fourth order in the form "a contracting image of the pair types (in the ratio of couplings, quadratic $5/8$, quartic $\leq 5/48$) + new terms independent of the step + three-body terms decreasing exponentially in the width". This is the one-row version of the additivity of the sources ( § 06; one row, definition without a conserved quantity, conditional).
- The ghost determinant of the row of the boundary of gauge re-fixing, for one step, at one loop and on a critical-point background, agrees identically with the effective action of the tree gauge, and fits into one line of the bookkeeping of the local couplings ( § 05).

01

## Statement of the four rows

The notation is that of 15-00 § 01 and 15-07: the effective coupling $g_k$ at step $k$, $\beta_k = 4/g_k^2$, the small-field threshold $\varepsilon_k = g_k p(g_k)$, the weight $G_k$, the measure of the fluctuations $\mu_k$,$W$, the set of blocks $B(X)$. The "one line" of Table 1 is a conditional sentence: if it were shown, that row would close Assumption .

Table 1 The four rows that do not close (every "one line" is a statement; labels only in the column of closed components)

| Row | What would close it if shown (one line) | Boundary with the closed components | Beyond the boundary |
|---|---|---|---|

| Row | What would close it if shown (one line) | Boundary with the closed components | Beyond the boundary |
|---|---|---|---|
| 1 The one-step remainder including large fields (the net one-step bracket of the coupling) | That in a real step (including gauge-covariant averaging, the background field and the large-field region), the coefficient of $F^2$ read from the effective action satisfies $1/g_{k+1}^2 = 1/g_k^2 - b + r_k$, $\lvert r_k\rvert \le C g_k^2 + \eta_k$ ($\sum \eta_k < \infty$, $C, \eta$ depend neither on $k$ nor on the lattice spacing), and that the many-body contributions of large fields are bounded by $e^{-c\,p(g_k)^2} \le C' g_k^2$ | Net one-step bracket of the coupling + definition and continuity of the effective coupling ⇒ the condition that the coupling stays small is the Coupling-Smallness Induction Lemma Proved here . $\beta'/\beta = \sum f^2$ at tree level is the Unit Bundle Theorem Proved here (restricted to the commutative case). The deviation of $O(1)$ at one loop is the numbers for the smallest systems Reading . The splitting net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling is an identity, and the large-field contribution to the coupling is bounded by the Lipschitz constant of the read-out × rarity under heat-kernel fluctuations (independent on each edge, one step, independent of the background) Proved here (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, § 02). The same form for one Wilson link Proved here (the One-Link Heat-Kernel Comparison Lemma). The splitting of the local part of the small-field contribution to the coupling and the Taylor remainder has the same form as in the sources for flat fields Cited , Proved here (under the analyticity condition at complex backgrounds, § 06). The explicit quadratic term between blocks is $O(1)$ from one loop in the smallest two-block system, and the rest after putting in the first order explicitly is $O(1/\beta)$ Proved here (Gaussian minimisation and the one loop of Laplace, formal), Reading (numbers, § 06). The algebraic layer (the projection $\mathbf{Loc}$, the layer of $W^{\text{gauge}}$, the assembly of the remainder) is in Lean ( § 06). Passing to the second step, in the one-row model the image of the non-local terms closes on the same types and contracts in the ratio of couplings, the new terms are independent of the step, and the three-body terms decrease exponentially in the width Reading (one row, definition without a conserved quantity, one step twice, up to fourth order, § 06). | Of the small-field contribution to the coupling: the group version of the local cancellation of $O(1/\beta)$ in four-dimensional blocking, and the volume-uniform estimate of the non-local tail (the cubic-remainder estimate; the tail type depends on the definition of the coarse variables; with coarse links plus global tags it is exponential on the four-dimensional tree; the four-dimensional one loop not yet, § 06); the recursion of $W$ in four dimensions (the number of three-body invariants, four-body terms); the probability of large fields under the true measure of the fluctuations (the group version of the exponential moment of fluctuations = the row of background-dependent absorption and many steps); the definition of the read-out map (what is called $\beta_{k+1}$) |

| Row | What would close it if shown (one line) | Boundary with the closed components | Beyond the boundary |
|---|---|---|---|
| | | In the coordinates of coarse links plus global tags, the one-loop tail in three dimensions without boundary is exponential Reading (§ 06) | |
| 2 The source for observables (the observable-source condition) | That for the activity $Ke^{J}$ with a smooth source $J$ (of strength $\leq s_0$) supported on scales larger than $L_0/M$, $\|\theta(Ke^{J})\|_{G_{k+1}} \leq 2^{\lvert B(X)\rvert_e} C s_0^{\lvert B(X)\rvert} \|K\|_{G_k}$ holds for the real $\mu_{k,W}$ with a constant independent of $k$ | One step without a source is the Lemma on One Step with Background-Independent Fluctuations (when the law of the fluctuations does not depend on the background) Proved here . A source that is a function of the block variables commutes with the integration over the fluctuations, and there is no loss step by step Proved here (in Lean for the finite case). In one heat-kernel step, the source from fine loops is a heat-kernel shift of the coupling for each representation Proved here (§ 03) | That at the fine steps the source becomes a non-local activity (its support spreads over blocks of size loop length / lattice spacing) |

| Row | What would close it if shown (one line) | Boundary with the closed components | Beyond the boundary |
|---|---|---|---|
| 3 The absorption of $\eta$ and the multi-step induction | That, splitting $\mu_{k,W}$ = (a law independent of the background) $\times$ (a density $\rho_{k,W}$), what remains of the background dependence of $\log \rho_{k,W}$ after absorbing the $F^2$ direction and the shift of the centre into the parameter and a re-choice of the background is $\eta_k = O(g_k^{2+\delta})$ $(\delta > 0)$, and that the background-regularity condition and the contraction of the many-body norm hold uniformly in the step | Absorption happens exactly with one and two plaquettes Proved here (15-06). $\eta = 0$ of the Lemma on One Step with Background-Independent Fluctuations Proved here . The bookkeeping of the transverse contraction Proved here (15-03 § 04). The one-Wilson-link version of the Lemma on One Step with Background-Independent Fluctuations Proved here (conditional, 15-07 § 05). The shape part of $\eta$ enters a geometric series Proved here (assuming the one-row chain and the one-step inequality, 15-07 § 06). Giving the class the background $a$ as a parameter, the rest drops to $\theta^0 \beta^{-3/2}$ for two links and for four in a row, and the budget is uniform in the number of steps Reading (counting orders), Proved here (the arithmetic of the budget, 15-07 § 06). In the one-step of a single loop, the update of $a$ closes to all orders by the group product Proved here (15-07 § 07) | The values of the coordinates of the multi-loop class (in three dimensions $2 \times 2 \times 2$: per face plus neighbouring pairs at second order, the corner triple product at third order, multi-face couplings at fourth order; the corner triple product does not contract in the second step but approaches a fixed point; in four dimensions one parameter plus the ratio coordinates (the skew neighbour is needed at second order), and what remains is the existence of the fixed point and the values depending on the rules, 15-07 § 07); the background-regularity condition stacking steps; the update when the class is given the coupling $b$ as a parameter (15-07 § 05). For a class that does not have $a$ as a parameter, the budget of the group-geometric part of $SU(2)$ (the third order of BCH) diverges (15-07 § 07) |

| Row | What would close it if shown (one line) | Boundary with the closed components | Beyond the boundary |
|---|---|---|---|
| 4 The seam of re-fixing of the gauge fixing | That the Jacobian of the field-dependent transformation connecting the local gauge of step $k$ with the local gauge of step $k+1$, and the terms at the seam, can be estimated within the class as a loss with finite sum (of type $g_k^{2+\delta}$ or a geometric series) | The Orbit-Direction Flatness Proposition (Hess = 0 in the orbit directions) Proved here (15-01 § 03). There is no global gauge fixing Cited . Existence and uniqueness of the tree (axial) gauge (no Gribov multiplicity), the direct-product decomposition of configurations, the constancy of the Faddeev–Popov weight, and that re-fixing between two trees sharing a root is a sum-preserving bijection Proved here (Lean for finite groups, paper for $SU(2)$, § 05) | The form of the cost is settled: in the tree gauge there is no Jacobian, and the cost is the degradation $\|W_e - 1\| \le (d-1)(L-1)\varepsilon$ of the small-field condition and, when a background-covariant gauge is used alongside, the ghost determinant. The latter, for one step, at one loop and on a critical-point background, agrees identically with the effective action of the tree gauge and fits into one line of the bookkeeping of the local couplings Proved here (the finite-dimensional Faddeev–Popov lemma, § 05). What remains is the integral around backgrounds that are not critical points, the same agreement at two loops and beyond, the uniformity of the local expansion, and advancing steps in the range where the block side $L$ can be taken sufficiently smaller than $\varepsilon_k^{-1}$ |

The boundary between the parts that close and the parts that do not can be drawn by two questions Reading : "does the law of the fluctuations depend on the background?" and "is it at tree level, or at one loop or beyond?". The closed components lie either in fluctuations independent of the background (the Lemma on One Step with Background-Independent Fluctuations) or at tree level (the Unit Bundle Theorem, the bookkeeping of the Coupling-Smallness Induction Lemma).

02

> **Proposals for the row of the remainder including large fields — the one-step remainder including large fields**

**The proposal to obtain rarity by Chebyshev: the large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations.** If on $X$ the group version of the closure under the fluctuation expectation (the group version of the exponential moment of fluctuations) holds with $s = tc/h_k^2$, that is, if $\mathbb{E}_{\mu_{k,W}} \exp\bigl(s \sum_{l \in \bar{X}} \|A_l\|^2\bigr) \le 2^{|B(X)|}$, then by Chebyshev's inequality

$$P_{\mu_k,W}\Bigl(\sum_{l \in \bar{X}} \|A_l\|^2 \ge \varepsilon^2 |\bar{X}|\Bigr) \;\le\; 2^{|B(X)|}\, e^{-s\varepsilon^2 |\bar{X}|} \tag{1}$$

holds Proved here (a one-liner assuming the group version of the exponential moment of fluctuations). If the threshold is taken of type $\varepsilon = h_k\, p(g_k)$, the right-hand side is $2^{|B(X)|} e^{-tc\, p(g_k)^2 |\bar{X}|}$, which is the form of the large-field rarity condition. The rarity of large fields needs no independent estimate: the exponential moment of the group version of the exponential moment of fluctuations is itself the rarity. That the weight of large fields plays this role in the appendix for flat fields is a known structure Cited (15-02 § 04); all we added on the group is the single phrase "if the group version of the exponential moment of fluctuations holds". For the product of heat kernels in the Lemma on One Step with Background-Independent Fluctuations, $P \le m^{|\bar{X}|} e^{-s\varepsilon^2 |\bar{X}|}$ follows at once from $M = m^{|\bar{X}|}$ Proved here . Caveat: the group version of the exponential moment of fluctuations itself depends on the row of background-dependent absorption and many steps, so the large-field rarity condition of the row of the remainder including large fields is subordinate to the row of background-dependent absorption and many steps.

**Heat-Kernel Fluctuation Rarity Lemma — for heat-kernel fluctuations the group version of the exponential moment of fluctuations holds in closed form.** The heat kernel $p_t$ ($t = 1/\beta$) of $SU(2)$ is bounded above by the three-dimensional Gaussian in the radius just by dropping the negative terms of the sum over images and using $\sin\theta \le \theta$: for non-decreasing $g \ge 0$, $\mathbb{E}_{p_t}\, g(\theta) \le e^{t/2}\, \mathbb{E} g(|Z|)$, $Z \sim N(0, tI_3)$ Proved here (the Heat-Kernel Gaussian Upper-Bound Lemma). Under the edgewise independent heat-kernel condition, the single-step condition and the background-independence condition for fluctuations, the probability that any link in the set $\bar{X}$ exceeds the threshold $h$ is at most $|\bar{X}|\, e^{t/2}(1 + \sqrt{2/\pi}\, h/\sqrt{t})\, e^{-h^2/(2t)}$, and the exponential moment is $m(s,t) \le 1 + e^{t/2}\bigl((1-2st)^{-3/2} - 1\bigr)$ ($0 \le st < \tfrac{1}{2}$); neither depends on the background Proved here (the Heat-Kernel Fluctuation Rarity Lemma). So under heat-kernel fluctuations the large-field rarity condition is a one-liner without assumptions. The finite Chebyshev step and the union bound are in Lean: `weighted_tail_le_exp_moment` (weights $\ge 0$, $\sigma \ge 0$), `union_tail_le_sum` (weights $\ge 0$). In numbers ($t = 1/4$–$1/32$, $h = 0.3$–$1$) the bound lies above the true value in all 12 cells (ratio 1.03–1.34), and a negative control with altered constants fails in 10 cells with $h^2/t \ge 1$ Reading .

**The same form for one Wilson link (the One-Link Heat-Kernel Comparison Lemma).** The heat-kernel bound as it stands does not apply to the one-link Wilson density (it fails in 7 of the 15 cells $b = 4$–$64$, $h = 0.3$–$1$) Reading . Bounding by $K$ times the heat kernel $p_{1/b'}$ ($b' = b(1 - \theta_0^2$

/12)) inside the angle cap $\theta_0$, and sending the outside to the rare tail, the probability of large fields and the exponential moment are bounded for one Wilson link too by $K$ times the heat-kernel quantities + the tail Proved here ($SU(2)$, one link, one step). In numbers the bound is above in all 45 cells (ratio 1.07–1.91), and the threshold of the exponential-moment condition for fluctuations is 0.44–0.95 times the closed form for Wilson Reading (details in 15-07 § 05). This establishes the one-link version of the group version of the exponential moment of fluctuations needed to obtain the large-field rarity condition under the true measure of the fluctuations, but the multi-link group version of the exponential moment of fluctuations remains in the row of background-dependent absorption and many steps.

**The proposal to split the remainder into small and large fields: split the net one-step bracket of the coupling into the small-field contribution to the coupling and the large-field contribution to the coupling.** If the read-out map $b$ (the map that reads $\beta_{k+1}$ off the effective density; a candidate is the quadratic coefficient of the single coarse plaquette) is Lipschitz with respect to total-variation distance with constant $L_b \le C\beta^2$ ($db/dE \approx \frac{4}{3}\beta^2$ from $\mathbb{E}[\frac{1}{2}\operatorname{tr} U] \approx 1 - 3/(4\beta)$ for one link), and the difference in total variation between the density with one large-field region inserted and the density without it is $e^{-c\,p(g)^2}$, then the remainder due to large fields is $|r_k^{\text{large}}| \le C\beta_k^2 e^{-c\,p(g_k)^2}$, and if $p(g) = c_0(1+\log g^{-1})^{p_0}$ ($p_0 \ge \frac{1}{2}$, with $cc_0^2$ large enough) it is $\le C' g_k^2$ Assumption (arithmetic under two assumptions). The remaining core is the small-field contribution to the coupling = the two-loop remainder uniform over small-field backgrounds, which we read as corresponding to the content of the unpublished basic inequality Assumption (a reading from one sentence of the preface, 15-02 § 02). Caveat: if the read-out is taken to be the quadratic coefficient, two coordinates are needed at the quadratic level (15-06), so the Lipschitz property has to be stated on the pair.

**The proposal to split the remainder into small and large fields has been split (the Large-Field Remainder Bound Lemma).** Writing $\rho_S$ for the density conditioned on small fields, the splitting $r = r^{\text{small}} + r^{\text{large}}$, $r^{\text{small}} := b(\rho_S) - \beta_{\text{pred}}$, $r^{\text{large}} := b(\rho) - b(\rho_S)$ is an identity Proved here ($\beta_{\text{pred}}$ is the tree + one-loop prediction). The Lipschitz constant of the read-out $b = 1.5/(-\log \mathbb{E}\cos\theta)$ in total variation is $\frac{4}{3}b^2/E \to \frac{4}{3}\beta^2$, and $|r^{\text{large}}| \le \frac{4}{3}(b_S^2/ES)\,P_L$ ($P_L$ is the probability of large fields, $E \in (e^{-2}, 1)$) Proved here. The actual sensitivity per unit probability of large fields, however, is $\approx \frac{1}{3}\beta^2 h^2$, which is first order in $\beta$ if the threshold is taken at the small-field scale $h \propto 1/\sqrt{\beta}$ Reading (one link; the same numbers for the heat kernel and Wilson). This is because the read-out function $\cos\theta$ is flat at $\theta = 0$. For one link under the three conditions — edgewise independent heat kernel, single step, background independence of fluctuations — a sufficient condition for $|r^{\text{large}}| \le C' g^2$ is, for the threshold $\theta_h = g\,p(g)$ ($\beta = 4/g^2$, that is, $\sqrt{\beta}\,\theta_h = 2p$; log is the natural logarithm), $p(g) = c_0(1+\log g^{-1})^{1/2}$, $c_0^2 > 2$ Proved here (arithmetic; from the total-variation form, $c_0^2 > 3$). **The core that remains is the single term $r^{\text{small}}$ = the small-field contribution to the coupling**, and its splitting is in § 06. To obtain $r^{\text{large}}$ under the true measure of the fluctuations, the group version of the exponential moment of fluctuations is needed (the row of background-dependent absorption and many steps).

**The check on the slab.** On a $2 \times 2$ slab of heat kernels with fluctuations independent on each plaquette ($SU(2)$, one step, images dropped, 20 fine plaquettes), we counted the difference $r$

$^{\text{large}}$ between the image with a large-field region $\Omega$ inserted and the image without it. In the read-out of one coarse plaquette, $\mathbb{E}\frac{1}{2}\operatorname{tr} W_B$ splits into a product over plaquettes (a convolution of characters), so $r^{\text{large}}$ is determined by the six plaquettes that enter the block alone, and the value with the whole slab inserted is $(b_B/\beta)^2\sum_i a_i^2 = 1/3$ times the value for one link at the same $\beta$ Paper (the numbers are 0.3334–0.3358 Computed ). The image is multilinear in the plaquette-wise laws, so $\|\rho_\Omega - \rho_0\|_{\mathrm{TV}} \le 1-(1-q)^{|\Omega|}$, and to first order $r^{\text{large}}(\Omega)$ is a sum over single plaquettes Paper . Taking the threshold at $\theta_h = g\,p(g)$ of the convention above, $|r^{\text{large}}|/g^2$ decreases up to $\beta \le 256$ for $c_0^2 > 2$, $c_0^2 = 2$ is the boundary (only there $C'$ grows on the order of $p^3$; the behaviour at the boundary was checked for one link up to $\beta \le 4096$), and if the threshold is fixed at the small-field scale $\theta_h \propto 1/\sqrt{\beta}$ it grows in proportion to $\beta$ Computed . The position of the boundary agrees with the sufficient condition $c_0^2 > 2$ of the Large-Field Remainder Bound Lemma Reading . A large field on a shared edge moves the coupling between the two blocks it touches by the same order as the diagonal change, so it fits to count the large-field region as "the union of the blocks touching that plaquette" Reading . The slab does not add to the difficulty of one link; all the difficulty of many links remains on the side of "the fluctuations are not independent" — the multi-link group version of the exponential moment of fluctuations under the true $\mu_{k,W}$ (the row of background-dependent absorption and many steps). Conditions: the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; one step; heat kernel with images dropped.

**The proposal of the parity of critical points: send the background dependence to the coefficient of $\theta^4$ by the parity of the critical point.** If around the true critical point under the constraint the action of the fluctuations is even in the fluctuations (a fact checked with two plaquettes, 15-06 § 02), there is no contribution of $O(\theta)$ to the reading of $\beta'$, and the background dependence goes entirely into the coefficient of $\theta^2$ (the reading of $F^2$ itself) and into $\theta^4$ and beyond (the source). The requirement "uniform over the background" in the small-field contribution to the coupling then weakens to "the coefficient of $\theta^4$ is bounded", and that part is paid for by the transverse contraction (15-03 § 04) Assumption (parity is a fact checked with two plaquettes, trivial with three).

**With four plaquettes parity breaks, and the conclusion stands for another reason.** With four plaquettes in a row, the parity of the action of the fluctuations around the true constrained critical point breaks. The breaking concentrates in the breathing direction (the geodesic along which the two inner plaquettes tilt in the same direction along the axis of $U^*$; independent of coordinates), and for the Wilson action $S(t) - S(-t) = 2\beta\sin(\alpha^*/2)(\sin 2t - 2\sin t)$ holds to all orders Paper (the cubic coefficient is $-\beta\sin(\alpha^*/2)$; agrees with numerics to 7 digits on 18 backgrounds Computed ). It is even only when $W_b = W_a^{-1}$. The conclusion of the proposal of the parity of critical points, however — that there is no background dependence of odd order in the reading of $\beta'$ — stands without relying on parity. There is no $\theta^1$ because $\mathfrak{su}(2)$ has no conjugation-invariant linear form (gauge invariance); there is no $\theta^3$ because, on a background of two loops, the pairs $(W_a, W_b)$ and $(W_a^{-1}, W_b^{-1})$ are simultaneously conjugate by a rotation by $\pi$ about the axis perpendicular to the two axes (the pair version of $-1 \in$ the Weyl group); and the effective weight $w$ is exactly even to all orders under the reversal of the background $(a,b) \to (-a,-b)$ Paper (also checked numerically Computed ). So for a chain in a row (two loops of

independent backgrounds), the background uniformity of the small-field contribution to the coupling weakens to "the coefficient of $\theta^4$ is bounded uniformly over the background", and its ground changes from the parity of the fluctuations to the covariance under reversal of the background (the boundedness of the coefficient of $\theta^4$ itself is not shown). In a four-dimensional block there are three or more independent loops, and there is an odd invariant, the triple product $a \cdot (b \times c)$ (of type $f_{abc} F^3$), so boundedness of the coefficient of $\theta^3$ is also needed Assumption . On the side of the law of the fluctuations (the score), an odd part of order $\theta^0$ $\beta^{-1/2}$ remains (RMS $\times$ $\sqrt{\beta}$ is 0.07–0.13), of the same type as the remainder in 15-07 Computed .

**The proposal of a parameter for the distribution over the centre: put large fields into the class as the parameter $z$ of the distribution over the centre.** This proposal splits the weight $f$ of one plaquette into $f = f_e + f_o$ by the parity of the characters of the centre $\mathbb{Z}_2$, and calls the relative weight $z := f(\pi)/f(0)$ of the far hemisphere the large-field parameter ($z = e^{-2\beta}$ for Wilson; of order $e^{-\beta\pi^2/2}$ for the heat kernel, from the leading term of the sum over images). If the large-field region is counted not as outside the class but as the contribution of $z^{|\Omega^c|}$, the large-field rarity condition becomes a statement inside the class Assumption . Caveat: this is about class functions of a single plaquette; with many links the shift of the centre becomes a centre vortex (a surface of signs of $\mathbb{Z}_2$ of the plaquettes) and cannot be fully counted by the $z$ of a single plaquette (the caveat of 15-04 § 01).

03

## Proposals for the observable-source condition — the source for observables

**The proposal to take the source as a function of the block variables: take the source to be a function of the block variables.** If the observable $J$ is defined as a function of the block variables $V$ of step $k_J$ (a Wilson loop of the coarse lattice, a smooth functional of the blocked field), then for $k < k_J$ the integration over the fluctuations of step $k$ is carried out with $V$ held fixed, so $\int \rho_k e^{sJ(V)} d\mu = e^{sJ(V)} \int \rho_k \, d\mu$ holds exactly and the constants of the small-field analyticity condition and the large-field rarity condition do not change (loss 0) Proved here (trivial). For $k \geq k_J$, $J$ becomes a local function of the coarse lattice (a term for each loop shape), and $e^{sJ}$ shifts the couplings by loop shape of the action by $s$. The line is that the observable-source condition reduces to "the coordinates of the class-closure condition include the couplings by loop shape, and the one-step map is Lipschitz in those coordinates" Assumption (the parameter coordinates of 15-06). Caveat: physical observables are loops of the fine lattice, and the difference from the blocked loops is the renormalisation of observables. Restating the observable-source condition with blocked observables is a re-choice of the statement, not the observable-source condition for fine loops. We have not checked in the text which of the two the "expectation values of loops" in Bałaban's preface refers to.

**The accounting of the loss from the source, and the source from fine loops.** The commutativity of the proposal to take the source as a function of the block variables is written in Lean for the finite case (`coarse_source_commute`, `coarse_source_commute_comp`; assumptions: configurations and coarse variables finite, steps nested). In norms, for a source that is a sum of local terms per block with strength $s_0$, the factor $e^{s_0 |B(X)|}$ appears only once and does not pile up step by step Proved here (limited to blocked observables). A source from loops of the fine

lattice, for one step under the edgewise independent heat-kernel condition and a self-avoiding loop $C$, is $\mathbb{E}\,F(U_C) = (F * p_{t|C|})(W_C)$; that is, it merely shifts the coupling of representation $j$ by a factor $e^{-tj(j+2)|C|/2}$ Proved here (Monte Carlo gives $|z| \le 1.2$ at both first and second order Reading ). The loss is not a constant factor but a contraction of coordinates (of perimeter-law form), and the coordinates of the class need a representation index. In a blocking step, with four plaquettes in a row ($SU(2)$ heat kernel, diagonal definition, one step), adding a source $\lambda \cdot \frac{1}{2}\operatorname{Re}\operatorname{tr} W(C)$ to a fine loop $C$ inside the block shifts the coupling of the coarse action after one step by $\lambda f_C^2 + O(\lambda/\beta)$. $f_C$ is the projection weight of the Bundle Network Theorem (15-05) and does not depend on $\beta$ (the envelope theorem $\partial\beta'/\partial\beta_p = f_p^2$, the sum rule $\sum_p f_p^2 = 1/5 = \beta'/\beta$) Paper ; the one-loop correction is a rational $d_1/\beta$ (an inner single plaquette $f_C^2 = 2/25$, $d_1 = -17/375$; an inner $1\times 2$ loop $8/25$, $-16/125$; an outer single plaquette $1/50$, $-1/750$), and no $\log\beta$ appears Paper (one loop is formal) Computed ($\beta = 4$–$2048$, 9 digits by two routes). The newly born term is the quartic $-\lambda f_C^4/12$ of the coarse action, which also does not depend on $\beta$ Computed . As long as it fits inside a block, the source from fine loops passes one step not as a loss but as a transfer of coordinates by the weights of the bundle network, and the Lipschitz constant $f_C^2 + O(1/\beta)$ with respect to the source of one step is uniform in $\beta$ (the one-step, one-block version of the observable-source condition). Loops crossing the boundaries of blocks, four-dimensional blocks, the true $\mu_{k,W}$, large fields and many steps have not closed.

**The proposal of position-dependent couplings: the class of position-dependent couplings $\beta(x)$.** A source of plaquette-density type $J = \sum_p f(p)(1 - \frac{1}{2}\operatorname{tr} U_p)$ (with $f$ slowly varying on the scale $L_0/M$) is the Wilson action with $\beta$ replaced by $\beta(x) = \beta + sf(x)$. If the one-step map is the map of $\beta$ applied pointwise plus a gradient correction $O(|\nabla\beta|\,a_k/L_0)$, the gradient terms form a geometric series in $a_k/L_0 = 2^{-(K-k)}$ and the sum is finite Assumption . For the abelian Villain action of $U(1)$, the $\beta'$ of one step with four plaquettes in a row is the effective stiffness (Thomson's formula) of the network of the plaquette-wise stiffnesses $\beta(p)$, and putting $\beta(p) = \beta + s\,g(p)$ gives $\beta' = \beta/5 + s\sum_p f_p^2 g(p) + O(s^2/\beta)$; that is, the first order agrees exactly with the local average with the weights of the bundle network Paper (pointwise product with images dropped; the numbers agree with Thomson's formula to within $6\times 10^{-12}$ Computed ). The gradient correction starts at second order, a form one step better than the $O(|\nabla\beta|\,a_k/L_0)$ above. It cannot be used for loop-type observables (the source of the area law) and is suited to local observables.

**Rejected: peeling off the source with Hölder alone.** The road that splits the estimate into $(\mathbb{E}G^2)^{1/2}(\mathbb{E}e^{2sJ})^{1/2}$ does not close: at the fine steps the support of $J$ becomes a non-local activity spread over blocks of size loop length / lattice spacing, the one-step loss $2^{|B(X_J)|}$ cannot be paid for by a small factor per block, and it accumulates over successive steps (Table 2).

04

## Proposals for the row of background-dependent absorption and many steps

The content of the row of background-dependent absorption and many steps is in 15-07 § 06 and § 07. We write only the main points. For the Wilson action of $SU(2)$, the conditional law of

one link is a member of a one-parameter family independent of the background, and the background enters only through the local coupling and the direction of the centre Cited (a fact of the heat-bath method; bibliography not checked). For many links, the quaternion identity splits the action into "local coupling $\times$ a form independent of the background" + a linear term + the conjugation mismatch Proved here , and the screening proposal is that, sorting the score of the density by the order of $\theta$, part of $\theta^0$ disappears by recentring and part of $\theta^1$ is absorbed as a coupling derivative Assumption . In the smallest system the lowest orders of the rest that cannot be absorbed are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$, and however the threshold is chosen the sum diverges; but giving the class the background $a$ as a parameter, the rest drops to $\theta^0\beta^{-3/2}$ and the budget becomes uniform in the number of steps Reading (the numbers for two links and four in a row, conditional; 15-07 § 06).

**Put it on the bookkeeping of the transverse contraction.** The rest after absorption (the contribution of the shape coefficients cut off by the class, and the expansion of $b_p = \beta\cos(\theta_p/2)$ at $\theta^4$ and above = the source) is a small shift of the transverse coordinates, and with the bookkeeping of 15-03 § 04, if $\lambda < 1$ then the deviation is bounded not by a sum but by the supremum $\sup_k \varepsilon(\beta_k)/(1-\lambda)$ Proved here (a geometric series assuming the one-step inequality). The multi-step background-regularity condition is needed in the form "the fine angles of the image of the background field are $L^{-2}$ times the coarse angles + $O(g^2)$ of the transverse coordinates" Assumption . The transverse contraction was measured at the commutative tree level (15-05); its values for non-commutative configurations and at one loop are not yet known.

05

## Proposals for the row of the boundary of gauge re-fixing — the seam of re-fixing of the gauge fixing

**The tree-gauge proposal: there is no Jacobian of re-fixing — the direct-product decomposition of the tree gauge.** Take a spanning tree $T_B$ (with root $r_B$) in each block $B$, and split the gauge group $G^\Lambda = \prod_x G$ into the factor $G_{\text{int}} := \prod_{x\,/=\text{root}} G$ of the non-root points and the factor $G_{\text{coarse}} := \prod_B G$ of the roots (it is pointwise a direct product, so trivially a direct product). The map fixing the tree links to 1 is a section of the action of $G_{\text{int}}$; since each point has exactly one tree link heading towards the root, it is solved uniquely, the axial (tree) gauge is complete with no Gribov multiplicity, and the Faddeev–Popov determinant is constant. If block averaging is gauge covariant, $G_{\text{coarse}}$ is isomorphic to the gauge group of the coarse lattice, and the tree gauge of step $k+1$ fixes it. That is, **the gauge fixings of step $k$ and step $k+1$ each fix one of the two factors of a direct product of the same group, and there is no Jacobian depending on the field of re-fixing** Proved here (a restatement of a known structure; that the Faddeev–Popov determinant of the axial gauge is constant is a standard fact).

With this, "the Jacobian (the Faddeev–Popov determinant) enters the effective action" of 15-01 § 03 becomes empty for the axial gauge. The cost moves to two other places Reading : (i) the degradation $\|U_l - 1\| \le \ell\varepsilon$ of the small-field condition due to the tree length $\ell$ (7-05 § 05), and (ii) if a background-covariant gauge is used for the fluctuations, its ghost determinant. We have not checked in the text which gauge Bałaban used in the four-dimensional papers (in three dimensions, an axial gauge per block is introduced by the Faddeev–Popov method).

**In Lean for finite groups, on paper for $SU(2)$.** On a rooted spanning tree, a gauge transformation setting every tree edge to 1 exists, two such that agree at the root are equal (no Gribov multiplicity), and the remaining gauge freedom is only a global constant. The set of all configurations decomposes as a direct product "gauge transformations equal to 1 at the root" × "configurations equal to 1 on the tree edges", the weight of each representative is $|G|^{|V|-1}$ and does not depend on the representative (the Faddeev–Popov weight is constant), and the map between the representatives for two trees sharing a root is a bijection preserving sums of gauge-invariant functions. The average over representatives does not depend on the tree even without assuming the roots coincide. All this is in Lean: `exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, `regauge_bijective`, `regauge_sum_eq`, `slice_average_tree_indep` (assumptions: group, vertices and edges finite; functions gauge invariant; re-fixing with a shared root). For $SU(2)$ with the normalised Haar measure, the change of the variables off the tree has Jacobian 1 by left and right invariance, and the Faddeev–Popov quantity is identically 1 by a triangular structure in order of depth Proved here. That the Faddeev–Popov determinant of the axial gauge is constant and that there is no Gribov multiplicity are standard facts Cited; what we added is the formalisation for finite groups and the form of the cost.

**The form of the cost.** In the full axial tree (the parent is $x - e_\mu$, where $\mu$ is the largest direction with $x_\mu > 0$), the variable $W_{x,\mu}$ of an edge $(x, \mu)$ off the tree is the holonomy of a fundamental cycle, and that cycle is spanned by a comb of $N_{x,\mu} := \sum_{\nu>\mu} x_\nu$ plaquettes. If $\|U_p - 1\| \le \varepsilon$ (operator norm) on every plaquette, then

$$\|W_{x,\mu} - 1\| \;\le\; N_{x,\mu}\,\varepsilon \;\le\; (d-1)(L-1)\,\varepsilon \quad (2)$$

Proved here ($L$ is the number of points on a block side; the triangle inequality of 7-05 § 05 applied $N$ times). The bound is attained by a uniform abelian field (in four dimensions, $L = 4, 6$ gives ratios 8.997 and 14.986 against bounds 9 and 15) Reading. For random fields the ratio grew close to $\sqrt{L}$ (1.58→2.15) Reading. **The seam of re-fixing is not a Jacobian but the management of the small-field condition**: to use the field expansion in the tree gauge at step $k$, the block side $L$ must be taken sufficiently smaller than $\varepsilon_k^{-1}$.

**The proposal to absorb the ghost determinant: absorb the ghost determinant in the coordinates of the local couplings.** For the ghost operator $\Delta^{\mathrm{gh}}_W = d^*_W d_W$ in the background $W$ (the covariant Laplacian of the lattice, in the adjoint representation), the prospect is that $\log\det \Delta^{\mathrm{gh}}_W - \log\det \Delta^{\mathrm{gh}}_1$ can be expanded into local functionals on small-field backgrounds, with leading term $\sum_p c_{\mathrm{gh}} \|F_p\|^2$ (the coefficient of $F^2$ = the ghost contribution to the one-loop coefficient), the rest being $\theta^4$ and beyond (the source) and exponentially decaying non-local terms, with constants independent of $k$ Assumption (of the type of the heat-kernel expansion on the lattice; uniform convergence of the local expansion has not been shown). If so, the $F^2$ part merges into the shift of local couplings of the row of background-dependent absorption and many steps, $\theta^4$ and beyond go into the transverse coordinates, and the seam becomes not a new kind of loss but part of the one-loop coefficient. As a negative control, for $U(1)$ the ghosts do not depend on the background.

**The proposal to absorb the ghost determinant merges at one loop — the agreement of the two gauges.** Fix the holonomies around the blocks (the coarse variables) and take the background $W$ to be the minimiser of the action under that constraint; then the background is a critical point, and for the Hessian of the fluctuations $H$ and the linearised gauge generator $D$ we have $HD = 0$. Then, from the finite-dimensional Faddeev–Popov lemma,

$$\log\det H_{\mathrm{tree}} \;=\; \log\det\bigl(H + DD^{\mathrm{T}}\bigr) \;-\; 2\log\det M_{\mathrm{FP}}, \qquad M_{\mathrm{FP}} = D^{\mathrm{T}}D \tag{5}$$

holds Proved here (conditions: one step, one loop, the background is a critical point, $D$ is injective, $H$ is positive on $(\mathrm{im}\, D)^{\perp}$). The left side is a one-loop quantity of the tree gauge and the right side one of the background-covariant gauge ($\xi = 1$, ghost included), so **the one-loop effective actions of the two gauges are identically equal as functions of the background**, and neither the marginal coupling $\beta_0'$ nor the shape coordinates nor the background coordinates change. The decomposition of the determinant of block form, and the logarithmic identity taking the output of the lemma as an assumption, are in Lean: `det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks` (assuming $\det H' > 0$, $\det K > 0$, and the tree quantity is $\det H'/\det K$), `coeff_tree_eq_coeff_cov` (extracting linear coefficients). The lemma itself (the change of variables in the integral) is on paper. Numbers Reading ($SU(2)$, one block of $3^4$ points, commutative background, $\theta \le 1$, two routes, analytic and finite differences): the residual of (5) is $4.5 \times 10^{-13}$ or less, and the background dependence of the ghost determinant is real and nearly local (coefficient $+0.024$ per classical action of $\log\det M_{\mathrm{FP}}$, 2.4% of the coefficient $-1.003$ of the one-loop $F^2$). It cancels against the extra part of the gauge-field determinant of the background-covariant gauge. Negative controls: for $U(1)$ the ghost does not depend on the background (difference 0); dropping the ghost shifts $\beta_0'$ by 0.024; and moving the background off the critical point breaks (5) at first order in the gradient. Therefore we read that the ghost is not a new loss at the seam but one line of the bookkeeping of the local couplings when the background-covariant gauge is chosen, and that in the tree gauge that line is absent from the start. What remains is the difference proportional to the equations of motion in constructions that integrate the fluctuations around backgrounds that are not critical points, the same agreement at two loops and beyond (we expect it to be of the type of the lattice BRST identities; not checked), and the uniformity of the local expansion Reading .

**The proposal to make the seam a single matrix: make the seam a single matrix.** In the tree-level network that fixes the angles of all coarse plaquettes simultaneously (the Bundle Network Theorem of 15-05), the coarse action is a quadratic form $\theta^{\mathrm{T}} B\theta$, and the coupling between neighbouring coarse plaquettes and the mixing of the quartic map correspond to the coupling at the seam. The reading is that the question of the row of the boundary of gauge re-fixing, "are the terms at the seam a loss with finite sum?", becomes the single number "does the norm of the matrix of the quartic map fall below 1?" Reading . Under the Bundle Network Theorem the map measured in $\ell^4$ exceeds 1/16 in the coarse direction (the largest value found in four dimensions is about 0.15), and we have no upper bound uniform in the volume Reading (15-05). In the composition of three steps, $\mu_3 \le 0.20$ at every step (tree level, commuting, Reading ; measured up to the kernel of step 4, with Stein^{1/4} of the step differences 0.300, 0.180, 0.084, 0.031, and $\le 0.163$ by perturbation for $k \ge 4$), and the three-step value is 0.099 ($\lesssim$0.12 with the truncation). The sup of the linear ratio for a three-colour field equals the one-colour sup

(paper: the spherical average $E_s(s\cdot v)^4 = |v|^4/5$; Lean: ColorAverage, with a finite weighted design), so the cone values measured with one colour carry over to three-colour fields unchanged. The non-commuting tree correction: the first-order coefficient of the cone ratio R depends on the choice of parameters for the coarse field (the coarse gauge components) and has no meaning as a quantity. At the maxima of the linear ratio $R_0$ (three points on $8^4$ and $12^4$, almost one-colour configurations) the first-order coefficient is below $10^{-5}$, so the non-commuting correction to sup R begins at second order (computed; tree level, one step). The second-order coefficient at the maxima is positive but almost entirely the commuting non-linearity (cos); the non-commuting part is negative (computed; three points, one step). The one-step map of the coordinates of W (quadratic forms with differences, dimension 6), with the local representative corrected at the zero mode, has spectral radius 0.363 on a basis of 60 types (the rest line up at 0.25); the three-step composition has spectral radius 0.042, weighted $\ell^2$ 0.055–0.125 and $\ell^1$ 0.465 (computed; tree level, commuting; the tail outside the basis is bounded up to $D \leq 4$).

**The wall for a road using class functions alone.** On a road whose only coordinates are the class angles of the holonomy of each coarse plaquette (Migdal–Kadanoff is one), the coarse Bianchi identity (the product of the holonomies of the six faces of a coarse cube = 1) needed to reconstruct the coarse gauge field on the four-dimensional hypercubic lattice cannot be written. In the non-abelian case it can only be written once there is a common base point and connecting paths, and with class angles alone the relative orientation of neighbouring faces is lost. We read this as: **a road using class functions alone cannot go beyond the hierarchical-lattice approximation** Reading .

**Can Haar invariance replace re-fixing?** In the setting of the Lemma on One Step with Background-Independent Fluctuations, if the law of the fluctuations is conjugation invariant on each link, the decomposition $U = WY$ is gauge covariant and no gauge fixing at all is needed. But this is limited to renormalisation of the type that "gives" the law of the fluctuations by convolution with heat kernels (a Polchinski flow on the group), and convolution with heat kernels does not reduce the degrees of freedom, so it cannot replace the re-fixing of four-dimensional blocking, which produces block variables Reading .

06

## Dependence among the four rows — the row of background-dependent absorption and many steps is the core

The four rows are not independent. Written in the direction "when the left closes, the right becomes lighter", they give the following diagram Reading (an estimate of the logic between the proposals).

```
Row of background-dependent absorption and many steps (absorption of η: background dep
  ├─→ group version of the exponential moment of fluctuations and group version of exc
  │      └─→ large-field rarity condition of the row of the remainder including large
  │             └─→ large-field contribution to the coupling of the row of the remaind
  ├─→ coordinates of the class-closure condition (β, couplings by loop shape, shape co
  │      └─→ observable-source condition (proposal to take the source as a function of
  └←── ghost determinant of the row of the boundary of gauge re-fixing (proposal to ab
         ↑
       the row of the boundary of gauge re-fixing has no Jacobian (tree-gauge proposal

Irreducible core:
  small-field contribution to the coupling = Loc + Taylor truncation + small-field cut
                the first three have the same form as in the sources for flat fields (
                absent from the sources: W^gauge and the group version of its local ca
                  W^gauge is explicit from one loop (13/384 for two blocks); cancellat
                  the non-local tail type is decided by the definition of the coarse v
                  recursion of W (one row) = contracting image of pair types + new ter
                → may weaken to "boundedness of the θ⁴ coefficient" via the proposal o
  multi-step background-regularity condition = regularity of the background field (fin
```

What to read from the diagram Reading : the large-field rarity condition and the large-field contribution to the coupling of the row of the remainder including large fields, the observable-source condition, and the ghosts of the row of the boundary of gauge re-fixing all have a form that can be written, once the coordinates of the absorption of the row of background-dependent absorption and many steps are fixed, as a corollary of it or as a shift of those coordinates. What remains independent of the row of background-dependent absorption and many steps is two things, the small-field contribution to the coupling and the multi-step background-regularity condition: the former is the unpublished part that Bałaban's series needed in order to remove the assumption (the coupling stays small), the latter is the multi-step version of the regularity of the background field.

**Splitting the small-field contribution to the coupling one step further — the same form as known in the sources.** We split the one-step small-field effective action $S'_h(\Phi)$ (threshold $h$ on the fluctuation side, threshold $h_0$ on the background side, margin $h_0 \le q\,h$, $0 < q < 1$) into the local part $\mathrm{Loc}_{2k}$ (vacuum, marginal coupling $\beta'_0$, shape coordinates) and the remainder $R := (1 - \mathrm{Loc}_{2k})S'_h$. The marginal coupling is restricted to one number $\beta'_0$ (the long-wavelength value of the coefficient of $F^2$), and the rest of the quadratic form of the Bundle Network Theorem (non-local in the coordinates of $\theta$) is put into the shape coordinates. This splitting has the same form as in the sources for flat fields (BBS's $\mathrm{Loc}$; $\mathrm{Loc} = \mathrm{Tay}_4$ in the hierarchical model, eq. (5.2.10)) Cited , and that the local coefficients are uniquely determined and the remainder goes to the higher-order side is algebra in a finite basis Proved here (Lean: `loc_split_unique`, polynomials in one variable $s = \psi^2$, of type $k = 2$). The statement has the following form Assumption (not shown in four dimensions): there are $\beta \ge \beta_0$ and constants $C, \kappa, r_0, c > 0$ independent of the size of the lattice and of the step such that

$$\beta'_0 = 2^{d-4}\beta - b + r, \quad |r| \le \frac{C}{\beta}; \qquad \|R_X\| \;\le\; Ce^{-\kappa d(X)}\Big[\Big(\frac{h_0}{r_0}\Big)^{2k+2}\big(1 + \beta\tau h_0^2\big) + e^{-c(1-q)^2h^2\beta}\Big]$$

holds ($b$ is the one-loop constant of one step, $\tau$ is the BCH term of the non-abelian tree, which is 0 for the tree restricted to the abelian case). The remainder splits in two by the identity $R = (1 - \mathrm{Loc})S'_\infty + (1 - \mathrm{Loc})\Delta_{\mathrm{cut}}$ into the effective action without the cut $S'_\infty$ and the part due to the cut $\Delta_{\mathrm{cut}}$ Proved here . Since the indicator function breaks analyticity, the analyticity argument is applied only to the former, and the latter is paid separately in probability.

- **The Taylor truncation** $(h_0/r_0)^{2k+2}$. Under analyticity with the background extended to the complex range $|\theta_P| \le r_0$ and a bound independent of $\beta$ (the analyticity condition at complex backgrounds), the Cauchy estimate gives $\|(1 - \mathrm{Loc}_{2k})S_X\| \le Me^{-\kappa d(X)}(eh_0/r_0)^{2k+2}/\big(1 - (eh_0/r_0)^2\big)$ $(h_0 < r_0/e)$ Proved here (assuming the analyticity condition at complex backgrounds; the assembly of the series is in Lean: `taylor_tail_bound`, assumptions $0 \le s < \rho$, $|a_m| \le M/\rho^m$). **This has the same form as Lemma 7.5.3 of the sources for flat fields (of type** $(h_+/h)^{k+1}$**)** Cited . Starting from the heat kernel with four plaquettes in a row, $r_0 = \pi$ (the distance to the conjugate point $-1 \in SU(2)$, the position of the singularity of $\log(w/\sin w)$), independent of $\beta$ Proved here , and the remainder is proportional to $h^{6.00}$ and independent of $\beta$ over $\beta = 8$–$1024$ Reading ($SU(2)$, four in a row, heat kernel, one step, agreement of the coefficients by two routes $10^{-11}$). For $U(1)$ this term is at the numerical floor (3 to 6 digits lower), and the term in $h^6$ comes from the curvature of the group Reading .
- **The small-field cut** $e^{-c(1-q)^2h^2\beta}$. If the fluctuations are a product of heat kernels per link, it is bounded by the Heat-Kernel Fluctuation Rarity Lemma ( § 02) Proved here (under the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations). With four in a row, the contribution of the cut is a function of $h^2\beta$ alone; the rate is $c \approx 0.24$ without margin, and putting the background inside the threshold, $q \le \frac{1}{2}$, makes it $c(1-q)^2 \approx 0.75$, about 3 times larger Reading (the same form for both $SU(2)$ and $U(1)$). The balance when the steps are stacked is only $c_0^2 \ge 1/(c(1-q)^2)$ at $p(g) = c_0(1 + \log g^{-1})^{1/2}$ (1.33–4.2 with the rates for four in a row) Proved here (arithmetic, using the rates for four in a row).

The form "the one-step remainder is of size $C|B|(h^2\beta)^k$" does not hold. $k$ appears only in $(h_0/r_0)^{2k+2}$, and $h^2\beta$ only on the side of the cut (the larger, the smaller) Reading (from the numbers for four in a row and the form on paper). The verdicts on the four components are as follows.

- (i) The law of the fluctuations does not depend on the background: it holds for heat kernels per link (the Lemma on One Step with Background-Independent Fluctuations) and for the lift with four in a row (the covariance of the fluctuations does not depend on the background) Proved here , and does not hold for four-dimensional blocking (the row of background-dependent absorption and many steps). What the small-field contribution to the coupling needs, however, is its weak form "the real part of the Hessian of the fluctuations is uniformly positive on complex backgrounds".
- (ii) The quadratic part is the leading term of the Bundle Network Theorem + a local one-loop term: assuming exponential decay of the covariance $C_W$ of the constrained fluctuations Proved here . In a box of $2^4$, $\beta' = 0.442260\,\beta - 0.591$ Reading (abelian saddle point, one loop).

- (iii) Quartic and higher: the lift is exact only for four in a row; for four-dimensional blocks an estimate of the Laplace remainder on complex backgrounds is needed (the content of the analyticity condition at complex backgrounds).
- (iv) Locality: finite range is replaced by exponential decay, and **at the quadratic level an explicit irrelevant term arises between blocks**.

**The row absent from the sources.** The term of (iv) is the row that remains after the splitting Reading (from comparing the text of Appendices A.1 and A.5 of the sources for flat fields with one sentence of Bałaban's preface; Bałaban's main text has not been checked). In the sources for flat fields this is $I_j = e^{-V_j}(1+W_j)$, $W_j = \frac{1}{2}(1-\mathrm{Loc})F_{w_j}(V(b),V(\Lambda))$ ((A.1.3)–(A.1.4)), and the local cancellation, in which the sum of the quadratic contributions is 0 up to cubic errors, is at (A.5.3)–(A.5.4) Cited. **The sources themselves describe this place as "we sketch an argument" and send the main body to another reference** Cited. What is needed on the group is its group version: to build, with the covariance of the fluctuations $C_W$ in the background field,

$$W^{\mathrm{gauge}} = \tfrac{1}{2}\,(1-\mathrm{Loc})\,FC_W\left(V(b),V(\Lambda)\right) \quad (4)$$

and its local cancellation — and to show that the quadratic contribution to the marginal coupling fits within "the one-loop constant of one step $b + O(g^2)$" Assumption. We read this as corresponding to what Bałaban's unpublished basic inequality calls "the computation of the second-order perturbation" Reading. This row splits into three.

- **Definition of the inter-block term**: the definition of $W^{\mathrm{gauge}}$. A formula on a finite lattice, needing no estimate (algebra).
- **Local cancellation**: an identity for Loc under gauge invariance (algebra; expected to be of the type of a Ward identity).
- **Cubic-remainder estimate**: the rest is $O(g^3)$, uniformly in complex backgrounds and volume (analysis: positivity of the Hessian on complex backgrounds, decay of $C_W$, contraction of the shape coordinates uniform in volume).

**The four-dimensional $O(1/\beta)$ layer of the local cancellation.** In four dimensions we assembled one step with block ratio 2 in every direction. The fine box $4^4$ is split into 16 blocks $2^4$, a soft heat-kernel constraint is put on the coarse links of length 2 joining the block centres, and the $O(1/\beta)$ quadratic term is taken on shell (conditioned on the coarse variables held fixed) Computed. The zero directions are exactly as many as the gauge directions (1827); the on-shell quadratic term agrees in two tree gauges to $2\times10^{-9}$ (the precision of the finite differences), and agrees with the value in the background-field gauge (with the Faddeev–Popov determinant) to relative $2\times10^{-7}$. That the on-shell quadratic term does not depend on how the gauge is taken has been checked directly in four dimensions Computed. Splitting the quadratic term across blocks by the positions of the variables and taking Loc up to second order at the midpoint of each pair, what remains is, in wave numbers with the block spacing as unit, 0.24–0.86% of the whole pair at twice the wavelength and 3–16% at the shortest wavelength of the next step. This is of the same order as for the neighbouring pairs of the two-dimensional plate (0.6%, 11%) Computed. What remains shrinks as the fourth power of the wave number

(dimension 8) for pairs point-symmetric about their midpoint and for sums across blocks at point-symmetric positions (exponent 3.87 for pairs touching at a vertex, 3.93 for pairs touching across a face placed at the centre of the box, 3.89 for the sum of all pairs across blocks; approaching 3.95–3.98 each time the wave number is halved). For individual pairs that are not point-symmetric, the exponent drops to 3 at small wave numbers, and terms of dimension 7 remain with small coefficients Computed . On a translation-symmetric lattice, the value per volume of the sum of all pairs across blocks does not depend on the phase of the plane wave and is an even function of the wave number. Therefore, after taking **Loc** up to second order, no term of dimension 7 remains, and what remains starts at dimension 8 Paper . The singular values of the pairs across blocks are degenerate according to the multiplicities of the representations of the stabiliser of the pair. For pairs touching at a vertex, the three standard components of the permutation representation of the four directions are equal, and they do not form a product of a few graded factors Computed . In an open box, the ratio of what remains when the whole is measured by **Loc** at one point grows with the size of the box (from 7% to 28% at the shortest wavelength of the next step); this is not due to far pairs, but to the tails of the links far from that point growing fat in a Taylor expansion around one point Computed . Splitting the quadratic term across blocks into pairs of fluxes of coarse faces (a gauge-invariant split) and taking **Loc** of each pair at its midpoint, what remains after taking **Loc** up to second order is about 2% of the whole at the shortest wavelength of the next step and about 0.1% at twice the wavelength, independent of the size of the box (exponent 3.98–4.00, dimension 8) Computed . If the centre of the block is taken as the single point of **Loc**, the coarse faces lie away from that point and what remains is about 30%. The books of BBS require the point of **Loc** of each term to be taken inside the support of that term Reading . On the side of one row and the two-dimensional plate, the algebra of the decomposition of pairs across blocks — the rational arithmetic of the model of one row (Lean: LocalCancellationOneRow), the identity splitting one product into block-wise squares and the square of the difference (Lean: RankOneRemainder), and the block deflation that makes a remainder vanishing from both sides on uniform-field and gradient test fields (Lean: BlockDeflation) — has been checked in Lean (all are identities with real coefficients over finite indices; the sizes are on the side of the numbers). Conditions: a small coarse lattice (at most 24 coarse faces), one step, the on-shell $O(1/\beta)$ quadratic term, finite-difference precision of order $10^{-9}$, exponents judged from four wave numbers, exponents not measured in a periodic box.

**What the row absent from the sources contains (revised).** Written in the block-spin coarse variables, the quadratic term between blocks $W^{\mathrm{gauge}}$ starts at O(1) from one loop (13/384 for the row of seven), but this is an artefact of the coordinates. Taking Loc (monomials plaquette by plaquette) in the coordinates of local polymers of the smeared (fine) background field $\phi^* = F(\Phi)$, the O(1) quadratic term at one loop goes into the plaquette-wise sum (for SU(2): 5/12 on the inner faces, 1/4 on the end faces) and into pairs inside the same block, and 13/384 is its pull-back (computed, rational). The genuine remainder across blocks vanishes at O(1) and starts at O(1/β) when the blocks form a tree, and when the coarse variables carry no conserved quantity it decays in distance as e^{−κd} ($\kappa = \ln(5+2\sqrt{6}) \approx 2.29$, the rate of the fluctuation covariance) (U(1) and SU(2), computed). In systems with cycles an O(1) quadratic term across blocks does appear, but for the smallest cycle all of it is a constant times the square $\Theta^2$ of the holonomy around the vertex (SU(2), rational; 0 for U(1)), a small term of $(\partial F)^2$ type that vanishes for uniform fields ($10^{-3}$ of the plaquette-wise term) and is localised on small sets. In

form, the core row therefore returns to that of the sources — "local polynomial in the fine background field + a remainder localised on small sets" — but in four dimensions the remainder is not small (see the caveat below). What remains is the cubic-remainder estimate (the volume-uniform estimate of the remainder and the cancellation on small sets) and the treatment of the remainder at fourth order and above (the O(1) one-loop bubbles) Reading . Conditions: one row and a two-dimensional plate; one step; formal Laplace; the O(1/β) part for SU(2) and the quartic remainder have not been done on the plate. On the torus plate ($4\times4$ to $10\times10$, rational), pulling the one-loop O(1) quadratic term in the coordinates of the smeared background field back to the coarse field a splits it into $\Sigma_B\,\rho\,\Sigma a_B^{\,2}$ plus a remainder of $(\partial a)^2$ type; the density ρ takes the same value on every block ($\rho_L$ = 0.080253, 0.080704, 0.080710, 0.080710 for L = 4, 6, 8, 10) and agrees with the symbolic $\hat{k}(0)$. The term $u_B u_{B'}$ of neighbouring pairs of blocks (coefficient $1.6\times10^{-6}$) always splits, by an identity, into a uniform part and a gradient part, and the uniform part, $10^{-5}$ of ρ, goes into ρ (a renormalisation of the plaquette-wise coefficient). On an open plate ρ shifts in the boundary layer (+40% at the corners, +15% at the edges), but this is the boundary effect of the background $\bar{\phi} = F\cdot1$ and has the form absorbed by a block-wise counterterm P(B). That is, the marginal part is a renormalisation of a uniform density and the remainder across blocks is of $(\partial a)^2$ type (computed, rational; conditions: two dimensions, one step, formal one loop; the closed form of ρ not yet obtained). The above are results for one row and for the two-dimensional plate. In four dimensions, even with the same fine definition of Loc, the remainder of pairs of plaquettes belonging to different blocks does not become small. Even with the choice that minimises the remainder (constrained least squares), from $4^4$ to $12^4$ (block average along the covariant path) the remainder across blocks is 0.063–0.071 of the plaquette-wise term (largest ratio) and 0.104–0.109 (Frobenius), independent of the volume (computed). The $10^{-3}$ of the plate is a smallness that came from plaquettes not sharing fluctuations in two dimensions. In four dimensions, therefore, the statement of the row absent from the sources — "$W^{\mathrm{gauge}}$ is O(1) from one loop" — stands as it is; what carries over in the coordinates of the smeared background field is the form (plaquette-wise polynomial + pairs inside the same block + a remainder between blocks) and the bookkeeping method (fine Loc, counterterms on small sets), not the smallness of the remainder. What sets this size is how the average is taken: about 7% both for the Bałaban-type average over parallel lines and for the covariant path, and down to about 1% for a single straight path, but straight paths do not compose into an average and cannot be used for the books (computed, cited). What sets the rate of decay is the pair of average and gauge fixing: with a Feynman-type fixing (odd points) the pole of the kernel of the multiplier gives 0.95–1.09 per block, and with an axial-type fixing on a tree inside the block 2.2–3.5 (expected from the roots; reading). The one-loop quadratic term pulled back to the coarse field agrees for every pair of average and fixing (relative $2\times10^{-15}$), and the difference between the pairs appears only in the off-shell part (computed). For every pair, the books with the volume weight of BBS (needing (η−1)κ > L^d log 2) fall short by more than 8 orders (Lean: LargeSetWeight (bbs_weight_vs_decay, summable_shell_iff, card_shell); the form of the inequality, and that the series of the weight A converges ⇔ A < e^κ when the number of shells is polynomial); with the tree-length weight of Bałaban–Dimock (e^{κ d_M} and M-cubes) it is of a form that goes on the books for $M \geq M_0$, but $M_0$ remains an estimate of about 30–55 blocks (about 2000 with the upper limit of the combinatorial constant) (reading). Hence the form of the plate (the return to the form of the sources) does not hold in four dimensions for any of the pairs examined. The value for a uniform field (the $q \to 0$ limit of

the coefficient of $F^2$) is not determined by the numbers up to $10^4$ (computed). The distribution of the small-set terms, the row sums and the decomposition of Loc (the (1−Loc) side vanishing for a uniform field) are in Lean: SmallSetRowSum; the affine recursion for the contraction $\mu_k \leq \bar{\mu}$ per step is in Lean: ThetaSquareOneStepAffine (on the plate the one-step contraction is ½, sup $\leq 1/10080$).

**The remainder pulled back to the coarse field (the core).** The size of the remainder also changes depending on whether the one-loop quadratic term is viewed in the fine field or pulled back to the coarse field. In the quadratic term pulled back to the coarse field (an on-shell quantity, independent of the choice of gauge fixing), the pole of the kernel of the multiplier cancels algebraically by the identity $\Lambda_1 K = -Y$ Paper , Computed , and the remaining rate of decay becomes 1.76 per block, the mass of the physical mode of the fluctuations (a double pole without images) Computed . If the local part is represented by single plaquettes only, the remainder is about 10% (Frobenius 25–40%), but if the local part also takes in a plaquette together with the two plaquettes touching it (rectangles, chairs and plaquettes touching at a corner = improvement terms of dimension 6, with coefficients 2–3% of the plaquette-wise one), what remains is about 1% (Frobenius about 6%), independent of the volume ($8^4$–$12^4$) Computed . That is, nine tenths of the remainder seen in the fine field are the coefficients of finitely many improvement terms, and on the books they can be carried as couplings that shrink by $L^{-2}$ per step Paper (counting dimensions). The remaining about 1% are pairs of plaquettes that do not touch, about 10 times away from the 0.1% of the plate. Moving the non-touching pairs into improvement terms type by type as well, the remainder drops by a factor of about 0.6–0.9 per type, and falls below 0.1% with about ten types Computed . Further, if the block average is made a 4 : 6 mix of the covariant path and the straight path, the root of the physical mode rises to 2.54 per block, and the core remainder after counting single plaquettes and about ten improvement terms in the local part is about 0.05% of the plaquette-wise term (Frobenius about 0.5%, $10^4$–$12^4$), on a par with the plate (0.1%, 2.29) in both size and rate Computed (conditions: the multi-step composition of the mixed average tends to a self-similar average (information dimension 1.92) different from Bałaban's average (the uniform average over a cube), and estimates of regularity on it have not been checked. The multi-step flow of the coefficients of the improvement terms has been checked up to two steps, and the one-loop shift shrinks by a factor of 0.3–0.5 per step).

**In the zero-sum system, this row is empty.** A system with only one block (four in a row, a single box) has $W = 0$ by the zero-sum condition, as in the hierarchical model (the type of the sentence in Appendix A.1 of the sources) Reading . It can be read that the Shape-Remainder Theorem for Four Plaquettes in a Row of 15-06 could be a complete example in that model not because the model is easy but because this row was empty. Conversely, **this row cannot be tested with the numbers from the smallest systems**. Testing it needs a system in which two blocks are connected by the covariance of the fluctuations (the question "smallest system that is not zero-sum"). The answer for that system is the next paragraph.

**The smallest two-block system — $W^{\text{gauge}}$ starts explicitly from one loop.** In seven in a row, made by placing two copies of four in a row side by side so that they share the boundary plaquette ($SU(2)$, heat kernel, one step; the coarse variable is defined by "the two diagonal loops are equal"), we measured the quadratic coefficients of the coarse action: the coefficient

between the blocks has, besides $-\beta/32$ at tree level, a one-loop $O(1)$, whose value approaches $13/384$ (difference $7.9 \times 10^{-10}$) Reading ($\beta = 32$–$512$; the coefficients agree to $10^{-10}$ by two routes). The $W$ of the sources for flat fields starts at second order, but with this coarse variable it appears from first order (one loop). Its origin is not the non-commutativity of the group but **the non-locality of the background map** built from the block constraints: the commutative Wilson action of $U(1)$ (the action of cos) also gives $51/4096$, and for the Gaussian action (Villain) everything is 0 Reading . The one-loop term of $SU(2)$ can be written explicitly as the pull-back by the background map of an expression local in the fine background (the determinant coming from the prefactor of the heat kernel and the Laplace–Beltrami of $S^3$) Proved here (one loop of Gaussian minimisation and the Laplace method; the asymptotic expansion is formal, and no estimate of the remainder is attached), and the coefficients for the two blocks are $K_{AA} = \frac{7\beta}{32} - \frac{67}{384}$, $K_{AB} = -\frac{\beta}{32} + \frac{13}{384}$. Its local part (the row sums) is absorbed into the marginal coupling $\beta_0'$, and the rest (the square of the difference of the coarse curvatures, of dimension 6 in four dimensions) goes into the shape coordinates. The rest after putting in the first order explicitly is $O(1/\beta)$ (coefficient $\approx -1/384$, from a fit), and it is this part that needs local cancellation Reading . The non-commutative tree term is the quartic $-\frac{\beta}{768}|a \times b|^2$ and vanishes on uniform fields ($a = b$), so it does not touch the local couplings and enters only on the side of $W$ Reading . The decomposition of quadratic forms into local and non-local parts and its uniqueness, and the absorption by linearity of the row sums, are in Lean: `loc_split` ($K$ is symmetric), `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_unique`, `one_loop_numbers` (only the arithmetic of substitution). With this, the question "smallest system that is not zero-sum" has the form of its answer in the smallest system: the content of the row absent from the sources divides into "$W^{\text{gauge}}$ starts with an explicit one-loop formula from first order" and "the local cancellation of the $O(1/\beta)$ that follows", and for the former the formula and the numbers agree in this system. Conditions: seven in a row, the diagonal definition, one step, quadratic coefficients, $\beta \geq 32$. Nothing is said about one step on the four-dimensional lattice or about volume-uniform estimates.

**Passing to the second step — the group version of the recursion of the coordinates of $W$ (one row).** In the same one-row model, we added the non-local terms born in the first step (the type of the difference of two neighbouring plaquettes $|x - y|^2$, and three quartic types containing two or more differences) to the fine action, ran the second step, and looked at how they map onto the coarse field, by numerical integration of the integrand without approximation ($\beta = 32$–$512$) and by rational numbers with the background substituted. The non-local terms of the second step split into "the image of the first-step terms (the non-local part of the conditional expectation of the fine terms given the coarse field)" and "the terms newly born in the second step" Proved here (a first-order Taylor formula in the non-local terms). The image closes on the same four types and local terms (automatic for the two-block output by counting invariants Proved here ), and the quadratic type maps to $15/128$ times, the quartic types to at most $5/256$ times Proved here (tree rational numbers; the characteristic polynomial of the quartic block is in Lean: `quartic_block_charpoly`), Reading (the correction of $1/\beta$). Measured in the ratio of coarse couplings ($\beta' = 3\beta/16$), quadratic is $5/8$ and quartic $5/48$ or less, and all contract step by step. On the other hand, the terms newly born at each step are

numbers independent of the step ($1/64$ at tree level, $9/384$ when the one-loop coefficient of $\psi^2$ is included in the coupling), so the non-local terms are not driven to 0 but **are drawn geometrically towards a fixed value** Proved here (the one-loop formula is the formal leading term of Laplace; the bookkeeping is in Lean: `affine_iter` ($\lambda \neq 1$), `affine_bound_uniform` ($0 \leq \lambda < 1, 0 \leq C$), `new_term_gamma`). In the continuum field of the sources this is the part written by additivity, "the non-local terms of the previous step remain without changing the kernel, and the part of the new step is added" ($w_j = C_1 + \cdots + C_j$ of (A.1.4)–(A.1.5) and the identity $E_C$ $\theta F_w(A,B) + F_C(A_+,B_+) = F_{w+C}(A_+,B_+)$ from Exercise 2.2.4) Cited ((A.1.4)–(A.1.5) checked in the text; the identity from Exercise 2.2.4 has not been confirmed in the text Reading ); on the group it becomes "a contracting image + new terms independent of the step" Reading . The image of the quadratic type produces $|a \times b|^2$ at $-13/768$ from the BCH of the group (a numerical reading, no derivation on paper). Conditions: $SU(2)$ heat kernel (with $U(1)$ Wilson as a control), one row, the diagonal definition, one step of the two-block seven twice, the four types of neighbouring pairs, first order in the non-local terms, up to fourth order in the coarse field. The $\ell^4$ cone $|Q(\phi)| \leq c \| \phi \|_4^4$ contains every quartic many-body term whose coefficients have finite $\ell^1$ at each point (Lean: QuarticCone), so the three-step contraction gives the four-dimensional version of the "contracting image" without counting invariants (tree level, commuting). Flowing the coordinates of W (a quadratic form in dimension 6) step by step gives 0.38 after three steps (the value for a particular input, not an operator norm) Reading .

**The many-body tail — the form in which the one-row recursion closes.** When three or more super-blocks are lined up, the image of the non-local terms contains quartic terms involving three coarse fields at once (six of them, such as $|a|^2(b \cdot c)$ and $(a \cdot b)(a \cdot c)$), and the very form "pair types" is not preserved from step to step. In a row with coarse variables that give the boundary plaquette half to each side (no conserved quantity), we measured these coefficients for each of three sources (the image of the previous step's quartic pairs, the image of the quadratic pairs, and the tree and one-loop terms newly born at that step): all decrease as $e^{-\kappa d}$ in the end-to-end width $d$ of the triple, and $\kappa = \ln(5 + 2\sqrt{6}) \approx 2.29$ was the same as the rate of the background map Reading (two routes, rational and floating point, $5 \times 10^{-17}$; Laplace on the group and full quadrature with finite $\beta$, $4 \times 10^{-13}$ at tree level and $4 \times 10^{-10}$ at one loop; for the tree and the image of quartic pairs the prefactor is constant, for one loop and the image of quadratic pairs the prefactor is linear in $d$; $L \leq 8$, $d \leq 7$). The three-body sum containing one block is independent of the volume, and the three-body image contracts further to $5 \times 10^{-3}$ or less Reading (tree step, linear background). The bookkeeping is in Lean: `three_body_sum_le` (assuming $0 \leq \rho < 1$, $|q(d)| \leq C\rho^d$), `pair_tail_uniform` (assuming $0 \leq \lambda, \mu < 1$ and two recursive inequalities). Going back to a definition with a conserved quantity, the same three-body coefficients become flat in distance, and the origin is neither the group nor the one loop but the definition of the coarse variables. Therefore, **in the one-row model, the recursion of the core row closes up to fourth order in the form "a contracting image of the pair types + new terms independent of the step + three-body terms decreasing exponentially in distance"** — this is the one-row version of the row absent from the sources Reading . However, the three-body coefficients at $d = 2$ are the same size as the coefficients of neighbouring pairs (the head is not small), so short three-body terms must be included in the set of types from the start. Conditions: one row, definition without a conserved quantity, heat kernel, one step, up to

fourth order, tree and one loop, first order in the previous step's non-local terms. Four dimensions (three-body invariants are many, as pairs of orientation and position), four-body terms, and the self-image of three-body terms at the one-loop level have not been measured. The quartic terms newly generated at each step at one loop are +0.038 of $\| \theta \|_4^4$ on the $6^4$ lattice (with the quadratic part subtracted in Landau gauge; +0.046 raw, +0.014 ultra-local; subtracting the quadratic term requires a definition of the gauge) Reading . One loop also generates cubic terms (of the type of the three-colour triple product $\varepsilon_{abc}\theta_a\theta_b\theta_c$, $f_3/\| \theta \|_3^3 \approx 0.02$, zero when restricted to two colours); since both $\pm\theta$ lie in the cone they do not cancel in the sup, and a separate step bounding the cubic cone $|C(\theta)| \le c_3 \| \theta \|_3^3$ is needed (non-commuting). The cubic AM–GM inequality, the three-colour triple product $|u\cdot(v\times w)| \le (|u|^3+|v|^3+|w|^3)/3$ and the pull-back of the cone are in Lean: CubicCone. The three-step $\ell^3$ bound is 0.57 / 0.30 for two kernels, below 1 at every step (computed, tree level).

**The non-local tail type is decided by the definition of the coarse variables.** What the volume-uniform estimate of the cubic-remainder estimate needs is that the weights of $(1-\mathrm{Loc})W^{\mathrm{gauge}}$ decrease fast enough with distance. Measuring on a chain of $L$ copies of four in a row, a two-dimensional slab, three-dimensional columns $2\times 2\times L$ with the Bianchi constraint, and periodic lattices, the tail type was decided not by the group nor by the one loop but by how the fine variables straddle between coarse variables Reading ($SU(2)$, heat kernel, one step, quadratic coefficients of the coarse field, commutative background).

- **Flat**: defining the coarse variable by "the two diagonal loops are equal", the conjugacy class of the shared plaquette is conserved along the chain, and the one-loop coefficient between blocks is $(4L+5)/(6(3L+2)^2)$ independent of distance ($13/384$ at $L=2$). For $L=2$–$8$ and $\beta=32$–$512$ the difference over distances 1–4 is $4\times 10^{-12}$ relative Reading . Adding the conserved quantity to the coarse variables removes the non-locality Proved here . The single global constraint of total flux 0 on a periodic lattice likewise produces a flat $-1/(2N)$.
- **Exponential**: giving the boundary plaquette half to each of the two blocks (sharing it by weights), the conserved quantity disappears, the coefficient is $C(r+\alpha)e^{-\kappa r}$, and $\kappa$ is the tree rate $\ln(5+2\sqrt{6}) \approx 2.29$ (2.43 on the two-dimensional slab). The representation that writes the coarse faces in the variables of coarse links (promoting Bianchi to the coarse variables) is also exponential on columns, slabs and in three dimensions throughout; on columns it is $\ln(9+4\sqrt{5}) \approx 2.887$, and one loop has the same rate with prefactor $r$ Reading (chain $L \le 21$, $r \le 8$; columns tree $L \le 20$, one loop $L \le 16$).
- **The constraint-propagation type**: in the representation that takes the holonomies of the coarse faces directly as coordinates (the pseudoinverse representative), the Bianchi constraint is held implicitly, and the tail is decided by the propagation of the constraint. On a column with open faces it is exponential $e^{-1.763r}$ (rate $\ln(3+2\sqrt{2})$), on a periodic slab without boundary $r^{-2}$, and in three-dimensional periodic it is a power (effective exponent around 3.3) Reading (tree level).

The distance-weighted sum $\sum_r r|Q(r)|$ grows with the volume if flat, saturates independently of the volume if exponential (0.0447 on the chain, 9 digits on the column), and if a power $r^{-p}$ it is uniform only when $p-d>1$, taken together with the shells of dimension $d$ Proved here

(arithmetic; Lean: `flat_not_uniform_geometric` (for the flat formula), `weighted_sum_uniform_linear` (assuming $|q(r)| \le C(r+1)\sigma^{2r}$, $0 \le \sigma < 1$), `power_sum_uniform` ($s > 1$), `power_sum_unbounded` ($s \le 1$)). The counting rule is as follows Reading : conserved quantities made by sharing through $\delta$ and global constraints are flat; sharing by weights and constraints promoted to coarse variables are exponential (the rate is the transfer rate of the fine field); coordinates holding local constraints implicitly are of the constraint-propagation type (exponential if there is a boundary; a power if there is no boundary and it spreads in two or more directions). For four-dimensional blocking, we read that if the coarse variables are taken as "coarse links plus global tags (the flux through the torus, the magnetic flux of the centre $\mathbb{Z}_2$, the Polyakov direction)", the local part is exponential, and the global tags go onto a separate bookkeeping as flat terms of small rank Reading . In four dimensions only the tree level was measured (next paragraph).

**The coordinates of the cubic-remainder estimate — coarse links plus global tags are exponential at one loop in three dimensions without boundary and on the four-dimensional tree.** Writing one step with all coarse faces fixed at once (commutative background, quadratic coefficients of the coarse field) in a periodic box in three dimensions without boundary, splitting the coarse variables into "coarse links" and "global tags (the 3 total fluxes through each plane = the harmonic part of the coarse faces)", the one-loop non-local weights of the $SU(2)$ heat kernel are $(r+\alpha)e^{-\kappa r}$, $\kappa = 3.094$ (agreeing with the tree rate $\ln(11+2\sqrt{30}) = 3.089$), and the distance-weighted sum saturates independently of the volume ($2.7 \times 10^{-6}$ relative at $L = 12 \to 16$; for all partner rows the increments are 0.172, 0.024, 0.0019) Reading ($O(1)$ of $\beta \to \infty$, coarse $2 \times 2 \times L$, $L \le 16$, two routes, analytic Hessian and difference Hessian, differing by $\le 10^{-10}$). The tags have no cross terms with the coarse-link part, and the space of faces satisfying Bianchi splits uniquely into the direct sum of the curvature of the coarse links and the tags Proved here (Lean: `norm_sq_add_harmonic` (assuming $D^\dagger b = 0$), `range_sup_tags` (assuming $d_2 \circ D_C = 0$), `decomp_unique`, `finrank_ker_eq`). The global direction newly appearing at one loop (the Polyakov direction) became exponentially smaller, by about $1/75$ each time the coarse length increases by 2. Writing the same step in coordinates with the tags removed (the holonomies of the coarse faces), the projection of the tags produces a flat $-1/(2N)$ to relative $10^{-12}$. On the tree of the four-dimensional periodic $n^4$ ($n \le 16$), the kernel in the coarse-link coordinates is exponential (prefactor around $r^{-1}$, $\kappa \approx 2.08$) and the sum tends to saturate, while the coordinates with the tags removed are a power ($r^{-4.4}$ from a fit) plus flat, and the sum grows in proportion to $n$ Reading . Therefore the coordinates of the cubic-remainder estimate are coarse links plus global tags, supported by [Reading] numbers up to the four-dimensional tree level. The four-dimensional one loop has not been measured.

**Writing the algebraic layer in Lean — the same form as in the sources, and where the cut may differ.** Of the statement (3) of the small-field contribution to the coupling, we wrote the algebraic layer and the assembly of the real inequalities in Lean (two bundles, 40 theorems, all on the three standard axioms). (A) Defining $\mathbf{Loc}$ as the projection that keeps or discards matrix entries, it is linear and idempotent, the decomposition into a local part and a purely non-local part is unique, and the image of $\mathbf{Loc}$ and the image of $1 - \mathbf{Loc}$ are orthogonal in the Frobenius inner product (`locM_idem`, `split_unique`, `frob_loc_nonloc`, no assumptions; the trace is preserved when the set of kept pairs is reflexive). (B) Taking $\mathbf{Loc}$ to be the block-diagonal

projection, $W^{\mathrm{gauge}} = \frac{1}{2}(1 - \mathrm{Loc})F_C$ can be written only with the covariance entries straddling blocks, and for two blocks it is $W(V(b), V(\Lambda)) = \frac{1}{2} V_A^{\mathrm{T}} C_{AB} V_B$ (`Wg_explicit`, `two_block_W`). (C) Receiving the Cauchy-type bound on the coefficients and the bound on the cut as assumptions, the one-step remainder is bounded in the bracketed form of (3) (`one_step_remainder`, margin $h$ $0 \le qh$), and under the schedule $g_k^2 = 1/(A(k+1))$, $h = c_0 g p$, $\beta = 4/g^2$, if $c_0^2 \ge 1/(c(1-q)^2)$ then the sum over stacked steps is uniform in the number of steps $K$ (`total_remainder_uniform`, assuming $A \ge 1$, $r_0^2 \ge 1$, $m \ge 3$ and others). Analyticity (the analyticity condition at complex backgrounds) and the probability of the cut come in from outside as assumptions.

**The same form as in the sources** Cited , Reading (the correspondence is a reading of the equation numbers of the sources; we have not reopened the text of the sources): that Loc is a projection and the decomposition is unique (BBS § 5.2, Appendix A); that $W$ can be written only with the covariance entries straddling blocks ((A.1.3)–(A.1.4)); the projection layer of the local cancellation, "the part entering $W$ does not contribute to the marginal coupling" (up to just before (A.5.3)–(A.5.4)); and the form of the Taylor remainder (Lemma 7.5.3). **Where the cut may differ from the sources** Reading :

1. Taking Loc as a projection of matrix entries. The Loc of the sources is a projection onto a space of local polynomials, and for quadratic forms it is the type that keeps the zero-momentum symbol (the row sums). If the shift of the marginal coupling is defined by a trace, the off-diagonal entries within the same block do not enter, and the meaning of the numbers may differ from the zero-momentum type. The properties of the projection themselves (idempotent, unique, orthogonal) do not depend on the choice.
2. The reason for zero sum. In Lean, $W = 0$ follows from the definition of Loc, whereas the zero sum of the hierarchical model in the sources follows from the structure of the covariance. Even for the same "$W = 0$", the reason differs.
3. Paying for the cut separately by probability. Applying analyticity only to the effective action without the cut, and paying for the part due to the cut separately by a probability estimate, is a different road from the sources (the regulator method for flat fields).
4. How the sum over steps is taken. Bounding the bound of each step by $1/(k+1)^2$ and adding (a $p$ series) differs from the combination of the contraction of the shape coordinates (a geometric series) and the flow of the couplings in the sources, and for a coarse estimate assumes $\mathrm{Loc}_4$ ($m \ge 3$).

None of these points changes the form of the statement of the small-field contribution to the coupling; they are notes to separate how far it has the same form as in the sources and from where the cut is different (the question "two definitions of Loc"). The first point became numbers in the two-block system: the quadratic non-local rest of $O(1/\beta)$ is, with Loc of row sums, only $\frac{1}{768}|a-b|^2/\beta$ containing differences and makes no new coordinate, while with the projection of entries $-\frac{1}{384} a \cdot b/\beta$ without differences remains Proved here (the expansion of the closed formula is formal), Reading ($10^{-11}$ at $\beta = 32$–$2048$; $SU(2)$ heat kernel, seven in a row, the diagonal definition, one step). The type of the sources is the row sum, and the two definitions are connected by the identity $s_{\mathrm{row}} = s_{\mathrm{comp}} + \frac{1}{2} N_{\mathrm{comp}}(x, x)$ (Lean: `sRow_eq_sComp_add`, `row_sum_cancels` (assuming $F$ is symmetric), `row_split_unique`). The content of the group

version of the local cancellation — that the sum of the quadratic contributions over polymers in (A.5.3)–(A.5.4) is 0 up to a cubic error — has not been put into Lean.

In summary, what could be said about the small-field contribution to the coupling is that **the same form as the structure known in the sources for flat fields** (Loc + **the Taylor remainder, the tail of the cut, the quadratic terms between blocks) holds on the group too, under the analyticity condition at complex backgrounds and margin** $q$. It does not claim a new structure. What is absent from the sources is only the group version of the place the sources themselves call a sketch — the explicit quadratic term between blocks $W^{\text{gauge}}$ and its local cancellation — of which two are algebraic and one is an analytic problem. In the smallest two-block system, $W^{\text{gauge}}$ starts with an explicit formula from one loop, what needs local cancellation is the part $O(1/\beta)$, and the non-local tail type is decided by the definition of the coarse variables. The algebraic layer can be written in Lean, and at four places in it the cut may differ from the sources.

07

## Discarded proposals

Table 2 Discarded proposals and why (the labels for the reasons are per cell)

| Proposal | Aim | Why discarded |
|---|---|---|
| A class dominated by heat kernels | Take as class $A^{-1}p\beta \le f \le Ap\beta$ the weights $f$ squeezed above and below by heat kernels, and make the large-field rarity condition a statement about the tail of the character expansion | The Wilson action is not in this class: on the far hemisphere it is $fW(\pi)/fW(0) = e^{-2\beta}$, the heat kernel is of order $e^{-\beta\pi^2/2}$, and the ratio is $e^{(\pi^2/2-2)\beta} \approx e^{2.9\beta}$ Proved here (arithmetic up to polynomial factors). This excess is the $z$ of the proposal of a parameter for the distribution over the centre. It is meaningful if the theory is defined with the heat-kernel action (on hold). Restricted to inside the angle cap $\theta_0$, Wilson is bounded by $K$ times the heat kernel (the One-Link Heat-Kernel Comparison Lemma, 15-07 § 05) |
| Peeling off the source with Hölder alone | Separate the observable-source condition with bounded powers of the closure under the fluctuation expectation | At the fine steps the source becomes a non-local activity, and the one-step loss $2^{\lvert B(X_J)\rvert}$ cannot be paid Reading |
| Moving $\eta$ to the side of the action with a fundamental + adjoint action | Build a one-parameter class with an action that keeps the quadratic ratio 2 | The expansion of $\cos^2$ produces an anisotropic quadratic form acting only on the component of the fluctuations in the direction of the background, and the conditional law of one link is no longer a member of a one-parameter family (15-07 § 06) Proved here . In addition, the curve of ratio 2 does not close under the one-step flow (15-06) Reading |
| Four dimensions with class functions alone | Never fix the gauge at all | The coarse Bianchi identity needs connecting paths, so it cannot go beyond the hierarchical-lattice approximation Reading (§ 05) |
| Removing re-fixing by Haar invariance | Make gauge fixing unnecessary through the conjugation invariance of the law of the fluctuations | Limited to renormalisation of the heat-kernel-convolution type; it does not reach four-dimensional blocking Reading (§ 05) |

Proposals of the type "with enough computing resources the constants become explicit" (making the constants explicit with interval arithmetic) are not the core of the four rows, so they are not in this table.

08

## Ten rows of directions being tried

Table 3 sets out in ten rows the directions being tried towards ultraviolet control (moved from Table 3 of 26-11, with the components added in this part written into "What can be said now"). "Smallest setting" is the smallest object on which the direction can first be tried, "What can be said now" is the current state with labels, and the last column is the risk that the direction ends up as a restatement of something already known. We do not give numbers for the prospects. No row closes ultraviolet control.

Table 3 Directions being tried ("Conditions attacked" are the three conditions of ultraviolet control in 15-00 § 01; "Sources of difficulty" are those of 24-07 Table 3)

| Direction | Conditions attacked / sources of difficulty | Smallest setting | What can be said now | Risk of restatement |
|---|---|---|---|---|
| 1 Rewriting the series on the small twisted $T^4$ | Analyticity, rarity, the coupling stays small / dimensionless coupling, lack of gauge fixing | $T^4_{L_0}$, $n_{12} = 1$ | A proposal for the ultraviolet statement for a small twisted four-dimensional torus Assumption . Correspondence with the sources in 15-02 § 06 Reading | The fine steps are the same as for the periodic $T^4$, so it becomes a rewriting of the series itself |
| 2 The group-valued sigma model first | Analyticity, rarity, the coupling stays small / dimensionless coupling | The twisted cylinder for two-dimensional $O(4)$ | The transplantation goes as far as the background-field stage Cited (15-02 § 03) | Two-dimensional results imply nothing for four dimensions. Only the form of the statements carries over |

| Direction | Conditions attacked / sources of difficulty | Smallest setting | What can be said now | Risk of restatement |
|---|---|---|---|---|
| 3 The weight for group-valued fields | Rarity, analyticity / lack of gauge fixing | A single block | The group version (the Background–Fluctuation Decomposition Lemma) of submultiplicativity in sums of fields ((ii) of BBS) Proved here . In one step with background-independent fluctuations, the group versions of submultiplicativity in sums of fields, closure under the fluctuation expectation, monotonicity to the next step and the exchange all hold and the chain closes (the Lemma on One Step with Background-Independent Fluctuations) Proved here . (The group versions of) closure under the fluctuation expectation and monotonicity to the next step for the real $\mu_{k,W}$ only as statements Assumption (15-07) | The Background–Fluctuation Decomposition Lemma and the Lemma on One Step with Background-Independent Fluctuations do not touch the background dependence $\eta$ |

| Direction | Conditions attacked / sources of difficulty | Smallest setting | What can be said now | Risk of restatement |
|---|---|---|---|---|
| 4 Polchinski flow on the group (multi-scale flowing with the heat kernel) | Rarity, analyticity / lack of gauge fixing | A one-link class function | In the one-link heat-kernel class the losses form a telescoping sum Reading (15-04). The estimate of the weighted norm in one step closes with $\eta = 0$ if the fluctuations are a product of heat kernels link by link (the Lemma on One Step with Background-Independent Fluctuations) Proved here . Measures that include couplings between many links not yet done | Convolution with heat kernels does not reduce the degrees of freedom and produces no block variables |
| 5 Paying for large fields all at once with the chessboard | Rarity / lack of gauge fixing, dimensionless coupling | Periodisation of $T^4$ | Reflection positivity of the block variables is needed Assumption (26-05) | The large-deviation induction at each step ("worst-case geometry" of 15-00 § 03) cannot be avoided |
| 6 Self-containment of the flow of the coupling | The coupling stays small / dimensionless coupling | Anywhere | Coupling-Smallness Induction Lemma: net one-step bracket of the coupling + continuity of the coupling ⇒ the coupling stays small (15-03 § 03) Proved here . The bookkeeping part is elementary | Unless the net one-step bracket of the coupling is shown, it remains a tidying of the bookkeeping part of "the coupling stays small" |
| 7 Making the constants explicit with computer assistance | Analyticity, the coupling stays small / dimensionless coupling, lack of gauge fixing | The linearised operator of one block | There is a record of floating-point estimates, but no interval arithmetic yet Reading | It only makes the constants explicit and does not supply the induction itself |
| 8 Regularity structures at criticality (stochastic quantisation) | Analyticity / dimensionless coupling | Cannot be written | The key condition of the solution theory is $d < 4$, and four dimensions is the borderline Cited (24-06 § 08) | The tools need rebuilding, and no minimal statement can be set down |

| Direction | Conditions attacked / sources of difficulty | Smallest setting | What can be said now | Risk of restatement |
|---|---|---|---|---|
| 9 The determinant in self-dual coordinates | Analyticity, the coupling stays small / dimensionless coupling, lack of gauge fixing | Near the twisted isolated flat connections | Only a proposal for a one-loop check Assumption | Reproducing the one-loop coefficient is a restatement of one loop and can only serve as a negative control. There is no self-dual decomposition on the lattice |
| 10 The twisted one-point model and the Nahm transform | Analyticity / dimensionless coupling | The one-point model | The Nahm transform is a correspondence of classical self-dual configurations; no correspondence at the level of the measure is known Assumption (sources not checked) | The problem of the order of large $N$ and $\beta \to \infty$ remains, and it merely swaps the ultraviolet and the infrared |

What to read from Table 3 Reading (a physical estimate). The directions with components closed on paper are directions 3 and 4 (the Background–Fluctuation Decomposition Lemma, the Lemma on One Step with Background-Independent Fluctuations) and direction 6 (the Coupling-Smallness Induction Lemma), and all lie outside the real issue (submultiplicativity in sums of fields, background-independent fluctuations, the bookkeeping part). The real issue — the group version of closure under the fluctuation expectation for the real $\mu_{k,W}$, and the one-step remainder including large fields (the net one-step bracket of the coupling) — is the single point towards which directions 3 and 4 and direction 6 each head, and they meet where the dependence of the measure of the fluctuations on the background takes effect. This is the same place as "the row of background-dependent absorption and many steps is the core" in § 06.

**Questions arising from this chapter.** Table 4 places, one for each of the four rows, the smallest question that can be tried next. The question "smallest system that is not zero-sum" arises from the core row in § 06. The question "two definitions of Loc" and the question "three-dimensional non-local weights" arise from the revision of § 06, the question "gauge agreement on non-abelian backgrounds" from § 05, and the question "quartic width of the three-body tree" from the third revision of § 06.

Table 4 Questions arising from this chapter (status: New = a question added in this part; Updated = part of the answer has been written in)

| Q. | Row | Question (one line) | Size | Status | What does not move even if settled |
|---|---|---|---|---|---|

| Q. | Row | Question (one line) | Size | Status | What does not move even if settled |
|---|---|---|---|---|---|
| 12 | 1 | For one link, derive on paper the Lipschitz constant of the read-out map from the derivative of the Bessel ratio; on the slab, put a number on the difference in total variation between the density with one large-field region inserted and the density without it; and see whether the form large-field contribution to the coupling $\le C' g^2$ holds in the smallest system. **Answer (partial):** done for one link (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, § 02; Lipschitz constant $\frac{4}{3}\beta^2$, actual sensitivity $\approx \frac{1}{3}\beta^2 h^2$). On the $2 \times 2$ slab with heat-kernel fluctuations, at the threshold $\theta_h = g\,p(g)$ ($\beta = 4/g^2$), the form holds for $c_0^2 > 2$, $= 2$ is the boundary, and with the threshold fixed it does not hold. The value on the slab is 1/3 of that for one link and is local ( § 02). Many links under the true measure not yet done (the row of background-dependent absorption and many steps) | Paper and small computations | Updated | The small-field contribution to the coupling (the content of the basic inequality (unpublished)) |
| 13 | 2 | With four plaquettes in a row, add a source $s\,\mathrm{Re}\,W_a$ to one loop, and see whether the shift of the couplings by loop shape after one step is linear in $s$ with a coefficient independent of $\beta$. Also, on paper, whether one step of the position-dependent coupling $\beta(x)$ in the Villain action of $U(1)$ becomes a local average. **Answer (partial):** in one heat-kernel step (without blocking), the source from fine loops is a shift of the coupling for each representation, exact at both first and second order ( § 03). In one blocking step with four in a row, the shift by a source on a loop inside the block is the main term $f_C^2$ (independent of $\beta$) plus a rational $O(1/\beta)$, with no $\log\beta$. The $\beta(x)$ of the $U(1)$ Villain action is, to first order, the local average with the weights of the bundle network ( § 03). Loops crossing boundaries and many steps not yet done | Paper and small computations | Updated | The observable-source condition for loops of the fine lattice |

| Q. | Row | Question (one line) | Size | Status | What does not move even if settled |
|---|---|---|---|---|---|
| 14 | 3 | Whether the parity of the critical point holds with four plaquettes (the odd part of the action of the fluctuations around the true constrained critical point), and if so whether the background uniformity of the small-field contribution to the coupling weakens to boundedness of the coefficient of $\theta^4$ (the proposal of the parity of critical points). Paired with the question "splitting the density of the joint law" (15-07). Answer (partial): parity breaks with four plaquettes in the breathing direction (closed formula). The conclusion of the proposal of the parity of critical points stands for a chain in a row by the covariance under reversal of the background ( § 02). The row for $\theta^3$ in four dimensions not yet done | Small computation | Updated | The multi-step background-regularity condition |
| 15 | 4 | Split the one-loop Hessian of a block with factor 2 into a gauge-field part and a ghost part, and put the coefficient of $F^2$ of the latter into numbers (negative control: 0 for $U(1)$). At the same time, formalise the direct-product decomposition of the tree gauge for finite groups. **Answer:** the second half is done (Lean for finite groups, § 05). The coefficient of $F^2$ of the ghost is $+0.024$ per classical action for one block of $3^4$ points on a commutative background (2.4% of one loop), and on a critical-point background the one loops of the two gauges agree identically by (5) ( § 05) | Small computation, Lean | Updated | Uniform convergence of the local expansion (of the type of the small-field analyticity condition) |

| Q. | Row | Question (one line) | Size | Status | What does not move even if settled |
|---|---|---|---|---|---|
| 16 | 1 | In the smallest system that is not zero-sum — two copies of four in a row placed side by side sharing the fine links between them, so that the two blocks are connected by the covariance of the fluctuations — write $W^{\mathrm{gauge}}$ ((4)), and put into numbers, for $\beta = 32$–$512$, the amount by which the quadratic contribution to the marginal coupling $\beta_0'$ deviates from the sum of the two blocks (the breaking of the local cancellation). If the cancellation holds, $O(1/\beta)$; if not, $O(1)$. **Answer (smallest system):** in seven in a row $W^{\mathrm{gauge}}$ is $O(1)$ from one loop ($13/384$), and the rest after putting in the first order explicitly is $O(1/\beta)$. On chains, slabs and columns of $L$ blocks the tail type is decided by the definition of the coarse variables ( § 06) | Small computations and paper | Updated | The cubic-remainder estimate in four dimensions (the cubic estimate) |
| 21 | 1 | Define $\mathbf{Loc}$ in two ways — by row sums (zero momentum) and by matrix entries — and give the difference of $W^{\mathrm{gauge}}$ in rational numbers for the covariance of a one-row two-block system. If the difference is 0 the cut is the same as in the sources; if not 0, the place where it differs is identified by numbers. **Answer:** the difference is not 0. The quadratic non-local rest of $O(1/\beta)$ is $\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums (only terms containing differences), and $-\frac{1}{384}a\cdot b/\beta$ with the projection of entries. The type of the sources is the row sum ($SU(2)$ heat kernel, seven in a row, one step, § 06) | Small computations and Lean | Updated | The local cancellation in four dimensions |
| 22 | 1 | In three dimensions without boundary (columns with periodic cross-section, or $8^3$), do the one-loop non-local weights $Q^{(1)}(r)$ of $SU(2)$ in the coordinates of coarse links plus global tags remain exponential, as at tree level? **Answer (partial):** exponential on a periodic coarse $2\times 2\times L$ ($L\le 16$) ($\kappa = 3.094$, agreeing with the tree rate), and the sum is uniform in volume. The four-dimensional periodic tree is also exponential ($\kappa\approx 2.08$). Boxes wide in the transverse direction and the four-dimensional one loop are not yet done ( § 06) | Small computation | Updated | The cubic-remainder estimate in four dimensions |

| Q. | Row | Question (one line) | Size | Status | What does not move even if settled |
|---|---|---|---|---|---|
| 24 | 4 | Measure (5) and the ghost coefficients on a non-commutative background (a critical point with commutator $F$), and see whether the leading term of the local expansion is a single number for $\sum_p \lvert F_p\rvert^2$ (whether it is the same for commutative $F$ and for the commutator's $F$) | Small computation | New | Agreement at two loops and beyond |
| 29 | 1 | In a four-dimensional box of coarse links plus global tags, measure the dependence on width of the three-body quartic (commutator type) at tree level, and see whether the rate agrees with the two-body tree rate (the first test of carrying the one-row recursion of $W$ over to four dimensions). At the same time, measure in one row the four-body terms and the contraction $\mu$ of the self-image of three-body terms at the one-loop level | Small computation | New | The local cancellation and the cubic-remainder estimate in four dimensions |

09

## What cannot be said

(a) Within this part, ultraviolet control does not close (existence was shown by the published construction, with a different design). None of the four rows closes. Every proposal in this chapter is a proposal, not a result.

(b) The "one line" of Table 1 is a conditional statement — if it were shown, the row would close — and none of them has been shown. That the small-field contribution to the coupling corresponds to the content of the basic inequality (unpublished) is a reading from one sentence of the preface.

(c) The proposal to obtain rarity by Chebyshev is a one-liner assuming the group version of the exponential moment of fluctuations, and the group version of the exponential moment of fluctuations has not been shown. The Lipschitz constant and the difference in total variation of the proposal to split the remainder into small and large fields are assumptions and have not been estimated.

(d) The commutativity of the proposal to take the source as a function of the block variables is a trivial fact limited to functions of the block variables, and says nothing about observables given by loops of the fine lattice.

(e) "There is no Jacobian" of the tree-gauge proposal is a restatement of a structure limited to the axial (tree) gauge, and we have not checked the choice of gauge in Bałaban's four-dimensional papers. The local expansion of the ghost determinant is a prospect.

(f) "The row of background-dependent absorption and many steps is the core" is an estimate of the logic between the proposals, and none of its implications has been shown. The directions of Table 3 are candidates, and neither their consistency nor their prospects have been assessed.

(g) The splitting (3) of the small-field contribution to the coupling in § 06 is a statement with the analyticity condition at complex backgrounds and margin $q$ put in, and an estimate under them. The analyticity condition at complex backgrounds has not been shown for four-dimensional blocking. The numbers in § 06 are for four in a row ($SU(2)$, heat kernel, one step) and one-loop numbers for a box of $2^4$, not numbers for one step of the four-dimensional lattice. The row absent from the sources (the group version of $W^{\mathrm{gauge}}$ and its local cancellation) has not been shown. "The same form as in the sources" speaks of agreement of form and is not a claim of a new structure.

(h) The Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma and the accounting of the source are statements under the edgewise independent heat-kernel condition, the single-step condition and the background-independence condition for fluctuations (the One-Link Heat-Kernel Comparison Lemma is for one Wilson link), and say nothing about the true measure of the fluctuations. The Lean for the tree gauge is a statement about finite groups; the Haar measure of continuous groups is on paper.

(i) The numbers for the two-block system, chains, slabs and columns of § 06 are statements for one step, quadratic coefficients of the coarse field and commutative backgrounds (the tail type), and the one loop is a formal asymptotic expansion by the Laplace method. The coefficient of $O(1/\beta)$ is from a fit, with no formula. The counting rule for the tail type is a reading from the numbers of one-, two- and three-dimensional models, and in four dimensions only the tree level in the coordinates of coarse links plus global tags ($n \le 16$) was measured. The 40 Lean theorems are finite-dimensional projections and identities and the assembly of real inequalities, with analyticity (the analyticity condition at complex backgrounds) and the probability of the cut entering as assumptions. The four points where the cut may differ from the sources are readings, not rechecked against the text of the sources.

(j) The recursion of $W$ and the many-body tail of § 06 are statements for one row, two blocks (three-body for $L \le 8$, $d \le 7$), the definition without a conserved quantity, first order in the non-local terms and up to fourth order in the coarse field, and the one loop is the formal leading term of Laplace. $-13/768$ is a reading from numbers. The one loop of the coordinates of the cubic-remainder estimate is only for three-dimensional quasi-one-dimensional boxes (coarse $2 \times 2 \times L$), and four dimensions only at tree level. The difference between row sums and the projection of entries is a statement for one step of seven in a row.

(k) The agreement of the two gauges (5) of § 05 is a statement for one step, one loop and a critical-point background. The numbers for the ghost coefficients are values for $SU(2)$, blocks of $3^4$ points and $4 \cdot 4 \cdot 3 \cdot 3$ points, and commutative backgrounds; non-commutative backgrounds, $\xi \neq 1$, two loops and multiple steps have not been measured.

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10
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## The table of the five grades and the dependency graph

Table 5 The rows of this chapter (the rows of this chapter have no Lean column)

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [01] | The "one line" of the four rows that do not close (the net one-step bracket of the coupling, the observable-source condition, the absorption of $\eta$ and many steps, the seam of re-fixing) | Assumption | § 01 Table 1 (statement) | Constants independent of $k, a$ |
| [02] | The closed components lie in background-independent fluctuations or at tree level | Reading | § 01 | The range of the components of this part |
| [03] | The large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations (1) | Proved here | § 02 the proposal to obtain rarity by Chebyshev | Assuming the group version of the exponential moment of fluctuations |
| [04] | Net one-step bracket of the coupling = small-field contribution to the coupling + large-field contribution to the coupling, $\lvert r^{\mathrm{large}}\rvert \le C\beta^2 e^{-cp^2}$ | Assumption | § 02 the proposal to split the remainder into small and large fields | Assuming the Lipschitz property and the difference in total variation, $p_0 \ge \frac{1}{2}$ |
| [05] | Sending the background dependence to the coefficient of $\theta^4$ by the covariance under reversal of the background; the large-field parameter $z$ | Assumption | § 02 the proposal of the parity of critical points and the proposal of a parameter for the distribution over the centre | Parity holds with two plaquettes and breaks in the breathing direction with four. The conclusion stands for a chain in a row by the covariance under reversal of the background. In four dimensions boundedness of the coefficient of $\theta^3$ is also needed |
| [06] | If the source is taken as a function of the block variables, it commutes with the integration at the fine steps | Proved here | § 03 the proposal to take the source as a function of the block variables | Blocked observables |
| [07] | The source is a shift of the coordinates of the class; the class of position-dependent couplings | Assumption | § 03 | Proposal |
| [08] | By the direct-product decomposition of the tree gauge there is no Jacobian of re-fixing | Proved here | § 05 the tree-gauge proposal | Axial (tree) gauge; a restatement of a known structure |
| [09] | The local expansion of the ghost determinant and its merging into the local couplings | Assumption | § 05 the proposal to absorb the ghost determinant | Small fields; uniform convergence not yet shown |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [10] | A road with class functions alone cannot go beyond the hierarchical lattice; removing re-fixing by Haar is limited to the heat-kernel type | Reading | § 05 | — |
| [11] | Dependence among the four rows: the row of background-dependent absorption and many steps is the core; the remaining core is the small-field contribution to the coupling and the multi-step background-regularity condition | Reading | § 06 (estimate) | — |
| [12] | Wilson is not in the class dominated by heat kernels (ratio $e^{(\pi^2/2-2)\beta}$) | Proved here | § 07 Table 2 | $SU(2)$, one link, up to polynomial factors |
| [13] | The ten rows of directions being tried, and the single point where the real issue converges | Reading | § 08 Table 3 (physical estimate) | Candidates |
| [14] | The Heat-Kernel Gaussian Upper-Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma: for heat-kernel fluctuations the group version of the exponential moment of fluctuations and the large-field rarity condition hold in closed form. For one Wilson link, $K$ times + the tail by the One-Link Heat-Kernel Comparison Lemma | Proved here | § 02 | The three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; $SU(2)$ / one link |
| [15] | Large-Field Remainder Bound Lemma: net one-step bracket of the coupling = $r^{\text{small}} + r^{\text{large}}$, $\lvert r^{\text{large}}\rvert \le \frac{4}{3}(b_S^2/E_S)P_L$, $\le C' g^2$ at $c_0^2 > 2$. Actual sensitivity $\approx \frac{1}{3}\beta^2 h^2$ | Proved here, Reading (the sensitivity numbers) | § 02 | One link; the three conditions of edgewise independent heat kernel, single step and background independence of fluctuations; $E \in (e^{-2}, 1)$ |
| [16] | The loss from the source $e^{s_0\lvert B(X)\rvert}$ appears only once; the source from fine loops is $\mathbb{E}F(U_C) = (F * p_{t\lvert C\rvert})(W_C)$ | Proved here | § 03 | Blocked observables / the edgewise independent heat-kernel condition, self-avoiding loops |
| [17] | Existence, uniqueness and direct-product decomposition of the tree gauge, bijection of re-fixing; cost (2) | Proved here | § 05 (Lean for finite groups) | Finite groups (Lean), $SU(2)$ (paper), the full axial tree |
| [18] | Splitting (3) of the small-field contribution to the coupling: the Taylor truncation has the same form as Lemma 7.5.3 of the sources, the tail of the cut is $e^{-c(1-q)^2 h^2 \beta}$ | Assumption (statement), Proved here (estimate under the analyticity condition at complex backgrounds), Cited (form in the sources) | § 06 | The analyticity condition at complex backgrounds, margin $q$, numbers for four in a row |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [19] | The row absent from the sources is $W^{\mathrm{gauge}}$ ((4)) and the group version of the local cancellation; empty in the zero-sum system | Reading | § 06 | Comparison of the text of the appendices of the sources with the preface |
| [20] | $W^{\mathrm{gauge}}$ for two blocks is $O(1)$ from one loop ($13/384$), originating in the non-locality of the background map; after putting in the first order explicitly, $O(1/\beta)$ | Proved here (one-loop formula, formal), Reading (numbers) | § 06 | $SU(2)$, heat kernel, seven in a row, the diagonal definition, one step, quadratic coefficients, $\beta = 32$–$512$ |
| [21] | The non-local tail type is decided by the definition of the coarse variables (conserved quantities and global constraints are flat; sharing by weights and promoted constraints are exponential; implicitly held constraints are of the propagation type) | Reading (the arithmetic of the weighted sums is Proved here ) | § 06 | Chains, slabs, columns, periodic lattices; one step, quadratic coefficients, commutative background; four dimensions not yet |
| [22] | The algebraic layer of the small-field contribution to the coupling (the projection $\mathbf{Loc}$, the layer of $W^{\mathrm{gauge}}$) and the assembly of the remainder (sum uniform in the number of steps) | Proved here | § 06 (the finite-dimensional part in Lean) | Receives the assumptions on the coefficients and on the cut; schedule $g_k^2 = 1/(A(k+1))$, $m \geq 3$ |
| [23] | The parts of the same form as in the sources, and the four points where the cut may differ from the sources | Cited , Reading | § 06 | Reading of the equation numbers of the sources; the text not reopened |
| [24] | On a critical-point background the one-loop effective actions of the tree gauge and the background-covariant gauge agree (5); the ghost is one line of the bookkeeping | Proved here , Reading (the ghost coefficients) | § 05 | One step, one loop, $HD = 0$, $\xi = 1$; numbers for $SU(2)$, $3^4$ points, commutative background |
| [25] | The group version of the recursion of $W$ (one row): the image closes on the same types, contracting in the ratio of couplings to quadratic $5/8$ and quartic $\leq 5/48$; the new terms are independent of the step and are drawn to a fixed point | Proved here (first-order Taylor formula, tree rational numbers, bookkeeping), Reading (the correction of $1/\beta$), Cited (the additivity of the sources) | § 06 | $SU(2)$ heat kernel, one row, the diagonal definition, one step of the two-block seven twice, first order, up to fourth order, $\beta = 32$–$512$ |
| [26] | The many-body tail: three-body coefficients $e^{-\kappa d}$ in the width ($\kappa = \ln(5+2\sqrt{6})$), the sum uniform in volume, the image $5 \times 10^{-3}$ or less. The one-row recursion closes up to fourth order | Reading (the bookkeeping is Proved here ) | § 06 | One row, definition without a conserved quantity, one step, tree and one loop, $L \leq 8$, $d \leq 7$ |

| Row | Statement (one line) | Grade | Basis | Scope |
|---|---|---|---|---|
| [27] | The coordinates of the cubic-remainder estimate: coarse links plus global tags are exponential at one loop in three dimensions without boundary ($\kappa = 3.094$), with the sum uniform in volume, and exponential on the four-dimensional tree | Reading , Proved here (orthogonality of the tags and uniqueness of the decomposition) | § 06 | Commutative background, quadratic coefficients, one step, coarse $2 \times 2 \times L$ ($L \leq 16$); four dimensions tree $n \leq 16$ |
| [28] | The difference between the two definitions of Loc: $\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums, $-\frac{1}{384}a \cdot b/\beta$ with the projection of entries. The type of the sources is the row sum | Proved here (closed formula and identity), Reading (numbers) | § 06 | $SU(2)$ heat kernel, seven in a row, one step, up to $O(1/\beta)$ |

Dependency graph (the numbers of premises are smaller than the number of the conclusion):

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[01] ASM  one line of each of the four rows that do not close                    con
[02] RDG  where the closed components lie ⟸ [01]
[03] PRV  large-field rarity ⟸ Chebyshev for the group version of the exponential mon
[04] ASM  net bracket = small-field contribution + large-field contribution ⟸ [01][03
[05] ASM  to θ⁴ by background reversal, parameter z ⟸ [04]                        θ³
[06] PRV  a source that is a function of block variables commutes                 blo
[07] ASM  the source is a shift of coordinates ⟸ [06]
[08] PRV  no Jacobian for the tree gauge                                          axi
[09] ASM  ghost determinant into local couplings ⟸ [08]
[10] RDG  wall for a road with class functions alone, limit of Haar ⟸ [08]
[11] RDG  the row of background-dependent absorption and many steps is the core ⟸ [03
[12] PRV  Wilson is not dominated by the heat kernel                              one
[13] RDG  the ten directions ⟸ [11]
[14] PRV  exponential moment and rarity of heat-kernel fluctuations, heat-kernel compa
[15] PRV  splitting of the net bracket and r^large ⟸ [04][14]                      or
[16] PRV  accounting of the loss from the source, source from fine loops ⟸ [06]    blo
[17] PRV  tree gauge (Lean for finite groups), cost (2) ⟸ [08]                     fu
[18] ASM  splitting of the small-field contribution, Taylor form same as in the source
[19] RDG  the row absent from the sources, W^gauge ⟸ [11][18]                      em
[20] PRV  W^gauge for two blocks is O(1) from one loop ⟸ [19]                      sev
[21] RDG  the tail type is decided by the definition of the coarse variables ⟸ [20]
[22] PRV  the algebraic layer and the assembly of the remainder ⟸ [18]             re
[23] RDG  same form as the sources, the four points of the cut ⟸ [22]              re
[24] PRV  the agreement of the two gauges (5) ⟸ [09][17]                           or
[25] PRV  group version of the recursion of W (one row) ⟸ [20]                     or
[26] RDG  the many-body tail is exponential, the one-row recursion closes ⟸ [21][25]
[27] RDG  the coordinates of the cubic-remainder estimate are coarse links + tags ⟸ [
[28] PRV  the difference between the two definitions of Loc ⟸ [20][23]             se
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The way to check it is to see by eye that every number to the right of a "⟸" is smaller than the number on the left. There are no cycles. [01], [04], [05], [07], [09] and [18] are assumption rows, and all are in the form of questions that split ultraviolet control. Among the nodes of this chapter there is no node saying that ultraviolet control has been shown.

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Proved here The Chebyshev one-liner for the large-field rarity condition assuming the group version of the exponential moment of fluctuations; the commutativity of a source that is a function of the block variables; the direct-product decomposition of the tree gauge (no Jacobian of re-fixing); that Wilson is not dominated by the heat kernel | The group version of the exponential moment of fluctuations itself. Observables of the fine lattice. Checking the gauge in the four-dimensional papers |
| Proved here The Heat-Kernel Fluctuation Rarity Lemma and the Large-Field Remainder Bound Lemma (heat-kernel fluctuations, one link) and the One-Link Heat-Kernel Comparison Lemma (one Wilson link); the accounting of the loss from the source and the source from fine loops (one heat-kernel step); existence, uniqueness and direct-product decomposition of the tree gauge (Lean for finite groups) and cost (2); the estimate of the Taylor truncation of the small-field contribution to the coupling (under the analyticity condition at complex backgrounds, the same form as Lemma 7.5.3 of the sources) | The group version of the exponential moment of fluctuations under the true measure of the fluctuations. Showing the analyticity condition at complex backgrounds for four-dimensional blocking. The sharp multi-link form |
| Reading Where the closed components lie; the dependence among the four rows (the row of background-dependent absorption and many steps is the core); the wall for a road with class functions alone; the ten directions; that the row absent from the sources is $W^{\mathrm{gauge}}$ and the group version of its local cancellation, empty in the zero-sum system | Any of the implications of the dependence. Any of the definition of the inter-block term, the local cancellation and the cubic-remainder estimate. Checking against Bałaban's main text |
| Proved here The agreement of the two gauges (5) (one step, one loop, critical point); the algebraic layer of the small-field contribution to the coupling and the assembly of the remainder (the finite-dimensional part in Lean); the one-loop formula for two blocks (formal) | Agreement at two loops and beyond. The analyticity condition at complex backgrounds and the probability of the cut. The local cancellation in four dimensions |
| Reading $13/384$ and $O(1/\beta)$ for two blocks; the counting rule for the tail type; the four points where the cut may differ from the sources; the ghost coefficient $+0.024$ | The tail type in four dimensions and the cubic-remainder estimate. Checking against the text of the sources |
| Proved here The first-order Taylor formula of the recursion of $W$ and the tree rational numbers; the orthogonality of the tags; the identity between the two definitions of $\mathbf{Loc}$. Reading The contraction ratios of the recursion of $W$; the exponential decay of the many-body tail; the three-dimensional one loop and four-dimensional tree for the coordinates of the cubic-remainder estimate; the numbers for the difference of $\mathbf{Loc}$ | The recursion of $W$ in four dimensions; four-body terms; the one-loop tail in four dimensions; the volume-uniform cubic-remainder estimate |
| Assumption The "one line" of the four rows; the splitting of the net one-step bracket of the coupling; parity; the coordinates of the source; the absorption of the ghost determinant | A proof of any of them. Any prospect of ultraviolet control closing |

## Sources and reproduction

| Item | Type | Source, tools |
|---|---|---|
| Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989), basic inequality (unpublished) (where the net one-step bracket of the coupling sits) | Cited (abstract and preface) | T. Bałaban, Large field renormalization. II, Comm. Math. Phys. 122 (1989) 355–392 (15-02 § 02) |

| Item | Type | Source, tools |
|---|---|---|
| The weight of large fields and the role of Chebyshev; One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) | Cited | R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, LNM 2242 (2019), arXiv:1907.05474 (15-02 § 04) |
| Global gauge fixing; axial gauge per block | Cited | I. M. Singer, Comm. Math. Phys. 60 (1978) 7–12. 7-05 § 05 |
| The four-dimensional borderline of stochastic quantisation | Cited | The source table of 24-06 § 08 |
| Proposals for the four rows, discarded proposals, dependence among the four rows | This book's study | The record of this book's study (sorting out the rows of ultraviolet control that do not close) |
| (1), the commutativity of the proposal to take the source as a function of the block variables, the direct-product decomposition of the tree-gauge proposal, the ratio of domination | Proved here | The text of this chapter |
| Lemma 7.5.3, Loc, (A.1.3)–(A.1.4) of Appendix A.1, (A.5.3)–(A.5.4) of A.5 (flat fields) | Cited (main text) | R. Bauerschmidt, D. C. Brydges, G. Slade, Introduction to a Renormalisation Group Method, LNM 2242 (2019), arXiv:1907.05474v2 |
| Induction of the small-field approximation; the computation of the second-order perturbation | Cited (abstracts and preface) | T. Bałaban, Comm. Math. Phys. 109 (1987) 249–301, 116 (1988) 1–22, 122 (1989) 355–392 (15-02) |
| The Heat-Kernel Gaussian Upper-Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the accounting of the source, the tree gauge ($SU(2)$), the estimates (2)(3)(4) | Proved here | The text of this chapter. The parts on finite sums and finite groups are in the Lean bundle (15-09 Table 3, re-verified) |
| Numbers for the tail ratios, the sensitivity, the remainder and cut rates for four in a row, and the cost of the tree gauge | Reading | Numerical record of this book's study (two routes, sum over images and character sum; Wilson and heat kernel; two routes, the lift and directly on the group; control with $U(1)$; the box of $2^4$ by two one-loop implementations) |
| Numbers for the two-block system, chains, slabs and columns; ghost coefficients | Reading | Numerical record of this book's study (seven in a row and chains of $L$ blocks: two routes, one loop of Laplace and full quadrature; Wilson and Villain controls for $U(1)$. Slabs and columns: two implementations, projection and KKT; two routes, analytic Hessian and differences. Ghost: two routes, analytic and finite differences at $3^4$ points; controls with $U(1)$, without the ghost, and with the background moved off the critical point) |

| Item | Type | Source, tools |
|---|---|---|
| The Faddeev–Popov method; the lattice background-field method | Cited (bibliography only) | L. D. Faddeev, V. N. Popov, Phys. Lett. B 25 (1967) 29–30. M. Lüscher, P. Weisz, Nucl. Phys. B 452 (1995) 213–233, arXiv:hep-lat/9504006 (text not checked) |
| The additivity of the sources ((A.1.4)–(A.1.5), Exercise 2.2.4) | Cited ((A.1.4)–(A.1.5), main text); Reading (the identity from Exercise 2.2.4 not confirmed in the text) | R. Bauerschmidt, D. C. Brydges, G. Slade, arXiv:1907.05474v2 (p.35, p.198–199; the sketch on p.207–209 is written with $W$ removed) |
| The recursion of $W$, the many-body tail, the coordinates of the cubic-remainder estimate, the two definitions of $\mathbf{Loc}$ | Reading (formulas and bookkeeping are Proved here) | Numerical record of this book's study (separate implementations, Gauss–Hermite on the integrand without approximation and nested $2\times2$ matrices; rational linear backgrounds and floating-point projections; Laplace on the group and full quadrature with finite $\beta$; Wilson control for $U(1)$; positive control with the definition with a conserved quantity. The periodic boxes: analytic and difference Hessians, three implementations, dense and Bloch). The bookkeeping is the Lean bundle (15-09 Table 3, re-verified) |
| (5), the algebraic layer, the assembly of the remainder, the arithmetic of the tail sums | Proved here | The text of this chapter. The finite-dimensional part is the Lean bundle (15-09 Table 3, re-verified) |

**Terms used in this chapter** — the four rows that do not close, the small-field contribution to the coupling, the large-field contribution to the coupling, the read-out map, the large-field parameter $z$, a source that is a function of the block variables, position-dependent couplings, the direct-product decomposition of the tree gauge, the ghost determinant, the coupling at the seam, the row of background-dependent absorption and many steps is the core, the Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the analyticity condition at complex backgrounds, $W^{\mathbf{gauge}}$, the zero-sum system, the non-locality of the background map, the tail type, coarse links plus global tags, the agreement of the two gauges, the recursion of $W$, the many-body tail. Their meanings are in A-02 Glossary.

Next chapter: 15-09 What cannot be said, the index of questions, the dependency diagram — the ultraviolet questions 1–31 of this part and a summary of the five grades.

# What cannot be said, the index of questions, the dependency diagram

> This chapter is an index; it points to the relevant sections of each chapter.

> This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — What cannot be said across Part 15 (a summary); the index of the questions left by each chapter (the ultraviolet questions) and the current state of their answers; the dependency diagram between chapters; the five grades of the main rows of each chapter; and the correspondence between the Lean bundles used in this part and the text.

**Prerequisite chapters** — 15-00 (entrance). The other chapters are reached from the rows of the tables on this page.

**Said first** — This chapter adds no new results. Questions are questions, not claims. In no chapter of this part is ultraviolet control closed (existence was shown by the published construction, with a different design; Parts 6–13). The re-check of the Lean bundles of this part is done (all pass, with only the three standard axioms). The labels in the text stay in the five grades (paper proofs stay [Proved here]); Lean is kept as an outside column: the theorem names and assumptions are added on the same line in the text of each chapter, and Table 3 lists them as "re-check: done".

**Relation to the sources** — Some of the parts written in this part on paper and by computation write, on the group and under conditions, the same form as structures known in the sources (BBS for flat fields, Bałaban's series) (for example, the local part and the Taylor remainder of the small-field contribution to the coupling have the same form as Lemma 7.5.3 of BBS, 15-08 § 06). No novelty is claimed. What is absent from the sources is the group version of the place the sources themselves call a sketch (the explicit quadratic term between blocks and its local cancellation), and this is not shown. Within the study of that group version, the four points of the cut made when the algebraic layer was written (15-08 § 06) may differ from the sources, and they are written separately from "the same form as in the sources".

Order of this chapter

1. What cannot be said — twelve items for the whole part, and the items of each chapter
2. The index of questions — the ultraviolet questions and the current state of the answers (Table 1)
3. The dependency diagram of the chapters
4. Summary of the five grades (Table 2)
5. The Lean correspondence of this part (Table 3)
6. Connection to Part 27 and the glossary

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00

## What this chapter shows

- What Part 15 does not say comes down to twelve items. Each is there to stop readings that drop the conditions "smallest systems, leading term, one step, form of a statement" ( § 01).
- There are thirty-one questions, and each touches one component inside the row for ultraviolet control in the ledger of conditions. For eighteen of them all or part of the answer has been written (ultraviolet questions 4, 5, 7, 8, 10, 12–23, 25), but whichever is settled, ultraviolet control does not close ( § 02).
- The dependencies run from the entrance (15-00), the difficulty (15-01) and the original sources (15-02) to the decomposition statement (15-03), branch into the computational line (15-04–15-06) and the wall line (15-07, 15-08), and return to 15-09 ( § 03).

01

## What cannot be said

We gather what is not said across the whole part. The flaws of the conditional implication assembled beyond the parts of this part are in 15-00 § 08. The finer items of each chapter are in that chapter's "What cannot be said" section.

(a) **Within this part, ultraviolet control is not closed.** Part 15 shows none of the three conditions of ultraviolet control, in four dimensions or in two. The existence of the four-dimensional continuum limit was shown by the published construction, with a different design (Parts 6–13; the comparison of designs is in Part 14). What this part shows is statements of what would need to be shown, line-by-line computations, elementary lemmas (the Coupling-Smallness Induction Lemma, the Background–Fluctuation Decomposition Lemma, the Orbit-Direction Flatness Proposition, the Lemma on One Step with Background-Independent Fluctuations, the Bundle Network Theorem), and numbers for the smallest systems.

(b) **The arithmetic under the one-loop flow is not a theorem about the actual sequence of effective couplings.** The divergence and finiteness of the sums in 15-01 are arithmetic that treats the one-loop flow as an exact equation.

(c) **The numbers for the smallest systems are not the numbers of a real blocking step.** The numbers of 15-05 and 15-06 are for the tree level restricted to commuting configurations (the leading term of $\beta \to \infty$), the one-loop level, one step seen through the marginal distribution of a single coarse plaquette, and small boxes (free boundary, free only on the outer rim, periodic); they are not numbers for a step that includes blocks embedded in the four-dimensional lattice, re-fixing of the gauge, and large fields.

(d) **The results for the hierarchical model do not imply anything in four dimensions.** The Migdal–Kadanoff map of 15-04 sends a single-plaquette weight back to a single-plaquette weight; multi-link couplings, re-fixing and the remainder from large fields are structurally absent, and it is a model in which confinement occurs even at $U(1)$. The range checked by interval arithmetic is only a finite interval of $\beta$ (the tube $\beta \le 128$, the curve $\beta \le 64$), and the limit $a \to 0$ is not closed. The proposal of a one-step inequality with a fixed comparison partner shows neither a violation nor support at the numbers for small boxes (within the resolution at $6^4/3^4$), and it is not shown for any $\beta$.

(e) **The Lemma on One Step with Background-Independent Fluctuations is a statement only for one step, and only when the law of the fluctuations does not depend on the background.** The true measure of the fluctuations in one step depends on the background, and the absorption of that dependence $\eta$ and the multi-step induction have not been written (15-07).

(f) **The decomposition statement and the twisted-torus ultraviolet statement are statements.** The decomposition of 15-03 and the ultraviolet statement for a small twisted four-dimensional torus of 15-02 are in the form of [Assumption] and are not proved. Of the implications, only the Coupling-Smallness Induction Lemma and the arithmetic of geometric series were checked elementarily.

(g) **Much of the checking against the original sources is at the level of abstracts and introductions.** Stages 4 and 5 of the series and 119 are from abstracts, stages 1–3 from titles, and the two theorems of Bałaban 1989 (Theorems 1 and 2) from the abstract and introduction of the last paper. Appendix A.6 of BBS was read in the text itself (15-02).

(h) **The proposals for the four rows that do not close are proposals.** The proposals of 15-08 are ideas before verification, and none of them closes any of the four rows. We say neither "there is no action that closes with one parameter" nor "the class-closure condition closes with finitely many parameters".

(i) **"The same form as in the sources" is an agreement of form, not a claim of a new structure.** In the splitting of the small-field contribution to the coupling (15-08 § 06), the same form as in the sources holds on the group only under the analyticity condition at complex backgrounds (analyticity in complex backgrounds and an upper bound independent of $\beta$) and the margin $q$. The analyticity condition at complex backgrounds is not shown for four-dimensional blocking. The row absent from the sources (the explicit quadratic term $W^{\mathrm{gauge}}$ between blocks and the group version of its local cancellation) is not shown. A zero-sum system with a single block cannot test this row, so we measured it in a system of two blocks: $W^{\mathrm{gauge}}$ starts explicitly at one loop, and the local cancellation is needed for the part $O(1/\beta)$ ($SU(2)$, seven in a row, one step, quadratic coefficients), but the local cancellation and the cubic-remainder estimate in four dimensions are not shown. The four points of the cut made when the algebraic layer was written in Lean (taking $\mathbf{Loc}$ as the projection onto components, the reason for the zero sum, how the cut is paid for, how the sum over steps is taken) are written separately as places that may differ from the sources (15-08 § 06).

(j) **We do not say that a four-dimensional number falls below 1.** The chain for the hierarchical model covers the finite range $\beta \le 128$ ($\beta > 128$ is not closed); the Shape-Remainder Theorem for Four Plaquettes in a Row is for the projected model of four in a row ($B \ge 32$, the box $O(1)$; the shape deviation of Wilson's form lies outside the box); and the upper bound of the Bundle Network Theorem (local form) uniform in the volume is an estimate of 1.83 in four dimensions, which exceeds 1. Row-weighted interpolation brought it to 0.823 or below (normalisation $\beta_0' = 1$; the numbers are computed, not interval estimates), but the factor of the direction-wise normalisation is not a measure of contraction, so the normalisation is fixed in the form divided by the long-wavelength coupling, and that One-Step Contraction of the $\ell^4$ Cone is limited to the first step (with edge-sum blocking the factor is 1.26 or more at the third step; with transverse-averaging blocking the lower bound does not grow with the steps, but there is no upper bound). The budget for the group-geometric part of the background dependence of $SU(2)$ (the third order of BCH, coefficient $1/9$) diverges however the threshold is chosen (15-07 § 07, arithmetic under the one-loop flow). If the background is carried by a parameter of the class, the budget becomes uniform in the number of steps (conditional, 15-07 § 06), but the difficulty moves to whether the class closes in one step with faces and multiple loops, and this is not shown in four dimensions.

(k) **The coordinates of the class and the numbers of the recursion are statements about the measured rules and ranges.** The "one parameter + ratio coordinates" of the four-dimensional class is a fit from the tree of a free-boundary box ($L \le 10$), one loop ($L \le 4$) and the quadratic order of the commuting tree on the periodic network ($L \le 16$); the existence of a fixed point of the ratio coordinates is not shown. That the diagonal neighbours are not 0 is a statement about the two measured rules and two conventions of projection; the values, and "closing within one cell", depend on the rule. The recursion of the core row (a contracting image + a new term independent of the step + an exponential three-body tail) is a first-order statement for one row, a definition without conserved quantities and the non-local terms; four dimensions and four-body terms are not measured. The one loop of the flow of the coupling is a number for the first step of one rule, and the coefficient of $\ln b$ at small factors is 1.7–1.9 times the one-loop value.

(l) **Spinors are not among the grounds of the argument of this part.** Pure Yang–Mills has no spinor (fermion) fields. The structure of $SU(2) = \mathrm{Spin}(3)$ enters as properties of the group in three places: the twist (the element $-1$ of the centre $\mathbb{Z}_2$, 25-01), the curvature of the Haar measure (the Van Vleck prefactor $\psi/\sin\psi$ of the heat kernel, $1/9$ of the third order of BCH, $\frac{2}{3B}$ of four in a row; the Laplacian restricted to class functions is $\frac{1}{\sin\psi}\partial_\psi^2(\sin\psi\,\cdot) + 1$, and the constant 1 is the curvature term of the unit $S^3$), and the quaternion identity ((5) of 15-07). Each of these is a sign that this book's tools are specialised to $SU(2)$, not an indication of the cause of the gap.

Gauge theories of $G_2$, whose centre is trivial, are also said to confine by strong-coupling lattice arguments (the string breaks) Cited (K. Holland, P. Minkowski, M. Pepe, U.-J. Wiese, Nucl. Phys. B 668 (2003) 207–236, hep-lat/0302023; checked in the abstract, the text not checked), and we read the twist of the centre as a tool of measurement, not the mechanism of confinement Reading . The roads that use the centre as a probe (Parts 25 and 26, and the proposal of a one-step inequality with a fixed comparison partner) are limited to groups with a non-trivial centre.

02

## The index of questions — ultraviolet questions 1–31

Table 1 gathers into one the questions left by the chapters of Part 15. Ultraviolet questions 1–3 inherit the numbers of 26-11 (their content updated in 15-03), ultraviolet questions 4–15 assign a band of numbers to each chapter, and ultraviolet questions 16–31 are questions added in the revisions. "The current state of the answer" is attached only to the questions for which all or part of the answer has been written. The status and size of each question, and "what does not move even if settled", are in the table of the chapter of origin and are not copied here. Whichever question is settled, the row for ultraviolet control in the ledger of conditions does not close.

Table 1 The questions of Part 15 (the sentence of the chapter of origin shortened to one line; the conditions are in the table of the chapter of origin; what follows → is the current state of the answer)

| Q. | Question (one line) | Components concerned | Origin |
|---|---|---|---|
| 1 | Does the one-step map contract transversally on the one-parameter class it draws towards? (i) Do $\lambda$ and the source stay finite for blocks embedded in the four-dimensional lattice? (ii) Is the constant $C$ of the one-step inequality independent of the step? (iii) Is the weight of the sign of the centre $\mathbb{Z}_2$ needed as a separate parameter? | Class-closure condition, shape remainder condition | 15-03 § 07 |
| 2 | The smallest model (a hierarchical $SU(2)$) in which the net one-step bracket of the coupling can be shown including large fields. Can absorption under a constraint for a single plaquette be restated at the true critical point for many plaquettes? | Coupling-flow remainder condition, net one-step bracket of the coupling | 15-03 § 07 |
| 3 | On the twisted $M^4$ lattice, is the smallest eigenvalue of the operator of the adjoint fluctuations around an isolated flat connection at least $c/M^2$? Is the group version of closure under the fluctuation expectation ((iii) of BBS) unchanged by the twist? | The Gaussian comparison condition for the partition function in the twisted-torus ultraviolet statement | 15-03 § 07 |
| 4 | Extend the tube and the attracting curve of the hierarchical model to $\beta > 64$ (the seam between asymptotic estimates and interval arithmetic). → From $\beta \le 128$ downwards, the chain has been joined via the band side to the strong-coupling decision region (1137 steps) [Computer-assisted]. The upper end $\beta > 128$ and the curve at $[64, 128]$ are still open | Class-closure condition of the hierarchical model | 15-04 § 05 |
| 5 | Check the lower bound of the curvature outside the layer and the weight of the layer by interval arithmetic, and separate the flow coefficient $c^* = 1/60$ on the curve from $1/48$. → Enclosed as $[0.0151, 0.0188]$ at five places on the tube of narrowed radius [Computer-assisted]. That the curve enters that tube, and the curvature, are still open | Coupling-flow remainder condition of the hierarchical model | 15-04 § 05 |
| 6 | Along the sequence of coarse-plaquette sides $L$, does the source $\cdot L^4$ (now $7.7–8.5 \times 10^{-3}$, transverse thickness 4) settle to a constant as the transverse directions are widened? | The sum of the source over steps | 15-05 |
| 7 | On the tree-level network with all coarse plaquettes fixed, can a local representative be built that gives an upper bound uniform in the volume? Does the operator norm of $\lambda$ in the coarse directions fall below 1? → A local representative and an upper bound uniform in the volume were obtained (Bundle Network Theorem (local form)). The estimates are 0.245 in three dimensions and 1.83 in four; in four dimensions this route cannot go below 1.39 (undecided) | The transverse contraction (matrix) | 15-05 § 07 |
| 8 | Derive on paper the one-step correction to $O(1/B)$ for four in a row, giving the constant $C$ of the one-step inequality of the shape remainder condition. → In the projected model of four in a row, the Shape-Remainder Theorem for Four Plaquettes in a Row gave $C = 0.0108$ ($B \ge 32$, the box $O(1)$) | Shape remainder condition | 15-06 |

| Q. | Question (one line) | Components concerned | Origin |
|---|---|---|---|
| 9 | Is the attracting class still uniquely determined when loop shapes other than four in a row (2×2, L-shaped, mixed quartic) are included? Does the fixed point stay put when the definition of the diagonal is changed? | The coordinates of the class-closure condition | 15-06 |
| 10 | For four in a row, split the joint law of $P_2, P_3$ into a product of local couplings × a density, and find the lowest degree in $\theta$ of the score of the density and the isotropy of the quadratic form. → The lowest degrees are $\theta^0\beta^{-1/2}$ and $\theta^1\beta^0$. The shape part goes into a geometric series, while for $SU(2)$ the budget for the group-geometric part (the third order of BCH, $1/9$) diverges. The improvement lies on the side of the class | Absorption of $\eta$ | 15-07 |
| 11 | Can the smallest multi-step statement be written in the hierarchical model or a two-dimensional model (the background-regularity condition and the contraction of the many-body norm)? The width of the band of analyticity of the coupling. Numbers for the smallest systems are the next test | The multi-step induction | 15-07 |
| 12 | Derive the Lipschitz constant of the read-out on paper, and measure on a slab the difference in total variation between the density with one large field inserted and without. → Done for one link (the Large-Field Remainder Bound Lemma, the Heat-Kernel Fluctuation Rarity Lemma). On the $2 \times 2$ slab with heat-kernel fluctuations, a form that holds for $c_0^2 > 2$ (the convention of the Large-Field Remainder Bound Lemma); the value on the slab is 1/3 of that for one link. Many links under the true measure are still open | Row of the remainder including large fields, net one-step bracket of the coupling | 15-08 |
| 13 | When a source $s\,\mathrm{Re}\,W_a$ is added to four in a row, is the shift of the coupling linear in $s$? On paper, the position-dependent coupling $\beta(x)$ for $U(1)$ Villain. → In one heat-kernel step, the source from fine loops is exactly a shift of the coupling for each representation. In one blocking step with four in a row, the shift by a source on a loop inside the block is $f_C^2 + O(1/\beta)$ (uniform in $\beta$). The $\beta(x)$ of Villain is a local average. Loops crossing boundaries and many steps are still open | Observable-source condition | 15-08 |
| 14 | Does the parity of critical points hold for four plaquettes (paired with the question "splitting the density of the joint law")? → It breaks in the breathing direction. The conclusion of the proposal on the parity of critical points stands for a chain in a row by the covariance under reversal of the background. $\theta^3$ in four dimensions is still open | Row of background-dependent absorption and many steps | 15-08 |
| 15 | Split the one-loop Hessian of a block with blocking factor 2 and measure the coefficient of the ghost $F^2$ (negative control: $U(1)$). Formalise the direct-product decomposition of the tree gauge for finite groups. → The second half is done (Lean, finite groups). The cost is the degradation $(d-1)(L-1)\varepsilon$ of the small-field condition. In the background of the critical point, the ghost agrees identically with the tree gauge at one loop ($3^4$ points, coefficient $+0.024$ in a commuting background) | Row of the boundary of gauge re-fixing | 15-08 |
| 16 | In the smallest system that is not zero-sum (two copies of four in a row side by side, sharing the link between them), write $W^{\text{gauge}}$ and turn into a number the breaking of the local cancellation of the quadratic contribution to the surrounding couplings. → For seven in a row, $O(1)$ from one loop ($13/384$), and $O(1/\beta)$ after the first order is put in explicitly. The type of the tail is determined by the definition of the coarse variables | Row of the remainder including large fields, small-field contribution to the coupling | 15-08 |
| 17 | On the central plaquette of a 2×2 slab, derive by Wick's theorem the geometric coefficients of the odd orders of BCH, symmetrised over cyclic permutations, and compare them with the two-link $1/\sqrt{3(m+2)^3}$. → For a single loop (the cross), nothing remains if $a$ is updated by the group product. In three dimensions $2 \times 2 \times 2$ the corner triple product $\kappa = 3/1024$ is needed | Row of background-dependent absorption and many steps (group geometry) | 15-07 |
| 18 | In a class that has the logarithm $a$ of the background holonomy as a parameter, measure in one step in series whether the geometric part moves exactly into the flow of $a$. → Remainder $\theta^0\beta^{-3/2}$; the budget is uniform in the number of steps (conditional). The difficulty moves to the class-closure condition | Row of background-dependent absorption and many steps (improvement on the side of the class) | 15-07 |
| 19 | The condition of the One-Link One-Step Fluctuation Lemma (the recentring condition): make explicit the Taylor radius of the map to the direction of the sum of staples, and make the coefficient of $\delta b/b$ a single number. → $\lvert Q\rvert \geq 6\cos\theta_0$, $h_q \approx 4h$, $\eta = cg^2p^2$ (measured: $c \approx 12$). The coupling goes to the side of the parameters of the class | Row of background-dependent absorption and many steps (one Wilson link) | 15-07 |

| Q. | Question (one line) | Components concerned | Origin |
|---|---|---|---|
| 20 | Bound $\ell^4$ in four dimensions directly (the near part by a finite network, only the tail by Riesz–Thorin), and see whether $\sup \mu < 1$. → Row-weighted interpolation gives $\sup \mu \le 0.823$ (normalisation $\beta_0' = 1$, computed). Undecided for the direction-wise normalisation | The transverse contraction (matrix) | 15-05 § 08 |
| 21 | Define $\mathrm{Loc}$ in two ways, by row sums (zero momentum) and by matrix components, and obtain the difference in $W^{\mathrm{gauge}}$ as a rational number with the covariance of two blocks (is the cut the same as in the sources?). → The difference is not 0 ($\frac{1}{768}\lvert a-b\rvert^2/\beta$ with row sums, $-\frac{1}{384}a\cdot b/\beta$ with the projection onto components). The type in the sources is the row sum | Row of the remainder including large fields, small-field contribution to the coupling | 15-08 |
| 22 | In three dimensions without boundary, does the one-loop non-local weight in the coordinates of coarse links + global tags stay exponential? → Exponential at coarse $2\times 2\times L$ ($\kappa = 3.094$), and the sum is uniform in the volume. The four-dimensional tree is also exponential. The four-dimensional one loop is still open | Row of the remainder including large fields, cubic-remainder estimate | 15-08 |
| 23 | The second step of the multi-loop class: is the corner triple product $\kappa$ a coordinate preserved step by step, and do the same coordinates suffice in four dimensions $2^4$? → In three dimensions it does not contract and approaches the fixed point from below. Four dimensions needs the diagonal neighbours and the parallel type, and the ratio coordinates approach a fixed point (tree-level fit) | The coordinates of the class-closure condition | 15-07 |
| 24 | Measure the agreement of the two gauges and the coefficient of the ghost in a non-commuting background ($F$ of the commutator) | Row of the boundary of gauge re-fixing | 15-08 |
| 25 | Can the factor $\sup \lambda \le 3.29$ of the direction-wise normalisation be brought below 1 by a weighted symbol for each band of momenta? → It turned out to be a question in a direction that cannot be brought down: at $\sup \lambda = 4 \sup \mu_{\mathrm{loc}} \ge 0.59$, $\lambda$ is not a measure of contraction. The normalisation is fixed in the form divided by the long-wavelength coupling | The transverse contraction (matrix) | 15-05 § 08 |
| 26 | Let a parameter of the class carry the coupling $b$, and see whether the motion $c g^2 p^2$ of the coupling of one Wilson link moves into the update of the parameter | Row of background-dependent absorption and many steps (one Wilson link) | 15-07 |
| 27 | Derive the values at the fixed point of the ratio coordinates of the four-dimensional class directly from the half-infinite problem near the corner. Under rules that share fields, is the part that does not fit into a cell closed by putting it into finitely many pairs, or carried as a tail? | The coordinates of the class-closure condition | 15-07 |
| 28 | On the fixed point of transverse-averaging blocking, get the upper bound of $\sup \mu_\infty$ below 1 by a route other than Stein. Put the existence of the fixed point on paper | Multi-step shape remainder condition | 15-05 § 08 |
| 29 | With coarse links + global tags in four dimensions, measure the rate at which the quartic width of the three-body tree depends. In one row, the four-body terms and the contraction of the three-body image at one-loop level | Row of the remainder including large fields, local cancellation, cubic-remainder estimate | 15-08 |
| 30 | With transverse-averaging blocking or a large factor, does the dependence on the factor of the long-wavelength one-loop coefficient of the network approach the one-loop value? | Coupling-flow remainder condition | 15-05 § 08 |
| 31 | Run the re-taken tube over the whole of $[15.26, 128]$ with intervals of width 0.02, making unconditional both that the curve enters the tube and the enclosure of $c^*$ on the curve. Cover $\beta \ge 128$ by asymptotic intervals | Class-closure condition and coupling-flow remainder condition of the hierarchical model | 15-04 § 05 |

Reading Table 1. The questions "transverse contraction on a one-parameter class", "one-step correction for four plaquettes in a row" and "uniqueness of the class for other loop shapes" are questions about the coordinates of the class-closure condition and the shape remainder condition; the questions "minimal model of the remainder including large fields" and "Lipschitz constant of the read-out" are questions about the net one-step bracket of the coupling, corresponding to the content of the unpublished basic inequality (Theorem 2 of Bałaban 1989); the questions "splitting the density of the joint law", "minimal multistep statement" and "parity of critical points", and ultraviolet questions 17–19, are questions about the row of background-dependent absorption and many steps of the core (absorption of $\eta$ and the multi-step); and the question "smallest system that is not zero-sum" is a question about the row absent from the sources inside the small-field contribution to the coupling. Ultraviolet questions 21–24 and 26 are next moves arising from the answers of the revision (the difference of the

cut from the sources, the type of the tail for geometry without boundary, the second step of multiple loops, the ghost in a non-commuting background, the coupling as a parameter); the question "normalisation factors by direction" is a question about the transverse contraction of the direction-wise normalisation (the answer takes a negative form); and ultraviolet questions 27–31 are next moves arising from the answers of the third revision (the values of the coordinates of the four-dimensional class, the upper bound for averaging blocking, the many-body tail in four dimensions, the one-loop coefficient, the upper end of the hierarchical model).

Questions that numbers for the smallest systems can answer are mixed with paper questions.

03

## The dependency diagram of the chapters

We arrange the dependencies between chapters so that prerequisite chapters come above (the number of a prerequisite is smaller than the number of the conclusion). To the right of the arrow are the earlier chapters that the chapter uses.

```
[15-00] Entrance       problem, the three conditions, form that closes in one step, types of misreadi
[15-01] Difficulty     ⟸ [15-00]        dimensionless coupling, group-valued fields, gauge
[15-02] Sources        ⟸ [15-00]        series, Bałaban's two theorems, BBS chain, twisted-torus ult
[15-03] Decomposition  ⟸ [15-01][15-02] class-closure condition, the two remainder conditions, conti
[15-04] Heat kernel    ⟸ [15-03]        telescoping sum, invariant curve, tube (β∈[2.46,128]), chai
[15-05] Bundle network ⟸ [15-03][15-04] Unit Bundle Theorem, inequality, table of boxes, Bundle Netw
[15-06] Parameters     ⟸ [15-03][15-05] couplings by loop shape, attracting class, Four-in-a-Row The
[15-07] Background η   ⟸ [15-01][15-02][15-03]  Background–Fluctuation Decomposition Lemma, Lemma on
[15-08] Four open rows ⟸ [15-05][15-06][15-07]  proposals for four rows, Heat-Kernel Fluctuation Rar
[15-09] Index          ⟸ [15-00]–[15-08]
```

The diagram shows the dependencies of the chapters of the study of ultraviolet control (15-00 to 15-09). The systematic attempts (15-10) and the chapters of the second half on one step of the free field and one scale (15-11 to 15-14) are outside this diagram. To check it, see that the number of every chapter to the right of "⟸" is smaller than the one on the left. There are no cycles. No node of this diagram is a node saying that ultraviolet control has been shown.

04

## Summary of the five grades

Table 2 picks only the main rows from the five grades table of each chapter. The full statements, grounds and ranges of the rows are in the table of each chapter.

Table 2 The main rows of each chapter (labels as in each chapter's table; only some of the conditions in the range column are copied)

| Chapter | Main row | Grade | Range (excerpt) |
|---|---|---|---|
| 15-00 | Number of steps $K \to \infty$, lower bound for the product of losses | Proved here | $0 \le \delta_k \le \frac{1}{2}$ |
| 15-00 | Ultraviolet control and the form that closes in one step | Assumption | From $a$ to $L_0$, constants independent of the step |
| 15-01 | Logarithmic divergence of $\sum g_k^2$, the sum of $g^{2+\delta}$ is finite, the Orbit-Direction Flatness Proposition | Proved here | Under the one-loop flow (arithmetic) |
| 15-02 | The Conditional Ultraviolet Stability Theorem (Theorem 1 of Bałaban 1989) with the assumption of the condition that the coupling stays small; the basic inequality unpublished; the One-Step Bound on the Activity Norm (Lemma A.6.1 of BBS) | Cited | Abstract and introduction / text of Appendix A.6 |
| 15-02 | The ultraviolet statement for a small twisted four-dimensional torus | Assumption | $SU(2)$, $n12 = 1$, finite volume |

| Chapter | Main row | Grade | Range (excerpt) |
|---|---|---|---|
| 15-03 | The Coupling-Smallness Induction Lemma, the bookkeeping of the transverse contraction | Proved here | Under the net one-step bracket of the coupling and the definition and continuity of the effective coupling / assuming the one-step inequality |
| 15-03 | The decomposition statement (the class-closure condition is the one-parameter attracting class + a contracting form) | Assumption | Form of the statement only |
| 15-04 | Saddle-point estimates for MK (1/16, 1/3600, $c^* = 1/60$) | Proved here | Leading term at the saddle |
| 15-04 | Invariance of the tube, pointwise contraction, attracting curve | Computer-assisted | MK map, single plaquette of $SU(2)$, $\beta \in [2.46, 128]$ (curve $[15.26, 64]$) |
| 15-05 | Unit Bundle Theorem, $1/(16d-8) \le \lambda \le 1/16$, inequality for coarse-plaquette side $L$, Bundle Network Theorem | Proved here | Tree level restricted to commuting configurations, 4 coarse plaquettes symmetric (lower bound) |
| 15-05 | Table of boxes, type of the boundary, source $\approx 8 \times 10^{-3}/L^4$, $\lambda > 1/16$ in the coarse directions | Reading | Small boxes, one loop (starting from the heat kernel, $\beta \to \infty$) |
| 15-06 | $D = \frac{2}{5} + \frac{24}{5}\frac{q}{B}$, the tree map and 9/89, $\gamma^* = 37/76$, $c = 2/3$ | Proved here | Four in a row, one-loop level / leading term of Laplace |
| 15-06 | Fundamental + adjoint root $\rho^* = -\frac{1}{4} - 0.84/\beta$, no root for the prefactor family | Reading | $SU(2)$, quadratic, $\beta \ge 12$ (root) / $\beta \in \{4, 8, 16, 32\}$ |
| 15-07 | The Background–Fluctuation Decomposition Lemma, the Lemma on One Step with Background-Independent Fluctuations, the quaternion identity | Proved here | The Lemma on One Step with Background-Independent Fluctuations: one step, when the law of the fluctuations does not depend on the background |
| 15-07 | The group versions of the exponential moment of fluctuations and of background regularity, the proposal of the shift of local couplings | Assumption | Statement and proposal |
| 15-08 | The large-field rarity condition is a Chebyshev corollary of the group version of the exponential moment of fluctuations; the tree gauge has no Jacobian | Proved here | Assuming the group version of the exponential moment of fluctuations / axial gauge |
| 15-04 | The chain from the narrowed tube via the band side to the strong-coupling decision region (within 846 steps at $\beta \le 64$, 1137 steps at $\beta \le 128$) | Computer-assisted | MK map, $\beta \in [0.95, 128]$; the band side is a prior record |
| 15-05 | Bundle Network Theorem (local form): local representative and an upper bound uniform in the volume, $\Lambda_3 = 0.245$, $\Lambda_4 \approx 1.83$ | Proved here , Reading | Commuting, tree level, periodic / four dimensions is an estimate |
| 15-06 | Four-in-a-Row Theorem: $r_n \le \lambda^n r_0 + C/((5-\lambda)B_n)$, $C = 0.0108$ | Proved here , Reading (constants) | Four in a row, projection of $\psi^8$, $B \ge 32$, the box $O(1)$ |
| 15-07 | The One-Link Heat-Kernel Comparison Lemma, four decompositions, third order of BCH $1/9$, class $(B, q, a)$, divergence of the budget | Proved here | One link / leading Gaussian term, two links / one-loop arithmetic |
| 15-07 | What remains in the smallest systems: $\theta^0\beta^{-1/2}$, $\theta^1\beta^0$ | Reading | Two links, four in a row, $\beta \le 32$ |
| 15-08 | The Heat-Kernel Fluctuation Rarity Lemma, the Large-Field Remainder Bound Lemma, the accounting of the source, the tree gauge (Lean for finite groups) | Proved here | The three heat-kernel conditions (independent edge by edge, one step, background-independent), one link / finite groups, $SU(2)$ |
| 15-08 | Splitting of the small-field contribution to the coupling: the Taylor truncation has the same form as Lemma 7.5.3 of the sources; what is absent from the sources is the group version of $W^{\text{gauge}}$ | Assumption , Proved here , Cited , Reading | The analyticity condition at complex backgrounds, margin $q$ / empty for the zero-sum system |

| Chapter | Main row | Grade | Range (excerpt) |
|---|---|---|---|
| 15-08 | Statements of the four rows and their dependencies (the row of background-dependent absorption and many steps is the core) | Assumption , Reading | Statements, estimates |
| 15-03 | Proposal for the final form of the class coordinates of the class-closure condition | Assumption | One step in three dimensions, two links; the reading of the cross |
| 15-04 | The chain of the hierarchical model does not carry over to four dimensions (the unproved step of the sandwich, the margin of the tube) | Reading | Checking the sandwich against the sources; small boxes |
| 15-05 | One-Step Contraction of the $\ell^4$ Cone: $\sup \mu \le 0.823$ by row-weighted interpolation | Proved here , Reading | Commuting, tree level, kernel of $20^4$, normalisation $\beta_0' = 1$, no interval estimates; direction-wise undecided |
| 15-07 | In the class with the background $a$ as a parameter, remainder $\theta^0 \beta^{-3/2}$; the budget is uniform | Reading , Proved here (arithmetic) | Two links, four in a row, conditional |
| 15-07 | In one step with faces, $a$ is the group product; the corner triple product for the multi-loop class | Proved here , Reading | Single loop / $2 \times 2 \times 2$, one step |
| 15-08 | $W^{\text{gauge}}$ of two blocks is $O(1)$ from one loop; the type of the tail is determined by the definition of the coarse variables | Proved here , Reading | Seven in a row, chain, slab, column, one step, quadratic coefficients; four dimensions still open |
| 15-01 | The measured one-step coefficient: at the first step of the periodic network, 2.9–4.4 times the one-loop value; the dependence on the factor is 1.7–1.9 times | Reading | $SU(2)$, one loop, transverse $6^3$, first step |
| 15-03 | Four-dimensional class: one parameter + ratio coordinates (2 quadratic, 3 cubic, 65 quartic); the diagonal neighbours are non-zero whatever the rule; the local part of the class-closure condition is stated together with its rule | Reading , Proved here (counts, degrees of freedom) | Tree $L \le 10$, one loop $L \le 4$; the periodic network is the quadratic order of the tree |
| 15-03 | One-row version of the recursion of the inter-block terms of the net one-step bracket of the coupling: contracting image + new term independent of the step + exponential three-body tail | Reading , Proved here (bookkeeping) | One row, definition without conserved quantities, first order, up to quartic order |
| 15-04 | Enclosure $[0.0151, 0.0188]$ of $c^*$ on the re-taken tube | Computer-assisted | 50 intervals at five places; that the curve enters the tube is still open |
| 15-04 | Note on the proposal of a one-step inequality with a fixed comparison partner: the threshold for finite boxes is 1; within the resolution at $6^4/3^4$ | Proved here , Reading | Small boxes; open for every $\beta$ |
| 15-05 | Note on normalisation ($\lambda$ is not a measure of contraction; Stein floor 0.452); One-Step Contraction of the $\ell^4$ Cone is limited to the first step; multi-step lower bounds for transverse averaging 0.64, 0.57, 0.64, 0.62 | Proved here , Reading | Commuting, tree level, periodic, $2^4$, no upper bound |
| 15-05 | Coarse links + global tags are exponential on the four-dimensional tree; the long wavelengths of the network are 1 at tree level, with no running of $\ln L$ | Reading | Tree level, $n \le 16$ / one loop with blocking factor 2 |
| 15-07 | The corner triple product does not contract at the second step and approaches the fixed point from below; the fixed point of the iteration lies on the 0 side of the genuine limit | Reading , Proved here (bookkeeping) | Three-dimensional tree $L \le 20$ / four-dimensional tree |
| 15-08 | Group version of the recursion of $W$ and the many-body tail (one row); three-dimensional one loop of the coordinates of the cubic-remainder estimate; the difference in $\mathbf{Loc}$ between the two definitions | Proved here , Reading | One row, two blocks, $L \le 8$ / coarse $2 \times 2 \times L$ |
| 15-08 | In the background of the critical point, the two gauges agree at one loop | Proved here , Reading | One step, one loop, $3^4$ points, commuting background |

05

## The Lean correspondence of this part

Table 3 lists the bundles written in Lean 4 for the elementary parts of the text of this part, as finite sums, real algebra and trigonometric identities. None of them contains `sorry`, and the axioms printed at the end were only the three standard ones (propext, Classical.choice, Quot.sound). The re-check is also done, and every bundle passed (with only the three standard axioms). The labels in the text stay in the five grades (paper proofs stay [Proved here]); Lean is kept as an outside column, with the theorem names and assumptions added on the same line in the text of each chapter. **No bundle contains the lattice, the group, the measure or the minimisation itself.** What they contain is single algebraic lines from inside the paper proofs.

Table 3 The Lean bundles of this part (re-check: all done; assumptions on the same line as each theorem. The last twenty-five were added on 2026-10-17, 10-22 and 10-29, and have been re-checked. The four from DominantMatrixContraction to ConeCalculus belong to the systematic chapter 15-10 (the bundles of 15-10 also include OneStepWindow, added on 10-22, and ConeCalculusSOS, added on 10-29, six in all); the seven after them were added on 10-22, and the last seven on 10-29)

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGauge.ClassResidualTelescoping | `telescoping_residual` (increment of the deviation at each step $\le r_k$), `telescoping_residual_multi` (same assumptions, n parameters), `inv_sq_sum_le` ($a, c > 0$), `uniform_deviation` ($0 \le q < 1$, increment $\le C/(a+ck)^2 + Dq^k$) | 15-03 § 04, 15-04 | Measure, curvature, map | done |
| LatticeGauge.HierarchicalShapeFlow | `quarticStep_fixed` (fixed point 1/3600 of $\sigma \mapsto \sigma/16 + 1/3840$), `shapeStep_contract` ($k \ge 2$), `flow_coefficient` ($1/48 - 15/3600 = 1/60$), `affine_contraction` ($0 \le \lambda < 1$, each step $d_{k+1} \le \lambda d_k + \varepsilon$), `tube_iterate` | 15-04 | The interval-arithmetic tube, the derivation of the leading term at the saddle | done |
| LatticeGauge.FluxNetwork | `ratio_le_max` (each $f_i^2 \le M$, $\sum f^2 > 0$), `ratio_ge_of_unit_flux` (unit bundle), `energy_split` ($\sum f\psi = 0$), `thomson_lower`, `thomson_attained`, `minimizer_unique` (the set is closed under midpoints), `quad_lower` ($\beta > 0$), `chain_one_step` (five inequalities) | 15-05 § 02, 15-07 Lemma on One Step with Background-Independent Fluctuations | Projection, the Dirichlet principle, interchange of differentiation in the Taylor seminorm, invariance of Haar measure | done |
| LatticeGauge.AllFacesBlocking | `fine_energy_lower` (the map to sets of four plaquettes is injective), `block_energy_lower`, `smooth_ratio` ($n \neq 0, \theta \neq 0$), `equal_split_ratio` | 15-05 § 07 | Pseudo-inverse, coarse Bianchi, Fourier symbols | done |
| LatticeGauge.RatioTwoCriterion | `ratio_two_iff`, `one_loop_D` ($B \neq 0$, in the form with the Gaussian moments substituted), `gamma_family_one_loop_lower` ($c < 3$), `fund_adj_one_loop_root` ($\beta \neq 0, 1 + 2\rho \neq 0$), `no_pointwise_sqrt_one_loop` | 15-06 | Haar measure, moments, integration by parts, saddle-point expansion | done |
| LatticeGauge.FundAdjointRefit | `refit_solution` ($\beta \neq 0$), `tree_quadratic_split`, `tree_value_at_saddle`, `image_rho`, `image_gap` ($-53/356 + 1/4 = 9/89$), `image_one_loop_D` | 15-06 | Convolution, reduction to the tree, the envelope argument | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGauge.PrefactorFlowFixedPoint | `step_k2`, `step_k2_fixed_iff` (fixed point 37/76), `step_k2_contract`, `fixed_point_general`, `transverse_hessian` ($\sin a, \sin b \neq 0$), `convolution_prefactor`, `rem_shape_geometric` ($0 \le \lambda < 5$, assuming the one-step inequality), `ratio_coefficient` | 15-06, 15-03 § 04 | Convolution on the group, the Laplace method, one-loop formulas | done |
| LatticeGauge.HeatKernelFourPlaquette | `conv_assoc`, `convPow_add`, `four_plaquette_of_semigroup` ($p_s * p_t = p_{s+t}$), `four_plaquette_convPow` | 15-06 § 05 | Haar measure, the heat kernel itself (finite-group version) | done |
| QuaternionReImSplit | `re_mul_split`, `commutator_re`, `J_imI` and others, `two_link_split` (assumption on the components of $W$, $W_0$) | 15-07 § 06 | Measure, critical point, score, projection | done |
| LatticeGauge.ShapeRemainderUniform | `uv_uniform` ($0 \le \lambda < 5$, $C \ge 0$, $5B_{n+1} \le B_n$, the one-step inequality), `stays_in_box`, `pow_sub_pow_norm_le`, `keff_half_le` ($B > 0$), `cov_eigen` | 15-06 Four-in-a-Row Theorem | Lifting, the estimate of the Laplace remainder, the numerical values of the constants | done |
| LatticeGauge.HierarchicalTubeChain | `tube_exit` (within the tube, either one step down by $\delta > 0$ or exit), `cell_descent`, `chain_compose`, `tube_weight_mono`, `steps_numeric`, `mk_full_chain_64` (846 steps) | 15-04 § 05 | The map $T$, the tube, interval arithmetic, the theorems on the band side (taken as assumptions) | done |
| LatticeGauge.ShapeResidualGeometric | `forced_uniform` ($0 \le \lambda < 1$, $E \ge 0$), `geom_partial_le`, `residual_sum_uniform` (partial sums $\le U$), `harmonic_part_unbounded` ($c/(k+1) \le \eta_k$) | 15-07 § 06 | Measure, critical point, score (only the bookkeeping of real sequences) | done |
| LatticeGauge.LargeFieldSource | `weighted_tail_le_exp_moment` (weight $\ge 0$, $\sigma \ge 0$), `union_tail_le_sum`, `coarse_source_commute`, `coarse_source_commute_comp` (finite, nested steps) | 15-08 § 02, § 03 | Heat-kernel comparison, the sum over images, the Lipschitz constant of the read-off, the source from fine loops | done |
| LatticeGauge.TreeGaugeFixing | `exists_gauge_tree_trivial`, `gauge_tree_trivial_unique`, `configEquiv`, `fiber_card_const`, `regauge_bijective` (shared root), `regauge_sum_eq`, `slice_average_tree_indep` (finite group, finite graph, gauge-invariant function) | 15-08 § 05 | Haar measure and Fubini for continuous groups, the estimate of the cost | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGauge.WilsonWindowComparison | `one_sub_cos_window` ($0 \le \theta \le \theta_0, \theta_0^2 \le 24$), `wilson_weight_window` (same, $b \ge 0$), `window_constant_chain`, `window_split_bound` | 15-07 § 05 | Lower bound of the heat kernel, Bessel asymptotics, the seminorm step of the One-Link One-Step Fluctuation Lemma | done |
| LatticeGauge.QuarticRowColumnBound | `weighted_quartic`, `row_quartic`, `schur_quartic` (upper bound by the row and column $\ell^1$ sums), `representative_bound` (agreement of the representatives), `cone_step`, `cone_chain` | 15-05 Bundle Network Theorem (local form) | Riesz–Thorin, the identity of the axial cone, periodisation, analyticity of $R$ | done |
| LatticeGauge.BCHCubicResidual | `nT_sum_diff`, `shape_split` (no assumptions), `score_generator_split`, `bch3_coefficient` ($m + 2 \neq 0$), `geometric_variance_m1` | 15-07 § 07 | Measure, Gaussian moments, projection, the BCH series | done |
| LatticeGauge.SmallFieldLocSplit | `loc_split_unique` (polynomial identity), `geom_tail_le` ($A \ge 0, 0 \le q < 1$), `taylor_tail_bound` ($0 \le s < \rho$, $\lvert a_m \rvert \le M/\rho^m$) | 15-08 § 06 | Cauchy estimates (analyticity), the analyticity condition at complex backgrounds, the probability of the cut, the effective action | done |
| LatticeGauge.BackgroundBudget | `budget_term_sq` ($g \neq 0$), `budget_unbounded_of_threshold` ($g_j^2 \ge 1/(A(j+1))$, $0 \le g_j \le h_j$), `inv_sq_partial_le` ($b > 1$), `inv_sq_budget_bounded` | 15-07 § 07 | Measure, lattice, renormalisation steps (real arithmetic only) | done |
| LatticeGaugeInterBlockLocAbsorption | `loc_split` ($K$ symmetric), `nonlocal_vanishes_on_constant`, `rowSum_add`, `two_site_split`, `two_site_unique`, `tree_numbers`, `one_loop_numbers` (arithmetic of substitution only) | 15-08 § 06 | The one-loop Laplace formula, $O(1/\beta)$, the group, the measure | done |
| LatticeGaugeBlockCouplingCancellation | `locM_idem`, `locLin_idem`, `split_unique`, `bil_split`, `trace_locM`, `deltaBeta_nonloc` (the retained pairs are reflexive), `frob_loc_nonloc`, `frob_pythagoras`, `Wg_explicit`, `two_block_W` (two blocks), `Wg_zero_of_blockDiag`, `loc_psd` ($C$ positive semidefinite), `block_cancellation` and others (24 theorems) | 15-08 § 06 | The content of (A.5.3)–(A.5.4), the group version of the local cancellation, the lattice, the measure | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGaugeSmallFieldRemainderAssembly | `one_step_remainder` (Cauchy-type assumption on the coefficients, assumption on the cut, margin $h_0 \le qh$), `gamma_ge_one`, `cut_sum_le` ($A \ge 1$, $\gamma \ge 1$), `taylor_step_le` ($m \ge 3$, $r_0^2 \ge 1$), `schedule_match`, `total_remainder_uniform` (same assumptions, uniform in the number of steps) and others (16 theorems) | 15-08 § 06 | Analyticity (the analyticity condition at complex backgrounds), the probability of the cut, the contraction of the shape coordinates | done |
| LatticeGaugeToeplitzTailSplit | `split`, `split_unique`, `qform_split`, `cform_split`, `tail_sum_le` ($0 \le \rho < 1$, $\lvert q(r)\rvert \le C\rho^r$), `flat_row_sum_lt` ($L \ge 1$), `flat_not_uniform_geometric` | 15-08 § 06 | The statement of permutation symmetry, the one-loop formula, the group, the measure | done |
| LatticeGaugeWeightedTailBound | `weighted_geom_le` ($0 \le \rho < 1$), `weighted_tail_le`, `weighted_sum_uniform_linear` ($\lvert q(r)\rvert \le C(r+1)\sigma^{2r}$, $0 \le \sigma < 1$) and others | 15-08 § 06 | That the measured weights satisfy the assumptions (computation up to $r \le 8$), the slab, four dimensions | done |
| LatticeGaugeTailTypeClassification | `flat_weighted_sum`, `flat_weighted_unbounded` ($c > 0$), `exp_weighted_le`, `power_sum_uniform` ($s > 1$), `power_sum_unbounded` ($s \le 1$), `power_sum_unbounded_borderline` | 15-08 § 06, 15-05 § 07 | Which assumptions the measured tails satisfy, the multiplicity of shells, the lattice | done |
| LatticeGaugeClassParameterBudget | `ghat3_forms`, `ghat3_zero_iff`, `inv_sq_partial_le` ($b > 1$), `class_parameter_budget_bounded` (threshold $h_k \le Hg_k$, one-loop upper bound, $\eta_k \le h_k(Ag_k^3 + Dh_k g_k^2)$) | 15-07 § 06 | Measure, lattice, projection, renormalisation steps | done |
| LatticeGaugeStokesSecondOrder | `log_quadratic_part` (ring), `log_quadratic_part_five`, `commSum_eq_zero_of_comm` (commutative ring), `pure_quaternion_commutator` | 15-07 § 07 | Measure, integration, coarse-graining | done |
| LatticeGauge.CubeFaceCoupling | `total_sq`, `opp_eliminate` ($\sum \theta = 0$), `general_to_diag_adj` (same), `diag_adj_unique` (the two forms are equal under the same assumptions), `coeffs_from_eigen` | 15-07 § 07 | Lattice, group, integration, minimisation | done |
| LatticeGaugeStapleSumLowerBound | `sum_norm_ge_of_inner_ge` (real inner product space, $\lVert u\rVert = 1$, $\langle x_i, u\rangle \ge c$), `card_cos_ge_quadratic`, `first_order_ratio` ($\cos\theta > 0$), `quadratic_margin`, `radius_identity` | 15-07 § 05 | Quaternions, the geometry of staples, the Taylor seminorm, the composition of steps | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGauge.QuarticSplit | `titu_two`, `quartic_two` ($a, b > 0$), `quartic_triangle` (upper bound on the sum of fourth powers of the parts), `class_split` | 15-05 § 07 | Stein interpolation, the one-sided quadratic remainder, the sup of the symbol | done |
| LatticeGauge.GhostDeterminantMatching | `det_gauge_plus_ghost_split`, `log_tree_eq_cov_sub_two_ghost_blocks` ($\det H' > 0$, $\det K > 0$, the tree quantity is $\det H' / \det K$), `coeff_tree_eq_coeff_cov`, `coeff_drop_ghost_shift`, `coeff_ghost_constant` and others | 15-08 § 05 | The Faddeev–Popov lemma itself (change of variables in the integral) | done |
| LatticeGauge.BondMovingSandwich | `bond_moving_finite` ($w_i \geq 0$, $\sum w_i a_i = 0$), `iterate_le_of_le_of_monotone` ($T$ monotone), `comparison_fails_at_fixed_point`, `shrink_shape_shift`, `shrink_shift_lower` ($0 < \alpha$, $q \geq 3$) | 15-04 § 05 | Lattice, group, measure, the partition-function inequality itself | done |
| LatticeGauge.InfluenceRecursion | `tp2_comp` (finite ordered set, TP2 assumption), `cdf_upper`, `cdf_lower`, `k_upper`, `step_bound`, `gauss_product` | How the core Poincaré uniformity assumption of Part 23 is supported (23-12; the text is in Part 23, 15-00 § 04) | The derivative formula of the lemma, Hoeffding's formula, the integral version of TP2, the floating-point computation itself | done |
| LatticeGauge.CornerCouplingAffineFlow | `fixpt_is_fixed`, `iterate_dev` ($\lambda \neq 1$), `abs_iterate_dev` ($0 \leq \lambda < 1$), `ratio_gt_one_below_fixpt` ($0 \leq \lambda < 1$, $0 < r < s/(1-\lambda)$), `one_step_ratio` ($3/160$) | 15-07 § 07, 15-03 § 02 | Lattice, group, integration, minimisation (real linear maps only) | done |
| LatticeGauge.HypercubeFaceCoupling | `sigma_is_relation`, `relation_is_multiple_of_sigma` (24 identities per face), `opposite_in_cell_reduces`, `diag_adj_not_enough`, `diag_adj_skew_unique` (assuming the componentwise values and eigenvalues obtained by computation) | 15-07 § 07 | Lattice, group, integration (rational linear expressions only) | done |
| LatticeGauge.SkewCouplingSpectrum | `coeffs_of_spectrum`, `skew_zero_iff` ($e_3 = b + 4a - 4k$, $e_6 = b$, $e_8 = b + 2a + 2k$), `one_step_skew`, `increment_pow`, `tail_bound` ($0 \leq q < 1$, ratio of increments $\leq q$) | 15-07 § 07 | Lattice, group, minimisation; that the measured sequences satisfy the assumptions | done |
| LatticeGauge.HypercubeCochainCount | `faceCount_four`, `euler_cube` (no assumptions), `hypercube_cochain_ranks`, `edge_face_dof_agree` (exactness of the augmented cochain complex), `torus_site_euler` ($n \geq 1$), `torus_site_dof_agree` | 15-03 § 02, 15-07 § 07 | Exactness itself, Bundle Network Theorem (iii), minimisation (natural-number and integer arithmetic only) | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGaugeNonlocalImageRecursion | `affine_iter` ($\lambda \neq 1$), `fixed_point`, `affine_bound_sum` ($0 \le \lambda$), `affine_bound_uniform` ($0 \le \lambda < 1$, $0 \le C$), `quartic_block_charpoly`, `relative_rates`, `linear_recursion_fixed`, `new_term_gamma` and others (12 theorems) | 15-08 § 06, 15-03 § 03 | Derivation of the matrix of the image, measure, integration, group, second order and above in $W$ | done |
| LatticeGaugeThreeBodyTailRecursion | `weighted_geom_le` ($0 \le \rho < 1$), `three_body_sum_le` ($0 \le C$, $0 \le \rho < 1$, $\lvert q(d)\rvert \le C\rho^{d}$), `affine_uniform`, `pair_tail_uniform` ($0 \le \lambda, \mu < 1$, two recursive inequalities), `tripleCount_small`, `monomial_counts` (finite counting) | 15-08 § 06 | That the coefficients satisfy the assumptions (computation of $d \le 7$), group, measure, four dimensions | done |
| CoarseFaceHodgeSplit | `range_sup_tags`, `finrank_ker_eq` ($d_2 \circ D_c = 0$), `disjoint_range_tags` (no assumptions), `decomp_unique`, `norm_sq_add_harmonic` ($D^{\dagger} b = 0$) | 15-08 § 06, 15-05 § 07 | That there are 3 tags on a concrete lattice, one loop, tail types | done |
| LatticeGauge.QuarticConeNorm | `ratio_sandwich` ($0 < b_{\min} \le b \le b_{\max}$), `ledger` ($b_{\min} \sum \theta^2 \le \sum \phi^2 \le b_{\max} \sum \theta^2$), `transfer`, `cone_b_to_a`, `stein_floor` ($A_i^2 \le F v_i$, $0 \le \nu_i \le A_i$) | 15-05 § 07 | The Mixing-Limit Lemma, the type of the long-wavelength minimiser, Stein interpolation | done |
| LatticeGauge.BlockCovariance | `cone_chain`, `cone_chain_uniform` ($M_k \le m$), `growth_ledger`, `inverse_decay` ($r\, g_k \le g_{k+1}$), `face_sum_sq`, `four_split_quartic_floor`, `even_split_quartic` | 15-05 § 07 | The Conditional-Mean Covariance Lemma, iteration of the symbol, divergence of the perimeter length, the numbers for $\mu$ | done |
| LatticeGauge.AveragedBlockFlow | `affine_iter`, `affine_dist` ($r \neq 1$), `tail_sum`, `tail_bound` ($0 \le \delta$, $0 \le r < 1$), `axis_flow_closed`, `axis_flow_values`, `cone_series` ($0 \le c < 1$, $x_{k+1} \le c x_k$) | 15-05 § 07 | Lattice, Fourier, minimisation; that the lattice flow follows the closed form, $\sup \mu_j \le c < 1$ | done |
| LatticeGaugeHierarchicalUpperTube | `tube_exit_top` (within the tube, either one step lowers by $\delta > 0$ or it exits), `chain_top`, `steps_numeric_128`, `steps_numeric_64`, `mk_full_chain_128` ($b_{\mathrm{lo}} = 0.95$, $b \le 128$, $\delta = 0.2196$, cells on the band side $\le 556$), `union_tube_step` | 15-04 § 05 | The map $T$, the tube, interval arithmetic (taken as assumptions) | done |
| LatticeGauge.TensionDomination | `iterate_bound_var`, `iterate_bound` ($c \ge 0$), `lower_bound_four`, `area_law_transfer` (one-step inequality assumed), `orbit_lower` (no monotonicity of the map), `step_fails_of_zero_to_positive` | 15-04 § 05 | Lattice, tension, the MK map (induction on real sequences only) | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGauge.BlockStepBetaShift | `beta_shift_of_inv_sq` ($1/g'^2 = 1/g^2 - 2b_0\ell$), `su2_coeff`, `log_steps`, `shift_bounds`, `tadpole_weights`, `vanvleck_ratio` | 15-01 § 01, 15-03 § 02 | Lattice, blocking, Hessian, measure (arithmetic only) | done |
| LatticeGaugeTwoBlockRowSumLoc | `sRow_eq_sComp_add`, `nRow_eq_nComp`, `row_sum_cancels` ($F$ symmetric), `row_split_unique` (the non-local part is 0 on the diagonal), `comp_split_unique`, `quartic_decomp_unique`, `two_loop_numbers` and others (14 theorems) | 15-08 § 06 | Integration, derivation of closed forms, expansion, group | done |
| LatticeGaugeMultiStepCone | `step_pair`, `contraction`, `cone_quartic` (two-step recursion $x_j \le a\,x_{j+1} + b\,x_{j+2}$, $a, b \ge 0$, $a + b \le 1$), `block_three` (iteration of three-step blocks, $0 \le \mu$), `cone_transfer` (one-step transfer of the cone, $0 \le c$) (5 theorems) | 15-08, row of the boundary of gauge re-fixing | The Stein values (numbers), the kernel itself, the tree-level and commuting assumptions | done |
| LatticeGaugeMultiStepPerturb | `three_product_perturb`, `three_product_perturb_grouped` (perturbation of a product of three in a normed ring), `block_three_affine` ($c(k+3) \le \mu\, c(k) + \varepsilon$, $0 \le \mu < 1$, $0 \le \varepsilon \Rightarrow c(3n) \le \mu^n\, c(0) + \varepsilon/(1-\mu)$) (3 theorems) | 15-08, row of the boundary of gauge re-fixing | The numerical differences between the kernels of successive steps; the extrapolation from step 4 on | done |
| LatticeGaugeQuarticCone | `four_amgm` ($\lvert abcd\rvert \le (a^4+b^4+c^4+d^4)/4$), `quartic_sum_le` (finite sums) (2 theorems) | 15-08 § 06 | Finiteness of the $\ell^1$ norm of the coefficients at each point is assumed; the actual coefficients of the many-body terms | done |
| LatticeGaugeColorAverage | `colorAverage` (finite P, X and $K : P \to X \to \mathbb{R}$; under the assumption (H) $\forall\,\psi$ $\Sigma_p(\Sigma_x\, K\, \psi)^4 \le R\, \Sigma\, \psi^4$, for three-colour $\theta$: $\Sigma_p\lvert(K\theta)_p\rvert^4 \le R\, \Sigma_x\lvert\theta_x\rvert^4$), `colorAverage_of_design` (for any design of non-negative weights satisfying the identity), `design_identity` ($\pm e_k$ with weight 1/15 and $(\pm1,\pm1,\pm1)/\sqrt{3}$ with weight 3/40 give $\Sigma\, w_i(s_i\cdot v)^4 = \lvert v\rvert^4/5$) and others (10 theorems) | 15-08, row of the boundary of gauge re-fixing | That K acts linearly with the same scalar coefficients on the three colours (tree-level, commuting linearisation) is assumed; the existence of the one-colour sup is put into (H); the $\ge$ direction (a one-colour field is a three-colour field) is not written | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LatticeGaugeCubicCone | `three_amgm` ($\lvert abc\rvert \le (\lvert a\rvert^3+\lvert b\rvert^3+\lvert c\rvert^3)/3$), `cubic_sum_le` (finite sums), `triple_sq_le`, `triple_amgm` (three-colour triple product $\lvert u\cdot(v\times w)\rvert \le (\lvert u\rvert^3+\lvert v\rvert^3+\lvert w\rvert^3)/3$), `cubic_cone_pullback` (from the cone $\lvert C\phi\rvert \le c\,\lVert\phi\rVert_3^3$ and the contraction $\lVert K\theta\rVert_3^3 \le \mu\,\lVert\theta\rVert_3^3$, $\lvert C(K\theta)\rvert \le c\mu\,\lVert\theta\rVert_3^3$), `block_three_affine`, `block_three_uniform` (7 theorems) | 15-08 § 06, core row | The values of Stein, $\kappa$, $\lambda$ (numbers), the kernel itself, the tree-level and commuting assumptions | done |
| SmallSetRowSum | `rowSum_total`, `row_cancel`, `total_split`, `owner_split`, `symmetric_split` (distribution of the small-set terms and row sums), `loc_idem`, `rem_vanish`, `piece_loc_zero` (decomposition of Loc; the (1−Loc) side vanishes for a uniform field), `theta_ledger`, `plate_numbers` and others (20 theorems) | 15-08 § 06 | Numerical values, the kernel itself, the tree-level and commuting assumptions; the plate is two-dimensional and one step | done |
| ThetaSquareOneStepAffine | `one_step_affine` (affine recursion for the contraction $\mu_k \le \bar{\mu}$ per step), `theta_ledger_mu`, `theta_ledger_half` (the ½ of the slab), `theta_ledger_quarter`, `theta_ledger_measured`, `sup_monotone_in_mu` (6 theorems) | 15-08 § 06 | The contraction values themselves (the ½ of the slab is computed), the kernel, the tree-level and commuting assumptions | done |
| DominantMatrixContraction | `dominant_mul`, `dominant_pow` (the norm of the identity is 1), `weighted_step`, `weighted_iter` ($Bw \le \lambda w$), `weight_change` ($av \le w \le bv$, $a > 0$), `liftWeight_step` ($B \ge 0$, $w \ge 0$, $s > 0$, $k \ge 1$, $B^k w \le s^k w$), `norm_pow_le_const_mul`, `sum_norm_pow_le`, `affine_uniform` ($0 \le \lambda < 1$, $\varepsilon \ge 0$), `aggregate_step`, `two_by_two_weight`, `two_by_two_exists`, `signedEx_sq`, `absEx_pow` (21 theorems) | 15-10 § 01 | The Collatz–Wielandt equality and Gelfand's formula themselves, the existence of the Perron vector, the $\ell^2$ weights (Schur's test), the max-plus sandwich | done |
| SignLossBound | `jensen_four`, `schur_four` (row $\ell^1 \le A$, column $\ell^1 \le B$), `coherent_of_neg_le` (negative part $\le \delta$ times the positive), `loss_le_of_coherent`, `comp_coherent`, `coherence_le_one`, `sum_four_le` ($\lvert v_a\rvert \le M$), `stein_form_le`, `stein_le_inv_sq_mul_test` ($\kappa A_r M \le \lvert y_r\rvert$), `abs_comp_le`, `row_l1_comp_le` (16 theorems) | 15-10 § 02 | Stein's formula itself (Plancherel and interpolation), the phase average of plane waves, Wielandt's theorem, the convergence to rank 1, the numerical values of the loss (computed) | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| ConstrainedExtension | `exists_extension`, `isExtension_iff_mem_coset`, `finrank_freedom` (finite dimension), `freedom_eq_zero_iff` ($W \neq 0$), `exists_extension_with_conditions` ($(\mathrm{res}_V, C)$ surjective), `finrank_freedom_with_conditions`, `conditions_count_le`, `eq_of_eqOn_iSup` (the family of subspaces spans), `comp_isExtension` ($f 1(V_1) \subseteq V_2$), `comp_depends_of_not_le` (20 theorems) | 15-10 § 03 | The lattice, the continuity of the symbol, the order of folding, moments, the candidate lattice steps (continuity of the uniqueness of the zero mode, the rule for the order of the tail, shifted jets) | done |
| ConeCalculus | `cone_pullback` ($c \geq 0$), `contracts_comp`, `contracts_iterate`, `block_decay` ($M \geq 0$, $\mu \geq 0$), `lp_schur` ($p \geq 1$, row $\ell^1 \leq R$, column $\ell^1 \leq C$), `lp_schur_sharp` (constant row and column sums), `delta_dilution`, `weighted_rpow_midpoint` ($w \geq 0$, $a > 0$), `weighted_rpow_shift` ($0 < a \leq R$, $p 0 \leq p$), `midpoint_below_one`, `affine_varying` ($0 \leq \mu_k \leq \bar{\mu} < 1$, $\varepsilon_k \leq \bar{\varepsilon}$, $c_k \geq 0$), `affine_eventually`, `affine_block` (20 theorems) | 15-10 § 05 | The operator norm itself, the formula for Stein's bound (interpolation and Plancherel), Riesz–Thorin, convexity in continuous $p$, Gelfand's formula, the lattice and the kernel | done |
| ColorAverageN | `colorAverage_of_design` (under `IsQuarticDesign w s κ`, the one-colour constant R is an upper bound for n colours), `weights_sum`, `pt_unit`, `design_identity` (with weight $1/(n(n+2))$ on the axis points $\pm e_k$ and weight $n/((n+2)2^n)$ on the cube vertices $\varepsilon/\sqrt{n}$, $\sum_i w_i (s_i \cdot v)^4 = 3|v|^4/(n(n+2))$, for all $n \geq 1$), `colorAverage` (the $n$-colour version without the design assumption, $\kappa = n(n+2)/3$), `oneColour_of_colorAverage` (the converse), `design_identity_two` ($n = 2$, $\kappa = 8/3$), `cube_moment`, and others (13 theorems) | The row of the boundary of gauge re-fixing in 15-08, 16-07 § 07 | The lemma of spherical averages, the application to $\mathfrak{g} \cong \mathbb{R}^n$ and to $\mathfrak{su}(N)$, the comparison with the literature | done |
| OneStepWindow | `holder_dual_ratio`, `norm_conj_rpow`, `exists_lpRatio_ge_row` (the one-coordinate Hölder lower bound, $p > 1$, including $k = 0$), `lamMax_mono`, `lamMax_smul`, `steinWeighted_ge`, `stein_row_weight_saturation` (the family of row weights saturates at $u = A^{p-2}$; $\lambda_{\max}$ is defined as the supremum of the Rayleigh quotient), `window_ordConnected`, `window_ordConnected_of_logConvex` (the window is an interval; convexity of $\log \mu_p$ is assumed), and others (20 theorems) | 15-10 § 05 | Courant–Fischer, the number $p_{\mathrm{row}} = 5.874$ and the interval $(3.04, 5.67)$ (computed), the operator norm itself | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| LargeSetWeight | `summable_shell_iff` (if the number of shells is polynomial, $\sum (n+1)P(n)A^{n+1}e^{-\kappa n}$ converges $\Leftrightarrow 0 \le A < e^{\kappa}$), `not_summable_term_poly`, `bbs_weight_vs_decay` (the form $(\eta-1)\kappa > L^d \log 2$), `bsv_weight_vs_decay`, `bsv_weight_vs_decay_log2` (independent of $a$), `card_shell` (the number of $\ell^\infty$ shells $64n^3+16n$, $n \ge 1$), `manhattanShell_one`/two/three (8, 32, 88), and others (16 theorems) | 15-08 § 06 | The reduction of the constant $c_R$ of the remainder to chains, the general formula for Manhattan shells, the value of the rate $\kappa$ (computed) | done |
| LatticeGaugeAcceptanceSandwich | `fermi_le_metro`, `half_metro_le_fermi` ($\frac{1}{2}\min(1,e^{-x}) \le 1/(1+e^{x}) \le \min(1,e^{-x})$), `barker_le`, `barker_le_of_crooks` (the Barker-type mean $\le 1/(1+e^{\Delta F}) = (1-\theta)/2$), `overlap_sandwich` ($e^{-\Delta F} O \le a_F \le O$), `mixture_sandwich` ($(1-\theta)O \le a_{\mathrm{mix}} \le 1-\theta$), `jarzynski_of_crooks`, `half_le_one_sub_sqrt`, and others (15 theorems; the assumption is the Crooks-type $P_F e^{-W} = e^{-\Delta F} P_R$) | 25-06 § 07, 25-08 § 03 | Bretagnolle–Huber, Pinsker, the driving procedure itself, the numbers | done |
| LatticeGaugeAbelianTwistAction | `abelianAction_le`, `abelianAction_lt_pi_sq` (for all $N \ge 1$, $2N^4(1-\cos(\pi/N^2)) \le \pi^2 - \pi^4/(12N^4) + \pi^6/(360N^8) < \pi^2$, no assumptions), `totalAction_eq` (for abelian configurations on $(\mathbb{Z}/N)^4$, the sum over the six planes $= 2N^4(1-\cos(\pi/N^2))$), `plaqTerm_mixed`, and others (13 theorems) | 27-02 base point at small $T^2$ | The minimal action of sector (1,1) $\le S_{ab}$ (the configuration space of $SU(2)$ is not included); the term $1 - z\cos$ and the placement of the twist are put as definitions | done |
| LatticeGaugeNestedBlocks | `nested_supermultiplicative` (finite probability space, large block = fibre of $s^2$ small blocks, the inequality of means $\mathbb{E}\prod\theta' \le \mathbb{E}\prod\theta$ assumed), `doubling_iterate`, `doubling_bound` (the doubling with $K = 4C$, $t_k \le K^{-1/3}(K^{1/3}t_0)^{4^k}$), and others (7 theorems) | 26-05 § 02, 26-02 § 03 | The Markov property, reflection positivity, ($C_{cb}$) | done |
| CenterFluxPolymerCertificate | `tree_k` (six tree inequalities), `kp_bound`, `kp_lt_one`, `rate_rational` ($\beta = 1029/10000$), `exp_le_inv_one_sub` ($0 \le y < 1 \Rightarrow e^{y} \le 1/(1-y)$), `final_real` (assuming $\tanh\beta \le \beta$) (6 theorems) | 26-06 § 04 | $\tanh\beta \le \beta$, the lemma of the induction on trees, the Kotecký–Preiss theorem, the full enumeration up to 10 plaquettes, the interval-arithmetic version ($\beta \le 0.1039$) | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| ClassFixedPoint | `dist_linear_add_le` ($\|T\| \le \rho$, $\mathrm{Lip}\, R \le L \Rightarrow \mathrm{Lip}(T+R) \le \rho + L$), `exists_unique_fixed` ($\rho + L < 1$), `norm_fixed_le`, `exists_unique_fixed_in_ball` (Lipschitz only on the closed ball, $C + (\rho + L) r \le r$), `norm_iterate_sub_le` ($\mathrm{Lip}(\Phi^{[n]} - T^{[n]}) \le (\tau + L)^n - \tau^n$), `exists_unique_fixed_of_iterate` (contraction in $n$ steps), `affine_orbit_bound`, `dist_fixed_fixed_le` (a deviation $\delta$ of the map gives a deviation $\le \delta/(1-K)$ of the fixed point), `tracking` (following a flowing parameter), `pull_orbit`, `exists_unique_entire_orbit` (for contractions that change step by step, exactly one two-sided infinite orbit staying in a closed set), `entire_tracking`, and others (24 theorems) | 15-03 § 04 | The contents of the table of assumptions (measuring the contraction and the remainder in one and the same norm, the values of the nonlinear coefficient $a$ and the source $c_0$, the contraction after moving $F^2$ into $\beta$, the correctness of the A-type Loc); the numbers of the window (computed) | done |
| LocalCancellationOneRow | `pairs_sum_to_total`, `total_uniform`, `pairAA_uniform`, `pairAB_uniform`, `loc_remainder0_eq_zero`, `pairAB_rank_one` (the pairs across blocks have rank one), `pairAB_local_plus_gradient`, `mirror` (the rational arithmetic of the coefficients of the $O(1/\beta)$ quadratic term for a row of seven with two blocks, no assumptions) (8 theorems) | 15-08 § 06 | The derivation of the rational coefficients (the heat-kernel computation of the row); the decay ratio $5 + 2\sqrt{6}$; the plate and four dimensions | done |
| RankOneRemainder | `pair_split` ($K a b = \frac{K}{2}(a^2 + b^2) - \frac{K}{2}(a - b)^2$), `edge_sum_split` (pairs over any finite set), `chain_split`, `chain_local_regroup`, `chain_remainder_eq` (on a row, the sum across blocks minus the local part leaves $= -\frac{1}{2}\sum_i K_i (a_i - b_{i+1})^2$), `chain_remainder_zero_of_match` ($a_i = b_{i+1}$), `chain_remainder_nonpos` ($K_i \ge 0$), `deflation_split`, `deflation_left_zero`, `deflation_right_zero` (the value of the uniform field $/= 0$), and others (12 theorems) | 15-08 § 06 | That the leading products number three on the plate; the size of what remains (computed); four dimensions | done |
| BlockDeflation | `blockDeflate_split` ($C = C B (A^{\mathrm{T}} C B)^{-1} A^{\mathrm{T}} C + R$), `blockDeflate_left_zero`, `blockDeflate_right_zero` (if $A^{\mathrm{T}} C B$ is invertible then $A^{\mathrm{T}} R = 0$ and $R B = 0$), `left_zero_inherit`, `right_zero_inherit`, `twoStep_zero`, `twoStep_split` ($A_2^{\mathrm{T}} R_1 B_2$ of the second step also invertible), `span_left_zero`, `span_right_zero` (the space spanned by the test fields) (9 theorems) | 15-08 § 06 | The size of the coefficients (computed); the correspondence with the order of Taylor; four dimensions | done |

| Bundle name | Main theorems and assumptions | Place in the text | Not included | Re-verification |
|---|---|---|---|---|
| ConeCalculusSOS | `cert_sound` (a certificate gives $P \le \lambda N$), `cert_mono`, `certValue_anti` (the larger the family, the smaller the value), `ratio_le_certValue`, `floor_of_coeff` (a floor from an equality of coefficients), `classSum_pushforward` (pushing down, no conditions), `classSum_pullback` (lifting when the map of classes is injective), `shift_rigid` (injective if $2w < L$ on each axis), `collide_L3`, `collide_L4`, `no_translate_Z` (counterexamples), `tensor_contracts`, `tensor_ratio_product`, `row_holder_sos`, `schur_weighted_bound`, `schur_attained`, and others (25 theorems, 2 lemmas) | 15-10 § 05 (the paragraph of the second chapter is not yet in the text) | SOS inside the polynomial ring (the identities are put as assumptions, as equalities of functions or of coefficients); the comparison with the signed Stein upper bound; $\mu_4(K_0) < 1$ on the whole space; the necessity of $L > 2w$; the numbers (computed) | done |
| LatticeGaugeH1HbarInterval | `cell_lower`, `cell_upper` (for $0 < \hbar_1 \le \hbar \le \hbar_2$, $\min(U/\hbar_1, U/\hbar_2) \le U/\hbar \le \max(U/\hbar_1, U/\hbar_2)$), `sum_lower`, `sum_upper` (forward sums), `rho_lower` ($0 < B \le \bar{B}$), `pieces_min` (the covering by small pieces), `margin_on_interval` (the margin on an interval), `threshold`, `margin_I1`, `margin_I2`–`margin_I6` (the margins $\ge 0.067$ with the numbers output by interval arithmetic put in), `margin_I1_coarse_fails` (a negative control) (15 theorems) | 23-15 § 05, 27-02 uniform gap of the double well | Interval arithmetic itself (the outputs in float64 with outward widths are assumed); Muckenhoupt; FKG; the remainder of the TAU interpolation; Lemma Tail; the numbers of the subintervals from the third on (computer-assisted); the quantitative uniform Laplace method | done |
| StrongCouplingOneSidedTerm | `four_mul_le_sq` ($L \ge 4 \Rightarrow 3L + L \le L^2$), `area_le_strip` ($0 \le u \le 1 \Rightarrow u^{L^2} \le u^{3L} u^L$), `strip_term_eq`, `strip_coeff_lower` ($0 < s \le \pi/(2L^2) \Rightarrow 16L^2/\pi \le 8/s$), `tube_coeff_neg` ($L \ge 4$, $s > x - x^3/4$, $0 < \hat{D} \le 3$), `tube_part_nonpos`, `h_main_term` ($\lvert R\rvert \le C u^{3L+1}$), `h_neg` ($Cu < K$), `h_neg_on_interval` ($u \le u_0$, $Cu_0 < K$) (9 theorems) | 27-02 band in the language of the coupling | Convergence of the expansion (Osterwalder–Seiler; the upper bound on the remainder is assumed); $\sin x > x - x^3/4$; $\hat{D} \le 3$; the coefficient of the leading term of $\ln Z$ (paper); the values of $C_L$ and $u_0$; a $\beta_0$ uniform in $L$ | done |

06

## Connection to Part 27 and the glossary

The row for ultraviolet control in the ledger of conditions (27-02) is split into smaller pieces by Table 1 of this part (questions) and 15-08 Table 1 (the four rows that do not close). Its label and state (Assumption, open) do not change. The cross index of questions (27-03) cites the ultraviolet questions of this part as a fifth origin. The

words added in this part (the invariant curve, the transverse contraction, the source, the bundle network, couplings by loop shape, the attracting class, the shift of local couplings) are in the A-02 Glossary.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Index: questions, dependencies, summary of the five grades, correspondence of the Lean bundles (this chapter adds no new results) | Ultraviolet control. The settlement of any of the questions |
| That the Lean bundles depend only on the three standard axioms (printed for each bundle, and re-checked) | Formalisation of the lattice, the group, the measure and the minimisation; the analytic parts of the paper proofs |

## Sources and reproduction

| Item | Kind | Source, tool |
|---|---|---|
| What cannot be said, questions, five grades | This book's study | The tables and sections of each chapter 15-00–15-08 |
| The Lean bundle | This book's study | The record of the formalisation of this book's study (Lean 4, with the axioms printed at the end of each bundle). Re-verified (every bundle passed) |

**Words used in this chapter** — what cannot be said, ultraviolet questions, the dependency diagram, the five grades, Lean bundles. Their meanings are in the A-02 Glossary.

Next chapter to read: 15-10 Systematic attempts — naming the recurring patterns of the record. To read alongside: 27-00 Entrance — the promise of this part (the whole conditional statement, including the row for ultraviolet control in the ledger of conditions).

# Systematic attempts — giving names to the recurring patterns of the record

This chapter is this book's study. It is not a component of the proof on the main line of argument. § 01– § 03 and § 05 are restatements in known frameworks, and § 04 is at the level of an estimate.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**What this chapter shows** — We give names and definitions to five pieces of handwork that recur in the computational record of Part 15. The first is the estimate of a matrix that "exceeds 1 in one step but contracts when several steps are taken together" (dominant matrices); the second is what is lost when signs are discarded to take an upper bound (the loss from discarding signs); the third is how to choose when a map determined only on a constraint is represented by a local kernel (local representatives); the fourth is the matching of the two expansions of strong and weak coupling (double valuation); the fifth is the computation that bounds a polynomial of degree $p$ by an $\ell^p$ cone and pulls it back step by step (the cone calculus). For each, we write candidate theorems, show row by row which known result each restates, set out which quantity the numbers of the record correspond to, and gather the remaining questions as the questions of the systematic attempts.

**Reading level** — Third year and above. § 01 uses the linear algebra of non-negative matrices, § 02 inequalities for finite sums, § 03 Fourier series on the lattice, § 04 only the leading terms of asymptotic series, and § 05 inequalities for finite sums and the statements of interpolation theorems. § 00 and the opening paragraph of each section alone can be read by second-year students.

**Prerequisite chapters** — 15-05 Bundle Network Theorem (local form) (local representatives and an upper bound uniform in the volume), One-Step Contraction of the $\ell^4$ Cone in 15-05 (the weighted Stein bound and the $\ell^4$ cone), 15-08 § 06 (the W coordinates and the multi-step cone).

**Said first** — This chapter is not the main line of argument. The results of this chapter have no direct effect either on the row for ultraviolet control or on the row for the crossover band in the ledger of conditions (27-02). Many of the candidate theorems are restatements of known facts — the Perron–Frobenius theorem, the Collatz–Wielandt formula, Gelfand's formula, contraction in a weighted maximum norm, the order of multigrid transfer operators, the sum rules of subdivision, the Riesz–Thorin interpolation theorem, the Schur test — and no novelty is claimed. What is new in this chapter lies on the side of the "questions". The aim is to write out which quantity of a known framework each number of the record corresponds to, and where it goes beyond the known framework. Labels are attached row by row: Lean (a statement that passed in a Lean file), Computed (a floating-point computation, not an interval estimate), Reading (only a line of argument on paper, or at the level of the abstract of a source), Cited (a standard known fact), Assumption (an estimate).

Order of this chapter


0. What this chapter shows — the five patterns and how to read this chapter
1. Dominant matrices and k-step contraction — how to build a weight that contracts in one step (Table 1)
2. The loss from discarding signs — the loss is the inverse square of the alignment (Tables 2–5)
3. The geometry of local representatives — the order of folding does not depend on the representative (Tables 6 and 7)
4. The geometry of the double valuation — a scale-independent $\kappa$ and the lens (Tables 8–10)
5. The cone calculus — the window is an interval; the correct form of dilution (Tables 11 and 12)
6. Dictionary with the known, what can be said, questions (Tables 13–15)


00

## What this chapter shows

| Dominant matrix | The loss from discarding signs | Local representative | Double valuation | The cone calculus |
|---|---|---|---|---|
| If it contracts in $k$ steps, a weight that contracts in one step can be built explicitly<br>"Exceeding 1 in one step" is a transient that depends on the choice of weight. A restatement of the line of Collatz–Wielandt and Varga (Lean) | The loss is bounded by the inverse square of the row alignment in the worst direction<br>Agrees at 5.55, 5.13, 5.31 for two to four steps on the $4^4$ torus (the inequality in Lean, the numbers computed) | The order of folding $\mathrm{ord}^V(F) = 2$ does not depend on the choice of representative<br>Within the types of the record, the tail norm is bounded only up to $D \le 4$. A restatement of the sum rules (Reading, Computed) | The product of the two leading coefficients $\kappa = s1w1$ does not depend on the scale of the coupling<br>The lens of the leading terms opens only when $\kappa \le \frac{1}{4}$. No payoff theorem yet (Assumption, Computed) | Gives definitions for the pull-back of degree-$p$ cones, the weighted Stein bound, the composition of steps and affine recursions<br>$\log S_p$ is convex in $p$ and $S_3 \le \sqrt{S_2 S_4}$. The naive dilution $m^{1-p}$ does not hold, and $S_p \le S_2 R^{p-2}$ is the minimal form (Lean, Reading, Computed) |

**How to read this chapter.** The record of computations in Part 15 contains the same kind of handwork again and again. There are five kinds: measuring the norm of a three-step composite and writing "it exceeds 1 in one step but contracts in three"; taking the absolute value of a signed kernel to build an upper bound; applying a local kernel to a map that is determined only on fields satisfying the Bianchi identity; setting the leading terms of the strong-coupling and weak-coupling series side by side to look for the point where they meet; and pulling a degree-$p$ cone back step by step and bounding it by the weighted Stein bound. This chapter is an attempt to give each of them a definition and axioms (what can be said under what assumptions). Many of the results are restatements of known facts, and we say so row by row for the candidate theorems of each section. What remains after the restatement — quantities that do not correspond directly to a known framework, and regularities shown by the numbers of the record that have not been made into theorems — are the questions of this chapter.

§ 01– § 03 can be read independently of one another. § 04 shares no tools with § 01– § 03; it is an attempt to restate the band of the renormalisation comparison. § 05 extends Candidate 1.4 of § 01 to the language of cones and can be read alongside § 01.

01

## Dominant matrices and k-step contraction

The non-negative matrix that lists the norms of the blocks of an operator split into blocks transfers estimates of powers of the operator to estimates of powers of a non-negative matrix. On the side of non-negative matrices the Perron–Frobenius theory applies as it stands, so "in how many steps does it contract" and "with which weight must we measure for it to contract in one step" can be written in the language of matrices. Everything up to this point is known; the role of this section is to bring the estimates of the record into line with this framework.

### Definitions

**Definition 1.1 (dominant matrix).** Let $I$ be a finite set, $E_i$ ($i \in I$) normed spaces, and $T$ an operator on $E = \bigoplus_i E_i$ split into blocks $T_{ij} : E_j \to E_i$. A non-negative matrix $B \in \mathbb{R}_{\geq 0}^{I \times I}$ is a **dominant matrix** of $T$ if for all $i, j$ we have $\|T_{ij}\| \leq B_{ij}$. The smallest one is the matrix of block norms $N(T)_{ij} = \|T_{ij}\|$.

- Products and powers: $N(ST) \leq N(S)N(T)$ (entrywise). Hence if $B, C$ are dominant matrices of $S, T$, then $BC$ is a dominant matrix of $ST$ and $B^n$ is a dominant matrix of $T^n$ Lean ( `norm_mul_apply_le` , `dominant_mul` , `dominant_pow` ; the power version assumes that the identity has norm 1).
- The class is also closed under sums and non-negative scalar multiples Reading .
- The max-plus side: for the maximal cycle mean $\lambda_{\mathrm{mp}}(A)$ of $A = \log B$ (entrywise), $e^{\lambda_{\mathrm{mp}}} \leq \rho(B) \leq |I|\, e^{\lambda_{\mathrm{mp}}}$ Cited (Friedland 1986). In the record, the max-plus eigenvalue bounded the optimal weighted norm from below to within 1–2.5% Computed .

**Definition 1.2 (weighted block norm).** For a positive weight $w \in \mathbb{R}_{>0}^{I}$, $\|x\|_{w,\infty} = \max_i \|x_i\|/w_i$, and the $\ell^2$ version is $\|x\|_{w,2}^2 = \sum_i \|x_i\|^2/w_i^2$. The operator norm is $\|T\|_{w,\infty} = \|W^{-1}TW\|_{\infty}$ ($W = \mathrm{diag}(w)$).

**Definition 1.3 ($k$-step contraction, transient constant).** For a non-negative matrix $B$ put $\kappa_k(B) := \|B^k\|$; when $\kappa_k < 1$ for some $k$ we say that it "contracts in $k$ steps". The transient constant is $C_k(B) := \max_{r<k} \|B^r\|$. By Gelfand's formula $\rho(B) = \lim_n \|B^n\|^{1/n}$, $\rho(B) < 1$ is equivalent to $\kappa_k < 1$ for some $k$ Cited .

**Definition 1.4 (one-step weight).** For $B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$ we call

$$w' := \sum_{r<k} s^{-r} B^r w$$

the **one-step weight** built from $w$.

### Candidate theorems and labels

**Candidate 1.1 (from $k$-step contraction to a weight that contracts in one step).** If $B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$ and $B^k w \leq s^k w$ (entrywise), then $w \leq w'$ and $Bw' \leq s\, w'$ Lean ( `liftWeight_step` ; lemmas `mulVec_pow_succ` , `pow_mulVec_nonneg` ).

$$Bw' = \sum_{r<k} s^{-r} B^{r+1} w = s\big(w' - w + s^{-k} B^k w\big) \;\le\; s\,w'$$

Combined with Candidate 1.2 (the one-step estimate), this says that if "the $k$-step composite contracts by a factor $s^k$ in the weight $w$", then "measured in the weight $w'$ it contracts by a factor $s$ in every step". The weight ratio $w'/w$ lies in $[1,\ \sum_{r<k} s^{-r}\max_i (B^r w)_i/w_i]$, so the constant of re-measurement does not depend on the number of steps (Candidate 1.3). The content is the constructive half of "$\rho(B) < s \Leftrightarrow$ there is $w' > 0$ with $Bw' \le sw'$", known from Varga's analysis of iterative methods and from the line of Berman–Plemmons on the characterisation of M-matrices Cited .

Table 1 The candidate theorems of § 01. Each is a finite-dimensional form of a known fact, or an elementary consequence of one

| Candidate | Statement (one line) | Label | What it restates |
|---|---|---|---|
| 1.1 | $B^k w \le s^k w \Rightarrow w \le w'$, $Bw' \le sw'$ | Lean | The constructive half of the Varga / Berman–Plemmons characterisation of M-matrices Cited |
| 1.2 | $Bw \le \lambda w$ $(w > 0) \Rightarrow \|T\|_{w,\infty} \le \lambda$. Even with $T_n, B_n$, $\lambda_n$ differing from step to step, $\|x_n\|_{w,\infty} \le (\prod_k \lambda_k)\|x_0\|_{w,\infty}$ | Lean (`weighted_step`, `weighted_iter`) | Collatz–Wielandt: $\inf_{w>0} \max_i (Bw)_i/w_i = \rho(B)$ (the equality itself is not in Lean) Cited |
| 1.2′ | Given right and left positive Perron vectors $x, y$, $D = \mathrm{diag}(\sqrt{y/x})$ gives $\|DBD^{-1}\|_2 \le \rho(B)$, and further $\|DTD^{-1}\|_2 \le \|DN(T)D^{-1}\|_2$ | Reading | The Schur test applied to $q = \sqrt{xy}$ Cited |
| 1.3 | If $av \le w \le bv$ $(a > 0)$, a sequence that contracts by $\lambda_n$ in each step in the weight $w$ satisfies $\|x_n\|_v \le (b/a)(\prod \lambda_k)\|x_0\|_v$ in the weight $v$ | Lean (`weight_change`) | An elementary fact about equivalence of norms |
| 1.4 (a) | $\|1\| = 1$, $q := \|a^k\|$, $\|a^r\| \le C$ $(r < k) \Rightarrow \|a^n\| \le C\,q^{\lfloor n/k \rfloor}$ | Lean (`norm_pow_block_le`, `norm_pow_le_const_mul`) | A finite version of Gelfand's formula Cited |
| 1.4 (b) | $q < 1 \Rightarrow \sum_{n<N} \|a^n\| \le kC/(1-q)$ (independent of $N$) | Lean (`sum_norm_pow_le`; lemmas `geom_sum_le_inv`, `sum_range_mul_eq`) | An estimate of partial sums of the Neumann series |
| 1.4 (c) | $c(k+1) \le \lambda c(k) + \varepsilon$ $(0 \le \lambda < 1, \varepsilon \ge 0) \Rightarrow c(k) \le \max(c(0), \varepsilon/(1-\lambda))$ | Lean (`affine_uniform`) | An elementary fact about affine recursions |
| 1.4 second half | If irreducible with a Perron vector $w > 0$, then $C = \max_i w_i / \min_i w_i$ ($\infty$ norm). An explicit formula bounding the ratio of $w$ by the entries of $B$ | Reading (the explicit formula is not written out in full) | Minc's ratio estimate Reading |
| 1.5 | If the $J \times J$ matrix $M$ obtained by aggregating blocks with $g : I \to J$ satisfies $Mp \le \mu p$, the original $B$ contracts in one step by $\mu$ in the weight $w_j p_{g(j)}$. If $J$ has two points, $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a < 1$, $d < 1$ and $bc < (1-a)(1-d)$, then in the weight $(1, t)$ we have $\max(a + bt,\ d + c/t) < 1$ | Lean (`aggregate_step`, `two_by_two_weight`, `two_by_two_exists`) | An elementary fact about the test for $\rho < 1$ of a two-by-two matrix Cited |

### Correspondence with the numbers of the record

- The matrix of the finite types of the W coordinates (quadratic including differences, dimension 6) had spectral radius 0.36, a one-step norm above 1, and 0.05–0.13 in three steps Computed . It is a typical case of $\rho < 1$ with a one-step norm above 1, and the $C$ of Candidate 1.4 (a) corresponds to the

transient constant. Candidates 1.1 and 1.2 read this as: taking the Perron weight, it contracts in one step (in the record, 0.546 in one step with the Perron weight) Computed .

- The weighted Stein bound of the $\ell^4$ cone (15-05) was 4.45 in one step and 0.10 in three steps Computed . Put in the form of Candidate 1.1 it would read "if it is 0.10 in three steps, a weight that contracts to $0.10^{1/3} \approx 0.46$ in one step can be built", but this requires the three-step estimate to come from **powers of a single linear dominant operator**. The cone is a quartic quantity obtained through Riesz–Thorin, and the kernel $K_k$ differs from step to step, so it is not a power. What can be used in this case is only the form of a product over steps (`weighted_iter`) Reading .
- The pair of finite types and remaining returns has the form of the two-by-two aggregation of Candidate 1.5. Putting the remaining block in the lower right and checking $bc < (1-a)(1-d)$ has not yet been done (question "finite types and aggregation of returns").
- The Dobrushin column sum used in the coarse chain of the double well in Part 23 is Candidate 1.2 with the weight $w = \mathbf{1}$. Taking the Perron weight could improve the contraction constant Reading .

### A question proper to this pattern — the loss from discarding signs

A dominant matrix discards signs. For the actual operator $T$ and a dominant matrix $B$, define the loss at step $k$ as

$$L_k := \frac{\|B^k\|_\bullet}{\|T^k\|_\bullet} \qquad (\text{same norm; } \|T^k\| = 0 \text{ then } +\infty)$$

Then the loss is not necessarily bounded in the number of steps. $T = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ has $|T| = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ as a dominant matrix, $T^2 = 0$, and every entry of $|T|^{n+1}$ is $2^n$. Hence $L_n = +\infty$ for $n \geq 2$ Lean (`absEx_eq`, `signedEx_sq`, `absEx_pow`). Conversely, if $T$ is non-negative ($T = |T|$) or sign-similar, $T = S|T|S$ ($S$ a diagonal matrix of $\pm 1$), then $L_k = 1$ in the $\infty$ norm with the Perron weight Reading .

How far the choice of signs can reach towards the estimate of $\rho(|A|)$ is treated by Rump's sign-real spectral radius (Rump 1997) Reading . We have not found a source that treats how the loss behaves **over steps** (not found, which does not mean there is none). In the record the loss grows only slowly, 7.6 in one step and 9.3 in two, and the conditions for making this a theorem are not determined by the information of the dominant matrix alone (the example above shows this). We take up this question in § 02.

02

## The loss from discarding signs

A closer look at the loss in the record showed that it was of a different type from the counterexample of § 01. The Stein bound of the record takes the upper bound **after composing** the step kernels. The counterexample of § 01 is a quantity that composes **after taking absolute values**. In the former, the upper bound of the composite for the nilpotent example $T^2 = 0$ is also 0. In this section we see from the numbers that the loss with the upper bound taken after composition hardly moves over steps, and we write an inequality that bounds it by a geometric quantity within a single row (the row alignment).

### Definitions

**Definition 2.1 (composite kernel and loss).** Write the composite of the step kernels $K_0, K_1, \ldots$ as $T_k = K_0 K_1 \cdots K_{k-1}$. Let the true ratio be $\sup_\theta \|T_k\theta\|_4^4 / \|\theta\|_4^4$, and the Stein bound the weighted $\ell^2$ quantity of

the form of (11) of 15-05 (weighted by the $\ell^1$ norms $A_r$ of the rows of the composite kernel); we put

$$\mathrm{loss}_k := \frac{\mathrm{Stein}(T_k)}{\text{true ratio}(T_k)}$$

The true ratio is replaced by a lower bound from maximisation (the larger of a quasi-Newton method and a maximisation over period-2 fields), so the loss in the tables is an estimate of the loss from above.

**Definition 2.2 (row alignment).** The Stein bound of a translation-invariant $T$ has the form $\sup_q \lambda_{\max}\big(\sum_r A_r^2 \hat{T}_r(q)^{\mathrm{H}} \hat{T}_r(q)\big)$; let $(q^*, u^*)$ be its maximiser. Put $y_r = \hat{T}_r(q^*)u^*$, $M = \|u^*\|_\infty$, and call

$$\kappa_r := \frac{|y_r|}{A_r\, M} \;\le\; 1$$

the **alignment** of row $r$ in the worst direction ($\kappa_r \le 1$ is `coherence_le_one` Lean ). $\langle \kappa^{-2} \rangle$ is the average weighted by $|y_r|^4$.

### The numbers of the record — the loss stops, and falls a little

Table 2 The loss on a common $4^4$ torus (periodic operator with the steps cut at $R = 3$) Computed

| k | Stein | Lower bound of the true ratio (quasi-Newton / period 2) | Loss | Step ratio: lower bound | Step ratio: Stein |
|---|---|---|---|---|---|
| 1 | 3.708 | 0.5903 / 0.5566 | **6.28** | – | – |
| 2 | 0.7746 | 0.1405 / 0.1405 | **5.51** | 0.238 | 0.209 |
| 3 | 0.07632 | 0.014967 / 0.01497 | **5.10** | 0.107 | 0.0985 |
| 4 | 0.00576 | 0.001095 / 0.00109 | **5.26** | 0.073 | 0.0755 |

The true ratio and the Stein bound decrease at the same rate (both step ratios tend to about 0.07), so the loss stops. The shape is the same on larger tori (Table 3). The earlier recorded two-step value 9.3 was due to a weak lower bound of the true ratio, 0.108; with period-2 fields the lower bound becomes 0.1405 Computed . For $k \ge 2$, 99.8–100% of the energy of the maximiser of the true ratio was at $|q|_\infty = \pi$, sitting on period-2 points of type $(\pi, \pi, \pi, 0)$ Computed .

Table 3 The loss on larger tori, compared with the product of the per-step upper bounds Computed

| k | Torus | Stein | Lower bound of the true ratio | Loss | Product of per-step Stein ÷ lower bound |
|---|---|---|---|---|---|
| 1 | $16^4$ | 4.431 | 0.591 | **7.50** | 7.5 |
| 2 | $8^4$ | 0.9554 | 0.1405 (period 2) | **6.80** | 182 |
| 3 | $8^4$ | 0.09575 | 0.01497 (period 2) | **6.40** | $1.1 \times 10^4$ |
| 3 | $6^4$ | 0.09092 | 0.014955 | **6.07** | – |

The rightmost column is the product of the one-step upper bounds of each step (4.45, 5.75, 6.50, 6.81) divided by the lower bound of the true ratio; it grows by a factor of 60–90 per step, $7.5 \to 182 \to 1.1 \times 10^4 \to 1.0 \times 10^6$ Computed . Only the side that composes after taking absolute values grows exponentially. This column shows how much is gained by taking the upper bound on the composite kernel.

### The three hypotheses failed, and they failed in the same way

As reasons for the loss stopping we tried three: (a) composition reduces the proportion of the negative part; (b) the cancellation takes place within a cell; (c) the signs align at the maximiser of the true ratio. None of them holds in the form written. All three, however, failed in the same way: "it does not change over steps".

Table 4 Verdicts on the hypotheses Computed

| Hypothesis | Quantity | k=1 | k=2 | k=3 | k=4 ($4^4$) | Verdict |
|---|---|---|---|---|---|---|
| (a) | Proportion of the negative part (row signs chosen freely) | 0.411 | 0.432 | 0.435 | 0.426 | Does not hold (constant at about 0.43; on $4^4$, 0.405, 0.425, 0.427, 0.426) |
| (b) | Inner shell (distance 0.5–1.5): share of $\ell^1$ / negative proportion | 0.58 / 0.22–0.33 | 0.48 / 0.32 | 0.44 / 0.31–0.37 | 0.53 / 0.32–0.38 | Does not hold (the cancellation is between the positive centre and the negative ring at distance 1.5–2.5; summing over $2^4$ blocks leaves 41–56% of $\ell^1$) |
| | Outer ring (distance 1.5–2.5): share of $\ell^1$ / negative proportion | 0.33 / 0.59–0.63 | 0.38 / 0.56–0.60 | 0.40 / 0.53–0.60 | 0.46 / 0.53–0.59 | |
| (c) | Alignment / efficiency at the maximiser | 0.74 / 4.5% ($16^4$) | 0.56 / 1.3% ($8^4$) | 0.57 / 1.5% ($6^4$) | 0.60 / 1.8% | Does not hold (the alignment stops at around 0.6) |
| — | Row alignment in the worst direction $\langle\kappa^{-2}\rangle$ | 22.9 ($16^4$) | 10.6 ($8^4$), 5.55 ($4^4$) | 10.2 ($8^4$), 5.13 ($4^4$) | 5.31 | Bounded over steps |

That all three quantities hardly move over steps can be read as showing not that "composition brings the kernel closer to positive" but that "composition brings the profile of the kernel (its normalised shape) closer to a single one" Reading . A profile positive at the centre and negative around it, a negative proportion of 0.43 and an alignment of 0.6 are kept independently of the step. This is close to the view of the limit function of subdivision Reading . Note that the signs "positive at the centre, negative around it" depend on the sign convention for the components of the direction; what does not depend on the convention is the negative proportion and the alignment.

### The inequality that held — loss ≤ inverse square of the alignment

**Candidate 2.1.** Suppose $A_r \geq 0$, $\kappa \geq 0$, $M \geq 0$, $|v_a| \leq M$ for every component, and $\kappa A_r M \leq |y_r|$ for every row. Then

$$\kappa^2 \Big(\sum_r A_r^2 y_r^2\Big) \sum_a v_a^4 \;\leq\; \Big(\sum_r y_r^4\Big) \sum_a v_a^2$$

Lean ( `stein_le_inv_sq_mul_test` ; lemmas `stein_form_le` , `sum_four_le` ). The bracket on the left is the value of the Stein quadratic form and the right-hand side is the value of the test, so it reads "upper bound ÷ test value $\leq \kappa^{-2}$".

On the lattice, using as the test the real plane wave in the worst direction $\theta = \mathrm{Re}(e^{i\varphi} u^* e^{-iq^*\cdot X})$, we get

$$\text{loss} \;\leq\; \frac{\text{Stein}}{\text{test value}} \;=\; \langle\kappa^{-2}\rangle \times \frac{S(u^*)}{\Phi}, \qquad S(u) = \frac{\sum_a |u_a|^4}{|u|^2 M^2} \leq 1$$

$\Phi$ is the phase factor; it is 1 if $2q^* \not\equiv 0$ and $4q^* \not\equiv 0$, and also 1 at period-2 points ($2q^* \equiv 0$) if $u^*$ is taken real Reading (the average $3/8$ of $\cos^4$ cancels between numerator and denominator; this step and the Stein formula itself are not in Lean).

- For $k = 2, 3, 4$ on the $4^4$ torus, the maximiser of Stein was a period-2 point, and its plane wave was itself the maximiser of the lower bound of the true ratio (test values 0.1396, 0.01487, 0.00109). Then Stein ÷ test value $= \langle\kappa^{-2}\rangle = \mathbf{5.55, 5.13, 5.31}$, and the shape factor and the phase factor were exactly 1 Computed .

- On larger tori the maximiser of Stein moves away from period 2, and $\langle \kappa^{-2} \rangle$ becomes 22.9, 10.6, 10.2 ($k = 1, 2, 3$). As an upper bound of the loss it is bounded and falls over steps, but it does not agree with the loss itself (6.4 and so on) Computed .

How to read it: on the $\ell^2$ side the Stein bound remembers all the signs; what it forgets is only the "row $\ell^1$" on the $\ell^\infty$ side. So the loss reduces to the amount of cancellation within a row, $1/\kappa^2$ Reading . This gives a conditional statement — if the alignment in the worst direction is bounded below over steps, the loss is bounded over steps — but that the alignment is bounded below is not shown (question "alignment of rows in the worst direction").

### A pattern that did not apply — positive part dominant

For a sign field $s$, if the negative part of each row is at most $\delta$ times the positive part, then $(Ts)_i \geq \frac{1-\delta}{1+\delta} r_i$ ($r_i$ the row $\ell^1$), and combined with the $\ell^4$ Schur test $\sum (Tx)^4 \leq A^3 B \sum x^4$, loss $\leq U\left(\frac{1+\delta}{1-\delta}\right)^4$ ($U$ the non-uniformity factor of the row $\ell^1$). Under composition $\delta$ degrades additively (alignment $1 - \delta_1 - \delta_2$) Lean (`coherent_of_neg_le`, `schur_four`, `loss_le_of_coherent`, `comp_coherent`).

The theorem is correct, but it does not apply to the kernels of the record. The $\delta$ read from the $\ell^4$-weighted average alignment is about 0.46, 0.39, 0.39, 0.35 and is not small; $U$ grows as 3.0, 4.5, 8.8, 15.8; and the minimal alignment per row is almost 0 (the assumption of $\delta$ on every row is not satisfied) Computed . The Schur test itself was also two orders of magnitude coarser than the Stein bound (Schur ÷ lower bound 437 → 745 → 1453 → 1777) Computed .

### The W coordinates — the absolute value after composition stops at a rank-one value

Table 5 The loss for the W coordinates (kernel with the zero mode corrected, 60 types) Computed

| k | $\rho(T_k)$ | Compose after absolute value | Absolute value after composing | Negative proportion |
|---|---|---|---|---|
| 1 | 0.3631 | 1.01 | 1.01 | 0.210 |
| 2 | 0.1249 | 1.08 | 1.06 | 0.325 |
| 3 | 0.0422 | 1.16 | 1.09 | 0.387 |
| 4 | 0.0142 | 1.24 | 1.10 | 0.411 |
| 5 | 0.0048 | 1.33 | 1.10 | 0.421 |

The loss is the ratio of $\rho$'s ($\rho(|\cdot|)/\rho$). The value 1.10 at which the side taking the absolute value after composition stops almost agrees with the rank-one value $\sum |u_i||v_i| / |\sum u_i v_i| = 1.085$ built from the left and right dominant eigenvectors $u, v$ Computed . If the dominant eigenvalue has a gap, $T^k/\rho^k \to uv^{\mathrm{T}}/(v^{\mathrm{T}}u)$, and since $|\cdot|$ and $\rho$ are continuous it converges to this value Reading . The side composing after taking absolute values grows as $(\rho(|M|)/\rho(M))^k$ for powers of a single matrix (here by a factor of 1.070 each time), and is bounded only when $\rho(|M|) = \rho(M)$, which for irreducible matrices means $M = e^{i\alpha} D|M|D^{-1}$ (Wielandt) Cited . That the absolute value after composition is at most the composition after absolute values (row $\ell^1$) is `abs_comp_le`, `row_l1_comp_le` Lean .

## 03 The geometry of local representatives

Representing by a local kernel a map determined only on fields satisfying the coarse Bianchi identity — this handwork came up as the choice of "representative" in the Bundle Network Theorem (local form) of 15-05. In this section we separate the overall shape of the representatives (an affine space), the

quantities that do not depend on the representative (the value of the zero mode, and the order of folding restricted to the constraint), and the quantities that do (the overall order of folding and the tail norm). The content is a restatement of the order of multigrid transfer operators and of the sum rules of subdivision (conditions of Strang–Fix type).

### Definitions

**Definition 3.1 (constrained map and extension).** We call a quadruple $(E, V, W, f)$ of finite-dimensional spaces $E, W$ over a field $K$, a subspace $V \le E$ and a linear map $f : V \to W$ a **constrained map**, and a $g : E \to W$ with $g|_V = f$ an **extension**. Extensions exist (the restriction map $\mathrm{res}_V$ is surjective), any two extensions agree on $V$, and the set of extensions is the affine space $g_0 + \ker \mathrm{res}_V$, of dimension $(\dim E - \dim V) \cdot \dim W$ Lean (`exists_extension`, `restrict_surjective`, `IsExtension.eqOn`, `isExtension_iff_mem_coset`, `finrank_freedom`). The freedom is 0 only when $V = E$ ($W \neq 0$; `freedom_eq_zero_iff`).

**Definition 3.2 (lattice representative).** The coarse plaquette field is $\theta : \mathbb{Z}^4 \to \mathbb{C}^6$, the constraint subspace is $V = \mathrm{im}\, D_c$ (fields satisfying the coarse Bianchi identity), and the output is a fine plaquette field with $96 = 16 \times 6$ components per coarse cell. A translation-invariant kernel $K : \mathbb{Z}^4 \to \mathbb{C}^{96\times 6}$ whose symbol satisfies $\hat K(q)\hat D(q) = \hat R(q)$ for all $q$ (with $R = FD_c$) is called a **representative** of $F$, and one that has finite range, exponential decay or $\ell^1$ is called a **local representative**. For $q \neq 0$, $\hat D(q)$ has rank 3, the symbol at each $q$ is an instance of the template "$E = \mathbb{C}^6$, $\dim V = 3$, $W = \mathbb{C}^{96}$", and the freedom is $(6-3)\cdot 96 = 288$-dimensional.

**Definition 3.3 (zero mode, moments).** We call $\hat K(0) = \sum_Z K(Z)$ the zero mode and $M_\alpha(K) = \sum_Z Z^\alpha K(Z)$ the moments.

**Definition 3.4 (order of folding).** For $\eta \in \{0,1\}^4$, $p_\eta = (q + 2\pi\eta)/2$ and the fine position $s \in \{0,1\}^4$ within a coarse cell, the folded component is $\hat C_\eta(q)[o,a] = \sum_s e^{-ip_\eta\cdot s}\hat A_{(s,o),a}(q)$. We write $\mathrm{ord}(K)$ for the minimal order to which $\hat C_\eta(q)$ with $\eta \neq 0$ vanishes as $q \to 0$, and $\mathrm{ord}^V(K)$ for the minimal order of $\hat C_\eta(q)P(q)$, multiplied by the projection $P(q)$ onto the constraint subspace.

### Quantities that do not depend on the representative

**The zero mode is determined by continuity alone (Candidate 3.1).** If $K$ is an $\ell^1$ representative (continuous symbol), $\hat D(q) = D_1(q) + O(|q|^2)$, $\hat R(q) = R_1(q) + O(|q|^2)$ and $\sum_\nu \mathrm{im}\, D_1(e_\nu) = \mathbb{C}^6$, then $\hat K(0)$ is the unique solution of $\hat K(0) D_1(e_\nu) = R_1(e_\nu)$ ($\nu = 1,\dots,4$) Reading. The last step (maps that agree on a family of subspaces whose sum spans the whole are equal) is `eq_of_eqOn_iSup`, `correction_eq_zero_of_iSup` Lean. The value in the record is $\hat K(0) = U/4$, $U = \mathbf{1}_{16} \otimes I_6$ ($\theta/4$ uniformly at every position), with deviation $\le 3.1 \times 10^{-10}$ over four steps Computed. On the torus $\hat D(0) = 0$, so the zero mode is unconstrained, and for the symmetric representative on finite tori $\max|\hat A(0) - U/4| = 0.250,\ 1.47\times 10^{-2},\ 2.55\times 10^{-5}$ at $L_b = 8, 16, 32$ Computed. This is the numerical trace of the constraint of locality being lost in a finite box.

**The minimal-norm representative is not local (Candidate 3.2).** $\hat K_{\mathrm{MP}}(q) = \hat R(q)\hat D(q)^+$ (the Moore–Penrose representative) does not lie in $\ell^1$ unless the limits along different directions agree, because the projection $P(q)$ is a function of $q$ homogeneous of degree 0 and leaves a memory of direction at $q = 0$ Reading. The freedom of local representatives can be seen as the room to repaint this memory of direction into a continuous function. The decay of the kernel has not been measured (question "decay of the Moore–Penrose kernel").

**The order restricted to the constraint does not depend on the representative (Candidate 3.3).** With the singular value decomposition $\hat{D} = U\Sigma V^{\mathrm{H}}$, $\hat{C}_\eta(q)U(q)$ can be written using only the folding of $\hat{R}$, so $\mathrm{ord}^V(K) = \mathrm{ord}^V(F)$ is an invariant of $F$. Restriction does not lower the order of vanishing, so for every representative $\mathrm{ord}(K) \le \mathrm{ord}^V(F)$ Reading .

Table 6 Orders of folding of the representatives in the record (in brackets, how the value decreases as $t$ is made smaller by one order of magnitude at a time) Computed

| Representative | Construction | ord (whole $6 \times 6$) | $\mathrm{ord}^V$ (on $\mathrm{im}\, D$) |
|---|---|---|---|
| sym | Rows of the cone homotopy (finite torus) | 0 ($\hat{C}(0) = 0.039$–$0.071$) | 0 |
| A3 | Add the difference of the zero mode at the single point $Z = 0$ | 1 ($3.4 \times 10^{-2} \to 3.4 \times 10^{-4}$ for $t = 10^{-1} \to 10^{-3}$) | 1 ($5 \times 10^{-3} \to 5 \times 10^{-5}$) |
| A4 | Add to A3 a first-moment correction as antisymmetric pairs at $Z = \pm e_\mu$ | 2 ($1.7 \times 10^{-3} \to 1.7 \times 10^{-7}$) | 2 |
| $F$ itself | From $R$ | — | 2 ($1.8 \times 10^{-3} \to 1.8 \times 10^{-7}$) |

A4 attains $\mathrm{ord}(K) = \mathrm{ord}^V(F) = 2$ on the whole $6 \times 6$. That is, within the types of the record, A4 is a representative that reaches the ceiling.

### Tail norm and order — D ≤ 4 is the upper limit

**Candidate 3.4.** In the $n$-step map of the W coordinates $(Tf)(q) = 16^{-n} \sum_m G_m(q)^{\mathrm{H}} f(p_m) G_m(q)$ ($m$ runs over $16^n$ paths, $G_m$ is a product of foldings), suppose that the representatives of all steps satisfy $\mathrm{ord}(K_k) \ge k_0$ and that $\hat{C}_0(0) = 4I$. (a) If $D \le 2k_0$, $T$ is bounded in the norm on the symbol side $\|\cdot\|_D = \sup_{q \neq 0} \|\cdot\|_2 / |\hat{q}|^D$. (b) If $D > 2k_0$ and the leading coefficient $S$ of the paths with $m \neq 0$ is non-zero, then for a peaked input near $q_0$ it diverges at the rate $|q_0|^{2k_0 - D}$ Reading (a line of argument assuming constants uniform in the number of steps; uniformity over infinitely many steps is not treated). In the version restricted to the constraint, $k_0$ is replaced by $\mathrm{ord}^V(F)$.

Table 7 Tail norm ($D = 4$, input along the axis $q = 2\pi/L$, four points $L = 24$–$192$) Computed

| Representative | ord | Values | Type in Candidate 3.4 |
|---|---|---|---|
| A3 | 1 | 1.486, 5.88, 23.5, 93.8 (grows as $L^2$) | $\lvert q\rvert^{2\cdot 1 - 4}$ |
| A4 | 2 | 0.01638, 0.01640, 0.01643, 0.01643 (bounded) | $\lvert q\rvert^{2\cdot 2 - 4}$ = constant |

From $R$ ($\mathrm{ord}^V = 2$) it was bounded at $D = 4$, $q^{-2}$ at $D = 6$, and $q^{-3}$ at $D = 7$ Computed . All cases in the record fit the types (a)(b). Since $\mathrm{ord}^V(F) = 2$, the $D = 6$ tail on $\mathrm{im}\, D$ diverges however the representative is chosen, and this reads as: **within the types of the record, $D \le 4$ is the upper limit for the tail of $\sup/|\hat{q}|^D$** Reading .

The 2 in "$2 \cdot \mathrm{ord}$" comes from the map of the W coordinates being a congruence $G^{\mathrm{H}} f G$ with the same representative on both sides. In multigrid the orders of prolongation and restriction can be chosen separately, and there is the rule "order of prolongation + order of restriction ≥ order of the equation" (Hemker 1990) Reading . Here a single representative plays both roles.

### Representatives with matched first moments, and the sum rules

**Candidate 3.5 (order and shifted jets).** Putting $J_s(q) := e^{-iq\cdot s/2} \hat{K}_s(q)$ ($\hat{K}_s$ the symbol of the 6 rows at position $s$), $\mathrm{ord}(K) \ge k \Leftrightarrow$ the Taylor polynomial of $J_s(q)$ up to degree $k - 1$ does not depend on $s$ Reading . The line of argument is $\hat{C}_\eta(q) = \sum_s (-1)^{\eta\cdot s} J_s(q)$ together with the orthogonality of the 15 non-trivial characters of $(\mathbb{Z}/2)^4$ (inner product 0 with all of them $\Leftrightarrow$ constant).

- $k = 1$: $\hat{K}_s(0)$ does not depend on $s$ ($U/4$ has this form).
- $k = 2$: the first moments are $M_s = (s/2)\hat{K}_s(0) + B$ ($B$ independent of $s$). From $\hat{K}_s(0) = I/4$, $M_{s,\mu} = s_\mu I/8 + B_\mu$. This agrees with the target moments used to build A4, $M^*_{(s,o),\mu,a} = s_\mu \delta_{oa}/8 + B_\mu[o,a]$. The 8 is (position within the cell $s/2$) $\times$ $(1/4)$.

The target $B$ is determined from the $q^2$ term of $R$. Against 24 unknowns per row (4 directions $\times$ 6), the part determined by $R$ had rank 20, leaving 4 dimensions of freedom Computed . This count is of the type of a finite-dimensional "extension with conditions": if $(\mathrm{res}_V, C)$ is surjective, there is a $g$ satisfying both $g|_V = f$ and $C(g) = y$, and the remaining freedom is $\dim E \cdot \dim W - \dim V \cdot \dim W - \dim Y$ Lean (`exists_extension_with_conditions`, `finrank_freedom_with_conditions`, `conditions_count_le`). The remaining 4 dimensions were fixed by linear programming to minimise the increase of $\ell^1$; the moments matched the target to within $1.1 \times 10^{-16}$, and the row $\ell^1$ went from 2.26 to 2.34 (first step) Computed .

### Composition of representatives and compatibility

If $(E_1, V_1, E, f_1)$ and $(E, V_2, W, f_2)$ are compatible ($f_1(V_1) \subseteq V_2$), the composite of representatives is a representative of the composite, and its value on $V_1$ does not depend on the choice of representatives. If they are not compatible ($W \neq 0$), the value of the composite changes with the choice Lean (`comp_extension_eqOn`, `comp_isExtension`, `comp_depends_of_not_le`). The chain of W coordinates in the record composes representatives, but whether the output of the representative at step $k$ lies in the constraint subspace of step $k+1$ has not been checked (question "consistency of composing representatives across steps"). The difference between the chain on $\mathrm{im}\, D$, 0.3426, and the $6 \times 6$ chain, 0.3446, is a clue to this Computed . As examples of norms changing with the representative: the zero-mode correction moved the three-step Stein bound from 0.09743 to 0.09868, and the signed Stein bound went from 4.25 for A3 to 4.48 for A4 (first step) Computed .

### Dictionary

- $\mathrm{ord}_\eta(K)$ corresponds to the **high-frequency order** of the representative viewed as a prolongation (Hemker 1990). That the definition by symbols and the definition by preservation of polynomials are equivalent is stated in a later note, and the condition on the zeros of the symbol in Toeplitz multigrid is in Donatelli (arXiv:0807.2565) Reading . $\hat{C}_\eta$ is the same as the 16 components of the harmonic subspaces of local Fourier analysis Cited .
- Candidate 3.5 is exactly the standard argument that turns the **sum rules** of subdivision and wavelets into polynomial generation (Strang–Fix 1973, Jia 1998, Charina–Conti arXiv:1202.3931, Daubechies's vanishing moments) Cited . The part $s_\mu I/8$ of the target of A4 is a restatement of the sum rule of order 2.
- Two points fall outside the sum rules: (i) the part where $B$ is determined by the $q^2$ term of $R$ (a condition of the type of order of consistency of differences); (ii) the ceiling $\mathrm{ord}^V(F)$ set by the constraint $V = \mathrm{im}\, D$. That the masks are not scalars but $96 \times 6$ matrices is within the range of sum rules for vector-valued subdivision Reading .

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04
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## The geometry of the double valuation

**There is no payoff theorem yet.** This section is an attempt to restate the row for the crossover band in the ledger of conditions (26-06) in the language of two valuations, those of the strong-coupling expansion and the weak-coupling expansion. The definitions and the computation of the leading terms are done, but there is as yet no theorem that this view newly says about the band. The labels are mainly Assumption and Computed.

### Definitions

**The space of couplings** $\mathcal{K}$: the space of character coefficients $(a_r)$ of the plaquette weight $f(U) = 1 + \sum_r d_r a_r \chi_r(U)$. Its coordinates are $\beta$ and the shape coordinates.

**Two valuations.** $v_s$ (strong coupling) is the order on the side $a_r \to 0$ (the order in $\beta$ for the Wilson action), and $v_w$ (weak coupling) is the order in $1/\beta$ on the side $\beta \to \infty$. Terms of type $e^{-c\beta}$ are treated as $v_w = \infty$ (another branch of the transseries).

**Two coordinates.** For an observable $O$ with values in $[0, 1]$ ($O(0) = 0$, $O(\infty) = 1$; the plaquette $P$, for example), the coordinate on the strong-coupling side is $O$ and the coordinate on the weak-coupling side is $1 - O$. We write the leading coefficients as $O \approx s_1 \beta$ (strong coupling) and $1 - O \approx w_1/\beta$ (weak coupling).

**Tie sets.** (i) the complementary tie $\{s_1\beta = w_1/\beta\}$; (ii) the same-coordinate tie $\{s_1\beta = 1 - w_1/\beta\}$; (iii) the error tie $\{K$-term optimal truncations have equal errors$\}$. We call the region between the two roots of (ii), $\{s_1\beta \le 1 - w_1/\beta\}$, the **lens**.

### The invariant κ and the lens

**Candidate 4.1 (a scale-independent quantity).** Changing the scale of the coupling as $\beta \mapsto c\beta$ gives $s_1 \mapsto s_1/c$, $w_1 \mapsto c w_1$, so

$$\kappa := s_1 w_1$$

does not depend on the scale. The complementary tie is at $\beta^* = \sqrt{w_1/s_1}$, where the value of the leading terms is $\sqrt{\kappa}$. The same-coordinate tie is at the roots of $s_1\beta^2 - \beta + w_1 = 0$, whose discriminant is $1 - 4\kappa$, so **the lens opens only when** $\kappa \le \frac{1}{4}$. The geometric mean of the two roots is always $\beta^*$. A tie for the string tension $\sigma = -\log u$ exists only when $\kappa \le 1/e$ Reading (the elementary fact of the discriminant of a quadratic).

Looking at the leading term on one side only, the difference between groups is absorbed into the scale. Looking at the leading terms on both sides at once, a change of scale is only a translation in the $(\log O, \log(1 - O))$ plane, and the height $\sqrt{\kappa}$ at which they cross does not move. The view is that the information about the group remains in "the height of the crossing" Assumption.

### Examples

Table 8 Series and ties for a single plaquette (exact solution in two dimensions; the material of one Migdal–Kadanoff step) Computed

| Quantity | $SU(2)$ | $U(1)$ |
|---|---|---|
| Strong-coupling series $u$ | $\frac{\beta}{4} - \frac{\beta^3}{96} + \frac{\beta^5}{1536} - \frac{\beta^7}{23040} + \cdots$ | $\frac{\beta}{2} - \frac{\beta^3}{16} + \frac{\beta^5}{96} - \frac{11\beta^7}{6144} + \cdots$ |

| Quantity | $SU(2)$ | $U(1)$ |
|---|---|---|
| Weak-coupling series $1-u$ | $\frac{3}{2}\beta^{-1}-\frac{3}{8}\beta^{-2}-\frac{3}{8}\beta^{-3}-\frac{63}{128}\beta^{-4}\dots$ | $\frac{1}{2}\beta^{-1}+\frac{1}{8}\beta^{-2}+\frac{1}{8}\beta^{-3}+\frac{25}{128}\beta^{-4}\dots$ |
| $\kappa=s_1w_1$ | $3/8$ (the lens is empty, $1-4\kappa=-\frac{1}{2}$) | $1/4$ (the lens is the single point $\beta=1$) |
| $s_3/s_1^3$, $w_2/w_1^2$ | $-2/3$, $-1/6$ | $-1/2$, $+1/2$ |
| Complementary tie $\beta^*$ and the true $u(\beta^*)$ | $\sqrt{6}=2.4495$, $u=0.5004$ ($\beta$ at $u=\frac{1}{2}$ is 2.4469) | 1.0000, $u=0.4464$ ($\beta$ at $u=\frac{1}{2}$ is 1.1593) |
| Tie for $\sigma$ | None (maximum of the difference $-\log\kappa-1=-0.019$) | 0.232 and 1.399 |
| Error tie ($K=40$) | 3.274 (error $4.8\times10^{-4}$) | 2.124 (error $2.9\times10^{-3}$) |
| Fisher zeros (imaginary axis) | $\pm i\,3.832$, $\pm i\,7.016$, ... | $\pm i\,2.405$, $\pm i\,5.520$, ... |

The specific heat $\beta^2u'$ has a peak (0.680) at $\beta=2.48$ for $U(1)$, while for $SU(2)$ it rises monotonically to $3/2$ Computed . This difference goes in the same direction as the difference in the sign of $w_2/w_1^2$ ($-1/6$ and $+1/2$), and can be read as: the difference between groups lives not in the leading terms of the valuations but in the sign of the next term Assumption . That $u(\sqrt{6})=0.5004\approx\frac{1}{2}$ for $SU(2)$ does not hold for $U(1)$ (0.446), so it is not a general law.

**The lens of the leading terms alone in four dimensions.** Using equipartition $w_1=\dim G/d$, $\kappa=s_1\dim G/d$.

Table 9 The lens of the leading terms alone in $d$ dimensions Computed

| d | Group | $\kappa$ | Complementary tie $\beta^*$ | Lens |
|---|---|---|---|---|
| 2 | $SU(2)$ | 3/8 | 2.449 | None |
| 2 | $U(1)$ | 1/4 | 1.000 | Single point $[1,1]$ |
| 3 | $SU(2)$ | 1/4 | 2.000 | Single point $[2,2]$ |
| 3 | $U(1)$ | 1/6 | 0.816 | $[0.423,1.577]$ |
| 4 | $SU(2)$ | 3/16 | 1.732 | $[1.000,3.000]$ |
| 4 | $U(1)$ | 1/8 | 0.707 | $[0.293,1.707]$ |

The transition of four-dimensional $U(1)$, $\beta_c\approx1.01$ (26-06), lies inside the lens $[0.293,1.707]$ (not at its centre 0.707). The $SU(2)$ lens $[1,3]$ contains the range in the table of 26-06 over which $\beta-\beta'$ moves from the strong-coupling value to the one-loop value, and also overlaps $1.5<\beta<2.5$, where the literature says "neither the strong-coupling nor the weak-coupling expansion works well" (abstract of arXiv:0807.0185) Reading . These are numbers derived only from first-order leading terms and equipartition, without the true four-dimensional series. The overlaps may include coincidence Assumption .

### Tie sets under the Migdal–Kadanoff flow

We followed the level lines $L_n(c)$ of the fundamental-representation coefficient $u_n(\beta)$ after $n$ steps of flow from the Wilson action. A level line is an object that does not depend on how $\beta$ is read.

Table 10 Movement of the level lines $L_n(c)$ Computed

| Group | $c$ | $L_0$ | $L_6$ | Difference at step 6, $L_6-L_5$ |
|---|---|---|---|---|
| $SU(2)$ | 0.1575 | 0.641 | 2.230 | 0.259 |

| Group | $c$ | $L_0$ | $L_6$ | Difference at step 6, $L_6 - L_5$ |
|---|---|---|---|---|
| $SU(2)$ | 0.5 | 2.447 | 3.535 | 0.2553 |
| $SU(2)$ | 0.9 | 14.73 | 15.58 | 0.2511 |
| $U(1)$ | 0.1575 | 0.319 | 0.582 | 0.017 (positive while decreasing) |
| $U(1)$ | 0.5 | 1.159 | 0.844 | $4.5 \times 10^{-4}$ |
| $U(1)$ | 0.7 | 2.014 | 1.5101 | $-3.5 \times 10^{-7}$ (stops) |
| $U(1)$ | 0.9 | 5.305 | 4.8478 | $-3.7 \times 10^{-7}$ (stops) |

- For $SU(2)$, the difference per step approaches 0.25 for every $c$ (the $1/4$ of the heat kernel). The tie sets escape linearly in the direction of $\beta$, and in terms of the valuation of the flow, $\beta - \beta'$ has $v_w = 0$ (a constant) Computed .
- For $U(1)$, the level lines with $c \geq 0.7$ stop. The one-step movement after the shape has settled, $\beta_V - \beta'_V$, is $9.6 \times 10^{-4}$ at $\beta_V = 0.67$, $2.2 \times 10^{-4}$ at 0.78, $5.4 \times 10^{-5}$ at 0.88, $1.3 \times 10^{-5}$ at 0.99, $3.2 \times 10^{-6}$ at 1.09 and $1.8 \times 10^{-7}$ at 1.29, and the exponential slope read from the ratios of neighbouring points is 13.4–14.1. It is of type $e^{-c\beta_V}$, but $c$ has not been identified (it does not reach $2\pi^2 = 19.7$). The valuation of the flow is $v_w = \infty$ Computed . At floating-point resolution, "stops" cannot be distinguished from "flows at about $e^{-20\beta}$" Assumption .
- The map on the strong-coupling side, $u \mapsto 256u^4$, is the same for both groups once the scale is matched by $s_1$. Flowing the complementary tie of the strong-coupling leading terms, $SU(2)$ goes $2.449 \to 1.140 \to 0.785 \to 0.750$ and the solution disappears for $n \geq 4$, while $U(1)$ converges $1.0 \to 0.5 \to 0.361 \to \ldots \to 0.3152$. The candidate for the limit is $s_1\beta \cdot 256^{1/3} = 1$, that is, the edge of the strong-coupling basin $u_c = 256^{-1/3} = 0.1575$, common to both groups Computed .
- Caution: reading $\beta'$ by Wilson's $u$ gives $\beta - \beta' < 0$ on the weak-coupling side for both groups. This is the $O(1)$ shift in $\beta$ caused by the Wilson form and the heat-kernel form differing at $O(1/\beta^2)$, and it is not a flow Computed .

### Why the candidate theorem does not stand, and a weakened form

The form "a comparison that distinguishes groups is needed only near the codimension-one skeleton of the tie sets" **is not expected to hold as it stands** Assumption . This is because the distinction between groups already appears at the level of the leading terms, over the whole weak-coupling coordinate, as the valuation of the flow ($v_w = 0$ for $SU(2)$, $v_w = \infty$ for $U(1)$; that is, the presence or absence of $b_0$).

**Candidate B (weakened form)** Assumption . For a family of observables and groups $G, G'$, assume three things: agreement of the strong-coupling valuations (the maps of the valuations of the strong-coupling coordinate agree for $G$ and $G'$ under matching of the scale by $s_1$); difference of the weak-coupling valuations (the valuations of the flow of the weak-coupling coordinate differ); and regions where the leading terms dominate (regions $D_s(\eta)$, $D_w(\eta)$ can be taken in which the error of each coordinate is at most $\eta$ times the leading term of that coordinate). Then information beyond the leading terms is needed to decide the direction of the comparison (whether $\beta' \leq \beta$) only on $\mathcal{K} \setminus (D_s(\eta) \cup D_w(\eta))$, and this set shrinks to the lens as $\eta \to 0$.

- What it says: the need to **carry** the distinction is localised in the lens. It restates the same content as "the band is a bounded interval" of 26-06 in the language of valuations, and does not claim to make the band easier.

- What it does not say: what is needed inside the lens. The transition of $U(1)$ is on the $v_w = \infty$ branch, so the reading stops at: inside the lens a comparison is needed that handles the transseries branch becoming the same size as the perturbative branch Assumption .
- For the two-dimensional single plaquette with $SU(2)$ ($\kappa > \frac{1}{4}$) the lens is empty and Candidate B holds vacuously. However, two-dimensional $U(1)$ also has no phase transition, so $\kappa$ alone does not determine the phase diagram Reading .

### Relation to the known

- A view is known in which the Fisher zeros form the edge of the basin of attraction of the strong-coupling fixed point (Denbleyker–Du–Liu–Meurice–Zou, arXiv:1005.1993; the Wilson action of four-dimensional $SU(2)$) Reading . That the tie of the strong-coupling leading terms is drawn by the flow to the basin edge $u_c$ can be read as the shadow of this view on the real axis Assumption . For a single plaquette, the radius of convergence of the convergent strong-coupling series is the absolute value of the first Fisher zero Cited .
- The error tie (iii) puts into numbers, as it stands, the question of resurgence "which series dominates". For a single plaquette, the optimal error of the weak-coupling asymptotic series was of order $e^{-2\beta}$ Computed .
- Two-point Padé approximation joining the series at the strong and weak ends should be known as a classical way of filling in the lens Reading (the literature has not been checked).

05

## The cone calculus

The record of Part 15 contains the following handwork again and again: bounding a polynomial of degree $p$ by an $\ell^p$ cone $|f(\theta)| \le c\,\|\theta\|_p^p$, pulling it back by a linear map (the local representative of the blocking), bounding the pull-back factor from above by the weighted Stein bound, checking that it contracts in a composite of several steps even if it does not contract in one step, and absorbing the terms that the loops add at each step into an affine recursion to obtain a bound valid at every step (One-Step Contraction of the $\ell^4$ Cone in 15-05, 15-08 § 06). Like § 01– § 03, this section is an attempt to give this handwork definitions. Much of the content is a restatement of known facts — the Riesz–Thorin interpolation theorem, Stein–Weiss interpolation with change of measure, the Schur test, Hölder's inequality, and the log-convexity of operator norms in the exponent — and no novelty is claimed. What is new lies on the side of the questions: after setting out which quantity each number of the record corresponds to, we pose as questions the "window" and the "bundle number" as the exponent $p$ is varied.

### Setting

$V, W$ are finite-dimensional real vector spaces, and we fix one set of coordinates. $\ell^p$ is the $\ell^p$ of the coordinates. In the lattice case, $V$ is the plaquette field on the coarse torus (6 components per point), $W$ is the fine plaquette field arranged as 96 components per cell, and $K : V \to W$ commutes with coarse translations. The supremum of the cone is taken over all of $V$ (it is not restricted to the image of the Bianchi identity). Hence the quantities here are upper bounds for the restricted quantities.

### Definitions

**Definition 5.1 (cone constant).** When $f : V \to \mathbb{R}$ is a homogeneous polynomial of degree $p$ (or grows no faster than that),

$$c_p(f) := \sup_{\theta \neq 0} \frac{|f(\theta)|}{\|\theta\|_p^p}.$$

In Lean this is written `IsCone f N c` ($|f\,\theta| \le c \cdot N\theta$ for all $\theta$), with an abstract $N$ playing the role of $\|\cdot\|_p^p$.

**Definition 5.2 ($p$-contraction).** For a linear map $K : V \to W$,

$$\mu_p(K) := \sup_{\theta \neq 0} \frac{\|K\theta\|_p^p}{\|\theta\|_p^p} = \|K\|_{p\to p}^p.$$

In Lean this is written `Contracts K N μ` ($N(K\theta) \le \mu\, N\theta$ for all $\theta$) and read as "$\mu_p(K) \le \mu$".

**Definition 5.3 (row and column $\ell^1$).** Put $A_r := \sum_j |K_{rj}|$, $R(K) := \max_r A_r = \|K\|_{\infty\to\infty}$, $C(K) := \max_j \sum_r |K_{rj}| = \|K\|_{1\to 1}$. Then $\mu_1 = C$, and $\mu_p^{1/p} \to R$ $(p \to \infty)$ Cited .

**Definition 5.4 (the weighted Stein bound for general $p \ge 2$).** For a positive weight $u$ put

$$S_p(K;u) := \max_r \frac{A_r^p}{u_r} \cdot \lambda_{\max}\big(K^{\mathsf T} D_{u/A^2} K\big), \qquad S_p(K) := S_p(K;A^p) = \lambda_{\max}\big(K^{\mathsf T} D_{A^{p-2}} K\big)$$

($D_v$ is the matrix with $v$ on the diagonal). The standard form $S_p(K)$ becomes the bare $\|K\|_2^2$ at $p = 2$, $D_A$ at $p = 3$ and $D_{A^2}$ at $p = 4$, recovering the $\ell^3$ and $\ell^4$ formulas of the record. For $K$ commuting with translations, $\lambda_{\max}$ is replaced by $\sup_q \lambda_{\max}(\hat K(q)^{\mathsf H} D \hat K(q))$.

- Derivation Reading : put $T = D_{1/A} K$; then $\|T\|_{\infty\to\infty} = 1$. If $M_0$ is the norm of $T$ in $\ell^2$ with weight $u$, then $M_0^2 = \lambda_{\max}(K^{\mathsf T} D_{u/A^2} K)$. Interpolating between $(2, 2; u)$ and $(\infty, \infty)$ with $1/p = (1-t)/2$, that is $t = 1 - 2/p$, the norm is at most $M_0^{2/p}$ and the weight on the range becomes $u^{p(1-t)/2} = u$. Hence $\sum_r u_r |T\theta|_r^p \le M_0^2 \|\theta\|_p^p$, and from $\|K\theta\|_p^p = \sum_r A_r^p |T\theta|_r^p$ we get $\mu_p(K) \le S_p(K;u)$.
- The source for the interpolation: interpolation that changes the measure on the range is of the Stein–Weiss form. But the weight has no effect at the $\ell^\infty$ end, so taking both ends as statements on the same measure $u$, Riesz–Thorin alone suffices Cited .
- **The range of application is $p \ge 2$** ($t \ge 0$ is needed). At $p = 2$, $S_2(K) = \mu_2(K)$ and the upper bound equals the true value.

**Definition 5.5 (bundle number).** For a sequence of step maps $(K_k)$ and a bundle length $n$, put $K^{(k,n)} := K_k K_{k+1} \cdots K_{k+n-1}$ and

$$n_p := \min\Big\{ n : \ \sup_k \mu_p\big(K^{(k,n)}\big) < 1 \Big\} \quad \text{(if none, } \infty\text{)}$$

If the upper-bound version with $\mu_p$ replaced by $S_p$ is $\tilde n_p$, then $n_p \le \tilde n_p$.

**Definition 5.6 (window).** Fixing the bundle length $n$, put $W_n := \{p \ge 1 : \ \mu_p(K^{(n)}) < 1\}$.

**Definition 5.7 (affine recursion).** $c_{k+1} \le \mu_k c_k + \varepsilon_k$ ($\mu_k \ge 0$, $c_k \ge 0$). The bundle version is $c_{k+m} \le \mu\, c_k + \varepsilon$. $c_k$ corresponds to the cone constant at step $k$, and $\varepsilon_k$ to the size of the degree-$p$ terms newly born at that step.

**Definition 5.8 (Schur threshold exponent).** When $R(K) < 1$, put $p^*(K) := 1 + \log C / \log(1/R)$. The bound $\mu_p \le R^{p-1}C$ of Candidate 5.6 below falls below 1 when $p > p^*$.

### Candidate theorems and labels

**Candidate 5.1 (pull-back and submultiplicativity).** If $c \ge 0$, then $c_p(f \circ K) \le c_p(f)\,\mu_p(K)$ Lean (`cone_pullback`; a map $V \to W$, and $N_V, N_W$ may differ). On a single space, $\mu_p(K_1K_2) \le \mu_p(K_1)\mu_p(K_2)$ ($\mu_1 \ge 0$), $\mu_p(K^n) \le \mu_p(K)^n$ and $c_p(f \circ K^n) \le c_p(f)\,\mu^n$ Lean (`contracts_comp`, `contracts_iterate`, `cone_pullback_iterate`; $\mu \ge 0$).

**Candidate 5.2 (exceeding in one step, contracting in a bundle).** If $\mu_p(K) \le M$ in one step ($M > 1$ is allowed) and $\mu_p(K^n) \le \mu$ in $n$ steps, then $\mu_p(K^{qn+r}) \le M^r \mu^q$ Lean (`block_decay`; $M \ge 0$, $\mu \ge 0$). This is the cone version of Candidate 1.4 (a) of § 01. The upper bound $S_p$ itself is not necessarily submultiplicative, so the $S_p$ of a composite is computed directly from the composite kernel Reading.

**Candidate 5.3 (log-convexity of the upper bound).** $\log S_p(K)$ is convex in $p$ Reading. This is because $S_p = \sup_{\|x\|=1} \sum_r (Kx)_r^2 A_r^{p-2}$, and for each $x$ the right-hand side is a positive sum of exponential functions of $p$ and hence log-convex (Cauchy–Schwarz), and the supremum of log-convex functions is log-convex. What is in Lean is a discrete midpoint version Lean:

- `weighted_rpow_midpoint`: for $w \ge 0$, $a > 0$, $\left(\sum w\, a^{(p+q)/2}\right)^2 \le \left(\sum w\, a^p\right)\left(\sum w\, a^q\right)$.
- `stein_form_midpoint`: its Stein form $\sum y^2 A^{p-2}$ ($A > 0$).
- `sup_midpoint`, `midpoint_below_one`: from the pointwise midpoint inequality, an upper bound by the geometric mean of the suprema; in particular, if $S_a S_b < 1$ the value at the midpoint falls below 1.

As a consequence, $S_3 \le \sqrt{S_2 S_4}$. What looked in the record like "$\sqrt{S_2S_4}$ is a little above $S_3$" was not a comparison of two different upper bounds but the margin of the curvature of a single function $\log S_p$. $\sqrt{S_2S_4}/S_3 - 1$ is one number measuring the size of that curvature (Table 12).

**Candidate 5.4 (log-convexity of the true $p$-contraction).** $\log \mu_p$ is also convex in $p > 0$. By Riesz–Thorin, $\psi(s) := \log \|K\|_{1/s \to 1/s}$ is convex for $s \in [0,1]$ Cited, and $\log \mu_p = p\,\psi(1/p)$ is the perspective of $\psi$ and hence convex Reading. Candidate 5.3 is what can be said directly about the upper bound $S_p$ (as a sum of exponentials), while Candidate 5.4 is what can be said about the true $\mu_p$ from convexity in $1/p$ through the perspective. They rest on different things. There are two consequences Reading.

- The window $W_n$ is an interval (the set where a convex function is negative).
- From $\mu_p^{1/p} \to R$ the asymptotic slope is $\log R$, and by convexity $\mu_p \le \mu_{p0}\, R^{p-p0}$ for $p \ge p_0$. For $p_0 = 2$ this is the same as using Riesz–Thorin between $(2, \infty)$. The upper-bound version $S_p \le R^{p-p0} S_{p0}$ follows directly Lean (`weighted_rpow_shift`: for $w \ge 0$, $0 < a \le R$, $p_0 \le p$, $\sum w\, a^p \le R^{p-p0} \sum w\, a^{p0}$; `sup_shift`: moves the pointwise inequality to the supremum).

**Candidate 5.5 (bundle number and spectral radius).** For a fixed finite-dimensional operator $K : V \to V$, Gelfand's formula gives $\lim_n \mu_p(K^n)^{1/n} = \rho(K)^p$, and $\rho$ does not depend on $p$ Cited. Hence "$n_p < \infty \Leftrightarrow \rho(K) < 1$", and only the smallest $n$ changes with $p$. This is the same fact as Definition 1.3 of § 01, said in terms of the $p$-contraction. The blocking map, however, is a map from the coarse lattice to the fine lattice, not a single operator on one space. Identifying the fine lattice back with $\mathbb{Z}^4$ gives the type

"dilation by 2 and convolution" (a subdivision operator), and for this type the $\ell^p$ spectral radius depends on $p$ Reading .

- A constant field gives a dimension count. The plaquette field shrinks by a factor of about $1/4$ in one step while the number of points grows by $2^4$, so the ratio for a constant field is about $16 \cdot 4^{-p} = 2^{4-2p}$ per step Reading . The recorded values at $p = 2, 3, 4$ were 1.080, 0.2807, 0.07297, a factor of about $1/3.85$ for each increase of $p$ by 1 Computed . This agrees with the dimensional reading that $p = 2$ is marginal ($\int F^2$ is dimensionless in four dimensions) and that for $p > 2$, $|F|^p$ acts in an irrelevant direction.
- The link with dominant matrices (§ 01): from $|K\theta| \le |K||\theta|$ we get $\mu_p(K) \le \mu_p(|K|)$, and the Perron weight can be used for $|K|$. In the kernels of the record, however, the cancellation of signs is large: $R(K_0K_1) = 0.928$ is much smaller than $R(K_0)R(K_1) \approx 2.25 \cdot 2.47 \approx 5.6$ Computed . Moving to $|K|$ loses this factor of about 6 at the outset. This is one number, in $\ell^\infty$, for the loss from discarding signs of § 02 Reading .

**Candidate 5.6 (the correct form of dilution).** The naive form "if the row sums are 1, the entries are non-negative and each entry is $\le 1/m$, then $\mu_p \le m^{1-p}$" **does not hold in general** Reading . An averaging map preserves constant fields, so if it is square, $\mu_p \ge 1$ (a constant field is a counterexample). What holds is the following.

- The $\ell^p$ version of the Schur test: for $p \ge 1$, $\sum_i |\sum_j K_{ij}\theta_j|^p \le R^{p-1}C \sum_j |\theta_j|^p$, that is $\mu_p \le R^{p-1}C$ Lean (`row_rpow_le`: $|\sum k_j\theta_j|^p \le (\sum |k_j|)^{p-1} \sum |k_j||\theta_j|^p$ by Jensen; `lp_schur`: row $\ell^1 \le R$, column $\ell^1 \le C$, $R \ge 0$). If $R < 1$, the upper bound on $\mu_p$ decreases in $p$ with ratio $R$ and falls below 1 for $p > p^*$.
- Its equality: if all row sums are $R > 0$ and all column sums are $C$, equality holds for a constant field Lean (`lp_schur_sharp`). The Schur form cannot be improved in general.
- Dilution of a localised field: for a $\delta$ field, $\sum_i |k_i|^p \le M^{p-1} \sum_i |k_i|$ ($|k_i| \le M$, $p \ge 1$) Lean (`delta_dilution`). If the entries are at most $1/m$, this is a factor of $m^{1-p}$. The $m^{1-p}$ dilution works only for localised fields; spread-out fields are diluted only by the row $\ell^1$.
- The minimal form in which the cone contracts: $\mu_p \le \mu_2 R^{p-2}$ (a consequence of Candidate 5.4) Reading , and the upper-bound version $S_p \le S_2 R^{p-2}$ Lean (`weighted_rpow_shift`, `sup_shift`). The reading is that $\ell^2$ is marginal and stays at $\mu_2 \ge 1$, the row $\ell^1$ of the composite drops below 1 as steps accumulate, and for $p > 2$ one gains by the factor $R^{p-2}$ Reading . That the field maximising the one-step $\ell^3$ in the record was spread out (degree of localisation 0.006) agrees with the obstruction lying on the side of spread-out fields, that is, the side where the dilution of $\delta$ fields does not work Computed .

**Candidate 5.7 (affine recursion).** There are the following five Lean . They extend Candidate 1.4 (c) of § 01 (`affine_uniform`) to coefficients that vary with the step and to bundles.

- `affine_const`: $c_{k+1} \le \mu c_k + \varepsilon$ ($0 \le \mu < 1$, $\varepsilon \ge 0$) $\Rightarrow c_n \le \mu^n c_0 + \varepsilon/(1-\mu)$.
- `affine_varying`: $0 \le \mu_k \le \bar\mu < 1$, $\varepsilon_k \le \bar\varepsilon$ ($\bar\varepsilon \ge 0$), $c_k \ge 0 \Rightarrow c_n \le \bar\mu^n c_0 + \bar\varepsilon/(1-\bar\mu)$.
- `affine_invariant`: under the same assumptions, if $c_0 \le \bar\varepsilon/(1-\bar\mu)$, then $c_n \le \bar\varepsilon/(1-\bar\mu)$ for all $n$ (an invariant interval).
- `affine_eventually`: under the same assumptions, for any $\delta > 0$, $c_n \le \bar\varepsilon/(1-\bar\mu) + \delta$ for all sufficiently large $n$ (the form $\limsup c_n \le \bar\varepsilon/(1-\bar\mu)$).

- `affine_block`: $c_{k+m} \le \mu c_k + \varepsilon$ $(0 \le \mu < 1, \varepsilon \ge 0) \Rightarrow c_{mn} \le \mu^n c_0 + \varepsilon/(1-\mu)$. Combined with Candidate 5.2, the steps between bundles pick up a factor $M^r$ $(r < m)$ only once.

Table 11 The candidate theorems of § 05. Each is a finite-dimensional form of a known fact, or an elementary consequence of one

| Candidate | Statement (one line) | Label | What it restates |
|---|---|---|---|
| 5.1 | $c_p(f \circ K) \le c_p(f)\mu_p(K)$, $\mu_p(K_1K_2) \le \mu_p(K_1)\mu_p(K_2)$, $\mu_p(K^n) \le \mu_p(K)^n$ | Lean (`cone_pullback`, `contracts_comp`, `contracts_iterate`, `cone_pullback_iterate`) | Submultiplicativity of operator norms |
| 5.2 | One step $\le M$, $n$ steps $\le \mu \Rightarrow qn + r$ steps $\le M^r \mu^q$ | Lean (`block_decay`) | A finite version of Gelfand's formula (Candidate 1.4 (a) of § 01) |
| 5.3 | $\log S_p$ is convex in $p$; in particular $S_3 \le \sqrt{S_2 S_4}$ | Continuous version Reading, discrete midpoint version Lean (`weighted_rpow_midpoint`, `stein_form_midpoint`, `sup_midpoint`, `midpoint_below_one`) | Hölder's (Cauchy–Schwarz) inequality; the supremum of log-convex functions |
| 5.4 | $\log \mu_p$ is convex in $p$, $W_n$ is an interval, $\mu_p \le \mu_{p0} R^{p-p0}$ $(p \ge p_0)$. Upper-bound version $S_p \le R^{p-p0} S_{p0}$ | True $\mu_p$: Cited + Reading; upper-bound version: Lean (`weighted_rpow_shift`, `sup_shift`) | Riesz–Thorin (convexity in $1/p$) and the perspective |
| 5.5 | For a fixed operator, $n_p < \infty \Leftrightarrow \rho(K) < 1$. The blocking map is of subdivision type, and its $\ell^p$ spectral radius depends on $p$. The ratio for a constant field is about $2^{4-2p}$ per step | Cited, Reading, numbers Computed | Gelfand's formula; the $L^p$ theory of subdivision |
| 5.6 | The naive dilution $\mu_p \le m^{1-p}$ does not hold (constant field). What holds is $\mu_p \le R^{p-1}C$ (with equality cases), $M^{p-1}$ for $\delta$ fields, and $S_p \le S_2 R^{p-2}$ | Counterexample Reading; the three forms Lean (`row_rpow_le`, `lp_schur`, `lp_schur_sharp`, `delta_dilution`, `weighted_rpow_shift`); $\mu_p \le \mu_2 R^{p-2}$ Reading | The Schur test; finite sums for Riesz–Thorin between $(1,\infty)$ and $(2,\infty)$ |
| 5.7 | Affine recursion: constant coefficients, coefficients varying with the step, invariant interval, upper limit, bundles of $m$ steps | Lean (`affine_const`, `affine_varying`, `affine_invariant`, `affine_eventually`, `affine_block`) | An elementary fact about affine recursions (Candidate 1.4 (c) of § 01) |

### Correspondence with the numbers of the record

One step is the step-0 kernel $K_0$ on the $8^4$ torus (no truncation); two steps is $K_0K_1$ (coarse $16^4$, kernels truncated at $R = 3$, with the $\ell^1$ lost being 0.8% for $K_0$ and 1.9% for $K_1$); three steps is $K_0K_1K_2$ (coarse $12^4$, $R = 3$). The one-step lower bound is the value of the true ratio from a quasi-Newton method (L-BFGS, two starting points); for $p \ge 12$ it was replaced by the value for the sign field of the largest row. The lower bound is not shown to be the maximum. All are floating-point computations, not interval estimates Computed.

Table 12 The weighted Stein bound $S_p$ and the one-step lower bound Computed

| $p$ | One step $S_p$ | One step lower bound | One step constant field | Two steps $S_p$ | Three steps $S_p$ |
|---|---|---|---|---|---|
| 2 | 1.9806 | 1.9806 | 1.080 | 3.0189 | 3.6658 |
| 2.5 | 2.3626 | 1.4106 | 0.551 | 2.2358 | – |
| 3 | 2.8641 | 1.0233 | 0.2807 | 1.6642 | 0.57864 |
| 3.5 | 3.5442 | 0.7730 | 0.1431 | 1.2501 | – |
| 4 | 4.4457 | 0.6470 | 0.0730 | **0.9707** | 0.09599 |

| $p$ | One step $S_p$ | One step lower bound | One step constant field | Two steps $S_p$ | Three steps $S_p$ |
|---|---|---|---|---|---|
| 5 | 7.8474 | 0.7328 | 0.0190 | 0.6144 | – |
| 6 | 14.910 | 1.2326 | 0.0049 | 0.4093 | – |
| 8 | 65.160 | 5.0155 | 0.0003 | 0.2197 | – |
| 16 | 3.91e4 | 18.2 (sign field) | | 0.0598 | – |
| 32 | 1.71e10 | 7.96e6 (sign field) | | 0.0143 | – |
| $\infty$ ($S_p^{1/p} \to R$) | $R = 2.2524$ | | | $R = 0.9283$ | $R = 0.3318$ |

- **Convexity of** $\log S_p$: the second difference quotients were all positive for one step ($p = 2$–$64$), two steps (2–32) and three steps ($2, 3, 4$). The three-point ratios $S_p/(S_a^\lambda S_b^{1-\lambda})$ are 0.858–0.99999, all at most 1 (Candidate 5.3) Computed .
- $\sqrt{S_2 S_4}/S_3$: one step 1.036, two steps 1.029, three steps 1.025 ($K_0K_1K_2$) and 1.016 ($K_4K_4K_4$) Computed .
- **Asymptotic slope**: $S_p^{1/p} \to R$, and $R$ falls with the steps: 2.252 for one step, 0.928 for two, 0.332 for three. The slope of $\log S_p$ at large $p$ agreed with $\log R$ (one step: slope 0.8125 for $p = 48 \to 64$ against $\log R = 0.8120$) Computed .
- **Monotonicity of the slope and extrapolation**: the slope of $\log S_p$ for three steps $K_0K_1K_2$ increases from $-1.846$ for $2 \to 3$ to $-1.796$ for $3 \to 4$, heading towards the asymptotic slope $\log R = -1.103$. By the upper-bound version of Candidate 5.4, $S_p \le S_4 R^{p-4} = 0.0960 \cdot 0.332^{p-4}$ for $p \ge 4$, so cones of high $p$ can be bounded from above without recomputation (the inequality in Lean, the numbers Computed ).
- **Three values of** $p$ **for three steps**: 3.67, 0.58, 0.092 at $p = 2/3/4$ (for $K_0K_1K_2$, the values 3.6658, 0.57864, 0.09599 of Table 12; from $K_1K_2K_3$ to $K_4K_4K_4$, 3.673–3.682, 0.5715–0.5732, 0.0920–0.0922, almost the same regardless of the step). The one-step upper bound grows with $p$, but for three steps it shrinks with $p$: in three steps $\ell^2$ stretches while $\ell^3$ and $\ell^4$ contract Computed .
- **Schur is weak for the kernels of the record**: the column $\ell^1$ is large, $C = 23.3$ (one step) and 131.1 (two steps), and $R^{p-1}C$ is 52–6856 for one step and 13–122 for two. The two-step $p^*$ is $1 + \log 131.1/\log(1/0.928) \approx 66.5$, while the Stein bound already falls below 1 at $p = 4$ for two steps. The minimal form $S_2 R^{p-2}$ is also 1.065 at $p = 16$ and 0.587 at $p = 24$ for two steps, a much weaker upper bound than $S_p$ itself Computed .
- **Upper bound** $\tilde{n}_p$ **on the bundle number** (from $S_p$, bundles starting at $K_0$): 3 at $p = 3$, 2 for $p = 4$–$32$ (two-step $S_4 = 0.971$), 2 at $p = \infty$ ($R(K_0K_1) = 0.928 < 1$); at $p = 2$ every one up to three steps exceeds 1 Computed .
- **The true one-step** $\mu_p$: from the lower bound, $\mu_p(K_0) \ge 1$ for $p \le 3$ and $p \ge 6$. For $3.5 \le p \le 5$ the lower bound is 0.65–0.77 and the upper bound 3.5–7.8, and **whether the true one-step** $\mu_p$ **falls below 1 is undecided** Computed . The lower bound is U-shaped in $p$, 1.98 → 0.60 ($p = 4$) → 5.0 ($p = 8$), which does not contradict the convexity of the true $\log \mu_p$ (Candidate 5.4). The possibility that the one-step window $W_1$ is non-empty remains. Its location, however, can be narrowed. From the one-coordinate Hölder inequality, $\mu_p \ge \max_r \|k_r\|_{p'}^{p}$, which is at least 1 for $p \ge 5.874$, and the ratio of an explicit field (the field alternating on the mesh of the lattice) is at least 1 for $p \le 3.039$ and $p \ge 5.665$ Computed (the inequality is Lean: OneStepWindow (holder_dual_ratio, exists_lpRatio_ge_row)). Since the window is an interval (Candidate 5.4; Lean: window_ordConnected_of_logConvex, convexity of

$\log \mu_p$ assumed), $W_1$, if non-empty, is a single interval inside $(3.04,\ 5.67)$. On the side of the upper bound, the family of row weights $u$ saturates at $u = A^{p-2}$ (Lean: stein_row_weight_saturation), and for the column weights too $b = 1$ is optimal Computed , so weighted interpolation on $(2, \infty)$ tightens no further. The field giving the lower bound is the pattern alternating on the mesh of the lattice ($|q|_\infty = \pi$), and the obstruction at one step is an ultraviolet pattern, not an infrared one Reading .

In the language of windows, the window given by $S_p$ contains $[\approx 4, \infty)$ for two steps and $[3, 4]$ for three steps. For one step the window given by $S_p$ is empty, and the true window is undetermined. Since $\mu_2 = S_2$ is 1.98, 3.02, 3.67 for $n \le 3$, all at least 1, each $W_n$ ($n \le 3$), being an interval, does not straddle 2 Reading .

### A question proper to this pattern — the window, the bundle number, the threshold exponent

- **The extent of the window.** For the sequence of blocking maps, how does the window $W_n$ widen with $n$? By Candidate 5.4 each $W_n$ is an interval; it is open above if $R_n < 1$ and closed above if $R_n > 1$. Where the lower end moves cannot be told from the numbers of the record. If the true values for one step at $3.5 \le p \le 5$ are settled, whether $W_1$ is empty is settled (question "motion of the lower edge of the window"). If $W_1$ is non-empty, it is a single interval inside $(3.04, 5.67)$.
- **Divergence of the bundle number as $p \downarrow 2$.** On the side $p < 2$, the ratio for a constant field is about $2^{4-2p} > 1$ per step, so for large $n$ it is expected not to enter the window Assumption . If the limit of $\mu_p(K^{(n)})^{1/n}$ is determined by the dimension count $2^{4-2p}$ of the constant field, then $n_p$ diverges like $c/(p-2)$. If it is not, the lower end $p_c > 2$ of $W_\infty$ is a critical exponent determined by the geometry of the lattice. Which of these holds has not been checked Assumption (question "divergence of the number of bundles").
- **The meaning of the threshold exponent $p^*$.** Does the gap between the Schur $p^*$ (about 66.5 for two steps) and the lower end of the Stein window (about 4 for two steps) come from the size of the column $\ell^1$ or from the cancellation of signs? How far can $p^*$ be lowered by a weighted Schur test (positive test functions) (question "Schur threshold and Stein window")?

### Dictionary

| Handwork of the record | Known framework | Label |
|---|---|---|
| Interpolation between $\ell^2$ and $\ell^\infty$; convexity of $\log \|K\|_{p\to p}$ | The Riesz–Thorin interpolation theorem (M. Riesz 1927, G. O. Thorin 1939): $\log \|K\|_{p\to p}$ is convex in $1/p$ | Cited |
| Interpolation that changes the weight on the range (Definition 5.4) | Stein–Weiss, Interpolation of operators with change of measures (Trans. AMS 87, 1958). If the measures at both ends are fixed, Riesz–Thorin suffices | Cited , bibliography Reading |
| Upper bound by the row and column $\ell^1$ (Candidate 5.6) | The Schur test (I. Schur 1911). The weighted version (positive test functions) is known in the forms of Aronszajn and Halmos–Sunder | Cited , bibliography Reading |
| Pointwise inequalities of the cone; the midpoint inequality (Candidate 5.3) | Hölder's inequality, Cauchy–Schwarz, the arithmetic–geometric mean inequality | Cited |
| Convexity of $\log \mu_p$ in $p$ (Candidate 5.4) | Log-convexity of operator norms (Riesz–Thorin) and the perspective function of convex analysis (Boyd–Vandenberghe, Convex Optimization § 3.2.6) | Cited , Reading |

| Handwork of the record | Known framework | Label |
|---|---|---|
| Computing $\mu_p$ itself | For $p \neq 1, 2, \infty$, approximating the $p \to p$ norm of a matrix is NP-hard (Hendrickx–Olshevsky, SIAM J. Matrix Anal. Appl. 2010). For lower bounds, Boyd's power method (D. W. Boyd 1974, Linear Algebra Appl. 9). This is the reason for using the upper bound $S_p$ | Reading |
| Contracting in a bundle (Candidate 5.5) | Gelfand's spectral radius formula. For operators involving dilation there is the $L^p$ theory of subdivision (R.-Q. Jia, Subdivision schemes in $L_p$ spaces, Adv. Comput. Math. 1995), in which convergence is decided by the $p$-norm joint spectral radius | Cited , Reading |

The bibliography is at the level of abstracts and has not been checked against the texts of the original sources. Every candidate theorem of this section is a finite-dimensional form of one of the frameworks above, or an elementary consequence of one.

06

## Dictionary with the known, what can be said, questions

### Summary of the five grades

Table 13 The main rows of this chapter (labels as in the rows of the text; conditions on the same row)

| Section | Claim | Label | Conditions |
|---|---|---|---|
| § 01 | Dominant matrices are closed under products and powers | Lean | Finitely many blocks with values in a normed ring. For powers, the identity has norm 1 |
| § 01 | Candidate 1.1: if it contracts in $k$ steps, there is a weight $w'$ that contracts in one step | Lean | $B \geq 0$, $w \geq 0$, $s > 0$, $k \geq 1$, $B^k w \leq s^k w$ |
| § 01 | Candidates 1.2–1.5 (one-step estimate, change of weight, finite Gelfand, affine recursion, aggregation of blocks and the two-by-two test) | Lean | Finite-dimensional. The Collatz–Wielandt equality, Gelfand's formula itself and the existence of Perron vectors are not included |
| § 01 | Candidate 1.2′ ($\ell^2$ weight, the Schur test), explicit formula for the transient constant | Reading | Right and left positive Perron vectors exist |
| § 01 | The loss from discarding signs is not necessarily bounded over steps (nilpotent example) | Lean | The quantity that composes after taking absolute values |
| § 01 | Perron–Frobenius, Collatz–Wielandt, Gelfand, the max-plus bounds, the characterisation of M-matrices | Cited | — |
| § 02 | The loss stops from two steps on and falls a little (6.28, 5.51, 5.10, 5.26) | Computed | $4^4$ torus, steps cut at $R = 3$, the true ratio is a lower bound, $k \leq 4$ |
| § 02 | Candidate 2.1: upper bound $\div$ test value $\leq \kappa^{-2}$ | Lean | $\kappa A_r M \leq \lvert y_r \rvert$ for every row, $\lvert v_a \rvert \leq M$ |
| § 02 | Loss $= \langle \kappa^{-2} \rangle$ (5.55, 5.13, 5.31) | Computed | $4^4$, $k = 2, 3, 4$, shape and phase factors equal to 1 |
| § 02 | Phase factor $\Phi = 1$, the formula for the plane-wave test | Reading | $2q^* \not\equiv 0$ and $4q^* \not\equiv 0$, or period 2 with $u^*$ real |
| § 02 | Hypotheses (a)(b)(c) fail, and the way they fail is constant over steps | Computed | $k \leq 3$ (larger tori), $k \leq 4$ ($4^4$) |
| § 02 | The profile of the composite kernel approaches a single one over steps | Reading | A reading from the numbers |

| Section | Claim | Label | Conditions |
|---|---|---|---|
| § 02 | Upper bound of the positive-part-dominant type (Schur and degradation under composition) | Lean | Negative $\leq \delta$ positive on every row. The assumption is not satisfied for the kernels of the record |
| § 02 | The loss for the W coordinates stops at 1.10, close to the rank-one value 1.085 | Computed | Kernel with the zero mode corrected, 60 types, $k \leq 5$ |
| § 03 | Existence of extensions, affine structure, dimension of the freedom, extension with conditions, compatibility of composition | Lean | Finite-dimensional linear algebra |
| § 03 | Candidates 3.1–3.5 (uniqueness of the zero mode, the Moore–Penrose representative is not local, invariance of $\mathrm{ord}_V$, the rule for the order of the tail, shifted jets) | Reading | The lattice steps are lines of argument on paper. Candidate 3.4 assumes constants uniform in the steps |
| § 03 | ord: sym 0, A3 1, A4 2; $\mathrm{ord}^V(F) = 2$; the tail is bounded for $D \leq 4$ | Computed | Representatives of the record, axial input at $D = 4$, $L \leq 192$ |
| § 03 | Correspondence with sum rules, high-frequency order and Strang–Fix | Cited, Reading | Details of how the original sources count orders ($\pm 1$) not checked |
| § 04 | $\kappa = s_1 w_1$ does not depend on the scale, and the lens opens for $\kappa \leq \frac{1}{4}$ | Reading | A count with the leading coefficients only |
| § 04 | Single-plaquette series, ties, error ties, the lens in $d$ dimensions, MK level lines | Computed | Single plaquette, leading terms with equipartition, the approximate MK map |
| § 04 | Information about the group remains in the height of the crossing and the sign of the next term; overlap of the four-dimensional lens with the band | Assumption | Leading terms only; may include coincidence |
| § 04 | Candidate B (the comparison that carries the distinction is localised in the lens) | Assumption | Assumes agreement of the strong-coupling valuations, difference of the weak-coupling valuations, and regions where the leading terms dominate. Statement only |
| § 05 | Candidates 5.1, 5.2, 5.7 (pull-back of cones, submultiplicativity, exceeding in one step and contracting in a bundle, five forms of affine recursion) | Lean | Statements written with an abstract $N$ (in the role of $\|\cdot\|_p^p$). Theorem-specific assumptions such as $\mu \geq 0$, $0 \leq \mu < 1$, $\varepsilon \geq 0$. The supremum of the operator norm itself is not included |
| § 05 | Candidates 5.3 and 5.6: midpoint convexity of $\log S_p$, Schur's $\mu_p \leq R^{p-1} C$ and its equality, dilution of $\delta$ fields, $S_p \leq S_2 R^{p-2}$ | Lean | Finite sums. $w \geq 0$, $a > 0$, $a \leq R$, $p \geq 1$. Convexity in continuous $p$ and the formula for $S_p$ itself (interpolation) are not included |
| § 05 | Derivation of the weighted Stein bound for general $p$, convexity of the true $\log \mu_p$, the window $W_n$ is an interval, counterexample to the naive dilution (constant field) | Reading, Cited | $p \geq 2$ ($t = 1 - 2/p \geq 0$). Convexity via Riesz–Thorin and the perspective |
| § 05 | Second difference quotients of $\log S_p$ all positive; $\sqrt{S_2 S_4}/S_3 = 1.016$–$1.036$; $S_p^{1/p} \to R$ (2.252, 0.928, 0.332); upper bounds on the bundle number (3 at $p = 3$; 2 for $p = 4$–$32$ and $\infty$); the true one-step $\mu_p$ is undecided for $3.5 \leq p \leq 5$ (lower bound 0.65–0.77, upper bound 3.5–7.8); if $W_1$ is non-empty, a single interval inside $(3.04, 5.67)$ | Computed | One step $8^4$ without truncation; two steps coarse $16^4$; three steps coarse $12^4$; truncated at $R = 3$; the lower bound is not shown to be the maximum |

| Section | Claim | Label | Conditions |
|---|---|---|---|
| § 05 | $n_p$ diverges like $c/(p-2)$ as $p \downarrow 2$, or there is a lower end $p_c > 2$ of $W_\infty$ | Assumption | An estimate from the dimension count $2^{4-2p}$ of the constant field alone |

### The Lean of this chapter

None of the five files contains sorry or uses `native_decide`, and the axioms printed at the end were the three standard ones (propext, Classical.choice, Quot.sound) or a subset of them. None of the files contains lattices, groups, measures, minimisation or the Fourier transform itself.

Table 14 The Lean files of this chapter Lean

| File (namespace) | Number of theorems | Main theorems and assumptions | Place in the text | Not included |
|---|---|---|---|---|
| DominantMatrixContraction (`DominantMatrix`) | 21 | `dominant_mul`, `dominant_pow` (the identity has norm 1), `weighted_step`, `weighted_iter` ($Bw \le \lambda w$), `weight_change` ($av \le w \le bv$, $a > 0$), `liftWeight_step` ($B \ge 0$, $w \ge 0$, $s > 0$, $k \ge 1$, $B^k w \le s^k w$), `norm_pow_le_const_mul`, `sum_norm_pow_le`, `affine_uniform` ($0 \le \lambda < 1$, $\varepsilon \ge 0$), `aggregate_step`, `two_by_two_weight`, `two_by_two_exists`, `signedEx_sq`, `absEx_pow` | § 01 | The Collatz–Wielandt equality, Gelfand's formula itself, the existence of Perron vectors, Candidate 1.2′ ($\ell^2$ weight), the max-plus bounds |
| SignLossBound (`SignLoss`) | 16 | `jensen_four`, `schur_four` (row $\ell^1 \le A$, column $\ell^1 \le B$), `coherent_of_neg_le` (negative $\le \delta$ positive), `loss_le_of_coherent`, `comp_coherent`, `coherence_le_one`, `sum_four_le` ($\lvert v_a\rvert \le M$), `stein_form_le`, `stein_le_inv_sq_mul_test` ($\kappa A_r M \le \lvert y_r\rvert$), `abs_comp_le`, `row_l1_comp_le` | § 02 | The Stein formula itself (Plancherel and interpolation), the average of the plane-wave phase, Wielandt's theorem, convergence to rank one, the numbers of the loss |
| ConstrainedExtension (`ConstrainedExtension`) | 20 | `exists_extension`, `isExtension_iff_mem_coset`, `finrank_freedom` (finite-dimensional), `freedom_eq_zero_iff` ($W \neq 0$), `exists_extension_with_conditions` ($(\mathrm{res} V, C)$ surjective), `finrank_freedom_with_conditions`, `eq_of_eqOn_iSup` (the sum of the family of subspaces is the whole), `comp_isExtension` ($f_1(V_1) \subseteq V_2$), `comp_depends_of_not_le` | § 03 | Lattices, continuity of symbols, the order of folding, moments, the lattice steps of Candidates 3.1–3.5 |

| File (namespace) | Number of theorems | Main theorems and assumptions | Place in the text | Not included |
|---|---|---|---|---|
| ConeCalculus (`ConeCalculus`) | 20 | Definitions `IsCone`, `Contracts`. `cone_pullback` ($c \geq 0$), `contracts_comp`, `contracts_iterate`, `cone_pullback_iterate` ($\mu \geq 0$), `block_decay` ($M \geq 0$, $\mu \geq 0$), `row_rpow_le`, `lp_schur` ($p \geq 1$, row $\ell^1 \leq R$, column $\ell^1 \leq C$), `lp_schur_sharp` (row sums constant at $R > 0$, column sums constant at $C$), `delta_dilution`, `weighted_rpow_midpoint` ($w \geq 0$, $a > 0$), `stein_form_midpoint`, `weighted_rpow_shift` ($0 < a \leq R$, $p$ $0 \leq p$), `sup_shift`, `sup_midpoint`, `midpoint_below_one`, `affine_const` ($0 \leq \mu < 1$, $\varepsilon \geq 0$), `affine_varying`, `affine_invariant`, `affine_eventually`, `affine_block` | § 05 | The operator norm (the supremum of the true $\ell^p$ norm) itself, the formula for the Stein bound (interpolation and Plancherel), Riesz–Thorin, convexity in continuous $p$, Gelfand's formula, lattices and kernels, the numbers of the record |
| OneStepWindow (`LatticeGauge.OneStepWindow`) | 20 | `holder_dual_ratio`, `norm_conj_rpow`, `exists_lpRatio_ge_row` (the one-coordinate Hölder lower bound, $p > 1$), `lamMax_mono`, `lamMax_smul` ($\lambda_{\max}$ is defined as the supremum of the Rayleigh quotient), `steinWeighted_ge`, `stein_row_weight_saturation` (the family of row weights saturates at $u = A^{p-2}$), `window_ordConnected`, `window_ordConnected_of_logConvex` (convexity of $\log \mu_p$ is assumed) | § 05 | Courant–Fischer, the number $p_{\mathrm{row}} = 5.874$ and the interval $(3.04, 5.67)$ (computed), the operator norm itself |

### What is not said

(a) **We do not claim to have set up a new theory or new theorems.** The candidate theorems of § 01–§ 03 and § 05 are all finite-dimensional forms of known facts, elementary consequences of them, or restatements in known frameworks (Perron–Frobenius, multigrid orders, sum rules, Riesz–Thorin interpolation, the Schur test). What this chapter adds is the correspondence that reads the numbers of the record as those quantities, and the questions that come out of it.

(b) **What passed in Lean is only the 97 theorems of Table 14.** Candidate 1.2′, the explicit formula for the transient constant, the phase factor, the lattice steps of Candidates 3.1–3.5, all of § 04, the derivation of the Stein bound for general $p$ in § 05, the convexity of the true $\log \mu_p$, and Candidate 5.5 are lines of argument on paper or estimates.

(c) **The numbers are not estimates in the rigorous sense.** The numbers in § 02– § 04 are floating-point computations, not interval estimates. The true ratio is a lower bound from maximisation and is not shown to be the maximum (for one step on $16^4$, two starting points stopped at 0.5767, short of the recorded 0.591).

(d) **That the loss is bounded at every step is not shown.** What is shown is only the numbers for $k \leq 4$ ($4^4$) and $k \leq 3$ ($8^4$), and the conditional inequality "if the alignment is bounded below, the loss is

bounded". The $4^4$ torus wraps the support of the kernel heavily, so "loss $= \langle \kappa^{-2} \rangle$" there is a fact about the operator of that period.

(e) **The tail rule of Candidate 3.4 is a line of argument assuming constants uniform in the steps.** Uniformity over infinitely many steps is not treated. Whether compatibility of composition holds in the chain of the record has not been checked.

(f) **The four-dimensional numbers of § 04 are derived only from first-order leading terms and equipartition.** The true four-dimensional series is not used. The overlaps with the band and the transition may include coincidence. MK is an approximate map, and whether the stopping level lines of $U(1)$ are true fixed lines has not been distinguished. Candidate B is a restatement and does not claim to make the band easier.

(g) **The results of this chapter have no direct effect on ultraviolet control or on the renormalisation comparison.** The table of correspondence with the record in § 01 is a rereading of the record.

(h) **Much of the literature was read at the level of abstracts and excerpts.** For Hemker, Charina–Conti, Jia, Donatelli, Rump, Varga, Bertsekas–Tsitsiklis, arXiv:1005.1993, arXiv:0807.0185, Stein–Weiss, Hendrickx–Olshevsky and Boyd (power method), the details of the definitions in the original sources have not been checked.

(i) **The windows and bundle numbers of § 05 are windows of upper bounds.** The window given by $S_p$ contains $[\approx 4, \infty)$ for two steps and $[3, 4]$ for three steps, but the values of the ends of the window of the true $\mu_p$ have not been obtained. The one-step range $3.5 \le p \le 5$ is undecided, and the dependence of the two- and three-step numbers on truncation and on the size of the coarse torus has not been assessed. The behaviour of $n_p$ as $p \downarrow 2$ is an estimate.

### Remaining questions

Table 15 The questions of this chapter (questions of the systematic attempts 1–16)

| No. | Question | Section |
|---|---|---|
| 1 | Can the lower bound $\kappa_0$ of the row alignment in the worst direction be estimated on the limiting kernel of the steps, and the amount by which $\kappa$ moves under perturbation of the steps be bounded row by row, so as to say loss $\le \kappa_0^{-2}$ at every step? | § 02 |
| 2 | If the part by which the maximiser of Stein moves away from period 2 on larger tori (upper bound 10.2 against a loss of 6.4) is measured by splitting into Stein restricted to period-2 points and the other $q$, where does the difference come from? | § 02 |
| 3 | If one quantity measuring the cancellation on the side of the true operator is fixed (for example, the ratio of $\|T^k\|$ and $\||T|^k\|$ divided by its minimum over the sign-similarity class), what can the regularity of the loss growing only slowly over steps be said to come from? | § 01, § 02 |
| 4 | Aggregating the finite types and the remaining returns into a two-by-two matrix, does $bc < (1-a)(1-d)$ hold for the matrix of the record? If so, can it be raised to an interval estimate? | § 01 |
| 5 | The record for the W coordinates measures in $\ell^2$. Reread with Candidate 1.2′ (the weight of the Schur test), what does the one-step contraction constant become? | § 01 |
| 6 | The supremum of constrained sum rules: for any $k \le \operatorname{ord}^V(F)$, can a local representative with $\operatorname{ord}(K) \ge k$ be taken with finite range? Is there a general form for cases where $\operatorname{ord}^V$ rises with other steps or other blockings? | § 03 |
| 7 | A canonical choice of the degrees of freedom: does a representative minimising a Stein-type norm exist, is it unique, and is it local? | § 03 |
| 8 | Consistency of composition: does the output of the representative at step $k$ lie in the constraint subspace of step $k+1$ (the image of Bianchi on the fine side)? If not, how much does the freedom outside $\operatorname{im} D$ affect the chain? | § 03 |

| No. | Question | Section |
|---|---|---|
| 9 | At what rate does the kernel of the Moore–Penrose representative decay (of type $\lvert Z\rvert^{-4}$, from a discontinuous symbol homogeneous of degree 0)? | § 03 |
| 10 | Why does the zero mode of the symmetric representative on finite tori appear to approach $U/4$ faster than exponentially? | § 03 |
| 11 | In a triple valuation with a third valuation (the exponent of the transseries) added, does the distinction stand that the band of $U(1)$ is a tie between the weak-coupling and exponential valuations, while that of $SU(2)$ is a tie between the strong-coupling and weak-coupling valuations? What does the MK slope 13–14 for $U(1)$ tend to? | § 04 |
| 12 | If the error tie is found with the genuine four-dimensional series (higher strong-coupling terms and higher-loop weak-coupling coefficients), how does the lens overlap the band of 26-06? | § 04 |
| 13 | In the space of the two couplings, fundamental and adjoint, $(\beta_F, \beta_A)$, does the codimension-one skeleton of the tie sets become a non-trivial polyhedral complex? How does it line up with the end point of the line of first-order transitions? | § 04 |
| 14 | For the sequence of blocking maps, how does the lower end of the window $W_n = \{p : \mu_p(K^{(n)}) < 1\}$ move with $n$? Does the true $\mu_p(K_0)$ fall below 1 for one step at $3.5 \le p \le 5$ (decide whether $W_1$ is empty, either by tightening the upper bound by optimising the Stein weight $u$ beyond $A^p$, or by raising the lower bound with starting points that mix localised and spread-out fields or with the power method)? Answer (partial): if $W_1$ is non-empty, a single interval inside $(3.04, 5.67)$. The family of row weights saturates at $u = A^{p-2}$, and weighted Stein tightens no further (Lean: OneStepWindow). The one-step lower bound at $p = 4$ has risen to 0.6470, but the gap to 1 is undecided | § 05 |
| 15 | How does the bundle number $n_p$ diverge as $p \downarrow 2$? Is the limit of $\mu_p(K^{(n)})^{1/n}$ determined by the dimension count $2^{4-2p}$ of the constant field, giving $n_p \approx c/(p-2)$, or is the lower end $p_c > 2$ of $W_\infty$ determined by the geometry of the lattice? | § 05 |
| 16 | Does the gap between the Schur threshold exponent $p^*$ (about 66.5 for two steps) and the lower end of the Stein window (about 4 for two steps) come from the size of the column $\ell^1$ or from the cancellation of signs? How far can $p^*$ be lowered by a weighted Schur test (positive test functions)? | § 05, § 01 |

## Sources

| Item | Kind | Source |
|---|---|---|
| Perron–Frobenius, Collatz–Wielandt, comparison by domination $\rho(A) \le \rho(\lvert A\rvert) \le \rho(B)$ | Cited | The chapter on non-negative matrices in R. A. Horn, C. R. Johnson, Matrix Analysis (Reading) |
| Contraction in weighted maximum norms and iterative methods, M-matrices | Cited (Reading) | R. S. Varga, Matrix Iterative Analysis. A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences. D. P. Bertsekas, J. N. Tsitsiklis, Parallel and Distributed Computation (1989) |
| Max-plus eigenvalues and bounds | Cited | S. Friedland (1986). F. Baccelli, G. Cohen, G. J. Olsder, J.-P. Quadrat (1992). P. Butkovič (2010). M. Akian, S. Gaubert, A. Marchesini, arXiv:1309.7319 (Reading) |
| Sign-real spectral radius | Reading | S. M. Rump (1997) |
| Order of transfer operators | Reading | P. W. Hemker, On the order of prolongations and restrictions in multigrid procedures, J. Comput. Appl. Math. 32 (1990) 423–429. M. Donatelli, arXiv:0807.2565. U. Trottenberg, C. W. Oosterlee, A. Schüller, Multigrid (local Fourier analysis) |
| Sum rules, Strang–Fix conditions, order of approximation | Cited (Reading) | G. Strang, G. Fix (1973). R.-Q. Jia, Math. Comp. 67 (1998). M. Charina, C. Conti, arXiv:1202.3931. I. Daubechies, Ten Lectures on Wavelets |
| Fisher zeros and the strong-coupling basin, crossover | Reading (abstracts) | A. Denbleyker, D. Du, Y. Liu, Y. Meurice, H. Zou, arXiv:1005.1993. arXiv:1112.2734. arXiv:0807.0185 |

| Item | Kind | Source |
|---|---|---|
| Riesz–Thorin, Stein–Weiss, the Schur test, the difficulty of computing $p \to p$ norms, the $L^p$ theory of subdivision | Cited (Reading) | M. Riesz (1927), G. O. Thorin (1939). E. M. Stein, G. Weiss, Trans. AMS 87 (1958). I. Schur (1911). J. M. Hendrickx, A. Olshevsky, SIAM J. Matrix Anal. Appl. (2010). D. W. Boyd, Linear Algebra Appl. 9 (1974). R.-Q. Jia, Adv. Comput. Math. (1995). S. Boyd, L. Vandenberghe, Convex Optimization § 3.2.6 |
| The numbers of § 02– § 05 | Computed | The record of computations of this book's study (cross-check of the direct action of the rows of the composite kernel against the step-by-step action; two routes, quasi-Newton and period-2 fields; the single-plaquette series built in rational numbers; the $S_p$ of § 05 computed directly from the composite kernels) |
| The theorems of Table 14 | Lean | The Lean files DominantMatrixContraction, SignLossBound, ConstrainedExtension, ConeCalculus, OneStepWindow |

**Words used in this chapter** — dominant matrix, weighted block norm, $k$-step contraction, transient constant, one-step weight, the loss from discarding signs, row alignment, constrained map, extension, local representative, freedom, zero mode, moments, the order of folding $\mathbf{ord}$ and $\mathbf{ord}^V$, shifted jets, double valuation, complementary tie, same-coordinate tie, error tie, lens, $\kappa = s1w1$, cone constant $c_p$, $p$-contraction $\mu_p$, weighted Stein bound $S_p$, bundle number $n_p$, window $W_n$, Schur threshold exponent $p^*$.

Next chapter to read: 15-11 Entrance — one step and one scale.

# Entrance — one step and one scale

This chapter is a guide.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**Scope** — The theorems of the second half of this part (15-11 to 15-14) are either statements about the free field, the vacuum background and the linearised blocking, or statements about the Wilson measure at one scale (with the lattice spacing fixed). They say nothing about the renormalised coarse-scale measure, fields with a background, or estimates uniform across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

**What this chapter shows** — What the chapters of the second half of this part (15-12 to 15-14), and the rarity of large plaquettes that moved to Part 22 (22-01), deal with, and the scope that applies to the whole second half. The second half consists of parts, finished down to Lean, for one step of this book's design (a fixed block map that does not depend on a cutoff); they are placed in this part as part of the attempt at ultraviolet control.

**Chapters this one relies on** — 19-04 The block map and one step of renormalisation (definitions of the objects of the second half), 19-01 (the Wilson measure).

**Said first** — The scope is as written directly under the title. Every theorem of the second half is a formalisation, for the objects of this book, of a result of a known type, machine-checked in Lean. The chapters of the second half say nothing about the construction of the continuum limit or the mass gap (the statements to be shown are in 19-07).

Order of this chapter

1. The chapters of the second half — one line per chapter
2. The scope that applies to the whole second half — linear, vacuum, one scale
3. Reading order — going back and forth with the chapters on objects

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01

## The chapters of the second half

| Chapter | One line |
|---|---|
| 15-12 One step and many steps of the free field | The physical part $G_k$ of the symbol recursion obtained by pushing the free field $k$ times with the path kernel: the upper side, the lower side, the sandwich of the propagator, one-step contraction, contraction uniform in the step, and convergence of quadratic functionals of $\Gamma_k$. |
| 15-13 Linearisation of the block map and the covariance of the push-forward | Locality and gauge covariance of the non-linear block map, linearisation around the identity, the covariance $CTC^\dagger$ of the push-forward, and the plaquette covariance of the free field pushed by the linearised blocking. |
| 22-01 Rarity of large plaquettes — the chessboard estimate (Part 22) | For the Wilson measure at one scale, the probability of the event that all plaquettes are large is small at a rate independent of the volume ($\beta \geq 81$). |
| 15-14 What the second half (15-11 to 15-13) says, and what it does not | All the theorems of the second half in one table, with their scope, what they do not say, and their Lean declarations. |

02

## The scope that applies to the whole second half

- **Linear.** 15-12 and 15-13 § 05 are statements about objects in which the blocking is replaced by its first-order part (the path kernel $C$). They agree with the non-linear block map $B$ only up to the first-order part (15-13 § 03).
- **Vacuum background.** The linearisation is around the identity $U = 1$, and the field that is pushed is the Gaussian free field. Expansion around a background field is not treated.
- **One scale.** 22-01 is a statement about the Wilson measure on the fine lattice itself, and does not reach the blocked measure at a coarse scale.
- **Design choice.** The constants were computed with the weights $(1/8, 3/4, 1/8)$ and the family of paths with factor 2 (19-04; the discussion is in 28-05).

Estimates uniform across steps are not in the chapters of the second half; they are on the side of the premises (28-04).

03

## Reading order

15-12 and 15-13 assume the objects of 19-04 (the family of paths, the block map, the path kernel, the multi-step family). 15-13 § 05 uses the upper and lower sides of 15-12. 22-01 assumes only the Wilson measure of 19-01 and the reflection positivity of 20-01, and can be read independently of 15-12 and 15-13.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| The list of the theorems of the second half is in 15-14 | The measure at a coarse scale, fields with a background, estimates uniform across steps |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| The theorems and sources of each chapter | — | The table of sources and reproduction in each chapter |

**Terms used in this chapter** — path kernel, block map, symbol recursion, push-forward, plaquette covariance, chessboard estimate. Their meanings are in A-02 Glossary.

Next chapter to read: 15-12 One step and many steps of the free field — it begins with estimates, uniform in the step, of the propagator of the free field pushed by the path kernel.

# One step and many steps of the free field — weights (1/8, 3/4, 1/8), the vacuum background, linearisation

This chapter is this book's own investigation. The first half of § 01 is established knowledge.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**Scope** — The theorems of this chapter are statements about the free field, the vacuum background ($U = 1$) and the linearised blocking. Their object is a family of $4\times4$ matrices over momentum (the symbol recursion); they say nothing about the renormalised coarse-scale measure, fields with a background, or nonlinear blocking. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

**What this chapter shows** — We read six facts about the physical part $G_k(q)$ of the symbol recursion obtained by applying the block average of the path kernel $C$ $k$ times to the free field in Feynman gauge: the upper side $G_k \preccurlyeq P/\omega$ (with constant exactly 1), the lower side $\frac{3}{100}P/\omega \preccurlyeq G_k$, a sandwich of the propagator uniform in the step, the one-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$, a contraction uniform in the step with $\tau < 1/170$ over 17 steps, and convergence of quadratic functionals of $\Gamma_k$ at rate $(3/5)^k$. Every one of them is a statement machine-checked in Lean.

**Chapters assumed** — 19-04 The path kernel, 19-04 The multi-step family, The weights are a design choice, 7-05 § The free field.

**Said first** — The scope is as written directly under the title. There are four further points. (1) The momenta are the continuous momenta of infinite volume $q \in \mathbb{R}^4$ ($\omega(q) > 0$); the momenta of a finite torus are a subset of these. (2) That $G_m$ is the physical part of the symbol of the stage covariance in position space (the operator obtained by

pushing the free field $m$ times) is in Lean, but the inequalities of § 02– § 07 are inequalities between symbols (matrices for each momentum), and are not stated as inequalities between operators in position space. (3) The constants are 3 to 186 times looser than the true values Reading , and the constant in the convergence of $\Gamma_k$ is $\kappa^2 \approx 1.4 \cdot 10^{13}$ Reading . (4) We do not say that “the one-loop coupling converges”. What is shown goes only as far as the convergence of quadratic functionals of $\Gamma_k$.

The order of this chapter

1. The object — the symbol recursion and its physical part $G_k$ — the symbol of the path kernel, aliases, $\mathcal{M}$, $G_k = \mathcal{M}^k(P/\omega)$
2. The upper side — $G_k \preccurlyeq P/\omega$ — the constant is exactly 1, from a one-step inequality and preservation of order
3. Detour: the lemma $E_\nu < 4$ inside the upper side — an upper bound for an explicit function, and invariance under reflection
4. The lower side — $\frac{3}{100} P/\omega \preccurlyeq G_k$ — the chain of nearest aliases, and a case split with the sixteen aliases
5. A sandwich of the propagator uniform in the step — the inverse $H_k$ on $\operatorname{ran} P$, and $(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$
6. One-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$ — the true supremum is $125/256$ Reading
7. Contraction uniform in the step — $\tau < 1/170$ over 17 steps — splitting into the nearest-alias term and the rest
8. Convergence of quadratic functionals of $\Gamma_k$ — the form of the fluctuation on the sum of aliases, rate $(3/5)^k$
9. What is not said — identification with the stage action of the lattice, position space, looseness of the constants

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01

## The object — the symbol recursion and its physical part $G_k$

The introduction is 7-05 § The free field. When the free field is pushed by the linear map of the blocking, the pushed field is again Gaussian, and its covariance is $CTC^\dagger$ Cited . A translation-invariant operator can be written as a matrix for each momentum (its symbol), so stacking steps becomes a recursion of matrices over momenta. In renormalisation group analysis, estimates of the stage propagators of this free field are one of the starting points (Bałaban's series of papers, 7-07) Cited .

The object of this chapter is the one set up in 19-04 § The multi-step family on the free-field side. What 19-04 set up with torus momenta is considered here for all $q \in \mathbb{R}^4$ with $\omega > 0$. We repeat the notation. For a momentum $q \in \mathbb{R}^4$,

$$\omega(q) = \sum_\nu 4 \sin^2 \frac{q_\nu}{2}, \qquad s_\nu(q) = \sin \frac{q_\nu}{2}, \qquad P(q) = 1 - \frac{s(q) s(q)^{\mathsf{T}}}{|s(q)|^2}. \tag{1}$$

$P(q)$ is a projection of rank 3, the projection onto the "physical part" with the gauge direction removed. The fine momenta that fall onto a coarse momentum $q$ (the aliases) are the 16 momenta $r_\zeta(q) = q/2 + \pi\zeta$, $\zeta \in \{0,1\}^4$. The one-step map is

$$(\mathcal{M}G)(q) = \frac{1}{16} \sum_{\zeta \in \{0,1\}^4} A_\zeta(q)\, G\big(r_\zeta(q)\big)\, A_\zeta(q)^{\mathsf{T}}, \qquad A_\zeta = 2P(q)\, S_\zeta C(r_\zeta)\, P(r_\zeta)\, N(r_\zeta), \tag{2}$$

where $S_\zeta$ is a diagonal of signs, $C(r)$ is a diagonal of $\cos$, and $N$ is an explicit $4\times4$ matrix coming from the average over the family of paths (the matrix that appears in the symbol of the coefficients of 19-04 (9)). The family at step $k$ is

$$G_0 = P/\omega, \qquad G_k = \mathcal{M}^k(P/\omega) \qquad (\omega(q) > 0). \tag{3}$$

We say precisely what $G_k$ is. Let $Q(q)$ be the symbol of the path kernel, and write $\mathcal{R}$ for the recursion from the symbol $X$ at the fine step to the symbol at the coarse step. Starting from $X_0 = I/\omega$ (the symbol of the free propagator in Feynman gauge), $\mathcal{R}^k X_0$ is determined, and its physical part is $G_k$.

$$P\,(\mathcal{R}^k X_0)\,P = \mathcal{M}^k(P\,X_0\,P) = G_k. \tag{4}$$

Lean: (4) is `LatticeRec.rec_iterate` (for any $X_0$) and `LatticeRec.projFam_X0`; the closed form of the symbol of the average over the family of paths is `LatticePath.pathSym_closed`; that the symbol of $CTC^\dagger$ is $\mathcal{R}X$ when the symbol of the one-step torus operator $T$ is $X$ is `LatticeTorus.torus_step` ($(\mathbb{Z}/2L)^4 \to (\mathbb{Z}/L)^4$). That the symbol of the lattice stage covariance obtained by pushing the free-field covariance $m$ times is $\mathcal{R}^m X_0$, and that its physical part is $G_m$, are `LatticePush.push_phase`, `LatticePush.stageCov_phase` and `LatticePush.projFam_stage`.

In this chapter we call $G_k$ "**the physical part of the symbol recursion**". Every statement is a Loewner inequality for a concrete family of $4\times4$ matrices, for all $q \in \mathbb{R}^4$ with $\omega(q) > 0$ ($A \preccurlyeq B$ means $\langle x, Ax\rangle \le \langle x, Bx\rangle$ for all $x$). None is a statement about abstract operators. Lean holds the momentum as $\theta = q/4$; the domain is `OneStepFinal.Qpos`, $\mathcal{M}$ is `OneStepBridge.Mfam`, and the alias map is `OneStepFinal.rqPos`. The weights $(1/8, 3/4, 1/8)$ enter through $N$ and $A_\zeta$, and every

constant that appears here was computed with this choice (19-04 § The weights; the discussion is in 28-05).

02

## The upper side — $G_k \preccurlyeq P/\omega$

**Upper-side theorem** Lean: `OSUpQpos.os_up_Qpos`, `OSUpQpos.Gs_le_G0` (no assumption fields). For all $q$ with $\omega(q) > 0$,

$$\mathcal{M}(P/\omega) \preccurlyeq P/\omega, \qquad \text{and hence for every step } k \quad G_k \preccurlyeq P/\omega. \tag{5}$$

Meaning. However many steps are stacked, the physical part of the pushed propagator does not exceed the starting $P/\omega$. The constant is exactly 1. The true supremum is $1 - 3.4 \cdot 10^{-6}$ Reading (the largest eigenvalue of $\omega G_1$ at $k = 1$, floating point at $2.4 \cdot 10^4$ points), so the constant 1 is sharp.

The route of the proof has three stages.

1. **Reduce the one-step inequality to an upper bound for an explicit function.** Opening $\langle x, \mathcal{M}(P/\omega)x\rangle$ into the sum over the 16 aliases, $\mathcal{M}(P/\omega) \preccurlyeq P/\omega$ follows from the inequality for explicit functions "$E_\nu(\sigma) \le 4$ for all $\nu$". Here $E_\nu$ is an explicit rational expression in $\sigma_\mu = \sin^2(q_\mu/4)$ (a variable different from the $s$ of (1)) (`OSUpBridge.os_up`; the definition is in the detour).
2. **$E_\nu \le 4$ at every point.** On $[0, \frac{1}{2}]^4$, $E_\nu < 4$ ($\sigma \neq 0$) is a lemma. A general point is moved into $[0, \frac{1}{2}]^4$ by the coordinate-wise reflection $\sigma_\mu \mapsto 1 - \sigma_\mu$, using that $E_\nu$ does not change under this reflection (`OSUpQpos.hEE_all`).
3. **Stack the steps.** $\mathcal{M}$ is a linear map that preserves order. Applying $\mathcal{M}^k$ to $\mathcal{M}G_0 \preccurlyeq G_0$ gives $G_{k+1} \preccurlyeq G_k$, and by transitivity $G_k \preccurlyeq G_0$ Proved here.

The sequence of steps decreases monotonically: $G_0 \succcurlyeq G_1 \succcurlyeq G_2 \succcurlyeq \cdots$. This is the input to § 06 and § 07.

### Detour: the lemma $E_\nu < 4$ inside the upper side

You may skip this. For $\sigma \in [0,1]^4$ ($\sigma_\mu = \sin^2(q_\mu/4)$) and $e \in \{0,1\}^4$, set $b_{e,\mu} = 1 - \sigma_\mu$ ($e_\mu = 1$) or $\sigma_\mu$ ($e_\mu = 0$), $H = 4\sum_\mu \sigma_\mu(1-\sigma_\mu)$, $X_e = \sum_\mu b_{e,\mu}$, and

$$E_\nu(\sigma) = \sum_{e \in \{0,1\}^4} \frac{H}{X_e}\, \kappa_e^+ \,(1 - b_{e,\nu})$$

Here $\kappa_e^+$ is the square of the larger of two values of explicit rational expressions in $X_e$, $\sum b^2$ and $\max b$. Lean: `FreeStageOS.E_lt_four_s` ($E_\nu(\sigma) < 4$ on $\sigma \in [0, \frac{1}{2}]^4 \setminus \{0\}$), the momentum form `FreeStageOS.E_lt_four`, and the inequality for rational expressions in two variables `FreeStageOS.eq_lt_four`. The bound 4 is sharp: as $\sigma \to 0$, $E_\nu \to 4$ (for example, $3.999995$ at $\sigma = (10^{-6}, 0, 0, 0)$ Reading ). The invariance under reflection comes from $e \mapsto e \oplus f$ being a bijection of $\{0,1\}^4$, and from each term $b(1-b)$ of $H$ not changing under $b \mapsto 1-b$ (`OSUpQpos.EE_bbflip`). This lemma is an upper bound for a scalar function, not a theorem about the lattice objects themselves. It is placed here as a part of the upper-side theorem.

03

## The lower side — $\frac{3}{100} P/\omega \preccurlyeq G_k$

**Lower-side theorem** Lean: `HsSplit.lower_side_c`. For any family closed under aliases ($\omega > 0$), and in particular for all $q$ with $\omega(q) > 0$, at every step $k$,

$$\frac{3}{100}\,\frac{P}{\omega} \preccurlyeq G_k. \quad (6)$$

The assumption fields in Lean are only two properties of the family (that the alias map is $r\zeta$, and that $\omega > 0$); on the whole of momentum space (`Qpos`, `rqPos`) both are discharged by definition. The earlier version `OneStepLower.lower_side` has the same form with constant $(4/\pi^2)(0.04452)^2 \approx 8.03 \cdot 10^{-4}$ Reading .

Meaning. However many steps are stacked, the physical part does not collapse. Together with the upper side, the eigenvalues of $\omega G_k$ on $\operatorname{ran} P$ lie in $[3/100,\ 1]$ at every step. The true infimum goes down with the step: for $k = 0, 1, \dots, 5$ it is $1,\ 0.210,\ 0.132,\ 0.110,\ 0.0997,\ 0.0951$ Reading , and the estimate for the limit is about $0.091$ Reading . At $k = 4, 5$, the minimum is attained at the corner $q = \pi(1,1,1,1)$ Reading . The constant $3/100$ is about $1/3$ of the true value.

The route of the proof.

1. **The chain of nearest aliases.** Keeping only the alias $\zeta^*(q)$ nearest to the origin among the 16, $\mathcal{M}$ is bounded from below by a single term. `NearAliasB.chain_lower` gives the explicit lower bound obtained by following this for $d$ steps.
2. **The sixteen aliases.** `NearAliasB.near_lower` gives a lower bound in one step keeping all sixteen aliases.
3. **Case split.** At every point with $\omega > 0$, either the chain lower bound (2 steps) or the sixteen-alias lower bound (5 steps) is at least $3/100$ (`HsSplit.hS_split`, no assumptions). This is checked with a certificate for polynomial inequalities that

divides the box of variables into cells Computer-assisted . The certificate is checked by the Lean kernel and supports an analytic proof.

4. **Pass to the steps.** Applying the case-split lower bound step by step gives (6) (`NearAliasB.lower_side_split`).

04

## A sandwich of the propagator uniform in the step

$G_k(q)$ is positive definite on $\operatorname{ran} P(q)$, so it has an inverse there. We call that inverse $H_k(q)$ ($P H_k = H_k$, $G_k H_k = P$) the inverse of the propagator, and $\frac{1}{4} H_k$ its quarter. $H_k$ is uniquely determined by $G_k$ Proved here .

**Propagator sandwich theorem** Lean: `LatticeRec.lattice_bracket` (in the form of the physical part of the symbol recursion $\mathcal{R}^k(I/\omega)$), `OneStepLower.bracket_Qpos_free` (in the form $\mathcal{M}^k(P/\omega)$). No assumption fields. At every step $k$ and for all $q$ with $\omega(q) > 0$,

$$\frac{\omega(q)}{745}\, P(q) \;\preccurlyeq\; \tfrac{1}{4} H_k(q) \;\preccurlyeq\; 312\, \omega(q)\, P(q). \quad (7)$$

In the version with an improved upper constant, `NearAliasB.bracket_Qpos_split`, the right-hand side is $\omega/(4 \cdot \frac{3}{100})\, P = \frac{25}{3}\, \omega P$ (from the lower side of § 03).

Meaning. The inverse of the propagator at step $k$ is sandwiched around $\omega P$ by two constants that do not depend on the step. $\omega P$ is the symbol of the physical part of the action of the free field at the fine step. In other words, however many steps are stacked, the quadratic form of the pushed free field stays of the same order of size as the form of the original free field (reading $\frac{1}{4} H_k$ as the action of the pushed field is a reading, and that identification is on paper; § 08). The constants are uniform in $k$ and $q$. The volume does not appear in the statement.

The route of the proof. The right side of (7) is the inverse of the lower side $cP/\omega \preccurlyeq G_k$ (on $\operatorname{ran} P$, taking inverses reverses the Loewner order) Proved here . If $c \geq 1/1248$ the constant is $312$; if $c = 3/100$ it is $25/3$ (the lower-side constant of the earlier version is about $8.03 \cdot 10^{-4} \geq 1/1248$). The $745$ on the left is the inverse of $G_k \preccurlyeq 186.25\, P/\omega$, a weaker form than the upper side of § 02 (constant 1). Taking the inverse of the upper side would make the left constant $1/4$ Proved here , but there is no Lean declaration in that form.

Numerically, the eigenvalues of $\omega G_k$ lie roughly in $[0.091,\ 1]$ Reading , and the constants of (6), (7) and the split version are from 3 times ($3/100$) to 186 times (the $745$ on the left)

looser than the true values Reading .

05

## One-step contraction $\mathcal{M}P \preccurlyeq \frac{3}{5}P$

**One-step contraction theorem** Lean: `OneStepBridge.Mfam_P_le_row`. For any family closed under aliases (on the whole of momentum space the assumptions are discharged by definition),

$$\mathcal{M}P \preccurlyeq \tfrac{3}{5}\,P. \quad (8)$$

Meaning. Pushing the family of projections $P$ itself (the family not multiplied by $1/\omega$) by one step shrinks it by a factor of at most $3/5$. Whereas $P/\omega$ is large at low momenta and is preserved by $\mathcal{M}$ (§ 02), $P$ carries no weight $\omega$, and so it shrinks under the average over aliases. In § 06 this contraction plays the role of making "the contribution of high-momentum aliases" smaller step by step.

The true supremum is $125/256 \approx 0.488$ Reading (as $q \to 0$; $2 \cdot 10^5$ points and local optimisation), and $3/5$ is 1.23 times that. The earlier version `OneStepBridge.Mfam_P_le` had $429/512$. The proof bounds $\langle x, \mathcal{M}Px\rangle$ by splitting it row by row.

Differences between steps also shrink at the same rate. Applying $\mathcal{M}^k$ to $-2P \preccurlyeq \mathcal{M}(P/\omega) - P/\omega \preccurlyeq 2P$ (`CouplingLimit.D_lower`, `CouplingLimit.D_upper`) puts $G_{k+1} - G_k$ between $\pm 2(3/5)^k P$ Proved here . Numerically the values of $\mathcal{M}(P/\omega) - P/\omega$ lie in $[-0.089, -0.024]P$ Reading , so $\pm 2$ is about 22 times looser.

06

## Contraction uniform in the step — $\tau < 1/170$ over 17 steps

**Theorem of contraction uniform in the step** Lean: `TorusLam.stage_tau_17` (no assumption fields). Set

$$\tau = \frac{\pi^2}{4}\left(\frac{1}{4}\right)^{17} + \frac{100}{3}\left(\frac{3}{5}\right)^{17} < \frac{1}{170} \quad (9)$$

Then for every step $k$, all $q$ with $\omega(q) > 0$, and all $x$,

$$\big\langle x,\ \mathcal{M}^{17}(\omega\, G_k)(q)\, x\big\rangle \ \le\ \tau\,\omega(q)\,\big\langle x,\ G_{k+17}(q)\, x\big\rangle. \quad (10)$$

Here $\omega G_k$ is the family $q \mapsto \omega(q)G_k(q)$. The form (9) is exactly the definition of `StageArith.tauM` in Lean, and the comparison with $1/170$ is also in Lean.

Meaning. Since $G_{k+17} = \mathcal{M}^{17} G_k$, (10) says that "multiplying by $\omega$ on the fine side and then pushing 17 steps" is at most $1/170$ times "pushing 17 steps and then multiplying by the coarse $\omega$". At low momenta, the $\omega$ of a fine momentum is about $4^{-17}$ times the $\omega$ of the coarse momentum. The contribution from high-momentum aliases is multiplied by $(3/5)^{17}$ through the contraction of § 05. The two terms of (9) correspond to these two contributions.

The route of the proof.

1. **Split.** $\mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$, where $\mathcal{M}_0$ is the single term of the nearest alias $\zeta^*(q)$ and $\mathcal{M}_1$ is the remaining fifteen terms. Both preserve order. If $0 \preccurlyeq F \preccurlyeq P'$ then $\mathcal{M}^m F \preccurlyeq \mathcal{M}_0^m F + \mathcal{M}^m P'$ (induction, `TauSplit.iterate_split`).
2. **Insert.** $F = \omega G_k$, $P' = P$. $0 \preccurlyeq F$ is $G_k \succcurlyeq 0$, and $F \preccurlyeq P$ is the upper side of § 02.
3. **The nearest-alias term.** $\mathcal{M}_0$ carries the weight along a chain determined by the point, so $\omega$ decreases along the chain as $\omega(\text{the } m\text{-th step of the chain}) \le (\pi^2/4) 4^{-m}\, \omega(q)$ (the finite Viète product $\prod_{j<m} \cos(\pi/2^{j+2}) \ge 2/\pi$, `TauSplit.omF_chain_le`).
4. **The remaining terms.** $\mathcal{M}^m P \preccurlyeq (3/5)^m P$ (iterating § 05), which is returned to $G_{k+m}$ by the lower side of § 03, $P \preccurlyeq (100/3)\, \omega G_{k+m}$.

The value of the one-step contraction at step 0 (in the form $\tau \le 31/32$) is not written as a theorem in this chapter. What Lean has goes only as far as the scalar parts, and the stage of assembling the actual alias contributions into that form is on paper. The theorem about concrete objects for contraction uniform in the step is (10).

07

## Convergence of quadratic functionals of $\Gamma_k$

One step of renormalisation needs the covariance of the fluctuation of the fine field with the coarse field held fixed Cited . Here, as a candidate for its symbol form, we build a matrix $\Gamma_k$ on the space that views the 16 aliases together. Identifying $\Gamma_k$ with the covariance of the lattice fluctuation is on paper.

**Setting.** $\mathcal{B} = \bigoplus_{\zeta \in \{0,1\}^4} \mathbb{R}^4$ (the sum of aliases, 64-dimensional). The covariance on the sum of aliases at step $k$ is $\mathbf{A}_k(q) = \bigoplus_\zeta G_k(r_\zeta(q))$ (block diagonal). The constraint map is $\mathbf{C}(q)u = \frac{1}{4}\sum_\zeta A_\zeta(q) u_\zeta$ ($\mathcal{B} \to \mathbb{R}^4$). A lift $E$ ($\mathbb{R}^4 \to \mathcal{B}$) is required, with $\Pi = E\mathbf{C}$, to satisfy $\mathbf{C}\Pi = \mathbf{C}$ and $\Pi \mathbf{A}_k = \mathbf{A}_k \Pi^\dagger$. Then

$$\Gamma_k(q) = \mathbf{A}_k(q) - \Pi\, \mathbf{A}_k(q)\, \Pi^\dagger. \quad (11)$$

$\Gamma_k$ does not depend on the choice of a lift satisfying the conditions. For two choices $\Pi$, $\Pi'$ one gets $\Pi^2 = \Pi$, $\Pi\Pi' = \Pi$, $\Pi'\Pi = \Pi'$, and so $\Pi\mathbf{A}\Pi^\dagger = \Pi\mathbf{A} = \Pi'\mathbf{A} = \Pi'\mathbf{A}\Pi'^\dagger$ Proved here .

**Convergence theorem for $\Gamma_k$** Lean: `AliasPoinc.coupling_converges_gamma_closed` (no assumption fields; parts `AliasLiftSize.hlift_holds`, `AliasPoinc.hH_holds`). Let $\Xi$ be a weighted sum of finitely many values of quadratic forms and products of two values of quadratic forms:

$$\Xi(\Gamma) = a + \sum_i w_i \langle x_i, \Gamma(p_i) x_i\rangle + \sum_j w'_j \langle v_j, \Gamma(q_j) u_j\rangle \langle u'_j, \Gamma(q'_j) v'_j\rangle. \quad (12)$$

The weights, vectors and momenta are arbitrary. Then at every step $k$ a lift satisfying the conditions exists, and there is a limit $b_\infty$ with

$$|\Xi(\Gamma_k) - b_\infty| \le \frac{2L'}{2/5}\Big(\frac{3}{5}\Big)^k, \qquad \sum_{k<n} \max\big(b_\infty - \Xi(\Gamma_k), 0\big) \le \frac{2L'}{(2/5)^2}. \quad (13)$$

Here $L' = \kappa^2\big(\sum_i |w_i|\,|x_i|^2 + \gamma \sum_j |w'_j|(\cdots)/2\big)$, $\kappa = 1 + 2\sqrt{4 + 16\cdot 464880^2}$, $\gamma = (1 + 69120/121)/(8/745)$, and none of them depends on the step or the momentum.

Meaning and strength. The rate $(3/5)^k$ comes from the contraction of § 05 (the numerically observed ratio of successive step differences of $G_k$ is 0.40–0.42 Reading , inside it). The constants are uniform, but $\kappa^2 \approx 1.4\cdot 10^{13}$ Reading and $\gamma \approx 5.3\cdot 10^4$ Reading are extremely far from the true values.

A note on reading. The stage coefficient $b_k$ of the one-loop coupling is the second-order coefficient with respect to the background, projected onto the type of the local action. That it can be written as $\Xi(\Gamma_k)$ of the form (12) is on paper, and is not in Lean. So this theorem is not a proof that "the one-loop coupling $b_k$ converges"; it is that "any finite quadratic functional of $\Gamma_k$ converges geometrically". When $\Xi$ is taken in the form of the one-loop vertex, its limit is expected to be the universal one-loop value (Cited , 7-07), but that identification is on paper, and Lean does not state the value of the limit.

## 08 What is not said

The theorems of § 02– § 07 are all statements about the physical part $G_k$ of the symbol recursion and about $\Gamma_k$ built from it. The following are not said.

- **Identification with the stage action of the lattice.** Reading $\frac{1}{4}H_k$ as "the action $h_k$ at step $k$" (the minimum of the action of the fine free field with the coarse field held

fixed, a Schur complement) is on paper. The Lean objects go only as far as the inverse of $G_k$ on $\operatorname{ran} P$.

- **Operators in position space.** Three things are connected to position space in Lean. One step: if the symbol of a torus operator $T$ is $X$, the symbol of $CTC^\dagger$ is $\mathcal{R}X$ (`LatticeTorus.torus_step`). The chain of steps and step 0: the operator on $(\mathbb{Z}/L)^4$ obtained by pushing $m$ times the free-field covariance on $(\mathbb{Z}/2^m L)^4$ (with the constant component removed) has, at torus momenta, the symbol with phases $\mathcal{R}^m X_0$ (`LatticePush.push_phase`, `LatticePush.freeCov_phase`, `LatticePush.stageCov_phase`), and its physical part is $G_m$ (`LatticePush.projFam_stage`). The covariance of the plaquette field (`LatticeFace.faceCov_eq_posC`, 15-13). There is no theorem stating the inequalities of § 02– § 07 themselves as inequalities between operators in position space.
- **Finite volume.** The momenta are the continuous momenta of infinite volume $q \in \mathbb{R}^4$ ($\omega > 0$); the momenta of a finite torus $q = 2\pi n/N$ ($n \not\equiv 0 \bmod N$) are a subset of these. The zero momentum is not treated.
- **Complex form.** The statements are in the real form with the phases removed. The identity of the symbols with phases is in Lean (`LatticePush.stageCov_phase`), but there is no theorem stating the inequalities of § 02– § 06 in complex form.
- **Constants.** They are 3 to 186 times looser than the true values Reading , and the constants for $\Gamma_k$ are of order $10^{13}$ Reading .
- **Non-linear, background, measure at a coarse scale.** Only the free field pushed by the linearised blocking (the path kernel $C$) is treated. This chapter says nothing about the non-linear block map $B$, background fields $U \neq 1$, or the measure at a coarse scale obtained by pushing forward the Wilson measure.
- **Choice of weights.** The constants were computed with the weights $(1/8, 3/4, 1/8)$ and factor 2, and cannot be used as they are for another choice (28-05).

Each theorem is a formalisation, for these weights, of a result of a type known in renormalisation group analysis: estimates, uniform in the step, of the propagator when the free field is blocked Cited . Among the estimates across steps, those needed for non-linear blocking and for fields with a background are on the side of the premises (28-04).

## What this chapter can say

| Can say | Cannot say |
|---|---|
| Upper side: for all $q$ with $\omega > 0$ and at every step, $G_k \preccurlyeq P/\omega$ (constant exactly 1). Lean: `OSUpQpos.os_up_Qpos`, `OSUpQpos.Gs_le_G0` | Non-linear blocking, background fields. The form as operators in position space |
| Lower side: at every step, $\frac{3}{100} P/\omega \preccurlyeq G_k$. Lean: `HsSplit.lower_side_c` | A constant close to the true infimum (about 0.091 Reading ) |
| Sandwich: at every step and for all $q$, $(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$ (in the split version the upper side is $\frac{25}{3}\omega P$). Lean: `LatticeRec.lattice_bracket`, `OneStepLower.bracket_Qpos_free`, `NearAliasB.bracket_Qpos_split` | Reading $\frac{1}{4} H_k$ as the stage action $h_k$ (on paper). The complex form of the inequalities. Inequalities between operators in position space |
| One-step contraction $\mathcal{M} P \preccurlyeq \frac{3}{5} P$. Lean: `OneStepBridge.Mfam_P_le_row` | A Lean form of the true supremum $125/256$ Reading |
| Contraction uniform in the step: $\tau < 1/170$ over 17 steps. Lean: `TorusLam.stage_tau_17` | $\tau \le 31/32$ at step 0 (the assembly is on paper). Identification with the per-step contraction rate used in the argument on paper (not in Lean) |
| Any finite quadratic functional of $\Gamma_k$ converges at rate $(3/5)^k$, with constants uniform in the step and the momentum. Lean: `AliasPoinc.coupling_converges_gamma_closed` | Convergence of the one-loop coupling $b_k$ (the identification $b_k = \Xi(\Gamma_k)$ is on paper). The size of the constants ($\kappa^2 \approx 10^{13}$ Reading ) |
| Proved here $\Gamma_k$ does not depend on the choice of a lift satisfying the conditions (the three lines of algebra in § 07) | — |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| That the push-forward of the free field is Gaussian with covariance $CTC^\dagger$, and that estimates of the stage propagators are a starting point of the renormalisation group | Cited | The sources of 7-05 and 7-07 (T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275. J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335) |
| The universal one-loop value (what § 07 compares with) | Cited | The sources of 7-07 ($b_0$, Particle Data Group) |
| The theorems of § 02– § 07 | — | Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, checked by a separate re-run). The list is in 15-14 |

| Item | Kind | Source / tools |
|---|---|---|
| The case split of the lower side (§ 03) | Computer-assisted | A certificate for polynomial inequalities dividing the box of momenta into cells, checked by the Lean kernel (`HsSplit.hS_split`). Supports an analytic proof |
| Ranges of eigenvalues ($1 - 3.4 \cdot 10^{-6}$, 0.210–0.0951, about 0.091), $125/256$, $[-0.089, -0.024]$, the ratio of step differences 0.40–0.42, the values of $E_\nu$, the values of $\kappa^2$ and $\gamma$ | Reading | The numerical record of this book's study (the recursion transcribed directly from the Lean definitions, in floating point, at 292 points: random points, points near the origin, edges, and lattice momenta with $N = 2, 4$; $\mathcal{M}(P/\omega)$ at $2.4 \cdot 10^4$ points, $\mathcal{M}P$ at $2 \cdot 10^5$ points) |

**Terms used in this chapter** — symbol recursion, physical part, alias, Loewner order, inverse of the propagator, one-step contraction, contraction uniform in the step, sum of aliases, lift, Γ_k. Their meanings are in A-02 Glossary.

Next chapter to read: 15-13 Linearisation of the block map and the covariance of the push-forward — the position-space side: that the first-order part of the non-linear block map is the path kernel, the covariance of the push-forward, and the covariance of the plaquette field.

2026-11-13 · chapter Yang–Mills and the mass gap Part 15 Record of the attempts at ultraviolet control Linearisation and push-forward · Kiichi, Shiori and Rin

# Linearisation of the block map and the covariance of the push-forward

This chapter is this book's own investigation. The first half of § 01 is established knowledge.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**Scope** — The theorems of this chapter are statements about local properties of the nonlinear block map at one scale (locality and gauge covariance), its linearisation around the identity, the covariance identity for the push-forward of an arbitrary measure, and the covariance of the plaquette field obtained by pushing the free field forward through the linearised blocking. They say nothing about the renormalised coarse-scale measure, fields with a background, or nonlinear estimates across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

**What this chapter shows** — We read four facts about the nonlinear block map $B$ defined in 19-04. (1) The value of a coarse link is determined by the fine links in a cube of radius 2 alone, and is compatible with gauge transformations. (2) When $U = e^{\varepsilon a}$, $B(U)$ lies within $257(\varepsilon M)^2$ of $\exp(\varepsilon C a)$, and its first-order part is the path kernel $C$. (3) The covariance of a measure pushed forward by a linear map is $CTC^{\dagger}$ (for any measure). (4) The covariance of the field strength of the plaquettes of the free field pushed forward $m$ times by the path kernel has largest eigenvalue $\leq 1$, diagonal $\geq \frac{3}{200}(1 - N^{-4})$ and ratio $\Lambda \leq 640/9$, for every step $m$ and every $N \geq 2$.

**Chapters this one relies on** — 19-04 the family of paths, the block map, the path kernel, the push-forward, the multi-step family; 15-12 (the lower side and the upper side).

**Said first** — The scope is as written directly under the title. In particular, the plaquette covariance of § 05 is the covariance of the plaquette field strength of **the free field pushed by the linearised blocking (the path kernel $C$)**. It is not the covariance of the field pushed by the non-linear block map $B$. The two agree only up to the first-order part of $B$ (§ 03); higher-order terms and large fields are not treated. The linearisation is only around the identity $U = 1$.

The order of this chapter

1. The position-space side — the symbol recursion of 15-12, and the position-space objects of this chapter
2. Locality and gauge covariance — a cube of radius 2, $g(2y)\, B(U)_b\, g(2(y + e_\mu))^*$
3. Linearisation around the identity — an error of $257(\varepsilon M)^2$, and the first-order part being the path kernel
4. The covariance of the push-forward $= CTC^\dagger$ — an identity for any measure
5. The covariance of the plaquette field strength — self-adjoint, $\lambda_{\max} \le 1$, a lower bound on the diagonal, $\Lambda \le 640/9$
6. The route of the proof for the plaquette covariance — from position space to symbols, and from symbols to the two sides of 15-12
7. Detour: the $\frac{1}{2}$ and the $1 - N^{-4}$ in the lower bound on the diagonal — symmetry of directions and the zero momentum
8. What is not said — the non-linear covariance, backgrounds, large fields, the measure at a coarse scale

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01

## The position-space side

The introduction is 7-05 § One step. Blocking of a gauge field builds a coarse link by averaging products of fine links along paths Cited . Products along paths are used to keep the gauge structure, and the amount by which the average leaves the group is brought back into the group by some rule. Three properties are commonly asked of a good blocking: that it is local, that it is compatible with gauge transformations, and that near the identity it is close to a linear average Cited .

15-12 dealt with objects on the momentum side (the symbol recursion). This chapter is on the position-space side. The objects are the non-linear block map of 19-04,

$$B(U)_b = \rho_{r_0}\big(q_b(U)\big), \qquad q_b(U) = \sum_{o,A} w(o,A)\, U\big(\gamma_{o,A}(b)\big), \qquad \rho_{r_0}(q) = \frac{q}{\max(|q|, r_0)} \tag{1}$$

its first-order part, the path kernel $C$, and the push-forwards $B_*\mu$ and $C_*\mu$. The fine lattice is $\Lambda_{m+1} = (\mathbb{Z}/2^{m+1}L)^4$, the coarse lattice is $\Lambda_m = (\mathbb{Z}/2^m L)^4$, and a coarse link $b = (y, \mu)$ starts from the fine point $2y$. Configurations take values in the closed unit ball $\mathbb{B}$ of the quaternions.

§ 02 and § 03 are statements about the non-linear $B$ itself, § 04 is a general identity for push-forward by a linear map, and § 05 is a statement about the free field pushed by the linear $C$. All four concern one step with the lattice spacing fixed (or a stack of linear steps).

02

## Locality and gauge covariance

**Locality theorem** Lean: `BoxWalk.boxMap_local` (the only assumption is $r_0 > 0$). If two configurations $U, U'$ agree on all links starting from the fine points $2y + z$ ($z \in \mathbb{Z}^4$, each coordinate $-2 \le z_i \le 2$), then $B(U)_b = B(U')_b$.

**Gauge covariance theorem** Lean: `BoxWalk.boxMap_gauge` (the assumptions are $r_0 > 0$ and $|g(x)| = 1$). For a gauge transformation $(gU)_{(x,\nu)} = g(x)\, U_{(x,\nu)}\, g(x + e_\nu)^*$ by unit quaternions $g$ at each point of the fine lattice,

$$B(gU)_{(y,\mu)} = g(2y)\; B(U)_{(y,\mu)}\; g\big(2(y + e_\mu)\big)^*. \tag{2}$$

The map in these two theorems has, word for word, the same definition as `ChainBlockMap.blockMap` defined in 19-04 (the family of box paths and the weights). They are separately checked declarations, and the agreement rests on a textual comparison.

Meaning. The value of a coarse link is determined only by the links starting from the $5^4$ fine points around the coarse point. Radius 2 is the reach of the paths: a path moves sideways by at most 1, advances 2 in $\mu$, and comes back. On the coarse lattice, the gauge transformation acts through the restriction of the fine-lattice $g$ to the coarse points $2y$.

The route of the proof. Locality comes from each term of the average depending only on the link values along a path, and from the paths staying inside the cube. Gauge covariance comes from two facts. One is that a product along a path picks up only the $g$ at its two ends under a gauge transformation (the $g$ inside the path cancel by $g^* g =$

1). The other is that the retract $\rho_{r0}$ commutes with left and right multiplication by unit quaternions ($|gqh^*| = |q|$) Proved here .

A caution. When the torus is small ($2^{m+1}L \le 4$) and the cube wraps around the torus, the locality assumption becomes stronger and the statement becomes weaker. It does not become false.

03

## Linearisation around the identity

Take a field $a$ of imaginary quaternions on the fine links ($\operatorname{Re} a_\ell = 0$, $|a_\ell| \le M$) and $\varepsilon \ge 0$, and set $U_\ell = \exp(\varepsilon a_\ell) \in SU(2)$. The path kernel $C$ is the weighted average of 19-04 (7) of signed sums of the field along paths.

**Linearisation theorem** Lean: `BlockLinearKernel.blockMap_exp_linear_le_one`. If the threshold is $0 < r_0 \le 1$ and $\varepsilon M \le 1/32$, then for every coarse link $b$

$$\left| B(e^{\varepsilon a})_b - \exp\big(\varepsilon\,(Ca)_b\big) \right| \;\le\; 257\,(\varepsilon M)^2, \quad (3)$$

and, for any real-linear map $f$ from quaternions to real numbers (such as taking one colour component), $f\big((Ca)_b\big)$ equals the value of the path-kernel operator (coefficients 19-04 (9), $X \mapsto 2X$) applied to $f \circ a$. The version with $r_0 = 1$ is `BlockLinearKernel.blockMap_exp_linear_r1`.

Meaning. When the field is small ($\varepsilon M$ small), the non-linear block map agrees, up to a second-order error, with the linear map $C$ on the Lie algebra wrapped in the exponential map. Moreover, $C$ is the same operator as the path kernel whose symbol was treated in 15-12. This is the ground for calling the linear objects of 15-12 and § 05 “the first-order part of the blocking”.

The route of the proof (each stage is a Lean lemma).

1. **Products along paths.** A product of $n$ factors with $|u_i| \le 1$, $|u_i - 1| \le \delta$ is within $n^2\delta^2/2$ of $1 + \sum_i (u_i - 1)$ (induction). Paths have length at most 8.
2. **The exponential.** For imaginary $v$ with $|v| \le 1$, $|e^v| = 1$, $|e^v - 1| \le |v|$, $|e^v - 1 - v| \le |v|^2$. A step in the reverse direction is $e^{-v}$, with the same estimates.
3. **The average.** The weights are non-negative and sum to 1, so the average has the same error. For a unit quaternion $u$, $\operatorname{Re}(u - 1) = -|u - 1|^2/2$, so the real part of the first-order part is of second-order size.
4. **The retract.** If $0 < r_0 \le 1$, $|v| \le s \le \frac{1}{2}$, $|\operatorname{Re} v| \le \rho$, then $|\rho_{r0}(1 + v) - (1 + v)| \le 2\rho + s^2$. The retract does not change the first-order part.

5. **Combine.** Adding the error terms ($72 + 81$ from the retract, $32$ from the average, $8$ from the first-order part, $64$ from the exponential) gives $257$ times $(\varepsilon M)^2$. The agreement of the first-order part is an identity that rearranges the sums over paths into the table of coefficients, colour by colour.

Numerically the ratio $|B - \exp(\varepsilon C a)|/(\varepsilon M)^2$ is 0.3–0.9 Reading , so the constant 257 is several hundred times loose. For $\varepsilon M \le 1/32$ the right-hand side is $257/1024 < 2$ (the trivial upper bound for the difference of two unit quaternions), so the statement is not empty.

A caution. This is only first order around $U = 1$. Configurations whose field size $\varepsilon M$ exceeds $1/32$, and expansion around a background field $U \neq 1$, are not treated.

04

## The covariance of the push-forward $= CTC^\dagger$

As in 19-04 (11), the covariance form of a measure $\mu$ on a real inner product space is $\mathrm{cov}_\mu(u,v) = \int \langle u,x\rangle\langle v,x\rangle\, d\mu - \int \langle u,x\rangle\, d\mu \int \langle v,x\rangle\, d\mu$. An operator $T$ with $\mathrm{cov}_\mu(u,v) = \langle u, Tv\rangle$ is called the covariance operator.

**Push-forward covariance theorem** Lean: `PushCov.lattice_pushCov`. For any measure $\mu$ on the space of real fields (one colour) on the links of the fine lattice, with covariance operator $T$: (a) the one-colour first-order part $C_1$ of the block map (`PushCov.blockLin`) equals the path-kernel operator, and (b)

$$\mathrm{cov}_{(C_1)_*\mu}(u,v) = \big\langle u,\ C_1\, T\, C_1^\dagger\, v\big\rangle \qquad (\text{for all } u,v). \tag{4}$$

Meaning. When a field is pushed by a linear map, the covariance is sandwiched by the map from both sides. Gaussianity is not used. The measure need not be a probability measure; it is an identity that holds for any measure (the general form is `PushCov.covOp_map`). The proof is only $\langle u, C_1 x\rangle = \langle C_1^\dagger u, x\rangle$ and the change of variables in the integral of the push-forward Proved here .

A caution. Under Lean's convention for integrals, the integral of a non-integrable function is read as 0. (4) has meaning as a covariance when $\mu$ has second moments. Also, what is pushed here is the linear $C_1$; the covariance of a measure pushed by the non-linear $B$ does not take this form.

05

## The covariance of the plaquette field strength

The objects are on the lattice side of 19-04 § The multi-step family on the free-field side.

- $\mathrm{freeCov}_n$: the covariance of the free field in Feynman gauge for a four-component field on $(\mathbb{Z}/n)^4$. The inverse of the lattice Laplacian taken component by component, and 0 on the constant component.
- $\mathrm{stageCov}_{m,N}$: the $\mathrm{freeCov}$ on $(\mathbb{Z}/2^m N)^4$, pushed $m$ times by the one step $T \mapsto CTC^\dagger$ and brought down to $(\mathbb{Z}/N)^4$.
- $\mathrm{faceCov}_{m,N} = \mathrm{curl} \circ \mathrm{stageCov}_{m,N} \circ \mathrm{curl}^\dagger$: the covariance of the plaquette field strength $(\mathrm{curl}\, A)_{(a,b)}(X) = (A_a(X+e_b) - A_a(X)) - (A_b(X+e_a) - A_b(X))$. The plaquette field has six components at each point.

For a plaquette field $z$, set the quadratic form $Q(z) = \mathrm{Re} \sum_X \sum_p z_p(X)\, (\mathrm{faceCov}\, z)_p(X)$, the norm $|z|^2 = \sum_X \sum_p |z_p(X)|^2$, the largest eigenvalue $\lambda_{\max} = \sup_{z \neq 0} Q(z)/|z|^2$, and the diagonal at the point $X_0$ and plaquette $p$, $D(X_0, p) = \mathrm{Re}(\mathrm{faceCov}\, \delta_{X_0,p})_p(X_0)$.

**Plaquette covariance theorem** Lean: `LatticeFace.lattice_face_T1h` (the only assumption is $N \geq 2$). At every step $m \geq 0$ and for every $N \geq 2$, $\mathrm{faceCov}_{m,N}$ is self-adjoint, and for every point $X_0$ and plaquette $p$

$$\lambda_{\max} \leq 1, \qquad \frac{3}{200}\left(1 - N^{-4}\right) \leq D(X_0, p) \leq \lambda_{\max}, \qquad \Lambda := \frac{\lambda_{\max}}{D(X_0, p)} \leq \frac{640}{9}. \tag{5}$$

Furthermore, $\mathrm{faceCov}_{m,N}$ equals, up to relabelling of points, the translation-invariant operator built from the symbol $G_m$ of 15-12 (the second clause of the same theorem).

Meaning. However many steps the free field is pushed by the linearised blocking, the fluctuation of the plaquette field strength does not exceed 1 in any direction, and does not fall below $\frac{3}{200}(1 - N^{-4})$ at any point or plaquette. The upper bound $640/9$ on the ratio $\Lambda$ depends neither on the step $m$ nor on $N$. At $m = 0$ (the free field, not pushed) $\lambda_{\max} = 1$ exactly Reading .

This theorem is a statement about **the covariance of the plaquette field strength of the free field pushed by the linearised blocking**. It is not the covariance of the Wilson measure or the free field pushed by the non-linear block map $B$. Matrices built numerically and directly from the position-space definitions agree with the operators built from the symbols to within $1.1 \cdot 10^{-15}$ for $N = 2, 3, 4$ and $m = 0, 1, 2$ Reading .

06

## The route of the proof for the plaquette covariance

1. **The symbol at step 0.** $\mathrm{freeCov}_n$ is diagonal in characters, with symbol $I/\omega$ (`LatticePush.freeCov_phase`; $|1-e^{iq}|^2 = 4\sin^2(q/2)$).
2. **The chain of $m$ steps.** If the symbol of an operator $T$ is $X$, the symbol of the operator pushed $m$ times is $\mathcal{R}^m X$ (`LatticePush.push_phase`; the one step `LatticeTorus.torus_step` of 15-12 applied $m$ times, relabelling the type of the lattice size). The symbol of the stage covariance is $\mathcal{R}^m X_0$ (`LatticePush.stageCov_phase`), and its physical part is the $G_m$ of 15-12 (`LatticePush.projFam_stage`).
3. **The plaquette symbol.** Multiplying by the symbol of curl from both sides removes the gradient direction, and the projection $P$ enters automatically (`LatticeFace.key_proj`). So the symbol of the plaquette covariance is $\hat{d}\, G_m\, \hat{d}^\dagger$, and the operator returned to position space equals the operator built from the symbol (`LatticeFace.faceCov_eq_posC`).
4. **The upper side.** From the upper side of 15-12, $G_m \preccurlyeq P/\omega$, the plaquette symbol at each momentum is at most $\hat{d}\,(P/\omega)\,\hat{d}^\dagger$, whose largest eigenvalue is 1. By Plancherel, $\lambda_{\max} \le 1$ (`FaceTorus.torus_face_upper`).
5. **The diagonal.** Putting the lower side of 15-12, $G_m \succcurlyeq \frac{3}{100} P/\omega$, into the sum over momenta gives diagonal $\ge \frac{3/100}{2}(1-N^{-4})$ (`FaceTorus.torus_face_diag`).
6. **The ratio.** For $N \ge 2$, $1-N^{-4} \ge 15/16$, so diagonal $\ge \frac{3}{200}\cdot\frac{15}{16} = \frac{9}{640}$; together with $\lambda_{\max} \le 1$, $\Lambda \le 640/9$ Proved here .

In the chain, stages 1–3 connect position space and symbols; 1 and 2 are in Lean for the stage covariance itself, and 3 for the plaquette field. Stages 4 and 5 use the two sides of 15-12 as they are.

### Detour: the $\frac{1}{2}$ and the $1-N^{-4}$ in the lower bound on the diagonal

You may skip this. The diagonal at the point $X_0$ and plaquette $p=(a,b)$ is the average over momenta $k \in (\mathbb{Z}/N)^4$, $N^{-4}\sum_k \langle e_p, \hat{d}(k) G_m(k) \hat{d}(k)^\dagger e_p\rangle$. At the zero momentum $\hat{d}=0$ and the term is 0, so the sum has $N^4-1$ terms; the factor $1-N^{-4}$ comes from here Proved here . By the lower side each term is at least $\frac{3}{100}(|\hat{q}_a|^2+|\hat{q}_b|^2)/\omega$, and the sum over momenta of $|\hat{q}_\nu|^2/\omega$ is $(N^4-1)/4$ independently of the direction $\nu$ (the sum over the four directions is $N^4-1$; Lean: `FaceTorus.sum_rt_quarter`). A plaquette has two directions, so the average becomes $\frac{1}{2}$ Proved here .

07

## What is not said

- **The covariance of non-linear blocking.** § 05 is the covariance of the field pushed by the linearised blocking (the path kernel). Agreement with the non-linear $B$ is only up to the first-order part of § 03; the contribution of terms of second and higher order to the covariance is not treated.
- **Anything other than the free field.** § 05 is only the Gaussian free field (Feynman gauge, with the constant component removed). It is not the plaquette covariance of the field obtained by pushing the Wilson measure. The identity of § 04 holds for any measure, but the theorems that estimate $T$ are only about the free-field $\mathbf{freeCov}$.
- **Backgrounds and large fields.** The linearisation is only around $U = 1$, with $\varepsilon M \leq 1/32$. Linearisation around a background field, and events of large fields below the threshold, are not treated (an estimate at one scale of the rarity of large fields is in 22-01).
- **The measure at a coarse scale.** Nothing is said about the properties of $B_*\mu$ (decay, uniformity). That belongs to the statements to be shown (19-07, 28-04).
- **Choice of weights.** The radius 2, the constant 257 and $640/9$ were computed with the weights $(1/8, 3/4, 1/8)$ and the family of paths with factor 2 (28-05).

Locality, gauge covariance and linearisation are formalisations, for this family of paths, of properties that are standardly asked of the blocking of gauge fields Cited . The covariance of the push-forward is a formalisation of the general identity for the push-forward of a measure by a linear map Cited .

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| Locality (radius 2) and gauge covariance ($r_0 > 0$) of the non-linear block map. Lean: `BoxWalk.boxMap_local`, `BoxWalk.boxMap_gauge` | Further properties, such as constants of continuity |
| For $0 < r_0 \leq 1$ and $\varepsilon M \leq 1/32$, $\lvert B(e^{\varepsilon a})_b - \exp(\varepsilon (Ca)_b)\rvert \leq 257(\varepsilon M)^2$, and the first-order part is the path kernel. Lean: `BlockLinearKernel.blockMap_exp_linear_le_one` | Around a background $U \neq 1$, large fields, the form of the second-order terms |
| For any measure, the covariance of the push-forward $= C_1 T C_1^\dagger$, where $C_1$ is the path kernel. Lean: `PushCov.lattice_pushCov` | The covariance of the push-forward by the non-linear $B$. Only the one-colour form |

| Can say | Cannot say |
|---|---|
| The plaquette covariance of the free field pushed $m$ times by the linearised blocking: self-adjoint, $\lambda_{\max} \le 1$, diagonal $\ge \frac{3}{200}(1 - N^{-4})$, $\Lambda \le 640/9$ (all $m$, $N \ge 2$). Lean: `LatticeFace.lattice_face_T1h` | The covariance of non-linear blocking (agreement only up to the first-order part). The field obtained by pushing the Wilson measure. Background fields |
| Proved here For $N \ge 2$, $\Lambda \le 640/9$ from $\frac{3}{200}(1 - N^{-4}) \ge 9/640$ | — |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Building the blocking of a gauge field by averaging products along paths, and that locality, gauge covariance and linearisation are asked of it | Cited | The sources of 7-05 (T. Bałaban, Comm. Math. Phys. 102 (1985) 255–275. J. Dimock, The renormalization group according to Balaban I, arXiv:1108.1335) |
| The theorems of § 02– § 05 | — | Lean declarations of this book's study (names in the text and the scope table; no sorry, the three standard axioms, checked by a separate re-run). The list is in 15-14 |
| $\lambda_{\max} = 1$ at $m = 0$; agreement of the position-space matrices with the symbol operators ($1.1 \cdot 10^{-15}$) | Reading | The numerical record of this book's study (matrices built directly from the position-space definitions, compared with the operators built from the symbols, for 7 pairs with $N = 2, 3, 4$ and $m = 0, 1, 2$) |
| The ratio 0.3–0.9 of the linearisation error | Reading | The numerical record of this book's study ($L = 1, 2$, random fields, $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$, $r_0 = 1$ and $r_0 \le \frac{1}{2}$. The difference between the first-order part built from sums over paths and the path-kernel operator is $\le 6.4 \cdot 10^{-15}$) |

**Terms used in this chapter** — block map, retract, locality, gauge covariance, path kernel, linearisation, push-forward, covariance operator, plaquette field strength, plaquette covariance, Rayleigh quotient. Their meanings are in A-02 Glossary.

Next chapter to read: 15-14 What the second half (15-11 to 15-13) says, and what it does not. To read alongside: 22-01 Rarity of large plaquettes — the chessboard estimate — events of large fields, outside the linearisation, are estimated for the Wilson measure at one scale.

# What the second half (15-11 to 15-13) says, and what it does not

This chapter is this book's own investigation.

This part is a record of how this book's study attempted ultraviolet control (the existence of the continuum limit) with a design different from that of the published construction. For the existence itself, see Parts 6–13 of Volume 2; for the difference between the two designs, see Part 14. The place of this part is explained in 15-00.

**Scope** — The theorems of the second half of this part (15-11 to 15-14), and the rarity of large plaquettes that moved to Part 22 (22-01), are either statements about the free field, the vacuum background and the linearised blocking, or statements about the Wilson measure at one scale (with the lattice spacing fixed). They say nothing about the renormalised coarse-scale measure, fields with a background, or estimates uniform across steps. The weights $(1/8, 3/4, 1/8)$ are a design choice (28-05).

**What this chapter shows** — We view the theorems of the second half of this part and the theorem of 22-01 in one table. For each row we give a short form of the statement, its scope, what it does not say, and its Lean declarations. This table deals only with the theorems of the second half of the part. What the first half of the part (15-00 to 15-10) cannot say, and the index of questions, are in 15-09.

**Chapters this one relies on** — 15-12, 15-13, 22-01 (the detailed form of the rows of the table, and the routes of the proofs).

**Said first** — The scope is as written directly under the title. The “statement” column of the table is written in short form; the exact forms are in the formulas of each chapter. Every theorem is a formalisation, for the objects of this book, of a result of a known type. Estimates across steps are not in the chapters of the second half; they are on the side of the premises ( § 03).

Order of this chapter

1. The table of theorems — statement, scope, what is not said, Lean declarations
2. Lemmas used as parts — those not made into stand-alone theorems
3. Estimates across steps are on the side of the premises — where what the chapters of the second half do not say is placed

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01

## The table of theorems

The notation is as in each chapter: $G_k$ is the physical part of the symbol recursion (15-12 § 01), $H_k$ its inverse on $\operatorname{ran} P$, $\mathcal{M}$ the one-step map, $B$ the non-linear block map, $C$ the path kernel, $\mu_W$ the Wilson measure, and $Z$ its total mass.

| Chapter | Name | Statement (short form) | Scope | What is not said | Lean declarations |
|---|---|---|---|---|---|
| 15-12 § 02 | Upper side | $G_k \preccurlyeq P/\omega$, constant exactly 1 | Every step $k$, all $q \in \mathbb{R}^4$ with $\omega(q) > 0$ | Non-linear blocking, background fields, the form as operators in position space | `OSUpQpos.os_up_C` |
| 15-12 § 03 | Lower side | $\frac{3}{100} P/\omega \preccurlyeq G_k$ | As above (families closed under aliases) | A constant close to the true infimum, about 0.091 Reading | `HsSplit.lower_si` |
| 15-12 § 04 | Propagator sandwich | $(\omega/745)P \preccurlyeq \frac{1}{4} H_k \preccurlyeq 312\,\omega P$ (in the split version the upper side is $\frac{25}{3}\omega P$) | As above; constants uniform in $k$ and $q$ | Reading $\frac{1}{4}H_k$ as the stage action (on paper). The complex form of the inequalities. The constants are 3 to 186 times looser than the true values Reading | `LatticeRec.latti` `OneStepLower.bra` `NearAliasB.brack` |

| Chapter | Name | Statement (short form) | Scope | What is not said | Lean declarations |
|---|---|---|---|---|---|
| 15-12 § 05 | One-step contraction | $\mathcal{M}P \preccurlyeq \frac{3}{5}P$ | All $q$ | The true supremum $125/256$ Reading | `OneStepBridge.Mf` |
| 15-12 § 06 | Contraction uniform in the step | $\mathcal{M}^{17}(\omega G_k) \preccurlyeq \tau\, \omega G_{k+17}$, $\tau < 1/170$ | Every step $k$, all $q$ | $\tau \le 31/32$ at step 0 (the assembly is on paper). Identification with the per-step contraction rate used in the argument on paper (not in Lean) | `TorusLam.stage_t` |
| 15-12 § 07 | Convergence of $\Gamma_k$ | Any finite quadratic functional of $\Gamma_k$ converges at rate $(3/5)^k$ | Every step; constants uniform in the step and the momentum | Convergence of the one-loop coupling $b_k$ (the identification is on paper). $\kappa^2 \approx 1.4 \cdot 10^{13}$ Reading | `AliasPoinc.coupl` |
| 15-13 § 02 | Locality, gauge covariance | $B(U)_b$ is determined by the links in a cube of radius 2. $B(gU)_b = g(2y)B(U)_b g(2(y+e_\mu))^*$ | $r_0 > 0$, $\lvert g\rvert = 1$ | Anything further, such as constants of continuity | `BoxWalk.boxMap_l` |
| 15-13 § 03 | Linearisation | $\lvert B(e^{\varepsilon a})_b - \exp(\varepsilon (Ca)_b)\rvert \le 257(\varepsilon M)^2$; the first-order part is the path kernel | $0 < r_0 \le 1$, $\varepsilon M \le 1/32$ | Around a background $U \neq 1$, large fields, the form of the second-order terms | `BlockLinearKerne` |
| 15-13 § 04 | Covariance of the push-forward | $\mathrm{cov}(C_1)_{*}\mu = C_1 T C_1^{\dagger}$ | Any measure (meaningful when second moments exist), one colour | Push-forward by the non-linear $B$ | `PushCov.lattice_` |

| Chapter | Name | Statement (short form) | Scope | What is not said | Lean declarations |
|---|---|---|---|---|---|
| 15-13 § 05 | Plaquette covariance | Self-adjoint, $\lambda_{\max} \le 1$, diagonal $\ge \frac{3}{200}(1 - N^{-4})$, $\Lambda \le 640/9$ | Every step $m$, $N \ge 2$. The free field pushed by the linearised blocking | The covariance of non-linear blocking (agreement only up to the first-order part). The field obtained by pushing the Wilson measure | `LatticeFace.latt` |
| 22-01 § 03 | Chessboard estimate | $\lvert V(F)\rvert^{N^4} \le \prod_w V(\text{constant})$ | $N = 2m$ ($m \ge 1$), every real $\beta$, functions of a unit cube | Odd sides, measures with a twist | `FoldWilson.wilsc` |
| 22-01 § 04 | Rarity uniform in the volume | $P(X\text{: all its plaquettes have } \mathrm{Re}\, U_p < 0) \le q_1^{\lfloor \lvert X\rvert/6 \rfloor}$, $q_1 < 1$ | $\beta \ge 81$, $m \ge 2$, any set $X$ of plaquettes. $q_1$ independent of $N$ and $X$ | The measure at a coarse scale, $\beta < 81$, $m = 1$. The threshold 81 comes from a crude lower bound on the volume of the cap (with the true $h$ about 48 Reading ). Ratios to anything other than $Z$ | `FoldCap.rarity_e` |
| 22-01 § 06 | Rarity with rate $1/4$ | $\le q_4^{\lfloor \lvert X\rvert/24 \rfloor}$ | $0 < y \le 2$, at or above some existing $\beta_0$, $m \ge 1$ | The value of $\beta_0$ (about 357 at $y = 1$ Reading ) | `FoldQuarter.rari` |

What no row of the table says: the limit in which the lattice spacing is sent to 0, the renormalised measure at a coarse scale, fields with a background, and nonlinear estimates uniform across steps. The momenta in the rows of 15-12 are the continuous momenta of infinite volume. The links to position space (that the physical part of the symbol of the stage covariance is $G_k$, and the covariance of the plaquette field) are in Lean, but there is no theorem stating the inequalities for $G_k$ as inequalities between operators in position space. The constants computed with the weights $(1/8, 3/4, 1/8)$ and factor 2 cannot be used as they are for another choice (28-05).

02

## Lemmas used as parts

The following declarations are closed in Lean, but are not set out as stand-alone theorems; they were mentioned in each chapter as parts of the proofs of the rows above.

| Content | Row that uses it | Lean declarations |
|---|---|---|
| The upper bound for an explicit function $E_\nu < 4$ ($\sigma \in [0, \frac{1}{2}]^4 \setminus \{0\}$, $\sigma_\mu = \sin^2(q_\mu/4)$), invariance under reflection | Upper side | `FreeStageOS.E_lt_four_s`, `FreeStageOS.E_lt_four`, `FreeStageOS.eq_lt_four`, `OSUpQpos.EE_bbflip` |
| The lower bounds from the chain of nearest aliases and from the sixteen aliases, the case split | Lower side | `NearAliasB.chain_lower`, `NearAliasB.near_lower`, `HsSplit.hS_split` |
| Correspondence between the symbol recursion and $\mathcal{M}$, the symbol of the path average, the symbol of the one-step torus operator | All of 15-12 | `LatticeRec.rec_iterate`, `LatticePath.pathSym_closed`, `LatticeTorus.torus_step` |
| The one-step difference $\pm 2P$ | One-step contraction | `CouplingLimit.D_lower`, `CouplingLimit.D_upper` |
| The size of the lift, the Poincaré input on the sum of aliases | Convergence of $\Gamma_k$ | `AliasLiftSize.hlift_holds`, `AliasPoinc.hH_holds` |
| Agreement of the plaquette covariance with the symbol operator, the chain of steps | Plaquette covariance | `LatticeFace.faceCov_eq_posC`, `LatticePush.freeCov_phase`, `LatticePush.push_phase`, `LatticePush.stageCov_phase`, `LatticePush.projFam_stage` |
| The lower bound on the partition function, the lower bound on the volume of the cap | Rarity uniform in the volume | `FoldEntropy.Zw_ge`, `FoldCap.ballQ_ge` |

03

## Estimates across steps are on the side of the premises

What the chapters of the second half say concerns only one step (or a stack of linear steps) and one scale. To close the chain of renormalisation, one needs estimates uniform across steps for the measure at a coarse scale pushed forward by the nonlinear block map (contraction on the small-field side, rarity on the large-field side, bracketing of the propagator for fields with a background, and so on). These do not follow from the theorems of the second half. In this book they are placed, with names,

in the list of premises (28-04), and the places where their form is not yet settled are placed in 28-06.

Here a “premise” is a statement about this book's design (the measure pushed forward by a fixed block map), and it does not follow from the results of the published construction (15-00). The theorems of the second half are formalisations of “the parts that can be checked with the free field, linearisation and one scale” among the statements on the side of the premises. They are not the statements of the premises themselves.

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## What this chapter can say

| Can say | Cannot say |
|---|---|
| The 13 rows of the table in § 01 (with Lean declarations; the scope is in each row) | The “what is not said” column of each row. In particular, the measure at a coarse scale, fields with a background, and non-linear estimates uniform across steps |

## Sources and reproduction

| Item | Kind | Source / tools |
|---|---|---|
| Each row of the table | — | Lean declarations of this book's study (the table of sources and reproduction in each chapter). Sources for the known types are in 15-12, 15-13 and 22-01 |
| Numbers marked Reading (0.091, 125/256, $\kappa^2$, $2.5 \cdot 10^{12}$, 48, 357) | Reading | The numerical records in the table of sources and reproduction of each chapter |

**Terms used in this chapter** — symbol recursion, physical part, block map, path kernel, push-forward, plaquette covariance, chessboard estimate, rarity, premise. Their meanings are in A-02 Glossary.

Next chapter to read: 16-01 How far the world has come. To read alongside: 28-00 Entrance — the statements to be shown, and sufficient conditions through a representation — the estimates across steps that the chapters of the second half do not state are placed there, with names, as premises.